ISOPERIMETRIC INEQUALITIES FOR THE HANDLEBODY GROUPS

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Abstract. We show that the mapping class group of a handlebody V of genus at least 2 has a Dehn function of at most exponential growth type.

1. Introduction

A handlebody of genus $g \geq 2$ is a compact orientable 3-manifold V whose boundary ∂V is a closed surface of genus g. The *handlebody group* $\mathrm{Map}(V)$ is the group of isotopy classes of orientation preserving homeomorphisms of V. Via the natural restriction homomorphism, the group $\mathrm{Map}(V)$ can be viewed as a subgroup of the mapping class group $\mathrm{Map}(\partial V)$ of ∂V. This subgroup is of infinite index, and it surjects onto the outer automorphism group of the fundamental group of V which is the free group with g generators.

The handlebody group is finitely presented. Thus $\mathrm{Map}(V)$ can be equipped with a word norm that is unique up to quasi-isometry. Hence, the handlebody group carries a well-defined large-scale geometry. However, this large scale geometry is not compatible with the large-scale geometry of the ambient group $\mathrm{Map}(\partial V)$. Namely, we showed in [HH11] that the handlebody group is an exponentially distorted subgroup of the mapping class group of the boundary surface for every genus $g \geq 2$. Here, a finitely generated subgroup $H < G$ of a finitely generated group G is called *exponentially distorted* if the following holds. First, the word norm in H of every element $h \in H$ can be bounded from above by an exponential function in the word norm of h in G. On the other hand, there is no such bound with sub-exponential growth rate.

As a consequence, it is not possible to directly transfer geometric properties from the mapping class group to the handlebody group. In

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this note we initiate an investigation of the intrinsic large-scale geometry of the handlebody group.

A particularly useful geometric invariant of a finitely presented group G is its *Dehn function*, which can be defined as the isoperimetric function of a presentation complex for G (see Section 4 for a complete definition). Although the Dehn function itself depends on the choice of a finite presentation of G, the growth type of the Dehn function does not. In fact, the growth type of the Dehn function is a quasi-isometry invariant of G.

The mapping class group $\mathrm{Map}(\partial V)$ of ∂V is automatic [Mo95] and hence has quadratic Dehn function. Since $\mathrm{Map}(V)$ is exponentially distorted in $\mathrm{Map}(\partial V)$, this fact does not provide any information on the Dehn function of $\mathrm{Map}(V)$. On the other hand, for $g \geq 3$ the Dehn function of the outer automorphism group $\mathrm{Out}(F_g)$ of a free group F_g on g generators is exponential [HV96, BV95, BV10, HM10]. However, since the kernel of the projection from the handlebody group to $\mathrm{Out}(F_g)$ is infinitely generated [McC85], this fact also does not restrict the Dehn function of $\mathrm{Map}(V)$.

The goal of this note is to give an upper bound for the Dehn function of $\mathrm{Map}(V)$. We show

Theorem 1.1. *The handlebody group* $\mathrm{Map}(V)$ *satisfies an exponential isoperimetric inequality, i.e. the growth of its Dehn function is at most exponential.*

The strategy of proof for Theorem 1.1 is similar to the strategy employed in [HV96] to show an exponential upper bound for the Dehn function of outer automorphism groups of free groups. We construct a graph which is a geometric model for the handlebody group (a similar construction is used in [HH11] in order to show exponential distortion of handlebody groups). Vertices of this graph correspond to isotopy classes of special cell decompositions of ∂V containing the boundary of a *simple disk system* in their one-skeleton. A simple disk system is a collection of pairwise disjoint, pairwise non-homotopic embedded disks in V which decompose V into simply connected regions. We then use a a surgery procedure for disk systems to define a distinguished class of paths in this geometric model. Although these paths are in general not quasi-geodesics for the handlebody group (see the example at the end of this note), they are sufficiently well-behaved so that they can be used to fill a cycle with area bounded by an exponential function in the length of the cycle.

The organization of this note is as follows. In Section 2 we introduce disk systems and special paths in the disk system graph. Section 3 discusses a geometric model for the handlebody group. This model is used in Section 4 for the proof of Theorem 1.1.

2. Disk exchange paths

In this section we collect some facts about properly embedded disks in a handlebody V of genus $g \geq 2$. In particular, we describe a surgery procedure that is central to the construction of paths in the handlebody group.

A disk D in V is called *essential* if it is properly embedded and if ∂D is an essential simple closed curve on ∂V. A *disk system* for V is a set of pairwise disjoint essential disks in V no two of which are homotopic. A disk system is called *simple* if all of its complementary components are simply connected. It is called *reduced* if in addition it has a single complementary component.

We usually consider disks and disk systems only up to proper isotopy. Furthermore, we will always assume that disks and disk systems are in minimal position if they intersect. Here we say that two disk systems $\mathcal{D}_1, \mathcal{D}_2$ are in *minimal position* if their boundary multicurves intersect in the minimal number of points in their respective isotopy classes and if every component of $\mathcal{D}_1 \cap \mathcal{D}_2$ is an embedded arc in $\mathcal{D}_1 \cap \mathcal{D}_2$ with endpoints in $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$. Note that minimal position of disks is not unique; in particular the intersection pattern $\mathcal{D}_1 \cap \mathcal{D}_2$ is not determined by the isotopy classes of $\mathcal{D}_1$ and $\mathcal{D}_2$.

The following easy fact will be used frequently throughout the article.

Lemma 2.1. *The handlebody group acts transitively on the set of isotopy classes of reduced disk systems. Every mapping class of ∂V that fixes the isotopy class of a simple disk system is contained in the handlebody group.*

Proof. The first claim follows from the fact that the complement of a reduced disk system in V is a ball with $2g$ spots and any two such manifolds are homeomorphic. The second claim is immediate since every homeomorphism of the boundary of a spotted ball extends to the interior. □

Let $\mathcal{D}$ be a disk system. An *arc relative to* $\mathcal{D}$ is a continuous embedding $\rho : [0, 1] \to \partial V$ whose endpoints $\rho(0)$ and $\rho(1)$ are contained in $\partial\mathcal{D}$. An arc ρ is called *essential* if it cannot be homotoped into $\partial\mathcal{D}$ with fixed endpoints. In the sequel we always assume that arcs are essential and that the number of intersections of ρ with $\partial\mathcal{D}$ is minimal in its isotopy class.

Choose an orientation of the curves in $\partial\mathcal{D}$. Since ∂V is oriented, this choice determines a left and a right side of a component α of $\partial\mathcal{D}$ in a small annular neighborhood of α in ∂V. We then say that an endpoint $\rho(0)$ (or $\rho(1)$) of an arc ρ *lies to the right (or to the left) of* α, if a small neighborhood $\rho([0,\epsilon])$ (or $\rho([1-\epsilon,1])$) of this endpoint is contained in the right (or left) side of α in a small annulus around α. A *returning arc relative to* $\mathcal{D}$ is an arc both of whose endpoints lie on the same side of the boundary ∂D of a disk D in $\mathcal{D}$, and whose interior is disjoint from $\partial\mathcal{D}$.

Let E be a disk which is not disjoint from $\mathcal{D}$. An *outermost arc* of ∂E relative to $\mathcal{D}$ is a returning arc ρ relative to $\mathcal{D}$, with endpoints on the boundary of a disc $D \in \mathcal{D}$, such that there is a component E' of $E \setminus \mathcal{D}$ whose boundary is composed of ρ and an arc $\beta \subset D$. The interior of β is contained in the interior of D. We call such a disk E' an *outermost component* of $E \setminus \mathcal{D}$.

For every disk E which is not disjoint from $\mathcal{D}$ there are at least two distinct outermost components E', E'' of $E \setminus \mathcal{D}$. There may also be components of $\partial E \setminus \mathcal{D}$ which are returning arcs, but not outermost arcs. For example, a component of $E \setminus \mathcal{D}$ may be a rectangle bounded by two arcs contained in $\mathcal{D}$ and two subarcs of ∂E with endpoints on $\partial\mathcal{D}$ which are homotopic to a returning arc relative to $\partial\mathcal{D}$.

Let now $\mathcal{D}$ be a simple disk system and let ρ be a returning arc whose endpoints are contained in the boundary of some disk $D \in \mathcal{D}$. Then $\partial D \setminus \{\rho(0), \rho(1)\}$ is the union of two (open) intervals γ_1 and γ_2. Put $\alpha_i = \gamma_i \cup \rho$. Up to isotopy, α_1 and α_2 are simple closed curves in ∂V which are disjoint from $\mathcal{D}$ (compare [St99] for this construction). Therefore both α_1 and α_2 bound disks in the handlebody which we denote by Q_1 and Q_2. We say that Q_1 and Q_2 are obtained from D by *simple surgery along the returning arc* ρ.

The following observation is well known (compare [HH11], [M86, Lemma 3.2], or [St99]).

Lemma 2.2. *If Σ is a reduced disk system and ρ is a returning arc with endpoints on $D \in \Sigma$, then for exactly one choice of the disks Q_1, Q_2 defined as above, say the disk Q_1, the disk system obtained from Σ by replacing D by Q_1 is reduced.*

The disk Q_1 is characterized by the requirement that the two spots in the boundary of $V \setminus \Sigma$ corresponding to the two copies of D are contained in distinct connected components of $H \setminus (\Sigma \cup Q_1)$. It only depends on Σ and the returning arc ρ. We call the interval γ_1 used in the construction of the disk Q_1 the *preferred interval defined by the returning arc.*

Definition 2.3. Let Σ be a reduced disk system. A *disk exchange move* is the replacement of a disk $D \in \Sigma$ by a disk D' which is disjoint from Σ and such that $(\Sigma \setminus D) \cup D'$ is a reduced disk system. If D' is determined as in Lemma 2.2 by a returning arc of a disk in a disk system $\mathcal{D}$ then the modification is called a *disk exchange move of* Σ *in direction of* $\mathcal{D}$ or simply a *directed disk exchange move*.

A sequence (Σ_i) of reduced disk systems is called a *disk exchange sequence in direction of* $\mathcal{D}$ (or *directed disk exchange sequence*) if each Σ_{i+1} is obtained from Σ_i by a disk exchange move in direction of $\mathcal{D}$.

The following lemma is an easy consequence of the fact that simple surgery reduces the geometric intersection number (see [HH11] for a proof).

Lemma 2.4. *Let* Σ_1 *be a reduced disk system and let* $\mathcal{D}$ *be any other disk system. Then there is a disk exchange sequence* $\Sigma_1, \ldots, \Sigma_n$ *in direction of* $\mathcal{D}$ *such that* Σ_n *is disjoint from* $\mathcal{D}$.

To estimate the growth rate of the Dehn function of the handlebody group we will need to compare disk exchange sequences starting in disjoint reduced disk systems. This is made possible by considering another type of surgery sequence for disk systems, which we describe in the remainder of this section.

To this end, let $\mathcal{D}$ be any simple disk system and let ρ be a returning arc. A *full disk replacement defined by* ρ modifies a simple disk system $\mathcal{D}$ to a simple disk system $\mathcal{D}'$ as follows. Let $D \in \mathcal{D}$ be the disk containing the endpoints of the returning arc ρ. Replace D by both disks Q_1, Q_2 obtained from D by the simple surgery defined by ρ. The disks Q_1, Q_2 are disjoint from each other and from $\mathcal{D}$. If one (or both) of these disks is isotopic to a disk Q from $\mathcal{D} \setminus D$ then this disk will be discarded (i.e. we retain a single copy of Q; compare [Ha95] for a similar construction). We say that a sequence $(\mathcal{D}_i)$ is a *full disk replacement sequence in direction of* $\mathcal{D}$ (or *directed full disk replacement sequence*) if each $\mathcal{D}_{i+1}$ is obtained from $\mathcal{D}_i$ by a full disk replacement along a returning arc contained in $\partial\mathcal{D}$.

The following two lemmas relate full disk replacement sequences to disk exchange sequences. Informally, these lemmas state that every directed disk exchange sequence may be extended to a full disk replacement sequence, and conversely every full disk replacement sequence contains a disk exchange sequence. To make this idea precise, we use the following

Definition 2.5. Let $\mathcal{D}$ be an arbitrary disk system. Suppose that $\mathcal{D}_0, \dots, \mathcal{D}_n$ is a full disk replacement sequence in direction of $\mathcal{D}$ and that $\Sigma_1, \dots, \Sigma_k$ is a disk exchange sequence in direction of $\mathcal{D}$.

We say that the sequences $(\mathcal{D}_i)$ and (Σ_i) are *compatible*, if there is a non-decreasing surjective map $r : \{0, \dots, n\} \to \{1, \dots, k\}$ such that $\Sigma_{r(i)} \subset \mathcal{D}_i$ for all i.

Lemma 2.6. *Let Σ be a reduced disk system, let $\mathcal{D}$ be a simple disk system containing Σ and let $\mathcal{D} = \mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_m$ be a full disk replacement sequence in direction of a disk system $\mathcal{D}'$. Then there is a disk exchange sequence $\Sigma = \Sigma_0, \Sigma_1, \dots, \Sigma_u$ in direction of $\mathcal{D}'$ which is compatible with (D_i).*

Proof. We proceed by induction on the length of the full disk replacement sequence $(\mathcal{D}_i)$. If this length equals zero there is nothing to show. Assume that the claim holds true whenever this length does not exceed $m - 1$ for some $m > 0$.

Let $\mathcal{D}_0, \dots, \mathcal{D}_m$ be a full disk replacement sequence of length m and let $\Sigma \subset \mathcal{D}_0$ be a reduced disk system. Let $D \in \mathcal{D}_0$ be the disk replaced in the full disk replacement move connecting $\mathcal{D}_0$ to $\mathcal{D}_1$.

If $D \in \Sigma$ then for one of the two disks obtained from D by simple surgery, say the disk D', the disk system $\Sigma_1 = (\Sigma \setminus D) \cup D'$ is reduced. However, $\Sigma_1 \subset \mathcal{D}_1$ and the claim now follows from the induction hypothesis.

If $D \notin \Sigma$ then $\Sigma \subset \mathcal{D}_1$ by definition and once again, the claim follows from the induction hypothesis. □

Lemma 2.7. *Let $\Sigma_0, \dots, \Sigma_m$ be a disk exchange sequence of reduced disk systems in direction of a disk system $\mathcal{D}'$. Then for every simple disk system $\mathcal{D}_0 \supset \Sigma_0$ there is a full disk replacement sequence $\mathcal{D}_0, \dots, \mathcal{D}_k$ in direction of $\mathcal{D}'$ which is compatible with (Σ_i).*

Proof. We proceed by induction on the length m of the directed disk exchange sequence.

The case $m = 0$ is trivial, so assume that the lemma holds true for directed disk exchange sequences of length at most $m - 1$ for some $m \geq 1$. Let $\Sigma_0, \dots, \Sigma_m$ be a directed disk exchange sequence of length m. Suppose Σ_1 is obtained from Σ_0 by replacing a disk $D \in \Sigma_0$. Let ρ be the returning arc with endpoints on D defining the disk replacement, and let D_1 be the disk in Σ_1 which is the result of the simple surgery.

We distinguish two cases. In the first case, $\rho \cap \mathcal{D}_0 = \rho \cap D$. Then ρ is a returning arc relative to $\mathcal{D}_0$. Let $\mathcal{D}_1$ be the disk system obtained from $\mathcal{D}_0$ by the full disk replacement defined by ρ. One of the two disks obtained by simple surgery along ρ is the disk D_1 and hence $D_1 \in \mathcal{D}_1$.

The claim now follows from the induction hypothesis, applied to the disk exchange sequence $\Sigma_1, \dots, \Sigma_m$ of length $m-1$ and the simple disk system $\mathcal{D}_1$ containing Σ_1.

In the second case, the returning arc ρ intersects $\mathcal{D}_0 \setminus D$. Then $\rho \setminus (\mathcal{D}_0 \setminus D)$ contains a component ρ' which is a returning arc with endpoints on a disk $Q \in \mathcal{D}_0 \setminus \{D\}$. A replacement of the disk Q by both disks obtained from Q by simple surgery using the returning arc ρ' reduces the number of intersection components of $\rho \cap (\mathcal{D}_0 \setminus D)$. Moreover, the resulting disk system contains D. In finitely many surgery steps, say $s \geq 1$ steps, we obtain a simple disk system $\mathcal{D}_s$ with the following properties.

(1) $\mathcal{D}_s$ contains D and is obtained from $\mathcal{D}_0$ by a full disk replacement sequence.
(2) $\rho \cap (\mathcal{D}_s - D) = \emptyset$.

Define $r(i) = 0$ for $i = 0, \dots, s$, where r is the function required in the definition of compatibility. We now can use the procedure from the first case above, applied to $\Sigma_0, \mathcal{D}_s$ and ρ to carry out the induction step.

This completes the proof of the lemma. $\square$

3. The graph of rigid racks

The goal of this section is to describe a construction of paths in the handlebody group whose geometry is easy to control. A version of these paths was already used in [HH11] to establish an upper bound for the distortion of the handlebody group in the mapping class group.

The main objects are given by the following

Definition 3.1. A *rack* R in V is given by a reduced disk system $\Sigma(R)$, called the *support system* of the rack R, and a collection of pairwise disjoint essential embedded arcs in $\partial V \setminus \partial\Sigma(R)$ with endpoints on $\partial\Sigma(R)$, called *ropes*, which are pairwise non-homotopic relative to $\partial\Sigma(R)$. At each side of a support disk $D \in \Sigma(R)$, there is at least one rope which ends at the disk and approaches the disk from this side. A rack is called *large* if the set of ropes decomposes $\partial V \setminus \partial\Sigma(R)$ into simply connected regions.

We will consider racks up to an equivalence relation called "rigid isotopy" which is defined as follows.

Definition 3.2. i) Let R be a large rack. The union of the support system and the system of ropes of R defines the 1–skeleton of a cell decomposition of the surface ∂V which we call the *cell decomposition induced by R*.

ii) Let R and R' be racks. We say that R and R' are *rigidly isotopic* if there is an isotopy of ∂V which maps the support system of R to the support system of R' and defines an isotopy of the cell decompositions induced by R and R'.

In particular, if T is a simple Dehn twist about the boundary of a support disk of a rack R, then R and $T^n(R)$ are not rigidly isotopic for $n \geq 2$. This observation and the fact that the stabilizer in the mapping class group of a reduced disk system is contained in the handlebody group imply the following

Lemma 3.3. *The handlebody group acts on the set of rigid isotopy classes of racks with finite quotient and finite point stabilizers.*

For simplicity of notation, we call a rigid isotopy class of a large rack simply a *rigid rack*. Lemma 3.3 allows us to use rigid racks as the vertex set of a $\mathrm{Map}(V)$–graph. More precisely, we make the following

Definition 3.4. The *graph of rigid racks* $\mathcal{RR}_K(V)$ is the graph whose vertex set is the set of rigid racks. Two such vertices are joined by an edge if up to isotopy, the 1–skeleta of the cell decompositions induced by the racks intersect in at most K points.

It follows easily from Lemma 3.3 that the number K may be chosen in such a way that the graph $\mathcal{RR}_K(V)$ is connected. In Lemma 7.3 of [HH11] such a number $K > 0$ is constructed explicitly. In the sequel, we will always use this choice of K and suppress the mention of K from our notation. It then follows from Lemma 3.3 and the Svarc-Milnor lemma that the graph $\mathcal{RR}(V)$ is quasi-isometric to $\mathrm{Map}(V)$.

Next we construct a family of distinguished paths in the graph of rigid racks. The paths are inspired by splitting sequences of train tracks on surfaces. To this end, we first define a notion of "carrying" for racks.

Definition 3.5.
(1) A disk system $\mathcal{D}$ is *carried* by a rigid rack R if it is in minimal position with respect to the support system $\Sigma(R)$ of R and if each component of $\partial\mathcal{D} \setminus \partial\Sigma(R)$ is homotopic relative to $\partial\Sigma(R)$ to a rope of R.
(2) An embedded essential arc ρ in ∂V with endpoints on $\partial\Sigma(R)$ is *carried* by R if each component of $\rho \setminus \partial\Sigma(R)$ is homotopic relative to $\partial\Sigma(R)$ to a rope of R.
(3) A *returning rope* of a rigid rack R is a rope which begins and ends at the same side of some fixed support disk D (i.e. defines a returning arc relative to $\partial\Sigma(R)$).

Let R be a rigid rack with support system $\Sigma(R)$ and let α be a returning rope of R with endpoints on a support disk $D \in \Sigma(R)$. By Lemma 2.2, for one of the components γ_1, γ_2 of $\partial D \setminus \alpha$, say the component γ_1, the simple closed curve $\alpha \cup \gamma_1$ is the boundary of an embedded disk $D' \subset H$ with the property that the disk system $(\Sigma \setminus D) \cup D'$ is reduced.

A *split* of the rigid rack R at the returning rope α is any rack R' with support system $\Sigma' = (\Sigma(R) \setminus D) \cup D'$ whose ropes are given as follows.

(1) Up to isotopy, each rope ρ' of R' has its endpoints in $(\partial\Sigma(R) \setminus \partial D) \cup \gamma_1 \subset \partial\Sigma(R)$ and is an arc carried by R.
(2) For every rope ρ of R there is a rope ρ' of R' such that up to isotopy, ρ is a component of $\rho' \setminus \partial\Sigma(R)$.

The above definition implies in particular that a rope of R which does not have an endpoint on ∂D is also a rope of R'. Moreover, there is a map $\Phi : R' \to R$ which maps a rope of R' to an arc carried by R, and which maps the boundary of a support disk of R' to a simple closed curve γ of the form $\gamma_1 \circ \gamma_2$ where γ_1 either is a rope of R or trivial, and where γ_2 is a subarc of the boundary of a support disk of R (which may be the entire boundary circle). The image of Φ contains every rope of R.

We are now ready to recall the construction of a distinguished class of edge-paths in the graph of rigid racks from [HH11]. These paths are sufficiently well-behaved to yield some geometric control of the handlebody group.

For a reduced disk system Σ let $\mathcal{RR}(V, \Sigma)$ be the complete subgraph of $\mathcal{RR}(V)$ whose vertices are marked rigid racks with support system Σ.

Definition 3.6. Let $\mathcal{D}$ be a simple disk system. A *$\mathcal{D}$-splitting sequence* of racks is an edge-path R_i in the graph of rigid racks with the following properties.

i) There is a disk exchange sequence Σ_i in direction of $\mathcal{D}$ and a sequence of numbers $1 = r_1 < \cdots < r_k$ such that the support system of R_j is Σ_i for all $r_i \leq j \leq r_{i+1} - 1$. The sequence Σ_i is called the *associated disk exchange sequence*.
ii) For $r_i \leq j \leq r_{i+1} - 1$, the sequence R_j is a uniform quasi-geodesic in the graph $\mathcal{RR}(V, \Sigma_i)$.

Here and in the sequel, we say that a path is a *uniform* quasigeodesic if the quasigeodesic constants of the path depend only on the genus of the handlebody. Similarly, we say that a number is *uniformly bounded*, if there is a bound depending only on the genus of V.

We showed in [HH11] that any two points in the graph of rigid racks can be connected by a splitting sequence. More precisely, the proof of Theorem 7.9 of [HH11] yields

Theorem 3.7. *Let R, R' be two rigid racks. Then there is a disk system $\mathcal{D}$ depending only on the support system of R' with the following property. Let $\Sigma(R) = \Sigma_1, \Sigma_2, \dots, \Sigma_n$ be a disk exchange sequence in direction of $\mathcal{D}$ such that Σ_n is disjoint from $\mathcal{D}$.*

Then there is a splitting sequence connecting R to R' whose associated disk exchange sequence is (Σ_i). The length of such a sequence is bounded uniformly exponentially in the distance between R and R' in the graph of rigid racks.

In Section 2 we saw that $\mathcal{D}$–disk exchange sequences starting in disjoint reduced disk systems can be compared using full disk replacement sequences. In the rest of this section we develop a slight generalization of racks, which will allow to similarly compare $\mathcal{D}$–splitting sequences starting in adjacent vertices of $\mathcal{RR}(V)$.

Namely, define an *extended rack* R in the same way as a rack except that now the support system $\mathcal{D}(R)$ of R may be any simple disk system instead of a reduced disk system. The cell decomposition induced by an extended rack is defined in the obvious way, and similarly we can talk about rigid isotopies between extended racks. The rigid isotopy class of an extended rack is called a *rigid extended rack*.

Rigid extended racks can be used in the same way as racks to define a geometric model for the handlebody group.

Definition 3.8. The graph of rigid extended racks $\mathcal{RER}_K(V)$ is the graph whose vertex set is the set of large rigid extended racks. Two such vertices are connected by an edge of length one if up to isotopy the 1–skeleta of the cell decompositions induced by the corresponding vertices intersect in at most K points.

Again, the constant K is chosen in such a way that the graph of rigid extended racks is connected. We denote the resulting graph by $\mathcal{RER}(V)$. For future use, we choose the constant K big enough such that in addition the following holds. For a simple disk system $\mathcal{D}$ let $\mathcal{RER}(V, \mathcal{D})$ be the complete subgraph of $\mathcal{RER}(V)$ whose vertices are rigid extended racks with support system $\mathcal{D}$. We may choose K large enough such that for any simple disk system $\mathcal{D}$ the subgraph $\mathcal{RER}(V, \mathcal{D})$ is connected.

An analog of Lemma 3.3 holds for rigid extended racks as well, and implies that the handlebody group acts on the graph of rigid extended racks with finite quotient. Thus, the graph of rigid extended racks is quasi-isometric to the handlebody group. Note also that every large rack is a large extended rack. Thus the graph of rigid extended racks embeds as a subgraph in the graph of rigid extended racks. This inclusion is a quasi-isometry.

A *full split* of a rigid extended rack is defined as follows. Let R be a rigid extended rack and let α be a returning rope of R. A rigid extended rack R' is called a full split of R at α if the support system of R' is obtained from $\Sigma(R)$ by a full disk replacement along α. Moreover, we require that the ropes of R' satisfy the analogous conditions as the ropes of a split of a rigid rack.

The following is a natural generalization of splitting paths to extended racks.

Definition 3.9. Let $\mathcal{D}$ be a simple disk system. A *full $\mathcal{D}$-splitting sequence* of racks is an edge-path (R_i) in the graph of rigid extended racks with the following properties.

i) There is a full disk exchange sequence $(\mathcal{D}_i)$ in direction of $\mathcal{D}$ and a sequence of numbers $1 = r_1 < \cdots < r_k$ such that the support system of R_j is $\mathcal{D}_i$ for all $r_i \leq j \leq r_{i+1} - 1$. The sequence $(\mathcal{D}_i)$ is called the *associated full disk exchange sequence*.
ii) For $r_i \leq j \leq r_{i+1} - 1$, the sequence (R_j) is a uniform quasi-geodesic in the graph $\mathcal{RER}(V, \Sigma_i)$.

The proof of Theorem 7.9 of [HH11] implies the following theorem which allows to connect two rigid racks with a full splitting sequence.

Theorem 3.10. *There is a number k_1 with the following property. Let R, R' be two rigid racks. Then there is a simple disk system $\hat{\mathcal{D}}$ depending only on the support system of R' with the following property. Let $\mathcal{D}(R) = \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$ be a full disk exchange sequence in direction of $\hat{\mathcal{D}}$ such that $\mathcal{D}_n$ is disjoint from $\hat{\mathcal{D}}$.*

Then there is an extended rigid rack $\hat{R}$ which is at distance at most k_1 to R' in $\mathcal{RER}(V)$, and there is a full splitting sequence connecting R to $\hat{R}$ whose associated full disk replacement sequence is $(\mathcal{D}_i)$. The length of any such sequence is bounded by $e^{k_1 d}$, where d is the distance between R and R' in the graph of rigid extended racks.

Combining Lemmas 2.6 and 2.7 with Theorem 3.10 above, we obtain the following.

Corollary 3.11. *There is a number $k_2 > 0$ with the following property.*

i) *Let $(R_i), i = 1, \dots N$ be a $\mathcal{D}$–splitting sequence of racks with associated disk exchange sequence (Σ_j). Let $(\mathcal{D}_j)$ be a full disk replacement sequence compatible with (Σ_j). Then there is a full $\mathcal{D}$–splitting sequence $\widetilde{R}_k, k = 1, \dots K$ such that the following holds. The associated full disk replacement sequence to $(\widetilde{R}_k)$ is $\mathcal{D}_j$. Furthermore, $\widetilde{R}_1 = R_1$ and the distance between $\widetilde{R}_K$ and R_N is at most k_1. The length K of any such sequence is at most $e^{k_2 d}$, where d is the distance between R_0 and R_K in the graph of rigid racks.*

ii) *Conversely, suppose that $\widetilde{R}_k, k = 1, \dots, K$ is a full $\mathcal{D}$–splitting sequence with associated full disk replacement sequence $\mathcal{D}_j$. Suppose further that (Σ_j) is a disk exchange sequence compatible with $(\mathcal{D}_i)$. If $\widetilde{R}_1$ is a large rack, then there is a $\mathcal{D}$–splitting sequence $R_1, R_2, \dots, R_N$ whose associated disk exchange sequence is (Σ_j) such that R_N is of distance at most k_1 to $\widetilde{R}_K$. The length N of any such sequence is at most $e^{k_2 d}$, where d is the distance between R_0 and R_N in the graph of rigid racks.*

4. The Dehn function of the handlebody group

In this section we prove the main result of this note.

Theorem 4.1. *The Dehn function of the handlebody group has at most exponential growth rate.*

To begin, we recall the definitions of the Dehn function and growth rate. Let G be a finitely presented group. Choose a finite generating $\mathcal{S}$ and let $\mathcal{R}$ be a finite defining set of relations for G. This means the following. The set $\mathcal{R}$ generates a subgroup R_0 of the free group $F(\mathcal{S})$ with generating set $\mathcal{S}$. Denote by R the normal closure of R_0 in $F(\mathcal{S})$. The set $\mathcal{R}$ is called a defining set of relations for G if the quotient $F(\mathcal{S})/R$ is isomorphic to G.

Every $r \in R < F(\mathcal{S})$ can be written as a product of conjugates of elements in $\mathcal{R}$:

$$r = \Pi_{i=1}^n r_i^{\gamma_i}, \qquad r_i \in \mathcal{R}, \gamma_i \in G.$$

We call the minimal length n of such a product *the area* $\mathrm{Area}(r)$ *needed to fill the relation* r. On the other hand, r can be written as a word in the elements of $\mathcal{S}$. We call the minimal length of such a word the *the length $l(r)$ of the loop r*.

The *Dehn function of G* is then be defined by

$$\delta(n) = \max\{\mathrm{Area}(r) | r \in R \text{ with } l(r) \le n\}.$$

The function δ depends on the choice of the generating set $\mathcal{S}$ and the set of relations $\mathcal{R}$. However, the Dehn function obtained from different generating sets and defining relations are equivalent in the following sense. Say that two functions $f, g : \mathbb{N} \to \mathbb{N}$ are *of the same growth type*, if there are numbers $K, L > 0$ such that

$$L^{-1} \cdot g(K^{-1} \cdot x - K) - L \leq f(x) \leq L \cdot g(K \cdot x + K) + L$$

for all $x \in \mathbb{N}$.

In this section we use the graph $\mathcal{RER}(V)$ of rigid extended racks as a geometric model for the handlebody group.

To estimate the Dehn function, we consider a loop γ in $\mathcal{RER}(V)$ of length $R > 0$. We have to show that there is a number $k > 0$ and that there are at most e^{kR} loops $\zeta_1, \dots, \zeta_m$ of length at most k so that γ can be contracted to a point in m steps consisting each of replacing a subsegment of ζ_i by another subsegment of ζ_i. This suffices, since each loop ζ_i as above corresponds to a cycle in the handlebody group which can be filled with uniformly small area.

Recall from Section 3 the definition of the graph $\mathcal{RER}(V, \mathcal{D})$. The following lemma allows to control the isoperimetric function of these subgraphs.

Lemma 4.2. *Let $\mathcal{D}$ be a simple disk system for V.*

i) $\mathcal{RER}(V, \mathcal{D})$ is a connected subgraph of $\mathcal{RER}(V)$ which is equivariantly quasi-isometric to the stabilizer of $\partial\mathcal{D}$ in the mapping class group of ∂V.

ii) $\mathcal{RER}(V, \Sigma)$ is quasi-isometrically embedded in $\mathcal{RER}(V)$.

iii) Any loop in $\mathcal{RER}(V, \Sigma)$ can be filled with area coarsely bounded quadratically in its length.

Proof. $\mathcal{RER}(V, \Sigma)$ is connected by definition of the graph of rigid extended racks (see Section 3).

Let G be the stabilizer of $\partial\mathcal{D}$ in the mapping class group of ∂V. The group G is contained in the handlebody group since every homeomorphism of the boundary of a spotted ball extends to the interior. The group G acts on $\mathcal{RER}(V, \mathcal{D})$ with finite quotient and finite point stabilizers. To show this, note that up to the action of the mapping class group, there are only finitely many isotopy classes of cell decompositions of a bordered sphere with uniformly few cells. Thus by the Svarc-Milnor lemma, $\mathcal{RER}(V, \mathcal{D})$ is equivariantly quasi-isometric to G, showing *i)*.

The stabilizer G of $\partial\mathcal{D}$ is quasi-isometrically embedded in the full mapping class group of ∂V (see [MM00] or [H09b, Theorem 2]). Hence G is also quasi-isometrically embedded in the handlebody group. Together with *i)* this shows *ii)*.

The group G is a Lipschitz retract of the mapping class group of ∂V (see [HM10] for a detailed discussion of this fact which is a direct consequence of the work of Masur and Minsky [MM00]). Mapping class groups are automatic [Mo95] and hence have quadratic Dehn function. Then the same holds true for G (compare again [HM10]). This implies claim *iii)*. □

As the next step, we use Corollary 3.11 to control splitting paths starting at adjacent points in the graph of marked racks. We show that these paths can be constructed in such a way that the resulting loop can be filled with controlled area. Together with the length estimate for marked splitting paths from Theorem 3.10 this will imply the exponential bound for the Dehn function.

The main technical tool in this approach is given by the following lemma.

Lemma 4.3. *For each $k > 0$ there is a number $k_3 > 0$ with the following property.*

Let $\mathcal{D}$ be a simple disk system. Let $R_i, i = 1, \ldots, N$ be a $\mathcal{D}$–splitting sequence of rigid racks and let $\widetilde{R}_j, j = 1, \ldots, M$ be a full $\mathcal{D}$–splitting sequence of extended racks such that the following holds.

- *i) The rigid extended racks R_1 and $\widetilde{R}_1$ (respectively R_N and $\widetilde{R}_M$) have distance at most k in the graph of rigid extended racks.*
- *ii) The associated disk exchange sequences of R_i and $\widetilde{R}_j$ are compatible.*

Then the loop γ in $\mathcal{RER}(V)$ formed by the sequences (R_i), $(\widetilde{R}_j)$ and geodesics between R_1 and $\widetilde{R}_1$ and R_N and $\widetilde{R}_M$ can be filled with area $k_3(N + M)^3$.

Proof. The idea of the proof is to inductively decompose the loop γ into smaller loops, each of which can be filled with area at most $k_3(N+M)^2$ for a suitable k_3.

Denote the disk exchange sequence associated to R_i by $\Sigma_i, i = 1, \ldots, n)$ and the full disk replacement sequence associated to $\widetilde{R}_j$ by $\mathcal{D}_j, j = 1, \ldots m$. Let $r : \{1, \ldots, m\} \to \{1, \ldots, n\}$ be the monotone non-decreasing surjective function given by compatibility, i.e. $\Sigma_{r(j)} \subset \mathcal{D}_j$ for all $j = 1, \ldots, m$.

We define

$$I(i) = \{k \mid \Sigma(R_k) = \Sigma_i\}$$

and

$$J(i) = \{k \mid \mathcal{D}(\widetilde{R}_k) = \mathcal{D}_l \text{ and } r(l) = i\}.$$

Put $i_k = \max I(k)$ and $j_k = \max J(k)$. We will inductively choose paths d_k connecting R_{i_k} to $\widetilde{R}_{j_k}$ and paths c_k connecting R_{i_k+1} to $\widetilde{R}_{j_k+1}$ with the following properties.

i) The path c_k is a uniform quasigeodesic in $\mathcal{RER}(V, \Sigma_{k+1})$.
ii) The path d_k is a uniform quasigeodesic in $\mathcal{RER}(V, \Sigma_k)$.
iii) The paths c_{k+1}, d_k are uniform fellow-travelers, i.e. the Hausdorff distance between c_{k+1} and d_k is uniformly bounded.

A family of paths with these properties implies the statement of the lemma in the following way.

The restriction of the sequence R_i to $I(k)$ and the restriction of $\widetilde{R}_j^{-1}$ to $J(k)$ form together with c_{k-1}^{-1} and d_k a loop γ_k in $\mathcal{RER}(V, \Sigma_k)$. The length of c_{k-1} and d_k is coarsely bounded by $N + M$ by the triangle inequality. Hence, the length of γ_k can be coarsely bounded by $4(N + M)$. Since $\mathcal{RER}(V, \Sigma_k)$ admits a quadratic isoperimetric function, this loop can thus be filled with area bounded by $k_3(N + M)^2$ for some uniform constant k_3.

Similarly, the paths d_k^{-1} and c_{k+1} , together with the edges connecting R_{i_k} to R_{i_k+1} and $\widetilde{R}_{j_k}$ to $\widetilde{R}_{j_k+1}$ form a loop δ_k. The length of δ_k can again be coarsely bounded by $2(N + M)$ using the triangle inequality. Since the paths d_k and c_k are fellow-travelers, δ_k can be filled with area depending linearly on its length.

There are at most $2\max(N, M)$ loops γ_k, δ_k. Hence, the concatenation of all the loops γ_i and δ_j can be filled with area at most $k_3(N+M)^3$ (after possibly enlarging the constant k_2). The paths c_i and d_i occur in the concatenation of γ_i and δ_j twice, with opposite orientations, except for c_0 and the last occurring arc d_L. As a consequence, the concatenation of the loops γ_i and δ_j is, after erasing these opposite paths, uniformly close to γ in the Hausdorff metric. Thus, γ may also be filled with area bounded by $k_3(N + M)^2$ (again possibly increasing k_3).

We now describe the inductive construction of the paths c_k and d_k. We set $c_0 = d_0$ to be the constant path R_1. Suppose that the paths c_i, d_i are already constructed for $i = 0, \ldots, k-1$.

The support systems of R_{i_k} and $\widetilde{R}_{j_k}$ both contain Σ_k. We first construct the path d_k connecting R_{i_k} and $\widetilde{R}_{j_k}$.

Namely, the reduced disk systems Σ_k and Σ_{k+1} are disjoint. The simple disk system $\Sigma_k \cup \Sigma_{k+1}$ is disjoint from the support systems of R_{i_k}, R_{i_k+1} and $\widetilde{R}_{j_k}, \widetilde{R}_{j_k+1}$ by definition of a split. Furthermore, the 1-skeleta of the cell decompositions of all four of these extended racks intersect $\partial\Sigma_k \cup \partial\Sigma_{k+1}$ in uniformly few points. Hence, there are rigid extended racks U_1, U_2 which have $\Sigma_k \cup \Sigma_{k+1}$ as their support system and such that U_1 is uniformly close to R_{i_k}, and U_2 is uniformly close to $\widetilde{R}_{j_k}$ in $\mathcal{RER}(V)$. Let e be a geodesic path in $\mathcal{RER}(V, \Sigma_k \cup \Sigma_{k+1})$ connecting U_1 and U_2. Since $\mathcal{RER}(V, \Sigma_k \cup \Sigma_{k+1})$ is undistorted in $\mathcal{RER}(V)$ by Lemma 4.2, the length of e is coarsely bounded by $N+M+1$. By adding uniformly short geodesic segments in $\mathcal{RER}(V, \Sigma_k)$ at the beginning and the end of e, we obtain the path d_k with property *ii)*.

By definition of i_k and j_k, we have $i_k + 1 \in I(k+1)$ and $j_k + 1 \in J(k+1)$. Hence, both R_{i_k+1} and $\widetilde{R}_{j_k+1}$ contain Σ_{k+1} in their support systems. We can thus define c_k with properties *i)* and *iii)* by adding uniformly short geodesic segments in $\mathcal{RER}(V, \Sigma_{k+1})$ to the beginning and the end of e. □

We have now collected all the tools for the proof of the main theorem.

Proof of Theorem 4.1. Recall that it suffices to show that every loop in the graph of rigid racks can be filled with area coarsely bounded by an exponential function of its length.

Let R_i be a loop of length L in the graph of rigid racks based at $R_0 = \hat{R}$. Let $\hat{\Sigma}$ be the disk system given by Theorem 3.10 applied to $R' = R_0$. Since the graph of rigid racks is quasi-isometric to the graph of extended rigid racks, we can consider R_i as a loop in $\mathcal{RER}(V)$ and it suffices to show that this loop can be filled in $\mathcal{RER}(V)$ with area bounded exponentially in its length.

The strategy of this proof is similar to the proof of Lemma 4.3: we will write the loop (R_i) as a concatenation of smaller loops whose area we can control.

We will define paths c_i in $\mathcal{RER}(V)$ with the following properties.

(1) The path c_i connects R_i to a rack which is uniformly close to $\hat{R}$ in $\mathcal{RER}(V)$.
(2) The path c_i is a $\hat{\Sigma}$–splitting sequence of racks.
(3) The loop formed by c_i, c_{i+1}, the edge between R_i and R_{i+1}, and a geodesic connecting other pair of endpoints of c_i, c_{i+1} can be filled with area bounded by $e^{k_3 L}$.

As a consequence, the loop (R_i) itself can be filled with area at most $Le^{k_3 L}$, proving the theorem.

The construction of the paths c_i is again by induction. We set c_0 to be the constant path R_0. Suppose now that the path c_k is already constructed. Since R_k and R_{k+1} are connected by an edge in the graph of rigid racks, their support systems Σ_k and Σ_{k+1} are disjoint. Let $\Sigma_k^{(i)}, i = 1, \dots, n$ be the disk exchange sequence associated to the splitting sequence c_k. Put $\mathcal{D}_1 = \Sigma_k \cup \Sigma_{k+1}$. Using Lemma 2.7 we obtain a full disk replacement sequence $(\mathcal{D}_i)$ compatible with $(\Sigma_k^{(i)})$. Corollary 3.11 part *i)* then yields a full splitting sequence $\widetilde{R}_k, k = 1, \ldots, M$ with associated full disk exchange sequence $(\mathcal{D}_i)$. By Lemma 4.2, the loop formed by c_k and $\widetilde{R}_k, k = 1, \ldots, M$ can be filled with area bounded by $k_2(N+M)^3$, where N is the length of the path c_k.

Using Lemma 2.6 on the sequence $(\mathcal{D}_i)$ and the initial reduced disk system (Σ_{k+1}) we obtain a $\hat{\Sigma}$–splitting sequence $(\Sigma_{k+1}^{(i)})$ compatible with $(\mathcal{D}_i)$, which starts in Σ_{k+1}. Corollary 3.11 part *ii)* now yields a $\hat{\Sigma}$–splitting sequence c_{k+1} starting in R_{k+1} and ending uniformly close to $\hat{R}$. Applying Lemma 4.3 again, we see that the loop formed by c_{k+1} and $\widetilde{R}_k, k = 1, \ldots, M$ can be filled with area bounded by $k_3(N'+M)^3$, where N' is the length of the path c_{k+1}.

Since both c_k and c_{k+1} are splitting sequences connecting points which are of distance at most L, their lengths can be bounded by $k_4 e^{k_4 L}$ for a suitable k_4 by Theorem 3.7. As a consequence, the paths c_k and c_{k+1} satisfy condition *iii)*. This concludes the inductive construction of c_i and the proof of the theorem. □

The proof of the theorem would give a polynomial bound for the Dehn function provided that the length of the splitting paths used to fill in loops had a length which is polynomial in the distance between their endpoints. Unfortunately, however, the following example show that such a bound does not exist. This is similar to the behavior of paths of sphere systems used in [HV96] to show an exponential upper bound for the Dehn function of $\mathrm{Out}(F_n)$,

For simplicity of exposition, we do not construct these paths in the graph of rigid racks (or the handlebody group), but instead in a slightly simpler graph. The example given below can be extended to the full graph of rigid racks in a straightforward fashion.

We define $\mathcal{RD}(V)$ to be the graph of reduced disk systems in V. The vertex set of $\mathcal{RD}(V)$ is the set of isotopy classes of reduced disk systems, and two such vertices are connected by an edge of length one if the corresponding disk systems are disjoint. Every directed disk exchange sequence defines an edge-path in $\mathcal{RD}(V)$. The following example shows that the length of these edge-paths may be exponential in the distance between their endpoints.

Example 4.4. Consider a handlebody V of genus 4. For each $n \in \mathbb{N}$ we will construct a disk exchange sequence $\Sigma_1^{(n)}, \ldots, \Sigma_{N(n)}^{(n)}$ such that on the one hand, the length $N(n)$ of the sequence growth exponentially in n. On the other hand, the distance between endpoints $\Sigma_1^{(n)}$ and

$\Sigma^{(n)}_{N(n)}$ in $\mathcal{RD}(V)$ grows linearly in n. To simplify the notation, in this example we will only construct the endpoint $\Sigma^{(n)}_{N(n)}$ and denote it by Σ_n.

We choose three disjoint simple closed curves $\alpha_1, \alpha_2, \alpha_3$ which decompose the surface ∂V into a pair of pants, two once-punctured tori and a once-punctured genus 2 surface (see Figure 1). We may choose the α_i such that they bound disks in V. We denote the two solid tori in the complement of these disks by T_1, T_2 and the genus 2 subhandlebody by V'.

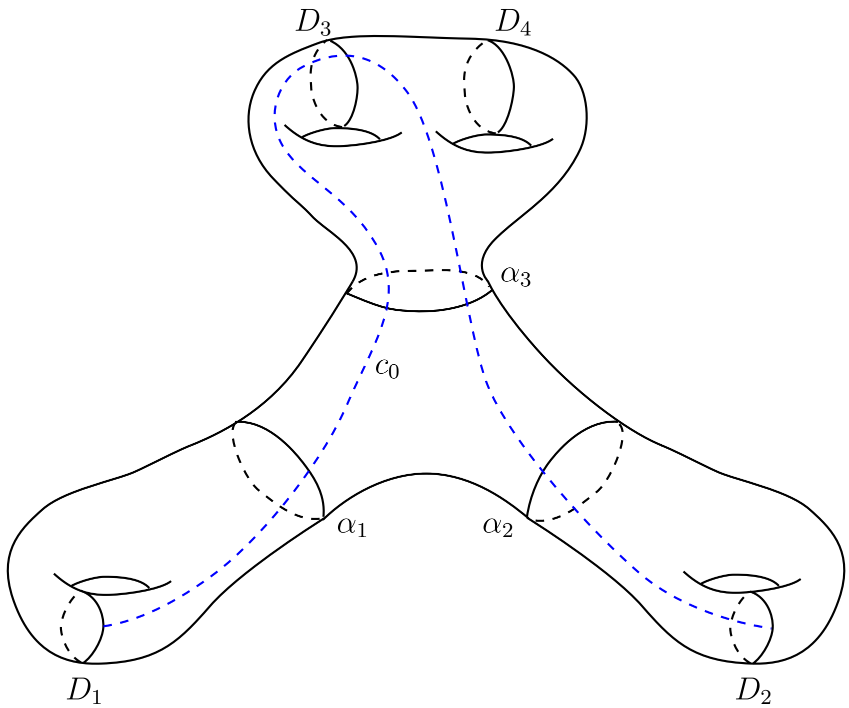

FIGURE 1. The setup for the example of a non-optimal disk exchange path. An admissible arc is drawn dashed.

Let $\Sigma_0 = \{D_1, D_2, D_3, D_4\}$ be a reduced disk system such that $D_1 \subset T_1, D_2 \subset T_2$ and $D_3, D_4 \subset V'$. Choose a base point p on α_3. Let γ_1, γ_2 be two disjointly embedded loops on $\partial V \cap V'$ based at p with the following property. The loop γ_1 intersects the disk D_3 in a single point and is disjoint from D_4, while γ_2 intersects D_4 in a single point and is disjoint from D_3. Since the complement of $D_3 \cup D_4$ in V' is simply connected, such a pair of loops generates the fundamental group of V'. Denote the projections of γ_1 and γ_2 to $\pi_1(V', p)$ by A_1 and A_2, respectively.

Let c be an embedded arc on ∂V. We say that c is *admissible* if the following holds. The arc c connects the disk D_1 to the disk D_2. The interior of c intersects α_1 and α_2 in a single point each. Furthermore it intersects α_3 in two points, and its interior is disjoint from both D_1 and D_2.

Let c be an admissible arc. The intersection of c with V' is an embedded arc c' connecting α_3 to itself. The arc c' may be turned into an embedded arc in V' based at p by connecting the two endpoints of c' to p along α_3. Since the curve α_3 bounds a disk in V', the image of this loop in $\pi_1(V', p)$ is determined by the homotopy class of the arc c relative to $\partial D_1, \partial D_2$. We call this image the *element induced by the arc c*.

Choose an admissible arc c_0 in such a way that it intersects the disk D_3 in a single point, and is disjoint from D_4 (see Figure 1 for an example). Up to changing the orientation of γ_1 we may assume that the element induced by c_0 is A_1.

We now describe a procedure that produces essential disks from admissible arcs. To this end, let c be an admissible arc. Consider a regular neighborhood U of $D_1 \cup c \cup D_2$. Its boundary consists of three simple closed curves. Two of them are homotopic to either ∂D_1 or ∂D_2. The third one we denote by $\beta(c)$. Note that $\beta(c)$ bounds a nonseparating disk in V.

Choose a fixed element φ of the handlebody group of V with the following properties. The mapping class φ fixes the isotopy classes of the curves α_1, α_2 and α_3. The restriction of φ to the complement of V' is isotopic to the identity. The restriction of φ to V' induces an automorphism of exponential growth type on $\pi_1(V')$. To be somewhat more precise, we may choose φ such that it acts on the basis A_i as the following automorphism Φ:

$$\begin{aligned} A_1 &\mapsto A_1 A_2 \\ A_2 &\mapsto A_1^2 A_2 \end{aligned}$$

Put $c_n = \varphi^n(c_0)$ and $\beta_n = \beta(c_n)$. We claim that a disk exchange sequence in direction of β_n that makes β_n disjoint from Σ_0 has length at least 2^n.

To this end, note that the arc c_n intersects the disks D_3 and D_4 in at least 2^n points. Namely, the element of $\pi_1(V', p)$ induced by $\varphi^n(c_0)$ is equal to $\Phi^n(A_1)$. The cyclically reduced word describing $\Phi^n(A_1)$ in the basis A_1, A_2 has length at least 2^n by construction of Φ.

Therefore, the curve β_n can be described as follows. Choose a parametrization $\beta_n : [0, 1] \to \partial V$. Then there are numbers $0 < t_1 < \cdots < t_N < t_{N+1} < \cdots < t_{2N} < 1$ such that the following holds. Each subarc $\beta_n([t_i, t_{i+1}])$ intersects Σ_0 only at its endpoints. The subarcs $\beta_n([t_N, t_{N+1}])$ and $\beta_n([0, t_1] \cup [t_{2N}, 1])$ are returning arcs to Σ_0. Furthermore, the arcs $\beta_n([t_i, t_{i+1}])$ and $\beta_n([t_{2N-i}, t_{2N+1-i}])$ are homotopic relative to Σ_0 for all $i = 1, \dots, N-1$. More generally, if there are numbers t_i with these properties for a reduced disk system Σ we say that β_n *is a long string of rectangles with respect to* Σ. The number N is then called *the length of the string of rectangles*. By construction, the length N of the string of rectangles β_n defines with respect to Σ_0 is at least 2^n.

The curve β_n has two returning arcs. Let a be one of them, say $\beta_n([t_N, t_{N+1}])$ and let $\sigma \in \Sigma_0$ denote the disk containing the endpoints of a. One of the disks obtained by simple surgery along a is isotopic to either D_1 or D_2 (depending on which returning arc we chose). The preferred interval defined by a contains every intersection point of β_n with σ except the endpoints of a.

Denote by Σ_1 the reduced disk system obtained by simple surgery along a. By construction, the subarc $\beta_n(t_{N-1}, t_{N+2})$ now defines a returning arc with respect to Σ_1. One of the disks obtained by simple surgery along this returning arc is still properly isotopic to D_1. Furthermore, the subarcs $\beta_n([t_i, t_{i+1}])$ and $\beta_n([t_{2N-i}, t_{2N+1-i}])$ are still arcs with endpoints on Σ_1 which are homotopic relative to Σ_1 for all $i = 1, \dots, N-2$. Each of these arcs cannot be homotoped into $\partial\Sigma_1$.

Hence the curve β_n has a description as a string of rectangles of length $N-1$ with respect to Σ_1 and the argument can be iterated. By induction, it follows that any disk exchange sequence starting in Σ_0 which ends in a disk system disjoint from β_n has length at least 2^n.

On the other hand, the growth of the distance between Σ_0 and $\varphi^n(\Sigma_0)$ in the graph of reduced disk systems is linear in n by the triangle inequality. The curve β_n intersects $\varphi^n(\Sigma_0)$ in uniformly few points, and thus the disk system $\varphi^n(\Sigma_0)$ is uniformly close to a reduced disk system that is disjoint from β_n. Thus the disk systems Σ_n have the properties described in the beginning of the example.

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