KRECK-STOLZ INVARIANTS FOR QUATERNIONIC LINE BUNDLES

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ABSTRACT. We generalise the Kreck-Stolz invariants s_2 and s_3 by defining a new invariant, the t-invariant, for quaternionic line bundles over closed spin-manifolds M of dimension $4k - 1$ with $H^3(M; \mathbb{Q}) = H^4(M; \mathbb{Q}) = 0$. The t-invariant classifies closed smooth oriented 2-connected rational homology 7-spheres up to almost-diffeomorphism and detects exotic homeomorphisms between such manifolds.

  The t-invariant also provides information about quaternionic line bundles over a fixed manifold and we use it to give a new proof of a theorem of Feder and Gitler about the values of the second Chern classes of quaternionic line bundles over $\mathbb{H}P^k$. The t-invariant for S^{4k-1} is closely related to the Adams e-invariant on the $(4k-5)$-stem.

INTRODUCTION

In [21], Kreck and Stolz introduced invariants s_1, s_2, $s_3 \in \mathbb{Q}/\mathbb{Z}$ of certain closed smooth oriented simply connected 7-manifolds M that completely characterise M up to diffeomorphism. The s-invariants are defect invariants based on index theorems for Dirac operators on 8-manifolds: the invariant s_1, which equals the Eells-Kuiper invariant μ of [13] if M is spin, is the defect of the untwisted Dirac operator, whereas s_2 and s_3 are the defects of the Dirac operator twisted by certain *complex line bundles*.

In this paper we define a defect invariant, the t-invariant, based on index theorems for Dirac operators twisted by *quaternionic line bundles*. Suppose now that M is a closed smooth spin $(4k-1)$-manifold such that the groups $H^3(M; \mathbb{Q})$ and $H^4(M; \mathbb{Q})$ vanish and let $\mathrm{Bun}(M)$ denote the set of isomorphism classes of quaternionic line bundles over M. The t-invariant is a function (see Definition 1.4),

$$t_M \colon \mathrm{Bun}(M) \longrightarrow \mathbb{Q}/\mathbb{Z} \,,$$

which as we explain in Section 2.c, is a precise generalisation of the invariants s_2 and s_3 in dimension 7. Another important invariant of quaternionic line bundles is their second Chern class which defines a function

$$c_2 \colon \mathrm{Bun}(M) \longrightarrow H^4(M) \,.$$

We shall say that M and N have *isomorphic t-invariants* if there is a group isomorphism $A \colon H^4(M) \to H^4(N)$ and a set bijection $B \colon \mathrm{Bun}(M) \to \mathrm{Bun}(N)$ which are compatible with second Chern class and the t-invariant.

Our first application of the t-invariant is in the realm of 2-connected 7-manifolds and was inspired by recent discoveries of Grove, Verdiani and Ziller [17]. The first named author established a complete classification of 2-connected rational homology 7-spheres in [8] using a certain *extrinsically defined* quadratic linking form $q_M \colon H^4(M) \to \mathbb{Q}/\mathbb{Z}$ whose values are calculated using a spin 8-manifold W with boundary $\partial W = M$. Theorem 0.1 below, which is a reformulation of Theorem 2.4, states that the t-invariant t_M is a refinement of q_M. Moreover, as a defect invariant, the t-invariant has an *intrinsic definition*, see (1.9), given via η-invariants arising from the Atiyah-Patodi-Singer index theorem of [4].

0.1. Theorem. *Let M be a closed smooth 2-connected oriented rational homology 7-sphere. Then*

    (1) *the map $c_2 \colon \mathrm{Bun}(M) \to H^4(M)$ is onto,*
    (2) *for all $E \in \mathrm{Bun}(M)$, $q_M(c_2(E)) = 12t_M(E)$,*
    (3) *the map $c_2 \times t_M \colon \mathrm{Bun}(M) \to H^4(M) \times \mathbb{Q}/\mathbb{Z}$ is injective.*

Applying [8, Theorem A] we obtain the following expanded version of Corollary 2.5.

0.2. Corollary. *Let N and M be closed smooth 2-connected oriented rational homology 7-spheres. Then N is diffeomorphic to M if and only if t_N is isomorphic to t_M and the Eells-Kuiper invariants of M and N agree: $\mu(N) = \mu(M)$.*

In [15] the second named author used Corollary 0.2 to classify manifolds constructed in [17] by directly calculating their t-invariants as defined in (1.9), see example 2.8.

A further feature of the t-invariant in dimension 7 is that it detects *exotic homeomorphisms*: these are homeomorphisms $h \colon N \cong M$ which are not homotopic to piecewise linear homeomorphisms. The work of Kirby and Siebenmann [22] implies that for a homeomorphism $h \colon N \cong M$ there is an invariant

$$\mathrm{KS}(h) \in H^3(M; \mathbb{Z}/2) \,,$$

depending only on the homotopy class of h, such that h is exotic if and only if $\mathrm{KS}(h) \neq 0$. The following shorter version of Theorem 2.9 combined with Theorem 0.1 (1) above shows that the Kirby-Siebenmann invariant of h can be computed using the induced map $h^* \colon \mathrm{Bun}(M) \to \mathrm{Bun}(N)$ and the t-invariants of N and M.

0.3. Theorem. *A homeomorphism $h \colon N \to M$ is exotic if and only if*

$$t_M \neq t_N \circ h^*.$$

More precisely, for all $E \in \mathrm{Bun}(M)$,

$$\big(\mathrm{KS}(h) \smile c_2(E)\big)[M]_2 = t_N(h^* E) - t_M(E) \in \mathbb{Z}/2 \subset \mathbb{Q}/\mathbb{Z} \,,$$

where $[M]_2$ generates $H_7(M; \mathbb{Z}/2)$.

We now change our focus from the base-space manifolds to the bundles themselves. For the simplest manifolds $M = S^{4k-1}$ we have the t-invariant

$$t_{S^{4k-1}} \colon \mathrm{Bun}(S^{4k-1}) \cong \pi_{4k-2}(S^3) \longrightarrow \mathbb{Q}/\mathbb{Z} \,.$$

Given its relationship to the Dirac operator and hence the $\widehat{A}$-genus one might expect that the t-invariant is related to the Adam's e-invariant $e \colon \pi^S_{4k-5} \to \mathbb{Q}/\mathbb{Z}$ of [1]. Theorem 1.12, restated immediately below, bears this expectation out.

0.4. Theorem. *Assume that $k \geq 3$. For any homotopy sphere Σ^{4k-1} the homomorphism*

$$t_\Sigma \colon \operatorname{Bun}(\Sigma) \cong \pi_{4k-2}(S^3) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

may be identified with the composition

$$-e \circ S \colon \pi_{4k-2}(S^3) \longrightarrow \pi^S_{4k-5} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

where $-e \colon \pi^S_{4k-5} \to \mathbb{Q}/\mathbb{Z}$ and $S \colon \pi_{4k-2} \to \pi^S_{4k-5}$ are respectively the the negative of the Adams e-invariant [1] and the stabilisation homomorphism.

A key feature of quaternionic bundles is that both the quaternions $\mathbb{H}$ and their group of units $S^3 \subset \mathbb{H}$ are non-abelian. As a result the classifying space $BS^3 = \mathbb{H}P^\infty$ is not an H-space, and for a general space X, $\operatorname{Bun}(X) \equiv [X, BS^3]$ does not have a naturally defined group structure. This leads to the fact that it is often a difficult problem to determine the image of c_2. For the case $X = \mathbb{H}P^k$ and the map

$$c_2 \colon \operatorname{Bun}(\mathbb{H}P^k) \longrightarrow H^4(\mathbb{H}P^k)$$

one says that an integer c is k-realisable if $c \cdot c_2(H) \in \operatorname{Im}(c_2)$ where H is the tautological bundle on $\mathbb{H}P^k$ and $c_2(H)$ is the universal second Chern class. In [14] Feder and Gitler proved

0.5. Theorem ([14, Theorem 1.1]). *If an integer c is k-realisable then*

$$\frac{a_j}{(2j)!} \prod_{i=0}^{j-1}(c - i^2) = 0 \in \mathbb{Q}/\mathbb{Z} \qquad \textit{for all } 2 \leq j \leq k \,.$$

Using the t-invariant, we give a new proof of the Feder-Gitler criteria of Theorem 0.5 based on Theorem 3.1.

The rest of the paper is organised as follows: in Section 1 we define the t-invariant from both the intrinsic analytic point of view and the extrinsic topological point of view. This section also introduces the basic concepts and tools required for our analysis of quaternionic line bundles. In section 2 we consider the t-invariant for 7-manifolds. We up-date the classification results of [8] in Theorem 2.4, we show that the t-invariant detects exotic homeomorphisms in Theorem 2.9. In Section 2.c we show how the t-invariant on simply connected 7-manifolds relates to the s-invariants and their generalisations by Hepworth in [19]. Section 3 gives our proof of the Feder-Gitler criteria as well as several explicit computations of the t-invariant for certain bundles over S^7, S^{11} and S^{15}.

1. Secondary Invariants

In this section we define the t-invariant which is an invariant of pairs (M, E) where M is a closed spin $(4k-1)$-manifold and E is a quaternionic line bundle over M. We will give both an extrinsic and an intrinsic definition of the t-invariant. Whereas the extrinsic definition using zero bordisms is easier to handle in many cases, the intrinsic definition using η-invariants is finer for manifolds that are only rationally zero bordant and may sometimes be computed when 0-bordisms cannot be found. We will state and prove properties of the t-invariant in both settings when we are able to because the extrinsic proofs are easier in most cases.

The section is organised as follows. Section 1.a briefly reviews quaternionic line bundles, Section 1.b defines the t-invariant and identifies some of its basic properties. Section 1.c introduces the notion of a *quaternionic divisor* used in the proof of Theorem 1.12 which occupies most of Section 1.d. Finally in Section 1.e we briefly consider dimension 3 and relate an invariant of Deloup and Massuyeau [11] to the Atiyah-Patodi-Singer ξ-invariant which is precisely analogous to the t-invariant.

1.a. **Quaternionic Line Bundles.** Let $\mathbb{H}, \mathbb{H}^\times$ and S^3 denote respectively the quaternions, the non-zero quaternions and the unit quaternions. A quaternionic line bundle, $E \to X$ or simply E, over a space X is a complex rank 2 vector bundle $V \to X$ together with a reduction of its structure group to $\mathbb{H}^\times$. An isomorphism of quaternionic line bundles is a vector bundle isomorphism respecting the quaternionic structures. As an example, consider the tautological bundle

$$H \longrightarrow \mathbb{H}P^\infty = BS^3 .$$

Since the structure group $\mathbb{H}^\times$ of a quaternionic line bundle can always be reduced to S^3, and this reduction is unique up to contractible choice, quaternionic line bundles are classified by the homotopy classes of maps to BS^3. So

$$\mathrm{Bun}(X) = [X, BS^3]$$

describes the set of isomorphism classes of quaternionic line bundles over a space X. By abuse of notation, we will sometimes write $E \in \mathrm{Bun}(X)$ to say that E is a quaternionic line bundle over X.

The integral cohomology of BS^3 is generated by the universal second Chern class $c_2(H) \in H^4(\mathbb{H}P^\infty)$, and so to each bundle $E \to X$ with classifying map $f \colon X \to BS^3$ we may associate the characteristic class

$$c_2(E) := f^* c_2(H) \in H^4(X) .$$

In this way we obtain a map

$$c_2 \colon \mathrm{Bun}(X) \longrightarrow H^4(X) , \qquad E \longmapsto c_2(E) .$$

Note that the theory of quaternionic bundles is not quite as straight forward as in the complex case. First, the tensor product of two quaternionic line bundles is merely a four-dimensional real vector bundle. In fact, every four-dimensional vector bundle that is orientable and spin arises this way. In particular, there is no canonical group structure on quaternionic line bundles.

Next, quaternionic line bundles are not classified by their second Chern class. The reason is that BS^3 is not an Eilenberg-MacLane space. For example, the quaternionic line bundles on S^7 are classified by

$$\pi_7(BS^3) \cong \pi_6(S^3) \cong \mathbb{Z}/12\mathbb{Z} \;,$$

but of course $H^4(S^7) = 0$. On the other hand, for a general space X, not all the elements in $H^4(X)$ are realised as $c_2(E)$ for a quaternionic line bundle $E \to X$: for example, as is well known and explained in Section 3, if $X = \mathbb{H}P^2$, then the image of c_2 in the group $H^4(\mathbb{H}P^2) \cong \mathbb{Z}$ is precisely all those classes $c \cdot c_2(H)$ such that $c(c-1)/2$ is congruent to 0 mod 12.

1.b. **Modified Kreck-Stolz Invariants.** Let M be a closed spin manifold of dimension $4k - 1$ and let $E \to M$ be a complex vector bundle over M. As observed by Kreck and Stolz, a source of $\mathbb{Q}/\mathbb{Z}$-valued invariants of the pair (M, E) is the following index theorem.

1.1. **Theorem** (Atiyah-Singer [5, III, Theorem 5.3]). *Let X^{4k} be a closed, smooth spin manifold and let E be a complex vector bundle over X. Then the index of D_X^E, the Dirac operator of X twisted by E, can be computed by the following equation:*

$$\mathrm{ind}\big(D_X^E\big) = \big(\widehat{A}(TX)\,\mathrm{ch}(E)\big)[X] \quad \in \quad \mathbb{Z}\,, \tag{1.1}$$

where $\widehat{A}(TX)$ and $\mathrm{ch}(E) \in H^*(X)$ denote respectively the $\widehat{A}$-genus of TX and the Chern character of E and $[X] \in H_{4k}(X)$ is the fundamental class of X.

Assume that (M, E) as above bounds a spin manifold and complex vector bundle $(W, \overline{E})^1$ and that the characteristic class $\widehat{A}(TX)\,\mathrm{ch}(\overline{E})$ has a natural interpretation in $H^{4k}(W, M; \mathbb{Q})$. Then evaluation on the relative fundamental cycle $[W, M]$ gives a rational number which, by Theorem 1.1 is an invariant of (M, E) mod $\mathbb{Z}$. The Atiyah-Patodi-Singer index theorem 1.2 below can also be used to identify this invariant as an intrinsic invariant of (M, E). In the remainder of this section, we carry out this program for quaternionic line bundles.

Recall that a *real structure s* (*quaternionic structure j*) on a complex vector bundle $E \to X$ is an anti-linear automorphism with $s^2 = 1$ ($j^2 = -1$). A quaternionic vector bundle $E \to X$ can be regarded as a complex vector bundle with complex structure i and quaternionic structure j. We can always choose a quaternionic-Hermitian metric on E and a connection such that the quaternionic structure is parallel.

If X^{4k} is a spin manifold and k is even then the complex spinor bundle ΣX of X carries a real structure s that is parallel with respect to the Levi-Civita connection. Then $s \otimes j$ is a quaternionic structure on $\Sigma X \otimes_{\mathbb{C}} E$ that commutes with the Dirac operator. This implies that $\mathbb{H}$ acts on all eigenspaces of D_X^E, in particular

$$\mathrm{ind}\big(D_X^E\big) \in 2\mathbb{Z} \qquad \text{if } \dim X \equiv 0 \text{ mod } 8. \tag{1.2}$$

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[1]Note that throughout $\overline{E}$ always denotes a bundle over a manifold W with boundary such that E is the restriction of $\overline{E}$ to the boundary ∂W. In particular $\overline{E}$ *is not* the complex conjugate of E.

Because c_2 generates the ring of characteristic classes of quaternionic line bundles, there exists a universal characteristic class ch' of quaternionic line bundles such that

$$2 - \mathrm{ch}(E) = c_2(E)\, \mathrm{ch}'(E) \,. \tag{1.3}$$

In fact, one computes that

$$\mathrm{ch}'(E) = 1 - \frac{1}{12}c_2(E) + \dots \,. \tag{1.4}$$

Now suppose that M bounds a compact spin manifold W such that E is the restriction of a bundle $\overline{E} \to W$. In other words,

$$[M, E] = 0 \quad \in \quad \Omega_*^{\mathrm{Spin}}(BS^3) \,. \tag{1.5}$$

Assume moreover that $H^3(M;\mathbb{Q}) = H^4(M;\mathbb{Q}) = 0$. Then because of the exact sequence

$$H^3(M;\mathbb{Q}) \longrightarrow H^4(W, M;\mathbb{Q}) \longrightarrow H^4(W;\mathbb{Q}) \longrightarrow H^4(M;\mathbb{Q}) \,,$$

the class $c_2(\overline{E}) \in H^4(W;\mathbb{Z})$ has a unique lift $\bar{c}_2(\overline{E}) \in H^4(W, M;\mathbb{Q})$.

As a preliminary definition of the t-invariant, let us consider the quantity

$$\tau_M(E) := \frac{-1}{a_{k+1}}\big(\widehat{A}(TW)\, \mathrm{ch}'(\overline{E})\, \bar{c}_2(\overline{E})\big)[W, M] \quad \in \quad \mathbb{Q}/\mathbb{Z} \,, \tag{1.6}$$

where a_j is 1 is j even and 2 if j is odd. Theorem 1.1 and property (1.2) above ensure that $\tau_M(E)$ is independent of the choice of the pair $(W, \overline{E})$.

We now want to define $\tau_M(E)$ intrinsically on M, so that we can drop condition (1.5). Also, given a particular M and a particular quaternionic line bundle $E \to M$, we can sometimes use the numeric value of $t(E)$ to derive information about M and E. Hence, we want to compute $t(E)$ before we know how to construct W and how to extend E to W.

We equip TM with a Riemannian metric g. It gives rise to a Levi-Civita connection ∇^{TM} on TM. Let D be the untwisted spin Dirac operator of (M, g) acting on the complex spinor bundle $\Sigma M \to M$. We also equip the quaternionic line bundle E with a quaternionic Hermitean metric h^E and a compatible connection ∇^E. As above, D_M^E denotes the spin Dirac operator twisted by (E, ∇^E, h^E). Let $\eta(D_M)$ and $\eta(D_M^E)$ denote the Atiyah-Patodi-Singer η-invariants and put $h(D_M) = \dim \ker D_M$ and $h(D_M^E) = \dim \ker D_M^E$.

Let $(\Omega^\bullet(M), d)$ denote the de Rham complex of M. We consider the characteristic Chern-Weil forms

$$\widehat{A}\big(TM, \nabla^{TM}\big) \,, \quad c_2(E, \nabla^E) \,, \quad \text{and} \quad \mathrm{ch}(E, \nabla^E) \quad \in \quad \ker d \subset \Omega^\bullet(M)$$

that represent the corresponding characteristic classes in de Rham cohomology. As in (1.3), there is a characteristic Chern-Weil form $\mathrm{ch}'(E, \nabla^E)$ such that

$$2 - \mathrm{ch}(E, \nabla^E) = c_2(E, \nabla^E)\, \mathrm{ch}'(E, \nabla^E) \,. \tag{1.7}$$

If $H^3(M;\mathbb{Q}) = H^4(M;\mathbb{Q}) = 0$, the de Rham cohomology of M vanishes in these degrees. Hence we find $\hat{c}_2(E, \nabla^E) \in \Omega^3(M)$ such that

$$d\hat{c}_2(E, \nabla^E) = c_2(E, \nabla^E) \,, \tag{1.8}$$

and $\hat{c}_2(E, \nabla^E)$ is uniquely determined modulo exact forms.

From the variation formulas for η-invariants and characteristic forms, it is easy to see that the expression

$$t_M(E) := \frac{1}{a_{k+1}} \left(\frac{\eta + h}{2}(D_M^E) - (\eta + h)(D_M) \right.$$
$$\left. + \int_M \widehat{A}\big(TM, \nabla^{TM}\big) \, (\hat{c}_2 \, \mathrm{ch}')\big(E, \nabla^E\big) \right) \quad \in \quad \mathbb{R}/\mathbb{Z} \quad (1.9)$$

does not depend on the choice of geometric data. Here we use that $\frac{\eta+h}{2}(D_M^E)$ jumps by even integers as the geometry varies if $\Sigma M \otimes_{\mathbb{C}} E$ carries a quaternionic structure, see (1.2) above.

In other words, $t_M(E)$ depends neither on the metric g on M nor on the connection ∇^E with parallel metric h^E on E. Alternatively, regard two sets of data on M and E. Because $\partial(M \times [0,1]) = M \sqcup (-M)$, we can extend these data to $W = M \times [0,1]$ and use Theorem 1.2 to see that $t_M(E)$ is indeed well-defined. The computation is similar as in the proof of Proposition 1.3 below.

Now assume that (M, g) bounds a Riemannian spin manifold $(W, \bar{g})$ and that (E, ∇^E, h^E) extends to $(\overline{E}, \nabla^{\overline{E}}, h^{\overline{E}}) \to W$. Then the pair

$$\bar{c}_2\big(\overline{E}, \nabla^{\overline{E}}\big) = \big(c_2\big(\overline{E}, \nabla^{\overline{E}}\big), \hat{c}_2\big(E, \nabla^E\big)\big) \quad \in \quad \Omega^4(W) \oplus \Omega^3(M) \qquad (1.10)$$

is closed in the mapping cone of the pullback $\Omega^\bullet(W) \to \Omega^\bullet(M)$. It represents a lift $\bar{c}_2(\overline{E})$ of $c_2(\overline{E})$ to $H^4(W, M; \mathbb{R})$. By the variation formulas for Chern-Simons classes, $\bar{c}_2(\overline{E})$ does not depend on ∇^E and h^E. Note that closed forms on W act on pairs as above by multiplication in both factors, and that evaluation on the fundamental cycle $[W, M]$ is given by

$$(\alpha, \beta)[W, M] = \int_W \alpha - \int_M \beta \, . \qquad (1.11)$$

Let us assume that $(W, \bar{g})$ and $(\overline{E}, \nabla^{\overline{E}}, h^{\overline{E}})$ are of product type near the boundary. In this case, the Atiyah-Patodi-Singer boundary conditions for the spin Dirac operator $D_W^{\overline{E}}$ twisted by $(\overline{E}, \nabla^{\overline{E}}, h^{\overline{E}})$ are defined, and we can use the following generalisation of Theorem 1.1.

1.2. Theorem (Atiyah-Patodi-Singer [4, I, (4.3)]). *Let $(M, g) = \partial(W, \bar{g})$ as above and let $(\overline{E}, \nabla^{\overline{E}}, h^{\overline{E}})$ denote a complex vector bundle with Hermitian connection and parallel metric over W. Then*

$$\mathrm{ind}\big(D_W^{\overline{E}}\big) = \int_W \widehat{A}\big(W, \nabla^W\big) \, \mathrm{ch}\big(\overline{E}, \nabla^{\overline{E}}\big) - \frac{\eta + h}{2}\big(D_M^E\big) \quad \in \quad \mathbb{Z} \, .$$

Again, if the Dirac operator commutes with a quaternionic structure as in (1.2), then the index above is even.

1.3. Proposition. *Let (E, ∇^E, h^E) be a quaternionic line bundle with quaternionic connection and parallel quaternionic-Hermitian metric over (M, g).*

(1) *Assume that (M, E) bounds $(W, \overline{E})$ as above. Then*

$$\tau_M(E) = t_M(E) \, .$$

(2) *The invariant $t_M(E)$ is always rational.*

Proof. For (1), we use the Atiyah-Patodi-Singer index theorem as in [12] and [21] together with (1.2), (1.4), (1.10) and (1.11). We find that

$$- \left(\widehat{A}(TW) \, \bar{c}_2(\overline{E}) \, \mathrm{ch}'(\overline{E}) \right)[W, M]$$

$$= - \int_W \widehat{A}\big(TW, \nabla^{TW}\big) \, (2 - \mathrm{ch})\big(\overline{E}, \nabla^{\overline{E}}\big)$$

$$+ \int_M \widehat{A}\big(TM, \nabla^{TM}\big) \, (\hat{c}_2 \, \mathrm{ch}')\big(E, \nabla^{E}\big)$$

$$\equiv \frac{\eta + h}{2}\big(D_M^E\big) - (\eta + h)\big(D_M\big)$$

$$+ \int_M \widehat{A}\big(TM, \nabla^{TM}\big) \, (\hat{c}_2 \, \mathrm{ch}')\big(E, \nabla^{E}\big) \qquad \mathrm{mod} \ a_{k+1}\mathbb{Z} \, .$$

By [2] the Spin bordism group $\Omega_{4j-1}^{\mathrm{Spin}}$ is a finite $\mathbb{Z}/2$-vector space for $j > 0$. From the Atiyah-Hirzebruch spectral sequence computing $\Omega_{4k-1}^{\mathrm{Spin}}(BS^3)$, the fact that $H^*(BS^3; \mathbb{Z}) = \mathbb{Z}[c_2]$ where c_2 has degree 4 and that each $\mathbb{H}P^k \subset BS^3$ is a spin manifold we see that $\Omega_{4k-1}^{\mathrm{Spin}}(BS^3)$ is also a finite $\mathbb{Z}/2$-vector space. Hence for any S^3-bundle (M, E), there is $n \in \{1, 2\}$ such that $(M, E)^{\sqcup n}$ (or equivalently $(M, E)^{\sharp n}$) bounds a spin manifold W with a quaternionic line bundle $\overline{E}$ that extends $E^{\sqcup n}$. By (1),

$$n \, t_M(E) = \tau_{M^{\sqcup n}}(E^{\sqcup n}) \quad \in \quad \mathbb{Q}/\mathbb{Z} \, . \qquad \square$$

The argument above also shows that in general, the extrinsic definition (1.6) gives the value of $t_M(E)$ only up to multiples of $\frac{1}{n}$ in $\mathbb{Q}/\mathbb{Z}$.

We finally give the definition of the t-invariant.

1.4. **Definition.** Let M be a closed smooth spin $(4k - 1)$-manifold such that $H^3(M; \mathbb{Q}) = H^4(M; \mathbb{Q}) = 0$, then we use equation (1.9) above to define the *t-invariant* of M as the function

$$t_M \colon \mathrm{Bun}(M) \longrightarrow \mathbb{Q}/\mathbb{Z}, \quad E \longmapsto t_M(E).$$

We shall say that M and N have *isomorphic t-invariants* if there is a group isomorphism $A \colon H^4(M) \to H^4(N)$ and a set bijection $B \colon \mathrm{Bun}(M) \to \mathrm{Bun}(N)$ which are compatible with the second Chern class and the t-invariant; i.e.

$$\forall E \in \mathrm{Bun}(M), \quad c_2(BE) = Ac_2(E) \text{ and } t_N(BE) = t_M(E) \, .$$

We conclude this subsection by recording some basic facts about the t-invariant.

1.5. **Proposition.** *The function $t_M \colon \mathrm{Bun}(M) \to \mathbb{Q}/\mathbb{Z}$ has the following properties:*

(1) *Triviality: for the trivial bundle $\varepsilon := M \times \mathbb{H}$, we have $t_M(\varepsilon) = 0$.*
(2) *Additivity with respect to connected sum of bundles and manifolds:*

$$t_{M_0 \sharp M_1}(E_0 \, \sharp \, E_1) = t_{M_0}(E_0) + t_{M_1}(E_1) \in \mathbb{Q}/\mathbb{Z}.$$

(3) *Naturality under almost-diffeomorphisms: if Σ is a homotopy sphere and $f \colon N \,\sharp\, \Sigma \cong M$ is a diffeomorphism then*

$$t_N \circ f^* = t_M$$

where we identify N and $N\sharp\Sigma$ as spaces so that $\mathrm{Bun}(N\sharp\Sigma) = \mathrm{Bun}(N)$.

1.6. *Remark.*   (1) Properties (1)–(3) hold for any generalised Kreck-Stolz invariants as we now explain. Let M a closed odd-dimensional manifold, equipped with a geometric Dirac operator D, and let P denote the corresponding local index class. For the t-invariant, we take the spin Dirac operator and the $\widehat{A}$-class, but one could also consider the signature operator and the L-class or something similar. Then let $E \to M$ be a vector bundle of a specific type. If there exists a natural class $\alpha \in \Omega^\bullet(M)/\operatorname{im} d$ such that $d\alpha = \mathrm{ch}(E) - \mathrm{rk}\,E$, then one can consider the invariant

$$\frac{\eta + h}{2}(D^E) - \frac{\eta + h}{2}(D)\,\mathrm{rk}(E) - \int_M P\big(TM, \nabla^{TM}\big)\,\alpha \in \mathbb{R}/a\mathbb{Z}\,,$$

where $a \in \mathbb{Z}$ depends on the index problem. Note that the Atiyah-Patodi-Singer ρ- and ξ-invariants [4, II] are special cases, where (E, ∇^E) is flat, so one can take $\alpha = 0$. The proof below will work equally well for all invariants of this type.
(2) We shall see later in Theorem 2.9 that the t-invariant is not in general preserved by homeomorphisms.

Proof of Proposition 1.5. For the trivial bundle ε, one can choose $\hat{c}_2 = 0$, so (1.9) gives $t_M(\varepsilon) = 0$, and (1) follows.

For (2), we choose points $p_i \in M_i$ and assume that (E_i, ∇^{E_i}) is isomorphic to a trivial bundle over a small disk U_i around p_i in M_i. Let

$$W = (M_0 \times [0,1])\,\natural\,(M_1 \times [0,1])$$

denote the boundary connected sum taken at the points $(p_0, 1)$ and $(p_1, 1)$. Put

$$M = \partial W = (M_0\,\sharp\,M_1) \sqcup (-M_0) \sqcup (-M_1)\,.$$

The boundary connected sum of the pullbacks of E_i with respect to the given trivialisations gives a bundle $\overline{E} \to W$. The pullback connections are trivial on $U_i \times [0,1]$, so we get a connection $\nabla^{\overline{E}}$ by gluing.

We note that $d\hat{c}^2(E_i, \nabla^{E_i}) = c^2(E_i, \nabla^{E_i})$ vanishes on U_i by assumption. Since U_i is contractible, there are representatives $\hat{c}_2(E_i, \nabla^{E_i}) \in \Omega^3(M_i)$ that vanish on U_0 and U_1 as well. By pullback and gluing, we obtain a global form $\hat{c}_2\big(\overline{E}, \nabla^{\overline{E}}\big) \in \Omega^3(W)$ with

$$d\hat{c}_2\big(\overline{E}, \nabla^{\overline{E}}\big) = c_2\big(\overline{E}, \nabla^{\overline{E}}\big)\,.$$

In particular, the lift $\bar{c}_2(\overline{E}, \nabla^{\overline{E}})$ is exact in the mapping cone of the pullback $\Omega^\bullet(W) \to \Omega^\bullet(M)$.

From the extrinsic description (1.6) and Stokes' theorem, we obtain

$$t_{M_0 \sharp M_1}(E_0 \,\sharp\, E_1) - t_{M_0}(E_0) - t_{M_1}(E_1) = \tau_M(E)$$

$$= - \int_W \widehat{A}\big(TW, \nabla^{TW}\big)\, (d\hat{c}_2 \,\mathrm{ch}')\big(\overline{E}, \nabla^{\overline{E}}\big)$$

$$+ \int_{\partial W} \widehat{A}\big(TM, \nabla^{TM}\big)\, (\hat{c}_2 \,\mathrm{ch}')\big(E, \nabla^{E}\big) = 0\,.$$

Naturality in (3) is clear for diffeomorphisms and follows from additivity for almost-diffeomorphisms: add the trivial bundle (Σ, ε) to (M, E). $\qquad\square$

1.c. **Divisors.** In this subsection we develop the concept of the *divisor* (Y, ν) of a quaternionic line bundle E in analogy with divisors of complex line bundles. The main result is Proposition 1.8 which explains how the η-invariants involved in the intrinsic definition of t_M localise near a divisor.

Complex line bundles on algebraic varieties are characterised by divisors, that is, by subvarieties of codimension 1. A similar construction still works for smooth manifolds. Let $L \to M$ be a complex line bundle, and let $s \in \Gamma(L)$ be a section that intersects the zero section $M \subset L$ transversally. Let $\nu \to Y$ be the normal bundle of the zero set $Y = s^{-1}(0)$ of s, then ν inherits a complex structure, which is equivalent to the choice of an orientation. Thus a divisor for a complex line bundle is a closed smooth submanifold of codimension 2 with a normal orientation. Two divisors give the same smooth complex line bundle iff they represent the same class in $H^2(M)$.

If $E \to M$ is a quaternionic line bundle, we choose a transversal section $s \in \Gamma(E)$ as above and put $Y = s^{-1}(0)$. Then $ds|_Y \colon \nu \to E|_Y$ is an isomorphism of real vector bundles, so ν inherits a quaternionic structure. In particular, the normal bundle is oriented, and hence defines a class $c_2(E) \in H^4(M)$.

1.7. **Definition.** Let M be a compact smooth manifold. A *quaternionic divisor* in M is a compact codimension-4 submanifold $Y \subset M$ together with a quaternionic structure on its normal bundle. We assume that Y meets ∂M transversally in ∂Y. If M is closed, two quaternionic divisors in $M \times \{0\}$, $M \times \{1\}$ are *equivalent* if they extend to a quaternionic divisor in $M \times [0, 1]$.

Note that $\partial Y \subset \partial M$, in particular $\partial Y = \emptyset$ if M is closed. The equivalence of quaternionic divisors can be defined similarly in the case where $\partial M \neq \emptyset$.

For example, $\mathbb{H}P^k \subset \mathbb{H}P^{k+1}$ with the obvious quaternionic structure on its normal bundle is a divisor for the tautological bundle $H \to \mathbb{H}P^{k+1}$. To reconstruct $E \to M$ from a divisor (Y, ν), let $\nu \to Y$ be classified by $\xi \colon Y \to BS^3$. Because M is compact, ξ factors through some $\mathbb{H}P^k \subset \mathbb{H}P^\infty = BS^3$. The Thom space of the tautological bundle over $\mathbb{H}P^k$ can be identified with $\mathbb{H}P^{k+1}$. Hence, the Pontrjagin-Thom construction gives a classifying map for E as in the following diagram.

$$
\begin{array}{ccccc}
M & \longrightarrow & \mathbb{H}P^{k+1} & \longrightarrow & \mathbb{H}P^\infty \\
\uparrow & & \uparrow & & \uparrow \\
Y & \xrightarrow{\;\xi\;} & \mathbb{H}P^k & \longrightarrow & \mathbb{H}P^\infty
\end{array}
$$

The vertical arrow on the right is given by inserting 0 as first coordinate and shifting all other coordinates one place to the right. One similarly checks that two quaternionic divisors in M give rise to the same quaternionic line bundle iff they are equivalent.

Representing a quaternionic line bundle $E \to M$ by a quaternionic divisor (Y, ν) in M allows us to replace the η-invariants in (1.9) by the η-invariant of an untwisted Dirac operator on Y. However, we will see in Remark 1.10 below that it is not possible to express $t_M(E)$ solely in terms of (Y, ν) without referring to the ambient manifold M.

For motivation, we assume first that (M, E) bounds. Hence let $\overline{E} \to W$ be an extension of $E \to M$, and let $s \in \Gamma(\overline{E})$ be a transversal section of $\overline{E}$ such that $s|_M \in \Gamma(E)$ is transversal as well. Then we obtain a divisor

$$(X, \nu) := \left(s^{-1}(0), \overline{E}|_{s^{-1}(0)} \right)$$

of $\overline{E}$ such that $Y := X \cap M$ is a divisor for M.

Let $\varepsilon \to W$ again denote the trivial quaternionic line bundle. The normal bundle $\nu \cong \overline{E}_{\mathbb{R}}|_X$ carries a natural spin structure with half spinor bundles $\Sigma_\nu^+ \cong \varepsilon|_X$ and $\Sigma_\nu^- \cong \overline{E}|_X$. This implies that $\varepsilon - \overline{E}$ is a K-theoretic direct image of $\varepsilon|_X$ under the inclusion $X \hookrightarrow W$ in the sense of [3] and [7]; in particular

$$\widehat{A}(\overline{E}_{\mathbb{R}}|_X) \, \mathrm{ch}'(\overline{E}|_X) = 1 \ . \tag{1.12}$$

This equation can also be derived directly using the splitting principle.

The bundles $\overline{E} \to M$ and $\nu \to X$ are naturally oriented as real vector bundles by their quaternionic structures. Let $\Omega_{(1)}^\bullet$ denote the space of L^1-forms, and let $\psi(\overline{E}, \nabla^{\overline{E}}) \in \Omega_{(1)}^3(\overline{E})$ denote the Mathai-Quillen current described in [6] with

$$d\psi\big(\overline{E}, \nabla^{\overline{E}}\big) = \pi^* c_2\big(\overline{E}, \nabla^{\overline{E}}\big) - \delta_0 \quad \in \Omega_{(1)}^4(\overline{E}) \ ,$$

where δ_0 denotes the distribution of integration over the zero section. This current can be pulled back to W by the transversal section s, and we have

$$d\big(s^*\psi(\overline{E}, \nabla^{\overline{E}})\big) = c_2\big(\overline{E}, \nabla^{\overline{E}}\big) - \delta_X \quad \in \Omega_{(1)}^4(W) \ , \tag{1.13}$$

where δ_X now denotes the current of integration over X. We conclude that

$$d\big((\hat{c}_2 - s^*\psi)(E, \nabla^E)\big) = \delta_Y \in \Omega_{(1)}^4(M) \ . \tag{1.14}$$

By our assumptions on M, this property determines $(\hat{c}_2 - s^*\psi)(E, \nabla^E)$ up to exact currents.

The connection $\nabla^{\overline{E}}$ induces a natural Riemannian metric on the total space of $\overline{E} \to W$, and we assume that ds induces an isometry of a neighbourhood of X

with a neighbourhood of the zero section in $\overline{E}|_X$. By (1.6), (1.12) and (1.13),

$$
\begin{aligned}
\tau_M(E) &= \frac{-1}{a_{k+1}} \int_W \widehat{A}\big(TW, \nabla^{TW}\big)\,(c_2\ \mathrm{ch}')(\overline{E}, \nabla^{\overline{E}}) \\
&\quad + \frac{1}{a_{k+1}} \int_M \widehat{A}\big(TM, \nabla^{TM}\big)\,(\hat{c}_2\ \mathrm{ch}')(E, \nabla^E) \\
&= \frac{-1}{a_{k+1}} \int_X \widehat{A}\big(TX, \nabla^{TX}\big) \\
&\quad + \frac{1}{a_{k+1}} \int_M \widehat{A}\big(TM, \nabla^{TM}\big)\,\big((\hat{c}_2 - s^*\psi)\ \mathrm{ch}'\big)(E, \nabla^E)\ .
\end{aligned}
\tag{1.15}
$$

Since we have assumed that M and W are spin, there are induced spin structure on X and $Y = X \cap M$. We consider the untwisted Dirac operators D_X and D_Y with respect to these spin structures. Note that ΣX and ΣY carry a quaternionic structure iff $\dim X \cong 4 \bmod 8$. But this is the case iff ΣW and ΣM carry real structures, and hence D_M^E respects a quaternionic structure on $\Sigma M \otimes E$.

1.8. **Proposition.** *Let (Y, ν) be a divisor of a quaternionic line bundle $E \to M$ and assume that a neighbourhood of Y in M is isometric to a neighbourhood of the zero section of ν, where the metric on the total space of ν is induced by a quaternionic metric and a compatible connection. Let $\alpha_Y \in \Omega^3_{(1)}(M)$ denote a current with $d\alpha_Y = \delta_Y$. Then*

$$
t_M(E) = \frac{-1}{a_{k+1}} \left(\frac{\eta + h}{2}(D_Y) - \int_M \widehat{A}\big(TM, \nabla^{TM}\big)\,\alpha_Y\ \mathrm{ch}'(E, \nabla^E) \right)\ .
$$

Proof. If (M, E) bounds, this follows from (1.14), (1.15) and the Atiyah-Patodi-Singer index theorem 1.2 applied to the untwisted Dirac operator D_X on X.

If (M, E) does not bound, we use Bismut-Zhang's formula for the η-invariant of direct images in [7] to derive the result directly from (1.9). $\qquad\square$

1.9. *Remark.* Note that Bismut and Zhang use a current $\gamma \in \Omega^\bullet_{(1)}(M)$ in [7] such that

$$
d\gamma = 2 - \mathrm{ch}\big(E, \nabla^E\big) - \widehat{A}\big(\nu, \nabla^\nu\big)^{-1}\delta_Y\ .
$$

By (1.3), (1.12) and (1.13), the current $(s^*\psi\ \mathrm{ch}')(E, \nabla^E) - \gamma$ is closed. Because in our special situation, the current γ of [7] is constructed directly in terms of s, there exists a natural current on E whose pullback by s becomes γ. In particular, we can work with universal currents on $\mathbb{H}P^N$ for N sufficiently large. Since $\mathbb{H}P^N$ has no cohomology in degrees $4k-1$, we conclude that $(s^*\psi\ \mathrm{ch}')(E, \nabla^E) - \gamma$ is exact, so that we replace γ by $(s^*\psi\ \mathrm{ch}')(E, \nabla^E)$.

1.10. *Remark.* There is a fundamental difference between the class $\hat{c}_2 \in \Omega^3(M)/\operatorname{im} d$ and the currents $s^*\psi$ and $\gamma \in \Omega^3_{(1)}(M)$. The latter are natural in E and s as we have seen in Remark 1.9. In particular, we can choose s in such a way that $\|s\| = 1$ outside a small neighbourhood U of Y. This allows us to choose ∇^E in such a way that $\nabla^E s = 0$ outside U. In other words, contributions coming from $s^*\psi$ or γ localise near Y and do not involve any other topological information about M.

On the other hand, the class $\hat{c}_2$ is constructed using the assumption that $H^3(M;\mathbb{Q}) = H^4(M;\mathbb{Q}) = 0$. In particular, the class $\hat{c}_2$ is not natural in E, but depends on the topology of M. This phenomenon will be exhibited again in Remark 1.14 below.

1.d. **The relationship with the Adams e-invariant.** Following the ideas above, we can identify certain cases where the t-invariant can be related to the Adams e-invariant.

1.11. Proposition. *Let M be a stably framed $(4k-1)$-manifold with $k \geq 3$ and $H^3(M;\mathbb{Q}) = H^4(M;\mathbb{Q}) = 0$. Assume that $E \to M$ is a quaternionic line bundle with a classifying map factoring through $\xi\colon M \to S^4 \cong \mathbb{H}P^1 \subset \mathbb{H}P^\infty$ and that $x_0 \in S^4$ is a regular value of ξ. If $Y = \xi^{-1}(x_0)$ then Y inherits a stable framing from M and ξ, and we have*

$$t_M(E) = -e(Y) \ .$$

Proof. By assumption, the manifold M is framed. The framing in particular selects a Chern-Weil type form $\widetilde{A} = \widetilde{A}(TM \oplus \mathbb{R}^N, \nabla^0, \nabla^{TM \oplus \mathbb{R}^N})$ such that

$$\widehat{A}(TM, \nabla^{TM}) = 1 + d\widetilde{A} \ .$$

We may assume that Y is totally geodesic in M. Because the normal bundle to Y is trivialised by $d\xi$, we also have

$$\widehat{A}(TY, \nabla^{TY}) = 1 + d\widetilde{A}|_Y \ .$$

Following [4, II, Theorem 4.14], we write the e-invariant of Y with the induced framing as

$$e(Y) = \frac{1}{a_k} \left(\frac{\eta + h}{2}(D_Y) - \int_Y \widetilde{A} \right) \ .$$

Consider the tautological bundle $\tau \to \mathbb{H}P^1$ and assume that s_0 is a transversal section with $s_0^{-1} = \{x_0\}$. Then $s = \xi^* s_0$ is a transversal section of E with divisor Y. Let us pull ∇^E back from a connection ∇^τ on $\mathbb{H}P^1$, then $\mathrm{ch}'(E, \nabla^E)|_Y = 1$. From Proposition 1.8 and with $d\alpha_Y = \delta_Y$, we have

$$e(Y) + t_M(E) = \frac{1}{a_k} \left(\int_M \widehat{A}(TM, \nabla^{TM})\, \alpha_Y\, \mathrm{ch}'(E, \nabla^E) - \int_Y \widetilde{A} \right)$$

$$= \frac{1}{a_k} \int_M \left(\alpha_Y\, \mathrm{ch}'(E, \nabla^E) + d(\widetilde{A}\,\alpha_Y\, \mathrm{ch}'(E, \nabla^E)) \right) = 0 \ ,$$

because $\mathrm{ch}'(E, \nabla^E) - 1 \in \Omega^4(M)$ by naturality, hence $\alpha_Y\, \mathrm{ch}'(E, \nabla^E)$ lives in degree $\leq 7 < 4k-1$. $\qquad\square$

The t-invariant is clearly unstable from the point of view of K-theory, because any invariant that is stable under addition of trivial vector bundles is generated by characteristic classes. On the other hand, the following result shows that the t-invariant on S^{4k-1} is stable in the sense of stable homotopy theory.

1.12. Theorem. *Assume that $k \geq 3$. For any homotopy sphere Σ^{4k-1} the homomorphism*

$$t_\Sigma\colon \mathrm{Bun}(\Sigma) \cong \pi_{4k-2}(S^3) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

may be identified with the composition

$$-e \circ S \colon \pi_{4k-2}(S^3) \longrightarrow \pi^S_{4k-5} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

where $-e \colon \pi^S_{4k-5} \to \mathbb{Q}/\mathbb{Z}$ *and* $S \colon \pi_{4k-2} \to \pi^S_{4k-5}$ *are respectively the the negative of the Adams e-invariant* [1] *and the stabilisation homomorphism.*

1.13. **Corollary.** *The t-invariant defines an injective homomorphism*

$$t_{S^{11}} \colon \operatorname{Bun}(S^{11}) \cong \pi_{10}(S^3) \longrightarrow \mathbb{Z}/15 \ .$$

Proof. In this case stabilisation is injective by [25, Ch. XIII] and the e-invariant is injective on the 11-stem by [1, Ex. 7.17].   □

Proof of Theorem 1.12. Recall that the boundary map associated to the Hopf fibration is a homomorphism $\partial \colon \pi_{j+1}(S^4) \to \pi_j(S^3)$. Because the fibre S^3 is contractible in the total space S^7, the long exact homotopy sequence implies that ∂ is surjective.

The Hopf fibration extends to $\mathbb{H}P^k$ for all k. From the diagram

$$
\begin{array}{ccccc}
\pi_{j+1}(S^7) & \longrightarrow & \pi_{j+1}(S^4) & \xrightarrow{\partial} & \pi_j(S^3) \\
\downarrow & & \downarrow & & \| \\
\pi_{j+1}(ES^3) & \longrightarrow & \pi_{j+1}(BS^3) & \xrightarrow[\sim]{\partial} & \pi_j(S^3)
\end{array}
$$

it follows that the homomorphism induced by the standard inclusion

$$\pi_{j+1}(S^4) \longrightarrow \pi_{j+1}(BS^3) \cong \pi_j(S^3)$$

is onto. So without loss of generality, the quaternionic line bundle $E \to S^{4k-1}$ is classified by a map $\xi \colon S^{4k-1} \to S^4 \subset BS^3$. Note that $E \cong \xi^* H$ where H is the tautological bundle over S^4.

We give S^{4k-1} the standard framing, and consider the class

$$[\xi] \in \pi^S_{4k-5} \cong \Omega^{\mathrm{fr}}_{4k-5} \cong \Omega^{\mathrm{fr}}_{4k-1}(S^4)$$

where Ω^{fr}_* denotes the framed bordism and we have applied the Pontrjagin-Thom isomorphism. By the Pontrjagin-Thom construction, this class $[\xi]$ is represented by the quaternionic divisor $Y = \xi^{-1}(x_0)$ for some regular value $x_0 \in S^4$ of ξ. In particular, the manifold Y inherits a framing from ξ. Then by Proposition 1.11, we have

$$t_{S^{4k-1}}(E) = -e(Y) = -e([\xi]) \ . \quad \square$$

1.14. *Remark.* Theorem 1.12 is void for $k = 0$ and 1 , and it does not hold for $k = 2$. For the case $k = 2$ consider the sequence of maps

$$S^7 \xrightarrow{H} S^4 \xrightarrow{F_c} S^4 \longrightarrow \mathbb{H}P^\infty$$

where the first map is the Hopf fibration H and F_c is a self-map of degree $c \in \mathbb{Z}$. Let $E_c \to S^7$ be the pullback of the tautological bundle $H_1 \to \mathbb{H}P^1$, then $t_{S^7}(E_c) = \frac{c(c-1)}{24}$ by Example 3.5.

On the other hand, let $x_0 \in S^4$ be a regular value of F_c and let $Y_c \subset S^7$ be its pre-image: Y_c if a framed 3-manifold which is a disjoint union of fibres of

the Hopf fibration. It is well known that the map H represents a generator of the stable 3-stem with $e([H]) = \pm\frac{1}{24}$. It follows that

$$e(Y_c) = \pm\frac{c}{24}$$

and in particular $t_{S^7}(E_c) \neq -e(Y_c)$ in general.

The following observations view the difference between the t-invariant and the e-invariant from the homotopic point of view: there is an isomorphism $\pi_7(S^4) \cong \mathbb{Z}[H] \oplus S\pi_6(S^3)$ where $S\pi_6(S^3) \cong \mathbb{Z}/12$ is the stabilised group. In these coordinates the stabilisation map $\pi_7(S^4) \to \pi_8(S^5) = \pi_3^S$ is isomorphic to

$$\mathbb{Z} \oplus \mathbb{Z}/12 \longrightarrow \mathbb{Z}/24 \ , \quad (a, [b]) \longmapsto [a - 2b] \ .$$

Moreover, in this basis $[F_c \circ H] = (c^2, [c(c-1)/2])$. In particular $[F_c \circ H]$ stabilises to $c^2 - c^2 + c = c$ and we have the equation

$$t_{S^7} = \pm\frac{\bar{e}(\bar{e} \pm \frac{1}{24})}{24} \colon \operatorname{Bun}(S^7) \cong \pi_6(S^3) \longrightarrow \mathbb{Q}/\mathbb{Z} \tag{1.16}$$

where $\bar{e} = e \circ S \colon \pi_6(S^3) \to \pi_3^S \to \mathbb{Q}/\mathbb{Z}$.

We sketch a proof that $e([H]) = \pm\frac{1}{24}$. The induced framing on a fibre of the Hopf fibration differs from the standard framing by the clutching function of the Hopf fibration itself: call this framed manifold (S^3, π_H). Now apply the bordism definition of the the e-invariant: as a spin manifold (S^3, π_H) bounds over D^4 as S^3 has a unique spin structure. We see that

$$e(S^3, \pi_H) = \frac{1}{2}\widehat{A}(TD; \pi_H)[D^4, S^3] = \frac{1}{2}\widehat{A}(H)[S^4] = \pm\frac{1}{24}$$

where we regard H as the Hopf bundle over S^4 with $p(H)$ a generator of $H^4(S^4)$ and $\widehat{A}_1 = -\frac{1}{12}p$. Here, $\widehat{A}(TD; \pi_H)$ denotes the relative $\widehat{A}$-class with respect to the prescribed framing at the boundary.

As pointed out in Remark 1.10, there are at least two different possible correction terms that construct a differential topological invariant out of the η-invariant of the untwisted Dirac operator on the divisor Y. For $k = 2$ different choices of such correction terms lead to different invariants.

1.e. **A related example in dimension 3.** Let (M, σ) be a compact oriented 3-manifold M with spin^c-structure. DeLoup and Massuyeau [11, Definition 2.2] defined an invariant

$$\phi_{M,\sigma} \colon H_1(M; \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z} \ .$$

In this subsection we give an analytic definition of $\phi_{M,\sigma}$ using the Atiyah-Patodi-Singer ξ-invariant [4, II] which is a precise analogue of the t-invariant in dimension 7 if M is a rational homology sphere.

Let W be a handlebody with $\partial W = M$. Then W carries a unique spin structure, which induces a spin structure on M. There is a one-to-one correspondence between complex line bundles L and spin^c-structures σ on M, given by twisting the fixed spin-structure above by L. The exact sequence of the pair (W, M) contains

$$0 \longrightarrow H^1(M) \longrightarrow H^2(W, M) \longrightarrow H^2(W) \longrightarrow H^2(M) \longrightarrow 0 \ ,$$

so L extends to a complex line bundle $\overline{L}$ on W that induces a spin^c-structure $\bar{\sigma}$ on W extending σ. We also fix a connection ∇^L on $\overline{L} \to W$. The Chern forms of these spin^c-structures are given by

$$c(\bar{\sigma}) = 2c_1\big(\overline{L}, \nabla^{\overline{L}}\big) \qquad \text{and} \qquad c(\sigma) = c(\bar{\sigma})|_M \; .$$

Let $a \in H^1(M; \mathbb{Q}/\mathbb{Z})$, then we construct a flat complex line bundle E_a over M with holonomy given by

$$a \in H^1(M; \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}\big(\pi_1(M), \mu^*\big) \; ,$$

where $\mu^* \subset \mathbb{C}$ denote the group of roots of unity and the isomorphism is induced by the isomorphism $\mathbb{Q}/\mathbb{Z} \to \mu^*$ sending q to $e^{2\pi i q}$. Let $\widetilde{M}$ denote the universal covering of M and let $\pi_1(M)$ act as the group of deck transformations. The bundle E_a is given by

$$\widetilde{M} \times_a \mathbb{C} \qquad \text{with} \qquad \big[x\gamma, e^{-2\pi i\, a(\gamma)} z\big] = [x, z] \qquad \text{for all } \gamma \in \pi_1(M) \; .$$

We define ∇^{E_a} to be the connection on E_a induced by the trivial connection on $\widetilde{M} \times \mathbb{C}$. If $\beta\colon H^1(M; \mathbb{Q}/\mathbb{Z}) \to H^2(M, \mathbb{Z})$ denotes the Bockstein homomorphism, then $c_1(E) = \beta(a)$.

By the exact sequence above, there exists a line bundle $\overline{E}$ on W that extends E, and we can choose an arbitrary connection ∇ on $\overline{E} \to W$, that will not be flat in general, but restricts to the given flat connection on $E \to M$. If $\gamma = \partial\Sigma$ is a curve in M, where $\Sigma \subset W$ is a closed immersed surface, then

$$a(\gamma) = c_1(\overline{E}, \nabla^{\overline{E}})[\Sigma, \partial\Sigma] \quad \mathrm{mod}\ \mathbb{Z} \; .$$

This shows in particular that the first Chern class

$$c_1(\overline{E}) = \big[c_1\big(\overline{E}, \nabla^{\overline{E}}\big)\big] \in H^2(W, M; \mathbb{R})$$

lifts the image of a in $H^2(W, M; \mathbb{Q}/\mathbb{Z})$. This lift is well-defined only up to an integral class.

We compute the Atiyah-Patodi-Singer ξ-invariant of E with respect to the fixed spin^c-structure σ on M and obtain

$$
\begin{aligned}
\xi_{M,\sigma}(E_a, \nabla^{E_a}) &= \frac{\eta+h}{2}\big(D_M^{L\otimes E_a}\big) - \frac{\eta+h}{2}\big(D_M^L\big) \\
&= \int_W \widehat{A}\big(TW, \nabla^{TW}\big)\, \mathrm{ch}\big(\overline{L}, \nabla^{\overline{L}}\big) \left(\mathrm{ch}\big(\overline{E}, \nabla^{\overline{E}}\big) - 1\right) \\
&= \int_W \left(1 + \frac{c(\bar{\sigma})}{2}\right) \left(c_1\big(\overline{E}, \nabla^{\overline{E}}\big) + \frac{1}{2}\, c_1\big(\overline{E}, \nabla^{\overline{E}}\big)^2\right) \\
&= \frac{c_1(\overline{E})\big(c(\bar{\sigma}) + c_1(\overline{E})\big)}{2}\, [W, M] \quad \in \mathbb{R}/\mathbb{Z} \; .
\end{aligned}
$$

Note that the last term is well-defined because $c_1(\overline{E})$ is well defined as a relative class modulo $\mathbb{Z}$. Comparing the above formula with [11, Lemma 2.6] we obtain

1.15. **Proposition.** *Let* (M, σ) *be a closed* spin^c*-manifold and let* $\wp$ *denote Poincaré duality. For all* $a \in H^1(M; \mathbb{Q}/\mathbb{Z})$

$$\phi_{M,\sigma}\big(\pm\wp\, a\big) = \pm\xi_{M,\sigma}(E_a, \nabla^{E_a}) \; .$$

2. THE t-INVARIANT OF 7-MANIFOLDS

In this section we investigate the t-invariant in dimension 7: in 2.a we show that the t-invariant classifies closed smooth 2-connected rational homology 7-spheres up to connected sum with homotopy spheres. In 2.b we show that the t-invariant detects homeomorphisms which are not homotopic to PL homeomorphisms on 2-connected rational homology 7-spheres. In Sections 2.c we relate the t-invariant to the s-invariants of Kreck and Stolz [21] and their generalisations in Hepworth [19].

2.a. **Classification results for 2-connected 7-manifolds.** Throughout this subsection M shall be a closed smooth oriented 2-connected 7-manifold. In addition we assume that M is a rational homology sphere which is equivalent to assuming that $\pi_3(M) \cong H_3(M) \cong H^4(M)$ are finite groups.

We first recall the classification of such 2-connected rational homology spheres started in [27] and completed in [8]. We then relate the t-invariant from Section 1 to this classification to obtain a classification theorem for such manifolds M using t_M and $\mu(M)$, the Eells-Kuiper invariant of M which we recall below.

Recall that Θ_7 denotes the group of diffeomorphism classes of oriented homotopy 7-spheres and that $\Theta_7 \cong \mathbb{Z}/28$. The homotopy 7-spheres can be detected by the Eells-Kuiper invariant $\mu(\Sigma) \in \mathbb{Q}/\mathbb{Z}$ which by [13, §6] defines an injective homomorphism

$$\mu \colon \Theta_7 \longrightarrow \mathbb{Q}/\mathbb{Z} \,.$$

Moreover the definition of μ can be extended to any 2-connected rational homotopy 7-sphere M to give $\mu(M) \in \mathbb{Q}/\mathbb{Z}$ with

$$\mu(M \,\sharp\, \Sigma) = \mu(M) + \mu(\Sigma) \in \mathbb{Q}/\mathbb{Z} \,. \tag{2.1}$$

The definition of $\mu(M)$ in this case is by now routine: from the analytic point of view it can be found in [12, 21]; using coboundaries the details are in [8, 2.12] where μ is called s_1. With either definition the additivity in (2.1) is clear.

This gives an effective strategy for classifying 2-connected rational homology 7-spheres. Firstly consider these manifolds up to *almost diffeomorphism* where an almost diffeomorphism is a homotopy sphere Σ and a diffeomorphism $f \colon N \,\sharp\, \Sigma \cong M$. If M and N are almost diffeomorphic then they are diffeomorphic if and only if $\mu(M) - \mu(N) = \mu(\Sigma) = 0$.

We turn to consider almost diffeomorphism invariants of M. Since M is 2-connected it possesses a unique equivalence class of spin structures. An important invariant is the first characteristic class of spin manifolds which generates $H^4(B\mathrm{Spin}) \cong \mathbb{Z}$. We choose the generator $p \in H^4(B\mathrm{Spin})$ so that that $2p = p_1$ where p_1 is the first Pontrjagin class. The class p is often called "half the first Pontrjagin class" and is sometimes denoted $\frac{p_1}{2}$. For any spin manifold X, we have $p_X = p(TX) \in H^4(X)$.

An important fact about closed smooth 2-connected 7-manifolds proven in [27, Theorem 4] is that every one is the boundary of a handlebody W: i.e. W is a 3-connected 8-manifold obtained from D^8 by attaching 4-handles,

and the boundary of W is identified with M. We consider the exact sequence

$$0 \longrightarrow H^4(W, M) \longrightarrow H^4(W) \xrightarrow{i^*} H^4(M) \longrightarrow 0 , \qquad (2.2)$$

which allows one to define the following quadratic linking function

$$q_M \colon H^4(M) \longrightarrow \mathbb{Q}/\mathbb{Z} , \qquad x \longmapsto \frac{1}{2}\big((\bar{x} + p_W) \smile \hat{x}\big)[W, M] , \qquad (2.3)$$

where $\bar{x} \in H^4(W)$ maps to $x \in H^4(M)$ and $\hat{x} \in H^4(W, M; \mathbb{Q})$ maps to $\bar{x}$ under the isomorphism $H^4(W, M; \mathbb{Q}) \cong H^4(W; \mathbb{Q})$.

2.1. **Lemma** (c.f. [8, Lemma 2.51]). *The function q_M is a well-defined almost diffeomorphism invariant of M.*

We shall say that q_M and q_N are isomorphic if there is an isomorphism $A \colon H^4(M) \cong H^4(N)$ such that if $q_M = q_N \circ A$. The following is also proven in [8].

2.2. **Theorem** (c.f. [8, Theorem A]). *Let N and M be 2-connected rational homology 7-spheres.*

  (1) *There is an almost diffeomorphism $f \colon N \sharp \Sigma \cong M$ with induced map $f^* = A \colon H^4(M) \cong H^4(N)$ if and only if $q_M = q_N \circ A$.*
  (2) *M is diffeomorphic to N if and only if q_M is isomorphic to q_N and $\mu(M) = \mu(N)$.*

2.3. *Remark.* Recall the linking form of M

$$b_M \colon H^4(M) \times H^4(M) \longrightarrow \mathbb{Q}/\mathbb{Z} .$$

which is a non-singular symmetric pairing. In the notation of (2.3), we have

$$b_M(x, y) = (\bar{x} \smile \hat{y})[W, M] = (\hat{x} \smile \bar{y})[W, M] .$$

From (2.3) we see that q_M refines b_M in the following sense:

$$q_M(x + y) = q_M(x) + q_M(y) + b_M(x, y) \quad \forall\, x, y \in H^4(M) . \qquad (2.4)$$

We also see that q_M need not be homogeneous; i.e. $q_M(x) \neq q_M(-x)$ in general. However, the homogeneity defect of q is determined by p_M:

$$q_M(x) - q_M(-x) = (p_W \smile \hat{x})[W, M] = b_M(p_M, x) .$$

Recall now that $\mathrm{Bun}(M)$ denotes the set of isomorphism classes of principal S^3-bundles over M. We have an action of $\mathrm{Bun}(S^7) \cong \pi_6(S^3) \cong \mathbb{Z}_{12}$ on $\mathrm{Bun}(M)$ by

$$\mathrm{Bun}(M) \times \mathrm{Bun}(S^7) \longrightarrow \mathrm{Bun}(M) , \quad (E, F) \longmapsto E \sharp F$$

which is defined by cutting and re-gluing a given principal S^3-bundle E over M along $S^6 = \partial D^7 \subset M$ using the clutching function of F. Next we relate q_M to the t-invariant t_M of Definition 1.4.

2.4. **Theorem.** *Let M be a 2-connected rational homology 7-sphere.*

  (1) *For $M = S^7$, the t-invariant defines an injective homomorphism*

$$t_{S^7} \colon \mathrm{Bun}(S^7) \cong \mathbb{Z}/12 \subset \mathbb{Q}/\mathbb{Z} .$$

(2) *The group* $\mathrm{Bun}(S^7)$ *acts freely on* $\mathrm{Bun}(M)$ *with*
$$\mathrm{Bun}(M)/\mathrm{Bun}(S^7) \xrightarrow[\cong]{c_2} H^4(M) \, ,$$
and for all $(E, F) \in \mathrm{Bun}(M) \times \mathrm{Bun}(S^7)$ *we have*
$$t_M(E \,\sharp\, F) = t_M(E) + t_{S^7}(F) \, .$$
(3) *The t-invariant determines* q_M*: for all* $E \in \mathrm{Bun}(M)$
$$q_M(c_2(E)) = 12 \, t_M(E) \in \mathbb{Q}/\mathbb{Z} \, .$$

As an immediate consequence of Theorem 2.2 (2) and Theorem 2.4 (3) we have

2.5. **Corollary.** *Let* N *and* M *be* 2*-connected rational homology* 7*-spheres. Then* N *is diffeomorphic to* M *if and only if* t_N *is isomorphic to* t_M *and* $\mu(N) = \mu(M)$.

Proof of Theorem 2.4. For (1) we use Proposition 2.6 below with $(n, p, k) = (1, 2, 1)$. One checks that this gives the bundle $E = p_2^* E_2$ of Example 3.5 which is a bundle over S^7 with $t_{S^7}(E) = \frac{1}{12}$. By Proposition 1.5 (2), t_{S^7} is a homomorphism with $t_{S^7}(E^{\sharp k}) = \frac{k}{12}$.

For part (2), let W be a handlebody with $\partial W = M$ as above. As W is homotopy equivalent to a wedge of 4-spheres and BS^3 is three-connected, we have an isomorphism $c_2 \colon \mathrm{Bun}(W) \to H^4(W)$. Hence $c_2 \colon \mathrm{Bun}(M) \to H^4(M)$ is onto by (2.2). There is a homotopy equivalence $M \simeq M^\bullet \cup e^7$, where $M^\bullet := M - \mathrm{int}(D^7)$ is homotopy equivalent to a degree 3-Moore space. Because BS^3 is three-connected, the map $c_2 \colon \mathrm{Bun}(M^\bullet) \to H^4(M^\bullet) \cong H^4(M)$ is an isomorphism. By the surjectivity of $c_2 \colon \mathrm{Bun}(M) \to H^4(M)$, each quaternionic line bundle $E^\bullet$ on $M^\bullet$ extends to M, so $E^\bullet|_{\partial e^7}$ is trivial and $\mathrm{Bun}(S^7)$ acts transitively on the set of possible extensions.

The formula for t_M given in (2) is a special case of the additivity formula of Proposition 1.5 (2). Together with (1), it proves that the action of $\mathrm{Bun}(S^7)$ on $\mathrm{Bun}(M)$ is free.

For part (3), let $\overline{E} \to W$ be a quaternionic line bundle with $c_2(\overline{E}) = \bar{x} \in H^4(W)$. Let $\hat{c}_2(\overline{E}) \in H^4(W, M; \mathbb{Q})$ be a lift of $c_2(\overline{E})$. We use equation (1.6), the fact that the $\widehat{A}$-genus begins $\widehat{A} = 1 - \frac{p}{12} + \dots$ and (1.4) to deduce the following formula
$$t_M(i^* E) = \frac{1}{24}\big(\hat{c}_2(\overline{E}) \smile (p_W + c_2(\overline{E}))\big)[W, M] \in \mathbb{Q}/\mathbb{Z} \, . \tag{2.5}$$
Comparing (2.5) with (2.3) we see that $q_M(c_2(E)) = 12 t_M(E)$. $\square$

We finish this subsection by recording calculations of t_M in some examples. Let n and p be integers with $n \neq 0$ and let $\pi \colon W_{n,p} \to S^4$ be the disc bundle of a vector bundle over the 4-sphere with Euler class $e(\pi) = n\bar{x}$ and with $p_W = p\bar{x}$: here we fix a generator $\bar{x}$ of $H^4(S^4)$ and identify $H^4(S^4) = H^4(W_{n,p})$. Let also $i \colon M_{n,p} \to W_{n,p}$ be the inclusion of the boundary so that $M_{n,p}$ is the total space of a 3-sphere bundle over S^4. We remark that these total spaces were classified up to diffeomorphism, homeomorphism and homotopy equivalence in [9] but using a different notation: the total space denoted $M_{m,n}$ in [9] is the total space denoted $M_{n,n+2m}$ in this work (see, e.g., [9, Fact 3.1]).

2.6. Proposition. *With $i\colon M_{n,p} \to W_{n,p}$ and $\bar{x} \in H^4(S^4)$ as above and k any integer, let $\overline{E}_k \to W_{n,p}$ be the quaternionic line bundle with $c_2(\overline{E}) = k\bar{x}$ and let $E_k := i^*\overline{E}_k$. Then for $M = M_{n,p}$ we have*

$$t_M(i^*E_k) = \frac{k(p+k)}{24n} \quad \in \quad \mathbb{Q}/\mathbb{Z} .$$

Moreover for $[k] := i^*(k\bar{x}) \in \mathbb{Z}/n\mathbb{Z} \cong H^4(M)$, the function q_M is given by

$$q_M([k]) = \frac{k(p+k)}{2n} \quad \in \quad \mathbb{Q}/\mathbb{Z} .$$

Proof. The intersection form $(H_4(W_{n,p}), \lambda)$ is isomorphic to $(\mathbb{Z}, n)$. We now simply apply the expression for t_M in (2.5). $\qquad\square$

2.7. *Remark.* While it is possible to use the analytic definition of the t-invariant to compute t_M in this case, the computations still require sophisticated technique and are somewhat lengthy compared to the topological definition.

In the search for new compact Riemannian manifolds of positive sectional curvature, Grove, Wilking and Ziller considered two families $M_{(p_-,q_-),(p_+,q_+)}$ and $N_{(p_-,q_-),(p_+,q_+)}$ of seven-manifolds of cohomogeneity one in [18]. The first family consists of two-connected manifolds, while the second is of the type considered by Kreck and Stolz in [21]. By [18], the manifolds $P_n = M_{(1,1),(2n-1,2n+1)}$ are the only members of the M-family that can carry metrics of positive sectional curvature. It was then proved independently by Dearricott [10] and Grove, Verdiani and Ziller [17] that the particular space $M_{(1,1),(3,5)}$, which is homeomorphic to the unit tangent bundle of S^4, carries a metric of positive sectional curvature. In [15], one of us determined the diffeomorphism types of the spaces P_n using Corollary 2.5 above.

Let (p_+, q_+) and (p_-, q_-) be two pairs of relative prime positive odd integers. According to [18], the manifold $M = M_{(p_-,q_-),(p_+,q_+)}$ has $H^4(M) \cong \mathbb{Z}/n\mathbb{Z}$ with $n = \left| p_-^2 q_+^2 - p_+^2 q_-^2 \right|/8$. One can write M as the total space of two Seifert fibrations π_1, π_2 over the base S^4. Both Seifert fibrations are singular over two disjoint submanifolds of S^4 diffeomorphic to $\mathbb{R}P^2$, and the type of the singularity is described by the two pairs $(p_\pm, q_\pm)$. Using a generalisation of the adiabatic limit formula for eta-invariants to Seifert fibrations, it is possible to compute $t_M(\pi_i^*E)$ for all quaternionic line bundles $E \to S^4$.

2.8. *Example.* Let $E \to S^4$ be a quaternionic line bundle with $c_2(E)[S^4] = k \in \mathbb{Z}$. By [15],

$$t_M(\pi_1^*E) = \frac{k(p_+^2 - p_-^2 - n + kp_-^2 p_+^2)}{24n} .$$

Swapping the roles of $p_\pm$ and $q_\pm$ gives the analogous formula for $t_M(\pi_2^*E)$.

For the manifolds P_n, the formula above specialises to

$$t_{P_n}(\pi_1^*E) = \frac{k(k-n)}{24n} .$$

By Theorems 2.2 and 2.4 and Proposition 2.6 above, these spaces P_n are almost diffeomorphic to the manifolds $M_{n,n}$ of Proposition 2.6: in other words, to the total spaces of the principal S^3-bundles over S^4 with Euler class given by n. The

classification is then completed by computing the Eells-Kuiper invariant $\mu(P_n)$. In particular,

$$P_n \cong M_{n,n} \,\sharp\, \Sigma^{\# \frac{n-n^3}{6}} \ ,$$

where Σ is a generator of Θ_7 with $\mu(\Sigma) = \frac{1}{28}$.

For the other spaces in the M family, one can to combine the formulas for $t_M(\pi_i^* E)$ for both fibrations with equation (2.4) to determine q_M completely.

2.b. **Detecting exotic homeomorphisms.** Throughout this subsection M and N will be closed smooth oriented 2-connected rational homology 7-spheres and all maps will be orientation preserving. We call a homeomorphism

$$h \colon N \cong M$$

exotic if it is not homotopic to a piecewise linear (PL) homeomorphism. The main result of this section is that h is exotic if and only if the induced map $h^* \colon \mathrm{Bun}(M) \to \mathrm{Bun}(N)$ does not preserve the t-invariants of M and N.

To make a more precise statement we first recall the following consequence of topological surgery from [22]. For a homeomorphism $h \colon N \cong M$ there is an invariant

$$\mathrm{KS}(h) \in H^3(M; \mathbb{Z}/2) \ ,$$

depending only on the homotopy class of h, such that h is exotic if and only if $\mathrm{KS}(h) \neq 0$.

2.9. **Theorem.** *A homeomorphism $h : N \to M$ is exotic if and only if*

$$t_M \neq t_N \circ h^*.$$

More precisely, for all $E \in \mathrm{Bun}(M)$,

$$\big(\mathrm{KS}(h) \smile c_2(E)\big)[M]_2 = t_N(h^* E) - t_M(E) \in \mathbb{Z}/2 \subset \mathbb{Q}/\mathbb{Z} \ ,$$

where $[M]_2$ generates $H_7(M; \mathbb{Z}/2)$.

We begin by showing that t_M is invariant under PL-homeomorphisms.

2.10. **Lemma.** *If $h \colon N \cong M$ is a PL-homeomorphism then $t_M = t_N \circ h^*$.*

Proof. It follows from smoothing theory, see [8, §6, Theorem 6.1], that there is a homotopy 7-sphere Σ such that h is homotopic to a diffeomorphism $g \colon N \,\sharp\, \Sigma \to M$. The lemma now follows by Proposition 1.5 (3). $\qquad\square$

2.11. *Remark.* Note that [8, Theorem 6.1] also proves that N is homeomorphic to M if and only if N is *PL*-homeomorphic to M. Hence the function t_M is a topological invariant in the weak sense that if $h \colon N \to M$ is a homeomorphism then there is a homeomorphism, indeed a *PL*-homeomorphism, $g \colon N \to M$ such that $t_M = t_N \circ g^*$.

Proof of Theorem 2.9. We will not explicitly construct exotic homeomorphisms but rather we use surgery theory to show that they exist. Hence we begin by briefly recalling some essential notions from surgery. Let $Cat = Top$ or PL denote respectively the topological and piecewise linear categories of manifolds. Recall the Cat-structure set of M,

$$\mathcal{S}^{Cat}(M) := \{g \colon N \simeq M \,|\, N \text{ a } Cat\text{-manifold}\}/ \sim \ ,$$

which consists of equivalence classes of *structures* which are homotopy equiv-
alences $g\colon N \to M$. Two structures g_0 and g_1 are equivalent if $g_1 \circ g_0^{-1}$ is
homotopic to a *Cat*-isomorphism. The base-point of $\mathcal{S}^{Cat}(M)$ is the trivial
element $[\mathrm{Id}\colon M \to M]$.

The *Cat*-structure set of M lies in the *Cat*-surgery exact sequence (see [26,
Chapter 10], [22] for more definitions and details):

$$L_8(e) \longrightarrow \mathcal{S}^{Cat}(M) \xrightarrow{\eta^{Cat}} [M, G/Cat] \longrightarrow L_7(e) \ .$$

Here $L_7(e) = 0$ and $L_8(e) \cong \mathbb{Z}$ are the simply connected surgery obstruction
groups, $G = \lim_n \mathrm{Map}_{\pm 1}(S^n, S^n)$ is the monoid of stable self-equivalences of the
n-sphere, *Top* and *PL* are the groups of stable homeomorphisms, respectively
PL-homeomorphisms, of Euclidean space and η^{Cat} denotes the *Cat*-normal in-
variant map. It is well known that the map $L_8(e) \to \mathcal{S}^{Cat}(M)$ vanishes in both
the topological or piecewise linear categories. So from the *Cat*-surgery exact
sequences we obtain the following commuting square:

$$
\begin{array}{ccc}
\mathcal{S}^{PL}(M) & \xrightarrow{\ \ F\ \ } & \mathcal{S}^{Top}(M) \\[4pt]
\Big\downarrow{\scriptstyle \eta^{PL}} & & \Big\downarrow{\scriptstyle \eta^{Top}} \\[4pt]
[M, G/PL] & \xrightarrow{\ \ F_*\ \ } & [M, G/Top]
\end{array}
\tag{2.6}
$$

Here each η^{Cat} is a bijection, F is the forgetful map and the canonical
map $G/PL \to G/Top$ induces F_*. By definition, an exotic homeomor-
phism $h\colon N \cong M$ defines a non-trivial element of $\mathcal{S}^{PL}(M)$ which maps to
the trivial element of $\mathcal{S}^{Top}(M)$.

2.12. **Lemma.** *There are bijections $\mathcal{S}^{PL}(M) \equiv H^4(M)$ and $\mathcal{S}^{Top}(M) \equiv H^4(M)$
such that the forgetful map F corresponds to the map $\times 2$ in the cohomology
Bockstein sequence for the coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$. In
particular, there is a short exact sequence of abelian groups*

$$0 \longrightarrow H^3(M; \mathbb{Z}/2) \longrightarrow \mathcal{S}^{PL}(M) \xrightarrow{\ F\ } \mathcal{S}^{Top}(M) \longrightarrow H^4(M; \mathbb{Z}/2) \ .$$

Proof. Using (2.6) if suffices to calculate $F_*\colon [M, G/PL] \to [M, G/Top]$ which
is routine: there is a homotopy equivalence

$$M \simeq M^\bullet \cup_\phi e^7$$

where $M^\bullet := M - \mathrm{int}(D^7)$, ϕ is the attaching map for the top cell of M
and $M^\bullet$ is homotopic to the Moore space $M(H, 3)$ where $H = H_3(M)$. We
write $H = \oplus_{i=1}^r C_{k(i)}$ as a sum of cyclic groups and so we have a homotopy
equivalence

$$M^\bullet \simeq \bigvee_{i=1}^r M(\mathbb{Z}/k(i), 3) \simeq \bigvee_{i=1}^r (S_i^3 \cup_{k(i)} e_i^4)$$

where $k(i)$ denotes a map $S_i^3 \to S_i^3$ of degree $k(i)$.

It is known that the homotopy groups of G/PL and G/Top vanish in odd
dimensions and that $\pi_4(G/PL) \to \pi_4(G/Top)$ is isomorphic to $\times 2\colon \mathbb{Z} \to \mathbb{Z}$.
We deduce that the homomorphism $[M^\bullet, G/PL] \to [M^\bullet, G/Top]$ is isomorphic
to $H^4(M^\bullet) \xrightarrow{\times 2} H^4(M^\bullet)$.

It remains to show that the inclusion $M^\bullet \to M$ of the 4-skeleton induces isomorphisms $[M, G/Cat] \cong [M^\bullet, G/Cat]$. Let

$$C \colon M^\bullet \longrightarrow \bigvee_{i=1}^{r} S^4 \qquad\qquad (2.7)$$

be the map collapsing the 3-skeleton and observe that the homomorphism

$$C^* \colon \oplus_{i=1}^{r} \pi_4(G/Cat) \longrightarrow [M^\bullet, G/Cat]$$

is onto because $\pi_3(G/Cat) = 0$. It follows that the induced homomorphism

$$\phi^* \colon [M^\bullet, G/Cat] \longrightarrow \pi_6(G/Cat)$$

vanishes since the composite $C \circ \phi \colon S^6 \to \vee S^4$ must be a wedge of trivial maps or the essential map $\eta^2 \colon S^6 \to S^4$ [2] and the induced homomorphism $(\eta^2)^* \colon \pi_4(G/Cat) \to \pi_6(G/Cat)$ factors through $\pi_5(G/Cat) = 0$. From the short exact sequence

$$0 = \pi_7(G/Cat) \longrightarrow [M, G/Cat] \longrightarrow [M^\bullet, G/Cat] \xrightarrow{\phi^*} \pi_6(G/Cat)$$

we deduce that $[M, G/Cat] \to [M^\bullet, G/Cat]$ is an isomorphism. $\qquad\square$

Now note that $H^3(M; \mathbb{Z}/2) = H^4(M; \mathbb{Z}/2) = 0$ if and only if $H^4(M)$ contains no 2-torsion. In this case the forgetful map $F \colon \mathcal{S}^{PL(M)} \to \mathcal{S}^{Top}(M)$ is a bijection and M admits no exotic homeomorphisms.

Henceforth we assume that $H^4(M)$ contains 2-torsion. We see that the PL normal invariants of exotic homeomorphisms lie in

$$\mathrm{Ker}\big([M, G/PL] \xrightarrow{F} [M, G/Top]\big) \cong H^3(M; \mathbb{Z}/2) \ .$$

Using Lemma 2.12 we shall regard $H^3(M; \mathbb{Z}/2)$ as a subset of $\mathcal{S}^{PL}(M) \equiv [M, G/PL]$.

2.13. Lemma. *Let $x \in H^3(M; \mathbb{Z}/2)$. Then there is a self-homotopy equivalence $p(x) \colon M \simeq M$ such that:*

(1) *the PL normal invariant of $p(x)$ is $x \in [M, G/PL]$,*
(2) *$t_M(p(x)^*E) - t_M(E) = \big(x \smile c_2(E)\big)[M]_2$ for all $E \in \mathrm{Bun}(M)$.*

Before proving Lemma 2.13 let us complete the proof of Theorem 2.9. Let $h \colon N \simeq M$ be a homeomorphism. If h is not exotic then $t_M = t_N \circ h^*$ by Lemma 2.10. Conversely, suppose that h is exotic: the PL normal invariant of h is a non-zero element $x \in H^3(M; \mathbb{Z}/2)$. By Lemma 2.13 (1), we have $[h] = [p(x)] \in \mathcal{S}^{PL}(M)$ and so by definition there is a PL homeomorphism $d \colon N \to M$ such that $h \simeq p(x) \circ d$. It follows for all $E \in \mathrm{Bun}(M)$ that

$$t_N(h^*E) = t_N(d^* p(x)^* E) = t_M(p(x)^*E) = t_M(E) + \big(x \smile c_2(E)\big)[M]_2 \ .$$

Here we used Lemma 2.10 for the second equality and Lemma 2.13 (2) for third equality. Finally, note that $p(x)$ has a trivial topological normal invariant and so is homotopic to an exotic self-homeomorphism $h \colon M \cong M$. This completes the proof of Theorem 2.9. $\qquad\square$

---

[2] In fact $c \circ \phi$ is null-homotopic since M is a manifold and the homotopy class of $c \circ \phi$ detects the exotic class of a 2-connected 7-dimensional Poincaré complex.

Proof of Lemma 2.13. We begin with the definition of $p(x)$ and the proof of part (1). Recall that $M^\bullet := M - \text{int}(D^7)$ and that $M^\bullet$ is homotopy equivalent to the Moore space $M(H, 3)$ where $H = H_3(M) \cong \oplus_{i=1}^r C_{k(r)}$ is a sum of cyclic groups. We assume that $k(r)$ is even if and only if $i \in \{1, \ldots, s\}$.

The self-equivalences $p(x) \colon M \to M$ we construct will be *pinch maps*. Pinch maps are described in [23, §4] for manifolds with boundary but to begin we shall consider the closed case. Take an element $\psi \in \pi_7(M^\bullet)$ and define $p(\psi)$ to be the composition

$$p(\psi) := (\text{Id} \vee \psi) \circ p \colon M \longrightarrow M \vee S^7 \longrightarrow M$$

where $p \colon M \to M \vee S^7$ collapses the boundary of a 7-disc $D^7 \subset M$ to a point and we identify $M = M/D^7$.

To build pinch maps $p(\psi)$ we shall need some knowledge of $\pi_7(M^\bullet)$: recall that $C \colon M^\bullet \to \bigvee_{i=1}^r S_i^4$ collapses the 3-skeleton and that $\pi_7(S_i^4) \cong \mathbb{Z} \oplus \mathbb{Z}/12$. Observe also that $M^\bullet = S(M(H, 2))$ is the suspension of the degree two Moore space.

2.14. **Lemma.** *For $i = 1, \ldots, s$ there are homotopy classes $\varphi_i \in \pi_6(M(H, 2))$ such that the suspensions $\psi_i \colon = S(\varphi_i) \in \pi_7(M^\bullet)$ satisfy*

$$C_*(\psi_i) = (0, 6) \in \mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \cong \pi_7(S_i^4) \subset \pi_7\left(\bigvee_{i=1}^r S_i^4\right).$$

Proof. The stabilisation homomorphism $S : \pi_6(M(H, 2)) \to \pi_7(M^\bullet)$ fits into the following commutative diagram of exact sequences [3].

$$\pi_6\left(\bigvee_{i=1}^r S^2\right) \longrightarrow \pi_6(M(H, 2)) \xrightarrow{j_*} \pi_6\left(M(H, 2), \bigvee_{i=1}^r S_i^2\right) \xrightarrow{\partial} \pi_5\left(\bigvee_{i=1}^r S_i^2\right)$$

$$\downarrow S \qquad\qquad \downarrow S \qquad\qquad \downarrow S \qquad\qquad \downarrow S$$

$$\pi_7\left(\bigvee_{i=1}^r S^3\right) \longrightarrow \pi_7(M^\bullet) \xrightarrow{j_*} \pi_7\left(M^\bullet, \bigvee_{i=1}^r S^3\right) \xrightarrow{\partial} \pi_6\left(\bigvee_{i=1}^r S_i^3\right)$$

Let $X_{4,i}$ generate the homotopy group $\pi_4(S_i^3 \cup_{k(i)} e_i^4, S^3) \cong \mathbb{Z}$, and let Ψ generate $\pi_7(D^4, S^3) \cong \pi_6(S^3) \cong \mathbb{Z}/12$. On classes of the form $X_{4,i} \circ \Psi$, the boundary map ∂ is given by multiplication by $k(i)$. An analogous statement holds for $X_{3,i}$, a generator of $\pi_3(S_i^2 \cup_{k(i)} e_i^3, S_i^2) \cong \mathbb{Z}$ and compositions $X_{3,i} \circ \Phi$ for Φ a generator of $\pi_6(D^3, S^2) \cong \pi_5(S^2) \cong \mathbb{Z}/2$. It is well known that the stabilisation map $S \colon \pi_5(S^2) \to \pi_6(S^3)$ maps Φ to 6Ψ. It follows for $i \leq s$, that the classes $X_{3,i} \circ \Phi$ and $X_{4,i} \circ (6\Psi)$ are in the image of the maps j_* and that $S(X_{3,i} \circ \Phi) = X_{4,i} \circ (6\Psi)$. We choose $\varphi_i \in \pi_6(M(H, 2))$ such that

$$j_*(\varphi_i) = X_{3,i} \circ \Phi$$

and define

$$\psi_i := S(\varphi_i) \in \pi_7(M^\bullet) \tag{2.8}$$

so that in particular $j_*(\psi_i) = X_{4,i} \circ (6\Psi)$.

---

[3] The lower exact sequence was comprehensively studied for $r = 1$ in [24].

To see that the homotopy classes ψ_i have the stated property, we pass to stable homotopy groups: denoted π_*^S with stabilisation $S : \pi_* \to \pi_*^S$. By excision we have an isomorphism

$$\pi_7^S\left(M^\bullet, \bigvee_{i=1}^r S_i^3\right) \cong \pi_7^S\left(\bigvee_{i=1}^r S_i^4\right)$$

and since $X_{4,i}$ stabilises to generate $\pi_4^S(S_i^3 \cup_{k(i)} e_i^4, S_i^3) \cong \mathbb{Z}$ we quickly deduce that $C_* S(\psi_i)$ is the element of order two in $\pi_7^S(S_i^4)$. The stabilisation map $\pi_6(S^3) \to \pi_3^S$ is isomorphic to the inclusion $\mathbb{Z}/12 \to \mathbb{Z}/24$ and this completes the proof. $\qquad\square$

2.15. **Definition.** Using the homotopy equivalence $M^\bullet \simeq \bigvee_{i=1}^r M(\mathbb{Z}/k(i), 3)$ define generators $\{x_1, \ldots, x_s\}$ of $H^3(M; \mathbb{Z}/2)$ where each x_i is the pullback of the generator of $H^3(M(\mathbb{Z}/k(i)); \mathbb{Z}/2) = \mathbb{Z}/2$. Given $x \in H^3(M; \mathbb{Z}/2)$ write $x = \Sigma_{i=1}^s \epsilon_i x_i$ where $\epsilon_i = 0$ or 1 and define

$$p(x) := p(\Sigma_{i=1}^s \epsilon_i \psi_i)$$

with $\psi_i \in \pi_7(M^\bullet)$ defined as in (2.8).

To complete the proof of part (1) of Lemma 2.13 it remains to show that $\eta^{PL}(p(x)) = x \in H^3(M; \mathbb{Z}/2) \subset [M, G/PL]$. We shall apply the methods and results of [23] which require us to consider *tangential surgery theory*. We shall not give the details here but rather refer to [23, §2]. Another feature of [23] is that they work with manifolds with boundary but this is fine in our setting: there are obvious maps $\mathcal{S}^{Cat}(M) \to \mathcal{S}^{Cat}(M^\bullet)$ which in our case are bijections. We shall also write $p(\psi_i) \colon M^\bullet \to M^\bullet$ for the corresponding pinch map on $M^\bullet$.

Choose a bundle isomorphism $b(\psi_i) \colon \nu_{M^\bullet} \cong \nu_{M^\bullet}$ covering $p(\psi_i)$ such that the pair $(p(\psi_i), b(\psi_i))$ defines an element in the *tangential structure set* of M, $\mathcal{S}^{Cat,t}(M)$. The tangential normal invariant set is in bijective correspondence with $[M, G]$ and the obvious forgetful map fits into the following commutative square (see [23, (2.4)]).

$$
\begin{array}{ccc}
\mathcal{S}^{Cat,t}(M^\bullet) & \xrightarrow{\;\;F\;\;} & \mathcal{S}^{Cat}(M^\bullet) \\
\Big\downarrow{\scriptstyle \eta^{Cat,t}} & & \Big\downarrow{\scriptstyle \eta^{Cat}} \\
[M^\bullet, G] & \longrightarrow & [M^\bullet, G/Cat]
\end{array}
$$

Moreover, for $Cat = PL$ we have that $[M^\bullet, G] \to [M^\bullet, G/PL]$ is isomorphic to the boundary map $\beta_{24} \colon H^3(M; \mathbb{Z}/24) \to H^4(M; \mathbb{Z})$ in the cohomology Bockstein sequence for $\mathbb{Z} \overset{\times 24}{\to} \mathbb{Z} \to \mathbb{Z}/24$. It is a simple matter to check that these facts and the following lemma complete the proof of part (1) of Lemma 2.13.

2.16. **Lemma.** *The tangential normal invariant of $(p(\psi_i), b(\psi_i))$ is independent of $b(\psi_i)$ and given by the equation*

$$\eta^{PL,t}(p(\psi_i), b(\psi_i)) = \iota(x_i) \in H^3(M; \mathbb{Z}/24) \cong [M^\bullet, G]$$

where x_i is as in Defintion 2.15 and $\iota \colon H^3(M; \mathbb{Z}/2) \to H^3(M; \mathbb{Z}/24)$ is induced by the inclusion of coefficients $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/24$.

Proof. The tangential normal invariant of $(p, b) := (p(\psi_i), b(\psi_i))$ is computed as follows: first use the Pontrjagin-Thom isomorphism to obtain an element in the stable homotopy group of the pair $(T(\nu_{M^\bullet}), T(\nu_{M^\bullet}|*))$ where $T(\xi)$ denotes the Thom space of a vector bundle which we assume has rank $k > 8$ and $* \in \partial M^\bullet$ is a base-point in the boundary of $M^\bullet$. We write:

$$PT(p, b) \in \pi_{7+k}(T(\nu_{M^\bullet}), T(\nu_{M^\bullet}|*)) \ .$$

Next one applies Spanier-Whitehead duality

$$D \colon \pi_{7+k}(T(\nu_{M^\bullet}), T(\nu_{M^\bullet}|*)) \longrightarrow [M, G]$$

and defines $\eta^{Cat,t}(p, b) := D(PT(p, b))$.

Now $M^\bullet = \Sigma M(H, 2)$ is a suspension and so $T(\nu_{M^\bullet})/T(\nu_{M^\bullet}|*) \simeq \Sigma^k M^\bullet$ (see the remark at the bottom of [23, p. 469]). Moreover, the pinch map $p(\psi_i)$ is defined using a suspension $\psi_i = S(\varphi_i)$, $\varphi_i \in \pi_6(M(H, 2))$. It follows by [23, Theorem 4.7 and Lemma 4.8] that

$$\eta^{PL,t}(p(\psi_i), b(\psi_i)) = D(S(\psi_i))$$

where $S(\psi_i) \in \pi_7^S(M^\bullet)$ is the stabilisation of ψ_i. But by the proof of Lemma 2.14 we know that $C_*S(\psi_i) \in \pi_7^S(\bigvee_{i=1}^r (S_i^4))$ is precisely the element of order two in the summand $\pi_7^S(S_i^4) \cong \mathbb{Z}/24$. Applying Spanier-Whitehead duality to this statement completes the proof. $\qquad\square$

We now prove part (2) of Lemma 2.16. If $x = 0$ then $p(x) \simeq \mathrm{Id}$ and the statement is obvious. Assume then that $x \neq 0 \in H^3(M; \mathbb{Z}/2)$. By choosing an appropriate set of generators for $H_3(M)$ we may assume that $x = x_1$ in the notation of Definition 2.15. Let $f \colon M \to BS^3$ classify $E \in \mathrm{Bun}(M)$. As $p(x_1)$ is the pinch map on $\psi_1 \in \pi_7(M^\bullet)$ the induced map $p(x_1)^*$ of $\mathrm{Bun}(M)$ is given by

$$p(x_1)^* E = E \,\sharp\, F_E \tag{2.9}$$

where $F_E \in \mathrm{Bun}(S^7)$ is classified by the composition

$$f|_{M^\bullet} \circ \psi_1 \colon S^7 \longrightarrow M^\bullet \longrightarrow BS^3 \ .$$

But BS^3 is 3-connected and so $f|_{M^\bullet}$ factors through the collapse map C of (2.7):

$$f|_{M^\bullet} \;=\; \bar{f} \circ C \colon M^\bullet \longrightarrow \bigvee_{i=1}^r S_i^4 \longrightarrow BS^3 \ .$$

By definition $\psi_1 \colon S^7 \to M^\bullet$ is such that $C_*(\psi_1) = (0, 6) \in \pi_7(S_1^4) \cong \mathbb{Z} \oplus \mathbb{Z}_{12}$. Recall that $H \to \mathbb{H}P^\infty$ is the tautological bundle and so $F_E = (f|_{M^\bullet} \circ \psi_1)^* H$ is determined by $c_2(E)$ as follows:

$$F_E = \begin{cases} 6 \in \pi_7(BS^3) \cong \mathbb{Z}/12\mathbb{Z}, & \text{if } c_2(E)[S_1^4] \text{ is odd}, \\[2mm] 0 \in \pi_7(BS^3) \cong \mathbb{Z}/12\mathbb{Z}, & \text{if } c_2(E)[S_1^4] \text{ is even}. \end{cases} \tag{2.10}$$

Applying Theorem 2.4 (1) and Proposition 1.5 (2) to (2.9) and (2.10) above we see that

$$t_M(p(x_1)^* E) - t_M(E) = t_{S^7}(F_E) = \big(x_1 \smile c_2(E)\big)[M]_2 \in \mathbb{Z}/2 \subset \mathbb{Q}/\mathbb{Z} \ . \qquad\square$$

2.c. **Remarks on simply-connected 7-manifolds.** In this subsection we make some remarks about the role of the t-invariant in the classification of simply connected spin 7-manifolds.

2.17. *Remark.* The invariant t_M is related to the Kreck-Stolz invariants s_2 and s_3 of [21]. To begin let M be a closed spin $(4k-1)$-manifold as in 1.4 and let $L \to M$ be a Hermitian line bundle with $c_1(L) = a \in H^2(M)$. In analogy with (1.3), we define a characteristic class ch' of complex line bundles such that

$$\text{ch}(L) = 1 + c_1(L) + c_1(L)^2 \ \text{ch}'(L) \ .$$

Assume that L extends to $\overline{L} \to W$, where W is a compact spin manifold with $M = \partial W$. The class $c_1(\overline{L})^2 \in H^4(W)$ lifts uniquely to $\bar{v} \in H^4(W, M; \mathbb{Q})$, so we can define

$$s_M(a) := \big(\widehat{A}(TW) \ \text{ch}'(\overline{L}) \ \bar{v}\big)[W, M] \in \mathbb{Q}/\mathbb{Z}$$

independent of the choices of W and $\overline{L}$. If $k = 2$ so that M is 7-dimensional and $H^2(M) \cong \mathbb{Z}$ with generator z, then Kreck and Stolz considered $s_2(M) := s_M(z)$ and $s_3(M) := s_M(2z)$.

A Hermitian metric on L identifies the dual bundle L^* with the complex conjugate of L, and this identification is parallel with respect to compatible Hermitian connections. Thus the bundle

$$E := L \oplus L^*$$

carries a natural quaternionic structure with $c_2(E) = -c_1(L)^2$. In particular, we have

$$\text{ch}(E) = \text{ch}(L) + \text{ch}(L^*) \qquad \text{and} \qquad \text{ch}'(E) = \text{ch}'(L) + \text{ch}'(L^*)$$

and $\bar{v} = -\bar{c}_2(E)$. Thus $a_{k+1}t_M(E) = 2s_M(a)$, or equivalently

$$t_M(E) = a_k \, s_M(a) \ .$$

Hence the t-invariant generalises the s-invariant if k is even. In particular for $k = 2$, t_M generalises s_2 and s_3. There is an analogous argument comparing the intrinsic definitions of s_M and t_M.

We may now unify the classification results of [21] and [8]. Let N and M be closed smooth simply-connected spin 7-manifolds and assume that $\pi_2(N) \cong \pi_2(M)$ is either trivial or infinite cyclic. In the latter case assume that $H^4(N) \cong H^4(M)$ are finite cyclic groups generated by the square of a generator of H^2 and extend the definition of the t-invariant to the $\hat{t}$-invariant which is the pair

$$\hat{t}_M = (t_M, A_M)$$

where $A_M \colon H^2(M) \cong \text{Bun}_{S^1}(M) \to \text{Bun}(M)$ maps L to $L \oplus L^*$ and t_M is the t-invariant.

2.18. **Theorem.** *Let N and M be simply connected spin 7-manifolds as above. Then N is diffeomorphic to M if and only if $\hat{t}_N$ is isomorphic to $\hat{t}_M$ and $\mu(N) = \mu(M)$.*

2.19. *Remark.* Expanding on [20, Theorem 6], Hepworth [19, Thereom 2.2.9] gave a classification theorem for simply connected compact spin 7-manifolds with $H^3(M) = 0$, such that $H^4(M)$ is torsion and generated by $p_M = \frac{p_1}{2}(TM)$ and cup products of elements of $H^2(M)$.

The group $H^2(M)$ is necessarily free, so there exists a $\mathbb{Z}$-basis $(x_1, \ldots, x_r)$. Apart from the Eells-Kuiper invariant, and the triple Massey products, Hepworth defines five more families of invariants, σ_i, σ_{ij}, τ_i, τ_{ij}, and $\tau_{ij,k}$ for i, j, $k \in \{1, \ldots, r\}$, based on the existence of a closed spin manifold W with $\partial W = M$ such that the inclusion $M \to W$ induces an isomorphism $H^2(W) \cong H^2(M)$. Together, these invariants give a complete set of diffeomorphism invariants. The σ- and τ-invariants can be expressed in terms of the function s_M of Remark 2.17 and some Massey products. As above, we may use $t_M(L \oplus L^*)$ instead of $s_M(L)$. Note that Hepworth also uses the linking form, but the linking form can be recovered from the t-invariant just as in Theorem 2.4.

If we now drop the assumption that $H^4(M)$ is generated by p_M and products of elements of $H^2(M)$, then as in the highly connected case, the invariant t_M contains potentially more information than s_M because one can define $t_M(E)$ even if $c_2(E)$ is not in the span of $H^2(M) \smile H^2(M)$. It therefore seems reasonable to hope that simply connected spin 7-manifolds M with $H^3(M) = 0$ and $H^4(M)$ finite can be classified via their Eells-Kuiper invariants, their Massey product structure, and their $\hat{t}$-invariants.

3. BUNDLES OVER $\mathbb{H}P^k$

In this section we investigate the map $c_2 \colon \mathrm{Bun}(\mathbb{H}P^k) \to H^4(\mathbb{H}P^k)$ using the t-invariant. Determining the image of c_2 is a difficult problem already investigated in [14, 16] and elsewhere. One calls an integer c *k-realisable* if there is a bundle $E \in \mathrm{Bun}(\mathbb{H}P^k)$ such that

$$c_2(E) = c \cdot c_2(H)$$

where we recall that H is the tautological bundle over $\mathbb{H}P^k$. Below we use the t-invariant to give a new proof of the necessary criteria of [14] for k-realisability.

Consider the Hopf fibration $p_k \colon S^{4k-1} \to \mathbb{H}P^{k-1}$ and let $\bar{p}_k \colon W_k \to \mathbb{H}P^{k-1}$ denote the corresponding D^4-bundle with boundary S^{4k-1}. Then

$$\mathbb{H}P^k = W_k \cup_{S^{4k-1}} D^{4k} .$$

One may therefore consider the problem of building bundles over $\mathbb{H}P^k$ inductively: assume that E_{k-1} is a quaternionic line bundle on $\mathbb{H}P^{k-1}$, then the pullback $\bar{p}_k^* E_{k-1}$ on W extends to $E_k \to \mathbb{H}P^k$ iff its restriction $\bar{p}_k^* E_{k-1}|_{S^{4k-1}} = p_k^* E_{k-1}$ to the boundary is trivial. If $p_k^* E$ is trivial then the group $\pi_{4k-1}(S^3)$ acts transitively on the set of possible extensions of E to $\mathbb{H}P^k$. The situation is summarised in the following commutative diagram.
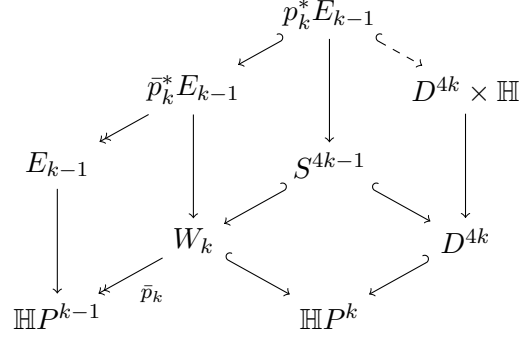
Determining whether $p_k^* E_{k-1}$ is trivial is a difficult problem in general. However Theorem 3.1 below determines the t-invariant of such pull-backs in terms of $c_2(E_{k-1})$. For ease of computations with signs we first define $a := -c_2(H)$ to be the other generator of $H^4(\mathbb{H}P^\infty) \cong \mathbb{Z}$ and for all j we identify $H^4(\mathbb{H}P^j) = H^4(\mathbb{H}P^\infty)$ in the obvious way.

3.1. Theorem. *If $k \geq 2$ and there exists a quaternionic line bundle E_c on $\mathbb{H}P^{k-1}$ with $c_2(E_c) = -c \cdot a$ in $H^4(\mathbb{H}P^{k-1}) \cong \mathbb{Z}a$, then*

$$t_{S^{4k-1}}\big(p_k^* E_c\big) = \frac{a_k}{(2k)!} \prod_{j=0}^{k-1} (c - j^2) \quad \in \quad \mathbb{Q}/\mathbb{Z} \,.$$

We defer the proof Theorem 3.1 to the end of the section.

3.2. Corollary ([14]Theorem 1.1)**.** *If an integer c is k-realisable then*

$$\frac{a_j}{(2j)!} \prod_{i=0}^{j-1} (c - i^2) = 0 \in \mathbb{Q}/\mathbb{Z} \qquad \text{for all } 2 \leq j \leq k \,.$$

Proof. Let $E_c \in \mathrm{Bun}(\mathbb{H}P^k)$ be a bundle with $c_2(E_c) = c \cdot c_2(\tau) = -c \cdot a$. Then for all $2 \leq j \leq k$, the bundle $E_c^j := p_j^*(E_c|_{\mathbb{H}P^{j-1}}) = E_c|_{S^{4j-1}}$ is trivial and in particular $t_{S^{4j-1}}(E_c^j) = 0$. Now apply Theorem 3.1. $\square$

3.3. Remark. Feder and Gitler proved the that the conditions of Corollary 3.2 are necessary using different methods in [14]. They also gave a proof that the conditions are sufficient if $k = \infty$ [4], where they are satisfied iff c is an odd square or zero. Feder and Gitler strongly suggested that their conditions are also sufficient for finite k, but so far, this has been proved only for $k = 2$, 3, 4, 5 (see [16, §2] for more details on this issue). Moreover, the bundle E_c is not unique in general, however Gonçalves and Spreafico [16, Theorem 3] proved that the number of non-isomorphic bundles with the same second Chern class is finite and depends only on k and the parity of c.

3.4. Remark. Theorem 3.1 and Theorem 1.12 shed some light on the Feder-Gitler conditions: as we saw above, the *a priori* obstruction to extending a bundle $E_c \to \mathbb{H}P^{j-1}$ to $\mathbb{H}P^j$ is the bundle $p_j^* E_c \in \mathrm{Bun}(S^{4j-1}) \cong \pi_{4j-2}(S^3)$.

---

[4]In [14] Feder and Gitler quote Sullivan who credits this statement to unpublished work of I. Bernstein, R. Stong, L. Smith and G. Cooke.

We see that the claim that the Feder-Gitler conditions are sufficient for c to be k-realisable is equivalent to the following claim: For each $2 \leq j \leq k$ such that c is $(j-1)$-realisable, the set

$$t_{S^{4j-1}}^{-1}(0) \ \cap \ p_j^* \big(c_2^{-1}(c \cdot c_2(H)) \big) \subset \mathrm{Bun}(S^{4j-1})$$

contains the trivial bundle $S^{4j-1} \times \mathbb{H}$. By Theorem 3.1, Remark 1.14 and Corollary 1.13 this is true if $k = 2$ or 3 but becomes surprising if $k > 3$. By Proposition 1.5 the t-invariant measures only minus the e-invariant of the stabilisation $S(p_j^* E_c) \in \pi_{4k-5}^S$ and for $k > 3$ the stabilisation homomorphism $S \colon \pi_{4k-2}(S^3) \to \pi_{4k-5}^S$ and the e-invariant $e \colon \pi_{4k-5}^S \to \mathbb{Q}/\mathbb{Z}$ both typically have non-trivial kernels.

We next discuss Theorem 3.1 for $k = 2, 3$ and 4.

3.5. *Example.* For $k = 2$, we recover a special case of Proposition 2.6. In particular, there exist quaternionic line bundles E_c on $\mathbb{H}P^1 = S^4$ with $c_2(E_c) = -c \cdot a$ for all $c \in \mathbb{Z}$, and we have

$$t_{S^7}\big(p_2^* E_c \big) = \frac{c(c-1)}{24} \quad \in \quad \frac{1}{12}\mathbb{Z} \ / \ \mathbb{Z} \ .$$

This confirms Theorem 2.4 (1) and Remark 1.14 stating that t_{S^7} is injective.

3.6. *Example.* A bundle $p_2^* E_c$ on S^7 is trivial iff $c(c-1) \equiv 0 \bmod 24$. This is the case iff

$$c \equiv 0, 1 \quad \bmod 8 \qquad \text{and} \qquad c \equiv 0, 1 \quad \bmod 3 \ .$$

Because $\pi_7(S^3) \cong \mathbb{Z}_2$, there exist at most two extensions of E_c to $\mathbb{H}P^2$ up to isomorphism. We pick one for each c and still denote it by E_c.

By Theorem 3.1,

$$t_{S^{11}}\big(p_3^* E_c \big) = \frac{2c(c-1)(c-4)}{720} = \frac{c(c-1)}{24} \cdot \frac{c-4}{15} \quad \in \quad \frac{1}{15}\mathbb{Z} \ / \ \mathbb{Z} \ ,$$

because the first factor is always an integer. Because E_c exists on $\mathbb{H}P^2$ for $c = 33$ and

$$t_{S^{11}}\big(p_3^* E_3 \big) = \frac{1}{15} \ ,$$

we conclude from Proposition 1.5 (2) that the t-invariant again gives an isomorphism

$$t_{S^{11}} \colon \ \mathrm{Bun}(S^{11}) \cong \pi_{10}(S^3) \longrightarrow \mathbb{Z}/15\mathbb{Z} \ .$$

This confirms Corollary 1.13 stating that $t_{S^{11}}$ is injective.

3.7. *Example.* A bundle $p_3^* E_c$ exists and is trivial on S^{11} iff

$$c \equiv 0, 1 \quad \bmod 8 \ , \qquad c \equiv 0, 1, 4, 7 \quad \bmod 9 \ , \qquad \text{and} \qquad c \equiv 0, 1, 4 \quad \bmod 5 \ .$$

In this case, it extends to $\mathbb{H}P^3$. Note that because $\pi_{11}(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ and because there were possibly two non-isomorphic bundles E_c on $\mathbb{H}P^2$ to start with, there can be up to four quaternionic line bundles on $\mathbb{H}P^3$ with second Chern class $-c \cdot a \in H^4(\mathbb{H}P^3)$. We pick one and again denote it by E_c.

By Theorem 3.1,

$$t_{S^{15}}\big(p_4^* E_c \big) = \frac{c(c-1)(c-4)(c-9)}{2^7 \cdot 3^2 \cdot 5 \cdot 7} \quad \in \quad \frac{1}{28}\mathbb{Z} \ / \ \mathbb{Z} \ ,$$

because the numerator is always divisible by $2^5 \cdot 3^2 \cdot 5$ for all $c \in \mathbb{Z}$ that can occur. For $c = 40$, we have

$$t_{S^{15}}\left(p_4^* E_{40}\right) = \frac{5}{28} \ ,$$

which generates $\frac{1}{28}\mathbb{Z} \ / \ \mathbb{Z}$.

Now by [25, (7.14), Theorem 13.9, (13.6)'] the stabilisation homomorphism $S \colon \pi_{14}(S^3) \to \pi_{11}^S$ is isomorphic to the homomorphism

$$\mathbb{Z}/84 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{(\times 36, 0, 0)} \mathbb{Z}/504 \ ,$$

and by [1] the e-invariant $e \colon \pi_{11}^S \to \mathbb{Q}/\mathbb{Z}$ is injective. Moreover, by Remark 3.3 the condition $c(c-1)(c-4)(c-9) \equiv 0 \bmod \frac{8!}{2}$ is sufficient for the existence of a bundle E_c on $\mathbb{H}P^4$, hence for the existence of a bundle $p_4^* E_c$ on S^{15} that is trivial. Thus the 3-torsion component in $\mathrm{Bun}(S^{15})$ cannot be generated by bundles of the type $p_4^* E_c$.

Proof of Theorem 3.1. We compute $t_{S^{4k-1}}$ using the zero bordism W_k as in Definition (1.6). Using the homotopy equivalence $\bar{p}_k \colon W_k \to \mathbb{H}P^k$ to identify $H^4(W_k) = H^4(\mathbb{H}P^{k-1})$ and the Thom isomorphism, we know that the generator $a \in H^4(W_k)$ lifts to a generator of $H^4(W_k, S^{4k-1}) \cong a\mathbb{Z}[a]/a^{k+1}\mathbb{Z}[a]$, In particular

$$\bar{c}_2(E_c) \operatorname{ch}'(E_c) = 2 - 2\cosh\left(\sqrt{ca}\right) \quad \in \quad H^\bullet(W_k, S^{4k-1}; \mathbb{Q}) \cong a\mathbb{Q}[a]/a^{k+1}\mathbb{Q}[a] \ .$$

We write $H_\mathbb{R}$ for H regarded as a real vector bundle. For the total Pontrjagin class of $H_\mathbb{R}$ we obtain

$$p(H_\mathbb{R}) = p(H)^2 = (1 + a)^2 \ .$$

Now let $\mathrm{End}_\mathbb{H}(H)$ denote the real 4-plane bundle of quaternionic bundles maps from H to itself. The Pontrjagin classes and Euler class of $\mathrm{End}_\mathbb{H}(H)$ are given by

$$p_1(\mathrm{End}_\mathbb{H}(H)) = 4a \qquad \text{and} \qquad e(\mathrm{End}_\mathbb{H}(H)) = p_2(\mathrm{End}_\mathbb{H}(H)) = 0 \ .$$

The tangent bundle of $\mathbb{H}P^k$ satisfies the relation

$$T\mathbb{H}P^k \oplus \mathrm{End}_\mathbb{H}(H) \cong H^{\oplus(k+1)} \ .$$

In particular, we have

$$\begin{aligned}
\widehat{A}(T\mathbb{H}P^k) &= \widehat{A}(H)^{k+1} \smile \widehat{A}(\mathrm{End}_\mathbb{H}(H))^{-1} \\
&= \left(\frac{\sqrt{a}/2}{\sinh(\sqrt{a}/2)}\right)^{2k+2} \left(\frac{\sqrt{a}}{\sinh\sqrt{a}}\right)^{-1} \\
&= \left(\frac{\sqrt{a}/2}{\sinh(\sqrt{a}/2)}\right)^{2k+1} \cosh\left(\sqrt{a}/2\right) \\
&\in \quad H^\bullet(\mathbb{H}P^k; \mathbb{Q}) \cong \mathbb{Q}[a]/a^{k+1}\mathbb{Q}[a] \ ,
\end{aligned}$$

and we know that $\widehat{A}(W_k) = \widehat{A}(T\mathbb{H}P^k)|_{W_k}$.

Let γ be a sufficiently small contour around 0 in $\mathbb{C}$, then γ^2 goes around 0 twice. Note that $a_k\,a_{k+1} = 2$ for all k. From the formula for $\widehat{A}(T\mathbb{H}P^k)$ above, we compute

$$
\begin{aligned}
t_{S^{4k-1}}\big(p_k^* E_c\big) &= -\frac{1}{a_{k+1}}\left(\widehat{A}(TW_k)\,\bar{c}_2(\bar{p}_k^* E_c)\,\mathrm{ch}'(\bar{p}_k^* E_c)\right)[W_k, S^{4k-1}] \\
&= \frac{a_k}{2\pi i}\int_\gamma \frac{\cosh(\sqrt{a}/2)\left(\cosh(\sqrt{ac})-1\right)}{2^{2k+1}\,\sqrt{a}\,\sinh(\sqrt{a}/2)^{2k+1}}\,da \\
&= \frac{a_k}{2\pi i}\int_{\gamma^2} \frac{\cosh(\sqrt{a}/2)\left(\cosh(\sqrt{ac})-1\right)}{2^{2k+2}\,\sqrt{a}\,\sinh(\sqrt{a}/2)^{2k+1}}\,da \\
&= \frac{a_k}{2\pi i}\int_\gamma \frac{\cosh\big(2\sqrt{c}\cdot\mathrm{ar\,sinh}\,\frac{z}{2}\big)-1}{z^{2k+1}}\,dz \quad \in \quad \mathbb{Q}/\mathbb{Z}\,.
\end{aligned}
$$

In the last step, we have substituted z for $2\sinh(\sqrt{a}/2)$. Note that the contour for z closes only after a has completed two cycles around the origin.

It is well known that there exist even polynomials P_n of degree at most $2n$ such that

$$
\cosh(2nx) = P_n(\sinh x) = \sum_{k=0}^{n} p_{n,k}\,\sinh^{2k} x\,,
$$

and $p_{n,n} = 2^{2n-1}$. On the other hand,

$$
\bar{t}_k(y) := \frac{1}{2\pi i}\int_\gamma \frac{\cosh\big(2y\cdot\mathrm{ar\,sinh}\,\frac{z}{2}\big)-1}{z^{2k+1}}\,dz
$$

is an even polynomial of degree at most $2k$ in y such that $\bar{t}_k(n) = 2^{-2k}\,p_{n,k}$. Because $p_{n,k} = 0$ if $n < k$, we conclude that $\bar{t}_k$ vanishes at $n = 1-k,\,\ldots,\,k-1$, and that $\bar{t}_k(k) = 2^{-2k}p_{k,k} = \frac{1}{2}$. Since $\bar{t}_k$ is even, this leads to the equations

$$
\bar{t}_k(y) = \frac{1}{2}\prod_{j=0}^{k-1}\frac{(y^2-j^2)}{(k^2-j^2)} = \frac{1}{(2k)!}\prod_{j=0}^{k-1}(y^2-j^2)\,.
$$

The proposition follows because $t_{S^{4k-1}}\big(p_k^* E_c\big) = a_k\,\bar{t}_k\big(\sqrt{c}\big) \bmod \mathbb{Z}$. $\qquad\square$

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