WILD KERNELS AND DIVISIBILITY IN K-GROUPS OF GLOBAL FIELDS

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Abstract. In this paper we study the divisibility and the wild kernels in algebraic K-theory of global fields F. We extend the notion of the wild kernel to all K-groups of global fields and prove that Quillen-Lichtenbaum conjecture for F is equivalent to the equality of wild kernels with corresponding groups of divisible elements in K-groups of F. We show that there exist generalized Moore exact sequences for even K-groups of global fields. Without appealing to the Quillen-Lichtenbaum conjecture we show that the group of divisible elements is isomorphic to the corresponding group of étale divisible elements and we apply this result for the proof of the lim^1 analogue of Quillen-Lichtenbaum conjecture. We also apply this isomorphism to investigate: the imbedding obstructions in homology of GL, the splitting obstructions for the Quillen localization sequence, the order of the group of divisible elements via special values of $\zeta_F(s)$. Using the motivic cohomology results due to Bloch, Friedlander, Levine, Lichtenbaum, Morel, Rost, Suslin, Voevodsky and Weibel, which established the Quillen-Lichtenbaum conjecture, we conclude that wild kernels are equal to corresponding groups of divisible elements.

1. Introduction

Let l be a prime number and let F be a global field of characteristic char $F \neq l$. If $l = 2$ we assume that $\mu_4 \subset F$. The main goal of this paper is to establish general results concerning divisibility and wild kernels in algebraic K-theory of global fields.

It has already been shown by Bass [B], Tate [Ta2] and Moore (see [Mi, p. 157]) that for a number field F the group $K_2(F)$ and in particular the group of divisible elements and the wild kernel in $K_2(F)$ are closely related to arithmetic of F and the Dedekind zeta $\zeta_F(s)$ at $s = -1$. The divisible elements for the Galois cohomology of number fields and local fields in the mix characteristic case were introduced in [Sch2]. The divisible elements and wild kernels for the odd torsion part for the even higher K-groups of number fields were introduced in [Ba1] and [Ba2]. The results of Bass [B], and Moore (see [Mi, p. 157]) concerning the divisible elements and the wild kernel for K_2 where extended in [Ba2] to higher even K-groups of number fields F and values of $\zeta_F(s)$ at negative odd integers. The étale wild kernel as a Shafarevich group in Galois cohomology of number fields was introduced in [Ng]. The work in [Ba1], [Ba2], [Ng] and [Sch2] was carried out under assumption $l > 2$. The wild kernels for the 2-primary part for the higher, even K-groups of number fields were introduced in [Os] and in [We3] were studied for all $l \geq 2$.

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In this paper we investigate the wild kernels and the divisible elements for even and odd K-groups of global fields and for all $l \geq 2$ (see Theorems 1.1 - 1.11 and Corollary 1.12 in this introduction for the statement of main results). These results are new in the char $F > 0$ case and some of them are new in the char $F = 0$ case for $l \geq 2$. Some of these results have already been known in the char $F = 0$ case, often for $l > 2$, so we make corresponding references in this introduction. The Theorems 1.1 - 1.8, Theorems 1.10 - 1.11 and Corollary 1.12 are proven without appealing to the Quillen-Lichtenbaum conjecture. The Theorem 1.9 is a consequence of the Quillen-Lichtenbaum conjecture that in turn resulted from the Voevodsky results [V1], [V2] and the motivic cohomology results due to Bloch, Friedlander, Levine, Lichtenbaum, Morel, Rost, Suslin, Voevodsky, Weibel, and others (see e.g. [BL] cf. [RW, Appendix B], [L], [MV], [R], [VSF], [We4])

It was shown in [Ba2] and [BGKZ] in the case of number fields that the Quillen-Lichtenbaum conjecture for odd torsion part of the even K-groups is equivalent to the isomorphism between corresponding wild kernels and divisible elements. C. Weibel [We3] and K. Hutchinson [Hu] worked on wild kernels and divisible elements in the K-theory of number fields for $l \geq 2$. C. Weibel [We3] computed the index of the the group of divisible elements in the corresponding wild kernel for even K-theory of number fields for the 2-primary part. The key ingredients in his proof were the results on motivic cohomology that led to the computations of the 2-primary part of Quillen K-theory (cf. [RW]). In this paper we show (see Theorem 1.9) that the wild kernel is isomorphic to the divisible elements in K-groups of global fields for all indexes $n > 1$ and $l \geq 2$ (assuming $\mu_4 \in F$ in the case of the 2-primary part). To prove this we show that, under our assumptions, the Quillen-Lichtenbaum conjecture is equivalent to the isomorphism between wild kernel and the divisible elements. In the number field case Theorem 1.9 follows from [Ba2] (for l odd) and [We3], [Hu] (for $l = 2$).

Recall that the divisible elements in K-groups of number fields are in the center of classical conjectures in algebraic number theory and algebraic K-theory. Indeed, the conjectures of Kummer-Vandiver and Iwasawa can be reformulated in terms of divisible elements in even K-groups of $\mathbb{Q}$ [BG1], [BG2]. We have already pointed out in [Ba2, p. 292], that the group of divisible elements in an even K-group of a number field F is an analogue of the class group of $\mathcal{O}_F$. Moreover, as shown in section 6 of this paper, there is a positive integer N_0 such that for every positive integer N, such that $N_0 \,|\, N$, there is the following exact sequence:

$$0 \to K_{2n}(\mathcal{O}_F) \to K_{2n}(F)[N] \to \bigoplus_v K_{2n-1}(k_v)[N] \to D(n) \to 0.$$

Recall that the class group $Cl(\mathcal{O}_F)$ appears in the classical exact sequence:

$$0 \to K_1(\mathcal{O}_F) \to K_1(F) \to \bigoplus_v K_0(k_v) \to Cl(\mathcal{O}_F) \to 0.$$

In addition, as already mentioned above, the conjecture of Quillen-Lichtenbaum can be reformulated in terms of the wild kernels and divisible elements (see also Theorem 5.10 for more detailed presentation). At the last but not the least we would like to point out the close relation of divisible elements and the Coates-Sinnott conjecture [Ba1], [BP].

The organization of the paper is as follows. In chapter 2 we introduce basic notation and recall some classical facts about the cohomological dimension. In chapter 3 we extend results of P. Schneider [Sch2] concerning the divisible elements in Galois cohomology (in [Sch2] the assumption was char $F = 0$ and $l > 2$). Namely we obtain Theorem 3.5 and Corollary 3.6 which lead us to the following analog of the classical Moore exact sequence (see [Mi, p. 157]) for higher étale K-theory:

Theorem 1.1. *Let $n \geq 1$. For every finite $S \supset S_{\infty,l}$ there are exact sequences:*

$$0 \to D^{et}(n) \to K_{2n}^{et}(\mathcal{O}_{F,S}) \to \bigoplus_{v \in S} W^n(F_v) \to W^n(F) \to 0.$$

$$0 \to D^{et}(n) \to K_{2n}^{et}(F)_l \to \bigoplus_{v} W^n(F_v) \to W^n(F) \to 0$$

where $D^{et}(n) := div\, K_{2n}^{et}(F)_l$. In particular:

$$\frac{|K_{2n}^{et}(\mathcal{O}_{F,S})|}{|D^{et}(n)|} = \frac{\big|\prod_{v \in S} w_n(F_v)\big|_l^{-1}}{|w_n(F)|_l^{-1}}.$$

In chapter 4 we investigate the divisibility in K-theory and étale K-theory of F. Let

$$D(n) := div\, K_{2n}(F).$$

For every $k > 0$ define:

$$D(n, l^k) := \ker\,(K_{2n}(\mathcal{O}_F,\, \mathbb{Z}/l^k) \to K_{2n}(F,\, \mathbb{Z}/l^k)),$$

$$D^{et}(n, l^k) := \ker\,(K_{2n}^{et}(\mathcal{O}_F[1/l],\, \mathbb{Z}/l^k) \to K_{2n}^{et}(F,\, \mathbb{Z}/l^k)).$$

The following result shows that the Dwyer-Friedlander homomorphism is an isomorphism when restricted to the groups $D(n, l^k)$ and $div\, K_{2n}(F)_l$:

Theorem 1.2. *If $l > 2$ then $\forall\, k \geq 1$ there is the following canonical isomorphism:*

$$D(n, l^k) \cong D^{et}(n, l^k)$$

If $l = 2$ then $\forall\, k \geq 2$ there is the following canonical isomorphism:

$$D(n, 2^k) \cong D^{et}(n, 2^k)$$

If $l \geq 2$ then there is the following isomorphism $D(n)_l \cong D^{et}(n)$ or more explicitly

$$div\, K_{2n}(F)_l \;\cong\; div\, K_{2n}^{et}(F)_l$$

The last isomorphism of Theorem 1.2 extends [Ba2, Theorem 3] which was my joint result with M. Kolster. The Theorem 3 of [Ba2] concerned the number field case and $l > 2$. By Theorem 1.2 and Corollary 3.6 (see section 3) the divisible elements are expressed in terms of Tate-Shafarevich groups for all $n > 0$:

$$D(n)_l \;\cong\; D^{et}(n) \;\cong\; D_{n+1}(F) \;=\; Ш^2_S(F, \mathbb{Z}_l(n+1)) = Ш^2(F, \mathbb{Z}_l(n+1)).$$

We also get the following lim^1 analogue of the Quillen-Lichtenbaum conjecture.

Theorem 1.3. *For every $n \geq 1$ there is the following isomorphism:*

$$\varprojlim_k{}^1 K_n(F, \mathbb{Z}/l^k) \xrightarrow{\cong} \varprojlim_k{}^1 K_n^{et}(F, \mathbb{Z}/l^k). \tag{1}$$

Moreover there is the following equality:

$$\varprojlim_k{}^1 K_{2n}(F, \mathbb{Z}/l^k) = 0 \tag{2}$$

and the exact sequence:

$$0 \to D(n)_l \to \varprojlim_k{}^1 K_{2n+1}(F, \mathbb{Z}/l^k) \to \varprojlim_k{}^1 \bigoplus_v K_{2n}(k_v, \mathbb{Z}/l^k) \to 0 \tag{3}$$

Theorem 1.3 for the number field case and $l > 2$ was proved in [BZ1].

In the end of chapter 4 we show that the natural maps:

$$H_{2n}(GL(\mathcal{O}_F), \mathbb{Z}/l^k) \;\to\; H_{2n}(GL(F), \mathbb{Z}/l^k)$$

are not injective in general. We have the following theorem:

Theorem 1.4. *For every $n \geq 1$, $k \geq 1$ and $l > n+1$ the kernel of the natural map*

$$H_{2n}(GL(\mathcal{O}_F), \mathbb{Z}/l^k) \;\to\; H_{2n}(GL(F), \mathbb{Z}/l^k) \tag{4}$$

contains a subgroup isomorphic to $D(n, l^k)$.

In particular we show that the maps:

$$H_{22}(GL(\mathbb{Z}), \mathbb{Z}/691) \;\to\; H_{22}(GL(\mathbb{Q}), \mathbb{Z}/691)$$

$$H_{30}(GL(\mathbb{Z}), \mathbb{Z}/3617) \;\to\; H_{30}(GL(\mathbb{Q}), \mathbb{Z}/3617)$$

are not injective. Moreover let $E/\mathbb{F}_p$ be the elliptic curve $y^2 = x^3 + 1$. For $p \geq 5$ and $p \equiv 2 \mod 3$ this curve is supersingular. If $F = \mathbb{F}_p(E)$ is the function field of E then we show that the following maps are not injective:

$$H_6(GL(\mathcal{O}_{\mathbb{F}_{29}(E)}), \mathbb{Z}/5) \;\to\; H_6(GL(\mathbb{F}_{29}(E)), \mathbb{Z}/5),$$

$$H_{10}(GL(\mathcal{O}_{\mathbb{F}_{41}(E)}), \mathbb{Z}/7) \;\to\; H_{10}(GL(\mathbb{F}_{41}(E)), \mathbb{Z}/7).$$

For $p \geq 3$ and $p \equiv 3 \mod 4$ the elliptic curve $y^2 = x^3 + x$ is supersingular. In particular we show that the following map is not injective:

$$H_6(GL(\mathcal{O}_{\mathbb{F}_{19}(E)}), \mathbb{Z}/5) \;\to\; H_6(GL(\mathbb{F}_{19}(E)), \mathbb{Z}/5).$$

In chapter 5 we define wild kernel $K_n^w(\mathcal{O}_F)_l$ for all $n > 0$:

$$K_n^w(\mathcal{O}_F)_l \;:=\; \ker\,(\,K_n(F)_l \to \bigoplus_v K_n^{et}(F_v)_l\,)$$

and observe that:

$$div\, K_n(F)_l \,\subset\, K_n^w(\mathcal{O}_F)_l \,\subset\, K_n(\mathcal{O}_F)_l.$$

Further, we obtain the analogue of the classical Moore exact sequence for higher K-groups:

Theorem 1.5. *For every $n \geq 1$ and every finite set $S \supset S_{\infty,l}$ there are the following exact sequences:*

$$0 \to K_{2n}^w(\mathcal{O}_F)_l \to K_{2n}(\mathcal{O}_{F,S})_l \to \bigoplus_{v \in S} W^n(F_v) \to W^n(F) \to 0. \tag{5}$$

$$0 \to K_{2n}^w(\mathcal{O}_F)_l \to K_{2n}(F)_l \to \bigoplus_{v} W^n(F_v) \to W^n(F) \to 0. \tag{6}$$

In particular:

$$\frac{|K_{2n}(\mathcal{O}_{F,S})_l|}{|K_{2n}^w(\mathcal{O}_F)_l|} = \Big|\frac{\prod_{v \in S} w_n(F_v)}{w_n(F)}\Big|_l^{-1}. \tag{7}$$

The exact sequences (5) and (6) where established for number fields in [Ba2] ($l > 2$) and [We3] ($l = 2$). In chapter 5 we define another wild kernel $WK_n(F)$ for all $n \geq 0$:

$$WK_n(F) \,:=\, \ker\,(\, K_n(F) \to \bigoplus_v \, K_n(F_v)\,)$$

We observe that for all $n \geq 0$:

$$WK_n(F) \subset K_n(\mathcal{O}_F)_{tor}$$
$$WK_n(F)_l \subset K_{2n}^w(\mathcal{O}_F)_l$$

and if $K_n(F_v)_l \xrightarrow{\cong} K_n^{et}(F_v)_l$ for every $v \in S_l$, then:

$$div\, K_n(F)_l \,\subset\, WK_n(F)_l \,\subset\, K_n(\mathcal{O}_F)_l.$$

The Dwyer-Friedlander homomorphisms [DF] which are surjective:

$$K_n(\mathcal{O}_{F,S})_l \to K_n^{et}(\mathcal{O}_{F,S})_l$$
$$K_n(F)_l \to K_n^{et}(F)_l,$$

(see also [Ba2]) are also split as follows by results of [Ba2], [Ca], [K] which can be extended also to the function field case. We use this to establish the following properties of wild kernels and divisible elements (see Theorems 5.1, 5.8 and 5.18).

Theorem 1.6. *For all $n \geq 1$ the Dwyer-Friedlander homomorphisms induce the following canonical map:*

$$K_n^w(\mathcal{O}_F)_l \to div\, K_n(F)_l$$

which is split surjective. The Dwyer-Friedlander homomorphisms induce the following canonical map:

$$WK_n(F)_l \to div\, K_n(F)_l$$

which is split surjective if $K_n(F_v)_l \xrightarrow{\cong} K_n^{et}(F_v)_l$ for every $v \in S_l$.

The first splitting map of the Theorem 1.6 in case of number fields and $l > 2$ was done in [Ng] by use of an argument from [Ba2]. Note that Theorem 1.6 is obvious for n odd since in this case $div\, K_n(F) = 0$.

Theorem 1.7 below shows that the Quillen-Lichtenbaum conjecture holds modulo the wild kernel:

Theorem 1.7. *The Dwyer-Friedlander homomorphisms induce the following canonical isomorphisms for all $n \geq 1$:*

$$K_n(\mathcal{O}_{F,S})_l/K_n^w(\mathcal{O}_F)_l \;\xrightarrow{\cong}\; K_n^{et}(\mathcal{O}_{F,S})_l/divK_n^{et}(F)_l$$
$$K_n(F)_l/K_n^w(\mathcal{O}_F)_l \;\xrightarrow{\cong}\; K_n^{et}(F)_l/divK_n^{et}(F)_l$$

Theorem 1.8 below shows that the difference between the wild kernels and the divisible elements is the obstruction to the Quillen-Lichtenbaum conjecture.

Theorem 1.8. *Let $n > 1$. The following two conditions are equivalent:*

$$K_n(\mathcal{O}_F) \otimes \mathbb{Z}_l \xrightarrow{\cong} K_n^{et}(\mathcal{O}_F[1/l]),$$
$$K_n^w(\mathcal{O}_F)_l = div K_n(F)_l.$$

Moreover assume that $K_n(F_v)_l \xrightarrow{\cong} K_n^{et}(F_v)_l$ for every $v \in S_l$. Then the two conditions above are equivalent to:

$$WK_n(F)_l = div K_n(F)_l.$$

At the end of chapter 5, by use of the Rost-Voevodsky theorem, we prove:

Theorem 1.9. *For every $n > 1$ we have the following equality:*

$$K_n^w(\mathcal{O}_F)_l = div K_n(F)_l.$$

Assume that $K_n(F_v)_l \xrightarrow{\cong} K_n^{et}(F_v)_l$ for every $v \in S_l$ and every $n > 1$. Then for every $n \geq 0$:

$$WK_n(F) = div\, K_n(F).$$

In the number field case the equality $K_n^w(\mathcal{O}_F)_l = div K_n(F)_l$, in Theorem 1.9, follows from [Ba2] (for l odd) and [We3], [Hu] (for $l = 2$). Observe that for $0 \leq n \leq 1$ we have $WK_n(F) = div\, K_n(F) = 0$ for obvious reasons.

In chapter 6 we investigate the obstructions for the splitting of the Quillen localization sequence and complete a statement of [Ba2, Cor. 1 and Prop. 1 p. 293]. Recall, that Tate (see [Mi, Theorem 11.6]) proved that there is the following isomorphism

$$K_2(\mathbb{Q}) \,\cong\, K_2(\mathbb{Z}) \,\oplus\, \bigoplus_p \, K_1(\mathbb{F}_p).$$

The results concerning the splitting of the Quillen exact sequence for higher K-groups of number fields were obtained in [Ba1] and [Ba2]. A very special case of results of [Ba1] is the following isomorphism:

$$K_{2n}(\mathbb{Q})_l \,\cong\, K_{2n}(\mathbb{Z})_l \,\oplus\, \bigoplus_p \, K_{2n-1}(\mathbb{F}_p)_l,$$

for $n = 3, 5, 7, 9$ and $l > 2$. Moreover for n odd and $l > 2$ the following conditions are equivalent [Ba2, Cor. 2 p. 294] (see also Corollary 6.6 in section 6):

$$K_{2n}(\mathbb{Q})_l \,\cong\, K_{2n}(\mathbb{Z})_l \,\oplus\, \bigoplus_p \, K_{2n-1}(\mathbb{F}_p)_l.$$
$$|w_{n+1}(\mathbb{Q})\zeta_{\mathbb{Q}}(-n)|_l^{-1} = 1.$$

In this paper, under the assumption that $l \geq 2$ and F is a global field with $\mathrm{char}\, F \neq l$ (if $l = 2$ we assume $\mu_4 \subset F$), we get the following result concerning the splitting of the Quillen localization sequence that extends the splitting results of [Ba1], [Ba2] and [Ca] in the number field case.

Theorem 1.10. *Let $n \geq 1$. The following conditions are equivalent:*

(1) $D(n, l^k) = 0 \quad$ *for every* $\quad 0 < k \leq k(l)$,
(2) $D^{et}(n, l^k) = 0 \quad$ *for every* $\quad 0 < k \leq k(l)$,
(3) $K_{2n}(F)_l \,\cong\, K_{2n}(\mathcal{O}_F)_l \,\oplus\, \bigoplus_v \, K_{2n-1}(k_v)_l$,
(4) $K_{2n}^{et}(F)_l \,\cong\, K_{2n}^{et}(\mathcal{O}_F[1/l])_l \,\oplus\, \bigoplus_v \, K_{2n-1}^{et}(k_v)_l$,

where $k(l)$ is defined by (59) and (60) in section 4.

The group of divisible elements is the obstruction to splitting of the following natural boundary map in the Quillen localization sequence:

Theorem 1.11. *Let $n > 0$ and let $k \geq k(l)$. The following conditions are equivalent:*

(1) $\partial_1 \,:\, K_{2n}(F)_l \to \bigoplus_{l^k \,|\, q_v^n - 1} K_{2n-1}(k_v)_l \quad$ *is split surjective,*
(2) $D(n)_l = 0$.

This implies the following corollary:

Corollary 1.12. *Let F be a totally real number field, n odd and $l > 2$ or let F be a global field of $\operatorname{char} F > 0$, $n \geq 1$ and $l \neq \operatorname{char} F$. Then for every $k \geq k(l)$ the following conditions are equivalent:*

(1) *The following surjective map splits*

$$\partial_1 \,:\, K_{2n}(F)_l \to \bigoplus_{l^k \,|\, q_v^n - 1} K_{2n-1}(k_v)_l$$

(2)

$$|\frac{w_n(F)\, w_{n+1}(F) \zeta_F(-n)}{\prod_{v \in S_{\infty, l}} \, w_n(F_v)}|_l^{-1} = 1.$$

Observe that $\left|w_n(\mathbb{R})\right|_l^{-1} = \left|w_n(F)\right|_l^{-1} = 1$ for F totally real, n odd and l odd.

2. Basic notation and set up

2.1. **Notation.**

(1) l is a prime number.
(2) $F \;:=$ a global field.
(3) $p := \operatorname{char} F$, if $\operatorname{char} F > 0$.
(4) $\mathcal{O}_F := \begin{cases} \text{the integral closure of } \mathbb{Z} \text{ in } F & \text{if} \quad \operatorname{char} F = 0 \\ \text{the integral closure of } \mathbb{F}_p[t] \text{ in } F & \text{if} \quad \operatorname{char} F > 0 \end{cases}$
(5) v a place of F.
(6) $S_\infty := \begin{cases} \{v \,:\, v | \infty\} & \text{if} \quad \operatorname{char} F = 0 \\ \{v \,:\, v | v_{t^{-1}}\} & \text{if} \quad \operatorname{char} F > 0 \end{cases}$
(7) $S_l := \begin{cases} \{v \,:\, v | l\} & \text{if} \quad \operatorname{char} F = 0 \\ \emptyset & \text{if} \quad \operatorname{char} F > 0 \end{cases}$
(8) $S_{\infty, l} := S_\infty \cup S_l$.
(9) S a finite set of places of F containing $S_{\infty, l}$.
(10) $\mathcal{O}_{F,S}$ the ring of S-integers in F. Note that $\mathcal{O}_{F, S_\infty} = \mathcal{O}_F$.
(11) F_v the completion of F at v.
(12) F_v^h the henselization of F at v (v nonarchimedean if $\operatorname{char} F = 0$)
(13) $\mathcal{O}_v := \begin{cases} \{\alpha \in F_v \,:\, v(\alpha) \geq 0\} & \text{if} \quad v \not| \infty \text{ and } \operatorname{char} F = 0 \\ \{\alpha \in F_v \,:\, v(\alpha) \geq 0\} & \text{if} \quad \operatorname{char} F > 0 \end{cases}$

(14) $k_v := \begin{cases} \mathcal{O}_F/v = \mathcal{O}_v/v & \text{if} \quad v \notin S_\infty \text{ and char } F \geq 0 \\ \mathcal{O}_v/v & \text{if} \quad v \in S_\infty \text{ and char } F > 0 \end{cases}$
(15) $\overline{F}_s$ the separable closure of F.
(16) $F_S \subset \overline{F}_s$ the maximal separable extension of F unramified outside S.
(17) $G_F := G(\overline{F}_s/F)$.
(18) $G_S := G(F_S/F)$.
(19) $W^n := W^n_l := \mathbb{Q}_l/\mathbb{Z}_l(n)$ for any $n \in \mathbb{Z}$.
(20) $W^n(L) := W^n_l(L) := H^0(G_L, \mathbb{Q}_l/\mathbb{Z}_l(n))$ for a field L with $\text{char}\, L \neq l$.
(21) $w_n(L) := \prod_{l \neq \text{char}\, L} |W^n_l(L)|$ whenever $|W^n_l(L)| < \infty$ for every $l \neq \text{char}\, L$ and $|W^n_l(L)| = 1$ for almost every l.
(22) $divA := \{a \in A : \ \forall_{m \in \mathbb{Z}} \ \exists_{a' \in A} \ ma' = a\}$ for an abelian group A,
(23) $DivA :=$ the maximal divisible subgroup of A,
(24) $A/Div := A/DivA$.

2.2. **Fields of cohomological dimension $\leq$ 2.** Let L be a field. If $L = \mathbb{F}_q$ is a finite field with q elements then $\text{cd}_l(\mathbb{F}_q) = 1$. If L is a local field it follows from [Se, II, sec. 4.3, Prop. 12,] that $\text{cd}_l(L) \leq 2$. If $L = F$ is a global field and $l > 2$ then $\text{cd}_l(F) \leq 2$ by [Se, II, sec. 4.3, Prop. 11 and Prop. 13,]. If $\text{char}\, F = 0$, then $\text{cd}_2(F) \leq 2$ iff $F_v = \mathbb{C}$ for every $v|\infty$. It is so because for any $m \geq 3$ and any 2-torsion, finite G_F-module M there is the following natural isomorphism [M1, Theorem 4.8 (c) Chap. I]:

$$H^m(F, M) \xrightarrow{\cong} \bigoplus_{v \text{ real}} H^m(F_v, M) \tag{8}$$

Hence if $\text{char}\, F = 0$, and F does not have real imbeddings then trivially $H^m(F_v, M) = 0$ for all $v|\infty$, all $G(\overline{F_v}/F_v)$-modules M and all $m > 0$. This will always be the case in this paper since for $l = 2$ we will assume that $\mu_4 \subset F$.

The localization sequence for étale cohomology [So1, pp. 267-268] shows that $\text{cd}_l(\mathcal{O}_v) \leq 2$ for all nonarchimedean v and $\text{cd}_l(\mathcal{O}_{F,S}) \leq 2$ for all finite $S \supset S_{\infty,l}$

Lemma 2.1. *Let L be a field such that $\text{char}\, L \neq l$ and $\mu_{l^\infty} \subset L$. Assume that $K_2(L')/DivK_2(L')$ is torsion for any algebraic extension L'/L. Then $cd_l(L) \leq 1$.*

Proof. Let L'/L be an algebraic extension. Since $\mu_l \subset L'$, by Merkurev-Suslin Theorem [MS]:

$$K_2(L')/lK_2(L') \xrightarrow{\cong} H^2(L', \mathbb{Z}/l(2)) \xrightarrow{\cong} Br(L')[l] \otimes \mathbb{Z}/l\mathbb{Z}(1). \tag{9}$$

By assumption $K_2(L')/lK_2(L') = K_2(L')_l/lK_2(L')_l$. By Suslin theorem [Su2, Theorem 1.8] if $\alpha \in K_2(L')[l^k]$ then there is $a \in L'$ such that $\alpha = \{\xi_{l^k},\, a\}$. Hence α is divisible by l in $K_2(L')_l$ because $\mu_{l^\infty} \subset L'$. This shows that $Br(L')[l] = 0$. Hence $\text{cd}_l(L) \leq 1$ by [Sh, Corollary 2, p. 100]. $\square$

Corollary 2.2. *Let L be an algebraic extension of a global or local field. Let $\text{char}\, L \neq l$ and $\mu_{l^\infty} \subset L$. Then $cd_l(L) \leq 1$.*

Proof. Let L be an algebraic extension of a global field. Observe that for $n > 0$ the K_{2n} groups of rings of integers in global fields are finite by results of Borel [Bo], Harder [Ha] and Quillen [Q2]. Hence by the Quillen localization sequence [Q1] the group $K_2(L')$ is torsion for every algebraic extension L'/L. If L is an algebraic extension of a local field then by [Ta3] and [Me] the group $K_2(L')/DivK_2(L')$ is torsion for every algebraic extension L'/L. Now the claim follows from Lemma 2.1. □

Let L be a field such that $\operatorname{char} L \neq l$. We have $G(L(\mu_{l^\infty})/L) \cong \Delta \times \Gamma$ where $\Delta := G(L(\mu_l)/L)$ and $\Gamma := G(L(\mu_{l^\infty})/L(\mu_l))$.

Lemma 2.3. *Let L be a field such that $\operatorname{char} L \neq l$. If $l = 2$ assume that $\mu_4 \subset L$. Then $cd_l(G(L(\mu_{l^\infty})/L) \leq 1$.*

Proof. By assumptions $\Gamma \cong \mathbb{Z}_l$ if $\mu_{l^\infty} \not\subset L$ and $\Gamma = 1$ if $\mu_{l^\infty} \subset L$. Moreover $\Delta \subset \mathbb{Z}/l^\times$ if $l > 2$ and $\Delta := 1$ if $l = 2$. Consider the spectral sequence for any l-torsion $G(L(\mu_{l^\infty})/L)$-module M.

$$E_2^{p,q} = H^p(\Delta,\, H^q(\Gamma,\, M)) \Rightarrow H^{p+q}(G(L(\mu_{l^\infty})/L),\, M). \tag{10}$$

$E_2^{p,q} = 0$ for all $p > 0$ and $q > 1$ because $l \not| |\Delta|$ and $\operatorname{cd}_l(\Gamma) \leq 1$ by [Ri, Chap. IV, Cor. 3.2]. Hence $H^m(G(L(\mu_{l^\infty})/L),\, M) = 0$ for all $m > 1$. □

The following two theorems are straightforward extensions of well know results of Tate [Ta1] and Schneider [Sch2] to the framework of general fields.

Theorem 2.4. *Let L be a field such that $\operatorname{char} L \neq l$ and $\mu_{l^\infty} \not\subset L$. If $l = 2$ assume that $\mu_4 \subset L$. Let M be a discrete $G(L(\mu_{l^\infty})/L)$-module. Then*

$$H^1(G(L(\mu_{l^\infty})/L),\, M \otimes_{\mathbb{Z}} W) = 0. \tag{11}$$

Proof. Is clear that $H^1(\Delta,\, M \otimes_{\mathbb{Z}} W) = 0$. Hence to get (11) it is enough to prove $H^1(\Gamma,\, M \otimes_{\mathbb{Z}} W) = 0$ as in [Ta1] and apply the inflation-restriction exact sequence. □

Theorem 2.5. *Let L be a field such that $\operatorname{char} L \neq l$ and $\mu_{l^\infty} \not\subset L$. If $l = 2$ assume that $\mu_4 \subset L$. Assume that $K_2(L')/DivK_2(L')$ is torsion for any algebraic extension $L'/L(\mu_{l^\infty})$. Then*

$$H^m(G_L,\, W^n) \;\cong\; \begin{cases} 0 & \text{if} \quad m > 2 \quad \text{and} \quad n \in \mathbb{Z} \\ 0 & \text{if} \quad m = 2 \quad \text{and} \quad n \neq 1 \\ Br(L)_l & \text{if} \quad m = 2 \quad \text{and} \quad n = 1 \end{cases} \tag{12}$$

Proof. Consider the spectral sequence:

$$E_2^{p,q} = H^p(G(L(\mu_{l^\infty})/L),\, H^q(G_{L(\mu_{l^\infty})},\, W^n)) \Rightarrow H^{p+q}(G_L,\, W^n) \tag{13}$$

Observe that $E_2^{p,q} = 0$ for all $p > 1$ or $q > 1$ by Lemmas 2.1 and 2.3. Hence $H^m(G_L,\, W^n) = 0$ for all $m > 2$ and all $n \in \mathbb{Z}$ and $E_2^{2,0} = E_2^{0,2} = 0$ for all $n \in \mathbb{Z}$. If $n \neq 1$ then by Theorem 2.4

$$\begin{aligned} E_2^{1,1} &= H^1(G(L(\mu_{l^\infty})/L),\, H^1(G_{L(\mu_{l^\infty})},\, W^n)) = \\ &= H^1(G(L(\mu_{l^\infty})/L),\; L(\mu_{l^\infty})^\times \otimes_{\mathbb{Z}} W^{n-1})) = 0. \end{aligned}$$

Hence $H^2(G_L,\, W^n) = 0$ for $n \neq 1$. For $n = 1$ the long cohomology exact sequence associated with the short exact sequence $1 \to \mu_{l^k} \to \mathbb{G}_m \to \mathbb{G}_m \to 1$ and the Hilbert 90 show that $H^2(G_L,\, \mu_{l^k}) \cong Br(L)[l^k]$ for each k. Hence $H^2(G_L,\, W) = Br(L)_l$. □

Corollary 2.6. *Let L be a global or local field with $\operatorname{char} L \neq l$. Assume that $\mu_4 \subset L$ if $l = 2$. Then the isomorphism (12) holds for L.*

Proof. It is shown in the proof of Corollary 2.2 that $K_2(L')/DivK_2(L')$ is torsion for any algebraic extension $L'/L(\mu_{l^\infty})$. Hence the claim follows from Theorem 2.5. □

2.3. **Some useful isomorphisms.** Let F be a global field such that $\mu_4 \subset F$ if $l = 2$. Then the l-cohomological dimension of any of the following rings $\mathcal{O}_{F,S}$, F, $\mathcal{O}_v$, and F_v is ≤ 2 for every l. Hence the Dwyer-Friedlander spectral sequence [DF] Proposition 5.1 shows that for $X = \operatorname{spec} \mathcal{O}_{F,S}$, $\operatorname{spec} F$, $\operatorname{spec} \mathcal{O}_v$, $\operatorname{spec} F_v$ there are natural isomorphisms:

$$K_{2n}^{et}(X) \cong H^2_{cont}(X, \mathbb{Z}_l(n+1)), \qquad K_{2n+1}^{et}(X) \cong H^1_{cont}(X, \mathbb{Z}_l(n+1)) \tag{14}$$

We will often use the following comparison isomorphisms (15), (16), (17) between étale cohomology of some affine schemes and corresponding Galois cohomology. For a commutative ring R with identity and an étale sheaf $\mathcal{F}$ on $\operatorname{spec} R$ we put:

$$H^*(R,\, \mathcal{F}) := H^*_{et}(\operatorname{spec} R,\, \mathcal{F}).$$

For a field K and an étale sheaf $\mathcal{F}$ on $\operatorname{spec} K$ let $M_{\mathcal{F}}$ is the discrete $G_K := G(\overline{K}_s/K)$-module corresponding to $\mathcal{F}$. Then [M2, Chap. II, Theorem 1.9, Chap. III Example 1.7]:

$$H^*(K,\, \mathcal{F}) \cong H^*(G_K,\, M_{\mathcal{F}}), \tag{15}$$

Let v be a nonarchimedean place of a global field F and let $\mathcal{F}$ be an étale sheaf on $\operatorname{spec} \mathcal{O}_v$. Let $G_v^{nr} := G_{F_v}/I_v \cong G(\overline{k_v}/k_v)$ where I_v is the inertia subgroup. Let $M_{\mathcal{F}}$ be the discrete G_v^{nr}-module corresponding to $\mathcal{F}$. Then [A, Chap. III, Th. 4.9]

$$H^*(\mathcal{O}_v,\, \mathcal{F}) \cong H^*(G_v^{nr},\, M_{\mathcal{F}}) \tag{16}$$

Let $\mathcal{F}$ be a constructible étale sheaf on $\operatorname{spec} \mathcal{O}_{F,S}$ and let $M_{\mathcal{F}}$ be the discrete $G_{F,S}$-module corresponding to $\mathcal{F}$. Then [M1, Chap. II, Prop. 2.9]:

$$H^*(\mathcal{O}_{F,S},\, \mathcal{F}) \cong H^*(G_S,\, M_{\mathcal{F}}). \tag{17}$$

From now on in this paper $\operatorname{char} F \neq l$ and $\mu_4 \subset F$ if $l = 2$.

3. Galois cohomology of local and global fields

3.1. **Tate-Shafarevich groups and Tate-Poitou duality.** The r-th Tate-Shafarevich group $Ш^r_S(F, M)$ for a $G_{F,S}$-module M is defined as follows [M1, p. 70]:

$$Ш^r_S(F, M) \ := \ \ker\, (H^r(G_{F,S},\, M) \ \longrightarrow \ \prod_{v \in S} H^r(G_{F_v},\, M)) \tag{18}$$

In this paper S is finite. In loc. cit. $Ш^r_S(F, M)$ is defined for any nonempty S (containing S_∞ if $\operatorname{char} F = 0$). In particular for a G_F-module M :

$$Ш^r(F, M) \ := \ \ker\, (H^r(G_F,\, M) \ \longrightarrow \ \prod_{v} H^r(G_{F_v},\, M)) \tag{19}$$

Observe that for an abelian variety A/F and $M = A(\overline{F}_s)$ we have $Ш^1(F, A(\overline{F})) \subset Ш(A/F)$ where $Ш(A/F)$ is the classical Tate-Shafarevich group. By Tate-Poitou duality (see eg. [M1, Theorem 4.10, Chap. I]) for any finite G_S-module M with order being a unit in $\mathcal{O}_{F,S}$ and for the G_S-module $M^D := Hom(M, \overline{F}^{\times})$ there is the following perfect pairing

$$\text{Ш}^1_S(F, M) \times \text{Ш}^2_S(F, M^D) \longrightarrow \mathbb{Q}/\mathbb{Z} \tag{20}$$

Since $\mathbb{Z}/l^k(n)^D \cong \mathbb{Z}/l^k(1-n)$ we get perfect pairing:

$$\text{Ш}^1_S(F, \mathbb{Z}/l^k(n)) \times \text{Ш}^2_S(F, \mathbb{Z}/l^k(1-n)) \longrightarrow \mathbb{Z}/l^k \tag{21}$$

Passing on the left and the target of (21) to the direct limit and on the right of (21) to the inverse limit we get perfect pairing:

$$\text{Ш}^1_S(F, \mathbb{Q}_l/\mathbb{Z}_l(n)) \times \text{Ш}^2_S(F, \mathbb{Z}_l(1-n)) \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l \tag{22}$$

Let M be a finite discrete $G(\overline{F}/F)$-module. If $\rho_M \, : \, G(\overline{F}/F) \to Aut_{\mathbb{Z}}(M)$, then we put $F(M) := \overline{F}^{\ker \rho_M}$. Let v be a place of F and let w denote a place of $F(M)$ over v. Consider the following commutative diagram with exact columns c.f. [Ne, p. 79]:

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 & & H^1(G(F(M)/F), M) & \longrightarrow & \prod_v \prod_{w|v} H^1(G(F(M)_w/F_v), M) \\
 & & \downarrow & & \downarrow \\
 & & H^1(G_F,\, M) & \longrightarrow & \prod_v \prod_{w|v} H^1(G_{F_v},\, M) \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1(G_{F(M)},\, M) & \longrightarrow & \prod_v \prod_{w|v} H^1(G_{F(M)_w},\, M)
\end{array}$$

By Chebotarev's density theorem the bottom horizontal arrow is a monomorphism, hence $\text{Ш}^1(F(M), M) = 0$. Hence it is clear that the upper horizontal arrow is a monomorphism if and only if $\text{Ш}^1(F, M) = 0$. If $F(M)/F$ is cyclic then by Chebotarev's theorem there are infinitely many places v such that $G(F(M)/F) = G(F(M)_w/F_v)$. So it is clear that in this case the top horizontal arrow is a monomorphism hence $\text{Ш}^1(F, M) = 0$.

The most interesting case for this paper is $M = \mathbb{Z}/l^k(n)$, where $l \neq \operatorname{char} F$ and $\mu_4 \in F$ if $l = 2$. In this case all horizontal arrows in the above diagram are monomorphisms. Hence for every $n \in \mathbb{Z}$ and every $k \geq 0$

$$\text{Ш}^1(F, \mathbb{Z}/l^k(n)) = 0. \tag{23}$$

By Tate-Poitou duality (21) for every $n \in \mathbb{Z}$ and every $k \geq 0$

$$\text{Ш}^2(F, \mathbb{Z}/l^k(n)) = 0. \tag{24}$$

The equalities (23) and (24), in the number field case, have already been observed by Neukirch [Ne, Satz 4.5]. Hence (24) and the exact sequence of G_F-modules

$$0 \longrightarrow \mathbb{Z}/l^k(n) \longrightarrow W^n \stackrel{l^k}{\longrightarrow} W^n \longrightarrow 0 \tag{25}$$

give the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(G_F, W^n)/l^k & \xrightarrow{\prod_v r_v} & \prod_v H^1(G_{F_v}, W^n)/l^k \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^2(G_F, \mathbb{Z}/l^k(n)) & \xrightarrow{\prod_v r_v} & \prod_v H^2(G_{F_v}, \mathbb{Z}/l^k(n)) \end{array}$$

This diagram gives the following equality:

$$div\, H^1(G_F, W^n) = \{h : \ r_v(h) \in div(H^1(G_{F_v}, W^n)), \text{ for all } v\} \tag{26}$$

For an abelian group M put $M^* := Hom(M, \mathbb{Q}/\mathbb{Z})$. For a finite S containing $S_{\infty,l}$ and for a finite G_S-module M Tate-Poitou duality gives the following exact sequence.

$$\begin{aligned} 0 \to H^0(G_S, M) &\longrightarrow \bigoplus_{v\in S} H^0(G_{F_v}, M) \longrightarrow H^2(G_S, M^D)^* \\ \to H^1(G_S, M) &\longrightarrow \bigoplus_{v\in S} H^1(G_{F_v}, M) \longrightarrow H^1(G_S, M^D)^* \\ \to H^2(G_S, M) &\longrightarrow \bigoplus_{v\in S} H^2(G_{F_v}, M) \longrightarrow H^0(G_S, M^D)^* \to 0 \end{aligned} \tag{27}$$

where for v archimedean $H^i(G_{F_v}, M)$ is $H^i_T(G_{F_v}, M)$. Taking $M := \mathbb{Z}/l^k(n)$ and passing to direct limits gives the exact sequence:

$$\begin{aligned} 0 \to H^0(G_S, W^n) &\longrightarrow \bigoplus_{v\in S} H^0(G_{F_v}, W^n) \longrightarrow H^2(G_S, \mathbb{Z}_l(1-n))^* \\ \to H^1(G_S, W^n) &\longrightarrow \bigoplus_{v\in S} H^1(G_{F_v}, W^n) \longrightarrow H^1(G_S, \mathbb{Z}_l(1-n))^* \\ \to H^2(G_S, W^n) &\longrightarrow \bigoplus_{v\in S} H^2(G_{F_v}, W^n) \longrightarrow H^0(G_S, \mathbb{Z}_l(1-n))^* \to 0 \end{aligned} \tag{28}$$

We notice that for v archimedean $H^i_T(G_{F_v}, M) = 0$ for an l-torsion G_{F_v}-module M since $G_{F_v} = G(\mathbb{C}/\mathbb{R})$ or trivial and $\mu_4 \subset F$ if $l = 2$. Hence in our case there is no contribution of archimedean part to the Tate-Poitou exact sequences with l-torsion Galois modules.

3.2. **Divisible elements in Galois cohomology of local fields.** For every prime v (nonarchimedean if $\operatorname{char} F = 0$) the Theorem 2.5 and Corollary 2.6 give:

$$H^m(G_{F_v}, W^n) \cong \begin{cases} 0 & \text{if} \quad m > 2 \quad \text{and} \quad n \in \mathbb{Z} \\ 0 & \text{if} \quad m = 2 \quad \text{and} \quad n \neq 1 \\ Br(F_v)_l \cong \mathbb{Q}_l/\mathbb{Z}_l & \text{if} \quad m = 2 \quad \text{and} \quad n = 1 \end{cases} \tag{29}$$

It follows from local Tate duality [M1, I Corollary 2.3] or [Se, II, sec. 5.2 Theorem 2 and the remark following it], that for each i such that $0 \leq i \leq 2$ there is a perfect pairing

$$H^i(F_v, \mathbb{Q}_l/\mathbb{Z}_l(n)) \times H^{2-i}(F_v, \mathbb{Z}_l(1-n)) \to \mathbb{Q}_l/\mathbb{Z}_l. \tag{30}$$

Hence the following group is finite for every $n \neq 1$:

$$H^2(G_{F_v}, \mathbb{Z}_l(n)) \cong H^0(G_{F_v}, W^{1-n})^* \cong W^{n-1}(F_v), \tag{31}$$

so by (29):

$$H^1(G_{F_v}, W^n)/Div \cong H^2(G_{F_v}, \mathbb{Z}_l(n)) \text{ for } n \neq 1. \tag{32}$$

Moreover by Hilbert 90 we have:

$$H^1(G_{F_v}, W^1) \cong F_v^\times \otimes \mathbb{Q}_l/\mathbb{Z}_l \cong Div\, H^1(G_{F_v}, W^1). \tag{33}$$

Hence for any local field F_v, any prime $l \neq \operatorname{char} F_v$ and any $n \in \mathbb{Z}$:

$$div\, H^1(G_{F_v}, W^n) = Div\, H^1(G_{F_v}, W^n). \tag{34}$$

By (16), (31) and (32), for any $v \notin S_l$, the boundary map ∂_v in the localization sequence for $\mathcal{O}_v$ gives the following isomorphism:

$$\partial_v \;:\; H^1(G_{F_v}, W^n)/Div \xrightarrow{\cong} H^0(G(\overline{k_v}/k_v), W^{n-1}) \tag{35}$$

Theorem 3.1. *There is the following isomorphism:*

$$Div\, H^1(G_{F_v}, W^n) \;\cong\; \begin{cases} (\mathbb{Q}_l/\mathbb{Z}_l)^{[F_v:\mathbb{Q}_l]+1} & \text{if} \quad v \in S_l, \;\; n \in \{0,1\} \\ (\mathbb{Q}_l/\mathbb{Z}_l)^{[F_v:\mathbb{Q}_l]} & \text{if} \quad v \in S_l, \;\; n \notin \{0,1\} \\ \mathbb{Q}_l/\mathbb{Z}_l & \text{if} \quad v \notin S_l, \;\; n \in \{0,1\} \\ 0 & \text{if} \quad v \notin S_l, \;\; n \notin \{0,1\} \end{cases} \tag{36}$$

Proof. For any finite G_{F_v}-module M with order $m := |M|$ prime to $\operatorname{char} F_v$ the Euler characteristic can be computed as follows [M1, chap. I Theorem 2.8]:

$$\chi(G_{F_v}, M) := \frac{|H^0(G_{F_v}, M)|\,|H^2(G_{F_v}, M)|}{|H^1(G_{F_v}, M)|} \;=\; |\; \mathcal{O}_v/m\mathcal{O}_v \;|^{-1} \tag{37}$$

Hence

$$\chi(G_{F_v}, \mathbb{Z}/l(n)) = \begin{cases} l^{-[F_v:\mathbb{Q}_l]} & \text{if} \quad v \in S_l \\ 1 & \text{if} \quad v \notin S_l \end{cases} \tag{38}$$

Taking log_l in (38) gives

$$\sum_{i=0}^{2} \dim_{\mathbb{Z}/l} H^i(G_{F_v}, \mathbb{Z}/l(n)) \;=\; \begin{cases} -[F_v : \mathbb{Q}_l] & \text{if} \quad v \in S_l \\ 0 & \text{if} \quad v \notin S_l \end{cases} \tag{39}$$

Observe that

$$H^0(G_{F_v}, W^n) \cong \begin{cases} \mathbb{Q}_l/\mathbb{Z}_l & \text{if} \quad n = 0 \\ \text{finite} & \text{if} \quad n \neq 0 \end{cases} \tag{40}$$

Computing the divisible rank of $H^1(G_{F_v}, W^n)$ by use of (29), (39), (40) and the following exact sequence gives the formula (36).

$$\begin{aligned} 0 \to H^0(G_{F_v}, \mathbb{Z}/l(n)) &\longrightarrow H^0(G_{F_v}, W^n) \xrightarrow{l} H^0(G_{F_v}, W^n) \\ \to H^1(G_{F_v}, \mathbb{Z}/l(n)) &\longrightarrow H^1(G_{F_v}, W^n) \xrightarrow{l} H^1(G_{F_v}, W^n) \\ \to H^2(G_{F_v}, \mathbb{Z}/l(n)) &\longrightarrow H^2(G_{F_v}, W^n) \xrightarrow{l} H^2(G_{F_v}, W^n) \to 0. \end{aligned} \tag{41}$$

□

Remark 3.2. Theorem 3.1 was proved by P. Schneider [Sch2, Satz 4 sec. 3] for $l > 2$ and $\operatorname{char} F_v = 0$.

3.3. **Divisible elements in Galois cohomology of global fields.** Consider any first quadrant spectral sequence $E_2^{i,j} \Rightarrow E^n$ with $n = i+j \geq 0$. The differentials $d_r^{i,j} \,:\, E_r^{i,j} \rightarrow E_r^{i+r,j-r+1}$ define $E_{r+1}^{i,j} := \ker d_r^{i,j} \,/\, \mathrm{im}\, d_r^{i-r,j+r-1}$ for every $r \geq 2$. For $r > n+1$ we have $d_r^{i,j} = 0$. Put $E_\infty^{i,j} := E_{n+2}^{i,j} = E_{n+3}^{i,j} = \dots$. The filtration $0 \subset E_n^n \subset E_{n-1}^n \subset \dots \subset E_0^n = E^n$ gives $E_i^n / E_{i+1}^n \cong E_\infty^{i,j}$. If $E_2^{i,j} = 0$ for every $i > 2$ then the exact sequence of lower terms extends to the following exact sequence:

$$0 \rightarrow E_2^{1,0} \rightarrow E^1 \rightarrow E_2^{0,1} \rightarrow E_2^{2,0} \rightarrow E_1^2 \rightarrow E_2^{1,1} \rightarrow 0 \tag{42}$$

Consider the Leray spectral sequence for the natural map $j \,:\, \mathrm{spec}\, F \rightarrow \mathrm{spec}\, \mathcal{O}_{F,S}$:

$$E_2^{i,j} = H^i(\mathcal{O}_{F,S},\, R^j j_* W^n) \Rightarrow H^{i+j}(F,\, W^n) \tag{43}$$

Since $\mathrm{cd}_l(G_S) = \mathrm{cd}_l(\mathcal{O}_{F,S}) = 2$ the exact sequence (42) gives the following localization exact sequence in cohomology:

$$0 \rightarrow H^1(\mathcal{O}_{F,S}, W^n) \longrightarrow H^1(F, W^n) \xrightarrow{\partial} \bigoplus_{v \notin S} H^0(k_v, W^{n-1}) \rightarrow \tag{44}$$

$$\rightarrow H^2(\mathcal{O}_{F,S}, W^n) \longrightarrow H^2(F, W^n) \xrightarrow{\partial} \bigoplus_{v \notin S} H^1(k_v, W^{n-1}) \rightarrow 0$$

By Theorem 2.5 and Corollary 2.6 we get $H^2(G_F, W^n) = 0$ for all $n \neq 1$. Hence for any $n \neq 1$ the sequence (44) has the following form:

$$0 \rightarrow H^1(\mathcal{O}_{F,S}, W^n) \longrightarrow H^1(F, W^n) \xrightarrow{\partial} \bigoplus_{v \notin S} H^0(k_v, W^{n-1}) \rightarrow H^2(\mathcal{O}_{F,S}, W^n) \rightarrow 0 \tag{45}$$

By [Ta2, Prop. 2.3] there is the following exact sequence:

$$0 \rightarrow H^1(G_S,\, W^n)/Div \rightarrow H^2(G_S,\, \mathbb{Z}_l(n)) \rightarrow H^2(G_S,\, \mathbb{Q}_l(n)) \rightarrow H^2(G_S,\, W^n) \rightarrow 0 \tag{46}$$

P. Schneider defined numbers $i_n(F)$ as follows:

$$i_n(F) := \dim_{\mathbb{F}_l} (Div H^2(G_S,\, W^n))[l]$$

In other words $i_n(F)$ is the number of copies of $\mathbb{Q}_l/\mathbb{Z}_l$ in $Div\, H^2(G_S,\, W^n)$. Note that $i_n(F) = 0$ iff $H^2(G_S,\, \mathbb{Q}_l(n)) = 0$ since $H^i(G_S,\, \mathbb{Z}_l(n))$ are finitely generated $\mathbb{Z}_l$-modules. It is immediate from (45) that $i_n(F)$ does not depend on the finite set S and for every finite S containing S_l :

$$Div\, H^1(G_{F,S}, W^n) = Div\, H^1(G_F, W^n). \tag{47}$$

Conjecture 3.3. *(P. Schneider)* $\quad i_n = 0$ *for all* $n \neq 1$.

The following lemma is well known. It was first proven by Soulé [So1, Théorème 5] in the number field case for $l > n$. We include here a proof that works for all global fields and $l \geq 2$ provided $\mu_4 \subset F$ if $l = 2$.

Lemma 3.4. *$H^2(\mathcal{O}_{F,S}, W^n) = 0$ for any $n > 1$ and any finite $S \supset S_{\infty,l}$. In particular $i_n = 0$ for all $n > 1$.*

Proof. Consider the following commutative diagram for each $n > 1$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_{2n-2}(\mathcal{O}_{F,S})_l & \longrightarrow & K_{2n-2}(F)_l & \xrightarrow{\partial} & \bigoplus_{v \notin S} K_{2n-3}(k_v)_l & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \cong & & \\
0 & \longrightarrow & H^1(\mathcal{O}_{F,S},\, W^n)/Div & \longrightarrow & H^1(F,\, W^n)/Div & \xrightarrow{\partial} & \bigoplus_{v \notin S} H^0(k_v,\, W^{n-1}) & \longrightarrow & 0
\end{array}$$

The top row is exact by Quillen localization sequence [Q1] and results of Soulé [So1, Théorème 3 p. 274], [So2, Théorème 1 p. 326]. The left and the middle vertical arrows are surjective by [DF, Theorems 8.7 and 8.9] and the right vertical arrow is an isomorphism by [DF, Corollary 8.6]. This implies that the bottom sequence is also exact. Hence (45) shows that $H^2(\mathcal{O}_{F,S}, W^n) = 0$ for all $n > 1$. So $i_n = 0$ for all $n > 1$. □

Put:

$$D_n(F) := div(H^1(G_F, W^n)/Div)$$

The following theorem extends [Sch2, Satz 8 sec. 4].

Theorem 3.5. *Assume that $i_n = 0$ for $n \neq 1$. There are the following exact sequences:*

$$0 \to D_n(F) \to H^1(G_S,\, W^n)/Div \to \bigoplus_{v\in S} W^{n-1}(F_v) \to W^{n-1}(F) \to 0. \tag{48}$$

$$0 \to D_n(F) \to H^1(G_F,\, W^n)/Div \to \bigoplus_{v} W^{n-1}(F_v) \to W^{n-1}(F) \to 0 \tag{49}$$

Proof. Let us prove the exactness of (48). Substituting n for $1-n$ in the first three terms of the exact sequence (28), dualizing and applying [Ta2, prop. 2.3] gives us the following exact sequence:

$$H^1(G_S,\, W^n)/Div \to \bigoplus_{v\in S} H^1(G_{F_v},\, W^n)/Div \to W^{n-1}(F) \to 0. \tag{50}$$

For every $S \supset S_{\infty,l}$ consider the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & D_n(F) & \longrightarrow & H^1(G_F,\, W^n)/Div & \xrightarrow{\partial} & \bigoplus_v H^1(G_{F_v},\, W^n)/Div \\
 & & \downarrow & & \downarrow \cong & & \downarrow \\
0 & \longrightarrow & H^1(\mathcal{O}_{F,S},\, W^n)/Div & \longrightarrow & H^1(F,\, W^n)/Div & \xrightarrow{\partial} & \bigoplus_{v\notin S} H^0(k_v,\, W^{n-1})
\end{array}$$

The exactness of the top sequence follows by (26) and (34). By (35) this gives the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & D_n(F) & \longrightarrow & H^1(G_F,\, W^n)/Div & \xrightarrow{\partial} & \bigoplus_v H^1(G_{F_v},\, W^n)/Div \\
 & & = \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & D_n(F) & \longrightarrow & H^1(\mathcal{O}_{F,S},\, W^n)/Div & \longrightarrow & \bigoplus_{v\in S} H^1(F_v,\, W^n)/Div
\end{array}$$

Hence the exact sequence (48) is obtained by connecting the bottom exact sequence of the last diagram and the exact sequence (50) and applying isomorphisms (31) and (32). The exact sequence (49) is obtained from (48) by passing to the direct limit over S. □

Corollary 3.6. *Let $n \neq 1$ and let $i_n = 0$. For every finite $S \supset S_{\infty,l}$:*

$$D_n(F) = Ш^2_S(F,\, \mathbb{Z}_l(n)) = Ш^2(F,\, \mathbb{Z}_l(n))_l. \tag{51}$$

Proof. Follows by Theorem (3.5) and [Ta2, Prop. 2.3]. □

Let (see (58) in the next chapter):

$$D^{et}(n) := div\, K_{2n}^{et}(F)_l. \tag{52}$$

Theorem 3.7. *Let $n > 0$. For every finite $S \supset S_{\infty,l}$ there are exact sequences:*

$$0 \to D^{et}(n) \to K_{2n}^{et}(\mathcal{O}_{F,S}) \to \bigoplus_{v \in S} W^n(F_v) \to W^n(F) \to 0. \tag{53}$$

$$0 \to D^{et}(n) \to K_{2n}^{et}(F)_l \to \bigoplus_{v} W^n(F_v) \to W^n(F) \to 0 \tag{54}$$

Moreover there is the following equality:

$$\frac{|K_{2n}^{et}(\mathcal{O}_{F,S})|}{|D^{et}(n)|} = \frac{\big|\prod_{v \in S} w_n(F_v)\big|_l^{-1}}{|w_n(F)|_l^{-1}}. \tag{55}$$

Proof. By Lemma 3.4 the sequences (48) and (49) are exact. Moreover by [DF, Prop. 5.1] and [Ta2, prop. 2.3] there are the following isomorphisms:

$$K_{2n}^{et}(\mathcal{O}_{F,S}) \cong H^2(G_{F,S},\, \mathbb{Z}_l(n+1))_l \cong H^1(G_{F,S},\, W^{n+1})/Div.$$

$$K_{2n}^{et}(F)_l \cong H^2(G_F,\, \mathbb{Z}_l(n+1))_l \cong H^1(G_F,\, W^{n+1})/Div$$

Hence $D^{et}(n) \cong D_{n+1}(F)$. Observe that the group $H^1(G_{F,S},\, W^{n+1})/Div$ is finite. Indeed by [Bo], [Ha] and [Q2] the group $K_{2n}(\mathcal{O}_{F,S})$ is finite and the Dwyer-Friedlander map $K_{2n}(\mathcal{O}_{F,S}) \to K_{2n}^{et}(\mathcal{O}_{F,S})$ [DF] is surjective. The formula (55) follows from (53) because all the terms of this exact sequence are finite. □

4. Divisible elements in K-groups of global fields

4.1. **General results on divisible elements.** Consider the following commutative diagram. The rows are localization sequences and the vertical maps are the Dwyer-Friedlander maps [DF].

$$\begin{array}{ccccccccc}
\longrightarrow & K_{2n+1}(F,\, \mathbb{Z}/l^k) & \xrightarrow{\partial} & \bigoplus_v K_{2n}(k_v,\, \mathbb{Z}/l^k) & \longrightarrow & K_{2n}(\mathcal{O}_F,\, \mathbb{Z}/l^k) & \longrightarrow \\
& \downarrow & & \downarrow \cong & & \downarrow & \\
\longrightarrow & K_{2n+1}^{et}(F,\, \mathbb{Z}/l^k) & \xrightarrow{\partial^{et}} & \bigoplus_{v \nmid l} K_{2n}^{et}(k_v,\, \mathbb{Z}/l^k) & \longrightarrow & K_{2n}^{et}(\mathcal{O}_F[\frac{1}{l}],\, \mathbb{Z}/l^k) & \longrightarrow
\end{array}$$

For every $k > 0$ define:

$$D(n, l^k) := \ker\left(K_{2n}(\mathcal{O}_F,\, \mathbb{Z}/l^k) \to K_{2n}(F,\, \mathbb{Z}/l^k)\right) \,=\, \mathrm{coker}\, \partial$$

$$D^{et}(n, l^k) := \ker\left(K_{2n}^{et}(\mathcal{O}_F[1/l],\, \mathbb{Z}/l^k) \to K_{2n}^{et}(F,\, \mathbb{Z}/l^k)\right) \,=\, \mathrm{coker}\, \partial^{et}$$

We do not consider $\mathrm{coker}\, \partial$ and $\mathrm{coker}\, \partial^{et}$ in the following commutative diagram:

$$\begin{array}{ccccccccc}
\longrightarrow & K_{2n}(F,\, \mathbb{Z}/l^k) & \xrightarrow{\partial} & \bigoplus_v K_{2n-1}(k_v,\, \mathbb{Z}/l^k) & \longrightarrow & K_{2n-1}(\mathcal{O}_F,\, \mathbb{Z}/l^k) & \longrightarrow \\
& \downarrow & & \downarrow \cong & & \downarrow & \\
\longrightarrow & K_{2n}^{et}(F,\, \mathbb{Z}/l^k) & \xrightarrow{\partial^{et}} & \bigoplus_{v \nmid l} K_{2n-1}^{et}(k_v,\, \mathbb{Z}/l^k) & \longrightarrow & K_{2n-1}^{et}(\mathcal{O}_F[\frac{1}{l}],\, \mathbb{Z}/l^k) & \longrightarrow
\end{array}$$

because the isomorphism $K_{2n-1}(\mathcal{O}_F) \cong K_{2n-1}(F)$ (resp. the isomorphism $K_{2n-1}^{et}(\mathcal{O}_F[\frac{1}{l}])_l \cong K_{2n-1}^{et}(F)_l$) for every $n > 1$ and the comparison of Bockstein sequences for $\mathcal{O}_F$ and F (resp. $\mathcal{O}_F[\frac{1}{l}]$ and F) show that:

$$\operatorname{coker} \partial \;=\; \ker\,(K_{2n-1}(\mathcal{O}_F,\, \mathbb{Z}/l^k) \rightarrow K_{2n-1}(F,\, \mathbb{Z}/l^k)) \;= 0$$

$$\operatorname{coker} \partial^{et} \;=\; \ker\,(K_{2n-1}^{et}(\mathcal{O}_F[1/l],\, \mathbb{Z}/l^k) \rightarrow K_{2n-1}^{et}(F,\, \mathbb{Z}/l^k)) \;= 0$$

Comparing the Bockstein exact sequences in K-theory for $\mathcal{O}_F$ and for F (resp. étale K-theory for $\mathcal{O}_F[1/l]$ and for F) we notice that for each $k > 0$:

$$D(n, l^k) \cong \ker\,(K_{2n}(\mathcal{O}_F)/l^k \rightarrow K_{2n}(F)/l^k) \cong \tag{56}$$

$$\cong K_{2n}(\mathcal{O}_F) \cap K_{2n}(F)^{l^k} / K_{2n}(\mathcal{O}_F)^{l^k}.$$

$$D^{et}(n, l^k) \cong \ker\,(K_{2n}^{et}(\mathcal{O}_F[1/l])/l^k \rightarrow K_{2n}^{et}(F)/l^k) \cong \tag{57}$$

$$\cong K_{2n}^{et}(\mathcal{O}_F[1/l]) \cap K_{2n}^{et}(F)^{l^k} / K_{2n}^{et}(\mathcal{O}_F[1/l])^{l^k}.$$

Hence for every $k \geq 1$ the group $D(n, l^k)$ (resp. $D^{et}(n, l^k)$) is a subquotient of $K_{2n}(\mathcal{O}_F)$ ($K_{2n}^{et}(\mathcal{O}_F[1/l])$ resp.). Following [Ba1] we will abbreviate our notation at some places as follows:

$$D(n) := div\, K_{2n}(F) \quad \text{and} \quad D^{et}(n) := div\, K_{2n}^{et}(F)_l\,. \tag{58}$$

Applying Bockstein sequences (cf. [Ba2, Diagrams 2.1 and 2.3, p. 289-290]) for all $k \gg 0$ gives:

$$K_{2n}(\mathcal{O}_F)/l^k = K_{2n}(\mathcal{O}_F)_l \qquad \text{and} \qquad D(n, l^k) \cong D(n)_l\,. \tag{59}$$

Moreover for all $k \gg 0$ we get by similar argument:

$$K_{2n}^{et}(\mathcal{O}_F[1/l])/l^k = K_{2n}^{et}(\mathcal{O}_F[1/l])_l \qquad \text{and} \qquad D^{et}(n, l^k) \cong D^{et}(n)\,. \tag{60}$$

Let $k(l)$ be the smallest k such that both conditions (59) and (60) hold. We observe that if $l \not| \, |K_{2n}(\mathcal{O}_F)|$ then $D(n, l^k) = D(n)_l = 0$ for all $k \geq 1$.

Theorem 4.1. *If $l > 2$ then $\forall\, k \geq 1$ there is the following canonical isomorphism:*

$$D(n, l^k) \cong D^{et}(n, l^k) \tag{61}$$

If $l = 2$ then $\forall\, k \geq 2$ there is the following canonical isomorphism:

$$D(n, 2^k) \cong D^{et}(n, 2^k) \tag{62}$$

If $l \geq 2$ then there is the following isomorphism $D(n)_l \cong D^{et}(n)$ or more explicitly

$$div\, K_{2n}(F)_l \;\cong\; div\, K_{2n}^{et}(F)_l \tag{63}$$

Proof. For every l odd and $k \geq 1$ (resp. for $l = 2$ and $k \geq 2$) consider the following commutative diagram.

$$\begin{array}{ccccccccc}
\longrightarrow & K_{2n+1}(F, \mathbb{Z}/l^k) & \xrightarrow{\partial} & \bigoplus_v K_{2n}(k_v, \mathbb{Z}/l^k) & \longrightarrow & D(n, l^k) & \longrightarrow & 0 \\
& \downarrow & & \downarrow \cong & & \downarrow \cong & & \\
\longrightarrow & K_{2n+1}^{et}(F, \mathbb{Z}/l^k) & \xrightarrow{\partial^{et}} & \bigoplus_{v \nmid l} K_{2n}^{et}(k_v, \mathbb{Z}/l^k) & \longrightarrow & D^{et}(n, l^k) & \longrightarrow & 0
\end{array}$$

The right vertical arrow is an isomorphism because the middle vertical arrow is an isomorphism [DF, Corollary 8.6] and the left vertical arrow is an epimorphism [DF, Theorem 8.5]. The isomorphism (63) follows from (59), (60), (61), (62). ☐

Corollary 4.2. *For all $n > 0$ there are the following isomorphisms:*

(64) $$D(n)_l \cong D^{et}(n) \cong D_{n+1}(F) = Ш^2_S(F, \mathbb{Z}_l(n+1)) = Ш^2(F, \mathbb{Z}_l(n+1)).$$

Proof. This follows by Lemma 3.4, Theorem 3.5, Corollary 3.6, Theorem 4.1 and by the following isomorphism $K_{2n}^{et}(F) \cong H^2(F, \mathbb{Z}_l(n+1))$ [DF, Prop. 5.1]. ☐

Theorem 4.3. *For every $n \geq 1$ there are the following isomorphisms:*

(65) $$\varinjlim_k D(n, l^k) = 0,$$

(66) $$\varprojlim_k D(n, l^k) \cong D(n)_l.$$

(67) $$\varinjlim_k D^{et}(n, l^k) = 0,$$

(68) $$\varprojlim_k D^{et}(n, l^k) \cong D^{et}(n)$$

Proof. The isomorphisms (65), (66), (67), (68) follow by comparing the Bockstein exact sequences in K-theory for $\mathcal{O}_F$ and for F (resp. in étale K-theory for $\mathcal{O}_F[1/l]$ and for F). ☐

Proposition 4.4. *For every $n \geq 1$:*

(69) $$\varprojlim_k \bigoplus_v K_{2n}(k_v, \mathbb{Z}/l^k) = \varprojlim_k \bigoplus_v K_{2n}^{et}(k_v, \mathbb{Z}/l^k) = 0$$

and the group

(70) $$\varprojlim_k{}^1 \bigoplus_v K_{2n}(k_v, \mathbb{Z}/l^k) = \varprojlim_k{}^1 \bigoplus_v K_{2n}^{et}(k_v, \mathbb{Z}/l^k)$$

is torsion free.

Proof. Notice that $K_{2n}(k_v, \mathbb{Z}/l^k) \cong K_{2n-1}(k_v)[l^k]$ and $K_{2n}^{et}(k_v, \mathbb{Z}/l^k) \cong K_{2n-1}^{et}(k_v)[l^k]$. Because $K_{2n}(k_v, \mathbb{Z}/l^k) \cong K_{2n}^{et}(k_v, \mathbb{Z}/l^k)$ by [DF] it is enough to make the proof for K-theory. Hence (69) follows because:

$$\varprojlim_k \bigoplus_v K_{2n}(k_v, \mathbb{Z}/l^k) \subset \varprojlim_k \prod_v K_{2n}(k_v, \mathbb{Z}/l^k) = 0.$$

Applying the $lim - lim^1$ exact sequence to the exact sequence:

$$0 \to \oplus_v K_{2n}(k_v, \mathbb{Z}/l^k) \to \prod_v K_{2n}(k_v, \mathbb{Z}/l^k) \to \prod_v K_{2n}(k_v, \mathbb{Z}/l^k)/ \oplus_v K_{2n}(k_v, \mathbb{Z}/l^k) \to 0$$

gives the natural isomorphism

$$\varprojlim_k \prod_v K_{2n}(k_v, \mathbb{Z}/l^k)/ \oplus_v K_{2n}(k_v, \mathbb{Z}/l^k) \cong \varprojlim_k{}^1 \oplus_v K_{2n}(k_v, \mathbb{Z}/l^k) \tag{71}$$

The group on the left hand side of (71) is clearly torsion free. □

Theorem 4.5. *For every $n \geq 1$ there is the following isomorphism:*

$$\varprojlim_k{}^1 K_n(F, \mathbb{Z}/l^k) \xrightarrow{\cong} \varprojlim_k{}^1 K_n^{et}(F, \mathbb{Z}/l^k). \tag{72}$$

Moreover there is the following equality:

$$\varprojlim_k{}^1 K_{2n}(F, \mathbb{Z}/l^k) = 0, \tag{73}$$

and the following exact sequence:

$$0 \to D(n)_l \to \varprojlim_k{}^1 K_{2n+1}(F, \mathbb{Z}/l^k) \to \varprojlim_k{}^1 \bigoplus_v K_{2n}(k_v, \mathbb{Z}/l^k) \to 0. \tag{74}$$

Proof. We are going to give a proof that works for all global fields satisfying our assumptions set up in section 2. Consider the following Bockstein exact sequences:

$$0 \to K_n(F)/l^k \to K_n(F, \mathbb{Z}/l^k) \to K_{n-1}(F)[l^k] \to 0 \tag{75}$$

$$0 \to K_n^{et}(F)/l^k \to K_n^{et}(F, \mathbb{Z}/l^k) \to K_{n-1}(F)^{et}[l^k] \to 0 \tag{76}$$

If $n = 2m$ then $K_{2m-1}(\mathcal{O}_{F,S})_l = K_{2m-1}(F)_l$ and $K_{2m-1}^{et}(\mathcal{O}_{F,S})_l \cong K_{2m-1}^{et}(F)_l$ are all finite groups. Since the natural maps $K_n(F)/l^{k+1} \to K_n(F)/l^k$ and $K_n^{et}(F)/l^{k+1} \to K_n^{et}(F)/l^k$ are surjective for all $n \geq 0$ and all $k \geq 0$, the equality (73) follows by applying the $lim - lim^1$ exact sequence to the Bockstein sequences (75) and (76).

Consider the natural maps

$$i \, : \, K_{2n+1}(\mathcal{O}_{F,S}, \mathbb{Z}/l^k) \to K_{2n+1}(F, \mathbb{Z}/l^k),$$

$$i^{et} \, : \, K_{2n+1}^{et}(\mathcal{O}_{F,S}, \mathbb{Z}/l^k) \to K_{2n+1}^{et}(F, \mathbb{Z}/l^k).$$

Since the groups $K_{2n+1}(\mathcal{O}_{F,S}, \mathbb{Z}/l^k)$ and $K_{2n+1}^{et}(\mathcal{O}_{F,S}, \mathbb{Z}/l^k)$ are finite, the $lim - lim^1$ exact sequence shows that

$$\varprojlim_k{}^1 K_{2n+1}(F, \mathbb{Z}/l^k)/i(K_{2n+1}(\mathcal{O}_{FS}, \mathbb{Z}/l^k)) \cong \varprojlim_k{}^1 K_{2n+1}(F, \mathbb{Z}/l^k)$$

$$\varprojlim_k{}^1 K_{2n+1}^{et}(F, \mathbb{Z}/l^k)/i^{et}(K_{2n+1}^{et}(\mathcal{O}_{F,S}, \mathbb{Z}/l^k)) \cong \varprojlim_k{}^1 K_{2n+1}^{et}(F, \mathbb{Z}/l^k).$$

Hence taking into account Theorem 4.3 (66, 68), Proposition 4.4 and applying the $lim - lim^1$ exact sequence to the rows of following commutative diagram :

$$\begin{array}{ccccccc}
0 \to K_{2n+1}(F, \mathbb{Z}/l^k)/i^{et}(K_{2n+1}^{et}(\mathcal{O}_{FS}, \mathbb{Z}/l^k)) & \longrightarrow & \bigoplus_v K_{2n}(k_v, \mathbb{Z}/l^k) & \longrightarrow & D(n, l^k) \to 0 \\
\downarrow & & \downarrow \cong & & \downarrow \cong \\
0 \to K_{2n+1}(F, \mathbb{Z}/l^k)/i^{et}(K_{2n+1}^{et}(\mathcal{O}_{FS}, \mathbb{Z}/l^k) & \longrightarrow & \bigoplus_{v \nmid l} K_{2n}^{et}(k_v, \mathbb{Z}/l^k) & \longrightarrow & D^{et}(n, l^k) \to 0
\end{array}$$

gives the natural commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D(n)_l & \longrightarrow & \varprojlim_k^1 \, K_{2n+1}(F,\, \mathbb{Z}/l^k) & \longrightarrow & \varprojlim_k^1 \bigoplus_v \, K_{2n}(k_v,\, \mathbb{Z}/l^k) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & D^{et}(n) & \longrightarrow & \varprojlim_k^1 \, K_{2n+1}^{et}(F,\, \mathbb{Z}/l^k) & \longrightarrow & \varprojlim_k^1 \bigoplus_{v \not| l} \, K_{2n}^{et}(k_v,\, \mathbb{Z}/l^k) & \longrightarrow & 0 \end{array}$$

Hence the top row of the diagram is the exact sequence (74) and the middle vertical arrow is the isomorphism (72) □

Theorem 4.1 gives the opportunity to compute the order of the group $D(n)_l$.

Example 4.6. Recall that for n odd, $l > 2$ and a totally real number field F [Ba2, Theorem 3 (ii) p. 289] there is the following formula:

(77) $$|D(n)_l| = \Big|\frac{w_{n+1}(F)\zeta_F(-n)}{\prod_{v|l} \, w_n(F_v)}\Big|_l^{-1}.$$

One gets this formula taking $S = S_l$, applying the equalities (55) and (63), using the theorem of Wiles which states that:

$$|H^2(\mathcal{O}_{S_l},\, \mathbb{Z}_l(n+1))| = |w_{n+1}(F)\zeta_F(-n)|_l^{-1}.$$

Observe that $|w_n(F)|_l^{-1} = 1$ for F totally real and l odd.

Example 4.7. Now let $\operatorname{char} F = p > 0$ and let $\mathbb{F}_q$ be the algebraic closure of $\mathbb{F}_p$ in F. Let $X/\mathbb{F}_q$ be a smooth curve corresponding to F. This curve is unique up to $\mathbb{F}_q$ isomorphism. Let $Z(X, t)$ denote the Weil zeta function for X. Then put $t = q^{-s}$ and define:

$$\zeta_F(s) := Z(X, q^{-s}).$$

The Leray spectral sequence for the natural map $i\,;\, \operatorname{spec} F \to X$:

$$E_2^{i,j} = H^i(X,\, R^j i_* W^{n+1}) \Rightarrow H^{i+j}(F,\, W^{n+1})$$

gives the following exact sequence of the lower terms:

(78) $$0 \longrightarrow H^1(X,\, W^{n+1}) \longrightarrow H^1(F,\, W^{n+1}) \xrightarrow{\partial} \bigoplus_v H^0(k_v,\, W^n)$$

By (35) and the exact sequences (49) and (54) we obtain for all $n \geq 1$ the following natural isomorphism:

(79) $$D^{et}(n) \cong H^1(X,\, W^{n+1})$$

Its is well known c.f. [Ko] p. 202 that $|H^0(X,\, W^{n+1})| = \big|q^{n+1} - 1\big|_l^{-1}$, $|H^2(X,\, W^{n+1})| = \big|q^n - 1\big|_l^{-1}$ and

(80) $$|H^1(X,\, W^{n+1})| = |(q^{n+1} - 1)(q^n - 1)\zeta_X(-n)|_l^{-1}.$$

Since $\mathbb{F}_q$ is the algebraic closure of $\mathbb{F}_p$ in F we have $W^k(F) = W^k(\mathbb{F}_q)$ for all $k \in \mathbb{Z}$. In particular for all $k > 0$ we have $|w_k(F)|_l^{-1} = |w_k(\mathbb{F}_q)|_l^{-1} = |q^k - 1|_l^{-1}$. If $q_v := Nv$, then $|w_k(F_v)|_l^{-1} = |w_k(k_v)|_l^{-1} = |q_v^k - 1|_l^{-1}$. Observe that:

$$\zeta_F(s) = \zeta_X(s) \prod_{v \,|\, \infty} (1 - Nv^{-s})$$

Hence by Theorem 4.1 we get:

$$|D(n)_l| = \Big|\frac{w_n(F)\, w_{n+1}(F)\, \zeta_F(-n)}{\prod_{v\,|\,\infty} w_n(F_v)}\Big|_l^{-1}. \tag{81}$$

4.2. **Application to the homology of GL.** Let us return to the general situation where $l \geq 2$ and F is a global field of characteristic $\operatorname{char} F \neq l$ such that for $l = 2$ it is assumed that $\mu_4 \subset F$.

Theorem 4.8. *For every $n \geq 1$, $k \geq 1$ and $l > n+1$ the kernel of the natural map*

$$H_{2n}(GL(\mathcal{O}_F),\, \mathbb{Z}/l^k) \;\to\; H_{2n}(GL(F),\, \mathbb{Z}/l^k) \tag{82}$$

contains a subgroup isomorphic to $D(n, l^k)$.

Proof. Let A be a commutative ring with identity. Comparing the Bockstein exact sequences for K-theory of A and for the homology of $GL(A)$ and applying the result of Arlettaz [Ar2, Cor. 7.19] (cf. [Ar1]) we observe that for all $l > n+1$ and $k \geq 1$ the Hurewicz homomorphism

$$h_{2n} \,:\, K_{2n}(A,\, \mathbb{Z}/l^k) \;\to\; H_{2n}(GL(A),\, \mathbb{Z}/l^k) \tag{83}$$

is injective. Hence the claim follows by the following commutative diagram.

$$\begin{array}{ccc}
K_{2n}(\mathcal{O}_F,\, \mathbb{Z}/l^k) & \longrightarrow & K_{2n}(F,\, \mathbb{Z}/l^k) \\
\downarrow{\scriptstyle h_{2n}} & & \downarrow{\scriptstyle h_{2n}} \\
H_{2n}(GL(\mathcal{O}_F),\, \mathbb{Z}/l^k) & \longrightarrow & H_{2n}(GL(F),\, \mathbb{Z}/l^k)
\end{array}$$

□

Corollary 4.9. *Let $n \geq 1$ and $l > n+1$. Assume that $K_{2n}^{et}(\mathcal{O}_F[1/l]) \cong D^{et}(n)_l$ and $l \,||\, |D(n)_l|$. Then kernel of the natural map*

$$H_{2n}(GL(\mathcal{O}_F),\, \mathbb{Z}/l) \;\to\; H_{2n}(GL(F),\, \mathbb{Z}/l) \tag{84}$$

contains a subgroup isomorphic to $D(n)_l$.

Proof. By Theorem 4.1 we have $D(n)_l \cong D^{et}(n)_l$ and $D(n, l) \cong D^{et}(n, l)$. Moreover by (57) and the assumptions we have the following isomorphism $D^{et}(n)_l \cong D^{et}(n, l)$, hence $D(n)_l \cong D(n, l)$. □

Corollary 4.10. *Let $F = \mathbb{Q}$ and let $n \geq 1$ be odd. Assume that $l > n+1$ is such that $l \,||\, |w_{n+1}(\mathbb{Q})\zeta_{\mathbb{Q}}(-n)|_l^{-1}$. Then the kernel of the natural map*

$$H_{2n}(GL(\mathbb{Z}),\, \mathbb{Z}/l) \;\to\; H_{2n}(GL(\mathbb{Q}),\, \mathbb{Z}/l) \tag{85}$$

contains a subgroup isomorphic to $\mathbb{Z}/l$.

Proof. Observe that $K_{2n}^{et}(\mathbb{Z}[1/l]) \cong D^{et}(n)_l$ by Theorem 3.7 equality (55) (see also [Ba2]). Moreover $|D(n)_l| = |D^{et}(n)_l| = |w_{n+1}(\mathbb{Q})\zeta_{\mathbb{Q}}(-n)|_l^{-1}$ by [Ba2, Theorem 3 p. 289]. Hence the claim follows by Corollary 4.9. □

Example 4.11. Let $F = \mathbb{Q}$, $n = 11$ and $l = 691$. Observe that $w_{12}(\mathbb{Q})\zeta_{\mathbb{Q}}(-11) = 2 \times 691$ cf. [Ba1, p. 343]. Then the kernel of the natural map:

$$H_{22}(GL(\mathbb{Z}),\, \mathbb{Z}/691) \;\to\; H_{22}(GL(\mathbb{Q}),\, \mathbb{Z}/691)$$

contains a subgroup isomorphic to $\mathbb{Z}/691$.

Example 4.12. Let $F = \mathbb{Q}$, $n = 15$ and $l = 3617$. Observe that $w_{16}(\mathbb{Q})\zeta_{\mathbb{Q}}(-15) = 2 \times 3617$ cf. [Ba1, Example, p. 358]. Hence the kernel of the natural map:

$$H_{30}(GL(\mathbb{Z}),\, \mathbb{Z}/3617) \;\rightarrow\; H_{30}(GL(\mathbb{Q}),\, \mathbb{Z}/3617)$$

contains a subgroup isomorphic to $\mathbb{Z}/3617$.

Let $E/\mathbb{F}_q$ be an elliptic curve given by the Weierstrass equation $y^2 = x^3 + Ax + B$ with $A, B \in \mathbb{F}_q$. Let $F := \mathbb{F}_q(E)$. Since $[F,\, \mathbb{F}_q(x)] = 2$ the finite field $\mathbb{F}_q$ is algebraically closed in F. There is only one point at ∞ and $F_\infty = \mathbb{F}_q((x))$. Moreover $|w_n(F_\infty)|_l^{-1} = |w_n(\mathbb{F}_q)|_l^{-1} = |q^n - 1|_l^{-1}$. It was proven by Weil (see [Sil]) that for $a := 1 + q - |E(\mathbb{F}_q)|$ there is the following formula:

$$Z(E, q^{-s}) = \frac{1 - aq^{-s} + q^{1-2s}}{(1 - q^{-s})(1 - q^{1-s})}$$

For an elliptic curve over $\mathbb{F}_q$ the formula (55) of the Theorem 3.7 yields the following equality $K_{2n}^{et}(\mathcal{O}_F) \cong D^{et}(n)_l$.

Now let $q = p$ a prime number and let $E/\mathbb{F}_p$ be supersingular. Since $|a| \leq 2\sqrt{p}$ by Hasse theorem and $a \equiv 0 \mod p$ then for $p > 3$ we have $a = 0$. Hence taking $s = -n$ and using (81) we get:

$$|D(n)_l| = \frac{\left|1 + p^{1+2n}\right|_l^{-1}}{\left|1 - p^n\right|_l^{-1}}. \tag{86}$$

Corollary 4.13. *Let $E/\mathbb{F}_p$ be a supersingular elliptic curve and let $F = \mathbb{F}_p(E)$. Let $n \geq 1$ be odd and l be a prime number such that $p \equiv -1 \mod l$, $l > n + 1$ and $l \not| \frac{(p+1)(2n+1)}{l}$. Then the kernel of the natural map:*

$$H_{2n}(GL(\mathcal{O}_F),\, \mathbb{Z}/l) \;\rightarrow\; H_{2n}(GL(F),\, \mathbb{Z}/l) \tag{87}$$

contains a subgroup isomorphic to $\mathbb{Z}/l$.

Proof. It is clear that for n odd and l such that $p \equiv -1 \mod l$ and $l \not| \frac{(p+1)(2n+1)}{l}$ the formula (86) yields $l \;||\; |D(n)_l|$. Hence the claim follows by Corollary 4.9. □

Example 4.14. The elliptic curve $y^2 = x^3 + 1$ over $\mathbb{F}_p$, for $p \geq 5$, is supersingular iff $p \equiv 2 \mod 3$ [Si, pp. 143-144]. In particular for $p = 29$, $l = 5$, n odd and $n \not\equiv 2 \mod 5$ we notice by (86) that $5 \;||\; |D(n)_5|$. Hence by Corollary 4.13 the kernel of the following map:

$$H_6(GL(\mathcal{O}_{\mathbb{F}_{29}(E)}),\, \mathbb{Z}/5) \;\rightarrow\; H_6(GL(\mathbb{F}_{29}(E)),\, \mathbb{Z}/5) \tag{88}$$

contains a subgroup isomorphic to $\mathbb{Z}/5$. Similarly for $p = 41$, $l = 7$ and any n odd such that $n \not\equiv 3 \mod 7$ we notice by (86) that $7 \;||\; |D(n)_7|$. Hence by Corollary 4.13 the kernel of the following map:

$$H_{10}(GL(\mathcal{O}_{\mathbb{F}_{41}(E)}),\, \mathbb{Z}/7) \;\rightarrow\; H_{10}(GL(\mathbb{F}_{41}(E)),\, \mathbb{Z}/7) \tag{89}$$

contains a subgroup isomorphic to $\mathbb{Z}/7$.

Example 4.15. The elliptic curve $y^2 = x^3 + x$ over $\mathbb{F}_p$, for $p \geq 3$, is supersingular iff $p \equiv 3 \mod 4$, [Si, pp. 143-144]. Again taking $p = 19$, $l = 5$, n odd and $n \not\equiv 2 \mod 5$ we notice by (86) that $5 \;||\; |D(n)_5|$. So by Corollary 4.13 the kernel of the following map:

$$H_6(GL(\mathcal{O}_{\mathbb{F}_{19}(E)}),\, \mathbb{Z}/5) \;\rightarrow\; H_6(GL(\mathbb{F}_{19}(E)),\, \mathbb{Z}/5) \tag{90}$$

contains a subgroup isomorphic to $\mathbb{Z}/5$.

5. The wild kernels and divisible elements

5.1. **Wild kernels and the Moore exact sequence.** The following theorem is basically known however the results are scattered over a number of papers. Namely, surjectivity of the map (91) is due to [DF] and surjectivity of (92) for number fields was proven in [Ba2]. The splitting of the map (91) in the number field case was settled in [Ba2] and the canonical splitting of the map (91) in global field case was settled in [K]. The splitting of the map (92) for the even K-groups of number field was proven in [Ca]. For the record we make a very short proof of Theorem 5.1 pointing out key ingredients.

Theorem 5.1. *For every $n \geq 1$ and every finite set $S \supset S_l$ the following natural maps are split surjective:*

$$K_n(\mathcal{O}_{F,S})_l \to K_n^{et}(\mathcal{O}_{F,S})_l. \tag{91}$$

$$K_n(F)_l \to K_n^{et}(F)_l. \tag{92}$$

Proof. If X denotes $\mathcal{O}_{F,S}$ or F then by [DF] Theorem 8.5 the left vertical arrow in the following commutative diagram is surjective.

$$\begin{array}{ccccc} K_{n+1}(X,\,\mathbb{Z}/l^k) & \longrightarrow & K_n(X)[l^k] & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ K_{n+1}^{et}(X,\,\mathbb{Z}/l^k) & \longrightarrow & K_n^{et}(X)[l^k] & \longrightarrow & 0 \end{array} \tag{93}$$

Hence the right vertical arrow is surjective so $K_n(X)_l \to K_n^{et}(X)_l$ is surjective cf. [Ba2, Theorem 1] . The surjectivity of the map (92) follows also by surjectivity of the maps (91) for all finite S upon taking the direct limit over S, (cf. [Ba2] the proof of Theorem 1). Since the groups $K_n(\mathcal{O}_{F,S})_l$, $K_n^{et}(\mathcal{O}_{F,S})_l$, are finite for all $n > 0$, and the groups $K_n(F)_l$ and $K_n^{et}(F)_l$ are finite for all n odd then the splitting for the map (91) for all $n > 0$ (resp. the splitting of the map (92) for all n odd) follows from the investigation of the right vertical arrow of the diagram (93), cf. the proof of [Ba2, Proposition 2] . For the splitting of the map (92) with n even we use the method of Luca Caputo [Ca]. Namely from the diagram (93) we find out that the kernel of the map (92) is a pure subgroup of $K_{2n}(F)_l$ and from the diagram of the proof of Lemma 3.4 we get that this kernel is finite. Hence by [Ka, Theorem 7] the map (92) is split surjective. Actually the splitting of both maps (91) and (92) for all $n \geq 1$ follows by this method by use of [Ka, Theorem 7]. □

The Wild kernels $K_n^w(\mathcal{O}_F)_l$ and $WK_n(F)$ are defined as the kernels of the natural localization maps to make the following sequences exact:

$$0 \to K_n^w(\mathcal{O}_F)_l \to K_n(F)_l \to \prod_v K_n^{et}(F_v)_l \tag{94}$$

$$0 \to WK_n(F) \to K_n(F) \to \prod_v K_n(F_v) \tag{95}$$

where the product is over all places of F.

Remark 5.2. By our general assumption in this paper, we will consider the 2-part of $WK_n(F)$ only for such F, for which $\mu_4 \subset F$, although the definition of $WK_n(F)$ is for any global field.

Lemma 5.3. *If F is a number field then $WK_n(F)_l$ maps to zero via*

$$K_n(F) \to \prod_{v \in S_\infty} K_n(F_v) \tag{96}$$

for every $l \geq 2$.

Proof. Choose an imbedding $\overline{F} \subset \mathbb{C}$. Take any $v' \notin S_\infty$.
If $l = 2$ then by assumption $\mathbb{Q}(i) \subset F$ so $F_v = \mathbb{C}$ for all $v \in S_\infty$. Then $F \subset F_{v'}^h \subset \overline{F} \subset F_v = \mathbb{C}$ for every $v \in S_\infty$. The map (100) factors as follows:

$$K_n(F) \to K_n(F_{v'}^h) \to \prod_{v \in S_\infty} K_n(F_v). \tag{97}$$

and since $K_n(F_{v'}^h) \to K_n(F_{v'})$ is a monomorphism [BZ2] then $WK_n(F)_2$ maps to zero via the left arrow of (97).
If $l > 2$ then for any $v \in S_\infty$ such that $F_v = \mathbb{R}$ consider the imbedding $F_v \subset \mathbb{C}$. The group $WK_n(F)_l$ maps to zero via the left map of composition of maps:

$$K_n(F) \to K_n(F_{v'}^h) \to \prod_{v \in S_\infty} K_n(\mathbb{C}). \tag{98}$$

The map $\prod_{v \in S_\infty} K_n(F_v) \to \prod_{v \in S_\infty} K_n(\mathbb{C})$ is an imbedding on the l-torsion part hence $WK_n(F)_l$ maps to zero via the left map of the composition

$$K_n(F) \to \prod_{v \in S_\infty} K_n(F_v) \to \prod_{v \in S_\infty} K_n(\mathbb{C}). \tag{99}$$

□

Lemma 5.4. *If F is a number field then $K_n^w(\mathcal{O}_F)_l$ maps to zero via*

$$K_n(F)_l \to \prod_{v \in S_\infty} K_n^{et}(F_v)_l \tag{100}$$

for every $l \geq 2$.

Proof. For every $v' \notin S_\infty$ there is a natural isomorphism $G_{F_{v'}^h} \cong G_{F_{v'}}$. Consequently there is the natural isomorphism $H^i(F_{v'}^h, \mathbb{Z}_l(j)) \cong H^i(F_{v'}, \mathbb{Z}_l(j))$. So the map $\operatorname{spec} F_{v'} \to \operatorname{spec} F_{v'}^h$ gives the isomorphism of Dwyer-Friedlander spectral sequences [DF, Prop. 5.1] which yields the isomorphism:

$$K_n^{et}(F_{v'}^h) \cong K_n^{et}(F_{v'}). \tag{101}$$

Now the proof is very similar to the proof of Lemma 5.3 applying (101) in place of [BZ1]. □

Remark 5.5. Lemmas 5.3 and 5.4 show that the wild kernels defined in this paper agree with the wild kernels defined in [Ba2] and [BGKZ] in the number field case .

Observe that $WK_n(F) \subset K_n(\mathcal{O}_F)$ for any global field F and $n \geq 0$ cf. [BGKZ]. In the number field case it was proved in [BGKZ] that $WK_n(F)$ is torsion. In the function field case $K_n(\mathcal{O}_F)$ is torsion for all $n > 1$ and it is clear that $WK_0(F) = WK_1(F) = 0$. Hence for any global field F and any $n \geq 0$ we get:

$$WK_n(F) \subset K_n(\mathcal{O}_F)_{tor} \tag{102}$$

Hence in particular if F is a number field then the group $K_{2n}^w(\mathcal{O}_F)_l$ has already been defined in [Ba2] and the group $WK_n(F)$ has been defined in [BGKZ].

Consider the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & WK_n(F)_l & \longrightarrow & K_n(F)_l & \longrightarrow & \prod_v K_n(F_v)_l & \longrightarrow & \\ & & \downarrow & & \downarrow = & & \downarrow & & \\ 0 & \longrightarrow & K_n^w(\mathcal{O}_F)_l & \longrightarrow & K_n(F)_l & \longrightarrow & \prod_v K_n^{et}(F_v)_l & \longrightarrow & \end{array}$$

From this diagram we notice that for any $n > 0$ and any $l \geq 2$ we have:

$$WK_n(F)_l \subset K_n^w(\mathcal{O}_F)_l \subset K_n(\mathcal{O}_F)_l. \tag{103}$$

Hence by [Ta2], Proposition 2.3 p. 261 we observe that

$$K_{2n}^{et}(F_v)_l \cong H^1(F_v, \mathbb{Q}_l/\mathbb{Z}_l(n+1))/Div \cong W^n(F_v), \tag{104}$$

$$K_{2n+1}^{et}(F_v)_l \cong H^0(F_v, \mathbb{Q}_l/\mathbb{Z}_l(n+1))/Div \cong W^{n+1}(F_v), \tag{105}$$

Hence the group $K_n^{et}(F_v)_l$ is finite for any v, any $n \geq 1$ and $l \geq 2$. This shows that

$$div\, K_n(F)_l \subset K_n^w(\mathcal{O}_F)_l \subset K_n(\mathcal{O}_F)_l. \tag{106}$$

Applying Quillen localization sequences for rings $\mathcal{O}_{F,S}$ and $\mathcal{O}_v$ it is also important to notice, that for any finite set $S \supset S_l$ there are the following exact sequences:

$$0 \to WK_n(F)_l \to K_n(\mathcal{O}_{F,S})_l \to \prod_{v \in S} K_n(\mathcal{O}_v)_l \tag{107}$$

$$0 \to K_n^w(\mathcal{O}_F)_l \to K_n(\mathcal{O}_{F,S})_l \to \prod_{v \in S} K_n^{et}(\mathcal{O}_v)_l \tag{108}$$

The following theorem gives the analog of the classical Moore exact sequence for higher K-groups of global fields.

Theorem 5.6. *For every $n \geq 1$ and every finite set $S \supset S_{\infty,l}$ there are the following exact sequences:*

$$0 \to K_{2n}^w(\mathcal{O}_F)_l \to K_{2n}(\mathcal{O}_{F,S})_l \to \bigoplus_{v \in S} W^n(F_v) \to W^n(F) \to 0. \tag{109}$$

$$0 \to K_{2n}^w(\mathcal{O}_F)_l \to K_{2n}(F)_l \to \bigoplus_{v} W^n(F_v) \to W^n(F) \to 0. \tag{110}$$

In particular:

$$\frac{|K_{2n}(\mathcal{O}_{F,S})_l|}{|K_{2n}^w(\mathcal{O}_F)_l|} = \Big|\frac{\prod_{v \in S} w_n(F_v)}{w_n(F)}\Big|_l^{-1}. \tag{111}$$

Proof. It results from Theorems 3.7 and 5.1. The equality (111) follows from (109) since all terms in this exact sequence are finite. □

Lemma 5.7. *For every $n \geq 1$ and every $l \geq 2$ there is the following exact sequence*

$$0 \to div\, K_n^{et}(F)_l \to K_n^{et}(F)_l \to \prod_v K_n^{et}(F_v)_l \tag{112}$$

Proof. Put $n = 2i - j$ for $j = 1, 2$. Hence by (14), by [Ja, Theorem 3.2] and by [Ta2, Proposition 2.3 p. 261], the exact sequence has the following form:

$$0 \to div\, K_{2i-j}(F)_l \to H^{j-1}(G_F, \mathbb{Q}_l/\mathbb{Z}_l(i))/Div \to \prod_v H^{j-1}(G_{F_v}, \mathbb{Q}_l/\mathbb{Z}_l(i))/Div. \tag{113}$$

Let $j = 1$. Then the map $H^0(G_F, \mathbb{Q}_l/\mathbb{Z}_l(i))/Div \to H^0(G_{F_v}, \mathbb{Q}_l/\mathbb{Z}_l(i))/Div$ is trivially injective for each v and $div\, K_{2i-1}(F) = 0$ so (113) is exact in this case. For $j = 2$ the exactness of (113) is the result of Theorem 3.5. □

Consider the following commutative diagrams:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_n^w(\mathcal{O}_F)_l & \longrightarrow & K_n(F)_l & \longrightarrow & \prod_v K_n^{et}(F_v)_l \\ & & \downarrow & & \downarrow & & \downarrow = \\ 0 & \longrightarrow & div\, K_n^{et}(F)_l & \longrightarrow & K_n^{et}(F)_l & \longrightarrow & \prod_v K_n^{et}(F_v)_l \end{array}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_n^w(\mathcal{O}_F)_l & \longrightarrow & K_n(\mathcal{O}_{F,S})_l & \longrightarrow & \prod_v K_n^{et}(F_v)_l \\ & & \downarrow & & \downarrow & & \downarrow = \\ 0 & \longrightarrow & div\, K_n^{et}(F)_l & \longrightarrow & K_n^{et}(\mathcal{O}_{F,S})_l & \longrightarrow & \prod_v K_n^{et}(F_v)_l \end{array}$$

The left vertical arrows in both diagrams are identical.

Theorem 5.8. *The left vertical arrows in the diagrams above are split surjective. The middle vertical arrows induce canonical isomorphisms for all $n > 1$:*

$$K_n(\mathcal{O}_{F,S})_l / K_n^w(\mathcal{O}_F)_l \xrightarrow{\cong} K_n^{et}(\mathcal{O}_{F,S})_l / div K_n^{et}(F)_l \tag{114}$$

$$K_n(F)_l / K_n^w(\mathcal{O}_F)_l \xrightarrow{\cong} K_n^{et}(F)_l / div K_n^{et}(F)_l \tag{115}$$

Proof. It follows since the middle vertical arrows in the diagrams above are split surjective by Theorem 5.1. □

5.2. **Divisible elements, wild kernels and Quillen-Lichtenbaum conjecture.** We keep working with global fields as stated in the introduction.

Conjecture 5.9. *(Quillen-Lichtenbaum) Let F be a global field and let $l \geq 2$. Assume that $\mu_4 \subset F$ if $l = 2$. Then for all $n > 1$ and $l \neq char F$ the natural map:*

$$K_n(\mathcal{O}_F) \otimes \mathbb{Z}_l \to K_n^{et}(\mathcal{O}_F[\frac{1}{l}]) \tag{116}$$

is an isomorphism.

It has been known for many years that the Quillen-Lichtenbaum conjecture can be reformulated in several ways. Theorem 5.10 below presents some of the ways to refermulate this conjecture.

Theorem 5.10. *The following conditions are equivalent:*

(1) $K_n(\mathcal{O}_F) \otimes \mathbb{Z}_l \xrightarrow{\cong} K_n^{et}(\mathcal{O}_F[\frac{1}{l}])$ *for all $n > 1$,*

(2) $K_n(F)_l \xrightarrow{\cong} K_n^{et}(F)_l$ *for all* $n > 1$,
(3) $K_n(\mathcal{O}_F, \mathbb{Z}/l^k) \xrightarrow{\cong} K_n^{et}(\mathcal{O}_F[\frac{1}{l}], \mathbb{Z}/l^k)$ *for all* $k > 0$ *and* $n > 1$.
(4) $K_n(F, \mathbb{Z}/l^k) \xrightarrow{\cong} K_n^{et}(F, \mathbb{Z}/l^k)$ *for all* $k > 0$ *and all* $n > 1$.
(5) $\varprojlim_k K_n(F, \mathbb{Z}/l^k) \xrightarrow{\cong} \varprojlim_k K_n^{et}(F, \mathbb{Z}/l^k)$ *for all* $n > 1$.
(6) $K_n^{cts}(F, \mathbb{Z}_l) \xrightarrow{\cong} K_n^{et}(F)$ *for all* $n > 1$.
(7) $K_n^w(\mathcal{O}_F)_l = div K_n(F)_l$ *for all* $n > 1$.

Proof. The equivalence of conditions (1), (2), (3) and (4) follows by finite generation of K-groups of $\mathcal{O}_F$ and by comparison of Bockstein and localization sequences for Quillen and étale K-theory. Clearly (4) implies (5). Consider the following commutative diagram cf. [BZ1]:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varprojlim_k{}^1 K_{n+1}(F, \mathbb{Z}/l^k) & \longrightarrow & K_n^{cts}(F, \mathbb{Z}_l) & \longrightarrow & \varprojlim_k K_n(F, \mathbb{Z}/l^k) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varprojlim_k{}^1 K_{n+1}^{et}(F, \mathbb{Z}/l^k) & \longrightarrow & K_n^{et}(F) & \longrightarrow & \varprojlim_k K_n^{et}(F, \mathbb{Z}/l^k) & \longrightarrow & 0 \end{array}$$

where $K_n^{cts}(F, \mathbb{Z}_l)$ is the continuous K-theory defined in [BZ1]. Hence (5) and Theorem 4.5 implies that the middle vertical arrow in this diagram is an isomorphism. By [BZ1] Theorem 1 this implies (2). Hence we proved that (5) implies (4). This diagram also shows that (5) and (6) are equivalent. By the diagram following the proof of Lemma 5.7 conditions (2) and (7) are equivalent. □

Base on the proof of Theorem 5.10 we easily prove the following theorem.

Theorem 5.11. *For every* $n > 1$ *the following conditions are equivalent:*

(1) $K_n(\mathcal{O}_F) \otimes \mathbb{Z}_l \xrightarrow{\cong} K_n^{et}(\mathcal{O}_F[\frac{1}{l}])$,
(2) $K_n(F)_l \xrightarrow{\cong} K_n^{et}(F)_l$,
(3) $\varprojlim_k K_n(F, \mathbb{Z}/l^k) \xrightarrow{\cong} \varprojlim_k K_n^{et}(F, \mathbb{Z}/l^k)$,
(4) $K_n^{cts}(F, \mathbb{Z}_l) \xrightarrow{\cong} K_n^{et}(F)$,
(5) $K_n^w(\mathcal{O}_F)_l = div K_n(F)_l$.

Proof. Exercise for the reader. □

It is well known that the important results on motivic cohomology due to S. Bloch, E. Friedlander, M. Levine, S. Lichtenbaum, F. Morel, M. Rost, A. Suslin, V. Voevodsky, C. Weibel and others (see e.g. [BL] cf. [RW, Appendix B], [L], [MV], [R], [V1], [V2], [VSF], [We4]) prove the Quillen-Lichtenbaum conjecture. Let us state this in the following theorem.

Theorem 5.12. *The Quillen-Lichtenbaum conjecture holds true for* F. *In particular the equivalent conditions in Theorems 5.10 and 5.11 hold true for* F.

Remark 5.13. The proof of Theorem 5.12 applies, as some of the key ingredients, the spectral sequence connecting the motivic cohomology and K-theory [BL] cf. [RW, Appendix B]:

$$E_2^{p,q} = H_{\mathcal{M}}^{p-q}(F, \mathbb{Z}/l^k(-q)) \Rightarrow K_{-p-q}(F, \mathbb{Z}/l^k) \tag{117}$$

and results of Voyevodsky ([V1, Theorem 7.9] for $l = 2$ and [V2, Theorem 6.16] for $l > 2$) giving:

(1) $H^j_{\mathcal{M}}(K, \mathbb{Z}/l^k(i)) \cong H^j_{et}(K, \mathbb{Z}/l^k(i))$ for all $j \leq i$,
(2) $H^j_{\mathcal{M}}(F, \mathbb{Z}/l^k(i)) = 0$ if $j > i$,

for any field K of characteristic not equal to l. Then one can connect these results with the Dwyer and Friedlander spectral sequence [DF, Proposition 5.2] to get the following isomorphism:

$$K_n(F, \mathbb{Z}/l^k) \cong K^{et}_n(F, \mathbb{Z}/l^k), \tag{118}$$

for all $l \geq 2$, which is equivalent with the Quillen-Lichtenbaum conjecture (see Theorem 5.10).

Remark 5.14. In 1992 M. Levine [L] observed that Bloch-Kato conjecture for fields implies the Quillen-Lichtenbaum conjecture. The Bloch-Kato conjecture is proved in [V1, Corollary 7.5] (for $l = 2$) and [V2, Theorem 6.16] (for all l). It is shown in [GL] that Bloch-Kato conjecture lead to the computation of the motivic cohomology in terms of étale cohomology, a property that also establishes the Quillen-Lichtenbaum conjecture as pointed out in Remark 5.13.

The Proposition 5.15 and its Corollary 5.16 are well known and follow from the Gabber rigidity, results of Suslin [Su1] and results of Dwyer and Friedlander [DF]. We include here short proofs.

Proposition 5.15. *Let l be prime to char k_v. There are natural isomorphisms:*

$$K_n(\mathcal{O}_v,\, \mathbb{Z}/l^k) \xrightarrow{\cong} K^{et}_n(\mathcal{O}_v,\, \mathbb{Z}/l^k) \tag{119}$$

$$K_n(\mathcal{O}_v)[l^k] \xrightarrow{\cong} K^{et}_n(\mathcal{O}_v)[l^k] \tag{120}$$

$$K_n(\mathcal{O}_v)/l^k \xrightarrow{\cong} K^{et}_n(\mathcal{O}_v)/l^k \tag{121}$$

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc} K_n(\mathcal{O}_v, \mathbb{Z}/l^k) & \xrightarrow{\cong} & K^{et}_n(\mathcal{O}_v, \mathbb{Z}/l^k) \\ \downarrow \cong & & \downarrow \cong \\ K_n(k_v, \mathbb{Z}/l^k) & \xrightarrow{\cong} & K^{et}_n(k_v, \mathbb{Z}/l^k) \end{array}$$

The left vertical arrow is an isomorphism by [Su1, Corollaries 2.5 and 3.9] . The bottom vertical arrow is an isomorphism by [DF, Corollary 8.6] . Hence the top horizontal arrow is a monomorphism. Moreover, the top horizontal arrow is an epimorphism by [DF] Theorem 8.5 and by comparison of K-theory and étale K-theory localization sequences with coefficients for the local ring $\mathcal{O}_v$. The maps (120), (121) are isomorphisms by comparison of Bockstein K-theory and étale K-theory sequences for $\mathcal{O}_v$ with corresponding Bockstein sequences for k_v cf. [BGKZ] Section 2. □

Corollary 5.16. *Let l be prime to char k_v. There are natural isomorphisms:*

$$K_n(F_v,\, \mathbb{Z}/l^k) \xrightarrow{\cong} K^{et}_n(F_v,\, \mathbb{Z}/l^k) \tag{122}$$

$$K_n(F_v)[l^k] \xrightarrow{\cong} K^{et}_n(F_v)[l^k] \tag{123}$$

$$K_n(F_v)/l^k \xrightarrow{\cong} K_n^{et}(F_v)/l^k \tag{124}$$

Proof. The isomorphism (122) follows by Proposition 5.15 and by comparison of K-theory and étale K-theory localization sequences with coefficients. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_n(\mathcal{O}_v)_l & \longrightarrow & K_n(F_v)_l & \longrightarrow & K_{n-1}(k_v)_l & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & K_n^{et}(\mathcal{O}_v)_l & \longrightarrow & K_n^{et}(F_v)_l & \longrightarrow & K_{n-1}^{et}(k_v)_l & \longrightarrow & 0 \end{array}$$

The bottom exact sequence is an appropriate étale cohomology exact sequence written in terms of étale K-theory. It follows by Proposition 5.15 that the map (123) is an isomorphism, hence by Bockstein sequence argument the map (124) is also an isomorphism. ☐

Remark 5.17. If $p = \text{char } k_v$ then it was proven in [HM] that:

$$K_n(F_v, \mathbb{Z}/p^k) \xrightarrow{\cong} K_n^{et}(F_v, \mathbb{Z}/p^k). \tag{125}$$

By Bockstein sequence argument the map

$$K_n(F_v)[p^k] \xrightarrow{\cong} K_n^{et}(F_v)[p^k] \tag{126}$$

is an epimorphism and the map

$$K_n(F_v)/p^k \xrightarrow{\cong} K_n^{et}(F_v)/p^k \tag{127}$$

is a monomorphism.

Consider the following commutative diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & WK_n(F)_l & \longrightarrow & K_n(F)_l & \longrightarrow & \prod_v K_n(F_v)_l \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & div\, K_n(F)_l & \longrightarrow & K_n^{et}(F)_l & \longrightarrow & \prod_v K_n^{et}(F_v)_l \end{array}$$

Theorem 5.18. *Assume that for every $v \in S_l$:*

$$K_n(F_v)_l \xrightarrow{\cong} K_n^{et}(F_v)_l. \tag{128}$$

Then for all $n \geq 1$ the left vertical arrow in the diagram above is split surjective. Moreover the following conditions are equivalent for all $n \geq 1$:

(1) $K_n(F)_l \xrightarrow{\cong} K_n^{et}(F)_l$,
(2) $WK_n(F)_l = divK_n(F)_l$

Proof. By theorem 5.1 the middle vertical arrow is split surjective. The right vertical arrow is an isomorphism by Corollary 5.16 and our assumption. This shows that the left vertical arrow is split surjective. Hence the left vertical arrow is an isomorphism if and only if the middle vertical arrow is an isomorphism. ☐

Under assumption (128) the Theorem 5.12 shows that the equivalent conditions in Theorem 5.18 hold true. Summing up results in this section concerning wild kernels and divisible elements, we state the following theorem.

Theorem 5.19. *Let $l \geq 2$. For every $n > 1$ and we have the following equality:*

$$K_n^w(\mathcal{O}_F)_l = div K_n(F)_l.$$

Assume that $K_n(F_v)_l \xrightarrow{\cong} K_n^{et}(F_v)_l$ for every $v \in S_l$ and every $n > 1$. Then for every $n \geq 0$:

$$WK_n(F)_l = div\, K_n(F)_l.$$

Remark 5.20. It is easy to observe that $WK_n(F) = div\, K_n(F) = 0$ for $0 \leq n \leq 1$.

6. Splitting obstructions to Quillen boundary map

Observe that the [Ba2, Diagram 2.5] and the corresponding diagram for étale K-theory and also [Ba2], Diagram 3.2 extend naturally to the global field case and $l \geq 2$. Hence by analogues arguments as the ones in loc. cit. we get for every $k \geq 1$, every $n \geq 1$ the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & K_{2n}(F)[l^k] & \longrightarrow & \oplus_v K_{2n-1}(k_v)[l^k] & \longrightarrow & D(n, l^k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \\
\dots & \longrightarrow & K_{2n}^{et}(F)[l^k] & \longrightarrow & \oplus_{v \not| l} K_{2n-1}^{et}(k_v)[l^k] & \longrightarrow & D^{et}(n, l^k) & \longrightarrow & 0
\end{array}$$

Actually the rows of this diagram have the following form:

$$0 \to K_{2n}(\mathcal{O}_F)[l^k] \to K_{2n}(F)[l^k] \to \oplus_v K_{2n-1}(k_v)[l^k] \to D(n, l^k) \to 0. \tag{129}$$

$$0 \to K_{2n}^{et}(\mathcal{O}_F)[l^k] \to K_{2n}^{et}(F)[l^k] \to \oplus_v K_{2n-1}^{et}(k_v)[l^k] \to D^{et}(n, l^k) \to 0. \tag{130}$$

Taking direct limit in (129) gives the l-part of the Quillen localization sequence

$$0 \to K_{2n}(\mathcal{O}_F)_l \to K_{2n}(F)_l \xrightarrow{\partial} \oplus_v K_{2n-1}(k_v)_l \to 0. \tag{131}$$

which also implies the property (65).

Recall the definition of the numbers $k(l)$ in section 4. Define

$$N_0 := \prod_{l \,\big|\, |K_{2n}(\mathcal{O}_F)|} l^{k(l)}. \tag{132}$$

The exact sequence (129) for every l shows that for every positive integer N such that $N_0 \,|\, N$ we have the following exact sequence:

$$0 \to K_{2n}(\mathcal{O}_F) \to K_{2n}(F)[N] \to \bigoplus_v K_{2n-1}(k_v)[N] \to D(n) \to 0. \tag{133}$$

The exact sequence (133) shows that the group $D(n)$ is an analog for higher K-groups of the class group $Cl(\mathcal{O}_F)$. Recall that the class group appears in the exact sequence:

$$0 \to K_1(\mathcal{O}_F) \to K_1(F) \to \bigoplus_v K_0(k_v) \to Cl(\mathcal{O}_F) \to 0. \tag{134}$$

Remark 6.1. To determine whether a map of two l-torsion abelian groups is split surjective I considered in [Ba2, p. 293 and p. 296] obstructions to the splitting via l^k truncations of this map. Working throughout [Ba2] with $k \gg 0$ I did not consider on p. 293 loc. cit. the cokernels of the l^k truncation of ∂ for $k < k(l)$. By (129) the cokernel of the l^k trancation of ∂ is $D(n, l^k)$ and in particular for $k \geq k(l)$ we have $D(n)_l = D(n, l^k)$. As a result in [Ba2, Corollary 1, p. 293] I have an incomplete statement. The Proposition 6.2 below completes the statement of [Ba2, Corollary 1, p. 293]. The proof of the Proposition 6.2 below is the same as the the proof of [Ba2, Corollary 1, p. 293] by considering l^k truncations for all $k > 0$ not just for $k \geq k(l)$. The gap in the statement of [Ba2] Corollary 1 p. 293 has been noticed by Luca Caputo in [Ca].

Proposition 6.2. *The following conditions are equivalent:*

(1) $D(n, l^k) = 0$ *for every* $0 < k \leq k(l)$,
(2) $K_{2n}(F)_l \cong K_{2n}(\mathcal{O}_F)_l \oplus \bigoplus_v K_{2n-1}(k_v)_l$

Proof. (2) implies $K_{2n}(F)[l^k] \cong K_{2n}(\mathcal{O}_F)[l^k] \oplus \bigoplus_v K_{2n-1}(k_v)[l^k]$ for every $k > 0$ hence $D(n, l^k) = 0$ for every $k > 0$ by (129).

Now assume (1). By definition of $k(l)$ we note that $D(n, l^k) = 0$ for every $0 < k \leq k(l)$ if and only if $D(n, l^k) = 0$ for every $k > 0$. Hence by (129) there is an exact sequence for every $k > 0$:

$$0 \to K_{2n}(\mathcal{O}_F)[l^k] \to K_{2n}(F)[l^k] \to \bigoplus_v K_{2n-1}(k_v)[l^k] \to 0. \tag{135}$$

The groups $K_{2n-1}(k_v)$ are finite cyclic. Hence for every v we can choose $k \geq 0$ that $K_{2n-1}(k_v)_l = K_{2n-1}(k_v)[l^k]$. Hence the exact sequence (135) allows us to construct a homomorphism $\Lambda_v : K_{2n-1}(k_v)_l \to K_{2n}(F)_l$ such that for every element $\xi_v \in K_{2n-1}(k_v)_l$ we get $\partial(\Lambda_v(\xi_v)) = (\dots, 1, \xi_v, 1, \dots) \in \bigoplus_v K_{2n-1}(k_v)_l$. Hence the map

$$\Lambda := \prod_v \Lambda_v,$$

$$\Lambda : \bigoplus_v K_{2n-1}(k_v)_l \to K_{2n}(F)_l$$

clearly splits ∂ in the Quillen localization sequence (131). □

Proposition 6.3. *The following conditions are equivalent:*

(1) $D^{et}(n, l^k) = 0$ *for every* $0 < k \leq k(l)$,
(2) $K_{2n}^{et}(F)_l \cong K_{2n}^{et}(\mathcal{O}_F[1/l])_l \oplus \bigoplus_v K_{2n-1}^{et}(k_v)_l$

Proof. The proof is precisely the same as the proof of Proposition 6.2 with use of the exact sequence (130). □

Theorem 6.4. *The following conditions are equivalent:*

(1) $D(n, l^k) = 0$ *for every* $0 < k \leq k(l)$,
(2) $D^{et}(n, l^k) = 0$ *for every* $0 < k \leq k(l)$,
(3) $K_{2n}(F)_l \cong K_{2n}(\mathcal{O}_F)_l \oplus \bigoplus_v K_{2n-1}(k_v)_l$,
(4) $K_{2n}^{et}(F)_l \cong K_{2n}^{et}(\mathcal{O}_F[1/l])_l \oplus \bigoplus_v K_{2n-1}^{et}(k_v)_l$.

Proof. It follows by Theorem 4.1, Propositions 6.2, 6.3 and the definition of $k(l)$. □

Observe that for any totally real number field F any odd $n > 0$ and any odd prime number l we have

$$|K_{2n}^{et}(\mathcal{O}_F[1/l])| = |w_{n+1}(F)\zeta_F(-n)|_l^{-1}.$$

The following corollary is a correction of [Ba2] Proposition 1 p. 293.

Corollary 6.5. *Let n be an odd positive integer and let l be an odd prime number. Let F be a totally real number field such that $\prod_{v|l} w_n(F_v) = 1$. The following conditions are equivalent:*

(1) *The following exact sequence splits*

$$0 \to K_{2n}(\mathcal{O}_F)_l \to K_{2n}(F)_l \xrightarrow{\partial} \oplus_v K_{2n-1}(k_v)_l \to 0.$$

(2)

$$|w_{n+1}(F)\zeta_F(-n)|_l^{-1} = 1$$

Proof. In our case $|D(n)_l| = |w_{n+1}(F)\zeta_F(-n)|_l^{-1}$, (see (77)). Moreover, by Theorem 4.1, for every $k > 0$ we have $D(n, l^k) \cong D^{et}(n, l^k)$ and $D^{et}(n, l^k)$ is a subquotient of $K_{2n}^{et}(\mathcal{O}_F[1/l])$. In addition, as we observed before, $D(n)_l \cong D(n, l^k)$ for $k \gg 0$. Hence in the assumptions of the corollary $D(n, l^k) = 0$ for all $k > 0$ iff $|w_{n+1}(F)\zeta_F(-n)|_l^{-1} = 1$. Hence the corollary follows by Corollary 6.2. □

Corollary 6.6. *Let n be an odd positive integer and let l be an odd prime number. The following conditions are equivalent:*

(1) *The following exact sequence splits*

$$0 \to K_{2n}(\mathbb{Z}) \to K_{2n}(\mathbb{Q}) \xrightarrow{\partial} \oplus_v K_{2n-1}(k_v) \to 0.$$

(2)

$$|w_{n+1}(\mathbb{Q})\zeta_{\mathbb{Q}}(-n)|_l^{-1} = 1$$

Proof. It follows from Corollary 6.5 since $|w_n(\mathbb{Q}_l)|_l^{-1} = 1$. □

Remark 6.7. Take $F = \mathbb{F}_p(x)$. Then $\mathcal{O}_F = \mathbb{F}_p[x]$. By the homotopy invariance [Q1] Corollary p. 122 we have $K_n(\mathbb{F}_p[x]) = K_n(\mathbb{F}_p)$. Hence the boundary map in the localization sequence gives the following isomorphism:

$$K_{2n}(\mathbb{F}_p(x)) \xrightarrow{\cong} \oplus_v K_{2n-1}(k_v)$$

In particular $D(n) = div K_{2n}(\mathbb{F}_p(x)) = 0$.

Let l^{k_0} be the exponent of the group $K_{2n}(\mathcal{O}_F)_l$.

Lemma 6.8. *For every $k \geq 1$ and every $k' \geq k + k_0$:*

(1) *the natural map* $D(n, l^k) \to D(n, l^{k'})$ *is trivial,*
(2) $\bigoplus_v K_{2n-1}(k_v)[l^k] \subset \partial(K_{2n}(F)[l^{k'}])$.

Proof. Statement (1) follows from the commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & D(n,l^k) & \longrightarrow & K_{2n}(\mathcal{O}_F)/l^k & \longrightarrow & K_{2n}(F)/l^k \\
 & & \downarrow & & \downarrow{\scriptstyle l^{k'-k}} & & \downarrow{\scriptstyle l^{k'-k}} \\
0 & \longrightarrow & D(n,l^{k'}) & \longrightarrow & K_{2n}(\mathcal{O}_F)/l^{k'} & \longrightarrow & K_{2n}(F)/l^{k'}
\end{array}$$

since the middle vertical map is trivial by definition of k_0.

Statement (2) follows from (1) and the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} K_{2n}(F)[l^k] & \xrightarrow{\partial} & \bigoplus_v K_{2n-1}(k_v)[l^k] & \longrightarrow & D(n,l^k) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow 0 & & \\ K_{2n}(F)[l^{k'}] & \xrightarrow{\partial} & \bigoplus_v K_{2n-1}(k_v)[l^{k'}] & \longrightarrow & D(n,l^{k'}) & \longrightarrow & 0 \end{array}$$

since the left and the middle vertical arrows are natural imbeddings. □

For any $k \geq k(l)$ let us define

$$\bigoplus_v^{(1)} K_{2n-1}(k_v)_l := \bigoplus_{l^k \,|\, q_v^n - 1} K_{2n-1}(k_v)_l$$

$$\bigoplus_v^{(2)} K_{2n-1}(k_v)_l := \bigoplus_{l^k \,\not|\, q_v^n - 1} K_{2n-1}(k_v)_l$$

Theorem 6.9. *Let F be a global field, $n \geq 1$ and l be any prime number. The following conditions are equivalent:*

(1) $D(n)_l = 0$,

(2) *the following surjective map splits*

$$\partial_1 \, : \, K_{2n}(F)_l \to \bigoplus_v^{(1)} K_{2n-1}(k_v)_l.$$

Proof. For each $k' \geq k$ consider the following exact sequence

$$K_{2n}(F)[l^{k'}] \xrightarrow{\partial} \bigoplus_v^{(1)} K_{2n-1}(k_v)[l^{k'}] \oplus \bigoplus_v^{(2)} K_{2n-1}(k_v)[l^{k'}] \to D(n)_l \to 0. \tag{136}$$

where $\partial = \partial_1 \oplus \partial_2$. Assume that $D(n)_l = 0$. Hence for each $k' \geq k$ the following map is surjective:

$$K_{2n}(F)[l^{k'}] \xrightarrow{\partial_1} \bigoplus_v^{(1)} K_{2n-1}(k_v)[l^{k'}]. \tag{137}$$

So for each v such that $l^k \,|\, q_v^n - 1$ we take $k' \geq k$ such that $l^{k'} || q_v^n - 1$ and we notice that there is a homomorphism

$$\Lambda_v \, : \, K_{2n-1}(k_v)_l \, \to \, K_{2n}(F)_l$$

such that

$$\partial_1 \circ \Lambda_v(\xi_v) \; = \; (\ldots, 1, \, \xi_v, \, 1, \ldots),$$

for any $\xi_v \in K_{2n-1}(k_v)_l$. It is clear that

$$\Lambda_1 \, : \, \bigoplus_v^{(1)} K_{2n-1}(k_v)_l \, \to \, K_{2n}(F)_l$$

$$\Lambda_1 \, := \, \prod_v^{(1)} \, \Lambda_v$$

splits ∂_1.

Assume now that ∂_1 is split surjective. Consider the exact sequence (136) for $k' = k + k_0$. For such k' by Lemma 6.8 we have

$$\partial_2(K_{2n}(F)[l^{k'}]) \subset \bigoplus_v^{(2)} K_{2n-1}(k_v)[l^{k'}] =$$

$$= \bigoplus_v^{(2)} K_{2n-1}(k_v)[l^k] \subset \bigoplus_v K_{2n-1}(k_v)[l^k] \subset \partial(K_{2n}(F)[l^{k'}]). \tag{138}$$

On the other hand ∂_1 is split surjective hence $\bigoplus_v^{(1)} K_{2n-1}(k_v)[l^{k'}] = \partial_1(K_{2n}(F)[l^{k'}])$. Since $\partial = \partial_1 \oplus \partial_2$ then by (138) we see that $\partial_1(K_{2n}(F)[l^{k'}]) \subset \partial(K_{2n}(F)[l^{k'}])$. Hence again by (138) the map ∂ in the exact sequence (136) is surjective. Hence $D(n)_l = 0$. □

Corollary 6.10. *Let F be a totally real number field. Let n be an odd positive integer and let l be an odd prime number. The following conditions are equivalent:*

(1) *The following surjective map splits*

$$\partial_1 \,:\, K_{2n}(F)_l \to \bigoplus_v^{(1)} K_{2n-1}(k_v)_l$$

(2)

$$|\frac{w_{n+1}(F)\zeta_F(-n)}{\prod_{v|l} w_n(F_v)}|_l^{-1} = 1$$

Proof. By [Ba2] Theorem 3 p. $|D(n)_l| = |\frac{w_{n+1}(F)\zeta_F(-n)}{\prod_{v|l} w_n(F_v)}|_l^{-1}$. Hence the corollary follows by Theorem 6.9. □

Corollary 6.11. *Let F be a global field of $\mathrm{char}\, F > 0$. Let $n > 1$ be an integer and let $l \neq \mathrm{char}\, F$. The following conditions are equivalent:*

(1) *The following surjective map splits*

$$\partial_1 \,:\, K_{2n}(F)_l \to \bigoplus_v^{(1)} K_{2n-1}(k_v)_l$$

(2)

$$|\frac{w_n(F)\, w_{n+1}(F)\, \zeta_F(-n)}{\prod_{v\,|\,\infty} w_n(F_v)}|_l^{-1} = 1.$$

Proof. Due to (81) the corollary follows by Theorem 6.9. □

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