THE THREE SMALLEST COMPACT ARITHMETIC HYPERBOLIC 5-ORBIFOLDS

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1. Introduction

Let $\mathrm{Isom}(\mathbf{H}^5)$ be the group of isometries of the hyperbolic space $\mathbf{H}^5$ of dimension five, and $\mathrm{Isom}^+(\mathbf{H}^5)$ its index two subgroup of orientation-preserving isometries. In [3] (see also [6]) the lattice of smallest covolume among cocompact arithmetic lattices of $\mathrm{Isom}^+(\mathbf{H}^5)$ was determined. This lattice was constructed as the image of an arithmetic subgroup Γ_0 of the spinor group $\mathrm{Spin}(1,5)$ (note that $\mathrm{Spin}(1,n)$ is a twofold covering of $\mathrm{SO}(1,n)^\circ \cong \mathrm{Isom}^+(\mathbf{H}^n)$). More precisely, Γ_0 is given by the normalizer in $\mathrm{Spin}(1,5)$ of a certain arithmetic group $\Lambda_0 \subset \mathrm{G}_0(k_0)$, where $k_0 = \mathbb{Q}(\sqrt{5})$ and G_0 is the algebraic k_0-group $\mathbf{Spin}(f_0)$ defined by the quadratic form

$$f_0(x) = -(3+2\sqrt{5})x_0^2 + x_1^2 + \cdots + x_5^2. \tag{1.1}$$

In [3] the index $[\Gamma_0 : \Lambda_0]$ was computed to be equal to 2. We note that it is easily checked that Λ_0 contains the center of $\mathrm{Spin}(1,5)$, so that the covolume of the action of Λ_0 on $\mathbf{H}^5$ is the double of the covolume of Γ_0.

In this article we construct a cocompact arithmetic lattice $\Gamma_2 \subset \mathrm{Spin}(1,5)$ of covolume slightly bigger than the covolume of Λ_0, and we prove that it realizes the third smallest covolume among cocompact arithmetic lattices in $\mathrm{Spin}(1,5)$. In other words, we obtain the second and third values in the volume spectrum of compact orientable arithmetic hyperbolic 5-orbifolds, thus improving the results of [3, 6] for this dimension. For notational reasons we put $\Gamma_1 = \Lambda_0$. Moreover, for $i = 0,1,2$, we denote by Γ_i' the image of Γ_i in $\mathrm{Isom}^+(\mathbf{H}^5)$.

Theorem 1. *The lattices Γ_0', Γ_1' and Γ_2' (ordered by increasing covolume) are the three cocompact arithmetic lattices in $\mathrm{Isom}^+(\mathbf{H}^5)$ of minimal covolume. They are unique, in the sense that any cocompact arithmetic lattice in $\mathrm{Isom}^+(\mathbf{H}^5)$ of covolume smaller than or equal to Γ_2' is conjugate in $\mathrm{Isom}(\mathbf{H}^5)$ to one of the Γ_i'.*

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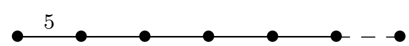
Lattice	Hyperbolic covolume
Γ_0'	$0.00153459236\ldots$
Γ_1'	$0.00306918472\ldots$
Γ_2'	$0.00396939286\ldots$

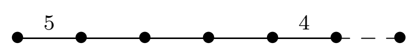
Coxeter group	Coxeter symbol	Hyperbolic covolume
Δ_0	$[5,3,3,3,3]$	$0.00076729618\ldots$
Δ_1	$[5,3,3,3,3^{1,1}]$	$0.00153459235\ldots$
Δ_2	$[5,3,3,3,4]$	$0.00198469643\ldots$

TABLE 1. Approximation of hyperbolic covolumes

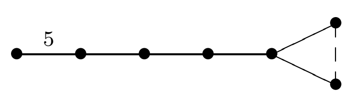
The precise formulas for the hyperbolic covolumes of these lattices are given below in Proposition 4. We list in Table 1 the corresponding numerical approximations.

A central motivation for Theorem 1 is that the lattices Γ_0', Γ_1' and Γ_2' can be related to concrete geometric objects. Namely, let P_0 and P_2 be the two compact Coxeter polytopes in $\mathbf{H}^5$ described by the following Coxeter diagrams, of respective Coxeter symbols $[5,3,3,3,3]$ and $[5,3,3,3,4]$ (see §4):

(1.2) P_0 : 

(1.3) P_2 : 

These two polytopes were first discovered by Makarov [10] (see also Im Hof [7]) (see §4). Combinatorially, they are simplicial prisms. Let $P_1 = DP_0$ be the geometric double of P_0 with respect to its Coxeter facet $[5,3,3,3]$. It follows that the Coxeter polytope P_1 can be characterized by the following Coxeter diagram, of symbol $[5,3,3,3,3^{1,1}]$:

(1.4) P_1 : 

We denote by $\Delta_i \subset \mathrm{Isom}(\mathbf{H}^5)$ the Coxeter group generated by the reflections through the hyperplanes delimiting P_i $(0 \le i \le 2)$. It is known, by Vinberg's criterion [14], that the lattices Δ_0 (thus Δ_1 as well) and Δ_2 are arithmetic.

Theorem 2. *For $i = 0,1,2$, let Δ_i^+ be the lattice $\Delta_i \cap \mathrm{Isom}^+(\mathbf{H}^5)$, which is of index two in Δ_i. Then Δ_i^+ is conjugate to Γ_i' in $\mathrm{Isom}(\mathbf{H}^5)$. In particular, Δ_0 realizes the smallest covolume among the cocompact arithmetic lattices in $\mathrm{Isom}(\mathbf{H}^5)$.*

The proof of Theorem 2 is obtained as a consequence of Theorem 1 (more exactly from the slightly more precise Proposition 6) together with an geometric/analytic computation of the volumes $\mathrm{vol}(P_0)$ and $\mathrm{vol}(P_2)$ that will be presented in §4. We note that the fact that Δ_2 and Γ_2' are commensurable lattices follows from the work of Bugaenko [4] where Δ_2 is constructed by applying Vinberg's algorithm on the same quadratic form (2.1) which we will use below to construct Γ_2. No arithmetic construction of Δ_0 and Δ_1 was known so far.

The approximations of the volumes of P_0, P_1 and P_2 are listed in Table 1. These volumes can be obtained by two completely different approaches: from the method given in §4, or from the covolumes of the arithmetic lattices Γ_i, which are essentially computed with Prasad's volume formula [13]. The comparison of these two approaches has some arithmetic significance that will be briefly discussed in §5.

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2. Construction and properties of Γ_2

We call an algebraic group *admissible* if it gives rise to cocompact lattices in $\mathrm{Spin}(1,5)$; see [3, §2.2] for the exact definition. We say that an admissible k-group G is *associated with* k/ℓ, where ℓ is the smallest field extension of k (necessarily quadratic) such that G is an inner form over ℓ, sometimes called "splitting field" of G. We use the same terminology for the arithmetic subgroups of G. Admissibility imposes that G is of type ${}^2\mathrm{A}_3$, the field k is totally real, and ℓ has signature $(2, d-1)$ where $d = [k:\mathbb{Q}]$ (cf. [3, Prop. 2.5]). Note that since we consider only cocompact lattices in this article, we assume that $k \neq \mathbb{Q}$. In the following, the symbol V_f will always refer to the set of finite places of the base field k (and not of ℓ).

Let G_2 be the algebraic spinor group $\mathbf{Spin}(f_2)$ defined over $k_0 = \mathbb{Q}(\sqrt{5})$, where f_2 is the following quadratic form:

$$f_2(x) = -\omega x_0^2 + x_1^2 + \cdots + x_5^2, \tag{2.1}$$

with $\omega = \frac{1+\sqrt{5}}{2}$. We have $\mathrm{G}_2(\mathbb{R}) \cong \mathrm{Spin}(1,5) \times \mathrm{Spin}(6)$, proving that G_2 is admissible. Its "splitting field" is given by (cf. [3, §3.2]):

$$\begin{aligned} \ell_2 &= \mathbb{Q}(\sqrt{\omega}) \\ &\cong \mathbb{Q}[x]/(x^4 - x^2 - 1), \end{aligned} \tag{2.2}$$

which has a discriminant of absolute value $\mathscr{D}_{\ell_2} = 400$. The following proposition shows an analogy between G_2 and G_0 (cf. [3, Prop. 3.6]).

Proposition 3. *The group* G_2 *is quasisplit at every finite place* v *of* k_0*. It is the unique admissible group associated with* k_0/ℓ_2 *with this property.*

Proof. Since ω is an integer unit in k_0 it easily follows that for at each nondyadic place $v \neq (2)$ the form f_2 has the same Hasse symbol as the standard split form of signature $(3,3)$. From the structure theory of **Spin** described in [3, §3.2] we conclude that G_2 must be quasisplit at every finite place v (note that at the place $v = (2)$, which is ramified in ℓ_2/k_0, the group G_2 is necessarily an outer form). Similarly to the proof of [3, Prop. 3.6], the second affirmation follows from [3, Lemma 3.4] together with the Hasse-Minkowski theorem. □

We write here $k = k_0$. By Proposition 3 we see that for every finite place $v \in V_\mathrm{f}$ there exists a special parahoric subgroup $P_v \subset \mathrm{G}_2(k_v)$. More precisely, P_v is hyperspecial unless v is the dyadic place (2) (the particularity of $v = (2)$ comes from the fact that this place is ramified in the extension ℓ_2/k_0). The collection $(P_v)_{v\in V_\mathrm{f}}$ of special parahoric subgroups can be chosen to be coherent, i.e., such that $\prod_v P_v$ is open in the group $\mathrm{G}_2(\mathbb{A}_\mathrm{f})$ of finite adelic points. We now consider the principal arithmetic subgroup associated with such a coherent collection:

$$\Lambda_2 = \mathrm{G}_2(k_0) \cap \prod_{v\in V_\mathrm{f}} P_v\,. \tag{2.3}$$

The covolume of Λ_2 can be computed with Prasad's volume formula [13]. If μ denotes the Haar measure on $\mathrm{Spin}(1,5)$ normalized as in [3] (which corresponds to the measure μ_S in [13]), then we obtain:

$$\mu(\Lambda_2 \backslash \mathrm{Spin}(1,5)) = \mathscr{D}_{k_0}^{15/2} \mathscr{D}_{\ell_2}^{5/2} C^2 \zeta_{k_0}(2)\zeta_{k_0}(4) L_{\ell_2/k_0}(3)\,, \tag{2.4}$$

where $C = 3 \cdot 2^{-7}\pi^{-9}$, the symbol ζ_k denotes the Dedekind zeta function associated with k, and $L_{\ell/k} = \zeta_\ell/\zeta_k$ is the L-function corresponding to a quadratic extension ℓ/k.

We can now construct the group Γ_2 and compute its hyperbolic covolume. In the same proposition we recall the value of the hyperbolic covolume of Γ_0, which was obtained in [3].

Proposition 4. *Let Γ_2 be the normalizer of Λ_2 in $\mathrm{Spin}(1,5)$. Then Λ_2 has index two in Γ_2. It follows that the hyperbolic covolume of Γ'_2 is equal to*

$$\frac{9\sqrt{5}^{15}}{2^3\pi^{15}} \zeta_{k_0}(2)\zeta_{k_0}(4) L_{\ell_2/k_0}(3) = 0.00396939286\ldots\,. \tag{2.5}$$

The hyperbolic covolume of Γ'_0 is equal to

$$\frac{9\sqrt{5}^{15}\sqrt{11}^{5}}{2^{14}\pi^{15}} \zeta_{k_0}(2)\zeta_{k_0}(4) L_{\ell_0/k_0}(3) = 0.00153459236\ldots\,, \tag{2.6}$$

where ℓ_0 is the quartic field with $x^4 - x^3 + 3x - 1$ as defining polynomial.

Proof. The relation between the measure μ and the hyperbolic volume is described in [3, §2.1], where it is proved that in dimension 5 the hyperbolic covolume corresponds to the covolume with respect to $2\pi^3 \times \mu$. Thus it remains to prove that $[\Gamma_2 : \Lambda_2] = 2$. Let $k = k_0$.

It follows from the theory developed in [3, §4] that the index $[\Gamma_2 : \Lambda_2]$ is equal to the order of the group denoted by A_ξ in *loc. cit.*, which can be identified as a subgroup of index at most two in $\mathbf{A}_4/(\ell_2^\times)^4$, where

$$\mathbf{A} = \left\{ x \in \ell_2^\times \;\middle|\; N_{\ell_2/k}(x) \in (k^\times)^4 \text{ and } x > 0 \right\} ; \tag{2.7}$$

$$\mathbf{A}_4 = \{ x \in \mathbf{A} \mid \nu(x) \in 4\mathbb{Z} \text{ for each normalized valuation } \nu \text{ of } \ell_2 \}. \tag{2.8}$$

Note that in particular, for the integers q and q' introduced in [3, §4.9], we have $q = \overline{q} = 1$. Moreover, if $v = (2)$ denotes the (unique) ramified place of ℓ_2/k, the subgroup A_ξ is proper of index two in $\mathbf{A}_4/(\ell_2^\times)^4$ if and only if there exists an element of $\mathbf{A}_4$ acting nontrivially on the local Dynkin diagram Δ_v of $\mathrm{G}_2(k_v)$. The action of $\mathbf{A}$ on Δ_v comes from its identification as a subgroup of the first Galois cohomology group $H^1(k, \mathrm{C})$ (where C is the center of G_2), which acts on every local Dynkin diagram associated with G_2. Since G_2 is of type A, we can use the results of [12, §4.2], which show that if $\pi_w \in \ell_2$ is a uniformizer for the ramified place $w|v$ of ℓ_2, then $s = \pi_w \overline{\pi_w}^{-1}$ is a generator of the group $\mathrm{Aut}(\Delta_v)$. Taking $\pi_w = 1 + \omega + \sqrt{\omega}$, we obtain a positive unit s acting nontrivially on Δ_v. Thus, A_ξ has index two in $\mathbf{A}_4/(\ell_2^\times)^4$. But the order of this latter group was computed in [3, §7.5] to be equal to 4. This gives $[\Gamma_2 : \Lambda_2] = 2$. □

The "uniqueness" part of Theorem 1 requires the following result.

Proposition 5. *Up to conjugacy, the image of* Γ_2 *in* $\mathrm{Isom}(\mathbf{H}^5)$ *does not depend on the choice of a coherent collection of special parahoric subgroups* $P_v \subset \mathrm{G}_2(k_v)$.

Proof. To prove this we can follow the same line of arguments as in [3, §6], where the result is proved for $\Gamma_0 \subset \mathrm{G}_0$ (our situation corresponding to the case of the type ${}^2\mathrm{D}_{2m+1}$). Thus, using [3, §6.5], the result follows by checking that $\mathbf{L}/\mathbf{A}$ and $U_\mathbf{L}/U_\mathbf{A}$ have the same order (equal to 2), where

$$\mathbf{L} = \left\{ x \in \ell_2^\times \;\middle|\; N_{\ell_2/k_0}(x) \in (k_0^\times)^4 \right\} \tag{2.9}$$

and $U_\mathbf{L}$ (resp. $U_\mathbf{A}$) is the intersection of $\mathbf{L}$ (resp. $\mathbf{A}$) with the integers units in ℓ_2. □

3. Proof of Theorem 1

In view of Proposition 4, Theorem 1 is a direct consequence of the following statement.

Proposition 6. *Let* $\Gamma' \subset \mathrm{Isom}^+(\mathbf{H}^5)$ *be a cocompact arithmetic lattice that is not conjugate to* Γ_0', Γ_1' *or* Γ_2'. *Then* $\mathrm{vol}(\Gamma' \backslash \mathbf{H}^5) > 4 \cdot 10^{-3}$.

Proof. Let $\Gamma \subset \mathrm{Spin}(1,5)$ be the full inverse image of Γ'. We suppose that Γ is an arithmetic subgroup of the group G defined, associated with ℓ/k. From the values given in (2.5) and (2.6), it is clear that if Γ is a proper subgroup of Γ_0, Γ_1 or Γ_2, then $\mathrm{vol}(\Gamma' \backslash \mathbf{H}^5) > 4 \cdot 10^{-3}$. Thus it suffices to prove the result assuming that Γ is a maximal arithmetic subgroup with respect to inclusion. In particular, Γ can be written as the normalizer of the principal arithmetic subgroup Λ associated with some coherent collection $P = (P_v)$ of parahoric subgroups $P_v \subset \mathrm{G}(k_v)$.

First we suppose that $k = k_0$, and $\ell = \ell_0$ or ℓ_2. By Proposition 3 and its analogue for G_0, if G is not isomorphic to G_0 or G_2 then at least one P_v is not special. In particular, a "lambda factor" $\lambda_v \geq 18$ appears in the volume formula of Λ [3, §7.1]. Together with [3, (15)] (note that we do not assume here that $\Gamma = \Gamma^{\mathfrak{m}}$, in the notation of *loc. cit.*) this shows that the covolume of Γ is at least 9 times the covolume of Γ_0. Now if G is isomorphic to G_0 or G_2, Proposition 5 and its analogue for G_0 show that at least one P_v is not special, and the same argument as above applies.

Now we consider the situation $(k,\ell) \neq (k_0,\ell_0)$ nor (k_0,ℓ_2). We will use the different lower bounds for the covolume of Γ given in [3, §7]. Note that in our case the rank r of G is equal to 3. The notations are the following: d is the degree of k, $\mathscr{D}_k$ and $\mathscr{D}_\ell$ are the discriminants of k and ℓ in absolute value, and h_ℓ is the class number of ℓ. Moreover, we set $a = 3^3 2^{-4}\pi^{-11}$. From [3, (37)] we have for $d \geq 7$ the following lower bound, which proves the result in this case (recall that the hyperbolic volume corresponds to $2\pi^3 \times \mu$, where μ is the Haar measure used by Prasad).

$$\begin{aligned} \mathrm{vol}(\Gamma' \backslash \mathbf{H}^5) &> \frac{2\pi^3}{32}\left(9.3^{5.5}\cdot a\right)^7 \\ &= 7.657\ldots \end{aligned} \tag{3.1}$$

The following bound corresponds to [3, (35)].

$$\mathrm{vol}(\Gamma' \backslash \mathbf{H}^5) > \frac{2\pi^3}{32}\mathscr{D}_k^{5.5} a^d \tag{3.2}$$

For each degree $d = 2,\ldots,6$ we can use (3.2) to prove the result for a discriminant $\mathscr{D}_k$ high enough (e.g., $\mathscr{D}_k \geq 27$ for $d = 2$). This leave us with a finite number of possible fields k to examine. From these bounds on $\mathscr{D}_k$ and the tables of number fields (such as [1] and [2]) we obtain a list of nineteen fields k (none of degree $d = 6$) that remain to check.

Let us further consider the two following bounds, corresponding to [3, (34) and (31)]. See (2.4) for the value of the symbol C.

$$\mathrm{vol}(\Gamma' \backslash \mathbf{H}^5) > \frac{2\pi^3}{32}\mathscr{D}_k^{2.5}\mathscr{D}_\ell^{1.5} a^d\ ; \tag{3.3}$$

$$\mathrm{vol}(\Gamma' \backslash \mathbf{H}^5) > \frac{2\pi^3}{h_\ell 2^{d+1}}\mathscr{D}_k^{7.5}(\mathscr{D}_\ell/\mathscr{D}_k^2)^{2.5} C^d\ . \tag{3.4}$$

For each of the nineteen fields k we easily obtain an upper bound b_k for $\mathscr{D}_\ell$ for which the right hand side of (3.3) is at most $4\cdot 10^{-3}$. Thus we only need to analyse the fields ℓ with $\mathscr{D}_\ell \leq b_k$. Let us fix a field k. The computational method described [5], based on class field theory, allows to determine all the quadratic extensions ℓ/k with $\mathscr{D}_\ell \leq b_k$ and with ℓ of right signature, that is, $(2, d-1)$ (cf. [3, Prop. 2.5]). More precisely, we obtain this list of ℓ/k

by programming a procedure in Pari/GP that uses the built-in functions `bnrinit` and `rnfkummer`. For each pair (k, ℓ) obtained, PARI/GP gives us the class number h_ℓ (checking its correctness with `bnfcertify`) and this information makes (3.4) usable. The inequality $\mathrm{vol}(\Gamma' \backslash \mathbf{H}^5) > 4 \cdot 10^{-3}$ follows then for all the remaining (k, ℓ) except for the two situations:

$$(\mathscr{D}_k, \mathscr{D}_\ell) = (8, 448)\,, \tag{3.5}$$

$$(\mathscr{D}_k, \mathscr{D}_\ell) = (5, 475)\,. \tag{3.6}$$

The case (3.6) follows from Proposition 7 below. Let us then consider the case associated with (3.5). The smallest possible covolume of a maximal arithmetic subgroup $\Gamma = N_{\mathrm{Spin}(1,5)}(\Lambda)$ associated with ℓ/k would be in the situation when all parahoric subgroup P_v determining Λ are special. In this case, by [3, Prop. 4.12] the index $[\Gamma : \Lambda]$ is bounded by 8, and together with the precise covolume of Λ by Prasad's formula, we obtain (using PARI/GP to evaluate the zeta functions):

$$\begin{aligned} \mathrm{vol}(\Gamma' \backslash \mathbf{H}^5) &\geq \frac{2\pi^3}{8} \mathscr{D}_k^{7.5} (\mathscr{D}_\ell / \mathscr{D}_k^2)^{2.5} C^2 \zeta_k(2) \zeta_k(4) \zeta_\ell(3) / \zeta_k(3) \\ &= 0.004997\ldots \end{aligned} \tag{3.7}$$

This conclues the proof. □

Proposition 7. *Let ℓ be the quadratic extension of $k_0 = \mathbb{Q}(\sqrt{5})$ with discriminant of absolute value $\mathscr{D}_\ell = 475$. There exists a cocompact arithmetic lattice in* $\mathrm{Isom}^+(\mathbf{H}^5)$ *associated with ℓ/k_0 whose approximate hyperbolic covolume is* $0.006094\ldots$*. This is the smallest covolume among arithmetic lattices in* $\mathrm{Isom}^+(\mathbf{H}^5)$ *associated with ℓ/k_0.*

Proof. Let $k = k_0$. The field ℓ can be concretely described as $\ell = k(\sqrt{\beta})$, where $\beta = -1 + 2\sqrt{5}$ (this is a divisor of 19). We consider the algebraic group $\mathrm{G} = \mathbf{Spin}(f)$ defined over $k = k_0$, with

$$f(x) = -\beta x_0^2 + x_1^2 + \cdots + x_5^2\,. \tag{3.8}$$

Similarly to [3, Prop. 3.6], we have that G is quasisplit at every finite place $v \in V_\mathrm{f}$ (note that the proof for the unique dyadic place can be simplified in *loc. cit.* by noting 2 is inert in ℓ and thus, G must be an outer form, necessarily quasisplit, cf. [3, §3.2]). It follows that there exist a coherent collection of special parahoric subgroups $P_v \subset \mathrm{G}(k_v)$, and by Prasad's volume formula the hyperbolic covolume of an associated principal arithmetic subgroup Λ is given by

$$\mathrm{vol}(\Lambda \backslash \mathbf{H}^5) = 2\pi^3 \mathscr{D}_k^{7.5} (\mathscr{D}_\ell / \mathscr{D}_k^2)^{2.5} C^2 \zeta_k(2) \zeta_k(4) \zeta_\ell(3) / \zeta_k(3) \tag{3.9}$$

The index $[\Gamma : \Lambda]$ of Λ in its normalizer Γ can be computed using the same method as in the proof of Proposition 4. That the group $\mathbf{A}_4 / (\ell^\times)^4$ has order 4 was already computed in [3, §7.5]. We use again [12, §4.2] to analyse the behaviour at the ramified place $v = (\beta)$: for the uniformizer $\pi_w = \frac{\sqrt{\beta} + \beta}{2}$ of the place $w | v$ we get that $s = \pi_w \overline{\pi_w}^{-1}$ is an element of $\mathbf{A}_4$ that acts nontrivially on the local Dynkin diagram Δ_v of $\mathrm{G}(k_v)$. As in the proof of Proposition 4 it follows that $[\Gamma : \Lambda] = 2$. From (3.9) we obtain the value $0.006094\ldots$ as the hyperbolic covolume of Γ. That no other arithmetic group associated with ℓ/k has smaller covolume follows from [3, §4.3] (since Λ is of the form Λ^{m}; cf. [6, §12.3] for more details).

□

4. Proof of Theorem 2

Consider the vector space model $\mathbb{R}^{1,5}$ for $\mathbf{H}^5$ as above and represent a hyperbolic hyperplane $H = e^{\perp}$ by means of a space-like unit vector $e \in \mathbb{R}^{1,5}$. A hyperbolic Coxeter polytope $P = \cap_{i\in I} H_i^-$ is the intersection of finitely many half-spaces (whose normal unit vectors are directed outwards w.r.t. P and) whose dihedral angles are submultiples of π. The group Δ generated by the reflections with respect to the hyperplanes $H_i, i \in I$, is a discrete subgroup of $\mathrm{Isom}(\mathbf{H}^5)$. If the cardinality of I is small, a Coxeter polytope and its reflection group are best represented by the *Coxeter diagram* or by the *Coxeter symbol*. To each limiting hyperplane H_i of a Coxeter polytope P corresponds a node i in the Coxeter diagram, and two nodes i, j are connected by an edge of weight p if the hyperplanes intersect under the (non-right) angle π/p. Notice that the weight 3 will always be omitted. If two hyperplanes are orthogonal, their nodes are not connected. If they admit a common perpendicular (of length l), their nodes are joined by a dashed edge (and the weight l is usually omitted). We extend the diagram description to arbitrary convex hyperbolic polytopes and associate with the dihedral angle $\alpha = \angle(H_i, H_j)$ an edge with weight α connecting the nodes i, j. For the intermediate case of *quasi-Coxeter polytopes* whose dihedral angles are rational multiples $p\pi/q$ of π, the edge weight will be q/p. The Coxeter symbol is a bracketed expression encoding the form of the Coxeter diagram in an abbreviated way. For example, $[p, q, r]$ is associated with a linear Coxeter diagram with 3 edges of consecutive markings p, q, r. The Coxeter symbol $[3^{i,j,k}]$ denotes a group with Y-shaped Coxeter diagram with strings of i, j and k edges emanating from a common node. However, dashed edges are omitted leaving a connected graph. The Coxeter symbol can be extended to the quasi-Coxeter case in an obvious way as well.

We are particularly interested in the quasi-Coxeter groups Δ_i and the polytopes P_i (see §1) as given in Table 2. In order to compute the volumes of P_i, we consider the 1-parameter sequence of compact 5-prisms with symbol

$$P(\alpha): \quad [5, 3, 3, 3, \alpha] \tag{4.1}$$

where $\alpha \in [\pi/4, 2\pi/5]$. Geometrically, they are compactifications of 5-dimensional orthoschemes by cutting away the ultra-ideal principal vertices by the associated polar hyperplanes. The sequence (4.1) contains the Coxeter polytopes $P_0 = [5, 3, 3, 3, 3]$ and $P_2 = [5, 3, 3, 3, 4]$ as well as the pseudo-Coxeter prism $[5, 3, 3, 3, 5/2]$. There is no closed volume formula for such polytopes known in terms of the dihedral angles. However, for certain non-compact

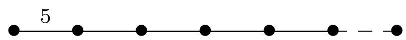
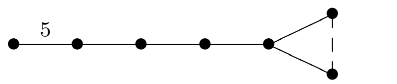
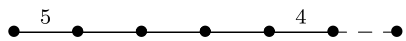
Group	Coxeter diagram	Coxeter symbol	Polytope
Δ_0		$[5,3,3,3,3]$	P_0
Δ_1		$[5,3,3,3,3^{1,1}]$	$P_1 = DP_0$
Δ_2		$[5,3,3,3,4]$	P_2

TABLE 2. Three hyperbolic Coxeter groups and their 5-polytopes

limiting cases and by means of scissors congruence techniques, exact volume expressions could be derived [8, §4.2]. For example,

$$\mathrm{vol}_5([5/2,3,3,5,5/2]) = \frac{13\zeta(3)}{9600} + \frac{11}{1152}\text{Л}_3(\frac{\pi}{5})\,, \tag{4.2}$$

$$\mathrm{vol}_5([5,3,3,5/2,5]) = -\frac{\zeta(3)}{4800} + \frac{11}{1152}\text{Л}_3(\frac{\pi}{5})\,, \tag{4.3}$$

and finally,

(4.4)

$$\mathrm{vol}_5(P(2\pi/5)) = \frac{1}{5}\left(\mathrm{vol}_5([5/2,3,3,5,5/2]) - \mathrm{vol}_5([5,3,3,5/2,5])\right) = \frac{\zeta(3)}{3200}\,.$$

Here,

$$\text{Л}_3(\omega) = \frac{1}{4}\sum_{r=1}^{\infty}\frac{\cos(2r\omega)}{r^3} = \frac{1}{4}\zeta(3) - \int_0^{\omega}\text{Л}_2(t)dt\,,\ \omega\in\mathbb{R}\,, \tag{4.5}$$

denotes the Lobachevsky trilogarithm function which is related to the real part of the classical polylogarithm $\mathrm{Li}_k(z) = \sum_{r=1}^{\infty} z^r/r^k$ for $k=3$ and $z = \exp(2i\omega)$ (see [8, §4.1] and (4.10)).

For the volume calculation of the prisms P_0 and P_2, we apply the volume differential formula of L. Schläfli (see [11], for example) with the reference value (4.4) in order to obtain the simple integral expression

$$\mathrm{vol}_5(P(\alpha)) = \frac{1}{4}\int_{\alpha}^{2\pi/5}\mathrm{vol}_3([5,3,\beta(t)])\,dt + \frac{\zeta(3)}{3200} \tag{4.6}$$

with a compact tetrahedron $[5,3,\beta(t)]$ whose angle parameter $\beta(t)\in]0,\pi/2[$ is given by

$$\beta(t) = \arctan\sqrt{2-\cot^2 t}\,. \tag{4.7}$$

Put

$$\theta(t) = \arctan\frac{\sqrt{1-4\sin^2\frac{\pi}{5}\sin^2\beta(t)}}{2\cos\frac{\pi}{5}\cos\beta(t)} \in]0,\frac{\pi}{2}[\,. \tag{4.8}$$

Then, the volume of the 3-dimensional orthoscheme face $[5,3,\beta(t)]$ as given by Lobachevsky's formula (see [8], (67), for example) equals

(4.9)
$$\begin{aligned}\mathrm{vol}_3([5,3,\beta(t)]) = \frac{1}{4}\{ & Л_2(\frac{\pi}{5}+\theta(t)) - Л_2(\frac{\pi}{5}-\theta(t)) - Л_2\left(\frac{\pi}{6}+\theta(t)\right) + \\ & + Л_2\left(\frac{\pi}{6}-\theta(t)\right) + Л_2(\beta(t)+\theta(t)) - Л_2(\beta(t)-\theta(t)) + 2Л_2\left(\frac{\pi}{2}-\theta(t)\right)\},\end{aligned}$$

where

$$Л_2(\omega) = \frac{1}{2}\sum_{r=1}^{\infty}\frac{\sin(2r\omega)}{r^2} = -\int_0^{\omega}\log|\,2\sin t\,|\,dt\,,\ \omega\in\mathbb{R}\,, \tag{4.10}$$

is Lobachevsky's function (in a slightly modified way).

The numerical approximation of the volumes of P_0 and P_2 can now be performed by implementing the data (4.7), (4.8) and (4.9) into the expression (4.6). We obtain, using the functions `intnum` and `polylog` in Pari/GP, that the three volumes of P_0, P_1 and P_2 (in increasing order) are clearly less than $2\cdot 10^{-3}$. Since the groups Δ_i $(i=0,1,2)$ are known to be arithmetic, it follows then from Proposition 6 that their subgroups of index two Δ_i^+ must coincide with the Γ_i'. This concludes the proof of Theorem 2.

5. Remarks on the identification of volumes

Although in the proof of Theorem 2 it suffices to use the rough estimate $\mathrm{vol}(P_2) < 2\cdot 10^{-3}$, the numerical approximations are much more precise. Namely, the equality $\mathrm{vol}(\Gamma_i'\backslash\mathbf{H}^5) = \mathrm{vol}(\Delta_i^+\backslash\mathbf{H}^5)$, proved by Theorem 2, yields for $i=0,2$:

$$\begin{aligned}\frac{9\sqrt{5}^{15}\sqrt{11}^{5}}{2^{14}\pi^{15}}\zeta_{k_0}(2)\zeta_{k_0}(4)L_{\ell_0/k_0}(3) &= 2\,\mathrm{vol}_5(P(\pi/3));\\ \frac{9\sqrt{5}^{15}}{2^{3}\pi^{15}}\zeta_{k_0}(2)\zeta_{k_0}(4)L_{\ell_2/k_0}(3) &= 2\,\mathrm{vol}_5(P(\pi/5)).\end{aligned} \tag{5.1}$$

Using Pari/GP, a computer checks within seconds that both sides of each equation coincide up to 50 digits (the right hand side being computed from (4.6) like in last step of §4).

The equalities (5.1) have also some arithmetic interest, due the presence on the left hand side of the special value $L_{\ell/k_0}(3)$ (with $\ell=\ell_0$ or ℓ_2). Since k_0 is totally real, it follows from the Klingen-Siegel theorem (see [9]; cf. also [12, App. C]) that $\zeta_{k_0}(2)\zeta_{k_0}(4)$ is up to a rational given by some power of π divided by $\sqrt{\mathscr{D}_{k_0}} = \sqrt{5}$. Thus, from (5.1) the nontrivial part $L_{\ell/k_0}(3)$ of $\mathrm{vol}(\Gamma_i'\backslash\mathbf{H}^5)$ can be expressed by a sum of integrals of Lobachevsky's functions. A related but much more significant idea is the possibility, predicted by Zagier's conjecture, to express $L_{\ell/k_0}(3)$ as a sum of trilogarithms evaluated at integers of k_0. We refer to [15] for more information on this subject.

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