K-theory of locally finite graph C^*-algebras

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Abstract

We calculate the K-theory of the Cuntz-Krieger algebra $\mathcal{O}_E$ associated with an infinite, locally finite graph, via the Bass-Hashimoto operator. The formulae we get express the Grothendieck group and the Whitehead group in purely graph theoretic terms.

We consider the category of finite (black-and-white, bi-directed) subgraphs with certain graph homomorphisms and construct a continuous functor to abelian groups. In this category K_0 is an inductive limit of K-groups of finite graphs, which were calculated in [4].

In the case of an infinite graph with the finite Betti number we obtain the formula for the Grothendieck group $K_0(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)+\gamma(E)}$, where $\beta(E)$ is the first Betti number and $\gamma(E)$ is the valency number of the graph E. We note, that in the infinite case the torsion part of K_0, which is present in the case of a finite graph, vanishes. The Whitehead group depends only on the first Betti number: $K_1(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)}$. These allow us to provide a counterexample to the fact, which holds for finite graphs, that $K_1(\mathcal{O}_E)$ is the torsion free part of $K_0(\mathcal{O}_E)$.

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1 Introduction

We start with defining, how the C^*-algebra is associated with a graph, in our setting. Namely, we explain that we deal with a C^*-algebras associated with a graph via the Bass-Hashimoto operator.

Let us consider first a non-directed graph $\hat{E}$, which is allowed to be infinite, to have loops, multiple edges and sinks.

We assume that the graph is *locally finite*, i.e. every vertex is connected only to finitely many vertices by edges, and that suppose that $\hat{E}$ is a connected non-directed graph (=geometrically connected graph).

We deal with the Cuntz-Krieger algebra $\mathcal{O}_E$ of the Bass-Hashimoto operator associated with the graph E (more precisely, its infinite, locally finite analogue). This operator (operator Φ_E defined below) was considered by Hashimoto [6] and Bass [2] and later studied in [4]. The algebra known as a *boundary operator algebra* (for example, cf. [8, 9, 10]) is Morita equivalent to the corresponding algebra associated with the Bass-Hashimoto operator. Namely, $\mathcal{O}_E \sim C^*(\delta\mathbb{E})/\Gamma$, where $\mathbb{E}$ is a universal covering tree of the graph E and Γ is a free group of rank β, where β is the first Betti number of E.

Note, that the Cuntz-Krieger algebra of the Bass-Hashimoto operator associated with a graph E (denoted here also by $\mathcal{O}_E$) should not be mixed with the Cuntz-Krieger algebra of the operator defined by the incidence matrix of the graph, as it is done, for example, in [1, 12]. These are two different ways to associate a Cuntz-Krieger algebra with a graph, via different operators. Although the operators appear very similar, the behaviour of the algebra changes dramatically. For example, as it can be seen from [1], K_0 of algebras associated with finite graphs via the incidence matrix are far from being defined by the first Betti number, as it is the case for algebras associated with a finite graph via the Bass-Hashimoto operator [4].

Our goal here is to calculate the K-theory of the Cuntz-Krieger algebra associated with the Bass-Hashimoto operator of an infinite, locally finite graph, purely in graph theoretic terms, as it was done in [4] in the case of a finite graph.

The idea of our calculations is essentially the same as in [4]: we use the fact that groups K_0 and K_1 do not change under the finite number of edge contractions. In case of the infinite graph, by the finite number of edge contractions we could arrive to a simpler, but still infinite graph. Namely, we arrive at a rose with outgoing trees. In the case of K_0, considering the presentation of this group as a quotient of an infinitely generated free group, we are able to write down relations for finite subsets of generators, and conclude that not only the Betti number, but also another characteristic of the graph, which we call the 'valency number', appears in the formula for K_0. In the case of K_1, calculating in the infinitely generated group, we prove, that an element from the kernel of the operator, associated with a graph (whose kernel is K_1), should contain only linear combinations of generators, corresponding to the edges of the rose, but not of the outgoing trees. Due to this, the formula for K_1, as in the finite case, contains only the Betti number of the graph.

For any infinite, locally finite graph, which is connected, we can define the first Betti number (cyclomatic number), extending the usual definition for a finite graph.

Definition 1.1. If $\widehat{E}$ is a finite geometrically connected graph, then the first

Betti number of $\widehat{E}$ is

$$\beta(\widehat{E}) = d_1 - d_0 + 1,$$

where d_0 is the cardinality of the set of vertices and d_1 is the cardinality of the set of (geometric) edges.

Note that for a finite graph with m connected components it would be

$$\beta(\widehat{E}) = d_1 - d_0 + m.$$

This number determines the number of cycles in $\widehat{E}$.

Definition 1.2. If $\widehat{E}$ is an infinite, locally finite geometrically connected graph, we define the first Betti number of $\widehat{E}$ as the limit of the sequence of Betti numbers of finite subgraphs $\widehat{E}_k$ of $\widehat{E}$ obtained in the following way: $\widehat{E}_0$ is an arbitrary connected finite subgraph of $\widehat{E}$ and for any n, $\widehat{E}_{n+1}$ is obtained from $\widehat{E}_n$ by adding to $\widehat{E}_n$ all edges of $\widehat{E}$ connected to the vertices of $\widehat{E}_n$ (together with the vertices on the other end of these edges). It will be a finite graph, since $\widehat{E}$ is locally finite.

Remark 1. This definition does not depend on the choice of the subgraph $\widehat{E}_0$, from which the sequence $\{\widehat{E}_n\}$ starts. Indeed, suppose one starts from another graph $\widehat{E}'_0$. At some step n, one will have the graph $\widehat{E}_0$ as a subgraph of $\widehat{E}'_n$ and $\widehat{E'}_0$ as a subgraph of $\widehat{E}_n$ (all vertices and edges will be 'eaten' due to the connectedness of $\widehat{E}$). It follows that the sequences $\beta(\widehat{E}_n)$ and $\beta(\widehat{E'}_n)$ have the same limit: either stabilize on the same positive integer or both grow to infinity.

Remark 2. It is clear that an infinite, locally finite graph with the finite first Betti number, has the shape of a finite graph (with the same first Betti number), with finite or infinite number of outgoing trees.

Now associate for convenience with any graph $\widehat{E}$ as above, an oriented *bi-directed* graph $E = (E^0, E^1, s, r)$ with the set of vertices E^0, set of edges E^1 and maps s, r from E^1 to E^0 which determine the source and the range of an arrow respectively (*source* and *range* maps). The graph E is obtained from $\widehat{E}$ by doubling the edges of $\widehat{E}$, so that each non-oriented edge of $\widehat{E}$ gives rise to the pair of edges of E, e and $\overline{e}$, equipped with opposite orientations.

For any such finite graph E one can associate a Cuntz–Krieger C^*-algebra $\mathcal{O}_E$, in the way it is done in [4]. Namely, there it is considered a Cuntz–Krieger C^*-algebra (as it is defined in the Cuntz–Krieger paper [5]) associated with the matrix A_E. The matrix A_E is obtained from the graph as a matrix of the following operator (homomorphism of countable direct sum $\mathbb{Z}^{(E^1)}$ of copies of $\mathbb{Z}$), written as follows in the basis labelled by the set E^1 of edges of E:

$$\Phi_E : \mathbb{Z}^{(E^1)} \to \mathbb{Z}^{(E^1)} : \quad e \mapsto -\overline{e} + \sum_{e' : r(e)=s(e')} e' \tag{$*$}$$

This operator was considered in [6] and [2] in connection with the study of the Ihara zeta function of a graph, and is called the *Bass-Hashimoto operator*.

The entries of the matrix A_E are in $\{0, 1\}$. If the graph E is finite, then A_E is an $2n \times 2n$ matrix, where n is the number of geometric edges of the graph E (=the number of edges of the corresponding non-oriented graph $\widehat{E}$).

Denote by $\mathcal{O}_E$ the Cuntz–Krieger C^*-algebra associated with this matrix A_E as in [5]. That is, $\mathcal{O}_E$ is the C^*-algebra generated by $2n$ partial isometries $\{S_j\}_{j=1}^{2n}$ which act on a Hilbert space in such a way that their support projections $Q_i = S_i^* S_i$ and their range projections $P_i = S_i S_i^*$ satisfy the relations

$$P_i P_j = 0 \ \text{ if } i \neq j \ \text{ and } \ Q_i = \sum_{j=1}^{2n} (A_E)_{ij} P_j \ \text{ for } \ 1 \leqslant j \leqslant 2n.$$

This definition of $\mathcal{O}_E$ surely makes sense for an arbitrary locally finite matrix A_E, since the relations still contain finite sums.

So, we have the following.

Definition 1.3. For an infinite, row finite graph $E = (E^0, E^1, s, r)$, its C^*-algebra $\mathcal{O}_E$ is generated by partial isometries $\{S_i : i \in E^1\}$ subject to the relations

$$S_i^* S_i = \sum_{j \in E^1} (A_E)_{ij} S_j S_j^*.$$

Note that there could be another way to associate a C^*-algebra with a graph, for example, the one which is considered in [1]. It should be distinguished from that described above. In [1] a C^*-algebra of the graph is defined as a Cuntz-Krieger algebra of another operator associated with a graph: the operator is represented by the edges adjacency matrix of an oriented graph.

In the paper [4] a formula was obtained for $K_0(\mathcal{O}_E)$ depending only on the first Betti number $\beta(E)$ of the graph E, for a finite graph, namely $K_0(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)} \oplus \mathbb{Z}/(\beta(E) - 1)\mathbb{Z}$. As a consequence, $K_1(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)}$, since it is well known in the finite graph setting ([11], [4]), that K_1 is a torsion free part of K_0.

It was mentioned in [4] that it would be interesting to extend these results to infinite, locally finite graphs. It turned out that the infinite case does indeed reveal some new phenomenon, which is not present in the finite case. We do answer this question from [4] here when the first Betti number $\beta(E)$ of an infinite graph is finite, and show that in the infinite case K_0 does not have a torsion. Moreover, in the infinite case the formula for K_0 involves not only the first Betti number, but also another combinatorial characteristic of the graph: the *valency number*.

Let us give a precise definition. Suppose we fix a finite subgraph in Γ with the Betti number $\gamma(\Gamma)$. (The definition obviously does not depend on this choice.)

The *root vertex* is a root of one of the infinite trees. Its *valency*, is the number of outgoing edges, continuing to infinity.

The *branching vertex* is the vertex of a tree with one incoming and more than one outgoing edges, which leads to an infinite path. If n is the number of such outgoing edges, the *valency* of that vertex is $n-1$.

Definition 1.4. *Valency number* $\gamma(\Gamma)$ *of an infinite, locally finite graph* E *with the finite Betti number* $\beta(E)$*, is the sum of valencies in all branching vertices and root vertices.*

Note, that in case, the valency number is finite, it coincides with the number of infinite chains outgoing from the finite subgraph of E (=the number of infinite ends). However, in general it is not true. This is demonstrated by the following example.

Example 1. Consider the full binary tree $\mathcal{B}$. The valency number $\gamma(\mathcal{B})$ is countable: all its vertices are branching vertices of valency one, so the valency number is equal to the number of vertices of that tree. The infinite ends however can be enumerated by all sequences of 0s and 1s. If for each vertex, we label the edge going to the right by 0, and the edge going to the left by 1, the infinite paths will be marked by all $0-1$ sequences, so there is a continuum of them.

Theorem 1.5. *Let* E *be an infinite, locally finite connected graph with the finite first Betti number* $\beta(E)$ *and the valency number* $\gamma(E)$*. Then* $K_0(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)+\gamma(E)}$.

Let us give here simple example to demonstrate this theorem.

Example 2. Consider the graph with one loop and one outgoing infinite chain. Denote generators of the group K_0 associated with a loop by u and $\bar{u}$, and those associated with the edges from the chain by $x_1, \bar{x}_1, x_2, \bar{x}_2, ...$ The group K_0, being the co-kernel of the operator $T = Id - \Phi_E$ is an abelian group with generators $u, \bar{u}, x_1, \bar{x}_1, x_2, \bar{x}_2, ...$ and relations $Tu, T\bar{u}, Tx_1, T\bar{x}_1, Tx_2, T\bar{x}_2, ...$ By definition of the operator T, $Tu = u-(u+x_1), T\bar{u} = \bar{u}-(\bar{u}+x_1), Tx_1 = x_1 - x_2, Tx_2 = x_2 - x_3, ..., T\bar{x}_1 = \bar{x}_1 - (u+\bar{u}), T\bar{x}_2 = \bar{x}_2 - \bar{x}_1, T\bar{x}_3 = \bar{x}_3 - \bar{x}_2....$ The group $cokerT$ is obviously isomorphic to the group generated by $u, \bar{u}, x_1, \bar{x}_1$, subject to relations $x_1 = 0, \bar{x}_1 = u + \bar{u}$, which means that $K_0(\mathcal{O}_E) = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^{\beta+\gamma}$.

To emphasize the nature of the new infinite phenomenon of torsion vanishing for K_0, we would like to give one more example here.

Example 3. Let us have two graphs: finite graph G_1- n-rose with finite outgoing path, consisting of edges x_1, x_2, x_3, and infinite graph G_2- n-rose with an infinite outgoing chain, consisting of edges $y_1, y_2, y_3, ...$ In $K_0(\mathcal{O}_{G_1})$ the relation corresponding to the last edge x_3 is $x_3 = 0$, for others in the path, these are $x_2 = x_3, x_1 = x_2$. So, in the quotient of the group we have $x_1 = x_2 = x_3 = 0$, which gives the torsion part of the group.

In the case of the infinite path in the graph G_2, where there is no 'last' edge, the relations are $x_1 = x_2 = x_3 = ...$, so we get just one extra variable out of any infinite path, and this variable is non zero.

Finally, in section 4, we calculate $K_1(\mathcal{O}_E)$ and express it in terms of the first Betti number.

Theorem 1.6. *Let E be an infinite, locally finite connected graph with the finite first Betti number $\beta(E)$, and $\mathcal{O}_E$ is the associated (via the Bass-Hashimoto operator) C^*- algebra. Then $K_1(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)}$.*

As a consequence of our results for K_0 and K_1 it turns out that $K_1(\mathcal{O}_E)$ is no longer a torsion free part of $K_0(\mathcal{O}_E)$ in the infinite graph setting, as it is the case for finite graphs, as shown in [11] or [4]. So, we have proved the following corllary.

Corollary 1.7. The K_0 group of any locally finite infinite graph E is the direct limit of groups corresponding to finite subgraphs from the category $\mathcal{E}$.

2 Category of black-and-white bi-directed graphs and a functor to abelian groups

Let E be an infinite, locally finite bi-directed graph as above. We define here a category $\mathcal{E}$ of black-and-white 'subgraphs' of E.

Let us note that this section is not necessary for the proof of the main results, so it can be considered as an independent part of the paper aiming to make a link and comparison to the category, considered by P.Ara [1] in the setting of another type of C^*-algebras associated with a graph (via a transcendency matrix) and with other graph categories. For example, the objects of our category, which we call 'black-and-white graphs', are the same as 'graphs with flags' used in the paper by Yu.Manin and D.Borisov [3] (flags there are our white edges), but there the set of morphisms is different. Analogously to [1], we prove the basic property of the functor from our category to abelian groups. This section also provides additional insight into the nature of our calculations for the main results.

To construct an object of $\mathcal{E}$, choose an arbitrary set Ω of vertices of E. The corresponding graph will contain all edges of E starting and ending on vertices from Ω. If the edge starts (ends) on a vertex from Ω, but ends (starts) outside, it will be called white, otherwise black. Thus the objects of $\mathcal{E}$ are certain (black-and-white) subgraphs of E. In particular, if we choose the whole set of vertices of E, as Ω, we arrive at the whole graph E (with only black edges present), as an element of $\mathcal{E}$.

Now we define the set of graph homomorphisms between those black-and-white bi-directed graphs. There exists a homomorphism $f : G \to F$, if F contains all vertices of G, and all (black and white) edges of G, as (black) edges of F. White edges of F are those which start at the ends of edges of G. So, these homomorphisms change white edges to a black ones and add, as a white, new edges which come out from the former white edges. Of course, the identity map from a black-and-white graph to itself is considered to be an elementary homomorphism as well. Any composition of the above defined elementary homomorphisms is also a homomorphism. These homomorphisms play a role analogous to the 'complete graph homomorphisms' in [1], however they are defined differently in our case.

Proposition 2.1. Every infinite, locally finite graph E is a direct limit of a sequence of finite graphs and homomorphisms in the category $\mathcal{E}$. Any finite subgraph of E can serve as a starting element of this sequence.

Proof. Take an arbitrary finite subgraph E_0 of E (as a black-and-white subgraph constructed on vertices of E_0) and consider a sequence of non-identity homomorphisms $\varphi_n : E_n \to E_{n+1}$ in $\mathcal{E}$. Due to the connectedness of the graph E, the union of edges of all elements of the sequence will coincide with the set of edges of E. Moreover, if an edge becomes black in the graph E_n from the sequence, then it will be a black edge in any E_N, for $N \geqslant n$. This means that E is indeed a limit of a sequence E_n, φ_n. □

After the category $\mathcal{E}$ is constructed, we define a functor from $\mathcal{E}$ to abelian groups $\mathcal{AG}$. We associate with a black-and-white graph $E \in \mathcal{E}$ the group $K_0(\tilde{\mathcal{O}}_E)$ with generators corresponding to all (black and white) edges $x_i \in E$ and relations $x_i = \sum_{y_i \in E^1} \lambda_{i,j} y_j$, for any black edge $x_i \in E$. Here y_j run over all edges, black and white of E, and $\lambda_{i,j}$ is equal to 1 if there is a path connecting directly x_i to y_j (they are adjacent in the directed graph), except from the case when y_i is the inverse of x_i. Otherwise $\lambda_{i,j}$ are equal to zero.

Let us describe how the functor $\mathcal{F} : \mathcal{E} \to \mathcal{AG}$ maps a black-and-white graph morphism $f : G \to F$ to the homomorphism of abelian groups $\mathcal{F}(f) : K_0(\mathcal{O}_G) \to K_0(\mathcal{O}_F)$. Since (due to the definition of graph morphisms) all edges of the graph G (black and white) are also edges of the graph F (black), and all relations of the group $K_0(\mathcal{O}_G)$ are present in the group $K_0(\mathcal{O}_F)$, the map $K_0(\mathcal{O}_G) \to K_0(\mathcal{O}_F)$ which sends generators of $K_0(\mathcal{O}_G)$ to themselves, in $K_0(\mathcal{O}_F)$, is a group homomorphism.

Theorem 2.2. The above defined functor $\mathcal{F}$ from $\mathcal{E}$ to $\mathcal{AG}$ is continuous, i.e. it commutes with direct limits.

Proof. Elements of the direct limit of groups $\mathcal{G} = \underline{\lim}\mathcal{G}_n$ are sequences of elements of corresponding graphs $\mathcal{G}_n$, mapped to each other by corresponding morphisms (modulo the equivalence relation). In particular, generators of the limit group $\mathcal{G}$ are (classes of) sequences $X = (x \to x \to x...)$ consisting of one generator x of a particular group $\mathcal{G}_n$, with trivial maps. If we take into account the way how summation on these (classes of) sequences is defined, we see that any relation on generators $X_1, ..., X_n$ (any particular relation contains a finite number of generators) of the limit group $\mathcal{G}$ is actually present for generators $x_1, ..., x_k$ of groups $\mathcal{G}_N, \mathcal{G}_{N+1}, ...$, starting from certain N. In other words, any relation of the limit group $\mathcal{G}$ appears as a relation of some group $\mathcal{G}_N$, and stays the same in $\mathcal{G}_l$, for $l \geqslant N$. Our definition of group associated with the black-and-white subgraph was constructed in a way, which ensures that K_0 of an infinite graph has the set of relations, obtained as a union of relations in groups associated with finite subgraphs. This means that two sets of relations for $\underline{\lim}K_0(\mathcal{O}_{F_n})$ and for $K_0(\underline{\lim}\mathcal{O}_{F_n})$ are coincide. □

Combining this theorem with Proposition 2.1 we have the following.

Corollary 2.3. Any K_0 group of a locally finite infinite graph E is a direct limit of groups corresponding to finite subgraphs from the category $\mathcal{E}$.

3 K_0 calculations in the case of finite Betti number

Now we turn to concrete calculations in the case when the Betti number of the graph is finite.

It is known for finite graphs and row-finite graphs (see, for example, [5] and [7]), that $K_0(\mathcal{O}_E) = coker(Id - \Phi)$, where $\Phi : \mathbb{Z}^{(E^1)} \to \mathbb{Z}^{(E^1)}$ is the homomorphism of countable direct sums of copies of $\mathbb{Z}$, defined for the graph E by the formula $(*)$.

First of all, we shall show that in any locally finite graph, we can perform any finite number of edge contractions, without changing K_0.

Theorem 3.1. *Let the graph E' be obtained from E by contraction of one non-loop edge x and its inverse $\bar{x}$. Then the groups $K_0(\mathcal{O}_E)$ and $K_0(\mathcal{O}_{E'})$ are isomorphic.*

Proof. We will obtain the fact that the group K_0 is preserved under the edge contraction as a corollary of the following general lemma, which might be interesting in its own right.

Lemma 3.2. *Let G, H be abelian groups and $T : G \oplus H \to G \oplus H$ a homomorphism, such that $Tx - x \in G$ for any $x \in H$. Let $P : G \oplus H \to G$ be a homomorphism such that $P|_G = id_G$ and $Px = x - Tx$, $x \in H$.*

Then for $\tilde{T} : G \to G = P \circ T|_G$ the following is true:
$G \oplus H/T(G \oplus H) \simeq G/\tilde{T}(G)$

Proof. (of lemma 3.2).

Define a homomorphism $J : G/\tilde{T}(G) \to G \oplus H/T(G \oplus H)$ as follows:

$$J(u + \tilde{T}(G)) = u + T(G \oplus H).$$

This map is well-defined, i.e. $\tilde{T}(G) \subseteq T(G \oplus H)$. Let $u \in G$ and $Tu = y + w, y \in H, w \in G$. Then $\tilde{T}u = P \circ Tu = P(y + w) = y - Tu + w = Tu - Ty \in T(G \oplus H)$.

The map J is injective, i.e. for $u \in G$, $u \in T(G \oplus H)$ implies that $u \in \tilde{T}(G)$. Indeed, let $u = T(w + y) \in G$ for $w \in G, y \in H$. Since $u = Tw+Ty = Tw+y+Ty-y$, we can present Tw as $Tw = (u+(y-Ty))-y$, and $u + (y - Ty) \in G, y \in H$. We showed above that for $w \in G$, $\tilde{T}w = Tw = Th$, for h being the H component of Tw: $Tw = w' + h$, $w' \in G, h \in H$. Due to the above presentation of Tw its H component is $-y$, so we have: $\tilde{T}w = Tw + Ty = T(w + y)$, hence indeed $T(w + y) \in \tilde{T}(G)$.

The map J is surjective, i.e. $G + T(G \oplus H) = G \oplus H$. Indeed, for $x \in H$, there exists $u \in G : u = Tx - x$. Then for $w + x \in G \oplus H$, $w + x = Tx - u + w$, where $w = u \in G$ and $Tx \in T(G \oplus H)$.

Thus J is the required isomorphism. □

Now to prove Theorem 3.1, for an edge x and its inverse $\bar{x}$ apply Lemma 3.2 for direct sum of copies of $\mathbb{Z}$: $G = \mathbb{Z}^{(E^1 \setminus \{x, \bar{x}\})}$ and $H = \mathbb{Z}^2$. As an operator T ($\tilde{T}$) we should take $T = Id - \Phi_E$ ($\tilde{T} = Id - \Phi_{E'}$), where Φ is defined by the formula (*).

□

We are now in a position to start the proof of the main theorem.

Theorem 3.3. *Let E be an infinite, locally finite connected graph with the finite first Betti number $\beta(E)$ and the valency number $\gamma(E)$.*

Then $K_0(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)+\gamma(E)}$.

Proof.
Type I. If the valency number $\gamma(E)$ is finite, by a finite number of steps we can reduce our graph to the rose with $\beta(E)$ petals and $\gamma(E)$ outgoing simple infinite chains. According to Theorem 3.1, K_0 will be preserved. In this case it is easy to calculate directly the group $K_0(\mathcal{O}_E) = coker(Id - \Phi)$, generated by relations readable from the graph. Indeed, let us denote variables corresponding to $\beta(E) = m$ petals (and their inverses) by $u_1, ..., u_m,\ \bar{u}_1, ..., \bar{u}_m$ and

variables corresponding to $\gamma(E) = n$ edges outgoing directly from the vertex of the rose (and their inverses) by $x_1, ..., x_n,\ \bar{x}_1, ..., \bar{x}_n$. Next, edges (and their inverses) in each chain will be $x_i^{(1)}, ..., x_i^{(k)}, ...,\ \bar{x}_i^{(1)}, ..., \bar{x}_i^{(k)} ...,\ i = \overline{1,n}$. Then K_0 will be the quotient of the free abelian group generated by the set $\Omega = \{x_i^{(k)},\ \bar{x}_i^{(k)}, i = \overline{1,n}, u_j, \bar{u}_j, j = \overline{1,m}\}$, subject to the relations defined by the formula $K_0(\mathcal{O}_E) = coker(Id - \Phi)$. For each edge $e \in E$ we will have one relation. Note that the relation written for edges belonging to chains will give $x_i^{(1)} = x_i^{(2)} = ...,\ i = \overline{1,n}$. So after that we actually have a finite number of relations for variables $x_i, \bar{x}_i,\ i = \overline{1,n},\ u_j, \bar{u}_j,\ j = \overline{1,m}$. Namely,

$$\sum_{j \neq k} (u_j + \bar{u}_j) + \sum_{l=1}^{n} x_l = 0, \quad 1 \leqslant k \leqslant m,$$

$$\bar{x}_l = \sum_{j=1}^{m} (u_j + \bar{u}_j) + \sum_{r \neq l} x_r, \quad 1 \leqslant l \leqslant n,$$

where the first group of relations corresponds to petals and the second to edges outgoing (incoming) from (to) the rose. It is a compete set of defining relations for K_0 on the set of generators Ω.

For convenience, let us denote by $w_j = u_j + \bar{u}_j$. Now write down the matrix of the above system of linear equations on variables $w_j, x_i, \bar{x}_i,\ j = \overline{1,m},\ i = \overline{1,n}$.

$$\begin{pmatrix} 0 & \dots & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ & \ddots & & & \dots & & & \dots & \\ 1 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ 1 & \dots & 1 & 0 & \dots & 1 & -1 & \dots & 0 \\ & \dots & & & \ddots & & & \ddots & \\ 1 & \dots & 1 & 1 & \dots & 0 & 0 & \dots & -1 \end{pmatrix}$$

By adding last n columns to the first m we can make zeros in the lower $n \times (m+n)$ block of the matrix. Then using the middle block of n columns we can transform the upper left $m \times (m+n)$ corner into

$$\begin{pmatrix} 1 & 0 & & \dots & & 0 & 0 & \dots & 0 \\ 0 & 1 & & \dots & & 0 & 0 & \dots & 0 \\ & & \ddots & & & & & & \\ 0 & 1 & & \dots & & 1 & 0 & \dots & 0 \end{pmatrix}$$

This shows that we have $m + n$ free variables: $u_1, ..., u_m, \bar{x}_1, ..., \bar{x}_n$. So we see, that $K_0(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)+\gamma(E)}$ in this case.

Type II. The second case is when the number $\gamma(E)$ is infinite. Here we can not write down a finite number of equations on the finite number of variables, which will define a group, but we can show what will be the system of free generators of the abelian group $K_0(\mathcal{O}_E)$. The group $K_0(\mathcal{O}_E)$ is defined by generators corresponding to all edges of the graph, consisting of one rose with $\beta(E)$ petals and finite number of outgoing infinite trees. The number of outgoing trees can not be infinite, because of the locally finiteness condition.

Let us consider generators corresponding to petals of the rose: $u_1, ..., u_m$, $\bar{u}_1, ..., \bar{u}_m$ and edges coming out directly from the rose vertex: $x_1, ..., x_n$, $\bar{x}_1, ..., \bar{x}_n$. We have the following equations on them:

$$\sum_{k\neq j}(u_k + \bar{u}_k) + \sum_{l=1}^{n} x_l = 0, \quad 1 \leqslant j \leqslant m$$

$$\bar{x}_j = \sum_{k=1}^{m}(u_k + \bar{u}_k) + \sum_{l\neq r} x_l, \quad 1 \leqslant j \leqslant n$$

These are the same as above and give us $n+m$ free variables: $u_1, ...u_m$, $x_1, ..., x_n$. Then consider for any branching vertex, a piece of the tree of the shape

$$\begin{array}{ccccc} & & & & b_1 \\ & & & \nearrow & \\ a & \longrightarrow & \bullet & & \vdots \\ & & & \searrow & \\ & & & & b_l \end{array}$$

Equations which we have to write for edges a, incoming for this vertex and b_j, $j = \overline{1, l}$, outgoing from it, and leading to an infinite path, form the following system.

$$a = b_1 + ... + b_l$$

$$\bar{b}_j = \bar{a} + b_1 + ... + \widehat{b_j} + ... + b_l$$

So, on such a step we get $l-1$ new free variables, corresponding to $l-1$ new infinite paths along the graph, we got in this vertex (which is equal to the valency of this branching vertex). If we sum up all new free variables, which we got from all vertices of outgoing trees, we arrive at $\gamma(E)$ additional variables. Note that again, if an outgoing chains are finite, then variables corresponding to their edges are just zero. So, we see that in this case also $K_0(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)+\gamma(E)}$, and here it is a direct sum of the countably infinite number of copies of $\mathbb{Z}$. By this the proof of the theorem is completed.

□

4 The Whitehead group expressed via the first Betti number

In the original paper due to Cuntz and Krieger [5] it was shown that K_0 and K_1 of the Cuntz-Krieger C^*- algebra $\mathcal{O}_A$, associated with any finite 0-1 matrix A are, respectively, co-kernel and kernel of the map $(Id - A^t) : Z^n \to Z^n$.

This fact was later generalized in [7] to the Cuntz-Krieger C^*- algebra $\mathcal{O}_A$, associated in the same way with an infinite 0-1 matrix A, with a finite number of 'ones' in any row. The graph algebra we consider, as mentioned in the Introduction, is a Cuntz-Krieger algebra associated with an infinite matrix, constructed from the graph by certain rules. So this result is applicable here and making use of this we will prove the following theorem.

Theorem 4.1. *Let E be an infinite, locally finite connected graph with the finite first Betti number $\beta(E)$, and $\mathcal{O}_E$ is an associated (via the Bass-Hashimoto operator) C^*- algebra. Then $K_1(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)}$.*

Proof. The proof will be divided into several steps.

First of all, we shall show that in any locally finite graph, we can perform any finite number of edge contractions, without changing K_1.

Theorem 4.2. *Let the graph E' be obtained from E by contraction of one non-loop edge x and its inverse $\bar{x}$. Then the groups $K_1(\mathcal{O}_E)$ and $K_1(\mathcal{O}_{E'})$ are isomorphic.*

Proof. Let us ensure the following fact of linear algebra.

Lemma 4.3. *Let G, H be abelian groups and $T : G \oplus H \to G \oplus H$ a homomorphism, such that $Tx - x \in G$ for any $x \in H$. Let $P : G \oplus H \to G$ be a homomorphism such that $P|_G = id_G$ and $Px = x - Tx$, $x \in H$.*

Then for $\tilde{T} : G \to G = P \circ T|_G$ the following is true: $\mathrm{Ker}T \simeq \mathrm{Ker}\tilde{T}$.

Proof. (of lemma 4.3)

Take an element $u + y \in \mathrm{Ker}T$, with $u \in G$, $y \in H$. Let $Tu = w + x$, where $w \in G$, $x \in H$. Then $\tilde{T}u = x - Tx + w = Tu - Tx$, so $Tu = \tilde{T}u + Tx$. Now substituting that to $T(u+y) = 0$, we have $0 = T(u+y) = \tilde{T}u + T(x+y)$. From this we see first that $\tilde{T}u = -T(x + y)$.

Denote the G and H components of an element $r \in G \oplus H$ by r_G and r_H respectively, so $r = r_G + r_H$, for $r_G \in G$ and $r_H \in H$. Now comparing the G and H components of the left and right hand side of $\tilde{T}u = -T(x + y) = -(x + y) - g_{x+y}$, we have that $x + y = 0$. Therefore $\tilde{T}u = -T(u + y)$ and $u + y \in \mathrm{Ker}T$ iff $u \in \mathrm{Ker}\tilde{T}$. □

The Proof of Lemma 4.3 will follow as a corollary from this lemma if we put $G = \mathbb{Z}^{|E^1 \setminus \{x,\bar{x}\}|}$ and $H = \mathbb{Z}^2$. As an operator T ($\tilde{T}$) we should take $T = Id - \Phi_E$ ($\tilde{T} = Id - \Phi_{E'}$), where Φ is defined by the formula (*).

□

Theorem 4.2 from the first step, allows us to reduce the calculation of $K_1(\mathcal{O}_E)$, where E is a locally finite graph with the Betti number $\beta(E)$ to the K_1 for the graph Γ, which is a rose with $\beta(E) = \beta(\Gamma)$ petals and a finite number of trees, rooted in the vertex of the rose, with a finite number of branches outgoing from each vertex, due to the locally finiteness condition.

Now the second step in the calculation of the Whitehead group will be a calculation for the graph Γ. We need to calculate $K_1(\mathcal{O}_\Gamma) = \operatorname{Ker} T_\Gamma = \operatorname{Ker}(Id - \Phi_\Gamma)$, where Φ_Γ is defined by formula (*).

For the graph Γ we can present the set of all edges as a disjoint union of three sets:

$$\Gamma^1 = \Gamma^\uparrow \sqcup \Gamma^\downarrow \sqcup R,$$

where $\Gamma^\uparrow$ is the set of edges of the tree, directed towards the rose, $\Gamma^\downarrow$ is the set of edges of the tree directed off the rose and R is the set of petals of the rose.

Let $\xi \in \operatorname{Ker} T_\Gamma$, $\xi = \sum_{e \in \Gamma_\xi} m_e e$, where Γ_ξ is a finite set of edges and $m_e \in \mathbb{Z} \setminus \{0\}$.

Let us show first that the following is true.

Lemma 4.4. *The set Γ_ξ does not contain tree edges in the direction towards the rose: $\Gamma_\xi \cap \Gamma^\uparrow = \emptyset$.*

Proof. Consider the projection $\pi : \mathbb{Z}^{(\Gamma)} \longrightarrow \mathbb{Z}^{(\Gamma)}/\mathbb{Z}^{(\Gamma')}$, where $\Gamma' = \Gamma^\downarrow \sqcup R$, then denote by T' the composition of our initial map T with π:

$$\mathbb{Z}^{(\Gamma)} \xrightarrow{T} \mathbb{Z}^{(\Gamma)} \xrightarrow{\pi} \mathbb{Z}^{(\Gamma)}/\mathbb{Z}^{(\Gamma')} \simeq \mathbb{Z}^{(\Gamma^\uparrow)}$$

Then $T'/\mathbb{Z}^{(\Gamma')} = 0$ and $T'e = e + f$, where f consists of edges which are higher in the tree (=closer to the rose) than e.

Suppose $\Gamma_\xi \cap \Gamma^\uparrow \neq \emptyset$. Consider $g \in \Gamma_\xi$ farthest away from the rose, $\xi = mg + \tilde{g}$, $m \in \mathbb{Z} \setminus \{0\}$.

Then

$$T'\xi = mg + f + T'(\tilde{g}).$$

Here f consists of terms corresponding to the edges, closer to the rose than g. $T'(\tilde{g})$ consists of terms corresponding to the edges, closer to the rose than $\tilde{g}$, which are in turn closer then g. This means that the term mg can not cancel, and $T'\xi \neq 0$, hence $T\xi \neq 0$. We arrive at a contradiction. □

Lemma 4.5. *The set* Γ_ξ *does not contain tree edges in the direction off the rose:* $\Gamma_\xi \cap \Gamma^\downarrow = \emptyset$.

Proof. Assume $\Gamma_\xi \cap \Gamma^\downarrow \neq \emptyset$, and take $h \in \Gamma_\xi \cap \Gamma^\downarrow$, farthest 1 away from the rose. Then

$$Th = h - (h_1 + h_2 + ...),$$

where all h_i are further away than h from the rose. Then if

$$\xi = mh + \sum g_i,$$

$m \in \mathbb{Z} \setminus \{0\}$, then $g_i \in \Gamma^\downarrow$, and since $g_i \in \Gamma^\xi$, they are closer than h. Therefore

$$T\xi = mh - m(h_1 + h_2 + ...) + T(\sum g_i),$$

and the farthest from the rose edge, which could be contained in $T(\sum g_i)$, is h. This means that the term mh_1 could not be cancelled, $T\xi \neq 0$ and we arrive at a contradiction. ☐

Now after the above two lemmas we are left with the only possibility, that

$$\Gamma_\xi \subseteq R = \{u_1, ..., u_n, \bar{u}_1, ..., \bar{u}_n\}.$$

Let $\xi = \sum_{j=1}^n (m_j u_j + n_j \bar{u}_j)$. We know that

$$T\bar{u}_j = Tu_j = -\sum_{k\neq j}(u_k + \bar{u}_k) + \sum_{l=1}^{s} x_l = w - (u_j + \bar{u}_j),$$

where for convenience we denote by $w = -\sum_{k=1}^{n}(u_k + \bar{u}_k) + \sum_{l=1}^{s} x_l$. Then

$$T\xi = (\sum_{j=1}^{n}(m_j + n_j))w - \sum_{j=1}^{n}(m_j + n_j)(u_j + \bar{u}_j) = 0.$$

Since all x_l appear in w with coefficient 1 they can disappear only if $\sum_{j=1}^n (m_j + n_j) = 0$. Hence

$$T\xi = -\sum_{j=1}^{n}(m_j + n_j)(u_j + \bar{u}_j) = 0.$$

Since each u_j appears in one term only, in order for the sum to be zero, we should have $m_j + n_j = 0$ for all j. This means that

$$\xi \in \mathrm{Ker}T \iff \xi = \sum_{j=1}^{n} m_j(u_j - \bar{u}_j),$$

$m_j \in \mathbb{Z}$, thus Ker$T = \mathbb{Z}^n$, where $n = \beta(\Gamma)$. This completes the Proof of Theorem 4.1.

□

As a consequence of our result we have the following corollary.

Corollary 4.6. *An infinite analogue of the statement that K_1 is a torsion free part of K_0, which holds for the case of finite graphs, is not true for infinite graphs.*

Proof. It is well known, that in the case of finite graphs or matrices K_1 of a Cuntz-Krieger C^*-algebra associated with a matrix is a torsion free part of K_0 (see [11] or [4]).

Let E be, for example, an infinite, locally finite graph with the finite first Betti number $\beta(E)$ and infinite valency number $\gamma(E)$. Then according to Theorem 3.3 $K_0(\mathcal{O}_E) \simeq \mathbb{Z}^\infty$ and $K_1(\mathcal{O}_E) \simeq \mathbb{Z}^{\beta(E)}$, according to Theorem 4.1. This shows that $K_1(\mathcal{O}_E)$ is not isomorphic to the torsion part of $K_0(\mathcal{O}_E)$. □

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