On the Complexity and Volume of Hyperbolic 3-Manifolds.

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Abstract

We compare the volume of a hyperbolic 3-manifold M of finite volume and the complexity of its fundamental group. [1]

1 Introduction.

Complexity of 3-manifolds and groups. One of the most striking corollaries of the recent solution of the geometrization conjecture for 3-manifolds is the fact that every aspherical 3-manifold is uniquely determined by its fundamental group. It seems to be natural to think that a topological/geometrical description of a 3-manifold M produces the simplest way to describe its fundamental group $\pi_1(M)$; on the other hand, the simplest way to define the group $\pi_1(M)$ gives rise to the most efficient way to describe M. More precisely, we want to compare the complexity of 3-manifolds and their fundamental groups.

The study of the complexity of 3-manifolds goes back to the classical work of H. Kneser [K]. Recall that the Kneser complexity invariant $k(M)$ is defined to be the minimal number of simplices of a triangulation of the manifold M. The main result of Kneser is that this complexity serves as a bound of the number of embedded incompressible 2-spheres in M, and bounds the numbers of factors in a decomposition of M as a connected sum. A version of this complexity was used by W. Haken to prove the existence of hierarchies for a large class of compact 3-manifolds (called since then Haken manifolds). Another measure of the complexity $c(M)$ for the 3-manifold M is due to S. Matveev. It is the minimal number of vertices of a special spine of M [Ma]. It is shown that in many important cases (e.g. if M is a non-compact hyperbolic 3-manifold of finite volume) one has $k(M) = c(M)$ [Ma].

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The rank (minimal number of generators) is also a measure of complexity of a finitely generated group. According to the classical theorem of I. Grushko [Gr], the rank of a free product of groups is the sum of their ranks. This immediately implies that every finitely generated group is a free product of finitely many freely indecomposible factors, which is an algebraic analogue of Kneser theorem.

For a finitely presented group G a measure of complexity of G was defined in [De]. Here is its definition :

Definition 1.1. *Let G be a finitely presented group. We say that $T(G) \leq t$ if there exists a simply-connected 2-dimensional complex P such that G acts freely and simplicially on P and the the number of 2-faces of the quotient $\Pi = P/G$ is less than t.*

If the group G is defined by a presentation $< a_1, ...a_r; R_1, ...R_n >$ the sum $\Sigma(|R_i| - 2)$ serves as a natural bound for $T(G)$.

Note that an inequality between Kneser complexity and this invariant is obvious. Indeed, by contracting a maximal subtree of the 2-dimensional skeleton of a triangulation of M one obtains a triangular presentation of the group $\pi_1(M)$. Since every 3-simplex has four 2-faces it follows

$$T(\pi_1(M)) \leq 4k(M).$$

In order to compare the complexity of a manifold and that of its fundamental group, it is enough to find a function θ such that $\theta(\pi_1(M)) \leq T(\pi_1(M))$. Note that the existence of such a function follows from G. Perelman's solution of the geometrization conjecture [Pe 1-3]. Indeed there could exist at most finitely many different 3-manifolds having the fundamental groups isomorphic to the same group G (for irreducible 3-manifolds with boundary this was shown much earlier in [Swa]). The question which still remains open is to describe the asymptotic behavior of the function θ.

Note that for certain lens spaces the following inequality is proven in [PP]:

$$c(L_{n,1}) \leq \ln n \approx \mathrm{const} \cdot T(\mathbb{Z}/n\mathbb{Z}).$$

However, the above problem remains widely open for irreducible 3-manifolds with infinite fundamental group. If M is a compact hyperbolic 3-manifold, D. Cooper showed [C]:

$$\mathrm{Vol}M \leq \pi \cdot T(\pi_1(M)) \qquad (C).$$

where Vol$M$ is the hyperbolic volume of M. Note that the converse inequality in dimension 3 is not true: there exists infinite sequences of different hyperbolic 3-manifolds M_n obtained by Dehn filling on a fixed finite volume hyperbolic manifold M with cusps such that $\mathrm{Vol}M_n < \mathrm{Vol}M$ [Th]. The ranks of the groups $\pi_1(M_n)$ are all bounded by rank$(\pi_1(M))$ and since $\pi_1(M_n)$ are not isomorphic, we must have $T(\pi_1(M_n)) \to \infty$. So the invariant $T(\pi_1(M))$ is not comparable with the volume of hyperbolic 3-manifolds. This difficulty can be overcome using the following relative version of the invariant T introduced in [De]:

Definition 1.2. *Let G be a finitely presented group, and $\mathcal{E}$ be a family of subgroups. We say that $T(G,\mathcal{E}) \le t$ if there exists a simply-connected 2-dimensional complex P such that G acts simplicially on P, the number of 2-faces of the quotient (an orbihedron) $\Pi = P/G$ is less than t, and the stabilizers of vertices of P are elements of $\mathcal{E}$.*

The main goal of the present paper is to obtain uniform constants comparing the volume of a hyperbolic 3-manifold M of finite volume and the relative invariant $T(\pi_1(M), E)$ where E is the family of its elementary subgroups.

To finish our historical discussion let us point out that the relative invariant $T(G, E)$ allows one to prove the accessibility of a finitely presented group G without 2-torsion over elementary subgroups [DePo1]. Using these methods it was shown recently that for hyperbolic groups without 2-torsion any canonical hierarchy over finite subgroups and one-ended subgroups is finite [Va]. The relative invariant T and the hierarchical accessibility was used in [DePo2] to give a criterion of the co-Hopf property for geometrically finite discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$.

Main Results. Let M be a hyperbolic 3-manifold of finite volume. We consider the family E_μ of all elementary subgroups of $\pi_1(M)$ having translation length less than the Margulis constant $\mu = \mu(3)$. The family E_μ includes all parabolic subgroups of G as well as cyclic loxodromic ones representing geodesics in M of length less than μ (see also the next Section).

The first result of the paper is the following:

Theorem A. *There exists a constant C such that for every hyperbolic 3-manifold M of finite volume the following inequality holds:*

$$C^{-1}T(G, E_\mu) \le \mathrm{Vol}(M) \le CT(G, E_\mu) \qquad (*)$$

■

The following are corollaries of Theorem A.

Corollary 1.3. *Suppose $M_n \xrightarrow{f_n} M$ is a sequence of finite coverings over a finite volume 3-manifold M such that $\deg f_n \to +\infty$. Then $T(\pi_1(M_n), E_n) \to +\infty$, where E_n is the above system of elementary subgroups of $\pi_1(M_n)$ whose translation length is less than μ.* ■

Proof: The statement follows immediately from the right-hand side of (*) since $\mathrm{Vol}(M_n) \to \infty$. *QED.*

Corollary 1.4. *Let M_n be a sequence of different hyperbolic 3-manifolds obtained by Dehn surgery on a cusped hyperbolic 3-manifold of finite volume M. Then*

$$T(\pi_1(M_n), E_n) \leq C \cdot \mathrm{Vol}(M) \ < +\infty.$$

■

Proof: The left-hand side of (*) gives

$$T(\pi_1(M_n), E_n) \leq C \cdot \mathrm{Vol}(M_n),$$

and by [Th] one has $\mathrm{Vol}(M_n) < \mathrm{Vol}(M)$. *QED.*

As it is pointed out in Corollary 1.3 above we must have $T(\pi_1(M_n)) \to +\infty$ for the absolute invariant. Our next result is the following :

Theorem B. *(Generalized Cooper inequality) Let E be the family of elementary subgroups of G, then one has*

$$\mathrm{Vol}(M) \leq \pi \cdot T(\pi_1(M), E) \qquad (**)$$

■

Note that Theorem B gives a generalization of the Cooper inequality (C) for the relative invariant $T(G, E)$. Furthermore, if one puts $E = E_\mu$, then Theorem B implies the right-hand side of (*) in Theorem A. Theorems A and B together have several immediate consequences:

Corollary 1.5. *For the constant C from Theorem A the following statements hold:*

i) *Let M be a finite volume hyperbolic 3-manifold and E_μ and E be the above families of elementary subgroups of $\pi_1(M)$. Then*

$$T(\pi_1(M), E_\mu) \leq C \cdot \pi \cdot T(\pi_1(M), E).$$

ii) *Let M be a hyperbolic 3-manifold such that $M = M_{\mu\mathrm{thick}}$, i.e. every loop in M of length less than μ is homotopically trivial. Then*

$$T(\pi_1(M)) \leq C \cdot \pi \cdot T(\pi_1(M), E).$$

Proof: i) By Theorems A and B we have

$$T(\pi_1(M), E_\mu) \leq C\text{Vol}(M) \leq C \cdot \pi \cdot T(\pi_1(M), E). \quad \textit{QED}.$$

ii) Since $E_\mu = \emptyset$ the result follows from i). *QED.*

Let us now briefly describe the content of the paper. In Section 2 we provide some preliminary results needed in the future. The proof of Theorem B is given in Section 3, it provides a "simplicial blow-up" procedure for an orbihedron. In Section 4 we prove the left-hand side of the inequality (*) using some standard techniques and the results of Section 2. In the last Section 5 we discuss some open questions related to the present paper.

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2 Preliminary results.

Let us recall few standard definitions which we will use in the future. We say that G *splits* as a graph of groups $X_* = (X, (C_e)_{e \in X^1}, (G_v)_{v \in X^0})$ (where C_e and G_v denote respectively edge and vertex groups of the graph X) if G is isomorphic to the fundamental group $\pi_1(X_*)$ in the sense of Serre [Se]. The Bass-Serre tree T is the universal cover of the graph $X = T/G$. When X has only one edge, we will say that G splits as an amalgamated free product (resp. an HNN-extension) if X has two vertices (resp. one vertex).

Definition 2.1. *Let G be a group acting on a tree T. A subset H of G is elliptic (resp. hyperbolic) in T (and in the graph T/G) if H fixes a point in T (resp. does not fix a point in T). If T is the Bass-Serre tree of a splitting of G as a graph of groups, H is elliptic if and only if it is conjugate into a vertex group of this graph.*
We say that G splits relatively to a family of subgroups $(E_1, ...E_n)$, or that the pair $(G, (E_i)_{1 \leq i \leq n})$ splits as a graph of groups, if G splits as a graph of groups such that all the groups E_i are elliptic in this splitting. A $(G, (E_i)_{1 \leq i \leq n})$-tree is a $G-$tree in which E_i are elliptic for all i. ■

Definition 2.2. *Suppose G splits as a graph of groups*

$$G = \pi_1(X, C_e, G_v) \tag{1}$$

relatively to a family of subgroups E_i $\{i = 1, ..., n\}$.

The decomposition (1) such that all edge groups are non-trivial is called **reduced** *if every vertex group G_v cannot be decomposed relatively to the subgroups $E_i \in G_v$ as a graph of groups having one of the subgroups C_e as a vertex group.*

The decomposition (1) is called **rigid** *if whenever one has a $(G, (E_i)_{i\in\{1,\dots,n\}})$-tree T^* such that the subgroup C_e contains a non-trivial edge stabilizer then C_e acts elliptically on T^*.* ■

It was shown in [De] that the sum of relative T-invariants of the vertex groups of a reduced splitting is less than or equal to the absolute invariant of G.

Recall that the Margulis constant $\mu = \mu(n)$ is a number for which any n-dimensional hyperbolic manifold M can be decomposed into thick and thin parts : $M = M_{\mu\text{thick}} \bigsqcup M_{\mu\text{thin}}$ such that the injectiviry radius at each point of $M_{\mu\text{thin}}$ is less than $\mu/2$, and $M_{\mu\text{thick}} = M \setminus M_{\mu\text{thin}}$. By the Margulis Lemma the components of $M_{\mu\text{thin}}$ are either parabolic cusps or regular neighborhoods (tubes) of closed geodesics of M of length less than μ. We will denote by $E = E(\pi_1(M))$ (respectively $E_\mu = E_\mu(\pi_1(M)))$ the system of elementary subgroups of $\pi_1(M)$ (respectively the systems of subgroups of $\pi_1 M_{\mu\text{thick}}$). We will need the following:

Lemma 2.3. *Let H be a group admitting the following splitting as a graph of groups:*

$$H = \pi_1(X, C_e, G_v), \tag{2}$$

where each vertex group G_v is a lattice in $\mathrm{Isom}(\mathbb{H}^n)$ $(n > 2)$ and $C_e \in E(G_v)$ $(n > 2)$.

Then (2) is a reduced and rigid splitting of the couple $(H, \mathcal{E})$ where $\mathcal{E} = \bigcup_v E(G_v)$.

Remark 2.4. *The above Lemma will be further used in a very particular geometric situation when the group H is the fundamental group of the double of the thick part $M_{\mu\text{thick}}$ of M along its boundary.* ■

Proof: We first claim that it is enough to prove that every vertex group G_v of the graph X cannot split non-trivially over an elementary subgroup. Indeed, if it is the case then obviously (2) is reduced. If it is not rigid, then the couple $(H, \mathcal{E})$ acts on a simplicial tree T^* such that one of the groups C_e contains an edge stabiliser C_e^* of T^* and therefore acts hyperbolically on T^*. It follows that the vertex group G_v containing C_e also acts hyperbolically on T^* and so is decomposable over elementary subgroups.

Let us now fix a vertex v and set $G = G_v$. The Lemma now follows from the following statement:

Sublemma 2.5. *[Be] Let G be the fundamental group of a Riemannian manifold M of finite volume of dimension $n > 2$ with pinched sectional curvature within $[a, b]$ for $a \le b < 0$. Then G does not split over a virtually nilpotent group.*

Proof: We provide below a direct proof of this Sublemma in the case of the constant curvature. Suppose, on the contrary, that

$$G = A *_C B \quad \text{or} \quad G = A*_C, \tag{3}$$

where C is an elementary subgroup. Let $\tilde{C}$ be the maximal elementary subgroup containing C. The group $\tilde{C}$ is virtually abelian and contains a maximal abelian subgroup $\tilde{C}_0$ of finite index. We have the following

Claim 2.6. *The group $\tilde{C}_0$ is separable in G.*

Proof: [2] Recall that the subgroup $\tilde{C}_0$ is said *separable* if $\forall g \in G \setminus \tilde{C}_0$ there exists a subgroup of finite index $G_0 < G$ such that $\tilde{C}_0 < G_0$ and $g \notin G_0$. Since $\tilde{C}_0$ is a maximal abelian subgroup of G, and $g \notin \tilde{C}_0$, it follows that there exists $h \in \tilde{C}_0$ such that $\gamma = gh_0g^{-1}h_0^{-1} \neq 1$. The group G is residually finite, so there exists an epimorphism $\tau : G \to K$ to a finite group K such that $\tau(\gamma) \neq 1$. Since $\tau(\tilde{C}_0)$ is abelian, $\tau(\gamma) \notin \tau(\tilde{C}_0)$ and the subgroup $G_0 = \tau^{-1}(\tau(\tilde{C}_0))$ satisfies our Claim. *QED.*

Denote $C_0 = C \cap \tilde{C}_0$ (the maximal abelian subgroup of C). We have $\tilde{C} = \bigcup_{i=1}^{m} c_iC_0 \cup C_0$. So by the Claim we can find a subgroup of finite index G_0 of G containing C_0 such that $c_i \notin G_0$ $(i = 1, ..., m)$. Then $G_0 \cap \tilde{C} = C_0$ is abelian group and by the Subgroup Theorem [SW] we have that G_0 splits as :

$$G_0 = A_0 *_{C_0'} B_0 \quad \text{or} \quad G_0 = A_0*_{C_0'}, \tag{3'}$$

where $C_0' < C_0$ is also abelian. Suppose first that $G_0 = A_0 *_{C_0'} B_0$, since G_0 is not elementary group, one of the vertex subgroups of this splitting, say A_0 is not elementary too. Then the map $\varphi : G_0 \to (cA_0c^{-1}) *_{C_0'} B_0,\ \ c \in C_0'$, such that $\varphi|_{A_0} = cA_0c^{-1}$ and $\varphi|_{B_0} = \mathrm{id}$ is an exterior automorphism (as c commutes with every element of C_0') of infinite order. So the group of the exterior automorphisms $\mathrm{Out}(G_0)$ is infinite. This contradicts to the Mostow rigidity as G_0 is still a lattice. In the case of HNN-extension $G_0 = A_0*_{C_0'} =< A_0, t \mid tC_0't^{-1} = \psi(C_0') >$ suppose first that t does not belong to the centralizer $Z(C_0')$ of C_0' in G_0. Then we put $\varphi|_{A_0} = cA_0c^{-1}$ for some $c \in C_0'$ such that $[c, t] \neq 1$ and $\varphi(t) = t$. Since $t \notin Z(C_0')$ we obtain again that φ is an infinite order exterior automorphism which is impossible. If, finally, $t \in Z(C_0')$ then put $\varphi|_{A_0} = id$ and $\varphi(t) = t^2$ and it is easy to see that $G_0' = \varphi(G_0)$ is a subgroup of index 2 of G_0 isomorphic to G_0. Then $\mathrm{Vol}(\mathbb{H}^n/\varphi(G_0)) < +\infty$ and again by Mostow rigidity we must have $\mathrm{Vol}(\mathbb{H}^n/G_0) = \mathrm{Vol}(\mathbb{H}^n/\varphi(G_0))$, and so $\varphi : G_0 \to G_0$ should be surjective. A contradiction. The Sublemma 2.5 and Lemma 2.3 follow. *QED.*

[2]The argument is due to M. Kapovich and one of the authors is thankful for sharing it with him (about 20 years ago).

3 Proof of the generalized Cooper inequality.

The aim of this Section is to prove Theorem B stated in the Introduction:

Theorem B. *Let E be an arbitrary family of elementary subgroups of G, then*

$$\mathrm{Vol}(M) \leq \pi \cdot T(\pi_1(M), E) \tag{1}$$

■

Proof: If $E = \emptyset$, then $\mathrm{Vol}(M) < \pi \cdot (L - 2n)$, where L is the sum of the word-lengths of the relations of $\pi_1(M)$ and n is the number of relations [C]. Let D be a disk representing a relation in the presentation complex R of $\pi_1(M)$. Then, triangulating D by triangles having vertices on ∂D, we obtain $|D| - 2$ triangles. So $L - 2n$ represents the total number of triangles in R. Thus Cooper's result implies $\mathrm{Vol}(M) \leq \pi \cdot T(\pi_1(M))$.

Suppose now that $M = \mathbb{H}^3/G$ where $G < \mathrm{Isom}(\mathbb{H}^3)$ is a lattice (uniform or not) and let E be a family of elementary subgroups of G. Let P be a simply-connected 2-dimensional polyhedron admitting a simplicial action of G such that the vertex stabilizers are elements of the system E. Let us also assume that the quotient $\Pi = P/G$ is a finite orbihedron. We will need the following:

Lemma 3.1. *There exists a G-equivariant simplicial continuous map $f : P \to \mathbb{H}^3 \cup \partial\mathbb{H}^3$ such that the images of the 2-simplices of P are geodesic triangles or ideal triangles of $\mathbb{H}^3$.*

Proof: Let us first construct a G-equivariant continuous map $f : P \to \overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \partial\mathbb{H}^3$ such that the image of the fixed points for the action G on P belong to $\partial\mathbb{H}^3$. To do it we apply the construction from [DePo, Lemma 1.6] where instead of a tree as the goal space we will use the hyperbolic space $\mathbb{H}^3$. Let us first construct a map $\rho : E \to \mathbb{H}^3$ as follows. Since the group G is torsion-free we can assume that all non-trivial groups in E are infinite. Then for every elementary group $E_0 \in E$ we put $\rho(E_0) = x \in \partial\mathbb{H}^3$ to be one of the fixed points for the action of E_0 on $\partial\mathbb{H}^3$ (by fixing a point $O \in \partial\mathbb{H}^3$ for the image of the trivial group $\rho(id)$). The map ρ has the following obvious properties :

a) $\forall E_1, E_2 \in E$ if $E_1 \cap E_2 \neq \emptyset$ then $\rho(E_1) = \rho(E_2)$;

b) if $\tilde{E}_0$ is a maximal elementary subgroup then $\rho(E_0) = \rho(\tilde{E}_0)$ and $\rho(g\tilde{E}_0 g^{-1}) = g\rho(\tilde{E}_0)$ $(g \in G)$.

We now choose the set of G-non-equivalent vertices $\{p_1, ..., p_l\} \subset P$ representing all vertices of $\Pi = P/G$. We first construct a map f on zero-skeleton $P^{(0)}$ of the complex P by putting $f(p_i) = \rho(E_i)$ and then extend it equivariantly $f(gp_i) = gf(p_i)(g \in G)$.

Suppose now $y = (q_1, q_2)$ $(q_1,\ q_2 \in P^{(0)})$ is an edge of P. To define f on y we distinguish two cases: 1) $H = \mathrm{Stab}(y) \neq 1$ and 2) $H = 1$.

In the first case we have necessarily that $E_{q_1} \cap E_{q_2} = H_0$ is an infinite elementary group where E_{q_i} is the stabilizer of q_i. Then there exist $g_i \in G$ such that $q_i = g_i(p_{k_i})$ $(i = 1, 2)$. So $E_{q_i} = g_i E_{p_{k_i}} g_i^{-1}$ and $g_1 E_{p_1} g_1^{-1} \cap g_2 E_{p_2} g_2^{-1} = H_0$. It follows that $E_{p_1} \cap g_1^{-1} g_2 E_{p_1} g_2^{-1} g_1$ is an infinite group and, therefore $f(p_1) = g_1^{-1} g_2(f(p_2))$ implying that

$$f(q_1) = f(g_1 p_1) = f(g_2 p_2) = f(q_2).$$

In the case 2) the stabilizer of the infinite geodesic $l =]f(q_1), f(q_2)[\subset \mathcal{P}$ is trivial so we extend $f : y \to l$ by a piecewise-linear homeomorphism. Having defined the map f as above on the maximal set of non-equivalent edges of $P^{(1)}$ under G, we extend it equivariantly to the 1-skeleton $P^{(1)}$ by putting $f(gy) = gf(y)$ $(g \in G)$. Finally we extend f piecewise linearly to the 2-skeleton $P^{(2)}$.

We obtain a G-equivariant continuous map $f : P \to \overline{\mathbb{H}^3}$ such that the all 2-faces of the simplicial complex $f(P) \cap \mathbb{H}^3$ are ideal geodesic triangles. The Lemma is proved. *QED.*

Remarks 3.2. *1. Note that the above Lemma is true in any dimension. We restricted our consideration to dimension 3 since the further argument will only concern this case.*

2. If the system E contains only parabolic subgroups one can claim that the action of G on $f(P) \cap \mathbb{H}^3$ is in addition proper. Indeed, using the convex hull $\mathcal{P} \subset \mathbb{H}^3$ of the maximal family of non-equivalent parabolic points constructed in [EP] the above argument gives the map $f : P \to \overline{\mathcal{P}} \subset \overline{\mathbb{H}^3}$. By [EP, Proposition 3.5] the set of faces of $\mathcal{P}$ is locally finite in $\mathbb{H}^3$. Since the boundary of each face of the 2-orbihedron $f(P)$ constructed above belongs to $\partial\mathcal{P}$, we obtain that the set of 2-faces of $f(P) \subset \mathbb{H}^3$ is locally finite in this case.

■

If now W is the set of the fixed points for the action of G on P, we put $P' = P \setminus W$ and $Q' = f(P') = f(P) \cap \mathbb{H}^3$. Let also $\nu : P \to \Pi$ and $\pi : \mathbb{H}^3 \to M = \mathbb{H}^3/G$ denote the natural projections. Then by Lemma 3.1 the map f projects to a simplicial map $F : (\Pi' = P'/G) \to Q'/G \subset M$ such that the following diagram is commutative:

$$\begin{array}{ccc} P' & \xrightarrow{f|_{P'}} & Q' \subset \mathbb{H}^3 \\ \nu \big\downarrow & & \pi \big\downarrow \\ \Pi' & \xrightarrow{F} & Q'/G \subset M \end{array}$$

Note that, if Π is a simplicial polyhedron, it is proved in [C] that the hyperbolic area of $F(\Pi)$ bounds the volume of the manifold M. This argument does not work if Π is an orbihedron but not a polyhedron. Indeed the complex Q' above is not necessarily simply connected. So the group G is not isomorphic to $\pi_1(Q'/G)$ but is a non-trivial quotient of it. Our goal now is to construct a new simplicial polyhedron Σ with the fundamental group G whose image into M has area arbitrarily close to that of $F(\Pi')$. So the main step in the proof of Theorem B is the following :

Proposition 3.3. *(simplicial blow-up procedure). For every $\varepsilon > 0$ there exists a 2-dimensional complex Σ_ε and a simplicial map $\varphi_\varepsilon : \Sigma_\varepsilon \to M$ such that*

1) The induced map $\varphi_\varepsilon : \pi_1\Sigma_\varepsilon \to M$ is an isomorphism.

and

2) For the hyperbolic area one has:

$$|\mathrm{Area}(\varphi_\varepsilon(\Sigma_\varepsilon)) - \mathrm{Area}(F(\Pi'))| < \varepsilon.$$

Proof of the Proposition: Let Π be a finite orbihedron with elementary vertex groups and such that $\pi_1^{\mathrm{orb}}(\Pi) \cong G$. Let us fix a vertex σ of Π and let $\widetilde{\sigma} \in \nu^{-1}(\sigma)$ be its lift in P. We denote by G_σ the group of the vertex σ in G. By Lemma 3.1 the point $f(\widetilde{\sigma}) \in \partial\mathbb{H}^3$ is fixed by the elementary group G_σ. We will distinguish between the two cases when the group G_σ is loxodromic cyclic or parabolic subgroup of rank 2.

Case 1. The group G_σ is loxodromic.

Let $V \subset \Pi$ be a regular neighborhood of the vertex σ. Then the punctured neighborhood $V \setminus \sigma$ is homotopically equivalent to the one-skeleton $L^{(1)}$ of the link L of σ.

We will call *realization* of L a graph $\Lambda \subset V \setminus \sigma$ such that the canonical map $L \to \Lambda$ is a homeomorphism. Let us fix a maximal tree T in Λ, and let y_i be the edges from $\Lambda \setminus T$ which generate the group $\pi_1(L)$ $(i = 1, ..., k)$.

By its very definition, the G-equivariant map $f : P \to \mathbb{H}^3$ sends the edges of P to geodesics of $\mathbb{H}^3$. So let $G_\sigma =< g >$ and let $\gamma \subset M$ be the corresponding closed geodesic in M. We denote by $A_g \subset \mathbb{H}^3$ the axis of the element g and by g^+, g^- its fixed points on $\partial\mathbb{H}^3$. Let us assume that $f(\widetilde{\sigma}) = g^+$. For $X \subset M$ we denote by diam(X) the diameter of X in the hyperbolic metric of M.

Recall that the map $f : P \to \mathbb{H}^3 \cup \partial\mathbb{H}^3$ constructed in Lemma 3.1 induces the map $F : \Pi' \to M$. We start with the following:

Step 1. *For every $\eta > 0$ there exists a realization Λ of L in Π such that for the maximal tree T of Λ one has*

$$\mathrm{diam}(F(T)) < \eta,$$

Furthermore, for every edge $y_i \in \Lambda \setminus T$ its image $F(y_i)$ is contained in a η-neighborhood $N_\eta(\gamma) \subset M$ of the geodesic γ (i=1,...,k).

Proof: We fix a sufficiently small neighborhood V of a vertex σ in Π (the "smalleness" will be specified later on). Let $\widetilde{\sigma} \in \nu^{-1}(\sigma)$ be its lift to P and let $\widetilde{\Lambda}$ and $\widetilde{T}$ be the lifts of Λ and T to a neighborhood $\widetilde{V} \subset \nu^{-1}(V)$ of $\widetilde{\sigma}$. We are going first to show that, up to decreasing V, the image $f(\widetilde{T})$ belongs to a sufficiently small horosphere in $\mathbb{H}^3$ centered at the point g^+.

Let α be an edge of Π having σ as a vertex and $\widetilde{\alpha}$ be its lift starting at a point $\widetilde{\sigma}$. Then $a = f(\widetilde{\alpha}) \subset \mathbb{H}^3$ is the geodesic ray ending at the point g^+, let $a(t)$ be its parametrization. For a given t_0 we fix a horosphere S_{t_0} based at g^+ and passing through the point $a(t_0)$. Suppose there is a simplex in P having two edges $\widetilde{\alpha} = [\widetilde{\sigma}, s], \widetilde{\alpha}_1 = [\widetilde{\sigma}, s_1]$ at the vertex $\widetilde{\sigma}$ and an edge $[s, s_1]$ in Λ. The horosphere S_{t_0} is the level set of the Busemann function β_{g^+} based at the point g^+. So for the geodesic rays $a = f(\widetilde{\alpha})$ and $a_1 = f(\widetilde{\alpha}_1)$ issuing from the point g^+ we have that the points $f(s) = a(t_0)$ and $f(s_1) = a_1(t_0)$ belong to the horosphere S_{t_0}. Proceeding in this way for all simplices whose edges share the vertex σ, we obtain that $f(\widetilde{T}^{(0)}) \subset S_{t_0} \subset \mathbb{H}^3$. Since Λ is finite, so is the tree $\widetilde{T}$. By choosing t_0 sufficiently large ($t_0 > \Delta$) we may assume that $d(\alpha_i(t_0), \alpha_j(t_0)) < \eta$ and $d(\alpha_i(t_0), A_g) < \eta$ $(i, j = 1, ..., k)$. We now connect all the vertices of $f(\widetilde{T})$ by geodesic segments $b_i \subset \mathbb{H}^3$. By convexity, and up to increasing the parameter t_0, we also have $d(b_i, A_g) < \eta$.

By Lemma 3.1 the map f sends the lifts $\widetilde{y}_i \in \widetilde{T}$ of the edges $y_i \in \Lambda \setminus T$ simplicially to b_i $(i = 1, ..., k)$; and f maps G_σ-equivariantly the preimage $\widetilde{\Lambda} = \nu^{-1}(\Lambda)$ to $\mathbb{H}^3$. Hence the map f projects to the map $F : \Lambda \to M$ satisfying the claim of Step 1. ∎

Step 2. *Definition of the polyhedron* $\Pi\check{}$

Using the initial orbihedron Π we will construct a new polyhedron $\Pi\check{}$ having the following properties :

a) $\Pi^{(0)} = \Pi\check{}^{(0)}$ and $\Pi = \Pi\check{}$ outside of V;

b) $\pi_1(L^*) = G_\sigma$, where L^* is the link of σ in $\Pi\check{}$;

c) $\pi_1(\Pi\check{}) \cong G$.

The graph Λ realizes the link of the vertex σ so there exists an epimorphism $\pi_1(\Lambda) \to < g >$. Every edge $y_i \in \Lambda \backslash T$ which is a generator of the group $\pi_1 \Lambda$ is mapped onto $g^{n_{y_i}}$ in G_σ $(i = 1, ..., k)$. We now subdivide each edge y_i by edges y_{ij} $(i = 1, ..., k, j = 1, ..., n_{y_i})$, and denote by Λ' the obtained graph. Let S be a circle considered as a graph with one edge e and one vertex u. Then there exists a simplicial map from Λ' to S mapping simplicially each edge y_{ij} onto S.

To construct polyhedron $\Pi\check{}$, we replace the neighborhood V by the cone of the above map. Namely, we first delete the vertex σ from Π as well as all edges connecting σ with L. Then we connect the vertices of the edge y_{ij} with the vertex $u \in S$ by edges which we call *vertical*

$(i = 1, ..., k, j = 1, ..., n_{y_i})$. So $\Pi^{\vee}$ is the union of $\Pi \setminus V$ and the rectangles R_{ij}, which are bounded by y_{ij}, two vertical edges and the loop S. The set of rectangles $\{R_{ij} \mid i = 1, ..., k,\ j = 1, ..., n_{y_i}\}$ realizes the epimorphism $\pi_1(L) \to G_\sigma$. By Van-Kampen theorem we have $\pi_1(\Pi^{\vee}) \cong G$, and the conditions a)-c) follow. ■

Step 3. *There exists a constant c (depending only on the topology of* Π*) such that for all* $\eta > 0$*, there exists a map* $F^{\vee} : \Pi^{\vee} \to M$ *such that*

1) $F^{\vee}$ *induces an isomorphism on the fundamental groups,*

2) $F^{\vee}|_{\Pi^{\vee} \setminus V} = F$,

3) $$\sum_{ij} \text{Area}(F^{\vee}(R_{ij})) < c \cdot \eta. \tag{2}$$

■

Proof: We choose a neighborhood V of the singular point σ and put $F^{\vee} = F|_{\Pi \setminus V}$. Using Step 2 we transform the orbihedron Π to $\Pi^{\vee}$ in the neighborhood V and let $P^{\vee}$ be the universal covering of $\Pi^{\vee}$. Note that, by construction, $P^{\vee}$ is obtained by adding the G-orbit of the rectangles R_{ij} to the preimage $\widetilde{\Lambda}' = \nu^{-1}(\Lambda')$ of the graph Λ' $(i = 1, ..., k,\ j = 1, ..., n_{y_i})$.

We will now extend the map f defined on $P \setminus V$ to the polyhedron $P^{\vee} \setminus P$ as follows. We first subdivide every segment b_i in n_{y_i} geodesic subsegments $b_{ij} \subset b_i$ corresponding to the edges y_{ij}. We now project orthogonally each b_{ij} to A_g and let $\widetilde{\gamma} \subset A_g$ denote its image. Let $\tau_{ij} \subset \mathbb{H}^3$ be the rectangle formed by b_{ij}, $\widetilde{\gamma}$ and these two orthogonal segments from b_{ij} to A_g whose lengths are by Step 1 less than η. We extend the map f simplicially to a map $f^{\vee}$ sending the rectangle $\nu^{-1}(R_{ij})$ to the rectangle τ_{ij} $(i = 1, ..., k,\ j = 1, ..., n_{y_i})$. Note that by construction the lift $\widetilde{S}$ of the circle S is mapped on $\widetilde{\gamma}$. The map $f^{\vee}$ descends to a map $F^{\vee} : \Pi_* \setminus \Pi \to N_\eta(\gamma)$. It induces the epimorphism $\pi_1 \Pi^{\vee} \to G$.

Let us now make the area estimates for the added rectangles τ_{ij}. Each rectangle $\tau = \tau_{ij}$ has four vertices A, B, C, D in $\mathbb{H}^3$ where $B = gA, D = g(C)$ and the segment $[A, B] \subset A_g$ is the orthogonal projection of $[C, D]$ on A_g. The rectangle τ is bounded by these two segments and two perpendicular segments $l_1 = [A, C]$ and $l_2 = [B, D]$ to the geodesic A_g $(l_2 = g(l_1))$. We have $\tau \subset ABC'D$ where $\angle BDC' = \frac{\pi}{2}$ and $\beta = \angle BC'D < \frac{\pi}{2}$. Then by [Be, Theorem 7.17.1] one has $\cos(\beta) \leq \sinh(d(B, D)) \cdot \sinh l(\gamma)$. Therefore $\text{Area}(\tau) < \frac{\pi}{2} - \beta$, and $\sin(\text{Area}(\tau)) \leq \sinh \eta \cdot \sinh l(\gamma)$. Summing up over all segments b_{ij} we arrive to the formula (2). This proves Case 1. ■

Case 2. The group G_σ is parabolic.

The proof is similar and even simpler in this case. Let again T be the maximal tree of the graph Λ realizing the link L of the vertex σ. We start by embedding a lift $\widetilde{T}^{(0)}$ of the zero-skeleton of T^0 into a horosphere $S_{t_0} \subset \mathbb{H}^3$ based at the parabolic fixed point $p \in \partial\mathbb{H}^3$ of the group $G_\sigma =< g_1, g_2 >\cong \mathbb{Z} + \mathbb{Z}$. Then, using Lemma 3.1, we construct an embedding $f : \widetilde{\Lambda}^{(0)} \to S_{t_0}$ of the zero-skeleton of the graph $\widetilde{\Lambda} = \nu^{-1}(\Lambda)$ into the same horosphere S_{t_0} invariant under G_σ (which was not so in the previous case). Since the number of vertices of $\widetilde{T}$ is finite, for any $\eta > 0$ we can choose a horosphere S_{t_0} $(t_0 > \Delta)$ such that diam$\widetilde{T} < \eta$. Fixing a point $O \in S_{t_0}$, we can also assume that $d(O, \widetilde{T}^{(0)}) < \eta$.

Now, let us modify the orbihedron Π in the neighborhood V of σ. First we delete the vertex σ from Π and all edges connecting σ with the graph Λ. We then add to the obtained orbihedron a torus $\mathcal{T}$ with two intersecting loops C_1 and C_2 representing the generators of $\pi_1(T, u)$ where $u \in C_1 \cap C_2$. To realize the epimorphism $\pi_1\Lambda \to G_\sigma$ in M we proceed as before. For any edge $y \in \Lambda \setminus T$ corresponding to the element $g = ng_1 + mg_2$ in G_σ we add a rectangle R bounded by y, two edges connecting the end points of y with u and a loop $C \subset \mathcal{T}$ representing the element g in $\pi_1(T, u)$. Let $\Pi\check{}$ denote the obtained orbihedron.

Coming back to $\mathbb{H}^3$, let us assume for simplicity that $p = \infty$ and the horosphere S_{t_0} is a Euclidean plane. By Lemma 3.1 the map f sends the edges $\widetilde{y}_i \in \widetilde{\Lambda} \setminus \widetilde{T}$ to the geodesic edges b_i connecting the vertices of $f(\widetilde{T})$.

We now construct the rectangles τ_i by projecting the end points of the edges b_i to the corresponding vertices of the Euclidean lattice given by the orbit $G_\sigma O$. Let us briefly describe this procedure in case of one rectangle τ. Suppose that the edge $y \in \Lambda \setminus T$ represents the element $g = ng_1 + mg_2 \in G_\sigma$. Let A and gA be vertices of $f(\widetilde{T})$ belonging to S_{t_0} connected by a geodesic segment b corresponding to y. Let $\tau \subset \mathbb{H}^3$ be the geodesic bounded by the edges $b, l = [O, A], gl, gb$. We extend the map $f\check{} : \widetilde{R} \to \tau$ where $\widetilde{R}$ is a lift of the corresponding rectangle R added to Π. The map $f\check{}$ descends now to a simplicial map $F\check{} : \Pi\check{} \to M$ sending the torus $\mathcal{T}$ into a cusp neighborhood of the manifold M. Since the rectangle τ belongs to η-neighborhood of the horosphere S_{t_0}, its area, being close to the Euclidean one, is bounded by $c \cdot \eta^2$ for some constant $c > 0$. Summing up over all edges y_i we obtain that the area of added rectangles does not exceed $k \cdot c \cdot \eta^2$. This proves Case 2. ■

To finish the proof of Proposition 3.3, we note that the initial orbihedron Π is finite, so it has a finite number of vertices $v_1, ..., v_l$ whose vertex groups are either loxodromic or parabolic. So for a fixed $\varepsilon > 0$, we apply the above simplicial "blow-up" procedure in a neighborhood of each vertex v_i $(i = 1, ..., l)$. Finally, we obtain a 2-complex Σ_ε; and the simplicial map $\phi_\varepsilon : \Sigma_\varepsilon \to M$ which induces an isomorphism on the fundamental groups and such that $|\text{Area}(\varphi_\varepsilon(\Sigma_\varepsilon)) - \text{Area}(f(\Pi'))| < \psi(\eta)$, where ψ is a continuous function such that $\lim_{\eta\to 0} \psi(\eta) = 0$. So for η sufficiently small we have $\psi(\eta) < \varepsilon$ which proves the Proposition. *QED.*

Proof of Theorem B. Let G be the fundamental group of a hyperbolic 3-manifold M of finite volume. Let $\Pi = P/G$ be a finite orbihedron realizing the invariant $T(G, E)$, i.e. $\pi_1^{\text{orb}}(\Pi) \cong G$, all vertex groups of Π are elementary and $|\Pi^{(2)}| = T(G, E)$. Hence $\text{Area}(F(\Pi')) = \pi \cdot T(G, E)$.

Then by Proposition 3.3 for any $\varepsilon > 0$ there exists a 2-polyhedron Σ_ε and a map $\psi_\varepsilon : \Sigma_\varepsilon \to M$ which induces an isomorphism on the fundamental groups and such that

$$\mathrm{Area}(\psi_\varepsilon(\Sigma_\varepsilon)) < \pi T(G, E) + \varepsilon$$

By [C] we have $\mathrm{Vol}M < \mathrm{Area}(\psi_\varepsilon(\Sigma_\varepsilon)) < \pi T(G, E) + \varepsilon$ ($\forall \varepsilon > 0$). It follows $\mathrm{Vol}M \leq \pi T(G, E)$. Theorem B is proved. *QED.*

4 Proof of Theorem A.

In this Section we finish the proof of

Theorem A. *There exists a constant C such that for every hyperbolic 3-manifold M of finite volume the following inequality holds*:

$$C^{-1}T(G, E_\mu) \leq \mathrm{Vol}(M) \leq CT(G, E_\mu) \qquad (*)$$

■

The right-hand side of the inequality (*) follows from Theorem B if one puts $E = E_\mu$. So we only need to prove the left-hand side of (*). We start with the following Lemma dealing with n-dimensional hyperbolic manifolds :

Lemma 4.1. *Let M be a n-dimensional hyperbolic manifold of finite volume. Then there exists a 2-dimensional triangular complex $W \subset M_{\mu\mathrm{thick}}$ such that $\pi_1(W) \hookrightarrow \pi_1 M_{\mu\mathrm{thick}}$ is an isomorphism and*

$$|W^2| \leq \sigma \cdot \mathrm{Vol}(M),$$

where $|W^2|$ is the number of 2-simplices of W and $\sigma = \sigma(\mu)$ is a constant depending only on μ.

Proof: The Lemma is a quite standard fact, proved for $n = 3$ in [Th] and more generally in [G], [BGLM], [Ge]. We provide a short proof of it for the sake of completeness. Consider a maximal set of points $\mathcal{A} = \{a_i \mid a_i \in M_{\mu\mathrm{thick}},\ d(a_i, a_j] > \mu/4\}$ where $d(\cdot, \cdot)$ is the hyperbolic distance of M restricted to $M_{\mu\mathrm{thick}}$. By the triangle inequality we obtain

$$B(a_i, \mu/8) \cap B(a_j, \mu/8) = \emptyset \ \text{ if } i \neq j,$$

where $B(a_i, \mu)$ is an embedded ball in M (isometric to a ball in $\mathbb{H}^n$) centered at a_i of radius μ. By the maximality of $\mathcal{A}$ we have $M_{\mu\mathrm{thick}} \subset \mathcal{U} = \bigcup_i B(a_i, \mu/4)$. Recall that the nerve $N\mathcal{U}$ of the covering $\mathcal{U}$ is constructed as follows. Let $N\mathcal{U}^0 = \mathcal{A}$ be the vertex set. The vertices $a_{i_1}, ..., a_{i_{k+1}}$

span a k-simplex if for the corresponding balls we have $\bigcap_{j=1}^{k+1} B(a_{i_j}, \mu/4) \neq \emptyset$. Since the covering $\mathcal{U}$ is given by balls embedded into M, the nerve $N\mathcal{U}$ is homotopy equivalent to $\mathcal{U}$ [Hat, Corollary 4G.3].

Note that $M_{\mu\text{thick}} \hookrightarrow \mathcal{U} \hookrightarrow M_{\frac{\mu}{2}thick}$. Indeed if $x \in \partial B(a_i, \mu/4)$ then by the triangle inequality we have $B(x, \mu/4) \subset B(a_i, \mu/2)$, and so both are embedded in M. Then $x \in M_{\frac{\mu}{2}thick}$. By the Margulis lemma, as the corresponding components of their thin parts are homeomorphic, the embedding $M_{\mu\text{thick}} \hookrightarrow M_{\frac{\mu}{2}thick}$ is a homotopy equivalence. It implies that the complex $N\mathcal{U}$ is homotopy equivalent to $M_{\mu\text{thick}}$. Let W denote the 2-skeleton of $N\mathcal{U}$. Then it is a standard topology fact that W carries the fundamental group of $N\mathcal{U}$ [Hat]. Therefore, $\pi_1 W \cong \pi_1 M_{\mu\text{thick}}$.

It remains to count the number of 2-faces of W. We have for the cardinality $|\mathcal{A}|$ of the set $\mathcal{A}$:

$$|\mathcal{A}| \leq \frac{\mathrm{Vol(M_{\mu thick})}}{\mathrm{Vol(B(\mu/8))}} \leq \frac{\mathrm{Vol(M)}}{\mathrm{Vol(B(\mu/8))}} \ ,$$

where $B(\mu)$ denotes a ball of radius μ in the hyperbolic space $\mathbb{H}^n$. The number of faces of W containing a point of $\mathcal{A}$ as a vertex is at most $m = \frac{\mathrm{Vol(B(\mu/2))}}{\mathrm{Vol(B(\mu/8))}}$. Then

$$|W^{(2)}| \leq C_m^2 \ \frac{\mathrm{Vol(M)}}{\mathrm{Vol(B(\mu/8))}} = \sigma \cdot \mathrm{Vol(M)} \ ,$$

where $\sigma = \sigma(\mu) = \frac{C_m^2}{\mathrm{Vol(B(\mu/8))}}$. This completes the proof of the Lemma. ■

Suppose now that M is a hyperbolic 3-manifold of finite volume and let $\mu = \mu(3)$ be the 3-dimensional Margulis constant. We are going to use a result of [De] which we need to adapt to our Definition 1.2 of the invariant T. So we start with the following:

Remark 4.2. *In the definition of the invariant T in [De] there is one more additional condition compared to our Definition 1.2. Namely, it requires that every element of a system E fixes a vertex of P. To be able to use the results of [De] we will denote by $T_0(G, E)$ the invariant defined in [De] and keep the notation $T(G, E)$ for that of our Definition 1.2. Notice that nothing changes for the absolute invariant $T(G)$.*

Let $l_1, ..., l_k$ be the set of closed geodesics in M of length less than μ. Then by [Ko] the manifold $M' = M \setminus \bigcup_{i}^{k} l_i$ is a complete hyperbolic manifold of finite volume and $\pi_1 M_{\mu\text{thick}} \cong \pi_1(M)'$.

Let $\mathcal{E}_\mu$ denote the system $\pi_1(\partial M_{\mu\text{thick}})$ of fundamental groups of the boundary components of the thick part $M_{\mu\text{thick}}$. We have the following :

Lemma 4.3.

$$T_0(\pi_1(M), \mathcal{E}_\mu) \le T_0(\pi_1(M'), \pi_1(\partial M')) \le T_0(\pi_1(M), \mathcal{E}_\mu) + 2k. \tag{5}$$

Proof: 1) Consider first the left-hand side. Let $G = \pi_1(M)$ and $G' = \pi_1(M')$. Let $\mathcal{E}'_\mu = \{E_{k+1}, ..., E_n\}$ be the set of fundamental groups of cusps of $M_{\mu\text{thin}}$. Let us fix a two-dimensional $(G', \mathcal{E}'_\mu)$-orbihedron P' containing $T_0(G', \mathcal{E}'_\mu)$ triangular 2-faces. The pair $(G', \mathcal{E}'_\mu)$ acts on its orbihedral universal cover P' [H]. Let $N(l_i)$ be a regular neighborhood of the geodesic $l_i \in M$ ($i = 1, ..., k$) and $H_i =< \alpha_i, \beta_i >$ be the fundamental group of the torus $T_i = \partial N(l_i)$ where α_i is freely homotopic to l_i in $N(l_i)$. The group H_i fixes a point $x_i \in P'$. We will now construct a 2-orbihedron P for the couple (G, E_μ) as follows. The group G is the quotient of G' by adding the relation $\beta_i = 1$ ($i = 1, ..., k$). We identify the vertices of P' equivalent under the groups generated by β_i ($i = 1, ..., k$). The natural projection map $P' \to P$ consists of contracting each edge of P' of the type $(y, \beta_i(y))$ ($y \in P'^{(0)}$) to a point. The projection has connected fibres so the 2-orbihedron P is simply connected and the pair (G, E_μ) acts on it. The procedure did not increase the number of 2-faces, and we have : $|\Pi^{(2)} = P/G| \le |\Pi'^{(2)} = P'/G'|$. Thus $T_0(\pi_1(M), E_\mu) \le T_0(\pi_1(M)', \pi_1(\partial M') = \mathcal{E}'_\mu)$.

2) Let Π be the 2-orbihedron which realizes $T_0(\pi_1(M), \mathcal{E}_\mu)$, and let P be its universal cover. To obtain a $(\pi_1(M)', \mathcal{E}'_\mu)$-orbihedron we modify P as follows. Let $H_i =< h_i >$ be the loxodromic subgroup corresponding to the geodesic $l_i \subset M$ of length less than μ ($i = 1, ..., k$). Let $x_i \in P$ be a vertex fixed by the subgroup H_i. Notice that the group G' is generated by G and elements β_i such that $[h_i, \beta_i] = 1$ ($i = 1, ..., k$). So we add to Π a new loop β_i (by identifying it with the corresponding element in G) and glue a disk whose boundary is the loop corresponding to $[h_i, \beta_i]$. By triangulating each such a disk we add $2k$ new triangles to $\Pi^{(2)}$. Thus the universal cover P' is obtained by adding to P a vertex y_i and its orbit $\{Gy_i\}$, so that the points $\beta_i h_i g y_i$ are identified with $h_i \beta_i g y_i$. We further add the rectangle gD_i ($g \in G$) whose vertices are $h_i g y_i, \beta_i h_i g y_i, \beta_i g y_i, g y_i$ and subdivide it by one of the diagonal edges, say $(h_i g y_i, \beta_i g y_i)$ ($i = 1, ..., k$). The construction gives a new 2-complex P' on which the pair $(G', \mathcal{E}'_\mu)$ acts simplicially. We claim that P' is simply connected. Indeed if α is a loop on it, since P is simply connected, α is homotopic to a product of loops belonging to the disks gD_i so α is a trivial loop. Since the 2-orbihedron $\Pi' = P'/G'$ contains $|\Pi^{(2)}|+2k$ faces, we obtain $T_0(\pi_1(M)', \pi_1(\partial M')) \le T_0(\pi_1(M), \mathcal{E}_\mu)+2k$ which was promised. *QED.*

Remark 4.4. *It is worth pointing out that in the context of volumes of hyperbolic 3-manifolds the following inequality (similar to (5)) is known:*

$$\mathrm{Vol}(M) < \mathrm{Vol}(M') < k \cdot (C_1(R) \cdot \mathrm{Vol}(M) + C_2(R)), \tag{\dagger}$$

where R is the maximum of radii of the embedded tubes around the short geodesics l_i ($i = 1, ..., k$) and $C_i(R)$ are functions of R ($i = 1, 2$). The left-hand side of (†) is classical and due to W. Thurston [Th], the right-hand side is proved recently by I. Agol, P. A. Storm, and W. Thurston [AST] ■

Proof of the left-hand side of the inequality (*): By Lemma 4.1 the thick part $M_{\mu\text{thick}}$ of M contains a 2-dimensional complex W such that $\pi_1 W \hookrightarrow \pi_1 M_{\mu\text{thick}}$ is an isomorphism and $|W^{(2)}| < \sigma \cdot \mathrm{Vol}(M)$ for some uniform constant σ. Consider now the double $N = DM_{\mu\text{thick}}$ of the manifold $M_{\mu\text{thick}}$ along the boundary $\partial M_{\mu\text{thick}}$. By repeating the argument of Lemma 4.1 to each half of N we obtain two complexes W and $\tau(W)$ embedded in N where $\tau : N \to N$ is the involution such that $M_{\mu\text{thick}} = N/\tau$. By Van-Kampen theorem the fundamental group of the complex $V = W \cup \tau(W)$ is generated by $\pi_1 W$ and $\pi_1(\tau(W))$ and is isomorphic to $\pi_1(N)$. Furthermore, for the number of two-dimensional faces in V we have $|N^{(2)}| = 2|W^{(2)}|$. So by Lemma 4.1 $T(\pi_1 N) \le |V^{(2)}| < 2\sigma \cdot \mathrm{Vol}(M)$. The group $\pi_1 N$ splits as the graph of groups whose two vertex groups are $\pi_1 M_{\mu\text{thick}}$. The edge groups of the graph of groups are given by the system $\mathcal{E}_\mu$. As $\pi_1 M_{\mu\text{thick}} \cong \pi_1(M)'$ and M' is a complete hyperbolic 3-manifold of finite volume it follows from Lemma 2.3 that the above splitting is reduced and rigid. So by [De] we have:

$$T(\pi_1 N) \ge 2T_0(\pi_1 M_{\mu\text{thick}}, \mathcal{E}_\mu). \tag{6}$$

Then by Lemma 4.3 $T_0(\pi_1 M_{\mu\text{thick}}, \mathcal{E}_\mu) \ge T_0(\pi_1(M), \mathcal{E}_\mu)$, and therefore

$$\sigma^{-1} \cdot T_0(\pi_1(M), \mathcal{E}_\mu) < \mathrm{Vol}(M).$$

Recall that the initial system E_μ of elementary subgroups includes all elementary subgroups of $\pi_1(M)$ whose translation length is less than μ. So $\mathcal{E}_\mu \subset E_\mu$ implying that $T(\pi_1(M), E_\mu) \le T_0(\pi_1(M), \mathcal{E}_\mu)$. We finally obtain

$$C^{-1} \cdot T(\pi_1(M), E_\mu) < \mathrm{Vol}(M),$$

where $C = \sigma$. The left-hand side of (*) is now proved. Theorem A follows. ■

5 Concluding remarks and questions.

The finiteness theorem of Wang affirms that there are only finitely many hyperbolic manifolds of dimension greater than 3 having the volume bounded by a fixed constant [W]. So it is natural to compare the volume of a hyperbolic manifold $M = \mathbb{H}^n/\Gamma$ with the absolute invariant $T(\Gamma)$. In the case $n > 3$ the inequality

$$\mathrm{const} \cdot T(\Gamma) \le \mathrm{Vol}(M)$$

follows from [Ge, Thm 1.7] (see also Section 2 above, where instead of $T(\pi_1(M), E)$ one needs to consider $T(\pi_1(M))$ and use the fact that $\pi_1 M_{\mu\text{thick}} \cong \pi_1(M)$). However, the result [C] is not known in higher dimensions. Thus we have the following :

Question 5.1. *Is there a constant C_n such that for every lattice Γ in $\mathrm{Isom}(\mathbb{H}^n)$ one has*

$$\mathrm{Vol}(\Gamma) \le C_n \cdot T(\Gamma) \ ?$$

Remark 5.2. *(M. Gromov) The answer is positive if M is a compact hyperbolic manifold of dimension 4. Indeed in this case by the Gauss-Bonnet formula one has* $\mathrm{Vol}(M) = \frac{\Omega_4}{2} \cdot \chi(M)$, *where* Ω_4 *is the volume of the standard unit 4-sphere. Hence* $\mathrm{Vol}(M) < \frac{\Omega_4}{2} \cdot (2 - 2b_1 + b_2)$ *where* $b_i = \mathrm{rank}(H_i(M, \mathbb{Z}))$ *is the i-th Betti number of* M $(i = 1, 2)$. *Since* $b_2 < T(\pi_1(M))$, *one has* $\mathrm{Vol}(M) < \frac{\Omega_4}{2} \cdot (2 + b_2) < \Omega_4 \cdot T(\pi_1(M))$ (as $T(\pi_1(M)) > 1$).

Recently it was shown by D. Gabai, R. Meyerhoff, and P. Milley that the Matveev-Weeks 3-manifold M_0 is the unique closed 3-manifold of the smallest volume [GMM]. Furthermore, C. Cao and R. Meyerhoff found cusped 3-manifolds $m003$ and $m004$ of the smallest volume [CM], [GMM]. In this context we have the following :

Question 5.3. *Is the invariant* $T(\pi_1(M), E_\mu)$ *on the set of compact hyperbolic 3-manifolds attained on the manifold* M_0 *? Is the minimal relative invariant* $T(\pi_1(M), E_\mu)$ *on the set of cusped finite volume 3-manifolds attained on the manifolds* $m003$ *and* $m004$ *?*

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