FLEXIBLE VARIETIES AND AUTOMORPHISM GROUPS

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Abstract. Given an irreducible affine algebraic variety X of dimension $n \geq 2$, we let $\mathrm{SAut}(X)$ denote the special automorphism group of X i.e., the subgroup of the full automorphism group $\mathrm{Aut}(X)$ generated by all one-parameter unipotent subgroups. We show that if $\mathrm{SAut}(X)$ is transitive on the smooth locus X_{reg} then it is infinitely transitive on X_{reg}. In turn, the transitivity is equivalent to the flexibility of X. The latter means that for every smooth point $x \in X_{\mathrm{reg}}$ the tangent space T_xX is spanned by the velocity vectors at x of one-parameter unipotent subgroups of $\mathrm{Aut}(X)$. We provide also various modifications and applications.

Introduction

All algebraic varieties and algebraic groups in this paper are supposed to be reduced and defined over an algebraically closed field $\Bbbk$ of characteristic zero. For such a variety X we say that a subgroup H of the automorphism group $\mathrm{Aut}(X)$ is *algebraic* if it admits a structure of an algebraic group such that the natural map $H \times X \to X$ is a morphism. We let $\mathrm{SAut}(X)$ denote the subgroup of $\mathrm{Aut}(X)$ generated by all algebraic one-parameter unipotent subgroups of $\mathrm{Aut}(X)$ i.e., algebraic subgroups isomorphic to the additive group $\mathbb{G}_a$. The group $\mathrm{SAut}(X)$ is called the *special automorphism group* of X; this is a normal subgroup of $\mathrm{Aut}(X)$. In this paper we study transitivity properties of the action of $\mathrm{SAut}(X)$ on an irreducible variety X.

For instance, the special automorphism group $\mathrm{SAut}(\mathbb{A}^1)$ of the affine line is an algebraic group that acts transitively but not 2-transitively. In contrast, for any $n \geq 2$ the group $\mathrm{SAut}(\mathbb{A}^n)$ is no longer an algebraic group. Indeed, it contains the infinite dimensional vector group of shears

$$(x_1, \ldots, x_{n-1}, y) \mapsto (x_1, \ldots, x_{n-1}, y + P(x_1, \ldots, x_{n-1}))\,,$$

where $P \in \Bbbk[x_1, \ldots, x_{n-1}]$ is an arbitrary polynomial. It is a well known and elementary fact that $\mathrm{SAut}(\mathbb{A}^n)$, $n \geq 2$, acts *infinitely transitively* on $\mathbb{A}^n$ that is, m-transitively for any $m \geq 1$ (see e.g. [28, Lemma 5.5] and references therein).

There is a number of further cases, where $\mathrm{SAut}(X)$ acts infinitely transitively. Consider, for instance, an equivariant projective embedding $Y \hookrightarrow \mathbb{P}^n$ of a flag variety $Y = G/P$. Then the special automorphism group of the affine cone X over Y acts infinitely transitively on the smooth locus X_{reg} of X [1, Theorem 1.1]. For non-degenerate toric affine varieties of dimension ≥ 2 a similar result is true [1, Theorem 2.1]. If Y is

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an affine variety on which $\mathrm{SAut}(Y)$ acts infinitely transitively, then the same holds for the suspension

$$X = \{uv - f(y) = 0\} \subseteq \mathbb{A}^2 \times Y \tag{1}$$

over Y, where $f \in \mathcal{O}(Y)$ is a non-constant function ([1, Theorem 3.2]; see also [28, §5] for the case $Y = \mathbb{A}^n$).

Transitivity properties of the special automorphism group are closely related to the flexibility of a variety, which was studied in the algebraic context in [1][1]. The variety X is called *flexible* if every point $x \in X_{\mathrm{reg}}$ is. We say that a point $x \in X_{\mathrm{reg}}$ is *flexible* if the tangent space T_xX is spanned by the tangent vectors to the orbits $H.x$ of one-parameter unipotent subgroups $H \subseteq \mathrm{Aut}(X)$. Clearly, X is flexible if one point of X_{reg} is and the group $\mathrm{Aut}(X)$ acts transitively on X_{reg}.

The following theorem[2] confirms a conjecture formulated in [1, §4].

Theorem 0.1. *For an irreducible affine variety X of dimension ≥ 2, the following conditions are equivalent.*

(i) The group $\mathrm{SAut}(X)$ *acts transitively on* X_{reg}.
(ii) The group $\mathrm{SAut}(X)$ *acts infinitely transitively on* X_{reg}.
(iii) X is a flexible variety.

The varieties studied in [1] are flexible. As a further example of a flexible variety one can consider the total space of a homogeneous vector bundle over a flexible affine variety (see Corollary 4.5). Every connected semisimple algebraic group and, more generally, every connected linear algebraic group G without non-trivial characters is generated by its unipotent 1-parameter subgroups, see [43, Lemma 1.1]. This implies that any affine homogeneous space G/H is flexible. Consequently in case $\dim G/H \geq 2$ the special automorphism group $\mathrm{SAut}(G/H)$ acts infinitely transitively. More generally, if a semisimple algebraic group acts with an open orbit on a smooth affine variety X then X is homogeneous with respect to a bigger affine algebraic group without non-trivial characters and so is flexible (see Theorem 5.5).

In contrast, a Lie group or an algebraic group G cannot act m-transitively on a variety X for $m > \dim G/\dim X$. Indeed, m-transitivity of the G-action on X is equivalent to the transitivity of the induced G-action on X^m minus the diagonals. According to A. Borel a much stronger fact is valid: a real Lie group cannot act even 3-transitively on a simply connected, non-compact real manifold (see Theorems 5 and 6 in [5]). The latter remains true, without the assumption of simple connectedness, for the actions of algebraic groups over algebraically closed fields [31, Korollar 2]; cf. [42] for related results.

Let us mention several applications. As an almost immediate consequence it follows that in a flexible irreducible affine variety X any finite subset $Z \subseteq X_{\mathrm{reg}}$ can be interpolated in X_{reg} by an $\mathbb{A}^1$-curve, that is by a curve isomorphic to $\mathbb{A}^1$ and contained in X_{reg}. Indeed, given a one-dimensional $\mathbb{G}_a$-orbit O in X_{reg} and a finite subset $Z' \subseteq O$ of the same cardinality as that of Z, by infinite transitivity there is an automorphism $g \in \mathrm{SAut}(X)$ which sends Z' to Z. Then $g(O) \cong \mathbb{A}^1$ is a $\mathbb{G}_a$-orbit passing through

[1] In the analytic context several other flexibility properties are surveyed in [13].

[2] Cf. Theorem 2.2 below.

every point of Z. In fact, X_{reg} is $\mathbb{A}^1$-rich in the sense of [29] (see Corollary 4.18). For the case $X = \mathbb{A}^n$ this is the Gromov-Winkelmann theorem, see [53].

An interesting class of flexible varieties is formed by degeneracy loci of generic matrices. These are the varieties $X_r \subseteq \mathbb{A}^{mn}$ consisting of $m \times n$-matrices of rank $\leq r$, where $1 \leq r \leq \min(n, m)$. The elementary transformations, which replace row i by row $i + t$·row j ($i \neq j$, $t \in \Bbbk$), and similarly for columns, constitute $\mathbb{G}_a$-actions on $X_r \setminus X_{r-1}$. By a standard fact of linear algebra each matrix can be transformed to a normal form by a sequence of elementary transformations. Since $X_r \setminus X_{r-1} = (X_r)_{\mathrm{reg}}$ for $1 \leq r < \min(n, m)$, in this range $\mathrm{SAut}(X_r)$ acts transitively and thus infinitely transitively on $X_r \setminus X_{r-1}$ by Theorem 0.1.

We establish this infinite transitivity even simultaneously for matrices of different ranks, see Theorem 3.3. This shows that any finite collection of $m \times n$ matrices can be diagonalized simultaneously by means of elementary row- and column transformations depending polynomially on the matrix entries. Similar statements also hold for symmetric and skew-symmetric matrices, see Theorems 3.5 and 3.6. A related result on collective infinite transitivity for conjugacy classes of matrices was established earlier by Z. Reichstein [47] using different methods.

The Gizatullin surfaces represent another interesting class of examples, where flexibility manifests (see Example 2.3). These are the normal affine surfaces which admit a completion by a chain of smooth rational curves.

We provide as well a version of infinite transitivity involving infinitesimally near points. More precisely we prove[3]:

Theorem 0.2. *Let X be a flexible irreducible affine variety of dimension $n \geq 2$ equipped with an algebraic volume form[4] ω. Then for every $m \geq 0$ and every finite subset $Z \subseteq X_{\mathrm{reg}}$ there exists an automorphism $g \in \mathrm{SAut}(X)$ with prescribed m-jets at the points $p \in Z$, provided these jets preserve ω and inject Z into X_{reg}.*

In the analytic context similar results were obtained in [6] and [25].

Let us give a short overview of the content.

In Section 1 we consider subgroups $G \subseteq \mathrm{Aut}(X)$ that are generated by collections of algebraic subgroups of $\mathrm{Aut}(X)$ and discuss their general properties. Special classes of such groups appeared earlier on different occasions, see e.g. [28, §5], [41], and [43]. Although G is not in general an algebraic group, we show that the G-orbits have the same properties as those of an algebraic group action (cf. [46], Lemma 2). We give an extension of Kleiman's Transversality Theorem in this context (see Theorem 1.15) and of the Rosenlicht Theorem on the separation of generic orbits by rational invariants (see Theorem 1.13). As an application we confirm a conjecture in [34] concerning the field Makar-Limanov invariant (see 5.1). We expect further development of invariant theory for algebraically generated groups.

In Section 2 we prove Theorem 0.1 (cf. Theorems 2.2 and 2.5). The methods developed there are applied in Section 3 to show infinite transitivity on several orbits simultaneously, see Theorem 3.1. This yields the aforementioned application to matrix varieties.

[3]See Theorem 4.14.

[4]By this we mean a nowhere vanishing n-form defined on X_{reg}.

Section 4 contains the results on interpolation of curves and automorphisms. In Section 5 we apply our techniques to homogeneous spaces and their affine embeddings.

In the Appendix we adopt a complex analytic point of view. We show in particular that the Oka-Grauert-Gromov Principle is available for smooth G-fibrations with flexible fibers, where G is an algebraically generated group of automorphisms (cf. Proposition 6.3 and Corollary 6.7). Besides, we generalize the notion of flexibility to the analytic setting, and compare our results with similar ones known in this setup.

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1. Flexibility versus transitivity

In this section X stands for a reduced and irreducible algebraic variety.

1.1. **Algebraically generated groups of automorphisms.** Recall from the introduction that a subgroup H of the automorphism group $\mathrm{Aut}(X)$ is *algebraic* if H has a structure of an algebraic group such that the natural action $H\times X \to X$ is a morphism. In the literature there are many attempts to define and to study more general classes of subgroups of $\mathrm{Aut}(X)$, see e.g. [46], [49][5]. For our purposes the following notion closely related to that in [43, Definition 1.36] turns out to be useful.

Definition 1.1. A subgroup G of $\mathrm{Aut}(X)$ is called *algebraically generated* if it is generated as an abstract group by a family $\mathcal{G}$ of connected algebraic[6] subgroups of $\mathrm{Aut}(X)$.

The following notation will be useful in the sequel.

Notation 1.2. 1. Let us introduce a partial order on the set of sequences in $\mathcal{G}$ defined via

$$(H_1,\ldots,H_m) \succcurlyeq (H'_1,\ldots H'_s) \Longleftrightarrow \exists i_1<\ldots<i_s : \quad (H'_1,\ldots,H'_s)=(H_{i_1},\ldots,H_{i_s})\,.$$

Clearly then any two sequences are dominated by a third one.

2. Given a sequence $\mathcal{H}=(H_1,\ldots,H_s)$ in $\mathcal{G}$ and a point $x\in X$ we consider the morphism

$$\Phi_{\mathcal{H},x} : H_1\times\ldots\times H_s \to X, \qquad (h_1,\ldots,h_s)\mapsto (h_1\cdot\ldots\cdot h_s).x\,. \tag{2}$$

Proposition 1.3. *If the subgroup $G\subseteq \mathrm{Aut}(X)$ is algebraically generated then for every point $x\in X$ the orbit $G.x$ is locally closed.*

Proof. Replacing X by the Zariski closure of the orbit $G.x$ we may assume that $X=\overline{G.x}$ i.e., the orbit of x is dense in X. Notice that for every finite sequence $\mathcal{H}=(H_1,\ldots,H_s)$ in $\mathcal{G}$ the subset $X_{\mathcal{H},x}=(H_1\cdot H_2\cdot\ldots\cdot H_s).x\subseteq X$ is constructible and irreducible, being the image of the irreducible variety $H_1\times\ldots\times H_s$ under the morphism $\Phi_{\mathcal{H},x}$. Taking a larger $\mathcal{H}$ we enlarge $X_{\mathcal{H},x}$ too (i.e. $X_{\mathcal{H},x}\subseteq X_{\mathcal{H}',x}$ for $\mathcal{H}'\succcurlyeq\mathcal{H}$). By assumption the union of all such sets $X_{\mathcal{H},x}$ is dense in X, hence also the union of the

[5] A thorough treatment of this approach can be found in [33, Chapt. 4], along with some historical remarks and bibliography.

[6] not necessarily affine.

closures $\bar{X}_{\mathcal{H},x}$ is. Since an increasing sequence of closed irreducible subsets becomes stationary, $X = \bar{X}_{\mathcal{H},x}$ for some $\mathcal{H}$. In particular, the interior $\mathring{X}_{\mathcal{H},x}$ of $X_{\mathcal{H},x}$ is a non-empty open subset in X. Now the transitivity of the G-action on $G.x$ implies that $G.x$ is open in X, as desired. □

Remark 1.4. We are grateful to one of the referees for pointing out to us that Proposition 1.3 is contained in Lemma 2 of [46].

For the next results it is useful to consider the map

$$\Phi_{\mathcal{H}} : H_1 \times \ldots \times H_s \times X \longrightarrow X \times X, \qquad (h_1, \ldots, h_s, x) \mapsto (x, (h_1 \cdot \ldots \cdot h_s).x)\,. \tag{3}$$

Proposition 1.5. *There are (not necessarily distinct) subgroups $H_1, \ldots, H_s \in \mathcal{G}$ such that*

$$G.x = (H_1 \cdot H_2. \cdot \ldots \cdot H_s).x \quad \forall x \in X. \tag{4}$$

Proof. The image $Z_{\mathcal{H}} = \Phi_{\mathcal{H}}(H_1 \times \ldots \times H_s \times X)$ in $X \times X$ is constructible and irreducible. As before, if $\mathcal{H}' \succcurlyeq \mathcal{H}$ then $Z_{\mathcal{H}} \subseteq Z_{\mathcal{H}'}$. Hence the union of closures $Z = \bigcup_{\mathcal{H}} \bar{Z}_{\mathcal{H}}$ stabilizes in $X \times X$ i.e., it coincides with $\bar{Z}_{\mathcal{H}}$ for $\mathcal{H}$ sufficiently large. In particular Z is closed.

Let $\mathring{Z}_{\mathcal{H}}$ be the interior of $Z_{\mathcal{H}}$ in Z. It follows as before that also $\{\mathring{Z}_{\mathcal{H}}\}$ becomes stationary and that the union $Z' = \bigcup_{\mathcal{H}} \mathring{Z}_{\mathcal{H}}$ is an open dense subset of Z.

Consider the G-action on $X \times X$ given by $g.(x, y) = (g.x, y)$. If $\mathcal{H} = (H_1, \ldots, H_s)$ and $H \in \mathcal{G}$ then for any $(h_1, \ldots, h_s) \in H_1 \times \ldots \times H_s$ and $h \in H$ we have

$$h.\Phi_{\mathcal{H}}(h_1, \ldots, h_s, x) = h.(x, (h_1 \cdot \ldots \cdot h_s).x) = \Phi_{(\mathcal{H},H)}(h_1, \ldots, h_s, h^{-1}, h.x).$$

Hence $h.Z_{\mathcal{H}} \subseteq Z_{(\mathcal{H},H)}$. It follows that Z and Z' are G-invariant.

Consider now for $\mathcal{H}$ sufficiently large the sets $Z_{\mathcal{H}}$, $Z' = \mathring{Z}_{\mathcal{H}}$, and $Z = \bar{Z}_{\mathcal{H}}$ as families over X via the first projection $p : (x, y) \mapsto x$. By [22, 9.5.3] there is an open dense subset V of X such that $Z'(x)$ is dense in $Z(x)$ for all $x \in V$, where for a subset $M \subseteq X \times X$ we denote by $M(x)$ the fiber of $p|M : M \to X$ over x. Since Z and Z' are invariant under the action of G and the projection p is equivariant, we may suppose that V is as well G-invariant.

In particular there is a sequence $\mathcal{H}_0$ such that $Z_{\mathcal{H}}(x) = (H_1 \cdot \ldots \cdot H_s).x$ is dense in $Z(x)$ for all $x \in V$ and all sequences $\mathcal{H} = (H_1, \ldots, H_s)$ dominating $\mathcal{H}_0$. It follows that $Z(x)$ is closure of the orbit $G.x$ and so $(H_1 \cdot \ldots \cdot H_s).x$ is dense in the orbit $G.x$ for all $x \in V$.

We claim that for every point $x \in V$

$$(H_s \cdot \ldots \cdot H_1 \cdot H_1 \cdot \ldots \cdot H_s).x = G.x\,.$$

Indeed, for any $y \in G.x$ the sets $(H_1 \cdots H_s).x$ and $(H_1 \cdots H_s).y$ are both dense in the orbit $G.x = G.y$. Hence they have a common point, say z. Thus

$$y \in (H_s \cdot \ldots \cdot H_1).z \subseteq (H_s \cdot \ldots \cdot H_1 \cdot H_1 \cdot \ldots \cdot H_s).x\,.$$

Replacing $\mathcal{H}$ by the larger sequence $(H_s, \ldots, H_1, H_1, \ldots, H_s)$ it follows that

$$(H_1 \cdot \ldots \cdot H_s).x = G.x \text{ for all } x \in V \text{ simultaneously.}$$

The complement $Y = X \backslash V$ is closed, G-invariant, and all its irreducible components are of dimension $< \dim X$. Using induction on the dimension of X we see that (4) holds for $\mathcal{H}$ sufficiently large and all $x \in X$ simultaneously, concluding the proof. □

Remark 1.6. Propositions 1.3 and 1.5 remain true with the same proofs for varieties over algebraically closed fields of arbitrary characteristic.

2. In the setup of Proposition 1.5, if $Y \subseteq X$ is constructible then so is its 'orbit' $G.Y$. Indeed, by Proposition 1.5 for some $\mathcal{H} = (H_1, \ldots, H_s)$ this orbit is the image of $H_1 \times \ldots \times H_s \times Y$ under the composition of $\Phi_{\mathcal{H}}$ and the natural projection $X \times X \to X$ to the second factor.

Definition 1.7. A sequence $\mathcal{H} = (H_1, \ldots, H_s)$ in $\mathcal{G}$ satisfying condition (4) of 1.5 will be called *complete*.

Proposition 1.8. *Assume that the generating family $\mathcal{G}$ of connected algebraic subgroups is closed under conjugation in G, i.e., $gHg^{-1} \in \mathcal{G}$ for all $g \in G$ and $H \in \mathcal{G}$. Then there is a sequence $\mathcal{H} = (H_1, \ldots, H_s)$ in $\mathcal{G}$ such that for all $x \in X$ the tangent space $T_x(G.x)$ of the orbit $G.x$ is spanned by the tangent spaces*

$$T_x(H_1.x), \ldots, T_x(H_s.x)\,.$$

Proof. We claim that $T_x(G.x)$ is spanned by the tangent spaces $T_x(H.x)$, where $H \in \mathcal{G}$. Indeed, consider a complete sequence $H_1, \ldots, H_s \in \mathcal{G}$ such that the map $\Phi_{\mathcal{H},x} : H_1 \times \ldots \times H_s \to G.x$ in (2) is surjective. By [23][III, Corollary 10.7] this map is generically smooth. Thus for some point $y = (h_1 \cdot \ldots \cdot h_s).x \in G.x$ the tangent map

$$d\Phi_{\mathcal{H},x} : T_{(h_1,\ldots,h_s)}(H_1 \times \ldots \times H_s) \longrightarrow T_y(G.x)$$

is surjective. Multiplication by $g = (h_1 \cdot \ldots \cdot h_s)^{-1}$ yields an isomorphism $\mu_g : G.x \to G.x$ which sends y to x. Hence the composition $\mu_g \circ \Phi_{\mathcal{H},x}$ has a surjective tangent map

$$d(\mu_g \circ \Phi_{\mathcal{H},x}) : T_{(h_1,\ldots,h_s)}(H_1 \times \ldots \times H_s) \cong \prod_{\sigma=1}^{s} T_{h_\sigma}(H_\sigma) \longrightarrow T_x(G.x)\,. \qquad (5)$$

Letting $g_\sigma := h_{\sigma+1} \cdot \ldots \cdot h_s$ the isomorphism

$$\tilde{H}_\sigma := h_1 \times \ldots \times h_{\sigma-1} \times H_\sigma \times h_{\sigma+1} \times \ldots \times h_s \longrightarrow H'_\sigma := g_\sigma^{-1} H_\sigma g_\sigma$$

with $(h_1, \ldots, h_{\sigma-1}, h, h_{\sigma+1}, \ldots, h_s) \mapsto g_\sigma^{-1} h_\sigma^{-1} h g_\sigma$ identifies the restriction $\mu_g \circ \Phi_{\mathcal{H},x} | \tilde{H}_\sigma$ with

$$\varphi : H'_\sigma = g_\sigma^{-1} H_\sigma g_\sigma \to G.x, \qquad h' \mapsto h'.x\,.$$

Thus the restriction of the tangent map $d(\mu_g \circ \Phi_{\mathcal{H},x})$ to the factor $T_{h_\sigma}(H_\sigma)$ in (5) can be identified with the map $d_e\varphi : T_e(H'_\sigma) \to T_x(G.x)$, where e denotes the identity element of H'_σ. Thus the claim follows.

Consider further the map $\Phi_{\mathcal{H}} : H_1 \times \ldots \times H_s \times X \to Z \subseteq X \times X$ as in (3) associated with a complete sequence $\mathcal{H}$, where $Z \subseteq X \times X$ is the closure of the image of $\Phi_{\mathcal{H}}$. Choose an invariant open subset $V \subseteq X_{\mathrm{reg}}$ such that the first projection $p : Z \to X$ is smooth over V. Note that the fiber of $Z_V = p^{-1}(V) \to V$ over x is just the orbit $G.x$. Let us consider the map of relative tangent bundles

$$d\Phi_{\mathcal{H}} : T(H_1 \times \ldots \times H_s \times V/V) \to \Phi_{\mathcal{H}}^*(T(Z_V/V))$$

and its restriction to $(e, \ldots, e) \times V \cong V$, where e is the identity element in G and therefore in each H_i,

$$d\Phi_{\mathcal{H}} : T_e H_1 \times \ldots \times T_e H_s \times V \to \Phi_{\mathcal{H}}^*(T(Z_V/V))|V\,.$$

The set $U_{\mathcal{H}}$ of points in V where this map is surjective, is open. By the above claim, the union $\bigcup_{\mathcal{H}} U_{\mathcal{H}}$ coincides with V. Any two sequences $\mathcal{H}_1$ and $\mathcal{H}_2$ are dominated by a third $\mathcal{H}_3$ in the partial order as in 1.2, and the corresponding subset $U_{\mathcal{H}_3}$ contains both $U_{\mathcal{H}_1}$ and $U_{\mathcal{H}_2}$. Thus the increasing union $\bigcup_{\mathcal{H}} U_{\mathcal{H}}$ stabilizes, that is, $V = U_{\mathcal{H}}$ for $\mathcal{H}$ sufficiently large. Induction on the dimension of X as in the proof of Proposition 1.5 ends the proof. $\square$

Remark 1.9. It may happen for a family $\mathcal{G}$ which is not closed under conjugation that for some point $x \in X$ the tangent spaces

$$T_x(H_1.x), \ldots, T_x(H_s.x)$$

do not span $T_x(G.x)$, whatever is the sequence $\mathcal{H} = (H_1, \ldots, H_s)$ in $\mathcal{G}$. For instance, the group $G = \mathrm{SL}_2$ is generated by the family $\mathcal{G} = \{U^+, U^-\}$, where $U^\pm$ are the subgroups of upper and lower triangular unipotent matrices. Letting SL_2 act on itself by left multiplication the tangent space T_eG of the orbit $G = G.e$ is $\mathfrak{sl}_2$, while for any sequence $\mathcal{H} = (H_1, \ldots, H_s)$ in $\mathcal{G}$ the tangent spaces $T_e(H_1), \ldots, T_e(H_s)$ are contained in the 2-dimensional subspace $T_e(U^+) + T_e(U^-)$.

Definition 1.10. Let $G \subseteq \mathrm{Aut}(X)$ be algebraically generated by a family $\mathcal{G}$ of connected algebraic subgroups, which is closed under conjugation. We say that a point $p \in X_{\mathrm{reg}}$ is *G-flexible* if the tangent space T_pX at p is generated by the subspaces $T_p(H.p)$, where $H \in \mathcal{G}$.

Corollary 1.11. *With G and $\mathcal{G}$ as in Definition 1.10 the following hold.*
(a) A point $p \in X_{\mathrm{reg}}$ is G-flexible if and only if the orbit $G.p$ is open in X.
(b) An open G-orbit (if it exists) is unique and consists of all G-flexible points in X_{reg}.

Proof. (a) By Proposition 1.8 the morphism $\Phi_{\mathcal{H},p} : H_1 \times \ldots \times H_s \to G.p$ is surjective and smooth for an appropriate choice of a sequence $\mathcal{H}$. Now (a) follows. Furthermore (b) follows from (a) since any two open G-orbits overlap and so must coincide. $\square$

Let us note that by Corollary 1.11(a) the definition of a G-flexible point only depends on G and not on the choice of the generating set $\mathcal{G}$.

Using the semicontinuity of the fiber dimension we can deduce the following semicontinuity result for the orbits of algebraically generated groups.

Corollary 1.12. *If a group $G \subseteq \mathrm{Aut}(X)$ is algebraically generated then the function $x \mapsto \dim G.x$ is lower semicontinuous on X. In particular, there is a Zariski open subset $U \subseteq X$ filled in by orbits of maximal dimension.*

Proof. We may suppose that $G = \langle \mathcal{G} \rangle$, where $\mathcal{G}$ is a family of connected algebraic subgroups of $\mathrm{Aut}(X)$ closed under conjugation in G. For a complete sequence $\mathcal{H} = (H_1, \ldots, H_s)$ consider the map $\Phi_{\mathcal{H}}$ from (3). By the semicontinuity of fiber dimension the function

$$X \ni x \longmapsto \dim_{\tau(x)} \Phi_{\mathcal{H}}^{-1}(x, x)$$

is upper semicontinuous on X, where $\tau(x) = (1, \ldots, 1, x) \in H_1 \times \ldots \times H_s \times X$. Here $\Phi_{\mathcal{H}}^{-1}(x, x)$ is just the fiber of the map $\Phi_{\mathcal{H},x} : H_1 \times \ldots \times H_s \to G.x$ over x.

Fix a point $x_0 \in X$. Enlarging $\mathcal{H}$ we may assume that $\Phi_{\mathcal{H},x_0}$ is a submersion. Thus for x in a suitable neighborhood U of x_0

$$\begin{aligned}\dim G.x_0 &= \sum_{\sigma=1}^{s} \dim H_\sigma - \dim \Phi^{-1}(x_0, x_0) \\ &\le \sum_{\sigma=1}^{s} \dim H_\sigma - \dim \Phi^{-1}(x, x) \\ &\le \dim G.x.\end{aligned}$$

It follows that $\dim G.x \ge \dim G.x_0$ for $x \in U$, as required. □

The following analog of the Rosenlicht Theorem on rational invariants holds for algebraically generated groups. The proof of this theorem given in [44, Theorem 2.3] works *mutatis mutandis* in our setting due to Proposition 1.5 and Corollary 1.12.

Theorem 1.13. *Let G be an algebraically generated group acting on X. Then there exists a finite collection of rational G-invariants which separate G-orbits in general position.*

Proof. Replacing X by a subset U as in Corollary 1.12 we may assume that all orbits of G are of maximal dimension. In particular then all G-orbits are closed in X. Let $\Gamma \subseteq X \times X$ consist of all pairs (x, x') such that x and x' are in the same G-orbit. Note that this is just the image of the map $G \times X \to X \times X$ with $(g, x) \mapsto (g.x, x)$. As we have seen in the proof of Proposition 1.5, Γ contains an open dense subset, say Γ_0, of the closure $\overline{\Gamma}$ in $X \times X$.

Letting G act on the first component of $X \times X$ we may assume that Γ_0 is G-invariant, since otherwise we can replace it by the union of all translates of Γ_0. If $p_2 : \Gamma_0 \to X$ denotes the second projection then for a general point $x \in X$ the fibre $p_2^{-1}(x) = G.x \times \{x\}$ is closed in $X \times \{x\}$. Hence there is an open dense subset $U \subseteq X$ such that $\Gamma_0 \cap p_2^{-1}(U)$ is closed in $X \times U$. In particular it follows that $\Gamma \cap X \times U$ is closed in $X \times U$. Shrinking U we may also assume that U is affine.

Let $\mathcal{I} \subseteq \mathcal{O}(X \times U)$ be the ideal sheaf of $\Gamma \cap X \times U$, and let J be the ideal generated by $\mathcal{I}$ in the algebra $\mathcal{M}er(X) \otimes \mathcal{O}(U)$[7]. The ideal J is G-invariant assuming that G acts on the first factor of $X \times X$. Moreover, J is generated as a $\mathcal{M}er(X)$-vector subspace by G-invariant elements (see [44, Lemma 2.4]). We can find a finite set of generators of J, say $F_1, ..., F_p$, among these elements. We have

$$F_i = \sum_s f_{is} \otimes u_{is}, \quad \text{where} \quad f_{is} \in \mathcal{M}er(X)^G \quad \text{and} \quad u_{is} \in \mathcal{O}(U)\,.$$

Let us show that the functions f_{is} separate orbits in general position.

Shrinking U once again we may assume that all the f_{is} are regular functions on U and that the elements F_i generate the ideal $\mathcal{I}$. Then the orbit of a point $x \in U$ is defined by the equations $F_i(x, y) = \sum_s f_{is}(x) u_{is}(y) = 0$, $i = 1, ..., p$. Consequently, the equalities $f_{is}(x_1) = f_{is}(x_2)$ for all i and s imply that $G.x_1 = G.x_2$ on U. □

As in [44, Corollary on p. 156] this theorem has the following consequence.

[7]Here $\mathcal{M}er(X)$ denotes the function field of X.

Corollary 1.14. *Let G be an algebraically generated group acting on X. Then*

$$\operatorname{trdeg}(\mathcal{M}er(X)^G : \mathbb{k}) = \min_{x\in X} \{\operatorname{codim}_X G.x\},$$

where $\mathcal{M}er(X)$ stands for the rational function field of X. In particular, G has an open orbit in X if and only if $\mathcal{M}er(X)^G = \mathbb{k}$.

1.2. **Transversality.** If a connected algebraic group G acts transitively on an algebraic variety X and Y, Z are smooth subvarieties of X then by Kleiman's Transversality Theorem [30] a general g-translate $g.Z$ $(g \in G)$ meets Y transversally. In this subsection we extend Kleiman's Theorem to the case of an arbitrary algebraically generated group.

Theorem 1.15. *Let a subgroup $G \subseteq \operatorname{Aut}(X)$ be algebraically generated by a system $\mathcal{G}$ of connected algebraic subgroups closed under conjugation in G. Suppose that G acts with an orbit O open in X.*

Then there exist subgroups $H_1, \ldots, H_s \in \mathcal{G}$ with the following property:
For any locally closed reduced subschemes Y and Z in O one can find a Zariski dense open subset $U = U(Y,Z) \subseteq H_1 \times \ldots \times H_s$ such that every element $(h_1, \ldots, h_s) \in U$ satisfies the following conditions.

(a) The translate $(h_1 \cdot \ldots \cdot h_s).Z_{\mathrm{reg}}$ meets Y_{reg} transversally.
(b) $\dim(Y \cap (h_1 \cdot \ldots \cdot h_s).Z) \leq \dim Y + \dim Z - \dim X$ [8].
In particular $Y \cap (h_1 \cdot \ldots \cdot h_s).Z = \varnothing$ if $\dim Y + \dim Z < \dim X$.

The proof is based on the following auxiliary result, which is complementary to Proposition 1.8.

Proposition 1.16. *Let the assumption of Theorem 1.15 hold. Then there is a sequence $\mathcal{H} = (H_1, \ldots, H_s)$ in $\mathcal{G}$ so that for a suitable open dense subset $U \subseteq H_s \times \ldots \times H_1$ [9] the map*

$$(6)\quad \Phi_s : H_s \times \ldots \times H_1 \times O \longrightarrow O \times O \quad \textit{with} \quad (h_s, \ldots, h_1, x) \mapsto ((h_s \cdot \ldots \cdot h_1).x, x)$$

is smooth on $U \times O$.

Proof. According to Proposition 1.5 there are subgroups $H_1, \ldots, H_s \subseteq G$ in $\mathcal{G}$ such that Φ_s is surjective. Hence there is an open dense subset $U_s \subseteq H_s \times \ldots \times H_1 \times O$ on which Φ_s is smooth. Assuming that U_s is maximal with this property we consider the complement $A_s = (H_s \times \ldots \times H_1 \times O) \backslash U_s$.

Let us study the effect of increasing the number of factors, i.e., passing to

$$\Phi_{s+1} : H_{s+1} \times \ldots \times H_1 \times O \longrightarrow O \times O$$

The map Φ_{s+1} is smooth on $H_{s+1} \times U_s$. Indeed, for every $h_{s+1} \in H_{s+1}$ we have a commutative diagram

$$\begin{array}{ccc} H_{s+1} \times \ldots \times H_1 \times O & \xrightarrow{\Phi_{s+1}} & O \times O \\ \uparrow{\scriptstyle h_{s+1}\times \mathrm{id}} & & \uparrow{\scriptstyle h_{s+1}\times \mathrm{id}} \\ \{1\} \times H_s \times \ldots \times H_1 \times O & \xrightarrow{\Phi_s} & O \times O \end{array}$$

[8]We let the dimension of the empty set be equal to $-\infty$.
[9]The inverse enumeration here is convenient for applying recursion.

where the lower horizontal map is smooth on U_s. In other words, $U_{s+1} \supseteq H_{s+1} \times U_s$ or, equivalently, $A_{s+1} \subseteq H_{s+1} \times A_s$. We claim that increasing the number of factors by $H_{s+1}, \ldots, H_{s+t}$ in a suitable way, we can achieve that

$$\dim A_{s+t} < \dim(H_{s+t} \times \ldots \times H_{s+1} \times A_s)\,. \tag{7}$$

If $(h_s, \ldots, h_1, x) \in A_s$ and $y = (h_s \cdot \ldots \cdot h_1).x$ then for suitable $H_{s+t}, \ldots, H_{s+1}$ the map

$$H_{s+t} \times \ldots \times H_{s+1} \times O \longrightarrow O \times O \quad \text{with} \quad (h_{s+t}, \ldots, h_{s+1}, x) \mapsto ((h_{s+t} \cdot \ldots \cdot h_{s+1}).x, x)$$

is smooth at all points $(e, \ldots, e, y)$, where e is the identity element of G; see Proposition 1.8. In particular Φ_{s+t} is smooth at all points $(e, \ldots, e, h_s, \ldots, h_1, x)$ with $x \in O$, i.e.

$$(e, \ldots, e) \times A_s \cap A_{s+t} = \varnothing.$$

Now (7) follows.

Thus increasing the number of factors suitably we can achieve that[10]

$$\dim A_s < \dim(H_s \times \ldots \times H_1)\,.$$

In particular, the image of A_s under the projection

$$\pi : H_s \times \ldots \times H_1 \times O \longrightarrow H_s \times \ldots \times H_1$$

is contained in a proper, closed subvariety of $H_s \times \ldots \times H_1$. Hence there is an open dense subset $U \subseteq H_s \times \ldots \times H_1$ such that $\Phi_s : U \times O \to O \times O$ is smooth. □

Proof of Theorem 1.15. Let us first show (a). Replacing Y and Z by Y_{reg} and Z_{reg}, respectively, we may assume that Y and Z are smooth. Applying Proposition 1.16 there are subgroups $H_1, \ldots, H_s$ in $\mathcal{G}$ such that $\Phi_s : U \times O \to O \times O$ is smooth for some open subset $U \subseteq H_1 \times \ldots \times H_s$. In particular $\mathcal{Y} = \Phi_s^{-1}(Y \times Z) \cap (U \times O) \subseteq U \times Z$ is smooth. By Corollary 10.7 in [23, Ch. III] the general fiber of the projection $\mathcal{Y} \to U$ is smooth as well. In other words, shrinking U we may assume that all fibers of this projection are smooth. Since for a point $h = (h_1, \ldots, h_s) \in U$ the fiber $\mathcal{Y} \cap \pi^{-1}(h)$ maps bijectively via Φ_s onto $Y \cap (h_1 \cdot \ldots \cdot h_s).Z$, (a) follows.

Now (b) follows by an easy induction on $l = \dim Y + \dim Z$, the case of $l = 0$ being trivial. Indeed, applying (a) and the induction hypothesis to Y_{sing} and Z and also to Y and Z_{sing}, for suitable connected algebraic subgroups $H_1, \ldots, H_s$ and general $(h_1, \ldots, h_s) \in H_1 \times \ldots \times H_s$ we have that Y_{reg} and $(h_1 \cdot \ldots \cdot h_s).Z_{\mathrm{reg}}$ meet transversally and that

$$\dim(Y_{\mathrm{sing}} \cap (h_1 \cdot \ldots \cdot h_s).Z) \le \dim Y_{\mathrm{sing}} + \dim Z - \dim X;$$
$$\dim(Y \cap (h_1 \cdot \ldots \cdot h_s).Z_{\mathrm{sing}}) \le \dim Y + \dim Z_{\mathrm{sing}} - \dim X\,.$$

This immediately implies the desired result. □

[10]In fact we can make the difference $\dim(H_s \times \ldots \times H_1) - \dim A_s$ arbitrarily large.

1.3. $\mathbb{G}_a$**-generated subgroups.** The following notion is central in the sequel.

Definition 1.17. A subgroup G of the automorphism group $\mathrm{Aut}(X)$ will be called $\mathbb{G}_a$*-generated*[11] if it is generated by a family of one-parameter unipotent subgroups i.e., subgroups isomorphic to $\mathbb{G}_a$.

We give two simple examples.

Example 1.18. (1) The group $\mathrm{SAut}(X)$ is $\mathbb{G}_a$-generated. The image of $\mathrm{SAut}(X)$ under the diagonal embedding $\mathrm{SAut}(X) \hookrightarrow \mathrm{SAut}(X^m)$ is also a $\mathbb{G}_a$-generated subgroup.

(2) A connected affine algebraic group acting regularly and effectively on X is a $\mathbb{G}_a$-generated subgroup of $\mathrm{Aut}(X)$ if and only if it does not admit nontrivial characters [43, Lemma 1.1]. In particular, every connected semisimple algebraic group is $\mathbb{G}_a$-generated.

It will be important to deal with the infinitesimal generators of algebraic subgroups of $\mathrm{Aut}(X)$ isomorphic to $\mathbb{G}_a$. Let us collect the necessary facts.

1.19. (1) If the group $\mathbb{G}_a$ acts on an affine variety $X = \operatorname{Spec} A$ then the associated derivation ∂ of A is locally nilpotent, i.e. for every $a \in A$ we can find $n \in \mathbb{N}$ such that $\partial^n(a) = 0$ [48]. It is immediate that for every $f \in \ker \partial$ the derivation $f\partial$ is again locally nilpotent [16, §1.4, Principle 7].

(2) Conversely, given a locally nilpotent $\Bbbk$-linear derivation $\partial : A \to A$ and $t \in \Bbbk$, the map $\exp(t\partial) : A \to A$ is an automorphism of A [16, 1.1.8]. Furthermore for $\partial \neq 0$, $H = \exp(\Bbbk\partial)$ is a subgroup of $\mathrm{Aut}(A)$ isomorphic to $\mathbb{G}_a$. Via the isomorphism $\mathrm{Aut}(A) \xrightarrow{\simeq} \mathrm{Aut}(X)$ given by $g \mapsto (g^{-1})^*$ this yields a one parameter unipotent subgroup of $\mathrm{Aut}(X)$, which we denote by the same letter H.

One can also consider ∂ as a vector field on X. If the ground field is $\mathbb{C}$ then the action of $H \cong \mathbb{G}_a$ on X is just the associated phase flow. We often use the term *locally nilpotent vector field* meaning that the associated derivation is locally nilpotent.

(3) The ring of invariants $\mathcal{O}(X)^H = \ker \partial$ has transcendence degree over $\Bbbk$ equal to $\dim X - 1$. For any H-invariant function $f \in \mathcal{O}(X)^H$ the one-parameter unipotent subgroup $H_f = \exp(\Bbbk f\partial)$ plays an important role in the sequel (cf. [40]). It will be called a *replica* of H.

(4) The H_f-action has the same general orbits as the H-action. However, the zero locus of f remains pointwise fixed under the H_f-action.

1.20. In order to illustrate the notions of a $\mathbb{G}_a$-generated group and a replica, the affine space $X = \mathbb{A}^n$ is a good choice, as this is done in [41, §§1,2]. For instance, a replica H_f of a one-parameter unipotent subgroup $H = \exp(\Bbbk\partial)$ generated by a directional partial derivative ∂ is a one parameter group of shears in the same direction. For $n = 3$, the famous Nagata automorphism[12] is actually a special value of the replica associated with the locally nilpotent derivation $\partial = X\frac{\partial}{\partial Y} + Y\frac{\partial}{\partial Z}$ of the polynomial ring $\Bbbk[X, Y, Z]$ and the invariant function $f = Y^2 - 2XZ \in \ker \partial$. The problem whether the subgroup $\mathrm{SAut}(\mathbb{A}^n)$ coincides with the group of all automorphisms of $\mathbb{A}^n$ with Jacobian

[11] These groups were introduced in the particular case $X = \mathbb{A}^n$ in [41, Definition 2.1], where they were called *∂-generated*. Cf. also Definition 1.36 in [43] for a more general notion of an *F-generated group*.

[12] Recall that the Nagata automorphism is wild, see [50].

determinant 1 is still widely open (see [41, Problem 2.1, Examples 2.3 and 2.5]; cf. also [17, Proposition 9]). Recall that this is indeed the case in dimension 2 due to the Jung-van der Kulk Theorem, see *ibid.*

Given an algebraic variety we denote by $\mathrm{LND}(X)$ the set of all locally nilpotent vector fields on X. If $G \subseteq \mathrm{Aut}(X)$ is any subgroup then the vector fields in $\mathrm{LND}(X)$ generating one-parameter unipotent subgroups of G form a subset $\mathrm{LND}(G)$ of $\mathrm{LND}(X)$. This set is a cone (i.e., $\Bbbk \cdot \mathrm{LND}(G) \subseteq \mathrm{LND}(G)$) stable under conjugations $\partial \mapsto g^*\partial(g^*)^{-1}$, where the automorphism $g^* : A \to A$ is induced by $g \in G$ (cf. [16, §1.4, Principle 1d]).

Let now $G \subseteq \mathrm{SAut}(X)$ be a $\mathbb{G}_a$-generated subgroup. In the sequel we consider subsets $\mathcal{N} \subseteq \mathrm{LND}(G)$ of locally nilpotent vector fields such that the associated one-parameter subgroups $\mathcal{G} = \{\exp(\Bbbk\partial) : \partial \in \mathcal{N}\}$ form a generating set of algebraic subgroups for G. By abuse of language, we often say that $\mathcal{N}$ is a generating set of G, and we write $G = \langle \mathcal{N} \rangle$.

¿From Proposition 1.8 we deduce the following result.

Corollary 1.21. *Given a $\mathbb{G}_a$-generated subgroup $G = \langle \mathcal{N} \rangle$ of $\mathrm{Aut}(X)$, where $\mathcal{N} \subseteq \mathrm{LND}(G)$ is stable under conjugation in G, there are locally nilpotent vector fields $\partial_1, \ldots, \partial_s \in \mathcal{N}$ which span the tangent space $T_p(G.p)$ at every point $p \in X$.*

For a point $p \in X$ we let $\mathrm{LND}_p(G) \subseteq T_pX$ denote the *nilpotent cone* of all tangent vectors $\partial(p)$, where ∂ runs over $\mathrm{LND}(G)$. By Corollary 1.21 we have $T_p(G.p) = \mathrm{Span}\,\mathrm{LND}_p(G)$.[13] Thus a point $p \in X_{\mathrm{reg}}$ is G-flexible (see Definition 1.10) if and only if the cone $\mathrm{LND}_p(G)$ spans the whole tangent space T_pX at p.

Applying Corollaries 1.11 and 1.21 to the special automorphism group $G = \mathrm{SAut}(X)$ yields the equivalence (i)⇔(iii) in Theorem 0.1 in the Introduction.

Corollary 1.22. *Given an affine variety X the action of* $\mathrm{SAut}(X)$ *on X_{reg} is transitive if and only if X is flexible.*

2. Infinite transitivity

2.1. **Main theorem.** In this section we show that the special automorphism group of a flexible irreducible affine variety X acts infinitely transitively on X_{reg}. We state this in a more general setup which turns out to be necessary for later applications. Let us first introduce the following useful notation.

Definition 2.1. Let X be an irreducible affine algebraic variety. A set $\mathcal{N}$ of locally nilpotent vector fields on X is said to be *saturated* if it satisfies the following two conditions.

(1) $\mathcal{N}$ is closed under conjugation by elements in G, where G is the subgroup of $\mathrm{SAut}(X)$ generated by $\mathcal{N}$.
(2) $\mathcal{N}$ is closed under taking replicas, i.e. for all $\partial \in \mathcal{N}$ and $f \in \ker \partial$ we have $f\partial \in \mathcal{N}$.

We note that replicas appear implicitely at many places of the literature, see for instance [28]; see also [41, Definition 2.1] for a related definition.

[13] Cf. Corollary 4.3 below.

Clearly every collection $\mathcal{N}^o$ of locally nilpotent vector fields on X can be extended to a saturated set $\mathcal{N}$. We note that in general this extended set generates a much larger group than $\mathcal{N}^o$. For instance, if $X = \mathbb{A}^2$ is equipped with coordinates (x, y) then $\mathcal{N}^o = \{\partial/\partial y\}$ generates the translations $(x, y) \mapsto (x, y + t)$ while its saturation $\mathcal{N}$ generates the group of all shears $(x, y) \mapsto (x, y + P(x))$ with $P \in \Bbbk[x]$.

The next result implies Theorem 0.1 in the Introduction.

Theorem 2.2. *Let X be an irreducible affine algebraic variety of dimension ≥ 2 and let $G \subseteq \mathrm{SAut}(X)$ be a subgroup generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields, which acts with an open orbit $O \subseteq X$. Then G acts infinitely transitively on O.*

Before starting the proof let us mention the following interesting class of examples.

Example 2.3. 1. (*Gizatullin surfaces.*) These are normal affine surfaces which admit a completion by a chain of smooth rational curves. Due to Gizatullin's Theorem ([20, II, Theorems 2 and 3][14]; see also [11]) a normal affine surface X different from $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})$ is Gizatullin if and only if the special automorphism group $\mathrm{SAut}(X)$ has an open orbit with a finite complement. By Theorem 2.2 the group $\mathrm{SAut}(X)$ acts infinitely transitively on this orbit. It was conjectured in [20, II] that in zero characteristic this orbit coincides with X_{reg} i.e., that every Gizatullin surface is flexible. This is definitely not true in positive characteristic, where the automorphism group $\mathrm{Aut}(X)$ of a Gizatullin surface X can have fixed points that are regular points of X [8]. The Gizatullin Conjecture is true for the Gizatullin surfaces given in $\mathbb{A}^3$ by equations[15] $xy - f(z) = 0$; see [1, Theorem 3.1] and [36, 37]. Yet another class of flexible Gizatullin surfaces consists of the Danilov-Gizatullin surfaces, see [19] (see also [9, Theorem 5]). We refer the reader to [12] and the references therein for a study of one-parameter groups acting on Gizatullin surfaces.

2.4. For subsets $\mathcal{N} \subseteq \mathrm{LND}(X)$ and $Z \subseteq X$ we let $\mathcal{N}_Z = \{\partial \in \mathcal{N} : \partial|Z = 0\}$ be the set of locally nilpotent vector fields in $\mathcal{N}$ vanishing on Z. If $G = \langle \mathcal{N} \rangle$ is the group generated by $\mathcal{N}$ then $\mathcal{N}_Z$ generates a subgroup denoted

$$G_{\mathcal{N},Z} = \langle H = \exp(\Bbbk\partial) : \partial \in \mathcal{N}_Z \rangle \subseteq G\,. \tag{8}$$

Clearly the automorphisms in $G_{\mathcal{N},Z}$ fix the set Z pointwise. In the case $\mathcal{N} = \mathrm{LND}(G)$ we simply write G_Z instead of $G_{\mathcal{N},Z}$.

We emphasize that if $\mathcal{N}$ is saturated then so is $\mathcal{N}_Z$. Hence in this case the group $G_{\mathcal{N},Z} = \langle \mathcal{N}_Z \rangle$ is again generated by a saturated set of locally nilpotent derivations.

With these notations our main technical result can be formulated as follows.

Theorem 2.5. *Let X be an irreducible affine algebraic variety of dimension ≥ 2 and let $G \subseteq \mathrm{SAut}(X)$ be a subgroup generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields, which acts with an open orbit $O \subseteq X$. Then for every finite subset $Z \subseteq O$ the group $G_{\mathcal{N},Z}$ acts transitively on $O \backslash Z$.*

[14] In [20, II] the result is stated in terms of $\mathrm{Aut}(X)$, but the proof applies to $\mathrm{SAut}(X)$.

[15] Over an arbitrary base field of characteristic zero.

Before embarking on the proof let us show how Theorem 2.2 follows from Theorem 2.5.

Proof of Theorem 2.2. Let $x_1, \ldots, x_m$ and $x'_1, \ldots, x'_m$ be sequences of points in O with $x_i \neq x_j$ and $x'_i \neq x'_j$ for $i \neq j$. Let us show by induction on m that there is an automorphism $g \in G$ with $g.x_i = x'_i$ for all $i = 1, \ldots, m$. As G acts transitively on O this is certainly true for $m = 1$. For the induction step suppose that there is already an automorphism $\alpha \in G$ with $\alpha.x_i = x'_i$ for $i = 1, \ldots, m-1$. Applying Theorem 2.5 to $Z = \{x'_1, \ldots, x'_{m-1}\}$ we can also find an automorphism $\beta \in G_{\mathcal{N},Z}$ with $\beta(\alpha(x_m)) = x'_m$. Clearly then $g = \beta \circ \alpha$ satisfies $g.x_i = x'_i$ for all $i = 1, \ldots, m$. □

2.2. **Proof of Theorem 2.5.** To deduce Theorem 2.5 we need a few preparations. As before X stands for an irreducible affine algebraic variety. Let us introduce the following technical notion.

Definition 2.6. Let $G \subseteq \mathrm{SAut}(X)$ be a $\mathbb{G}_a$-generated subgroup and let $\Omega \subseteq X$ be a subset invariant under the G-action, i.e. $G.\Omega \subseteq \Omega$. We say that a locally nilpotent vector field $\partial \in \mathrm{LND}(G)$ with associated one-parameter subgroup $H = \exp(\Bbbk\partial)$ satisfies the *orbit separation property* on Ω, if there is an H-stable subset $U(H) \subseteq \Omega$ such that

(a) for each G-orbit O contained in Ω, the intersection $U(H) \cap O$ is open and dense in O, and
(b) the global H-invariants $\mathcal{O}(X)^H$ separate all one-dimensional H-orbits in $U(H)$.

The reader should note that we allow $U(H) \cap O$ to contain or even to consist of 0-dimensional H-orbits. We also emphasize that Ω can be *any* union of orbits and can e.g. contain orbits in the singular part of X. As a trivial case, if $\partial = 0$ then $H = \{1\}$ and the orbit separation property is trivially satisfied with $U(H) = \Omega$.

Similarly we say that a set of locally nilpotent vector fields $\mathcal{N}$ satisfies the *orbit separation property* on Ω if this holds for every $\partial \in \mathcal{N}$.

As we shall see in Example 2.14 the orbit separation property is not necessarily satisfied on every G-stable subset. However, there are interesting geometric situations where this property holds for arbitrary subsets stabilized by G, see Subsection 3.2. In the following remarks we indicate possible choices of a good set Ω.

Remarks 2.7. 1. Let ∂ be a locally nilpotent vector field on X and let $H = \exp(\Bbbk\partial)$ be the subgroup of $\mathrm{SAut}(X)$ generated by ∂. According to [44, Theorem 3.3] the field of rational invariants $\mathcal{M}er(X)^H$ is the quotient field of the ring $\mathcal{O}(X)^H$ of regular invariants. Hence by a corollary of the Rosenlicht theorem on rational invariants (see [44, Proposition 3.4]) the regular invariants $\mathcal{O}(X)^H$ separate orbits on an H-invariant open dense subset $U(H)$ of X. With such a set $U(H)$ condition (b) in Definition 2.6 is automatically satisfied.

If furthermore $\Omega = O$ is an open G-orbit in X then $U(H)$ can be chosen to be contained in O so that the other requirements of Definition 2.6 are as well satisfied. It follows that *every* $\partial \in \mathrm{LND}(G)$ *satisfies the orbit separation property on an open* G*-orbit* $\Omega = O$ *in* X.

2. In a similar fashion, given a locally nilpotent derivation $\partial \in \mathrm{LND}(G)$, it satisfies the orbit separation property on any set Ω which is a union of G-orbits meeting $U(H)$,

where $H = \exp(\Bbbk\partial)$. In particular this property holds for general G-orbits (cf. Corollary 1.12).

3. Suppose that Ω in Definition 2.6 consists of a single G-orbit O. Let $\partial \in \mathrm{LND}(G)$ and $U(H)$ be as in 2.6. Shrinking $U(H)$ if necessary we can achieve that $U(H)$ has a geometric quotient $U(H)/H$, which admits a locally closed embedding into some $\mathbb{A}^N$ by regular invariants in $\mathcal{O}(X)^H$ (cf. also [44, Theorem 4.4]).

We need the following simple Lemma.

Lemma 2.8. *If a locally nilpotent vector field $\partial \in \mathrm{LND}(G)$ satisfies the orbit separation property on a G-stable subset $\Omega \subseteq X$ then also every replica $f\partial$, $f \in \ker \partial$, and every g-conjugate $g^*(\partial) = g \circ \partial \circ g^{-1}$, $g \in G$, has this property.*

Proof. Let $\partial' = f\partial$ be a replica of ∂ with associated one-parameter subgroup H'. In the case $f = 0$ the assertion is obvious. Otherwise the one dimensional orbits of H' are also one dimensional orbits of H, and the H and H' invariant functions are the same. Hence setting $U(H') = U(H)$, (a) and (b) in Definition 2.6 are again satisfied for H'. The fact that any g-conjugate of ∂ has again the orbit separation property can be left to the reader. □

For the remaining part of this subsection we fix the following notation.

2.9. Let $G \subseteq \mathrm{SAut}(X)$ be a $\mathbb{G}_a$-generated subgroup generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields. Let $\Omega \subseteq X$ be a G-stable subset. We choose $\partial_1, \ldots, \partial_s \in \mathcal{N}$ with associated one-parameter subgroups $H_\sigma = \exp(\Bbbk\partial_\sigma)$. We assume in Lemma 2.10 below that the following two conditions are satisfied.

(1) $\partial_1, \ldots, \partial_s \in \mathcal{N}$ span $T_x(G.x)$ for every point $x \in \Omega$ (see 1.21), and
(2) ∂_σ has the orbit separation property on Ω for all $\sigma = 1, \ldots, s$.

Consequently there are subsets $U(H_\sigma) \subseteq \Omega$ such that conditions (a) and (b) in Definition 2.6 are satisfied with $H = H_\sigma$. We let

$$V = \bigcap_{\sigma=1}^{s} U(H_\sigma).$$

In particular,

(i) $V \cap O$ is open and dense in O for every orbit O contained in Ω, and
(ii) any two points in V in different one dimensional H_σ-orbits can be separated by an H_σ-invariant function on X for all $\sigma = 1, \ldots, s$.

Lemma 2.10. *With the notation and assumptions as in 2.9 above, for any pair of distinct points $x, y \in \Omega$ lying in G-orbits of dimension ≥ 2 there exists an automorphism $g \in G$ such that*

(a) $g.x,\ g.y \in V$, and
(b) $g.x$ and $g.y$ are lying in different H_σ-orbits[16] *for all $\sigma = 1, \ldots, s$.*

Proof. (a) Since G acts transitively on every G-orbit O in Ω and $V \cap O$ is dense in O, we can find $g \in G$ with $g.x \in V$. Replacing x by $g.x$ we may assume that $x \in V$. For

[16]Possibly of dimension 0.

some $\sigma \in \{1, \dots, s\}$ we have $H_\sigma.y \cap V \neq \varnothing$. Taking $h \in H_\sigma$ general we have $h.x \in V$ and $h.y \in V$, as required.

(b) By (a) we may assume that $x, y \in V$. The property that $g.x$ and $g.y$ are in different H_σ-orbits is an open condition for (x, y) running over the space $V \times V$. Thus by recursion it suffices to find $g \in G$ such that (b) is satisfied for a fixed σ. If x and y are already in different H_σ-orbits then there is nothing to show.

So suppose that this is not the case and so x, y are sitting on the same H_σ-orbit, which is then necessarily one dimensional. By assumption the vector fields $\partial_1, \dots, \partial_s$ span the tangent space $T_x(G.x)$ at x, and the G-orbit of x has dimension ≥ 2. Hence ∂_τ is not tangent to $H_\sigma.x$ at x for some τ. In particular the orbits $H_\sigma.x$ and $H_\tau.x$ are both of dimension one and have only finitely many points in common.

If x and y are in different H_τ-orbits then we can choose a global H_τ-invariant f with $f(x) = 1$ and $f(y) = 0$. The group $H = \exp(\Bbbk f\partial_\tau)$ is contained in G, fixes y and moves x along $H_\tau.x$. Hence for a general $g \in H_\tau$ the points $g.x$ and $g.y = y$ lie on different H_σ-orbits.

Assume now that x and y belong to the same H_τ-orbit. We claim that again $g.x$ and $g.y$ are in different H_σ-orbits for a general $g \in H_\tau$.

To show this claim we consider $h_t = \exp(t\partial_\tau) \in H_\tau$. By assumption $h_a.x = y$ for some $a \in \Bbbk$, $a \neq 0$. We can find an H_σ-invariant function f on X, which induces a polynomial $p(t) = f(h_t.x)$ of positive degree in $t \in \Bbbk$. If $h_t.x$ and $h_t.y$ are in the same H_σ-orbits for a general $t \in \Bbbk$ then

$$p(t) = f(h_t.x) = f(h_t.y) = f(h_t.(h_a.x)) = f(h_{a+t}.x) = p(t + a),$$

which is impossible. Hence for a general $g = h_t \in H_\tau$ the points $g.x$ and $g.y$ are in different H_σ-orbits, as desired. □

Lemma 2.11. *With the notations as in 2.9 assume that $x, y \in V$ are distinct points lying in different (possibly zero dimensional) H_σ-orbits for all $\sigma = 1, \dots, s$. Then the vector fields $\partial \in \mathcal{N}$ vanishing at x span $T_y(G.y)$.*

Proof. The vectors $\partial_\sigma(y)$ with $1 \leq \sigma \leq s$ span the tangent space $T_y(G.y)$. Thus it suffices to find replicas $\partial'_1, \dots, \partial'_s$ of $\partial_1, \dots, \partial_s$, which vanish at x and are equal to ∂_σ at the point y.

If the H_σ-orbit of x is a point, then necessarily ∂_σ vanishes at x and we can choose $\partial'_\sigma = \partial_\sigma$. If the H_σ-orbit of y is a point then $\partial_\sigma(y) = 0$ and so we can take $\partial'_\sigma = 0$. Assume now that both $H_\sigma.x$ and $H_\sigma.y$ are one dimensional. By our construction of V there is an H_σ-invariant function f_σ on X with $f_\sigma(x) = 0$ and $f_\sigma(y) = 1$. Hence $\partial'_\sigma = f_\sigma \partial_\sigma$ is a locally nilpotent vector field on X vanishing in x and equal to ∂_σ at y. □

Corollary 2.12. *For each $x \in \Omega$ and every G-orbit $O \subseteq \Omega$ the group $G_{\mathcal{N},x}$ as in 2.4 acts transitively on $O \backslash \{x\}$.*[17]

Proof. Let y be a point in $O \backslash \{x\}$. With the notations as in 2.9, according to Lemma 2.10 there is an automorphism $g \in G$ with $g.x,\ g.y \in V$ such that $g.x,\ g.y$ are in different (possibly 0-dimensional) H_σ-orbits for $i = 1, \dots, s$. By Lemma 2.11 the

[17]In particular, it is transitive on O if $x \notin O$.

vector fields $\partial \in \mathcal{N}$ vanishing at $g.x$ span $T_{g.y}(O)$. Using the fact that $\mathcal{N}$ is stable under conjugation by elements $g \in G$ it follows that the vector fields in $\mathcal{N}$ vanishing at x span the tangent space $T_y(O)$. In other words, y is a $G_{\mathcal{N},x}$-flexible point on the orbit closure $\bar{O}$. Applying 1.11 we obtain that $G_{\mathcal{N},x}$ acts transitively on $O\backslash\{x\}$. □

Proof of Theorem 2.5. By Remark 2.7(1) the orbit separation property is satisfied on the open orbit $\Omega = O$. Given a set $Z = \{x_1, \ldots, x_m\} \subseteq O$ of m distinct points we consider $Z_\mu = \{x_1, \ldots, x_\mu\}$ for $\mu = 1, \ldots, m$. Let us show by induction on μ that $G_{\mathcal{N},Z_\mu}$ acts transitively on $O\backslash Z_\mu$. For $\mu = 1$ this follows from Corollary 2.12. Assuming for some $\mu < m$ that $G_{\mathcal{N},Z_\mu}$ acts transitively on $O\backslash Z_\mu$, Corollary 2.12 implies that $(G_{\mathcal{N},Z_\mu})_{x_{\mu+1}} = G_{\mathcal{N},Z_{\mu+1}}$ acts transitively on $O\backslash Z_{\mu+1}$. □

2.3. **Examples of non-separation of orbits.** Suppose as before that a subgroup $G \subseteq \mathrm{SAut}(X)$ is generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields. Then $\mathcal{N}$ satisfies the orbit separation property 2.6 on a general G-orbit (see Remark 2.7(2)), while this is not always true on an arbitrary G-orbit. Furthermore, the following example shows that on the union of two G-orbits this property might fail although it is satisfied on every single orbit.

Example 2.13. On the affine 4-space $\mathbb{A}^4 = \mathrm{Spec}\,\Bbbk[X, Y, Z, U]$ let us consider the locally nilpotent vector fields

$$\partial_1 = Y\frac{\partial}{\partial X} + Z\frac{\partial}{\partial Y} \quad \text{and} \quad \partial_2 = \frac{\partial}{\partial U}\,.$$

Let $G \subseteq \mathrm{SAut}(\mathbb{A}^4)$ be the $\mathbb{G}_a$-generated subgroup generated by ∂_1, ∂_2 and all their replicas, and let $\mathcal{N} \subseteq \mathrm{LND}(G)$ denote the saturated set generated by ∂_1 and ∂_2. Note that $O_\pm = \{Y = \pm 1,\ Z = 0\}$ is naturally isomorphic to the (X, U)-plane with the restrictions of ∂_1 and ∂_2 given by $\pm\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial U}$, respectively. Therefore $O_\pm$ is a G-orbit. It is easily seen that $\ker \partial_1 = \Bbbk[Z,\, Y^2 - 2XZ, U]$ and $\ker \partial_2 = \Bbbk[X, Y, Z]$. Hence the G-orbits O_+ and O_- are not separated by H_i-invariants, where $H_i = \exp(\Bbbk\partial_i)$, $i = 1, 2$. In particular, $\mathcal{N}$ does not satisfy the orbit separation property on $O_+ \cup O_-$. However, this property is satisfied on $\Omega = O_+$ and also on $\Omega = O_-$ separately as this is the case for ∂_1 and ∂_2 (cf. Lemma 2.8).

We note also that the isomorphism $\sigma : O_+ \to O_-$ with $\sigma(x, 1, 0, u) = (-x, -1, 0, u)$ commutes with the actions of H_1 and H_2. Hence there is no collective transitivity on $O_+ \cup O_-$ in the sense of Theorem 3.1 below, while G acts on every single orbit $O_\pm$ indeed infinitely transitively.

According to our next example one cannot expect infinite transitivity of G on an arbitrary G-orbit O of dimension ≥ 2 without assuming the orbit separation property on O. However, compare Theorem 3.1 below for a positive result.

Example 2.14. Consider the locally nilpotent derivations

$$\partial_1 = Y\frac{\partial}{\partial X} + Z\frac{\partial}{\partial Y} + U\frac{\partial}{\partial Z} \quad \text{and} \quad \partial_2 = Z\frac{\partial}{\partial X} + X\frac{\partial}{\partial Y} + U\frac{\partial}{\partial Z}$$

of the polynomial ring $\Bbbk[X,Y,Z,U]$. We claim that $\ker\partial_1 = \Bbbk[p_1,p_2,p_3,p_4]$, where

$$p_1 = U, \quad p_2 = Z^2 - 2YU, \quad p_3 = Z^3 - 3YZU + 3XU^2, \quad \text{and}$$
$$p_4 = \frac{p_2^3 - p_3^2}{p_1^2} = 9X^2U^2 - 18XYZU + 6XZ^3 - 3Y^2Z^2 + 8Y^3U\,.$$

Indeed, the image of the map

$$\rho = (p_1,\ldots,p_4) : \mathbb{A}^4 \to \mathbb{A}^4$$

is contained in the hypersurface

$$F = \{X_1^2X_4 - X_2^3 + X_3^2 = 0\}$$

which is singular along the line $F_{\rm sing} = \{X_1 = X_2 = X_3 = 0\}$. Being regular in codimension one, F is normal. We have

$$\bar{0} \in F_{\rm sing} \quad \text{and} \quad \rho^{-1}(\bar{0}) = \rho^{-1}(F_{\rm sing}) = \{Z = U = 0\} =: L \subseteq \mathbb{A}^4\,.$$

By the Weitzenböck Theorem (see e.g. [32]) $E = \operatorname{Spec}(\ker\partial_1)$ is an affine algebraic variety. The inclusions

$$\Bbbk[p_1,p_2,p_3,p_4] \subseteq \ker\partial_1 \subseteq \Bbbk[X,Y,Z,T]$$

lead to morphisms

$$\mathbb{A}^4 \stackrel{\pi}{\longrightarrow} E \stackrel{\mu}{\longrightarrow} F\,, \qquad \text{where} \quad \mu\circ\pi = \rho\,.$$

We claim that μ is an isomorphism. Since both E and F are normal affine threefolds, by the Hartogs Principle [7, Proposition 7.1] μ is an isomorphism if it is so in codimension one. In turn, it suffices to check that μ admits an inverse morphism defined on $F_{\rm reg}$. The latter follows once we know that ρ separates the H_1-orbits in $\mathbb{A}^4$ outside the plane L and that $F_{\rm reg} = \rho(\mathbb{A}^4\backslash L)$. Indeed, then π separates them as well, and μ induces a bijection between $\pi(\mathbb{A}^4\backslash L) \subseteq E$ and $F_{\rm reg}$.

The action of H_1 on $\mathbb{A}^4$ is given by

$$(-t).\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} x + ty + \frac{t^2}{2}z + \frac{t^3}{6}u \\ y + tz + \frac{t^2}{2}u \\ z + tu \\ u \end{pmatrix}. \tag{9}$$

Let O be an H_1-orbit contained in $\mathbb{A}^4\backslash L$. Suppose first that $p_1|O = U|O = u \neq 0$. Letting $t = -z/u$ in (9) we get a point $A = (x,y,0,u) \in O$. Since

$$p_2(A) = -2yu \quad \text{and} \quad p_3(A) = 3xu^2$$

we can recover the coordinates

$$y = -(p_2|O)/2u \quad \text{and} \quad x = (p_3|O)/3u^2\,. \tag{10}$$

Thus O is uniquely determined by the image $\rho(O) \in F$.

Suppose further that $p_1|O = U|O = 0$. Since $O \cap L = \varnothing$ then $Z|O = z \neq 0$. Taking $t = -y/z$ in (9) yields a point $A = (x,0,z,0) \in O$. Since

$$p_2(A) = z^2,\ p_3(A) = z^3, \quad \text{and} \quad p_4(A) = 6xz^3$$

we can recover the values[18]

$$z = (p_3|O)/(p_2|O) \quad \text{and} \quad x = (p_4|O)/6z^3 \,. \tag{11}$$

Now both claims follow.

The generators $p_1, \ldots, p_4$ of the algebra of H_1-invariants vanish on the plane $L = \{Z = U = 0\}$ so that every H_1-invariant is constant on L. Since ∂_2 is obtained from ∂_1 by interchanging X, Y, by symmetry also every H_2-invariant is constant on L. Letting $G = \langle \mathrm{Sat}(H_1, H_2) \rangle$ be the subgroup generated by H_1, H_2, and all their replicas, it easily follows that G stabilizes L and that the G-action on L factors through the SL_2-action associated with the $\mathfrak{sl}_2$-algebra generated by the vector fields $\partial_1|_L = Y\frac{\partial}{\partial X}$ and $\partial_2|_L = X\frac{\partial}{\partial Y}$. In particular, the action of G on its orbit $L \backslash \{0\}$ is not even 2-transitive, the linear dependence being an obstruction.

Observe finally that the three dimensional G-orbits in $\mathbb{A}^4$ are separated by the G-invariant function U.

2.4. **G_Y-orbits.** Let as before $G \subseteq \mathrm{SAut}(X)$ be generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields on an irreducible affine variety $X = \mathrm{Spec}\, A$. For a subvariety $Y \subseteq X$ we consider the subgroup $G_{\mathcal{N},Y}$ of G as defined in (8).

Theorem 2.15. *Under notation as above, assume that* $\dim X \geq 2$ *and that G acts on X with an orbit O open in X. Then letting $Y = X \backslash O$ the subgroup $G_{\mathcal{N},Y}$ acts transitively and hence infinitely transitively on O.*

Proof. Since $G_{\mathcal{N},Y}$ is generated by a saturated set of locally nilpotent derivations, by Theorem 2.2 it suffices to show that $G_{\mathcal{N},Y}$ acts transitively on O.

Using Corollary 1.21 we can choose $\partial_1, \ldots, \partial_s \in \mathcal{N}$ spanning the tangent space $T_x X$ at each point $x \in O$. Letting I denote the ideal of Y in A, we claim that for every $\sigma = 1, \ldots, s$ there is a nonzero function $f_\sigma \in I \cap \ker \partial_\sigma$. Let $H_\sigma = \exp(\Bbbk \partial_\sigma) \subseteq G$, $\sigma = 1, \ldots, s$. The set Y being G-invariant, for every nonzero function $f \in I$ the orbit $H_\sigma . f$ spans in A an H_σ-invariant finite dimensional subspace E_σ contained in I. By the Lie-Kolchin Theorem there is a nonzero element $f_\sigma \in E_\sigma$ fixed by H_σ. This proves the claim.

Let $p \in X$ be a general point so that $f_\sigma(p) \neq 0$ for $\sigma = 1, \ldots, s$. We can normalize the invariants f_σ so that $f_\sigma(p) = 1$ and $f_\sigma|Y = 0$. The derivation $f_\sigma \partial_\sigma$ then vanishes on Y and so the replica $H_{\sigma, f_\sigma} = \exp(\Bbbk f_\sigma \partial_\sigma) \subseteq G_{\mathcal{N},Y}$ of H_σ fixes Y pointwise while moving p in the direction of $\partial_\sigma(p)$. It follows that the $G_{\mathcal{N},Y}$-orbit of p is open in O.

Let now $q \in O$ be an arbitrary point. Choose $g \in G$ with $g.p = q$. Since g stabilizes Y the subgroup $H'_{\sigma, f_\sigma} = g H_{\sigma, f_\sigma} g^{-1} \subseteq G_{\mathcal{N},Y}$ fixes Y pointwise and moves q into the direction of $dg(\partial_\sigma)(q)$. It follows that also the $G_{\mathcal{N},Y}$-orbit of q is open. Finally $G_{\mathcal{N},Y}$ has O as an open orbit. □

Remark 2.16. [19] The conclusion of the theorem remains valid if we replace the group $G_{\mathcal{N},Y}$ by its subgroup $G^m_{\mathcal{N},Y}$ generated by the replicas in $\mathcal{N}$ which vanish on the mth infinitesimal neighborhood of Y in X, where $m \in \mathbb{N}$. Indeed, it suffices to replace the functions f_σ in the proof by their mth powers f^m_σ.

[18] Formulas (10) and (11) define sections of ρ in the open sets $U \neq 0$ and $Z \neq 0$, respectively. This shows that $\rho : \mathbb{A}^4 \backslash L \to F_{\mathrm{reg}}$ is a principal $\mathbb{A}^1$-bundle.

[19] We are grateful to M.H. Gizatullin for this observation.

3. Collective infinite transitivity

3.1. **Collective transitivity on G-varieties.** By *collective infinite transitivity* we mean a possibility to transform simultaneously (that is, by the same automorphism) an arbitrary finite set of points along their orbits into some given position. Applying the methods developed in Section 2 we can deduce the following generalization of Theorem 2.5. Below X stands as usual for an irreducible affine algebraic variety.

Theorem 3.1. *Let $G \subseteq \mathrm{SAut}(X)$ be a subgroup generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields, which has the orbit separation property on a G-invariant subset $\Omega \subseteq X$. Suppose that $x_1, \ldots, x_m$ and $x'_1, \ldots, x'_m$ are points in Ω with $x_i \neq x_j$ and $x'_i \neq x'_j$ for $i \neq j$ such that for each j the orbits $G.x_j$ and $G.x'_j$ are equal and of dimension ≥ 2. Then there exists an element $g \in G$ such that $g.x_j = x'_j$ for $j = 1, \ldots, m$.*

As in Section 2 this will be deduced from the following more technical result.

Theorem 3.2. *Let $G \subseteq \mathrm{SAut}(X)$ be a subgroup generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields. Suppose that $\mathcal{N}$ has the orbit separation property on a G-invariant subset Ω. If $Z \subseteq \Omega$ is a finite subset and $O \subseteq \Omega$ is an orbit of dimension ≥ 2, then the group $G_{\mathcal{N},Z}$ acts transitively on $O \backslash Z$.*

Proof. With $Z_\mu = \{x_1, \ldots, x_\mu\}$ let us show by induction on μ that $G_{\mathcal{N},Z_\mu}$ acts transitively on $O \backslash Z_\mu$ for every G-orbit $O \subseteq \Omega$ of dimension ≥ 2. For $\mu = 1$ this is just Corollary 2.12. Assuming for some $\mu < m$ that $G_{\mathcal{N},Z_\mu}$ acts transitively on $O \backslash Z_\mu$, Corollary 2.12 also implies that $(G_{\mathcal{N},Z_\mu})_{x_\mu+1} = G_{\mathcal{N},Z_{\mu+1}}$ acts transitively on $O \backslash Z_{\mu+1}$. Note that by Lemma 2.8 at each step the set $\mathcal{N}_{Z_\mu}$ has again the orbit separation property on Ω so that Corollary 2.12 is indeed applicable. □

Proof of Theorem 3.1. As in the proof Theorem 2.2 we proceed by induction on m, the case $m = 1$ being trivial. For the induction step suppose that there is already an automorphism $\alpha \in G$ with $\alpha.x_i = x'_i$ for $i = 1, \ldots, m-1$. Applying Theorem 3.2 to $Z = \{x'_1, \ldots, x'_{m-1}\}$ we can also find an automorphism $\beta \in G_{\mathcal{N},Z}$ with $\beta(\alpha(x_m)) = x'_m$. Clearly then $g = \beta \circ \alpha$ has the required property. □

3.2. **Infinite transitivity on matrix varieties.** In this subsection we apply our methods in a concrete setting where $X = \mathrm{Mat}(n, m)$ is the set of all $n \times m$ matrices over $\Bbbk$ endowed with the natural stratification by rank. We assume below that $mn \geq 2$. Let us introduce the following terminology.

Let $X_r \subseteq X$ denote the subset of matrices of rank $\leq r$. The product $\mathrm{SL}_n \times \mathrm{SL}_m$ acts naturally on X via the left-right multiplication preserving the strata X_r. For every $k \neq l$ we let $E_{kl} \in \mathfrak{sl}_n$ and $E^{kl} \in \mathfrak{sl}_m$ denote the nilpotent matrices with $x_{kl} = 1$ and the other entries equal zero[20]. Let further $H_{kl} = I_n + \Bbbk E_{kl} \subseteq \mathrm{SL}_n$ and $H^{kl} = I_m + \Bbbk E^{kl} \subseteq \mathrm{SL}_m$ be the corresponding one-parameter unipotent subgroups in the first and the second factor of $\mathrm{SL}_n \times \mathrm{SL}_m$, respectively, acting on the stratification $X = \bigcup_r (X_r \setminus X_{r-1})$ in a natural way. We also let δ_{kl} and δ^{kl}, respectively, denote the corresponding locally nilpotent vector fields on X tangent to the strata.

[20]Notice that $E_{kl} = E^{kl}$ if $m = n$.

We call *elementary* the one-parameter unipotent subgroups H_{kl}, H^{kl}, and all their replicas. In the following theorem we establish the collective infinite transitivity on the above stratification of the subgroup G of $\mathrm{SAut}(X)$ generated by the two sides elementary subgroups (cf. [47]).

By a well known theorem of linear algebra, the subgroup $\mathrm{SL}_n \times \mathrm{SL}_m \subseteq G$ acts transitively on each stratum $X_r \setminus X_{r-1}$ (and so these strata are G-orbits) except for the open stratum $X_n \setminus X_{n-1}$ in the case where $m = n$. In the latter case the G-orbits contained in $X_n \setminus X_{n-1}$ are the level sets of the determinant.

Theorem 3.3. *Given two finite ordered collections $\mathcal{B}$ and $\mathcal{B}'$ of distinct matrices in* $\mathrm{Mat}(n,m)$ *of the same cardinality, with the same sequence of ranks, and in the case where $m = n$ with the same sequence of determinants, we can simultaneously transform $\mathcal{B}$ into $\mathcal{B}'$ by means of an element $g \in G$, where $G \subseteq \mathrm{SAut}(\mathrm{Mat}(n,m))$ is the subgroup generated by all elementary one-parameter unipotent subgroups.*

Choosing in particular $\mathcal{B}'$ consisting of diagonal matrices we obtain a simultaneous diagonalization of the matrices in $\mathcal{B}$.

Theorem 3.3 is an immediate consequence of Theorem 3.1 and Lemma 3.4 below. To formulate this lemma, we let $\mathcal{N}$ be the saturated set of locally nilpotent derivations on X generated by all locally nilpotent vector fields δ_{kl} and δ^{kl} $(k \neq l)$ that is, the set of all conjugates of these derivations along with their replicas. The important observation is the following lemma.

Lemma 3.4. *$\mathcal{N}$ has the orbit separation property on $\Omega = X$.*

Proof. In view of Lemma 2.8 it suffices to show that the derivations δ_{kl} and δ^{kl} have the orbit separation property. Clearly it suffices to prove this for δ^{kl}. The action of the corresponding one-parameter subgroup $H^{kl} = \exp(\Bbbk\delta^{kl})$ on a matrix $B = (b_1, \ldots, b_m) \in X$ with column vectors $b_1, \ldots, b_m \in \Bbbk^n$ is explicitely given by

$$\exp(t\delta^{kl}).B = (b_1, \ldots, b_l + tb_k, \ldots, b_n),$$

where $b_l + tb_k$ is the lth column of the matrix on the right. Thus the H^{kl}-orbit of B has dimension one if and only if $b_k \neq 0$. The functions

$$B \mapsto b_{ij}\ (j \neq l) \quad \text{and} \quad B \mapsto \begin{vmatrix} b_{ik} & b_{il} \\ b_{jk} & b_{jl} \end{vmatrix}\ (i \neq j)$$

on $\mathrm{Mat}(n,m)$ are H^{kl}-invariants that obviously separate all H^{kl}-orbits of dimension one, as the reader may easily verify. □

3.3. **The case of symmetric and skew-symmetric matrices.** We can apply the same reasoning to the varieties

$$X = \mathrm{Spec}(\Bbbk[T_{ij}]_{1\le i,j\le n}/(T_{ij} - T_{ji})_{1\le i,j\le n})$$

of symmetric $n \times n$ matrices over $\Bbbk$ and to the variety

$$Y = \mathrm{Spec}(\Bbbk[T_{ij}]_{1\le i,j\le n}/(T_{ij} + T_{ji})_{1\le i,j\le n})$$

of skew symmetric matrices. The group SL_n acts on both varieties via

$$A.B = ABA^T \text{, where } A \in \mathrm{SL}_n \text{ and } B \in X (\in Y, \text{ resp.}).$$

The subvariety X_r of symmetric matrices of rank r in X is stabilized by this action, and also the determinant of a matrix is preserved. In the skew symmetric case again the subvarieties Y_r of matrices of rank r are stabilized, and also the Pfaffian $\mathrm{Pf}(B)$[21] of a matrix $B \in Y$ is preserved.

By a well known theorem in linear algebra the orbits of the SL_n-action on X are the subsets X_r of matrices of rank r in X for $r < n$, whereas for $r = n$ the orbits are the level sets of the determinant. Similarly, the orbits of the SL_n-action on Y are the subsets Y_r for $r < n$, whereas for $r = n$ the orbits are the level sets of the Pfaffian.

As in subsection 3.2 the elementary matrix E_{ij} $(i \neq j)$ generates a one-parameter subgroup $H_{ij} = I_n + \Bbbk E_{ij}$ that acts on X and on Y. The corresponding locally nilpotent vector field will be denoted by δ_{ij}.

Let $G_{\mathrm{sym}} \subseteq \mathrm{SAut}(X)$ and $G_{\mathrm{skew}} \subseteq \mathrm{SAut}(Y)$, respectively, be the subgroups generated by all H_{ij} along with their replicas. With these notations the following results hold.

Theorem 3.5. *Let $M_1, \dots, M_k$ be a sequence of pairwise distinct symmetric matrices of order $n \geq 2$ over $\Bbbk$. Assume that $M'_1, \dots, M'_k$ is another such sequence with*

$$\mathrm{rk}(M_i) = \mathrm{rk}(M'_i) \geq 2 \quad \textit{and} \quad \det(M_i) = \det(M'_i) \quad \forall i = 1, \dots k\,.$$

Then there exists an automorphism $g \in G_{\mathrm{sym}}$ with $g.M_i = M'_i$ for $i = 1, \dots k$.

A similar result holds in the skew symmetric case.

Theorem 3.6. *Let $M_1, \dots, M_k$ be a sequence of pairwise distinct skew-symmetric matrices of order $n \geq 2$ over $\Bbbk$. Assume that $M'_1, \dots, M'_k$ is another such sequence with*

$$\mathrm{rk}(M_i) = \mathrm{rk}(M'_i) \quad \textit{and} \quad \mathrm{Pf}(M_i) = \mathrm{Pf}(M'_i) \quad \forall i = 1, \dots k\,.$$

Then there exists an automorphism $g \in G_{\mathrm{skew}}$ with $g.M_i = M'_i$ for $i = 1, \dots k$.

We give a sketch of the proof in the symmetric case only and leave the skew-symmetric one to the reader. As in the case of generic matrices (see Theorem 3.3) Theorem 3.5 is an immediate consequence of Theorem 3.2 and Lemma 3.7 below. In this lemma we let $\mathcal{N}$ be the saturated set of locally nilpotent derivations on X generated by all locally nilpotent vector fields δ_{kl}.

Lemma 3.7. *$\mathcal{N}$ has the orbit separation property on $\Omega = X$.*

Proof. In view of Lemma 2.8 it suffices to show that the derivations δ_{kl} have the orbit separation property. We only treat the case $k < l$ the other one being similar. The action of the corresponding one-parameter subgroup $H_{kl} = \exp(\Bbbk\delta_{kl})$ on a matrix $B \in X$ with entries $b_{ij} = b_{ji}$ is explicitly given by $\exp(t\delta_{kl}).B = (b'_{ij})$, where

$$b'_{ij} = b_{ij} \text{ if } i, j \neq k\,, \quad b'_{ki} = b'_{ik} = b_{ik} + tb_{il} \text{ if } i \neq k\,, \quad \text{and} \quad b'_{kk} = b_{kk} + 2tb_{kl} + t^2 b_{ll}\,.$$

Thus the H_{kl}-orbit of B has dimension 0 if and only if $b_{il} = 0\,\forall i$. The functions

$$B \mapsto b_{ij}\ (i, j \neq k)\,, \quad B \mapsto \begin{vmatrix} b_{ik} & b_{il} \\ b_{jk} & b_{jl} \end{vmatrix}\ (i, j \neq k)\,, \quad \text{and} \quad B \mapsto \begin{vmatrix} b_{kk} & b_{kl} \\ b_{lk} & b_{ll} \end{vmatrix}$$

are H_{kl}-invariants that are easily seen to separate all H_{kl}-orbits of dimension one. □

[21]We keep the usual convention that the Pfaffian of a matrix of odd order equals zero.

4. Tangential flexibility, interpolation by automorphisms, and $\mathbb{A}^1$-richness

4.1. **Flexibility of the tangent bundle.** We start with the following fact (see the Claim in the proof of Corollary 2.8 in [27]).

Lemma 4.1. *Let ∂ be a locally nilpotent vector field on the affine $\Bbbk$-scheme $X = \operatorname{Spec} A$ and let $p \in X$ be a point. Assume that $f \in \ker\partial$ is an invariant of ∂ with $f(p) = 0$. If $\Phi = \exp(f\partial)$ is the automorphism associated with the locally nilpotent vector field $f\partial$, then*

$$d_p\Phi(w) = w + df(w)\partial(p) \quad \text{for all } w \in T_pX\,. \tag{12}$$

Proof. The tangent space T_pX is the space of all derivations $w : A \to \Bbbk$ centered at p. For such a tangent vector w its image $d\Phi(w) \in T_pX$ is the derivation

$$\begin{aligned} A \ni g \mapsto w(\Phi(g)) &= w\left(\sum_{i\geq 0} \frac{f^i\partial^i(g)}{i!}\right) = \sum_{i\geq 0}\frac{1}{i!}w(f^i\partial^i(g)) \\ &= w(g) + w(f)\partial(g)(p)\,, \end{aligned}$$

as $f(p) = 0$. Since by definition $w(f) = df(w)$, the result follows. □

Now we can show the following result.

Theorem 4.2. *Let X be an irreducible affine algebraic variety and let $G \subseteq \operatorname{Aut}(X)$ be a subgroup generated by a saturated set $\mathcal{N}$ of locally nilpotent vector fields. Assume that $\mathcal{N}$ satisfies the orbit separation property on a G-orbit O. Then for each point $p \in O$, associating to an automorphism $g \in G_{\mathcal{N},p}$ its tangent map $dg(p)$ yields a representation*

$$\tau : G_{\mathcal{N},p} \longrightarrow \operatorname{GL}(T_pO) \quad \textit{with} \quad \tau(G_{\mathcal{N},p}) = \operatorname{SL}(T_pO)\,.$$

Proof. The assertion is trivially true if $\dim O = 1$. Let us assume for the rest of the proof that $\dim O \geq 2$. For any one-parameter unipotent subgroup H in $G_{\mathcal{N},p}$ the image $\tau(H)$ is a subgroup of $\operatorname{SL}(T_pO)$. Hence also $\tau(G_{\mathcal{N},p}) \subseteq \operatorname{SL}(T_pO)$. Let us show the converse inclusion.

According to Proposition 1.8 there are locally nilpotent vector fields $\partial_1, \ldots, \partial_s \in \mathcal{N}$ spanning T_xO at every point $x \in O$. Let $H_j = \exp(\Bbbk\partial_j)$ be the one-parameter subgroup associated with ∂_j. Using Remark 2.7(3) there are H_j-invariant open subsets $U(H_j) \subseteq O$, $j = 1, \ldots, s$ such that the geometric quotients $\varrho_j : U(H_j) \to U(H_j)/H_j$ exist and satisfy the same properties as in 2.7(3). In particular, the image $\varrho_j(x)$ of a generic point $x \in \bigcap_{j=1}^s U(H_j)$ is a smooth point of $U(H_j)/H_j$, and ϱ_j has maximal rank at x.

We may assume that $\partial_1(x), \ldots, \partial_m(x)$, where $m = \dim O \leq s$, form a basis of T_xO. Hence for $j, \mu \in \{1, \ldots, m\}$ with $\mu \neq j$ there exist ∂_j-invariant functions $f_{\mu j}$ on X such that $f_{\mu j}(x) = 0$ and $d_xf_{\mu j}(\partial_i(x)) = \delta_{\mu i}$. Consider the automorphism $\Phi^t_{\mu j} = \exp(t \cdot f_{\mu j}\partial_j) \in G_{\mathcal{N},x}$ for $t \in \Bbbk$. According to Lemma 4.1 its tangent map at x is

$$d_x\Phi^t_{\mu j}(\partial_i(x)) = \partial_i(x) + t \cdot d_xf_{\mu j}(\partial_i(x)) \cdot \partial_j(x) = \partial_i(x) + t\delta_{\mu i}\partial_j(x)\,.$$

Thus representing the elements in $\operatorname{GL}(T_xO)$ by matrices with respect to the basis $\partial_1(x), \ldots, \partial_m(x)$, the elements $d_x\Phi^t_{\mu j} \in \operatorname{GL}(T_xO)$, $t \in \Bbbk$, form just the one-parameter unipotent subgroup generated by the elementary matrix $E_{j\mu}$. Since such one-parameter subgroups generate $\mathrm{SL}(T_xO)$, the image of $G_{\mathcal{N},x}$ in $\mathrm{GL}(T_xO)$ contains $\mathrm{SL}(T_xO)$ for a general point $x \in X$. Now the transitivity of G on O implies that the same is true for every point $p \in O$. □

The following corollary is immediate.

Corollary 4.3. *Under the assumptions of Theorem 4.2 for each point $p \in O$ we have*

$$\mathcal{N}(p) := \{\partial(p) \in T_pX : \partial \in \mathcal{N}\} = T_pO\,.$$

In particular, the nilpotent cone $\mathrm{LND}_p(G)$ coincides with the tangent space T_pO for each $p \in O$.

Proof. Indeed, the group $G_{\mathcal{N},p}$ stabilizes $\mathcal{N}(p)$ and for $m = \dim O \geq 2$ the group $\mathrm{SL}(T_pO)$ acts transitively on $T_pO\backslash\{0\}$. □

Remark 4.4. The last assertion in Corollary 4.3 does not hold any more for a general $\mathbb{G}_a$-generated subgroup $G \subseteq \mathrm{SAut}(X)$ which is not generated by a saturated set of locally nilpotent vector fields. For instance, if a semisimple algebraic group G acts on itself via left multiplications (i.e., $X = G$), then the cone $\mathrm{LND}_e(G)$ is just the usual nilpotent cone in the Lie algebra $\mathrm{Lie}(G) = T_eX$, which is a proper subcone.

We also have the following result on tangential flexibility.[22]

Corollary 4.5. *Let $\pi : E \to X$ be an irreducible and reduced linear space*[23] *over a flexible variety X, which is over X_{reg} a vector bundle. Assume that there is an action of $G := \mathrm{SAut}(X)$ on E such that the action of every 1-parameter subgroup of G is algebraic on E and π is equivariant. Then the total space E is also flexible. In particular, the tangent bundle TX and all its tensor bundles $E = (TX)^{\otimes a} \otimes (T^*X)^{\otimes b}$ are flexible.*

Proof. It suffices to check that the special automorphism group $G' = \mathrm{SAut}(E)$ acts transitively on $E_{\mathrm{reg}} = \pi^{-1}(X_{\mathrm{reg}})$. By our assumptions G can be considered as an algebraically generated subgroup of G'. Since X is flexible and π is equivariant, this subgroup acts transitively on the set of fibers of $E_{\mathrm{reg}} \to X_{\mathrm{reg}}$. Moreover, X being affine for any point $e \in E_{\mathrm{reg}}$ there is a section $V : X \to E$ with $V(\pi(e)) = e$. This section generates a $\mathbb{G}_a$-action $w \mapsto w + tV(\pi(w))$. Hence G' acts transitively on every fiber of E over a regular point, and the result follows. □

Corollary 4.6. *Let X be a flexible irreducible affine variety of dimension ≥ 2. Consider the special automorphism group $G = \mathrm{SAut}(TX)$ of the tangent bundle TX, and let $Z \subseteq TX$ be the zero section. Then the group G_Z acts infinitely transitively on $TX_{\mathrm{reg}}\backslash Z$.*

Proof. The special automorphism group $\mathrm{SAut}(X)$ induces a $\mathbb{G}_a$-generated subgroup $\tilde{G} \subseteq G$ acting on TX_{reg}. Since X is flexible this action is transitive on the zero section, hence also on the set of fibers of $TX \to X$ over X_{reg}. On the other hand, by Theorem 4.2 the stationary subgroup $\tilde{G}_p$ of a given point $p \in X_{\mathrm{reg}}$ acts on T_pX as $\mathrm{SL}(T_pX)$.

[22] We are grateful to Adrien Dubouloz whose observation allowed us to remove an inaccuracy in the original formulation.

[23] in the sense of [22] Chap. II, 1.7.

Since $\dim T_pX > 1$, it acts transitively off the origin. Finally the action of $\tilde{G}$ on TX_{reg} is transitive off the zero section. Hence by Theorem 2.2 the group G_Z, being $\mathbb{G}_a$-generated and generated by a saturated set of locally nilpotent derivations, acts infinitely transitively on $TX_{\mathrm{reg}}\backslash Z$. □

4.7. For later use let us mention the following slightly more general version of Theorem 4.2. For a finite subset $Z \subseteq X$ and $p \in O$ we let $\mathcal{N}^M_{p,Z} \subseteq \mathcal{N}$ denote the set of all locally nilpotent vector fields $\partial \in \mathcal{N}$ such that ∂ has a zero at p and a zero of order $\geq M+1$ at all points of $Z\backslash\{p\}$. Let further $G^M_{p,Z}$ be the subgroup of G generated by all exponentials of elements in $\mathcal{N}^M_{p,Z}$. Replacing in Theorem 4.2 $G_{\mathcal{N},p}$ by $G^M_{p,Z}$ the following result holds.

Proposition 4.8. *If* $\dim O \geq 2$ *then the image of the group* $G^M_{p,Z}$ *in* $\mathrm{GL}(T_pO)$ *coincides with* $\mathrm{SL}(T_pO)$.

Proof. With the notation as in the proof of *loc.cit.*, by infinite transitivity (see Theorem 3.2) it suffices to show the assertion for the case that $x = p$ is general and Z consists of general points. Under this assumption we can find ∂_j-invariant functions h_j with $h_j(x) = 1$ which vanish in all points of $Z\backslash\{p\}$. Replacing in the proof of 4.2 $f_{\mu j}$ by $h_j^{M+1} f_{\mu j}$, the automorphisms $\Phi^t_{\mu j}$ are the identity up to order M at the points of Z and remain unchanged at x. Now the same arguments as before give the conclusion. □

Let further G^M_Z have the same meaning as $G^M_{p,Z}$ above, but without any constraint imposed at p. That is, G^M_Z is the subgroup of G generated by the saturated set $\mathcal{N}^M_Z$ of locally nilpotent vector fields vanishing to order $M+1 \geq 1$ at all points of Z. Then the same argument as before proves the following proposition.

Proposition 4.9. *Every point* $p \in O\backslash Z$ *is* G^M_Z*-flexible, hence* $G^M_Z.p = O\backslash Z$.

4.2. **Prescribed jets of automorphisms.** Let us start with the following standard fact (see Proposition 6.4. in [26], cf. also Theorem 4.2). Recall that a *volume form* ω on a smooth algebraic variety X is a nowhere vanishing top-dimensional regular form on X; it does exist if and only if $K_X = 0$ in $\mathrm{Pic}(X)$.

Lemma 4.10. *If* X *is an irreducible affine algebraic variety and* $\omega \in \Omega^n_X$ *a volume form on* X_{reg}*, then* ω *is preserved under every automorphism* $g \in \mathrm{SAut}(X)$.

Proof. It suffices to show that for every locally nilpotent vector field ∂ the form ω is invariant under an automorphism of $H = \exp(\mathbb{k}\partial)$. If $h_t = \exp(t\partial)$ then for every $x \in X_{\mathrm{reg}}$ the pullback $h_t^*(\omega)(x)$ is a multiple of $\omega(x)$, i.e. $h_t^*(\omega)(x) = f(x,t)\omega(x)$, where $f(x,t) \neq 0$ for all x,t. For a fixed x the function $f(x,t)$ is thus a polynomial in one variable without zero. Hence f is independent of t and is equal to $f(x,0) = 1$. □

4.11. We adopt the following notation and assumptions. If $\varphi : X \to X$ is a morphism then its m-jet $j^m_p\varphi$ at $p \in X$ can be regarded as a map of $\mathbb{k}$-algebras

$$j^m_p\varphi : \mathcal{O}_{X,\varphi(p)}/\mathfrak{m}^{m+1}_{\varphi(p)} \longrightarrow \mathcal{O}_{X,p}/\mathfrak{m}^{m+1}_p\,,$$

where $\mathcal{O}_{X,x}$ denotes the local ring at a point $x \in X$ and $\mathfrak{m}_x$ its maximal ideal.

We assume in the sequel that $p \in X_{\mathrm{reg}}$ is a regular point and $\varphi(p) = p$. Letting $A_m = \mathcal{O}_{X,p}/\mathfrak{m}^{m+1}_p$ the m-jet of φ yields a map of k-algebras

$$j^m\varphi = j^m_p\varphi : A_m \longrightarrow A_m\,,$$

which stabilizes the maximal ideal $\mathfrak{m}$ of A_m and all of its powers $\mathfrak{m}^k$.

For $m \geq 1$ we let $\mathrm{Aut}_{m-1}(A_m)$ denote the set of $\mathbb{k}$-algebra isomorphisms $f : A_m \to A_m$ with $f \equiv \mathrm{id} \mod \mathfrak{m}^m$. For every $f \in \mathrm{Aut}_{m-1}(A_m)$ the map $f - \mathrm{id}$ sends A_m into $\mathfrak{m}^m$ and vanishes on the constants $\mathbb{k}$. As it vanishes as well on $\mathfrak{m}^2$ it induces a $\mathbb{k}$-linear map

$$\psi_f : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{m}^m = \mathfrak{m}_p^m/\mathfrak{m}_p^{m+1}\,.$$

Note that $\mathfrak{m}^m$ is naturally isomorphic to the mth symmetric power S^mV of the $\mathbb{k}$-vector space $V = \mathfrak{m}/\mathfrak{m}^2$. For every $m \geq 1$ our construction yields a map

$$\psi : \mathrm{Aut}_{m-1}(A_m) \longrightarrow \mathrm{Hom}_{\mathbb{k}}(\mathfrak{m}/\mathfrak{m}^2, \mathfrak{m}^m) \cong V^\vee \otimes S^mV\,, \tag{13}$$

where $V^\vee$ stands for the dual module of V. For $m = 1$ this map associates to $f = j^1\varphi$ just the cotangent map $d\varphi(0)^\vee$.

In terms of local coordinates this construction can be interpreted as follows. The $\mathbb{k}$-algebra A_m is isomorphic to the quotient $A/\mathfrak{m}_A^{m+1}$, where $A = \mathbb{k}[[x_1, \ldots, x_n]]$ is the $\mathbb{k}$-algebra of formal power series and $\mathfrak{m}_A$ is its maximal ideal. Any map $f \in \mathrm{Aut}_{m-1}(A_m)$ is represented by an m-jet of an n-tuple of power series $F = (F_1, \ldots, F_n) \in A^n$ with $F_i \equiv x_i \mod \mathfrak{m}_A^m$. Clearly for any $m \geq 1$ the m-form ψ_f corresponds to the mth order term of F.

With this notation we have the following lemma.

Lemma 4.12. *(a) For every $m \geq 1$ the map ψ in* (13) *is bijective.*
(b) If $m = 1$ then $\psi_{f\circ g} = \psi_f \circ \psi_g$ while for $m \geq 2$ we have $\psi_{f\circ g} = \psi_f + \psi_g$.
(c) If ∂ is a locally nilpotent vector field on X with a zero of order $m \geq 2$ at p then $\psi_{\exp(t\partial)} = t\psi_{\exp(\partial)}$.

Proof. (a) is immediate using the coordinate description above.

(b) is easy and can be left to the reader. To deduce (c) we note that $\exp(t\partial) \in \mathrm{SAut}(X)$ induces the map $\mathrm{id} + t\hat{\partial} \in \mathrm{Aut}_{m-1}(A_m)$, where $\hat{\partial}$ denotes the derivation on A_m induced by ∂. Hence $\psi_{\exp(t\partial)} = t\hat{\partial}$, proving (c). □

An n-tuple $F = (F_1, \ldots, F_n) \in A^n$ as in 4.11 representing an m-jet $f = j^mF \in \mathrm{Aut}_{m-1}(A_m)$ preserves a volume form ω on X_{reg} (or on (X,p)) if and only if the Jacobian determinant J_F of F is equal to 1. Modulo $\mathfrak{m}^m$ this determinant depends only on f and not on the representative F of f. Hence we can set $J_f := J_F \mod \mathfrak{m}^m$.[24] We say in the sequel that an m-jet $f \in \mathrm{Aut}_{m-1}(A_m)$ with $m \geq 1$ preserves a volume form if $J_f \equiv 1 \mod \mathfrak{m}^m$. The latter condition can be detected in terms of ψ_f as follows.

Lemma 4.13. *(a) If $m = 1$ then $f \in \mathrm{Aut}(A_1)$ preserves a volume form if and only if $\psi_f \in \mathrm{SL}(V)$.*

In case $m \geq 2$ the map $f \in \mathrm{Aut}_{m-1}(A_m)$ preserves a volume form if and only if ψ_f is in the kernel of the natural contraction map

$$\kappa_m : \mathrm{Hom}_{\mathbb{k}}(V, S^mV) \cong V^\vee \otimes S^mV \longrightarrow S^{m-1}V\,,$$

$$\lambda \otimes v_1 \cdot \ldots \cdot v_m \longmapsto \sum_{\mu=1}^{m} \lambda(v_\mu) \cdot v_1 \cdot \ldots \cdot \hat{v}_\mu \cdot \ldots \cdot v_m.$$

[24]However J_f is not an element in A_m since it is not well defined modulo $\mathfrak{m}^{m+1}$.

(b) $\ker \kappa_m$ *is an irreducible* $\mathrm{SL}_n(V)$*-module for all* $m \geq 1$.

Proof. In case $m = 1$ (a) is immediate. Suppose that $m \geq 2$. If $f = \mathrm{id} + f_m \mod \mathfrak{m}^{m+1}$ with an n-tuple of m-forms $f_m = (f_{m1}, \ldots, f_{mn})$, then J_f is easily seen to be equal to

$$1 + \operatorname{div} f_m = 1 + \frac{\partial f_{m1}}{\partial x_1} + \ldots + \frac{\partial f_{mn}}{\partial x_n} \qquad \mod \mathfrak{m}^m\,,$$

where $\operatorname{div} f_m$ is the divergence of f_m. Thus $J_f \equiv 1 \mod \mathfrak{m}^m$ if and only if $\operatorname{div} f_m = 0$. Writing $f_m \in V^\vee \otimes S^m V$ as $f_m = \sum_{i=1}^n \frac{\partial}{\partial x_i} \otimes f_{mi}$ the element $\operatorname{div} f_m$ in $S^{m-1}V$ corresponds just to the contraction $\kappa_m(f_m)$, proving (a).

(b) is a standard fact in representation theory, see e.g. [45, §IX.10.2]. □

Now we can state our main result in this subsection.

Theorem 4.14. *Let X be an irreducible affine algebraic variety of dimension $n \geq 2$ equipped with an algebraic volume form ω defined on X_{reg}, and let $G \subseteq \mathrm{SAut}(X)$ be a subgroup generated by a saturated set $\mathcal{N}$ of locally nilpotent derivations. If G acts on X with an open orbit O, then for every $m \geq 0$ and every finite subset $Z \subseteq O$ there exists an automorphism $g \in G$ with prescribed m-jets j_p^m at the points $p \in Z$, provided these jets preserve ω and inject Z into O.*

The proof will be reduced to the following lemma.

Lemma 4.15. *With the notation and assumptions of Theorem 4.2, suppose that j_p^m is an m-jet of an automorphism at a given point $p \in Z$, which is the identity up to order $m - 1 \geq 0$. Then for every $M > 0$ there is an automorphism $g \in G$ such that its m-jet at p is j_p^m while its M-jet at each other point $q \neq p$ of Z is the identity.*

Before proving Lemma 4.15 let us show how Theorem 4.14 follows.

Proof of Theorem 4.14. We proceed by induction on m. If $m = 0$ the assertion follows from the fact that G acts infinitely transitively on O. For the induction step suppose that we have an automorphism $g \in G$ with the prescribed jets up to order $m - 1 \geq 0$. Thus the m-jets $j_p'^m = j_p^m \circ g^{-1}$ are up to order $m-1$ the identity at every point $p \in Z$. If we find an automorphism $h \in G$ with m-jet equal to $j_p'^m$ for all $p \in Z$, then obviously the automorphism $h \circ g$ has the desired properties.

Thus replacing j_p^m by $j_p'^m$ we are reduced to show the assertion in the case that for all $p \in Z$ the m-jets j_p^m are the identity up to order $m - 1$, where $m \geq 1$.

Applying Lemma 4.15, for every point $p \in Z$ there is an automorphism $g_p \in G$ whose m-jet at p is the given one while its m-jets at all other points $q \in Z \backslash \{p\}$ are the identity. Obviously then the composition (in arbitrary order) $g = \prod_{p \in Z} g_p$ will have the required properties. □

Proof of Lemma 4.15. In the case $m = 1$ the assertion follows from Theorem 4.2 and Proposition 4.8. So we may assume for the rest of the proof that $m \geq 2$.

Consider the set $\mathcal{N}_{mp,Z}^M$ of all locally nilpotent derivations in $\mathcal{N}$ with a zero of order m at p and of order $M+1$ at all other points $q \in Z \backslash \{p\}$. Let $G_{mp,Z}^M$ be the subgroup of G generated by the exponentials of elements in $\mathcal{N}_{mp,Z}^M$ so that an automorphism in

$G^M_{mp,Z}$ is the identity up to order $(m-1)$ at p and up to order M at all other points $q \in Z\backslash\{p\}$. With the notation as introduced in 4.11 let us consider the composed map

$$\Psi : G^M_{mp,Z} \longrightarrow \mathrm{Aut}_{m-1}(A_m) \xrightarrow{\psi} \mathrm{Hom}_{\Bbbk}(V, S^mV)\,,$$

where ψ is as in (13) and the first arrow assigns to an automorphism its m-jet at p. Using Lemma 4.13(a) it suffices to show that Ψ maps $G^M_{mp,Z}$ surjectively onto the subspace $\ker \kappa_m$.

The group $G^M_{mp,Z}$ is generated by exponentials of vector fields in $\mathcal{N}^M_{mp,Z}$. Thus using Lemma 4.12(b), (c) the image $\mathrm{im}(\Psi)$ of Ψ is a linear subspace of $\mathrm{Hom}_{\Bbbk}(V, S^mV)$. We claim that this subspace is nonzero.

Indeed, consider a vector field $\partial \in \mathcal{N}$ with $\partial(p) \neq 0$ and the one-parameter subgroup $H = \exp(\Bbbk\partial)$. According to Remark 2.7(3) there is an open dense H-invariant subset $U(H) \subseteq O$ which admits a quasi-affine geometric quotient $U(H)/H$ with the same properties as in 2.7(3). By infinite transitivity of the action of G on O we may assume that $Z \subseteq U(H)$ is such that the image of Z in the quotient $U(H)/H$ is contained in the regular part of $U(H)/H$, has the same cardinality as Z, and the projection $U(H) \to U(H)/H$ is smooth in the points of Z. Thus we can find a regular H-invariant function f on X with a simple zero at p, and another such function h with $h(p) = 1$ and $h(q) = 0$ for all $q \in Z\backslash\{p\}$. Replacing f by $h^{M+1}f$ we may assume that f has a zero of order $\geq M + 1$ at all points of $Z\backslash\{p\}$ and a simple zero at p. Then $g = \exp(f^m\partial)$ is an automorphism in $G^M_{mp,Z}$ with $\Psi(g) = f^m\hat{\partial} \neq 0$, where $\hat{\partial}$ is the derivation of A_m induced by ∂ (cf. Lemma 4.12(c) and its proof). This proves the claim.

The group $G^M_{p,Z}$ acts on $G^M_{mp,Z}$ by conjugation $g.h = g \circ h \circ g^{-1}$, where $g \in G^M_{p,Z}$ and $h \in G^M_{mp,Z}$. If we write $h = \mathrm{id} + h_m \mod \mathfrak{m}^{m+1}$ with a map $h_m \in \mathrm{Hom}_{\Bbbk}(V, S^mV)$ then $g.h = \mathrm{id} + g \circ h_m \circ g^{-1} \mod \mathfrak{m}^{m+1}$. The map g induces the cotangent map $(d_pg)^\vee$ on $V = (T_pX)^\vee$ and its mth symmetric power $S^m((d_pg)^\vee)$ on S^mV. Hence there is a commutative diagram

$$\begin{array}{ccc} G^M_{p,Z} \times G^M_{mp,Z} & \longrightarrow & G^M_{mp,Z} \\ \Big\downarrow {\scriptstyle d_p^\vee - \times \Psi} & & \Big\downarrow {\scriptstyle \Psi} \\ \mathrm{SL}(V) \times \mathrm{Hom}_{\Bbbk}(V, S^mV) & \longrightarrow & \mathrm{Hom}_{\Bbbk}(V, S^mV)\,, \end{array}$$

where the lower horizontal map is induced by the standard representation of $\mathrm{SL}(V)$ on S^mV. Since the map $G^M_{p,Z} \to \mathrm{SL}(V)$ is surjective (see Theorem 4.2 and Remark 4.7), the image $\mathrm{im}(\Psi)$ of Ψ is a non-trivial $\mathrm{SL}(V)$-module. By Lemma 4.12 this representation is contained in the kernel of the contraction map κ_m. Since the latter kernel is irreducible (see Lemma 4.13(b)), it follows that $\mathrm{im}(\Psi) = \ker \kappa_m$, as required. □

Remark 4.16. If in the situation of Theorem 4.14 each of the jets j^m_p, $p \in Z$, fixes the point p and preserves a volume form,[25] then the conclusion of Theorem 4.14 remains valid without the requirement that there is a global volume form on X_{reg}.

Remark 4.17. If X_{reg} does not admit a global volume form i.e., $K_{X_{\mathrm{reg}}} \neq 0$, one can still formulate a necessary condition for interpolation of jets by an automorphism from

[25]Note that this is a local condition, see the discussion before Lemma 4.13.

a $\mathbb{G}_a$-generated group G, namely in terms of the 'volume form monodromy' of G. To define it we fix a volume form ω_x on the tangent space T_xX at some point $x \in X_{\text{reg}}$, and consider the stabilizer subgroup $G_x \subseteq G$. Every element $g \in G_x$ transforms ω_x into $\chi_x(g) \cdot \omega_x$, where $\chi_x(g) \in \mathbb{G}_m = \mathbb{G}_m(\Bbbk)$. The map

$$\chi_x : G_x \longrightarrow \mathbb{G}_m$$

is then a character on G_x which equals 1 on $G_{\mathcal{N},x}$, see Theorem 4.2. If $y \in X$ is a second point and $h \in G$ is an automorphism with $h.x = y$ then $hG_xh^{-1} = G_y$ and h transforms ω_y into ω_x. Hence $\chi_y(hgh^{-1}) = \chi_x(g)$ for all $g \in G_x$. In particular the image of χ_x forms a subgroup Γ of $\mathbb{G}_m$ independent of $x \in O$, which is called the *volume form monodromy* of G.

The volume form monodromy can be a nontrivial discrete group as in the case of $X = \mathrm{SL}_2/N(\mathbb{T})$ and $G = \mathrm{SAut}(X)$, where $N(\mathbb{T}) \subseteq \mathrm{SL}_2$ is the normalizer of the maximal torus $\mathbb{T} \subseteq \mathrm{SL}_2$. Note that in this case $X \simeq \mathbb{P}^2 \backslash C$, where C is a smooth conic in $\mathbb{P}^2$, see [39]. Using technique from [26] one can show that here $\Gamma = \{\pm 1\}$.

4.3. **$\mathbb{A}^1$-richness.** An irreducible affine variety X is called *$\mathbb{A}^1$-rich* if for every closed subset Y of codimension ≥ 2 and every finite subset $Z \subset X \setminus Y$ there is a regular map $\mathbb{A}^1 \to X$ whose image contains Z and omits Y [29, §2].

The following corollary is immediate from the Transversality Theorem 1.15. In particular this result shows that a flexible irreducible affine variety is $\mathbb{A}^1$-rich. In the special case where $X = \mathbb{A}^n_{\mathbb{C}}$ the latter also follows from the Gromov-Winkelmann theorem, see [53, §2, Proposition 1].

Corollary 4.18. *Let as before X be an irreducible affine variety and let $G \subseteq \mathrm{SAut}(X)$ be a subgroup generated by a saturated set $\mathcal{N}$ of locally nilpotent derivations, which acts with an open orbit $O \subseteq X$. Then for any finite subset $Z \subseteq O$ and any closed subset $Y \subseteq X$ of codimension ≥ 2 with $Z \cap Y = \varnothing$ there is an orbit $C \cong \mathbb{A}^1$ of a $\mathbb{G}_a$-action on X which does not meet Y and passes through each point of Z having prescribed jets at these points.*

Proof. In the case $\dim X = 1$ this is trivially true. So assume that $\dim X \geq 2$. Let C be an orbit of a $\mathbb{G}_a$-action on O. Since G acts infinitely transitively on O we may assume that $Z \subseteq C$. By Theorem 4.14 and Remark 4.16, applying an appropriate automorphism $g' \in G$ we may suppose as well that C has prescribed m-jets at the points of Z. Indeed, the m-jets of automorphisms stabilizing a given point $p \in O$ and having at this point the jacobian determinant equal to 1 modulo $\mathfrak{m}^m$ act transitively on the set of all m-jets of smooth curves at p.

By Proposition 4.9, using the notation as in 4.7, the $\mathbb{G}_a$-generated group G_Z^m acts transitively in $O \backslash Z$. Applying now the Transversality Theorem 1.15(b) to G_Z^m, $C \cap (O \backslash Z)$, and $Y \cap (O \backslash Z)$ we can find an element $g \in G_Z^m$ with $g.C \cap Y = \varnothing$. Thus the $\mathbb{G}_a$-orbit $g.C$ contains Z, has the prescribed jets at the points of Z, and does not meet Y. □

We can deduce also the following fact.

Proposition 4.19. *Let $G \subseteq \mathrm{SAut}(X)$ be a subgroup generated by a saturated set $\mathcal{N}$ of locally nilpotent derivations, which acts with an open orbit $O \subseteq X$. Then for any* *closed subset $Y \subseteq O$ of codimension ≥ 2 and for any $m \in \mathbb{N}$ the group $G^m_{\mathcal{N},Y}$ as in Remark 2.16 acts with an orbit open in X.*

Proof. According to Proposition 1.8 there are locally nilpotent vector fields $\partial_1, \ldots, \partial_s$ generating T_pX for all $p \in O$. Let $H_\sigma \subseteq G$ be the one-parameter subgroup associated to ∂_σ. By Remark 2.7(3) for suitable open dense H_σ-invariant subsets $U(H_\sigma)$ in O there are geometric quotients $U(H_\sigma)/H_\sigma$ as in 2.7(3). Using the same reasoning as in the proof of Theorem 2.15 there is an H_σ-invariant function $f_\sigma \in \mathcal{O}(X)$ vanishing on $\overline{H_\sigma.Y}$ and equal to 1 at a given general point $p \in U(H_\sigma)\backslash\overline{H_\sigma.Y}$. Consequently the replica $\exp(\Bbbk f^m_\sigma \partial_\sigma)$ fixes the mth infinitesimal neighborhood of Y in X and moves p in direction $\partial_\sigma(p)$. In other words, p is a $G^m_{\mathcal{N},Y}$-flexible point. Applying Corollary 1.11(a) the result follows. □

Problem 4.20. Is it true that in the situation of Proposition 4.19 the group $G^m_{\mathcal{N},Y}$ acts transitively on $O\backslash Y$?

In the case that $X = \mathbb{A}^n_{\mathbb{C}}$ and $G = \mathrm{SAut}(X)$ the answer is affirmative, see [53], §2, Proposition 1 and its proof.

Remarks 4.21. 1. Every algebraic variety X contains a divisor Y such that the logarithmic Kodaira dimension $\bar{\kappa}(X\backslash Y)$ is ≥ 0. In this case $X\backslash Y$ cannot carry a $\mathbb{G}_a$-action and so $G_Y = \{\mathrm{id}\}$ although X might be flexible. The simplest example of such a situation is given by the hypersurface $Y = \{X_1 \cdot \ldots \cdot X_n = 0\}$ in $X = \mathbb{A}^n$, see also [24].

2. If the group $\mathrm{SAut}(X)$ of an irreducible normal affine surface X acts with an orbit O open in X then X is a Gizatullin surface and $Y = X \setminus O$ is a finite set, see [20, II, Theorem 3], [11]. Such a surface X is usually non-$\mathbb{Q}$-factorial. In higher dimensions this complement may contain a divisor (see Example 5.10 below). However, such examples cannot exist if X is $\mathbb{Q}$-factorial (see Corollary 4.23 below).

We need the following auxiliary result.

Proposition 4.22. *Let X be an irreducible normal affine variety, and let G be a $\mathbb{G}_a$-generated subgroup of $\mathrm{SAut}(X)$ acting on X with an open orbit $O \subseteq X$. If the complement $X\backslash O$ contains a divisor D then D generates a nonzero element $[D]$ in the divisor class group $\mathrm{Cl}(X)_{\mathbb{Q}} = \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$.*

Proof. Assume to the contrary that $[D] = 0$ in $\mathrm{Cl}(X)_{\mathbb{Q}}$. Then there is a function f on X with $D = \mathbb{V}(f)$ set theoretically. For every one dimensional unipotent subgroup $H \subseteq G$ and $x \in O$ the function $f|H.x$ is a polynomial on $H.x \cong \Bbbk$. As $H.x \subseteq O$ and so $D \cap H = \varnothing$, this polynomial has no zero and so is constant equal to $a := f(x)$. Hence $H.x$ is contained in the level set $f^{-1}(a)$ of f. Since G is generated by such subgroups, the whole orbit $O = G.x$ is contained in $f^{-1}(a)$ and so it cannot be open, a contradiction. □

Corollary 4.23. *Let G be a $\mathbb{G}_a$-generated subgroup of $\mathrm{SAut}(X)$. If X is $\mathbb{Q}$-factorial and a closed subset $Y \subseteq X$ contains a divisor, then the group $G_{\mathcal{N},Y}$ has no open orbit.*

5. Some applications

5.1. **Unirationality, flexibility, and triviality of the Makar-Limanov invariant.** Recall [16] that the *Makar-Limanov invariant* $\mathrm{ML}(X)$ of an affine variety X is the intersection of the kernels of all locally nilpotent derivations on X. In other words $\mathrm{ML}(X)$ is the subalgebra of the algebra $\mathcal{O}(X)$ consisting of all $\mathrm{SAut}(X)$-invariants. Similarly [35] the *field Makar-Limanov invariant* $\mathrm{FML}(X)$ is defined as the subfield of $\mathcal{M}er(X)$ which consists of all rational $\mathrm{SAut}(X)$-invariants. If it is trivial i.e., if $\mathrm{FML}(X) = \Bbbk$ then so is $\mathrm{ML}(X)$, while the converse is not true in general, see Example 5.3(2) below. The next proposition confirms, in particular, Conjecture 5.3 in [35] (cf. also [43]).

Proposition 5.1. *An irreducible affine variety X possesses a flexible point if and only if the group $\mathrm{SAut}(X)$ acts on X with an open orbit, if and only if the field Makar-Limanov invariant $\mathrm{FML}(X)$ is trivial. In the latter case X is unirational.*

Proof. The first equivalence follows from Corollary 1.11(a) and the second from Corollary 1.14. As for the last assertion, see the next remark. □

Remark 5.2. As follows from Proposition 1.3(b) for every G-orbit O of a $\mathbb{G}_a$-generated group $G \subseteq \mathrm{SAut}(X)$ there is a surjective morphism $\mathbb{A}^s \to O$. Hence any two points in O are contained in the image of a morphism $\mathbb{A}^1 \to O$. In particular O is $\mathbb{A}^1$-*connected* in the sense of [27, §6.2].

Examples 5.3. 1. Flexibility implies neither rationality nor stable rationality. Indeed, there exists a finite subgroup $F \subseteq \mathrm{SL}(n, \mathbb{C})$, where $n \geq 4$, such that the smooth unirational affine variety $X = \mathrm{SL}(n, \mathbb{C})/F$ is not stably rational, see [43, Example 1.22]. However, by Proposition 5.4 below X is flexible and the group $\mathrm{SAut}(X)$ acts infinitely transitively on X.

2. There are non-unirational affine threefolds X with $\mathrm{ML}(X) = \Bbbk$ birationally equivalent to $C \times \mathbb{A}^2$, where C is a curve of genus $g \geq 1$, see [34, §4.2]. For such a threefold X the general $\mathrm{SAut}(X)$-orbits have dimension two, the field Makar-Limanov invariant $\mathrm{FML}(X)$ is non-trivial, and there is no flexible point in X.

5.2. **Flexible quasihomogeneous varieties.** An important class of flexible algebraic varieties consists of homogeneous spaces of semisimple algebraic groups. More generally, the following hold (cf. [43, §1.1]).

Proposition 5.4. *Let G be a connected affine algebraic group without non-trivial characters, and let H be a closed subgroup of G. Then the homogeneous space G/H is flexible. In particular, if G/H is affine of dimension $n \geq 2$ then the group $\mathrm{SAut}(G/H)$ acts infinitely transitively on G/H.*

Proof. The image of G in $\mathrm{SAut}(G/H)$ is a $\mathbb{G}_a$-generated subgroup (see Example 1.18 (2)). Thus the group $\mathrm{SAut}(G/H)$ acts on the quotient G/H transitively and G/H is flexible; see Proposition 1.1 in [1]. The second assertion follows from the first one in view of Theorem 0.1 and Corollary 1.22. □

The following problem arises.

Problem 5.5. *Characterize flexible varieties among affine varieties admitting an action of a semisimple algebraic group with a dense open orbit.*

For instance, if such a quasihomogeneous variety is smooth then in fact it is flexible. In the particular case $G = \mathrm{SL}_2$ this was actually established in [39, III], where we borrowed the idea of the proof of the following theorem.

Theorem 5.6. *Suppose that a connected semisimple algebraic group G acts on a smooth irreducible affine variety $X = \operatorname{Spec} A$ with an open orbit. Then X is homogeneous with respect to a connected affine algebraic group $\tilde{G} \supseteq G$ without non-trivial characters. In particular, X is flexible.*

Proof. Since by our assumption $A^G = \Bbbk$, the variety X contains a unique closed G-orbit $Z \subseteq X$ and the stabilizer of a point on this orbit is a reductive subgroup H of the group G (see Theorems 4.17 and 6.7 in [44]). Moreover, it follows from Luna's Étale Slice Theorem that there is a finite dimensional rational H-module W such that the variety X is G-equivariantly isomorphic to the total space $G \times^H W$ of the homogeneous vector bundle over G/H with the fiber W, see Theorem 6.7 in [44].

According to [3] there exists a finite dimensional G-module V such that $V = W \oplus W'$, where $W' \subseteq V$ is a complementary H-submodule. Letting

$$\tilde{G} = G \ltimes V \quad \text{with} \quad (g_1, v_1).(g_2, v_2) = (g_1 g_2, g_2^{-1} v_1 + v_2)\,,$$

$\tilde{H} = H \ltimes W'$, and $\tilde{H}_0 = \{e\} \ltimes W'$ we can identify $\tilde{G}/\tilde{H}_0$ and $G \times W$ as H-varieties. Since the subgroup $H \subseteq G$ normalizes $\tilde{H}_0$ in $\tilde{G}$ it acts $\tilde{G}$-equivariantly on the right on $\tilde{G}/\tilde{H}_0$. The latter fact can be used to deduce the isomorphisms of varieties

$$\tilde{G}/\tilde{H} \cong (\tilde{G}/\tilde{H}_0)/H \cong X\,.$$

By Proposition 5.4 X is flexible being a homogeneous variety of a connected affine algebraic group $\tilde{G}$ without non-trivial characters (indeed, $\tilde{G} = (G, 0) \cdot (e, V)$, where both groups admit no non-trivial characters). Now the proof is completed. ☐

In the next theorem we provide a complete solution of Problem 5.5 for $G = \mathrm{SL}_2 := \mathrm{SL}_2(\Bbbk)$ and X normal.

Theorem 5.7. *Every normal irreducible affine variety E admitting an SL_2-action with an open orbit is flexible.*

For a homogeneous affine variety $E = \mathrm{SL}_2 / H$ the result follows from Proposition 5.4. The proof in the general case given below is based on a description of normal SL_2-varieties due to Popov [39, I] (see also [32, Chapter III, §4]) and a Cox ring SL_2-construction due to Batyrev and Haddad [2]. Recall [39, I] that every non-homogeneous normal affine SL_2-threefold with an open orbit is uniquely determined by a pair (h, m), where m is the order of the generic isotropy group[26] and $h = p/q \in]0, 1] \cap \mathbb{Q}$ is the so called *height* of X. The SL_2-threefold with invariant (h, m) is denoted by $E_{h,m}$. Notice that $E_{h,m}$ is smooth for $h = 1$ and singular for $h < 1$.

Assuming in the sequel that p and q are coprime positive integers we let

$$(14) \qquad a = m/k \quad \text{and} \quad b = (q - p)/k, \quad \text{where} \quad k = \gcd(q - p, m)\,.$$

[26] Which is a cyclic group.

Let $\mu_a = \langle \xi_a \rangle$ denote the cyclic group generated by a primitive root of unity $\xi_a \in \mathbb{G}_m = \mathbb{G}_m(\Bbbk)$ of degree a. The SL_2-variety $E_{h,m}$ is isomorphic to the categorical quotient of the hypersurface $D_b \subseteq \mathbb{A}^5$ with equation

$$Y^b = X_1X_4 - X_2X_3 \tag{15}$$

modulo the diagonal action of the group $\mathbb{G}_m \times \mu_a$ on $\mathbb{A}^5 = \operatorname{Spec} \Bbbk[X_1, X_2, X_3, X_4, Y]$ via

$$\operatorname{diag}(t^{-p}, t^{-p}, t^q, t^q, t^k) \times \operatorname{diag}(\xi^{-1}, \xi^{-1}, \xi, \xi, 1)\,, \quad t \in \mathbb{G}_m,\, \xi \in \mu_a\,.$$

Here the SL_2-action on D_b is induced by the trivial action on the coordinate Y, while $\langle X_1, X_2\rangle$ and $\langle X_3, X_4\rangle$ are simple SL_2-modules. This SL_2-action on D_b commutes with the $(\mathbb{G}_m \times \mu_a)$-action and so descends to the quotient. This gives a simple and uniform description of all non-homogeneous singular normal affine SL_2-threefolds with an open orbit $E_{h,m}$ via the Cox realization as the quotient of the spectrum of the corresponding Cox ring by the action of the Neron-Severi quasitorus, see [2].

Proof of Theorem 5.7. Let E be a non-homogeneous singular normal irreducible affine SL_2-variety with an open orbit. If $\dim E = 2$ then E is a toric surface, in fact a Veronese cone, and the group SL_2 is transitive off the vertex (see [39, II] or, alternatively, Theorem 0.2 in [1]). Now the assertion follows by Theorem 0.1.

If further E as above is an SL_2-threefold then according to Popov's classification $E = E_{h,m}$ for some pair (h, m).

In the case where $E = E_{h,m}$ is smooth that is, $h = 1$ the result follows from Theorem 5.6.

In the case where $E = E_{h,m}$ is singular i.e., $h = p/q < 1$, there is a unique singular point, say, $Q \in E$. The complement $E\backslash\{Q\}$ consists of two SL_2-orbits O_1 and O_2, where $O_1 \cong \mathrm{SL}_2/\mu_m$ while $O_2 \cong \mathrm{SL}_2/U_{a(p+q)}$ has the isotropy subgroup

$$U_{a(p+q)} = \left\{ \begin{pmatrix} \xi & \eta \\ 0 & \xi^{-1} \end{pmatrix} \mid \eta \in \Bbbk,\, \xi^{a(p+q)} = 1 \right\}.$$

Consider the hypersurface $D_b \subseteq \mathbb{A}^5$ as in (15). We can realize $\mathbb{A}^5$ as a matrix space:

$$\mathbb{A}^5 = \left\{ (X, Y) \mid X = \begin{pmatrix} X_1 & X_3 \\ X_2 & X_4 \end{pmatrix}, X_i, Y \in \mathbb{A}^1 \right\}.$$

Then according to [2] the 3-fold $E = E_{h,m}$ admits a realization as the categorical quotient of D_b by the action of the group $G_m \times \mu_a$ via

$$(t, \xi).(X, Y) = \left(\begin{pmatrix} \xi^{-1}t^{-p}X_1 & \xi t^q X_3 \\ \xi^{-1}t^{-p}X_2 & \xi t^q X_4 \end{pmatrix}, t^k Y \right).$$

This action commutes with the natural SL_2-action on D_b given by

$$A.(X, Y) = (AX, Y)\,.$$

Hence the SL_2-action on D_b descends to the quotient $E = E_{h,m}$. The hypersurface $Z = \{Y = 0\}$ in D_b is the inverse image of the unique two dimensional SL_2-orbit closure in $E_{h,m}$. To show the transitivity (or the flexibility) of the group $\mathrm{SAut}(X)$ in E_{reg} it suffices to find a locally nilpotent derivation ∂ of the algebra $\mathcal{O}(D_b)$ with $\partial(Y) \neq 0$ which preserves the $(\mathbb{Z} \times \mathbb{Z}_a)$-bigrading on $\mathcal{O}(D_b)$ defined via

$$\deg X_1 = \deg X_2 = (-p, -\bar{1}),\ \ \deg X_3 = \deg X_4 = (q, \bar{1}), \quad \text{and} \quad \deg Y = (k, \bar{0})\,.$$

Indeed, such a derivation induces a locally nilpotent derivation on $\mathcal{O}(E)$. Since $\partial(Y) \neq 0$ the restriction of the corresponding vector field to the image Z' of Z in E is nonzero and so the points of Z' with $\partial \neq 0$ are flexible. By transitivity, every point of $Z' \setminus \{Q\}$ is.

The variety D_b can be regarded as a suspension over $\mathbb{A}^3 = \operatorname{Spec} \Bbbk[X_2, X_3, Y]$, see (1). Namely,

$$D_b = \{X_1 X_4 = f(X_2, X_3, Y)\} \quad \text{where} \quad f = X_2 X_3 + Y^b\,.$$

According to [1] (see also Lemma 3.3 in [28, §5]) a desired bihomogeneous locally nilpotent derivation ∂ can be produced starting with a locally nilpotent derivation $\delta \in \operatorname{Der} \Bbbk[X_2, X_3, Y]$. For instance, let δ be given by

$$\delta(X_2) = \delta(X_3) = 0, \quad \delta(Y) = X_2^c X_3^d\,.$$

Then ∂ can be defined via

(16)
$$\partial(X_1) = \partial(X_2) = \partial(X_3) = 0,\ \partial(X_4) = \delta(f) = bX_2^c X_3^d Y^{b-1},\ \partial(Y) = X_1 X_2^c X_3^d$$

with a, b as in (14) and with appropriate values of the natural parameters c, d. Such a derivation ∂ preserves the $(\mathbb{Z} \times \mathbb{Z}_a)$-bigrading[27] if and only if

$$\begin{aligned} -p - cp + dq &= k \\ k(b-1) - cp + dq &= q \\ -1 - c + d &\equiv 0 \mod a \\ -c + d &\equiv 1 \mod a\,. \end{aligned}$$

By virtue of (14) the second relation follows from the first one, while the last two are equivalent. Letting $c = s - 1$ we can rewrite the remaining relations as

$$\begin{aligned} dq - sp &= k \\ s &\equiv d \mod a\,. \end{aligned} \tag{17}$$

Since $\gcd(p, q) = 1$ the first equation admits a solution (d_0, s_0) in natural numbers. For every $r \in \mathbb{N}$, the pair $(d_0 + rp, s_0 + rq)$ also represents such a solution. The second relation in (17) becomes

$$r(q - p) \equiv d_0 - s_0 \mod a\,. \tag{18}$$

By (14) $k = \gcd(m, q - p)$, hence $\gcd(k, p) = 1$. The first equation in (17) written as

$$d_0(q - p) - p(s_0 - d_0) = k$$

implies that $k \,|\, (s_0 - d_0)$.

Let $l = \gcd(a, q - p) = \gcd(a, bk)$. Since $\gcd(a, b) = 1$ then $l \,|\, k$ and so (18) is equivalent to the congruence

$$r \cdot \frac{q - p}{l} \equiv \frac{d_0 - s_0}{l} \mod \frac{a}{l}\,.$$

Since $\frac{q-p}{l}$ and $\frac{a}{l}$ are coprime the latter congruence admits a solution, say, r_0. Letting finally

$$c = s_0 + r_0 q - 1,\ d = d_0 + r_0 p$$

[27] I.e. $\deg \partial(Y) = \deg Y$ and $\deg \partial(X_4) = \deg X_4$.

the locally nilpotent derivation ∂ as in (16) becomes homogeneous of bidegree $(0,\bar{0})$, as needed. Now the proof is completed. $\square$

The question arises whether the smooth loci of singular affine SL_2-threefolds are homogeneous as well, cf. Theorem 5.6. The answer is negative; the following proposition gives a more precise information.

Proposition 5.8. *Let $E = E_{h,m}$, where $h = p/q < 1$ with $\gcd(p,q) = 1$. The following conditions are equivalent:*

(i) The SL_2-action on E extends to an action of a bigger affine algebraic group G on E which is transitive in E_{reg};
(ii) The variety E is toric;
(iii) $(q-p)\,|m$ or, equivalently, $b = 1$ in (14).

Proof. Implication (i)⇒(ii) follows from Theorem 1 in [39, III]. According to this theorem, a normal affine threefold X with a unique singular point Q which admits an action of an affine algebraic group transitive on $X\backslash\{Q\}$, is toric.

The equivalence (ii)⇔(iii) follows from the results of [2] and [18]. Let us show the remaining implication (iii)⇒(i). If $b = 1$ in (14) then $D_b \cong \mathbb{A}^4 = \operatorname{Spec} \Bbbk[X_1, \dots, X_4]$. Hence the toric variety $E_{h,m}$ can be obtained as the quotient $\mathbb{A}^4/(G_m \times \mu_a)$, where the group $G_m \times \mu_a$ with $a = m/(q-p)$ as in (14) acts diagonally on $\mathbb{A}^4$ via

(19)
$$(X_1, X_2, X_3, X_4) \longmapsto (\xi^{-1}t^{-p}X_1, \xi^{-1}t^{-p}X_2, \xi t^q X_3, \xi t^q X_4), \quad (t,\xi) \in G_m \times \mu_a \,.$$

Consider the action of the group $\mathrm{SL}_2 \times \mathrm{SL}_2$ on $\mathbb{A}^4$ via

$$(A_1, A_2).(X_1, X_2, X_3, X_4) = \left(A_1 \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},\ A_2 \begin{pmatrix} X_3 \\ X_4 \end{pmatrix} \right) .$$

This action commutes with the $(G_m \times \mu_a)$-action (19) and so descends to the quotient $E_{h,m}$. The induced $(\mathrm{SL}_2 \times \mathrm{SL}_2)$-action on the quotient $E_{h,m}$ is transitive in the complement of the unique singular point Q. This yields (i). Now the proof is completed. $\square$

Corollary 5.9. *None of the non-toric affine threefolds $E = E_{h,m}$ with $h < 1$ admits an algebraic group action transitive in E_{reg}. However, the group $\mathrm{SAut}(E)$ acts infinitely transitively in E_{reg}.*

Let us finish this subsection with an example of a flexible non-normal irreducible affine variety with singular locus of codimension one.

Example 5.10. Consider the standard irreducible representation of the group SL_2 on the space of binary forms of degree three

$$V = \left\langle X^3, X^2Y, XY^2, Y^3 \right\rangle .$$

Restriction to the subvariety

$$E = \mathrm{SL}_2 .X^2Y \cup \mathrm{SL}_2 .X^3 \cup \{0\} \subseteq V$$

of forms with zero discriminant yields a non-normal SL_2-embedding, see [32]. Since for a hypersurface in a smooth variety normality is equivalent to smoothness in codimension one, the divisor $D = \mathrm{SL}_2 .X^3 \cup \{0\} \subseteq E$ coincides with the singular locus E_{sing}. The complement $E_{\rm reg} = \mathrm{SL}_2 .X^2Y$ is the open SL_2-orbit consisting of all flexible points of E. Hence E is flexible.

The normalization of E is isomorphic to $E_{\frac{1}{2},1}$. Indeed $m = 1$ because the stabilizer in SL_2 of a general point in E is trivial. On the other hand, the order of the stabilizer of the two dimensional orbit equals $p + q = 3$, hence $p = 1$ and $q = 2$.

For any admissible pair (h, m) the affine threefold $E_{h,m}$ is a union of an open SL_2-orbit O and an invariant prime divisor Y [39, I, Lemma 4 and Corollary 1]. Choosing a generating set of one-parameter unipotent subgroups of SL_2 we let $\mathcal{N}$ be the corresponding saturated set of locally nilpotent vector fields on E. Consider further the subgroup $G \subseteq \mathrm{SAut}(E)$ generated by $\mathcal{N}$. Clearly, G again stabilizes the divisor Y and acts transitively on its complement O. According to Theorem 2.15 the group G_Y also acts on E with an orbit O whose complement Y is a divisor. This shows that the assumption of $\mathbb{Q}$-factoriality in Corollary 4.23 is essential.

6. Appendix: Holomorphic flexibility

In this appendix we extend the notion of a flexible affine variety to the complex analytic setting (cf. [13]). We survey relations between holomorphic flexibility, Gromov's spray and the Andersen-Lempert theory. In particular, we show that every flexible variety admits a Gromov spray. This provides a new wide class of examples to which the Oka-Grauert-Gromov Principle can be applied. We refer the reader to [15] and the survey articles [14, §3] and [25] for a more thorough treatment and historical references.

6.1. **Oka-Grauert-Gromov Principle for flexible varieties.** The following notions were introduced in [21, §1.1.B].

Definition 6.1. (i) Let X be a complex manifold. A *dominating spray* on X is a holomorphic vector bundle $\rho : E \to X$ together with a holomorphic map $s : E \to X$, such that s restricts to the identity on the zero section Z while for each $x \in Z \cong X$ the tangent map $d_x s$ sends the fiber $E_x = \rho^{-1}(x)$ (viewed as a linear subspace of $T_x E$) surjectively onto $T_x X$.

(ii) Let $h : X \to B$ be a surjective submersion of complex manifolds. We say that it admits a *fiber dominating spray* if there is a holomorphic vector bundle E on X together with a holomorphic map $s : E \to X$ such that the restriction of s to each fiber $h^{-1}(b)$, $b \in B$, yields a spray on this fiber.

In these terms, the Oka-Grauert-Gromov Principle can be stated as follows.

Theorem 6.2. ([21, §4.5]) *Let $h : X \to B$ be a surjective submersion of Stein manifolds. If it admits a fiber dominating spray then the following hold.*
(a) Any continuous section of h is homotopic to a holomorphic one; and
(b) any two holomorphic sections of h that are homotopic via continuous sections are also homotopic via holomorphic ones.

Due to the following proposition, smooth affine algebraic G-fibrations with flexible fibers are appropriate for applying this principle (cf. [14, 3.4], [21]).

Proposition 6.3. *(a) Every flexible smooth irreducible affine algebraic variety X over $\mathbb{C}$ admits a dominating spray.*

(b) Let $h : X \to B$ be a surjective submersion of smooth irreducible affine algebraic varieties over $\mathbb{C}$ such that for some algebraically generated subgroup $G \subseteq \mathrm{Aut}(X)$ the orbits of G coincide with the fibers of h [28]. Then $X \to B$ admits a fiber dominating spray.

Proof. It suffices to show (b). Indeed, due to Corollary 1.22, (a) is a particular case of (b).

By Proposition 1.8 there is a sequence of algebraic subgroups $\mathcal{H} = (H_1, \ldots, H_s)$ of G such that the tangent space to the orbit $G.x$ at each point $x \in X$ is spanned by the tangent spaces at x to the orbits $H_i.x$, $i = 1, \ldots, s$. Let $\exp : T_1(H_i) \to H_i$ be the exponential map. Letting $E = X \times \prod_{i=1}^{s} T_1(H_i)$ be the trivial vector bundle over X we consider the morphism

$$s : E \to X, \qquad (x, (h_1, \ldots, h_s)) \longmapsto \Phi_{\mathcal{H},x}(\exp h_1, \ldots, \exp h_s)\,,$$

where $\Phi_{\mathcal{H},x}$ has the same meaning as in (2). This yields the desired dominating spray. □

To extend Proposition 6.2 to the analytic setting we introduce below the notions of holomorphic flexibility. Recall that a holomorphic vector field on a complex manifold X is *completely integrable* if its phase flow defines a holomorphic action on X of the additive group $\mathbb{C}_+ = \mathbb{G}_a(\mathbb{C})$.

Definitions 6.4. (i) We say that a Stein space X is *holomorphically flexible* if the completely integrable holomorphic vector fields on X span the tangent space T_xX at every smooth point of X.

(ii) Given a holomorphic submersion $h : X \to B$ of Stein manifolds, we say that X is *holomorphically flexible over* B if the completely integrable relative holomorphic vector fields on X span the relative tangent bundle of $X \to B$ at any point of X. In the latter case each fiber $h^{-1}(b)$, $b \in B$, is a holomorphically flexible Stein manifold.

Remarks 6.5. 1. The vector field $\delta = z\frac{d}{dz}$ on $X = \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ is completely integrable. However, the derivation $\delta \in \mathrm{Der}(\mathcal{O}(X))$ is not locally nilpotent. Hence $X = \mathbb{C}^*$ is not flexible in the sense used in this paper, while it is holomorphically flexible.

2. In the terminology of [51], a complex manifold X admits an *elliptic microspray* if the $\mathcal{O}_{\mathrm{an}}(X)$-module generated by all completely integrable holomorphic vector fields on X is dense in the $\mathcal{O}_{\mathrm{an}}(X)$-module of all holomorphic vector fields on X with respect to the compact-open topology.

We claim that a Stein manifold X admits an elliptic microspray if and only if X is holomorphically flexible. Indeed, admitting an elliptic microspray implies the holomorphic flexibility, because the holomorphic vector fields on a Stein manifold X span the tangent space at every point. As for the converse, we observe that on a holomorphically flexible manifold X the sheaf of germs of holomorphic vector fields is spanned by the sheaf of germs of holomorphic vector fields generated by completely integrable such fields. By Cartan's Theorem B, on a Stein manifold X the corresponding $\mathcal{O}_{\mathrm{an}}(X)$-modules coincide.

[28] We say in this case that X is *G-flexible over B*.

In the analytic setting, the following analog of Corollary 1.21 holds.

Lemma 6.6. *If a connected Stein manifold X is holomorphically flexible over a Stein manifold B then the relative tangent bundle of X over B is spanned by a finite number of completely integrable relative holomorphic vector fields on X.*

Proof. In the absolute case i.e., when B is a point, the assertion is just that of Lemma 4.1 in [25]. The proof of this lemma in [25] works without changes in the relative case as well. □

With the same arguments as in the proof of Proposition 6.3 this implies that a Stein manifold X, which is holomorphically flexible over another Stein manifold B, admits a fiber dominating spray. Thus we obtain the following result.

Corollary 6.7. *Every connected Stein manifold X holomorphically flexible over another Stein manifold B admits a fiber dominating spray. Consequently, the Oka-Grauert-Gromov Principle is valid for $X \to B$.*

In particular, the Oka-Grauert principle holds for any holomorphically flexible connected Stein manifold X.

Comparing with the algebraic setting, in the analytic case we know little about invariants of completely integrable holomorphic vector fields. This leads to the following question.

Problem 6.8. *Does the group $\mathrm{Aut}_{\mathrm{an}}(X)$ of holomorphic automorphisms of a flexible connected Stein manifold X act infinitely transitively on X?*

This group is transitive on X. Indeed, by the implicit function theorem every orbit of the group $\mathrm{Aut}_{\mathrm{an}}(X)$ is open in X with respect to the standard Hausdorff topology. On the other hand, such an orbit is the complement of the union of all other orbits, thus it is closed. Hence there is only one orbit.

However, the infinite transitivity holds under a stronger assumption. We need the following notion from the Andersen-Lempert theory.

Definitions 6.9. (see [25], [52]) (i) We say that a complex manifold X has the *density property* if the Lie algebra generated by all completely integrable holomorphic vector fields on X is dense in the Lie algebra of all holomorphic vector fields on X in the compact-open topology.

(ii) Similarly, we say that an affine algebraic manifold X has the *algebraic density property* if the Lie algebra generated by all completely integrable algebraic vector fields on X coincides with the Lie algebra of all algebraic vector fields on X.

An analytic version of Theorem 0.1 can be stated as follows (cf. Theorem 5.5 in [14]).

Theorem 6.10. *([25, 2.13], [52]) If a connected Stein manifold X of dimension ≥ 2 has the density property then the group $\mathrm{Aut}_{\mathrm{an}}(X)$ of holomorphic automorphisms of X acts infinitely transitively[29] on X. Moreover, for any discrete subset $Z \subseteq X$ and*

[29] By 'infinite transitivity' we mean, as before, m-transitivity for all $m \in \mathbb{N}$. Note however that transitivity for arbitrary discrete subsets does not hold already in $X = \mathbb{A}^n_{\mathbb{C}}$, as shows the famous example of Rosay and Rudin, see e.g., [14].

for any Stein space Y of positive dimension which admits a proper embedding into X, there is another proper embedding $\varphi : Y \hookrightarrow X$ which interpolates Z i.e., $Z \subseteq \varphi(Y)$.

We refer the reader to [6] for a result on interpolation of a given discrete set of jets of automorphisms by an analytic automorphism of an affine space, similar to our Theorem 4.5.

6.2. **Volume density property.** As usual a holomorphic volume form ω on a complex manifold X is a nowhere vanishing top-dimensional holomorphic form on X. We need the following notions.

Definitions 6.11. (i) Given a submersion $X \to B$ of connected Stein manifolds and a volume form ω on X we say that X is *holomorphically volume flexible over B*, if Definition 6.4(ii) holds with all relative holomorphic vector fields considered there being ω-divergence-free. The latter means that the corresponding phase flow preserves ω.

In the absolute case i.e., B is a point, we simply call the space X holomorphically volume flexible.

(ii) We say that X has the *volume density property* if Definition 6.9 holds with all fields in consideration being ω-divergence-free. The *algebraic volume density property* is defined likewise.

The holomorphic volume flexibility of a Stein manifold X is equivalent to the existence on X of an elliptic volume microspray as introduced in [51]. Lemma 6.6 and Corollary 6.7 admit analogs in this new context. However, the proofs become now more delicate. We address the interested reader to [25, 26].

The algebraic volume density property implies the usual volume density property [26]. However, we do not know whether a holomorphically volume flexible connected Stein manifold has automatically the volume density property (cf. [51]).

Concerning infinite transitivity, the following theorem is proven in [25, 2.1-2.2].

Theorem 6.12. *Let X be a connected Stein manifold of dimension ≥ 2 equipped with a holomorphic volume form. If X satisfies the holomorphic volume density property, then the conclusions of Theorem 6.10 hold, with volume preserving automorphisms.*

Given an algebraic volume form ω on a smooth affine algebraic variety X, every locally nilpotent vector field on X is automatically ω-divergence-free. Thus the usual flexibility implies the algebraic volume flexibility. Let us formulate the following related problem.

Problem 6.13. *Let X be a flexible smooth connected affine algebraic variety over $\mathbb{C}$ equipped with an algebraic volume form. Does the algebraic volume density property hold for X?*

We conclude with yet another problem.

Problem 6.14. *Does there exist a flexible exotic algebraic structure on an affine space that is, a flexible smooth affine variety over $\mathbb{C}$ diffeomorphic but not isomorphic to an affine space $\mathbb{A}^n_{\mathbb{C}}$?*

Notice that for all exotic structures on $\mathbb{A}^n_{\mathbb{C}}$ known so far the Makar-Limanov invariant is non-trivial, whereas for a flexible such structure, by Proposition 5.1 even the field Makar-Limanov invariant must be trivial.

Remark 6.15. The preprint of the present paper inspired some further related results and interesting conjectures, see [4], [10], and [38]. In particular, according to [38] the affine cones over smooth del Pezzo surfaces of degree ≥ 4 are flexible. In [4] a stable birational version of infinite transitivity is proposed. Conjecture 1.4 in [4] relates this property to unirationality in the sense converse to that of Proposition 5.1. This should give a characterization of unirationality versus rational connectedness.

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