MODULI SPACES AND GRASSMANNIAN

JIA-MING (FRANK) LIOU AND A. SCHWARZ

Abstract. We calculate the homomorphism of the cohomology induced by the Krichever map of moduli spaces of curves into infinite-dimensional Grassmannian. This calculation can be used to compute the homology classes of cycles on moduli spaces of curves that are defined in terms of Weierstrass points.

Mathematical Subject Classification (2010) 14H10,14M15,3E130

Keywords: Grassmannian, Krichever maps, lambda-classes, moduli Spaces, Schubert Classes.

1. Introduction

We study the relation between the topology of Sato Grassmannian and the topology of the moduli space of compact complex curves. The Sato Grassmannian (or, better to say the Segal-Wilson [14] version of Sato Grassmannian) associated with a polarized Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is an infinite dimensional Banach manifold $\mathrm{Gr}(\mathcal{H})$ modeled on the space of compact operators from $\mathcal{H}_-$ to $\mathcal{H}_+$. Its path components are parametrized by the set of integers. The cohomology of each path component can be identified with the cohomology of the infinite classical Grassmannian [13]. Let $\mathrm{Gr}_d(\mathcal{H})$ be a path connected component of $\mathrm{Gr}(\mathcal{H})$, where d is an integer. The cohomology ring of $H^*(\mathrm{Gr}_d(\mathcal{H}))$ is isomorphic to the polynomial ring $\mathbb{C}[c_1, c_2, \dots]$ with variables c_k whose degrees are $2k$. Since S^1 acts naturally on $\mathrm{Gr}(\mathcal{H})$, we can also consider the S^1-equivariant cohomology of $\mathrm{Gr}(\mathcal{H})$.

The moduli space $\widehat{\mathcal{F}}_{g,h}$ is the space of quintuples (C, p, z, L, ϕ) where C is a compact complex curve, z is a local coordinate in the disk D centered at the point $p \in C$, and L stands for a line bundle over C having a trivialization ϕ over D. This space can be mapped into the Sato Grassmannian $\mathrm{Gr}(\mathcal{H})$ by means of Krichever construction sending $x = (C, p, z, L, \phi) \in \widehat{\mathcal{F}}_{g,h}$ to the closed subspace of $\mathcal{H}$ consisting of functions $f : S^1 \to \mathbb{C}$ that can be obtained as restrictions of holomorphic sections of L over $C \backslash D$. (See, for example [10].) This construction determines an embedding $k : \widehat{\mathcal{F}}_{g,h} \to \mathrm{Gr}(\mathcal{H})$. The image of $\widehat{\mathcal{F}}_{g,h}$ in $\mathrm{Gr}(\mathcal{H})$ via k is called the Krichever locus. The continuous map $k : \mathcal{F}_{g,h} \to \mathrm{Gr}(\mathcal{H})$ induces a homomorphism on cohomology ring $k^* : H^*(\mathrm{Gr}(\mathcal{H})) \to H^*(\widehat{\mathcal{F}}_{g,h})$. The group $U(1) = S^1$ acts in natural way on $\widehat{\mathcal{F}}_{g,h}$ and on $\mathrm{Gr}(\mathcal{H})$; this action commutes with Krichever map hence we can talk about corresponding homomorphism of equivariant cohomology. We analyze the induced map k^* both for conventional cohomology and equivariant cohomology. We express this map in terms of lambda-classes, introduced by Mumford [9], and their generalizations.

The work was partially supported by NSF grant DMS-0805989.

The cohomology of Sato Grassmannian can be represented by finite-codimension subvarieties that are called Schubert cycles. [1]The intersections of Schubert cycles with Krichever locus can be described as cycles on moduli spaces that are defined in terms of Weierstrass points [1]. Our calculations can be interpreted as calculations of (co)homology classes of these cycles provided that the Schubert cycle and Krichever locus are in general position.

The results of the present paper should be important in the analysis of Grassmannian string theory suggested in [16] as a version of nonperturbative string theory. The ideas of [16] should be combined with ideas of [15]; this leads to the analysis of BV-algebra of equivarianted chains on Grassmannian. The present paper is a first step in this direction.

2. Preliminaries

2.1. Sato Grassmannian. A semi-infinite structure on an infinite dimensional separable Hilbert space $\mathcal{H}$ is a triple $(\mathcal{H}_+, \mathcal{H}_-, \kappa)$, where $\mathcal{H}_+$ and $\mathcal{H}_-$ are infinite dimensional closed subspace of $\mathcal{H}$ with $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\kappa : \mathcal{H} \to \mathcal{H}$ is an invertible map so that $\kappa : \mathcal{H}_\pm \to \mathcal{H}_\mp$. A polarized Hilbert space is a Hilbert space $\mathcal{H}$ together with a semi-infinite structure $(\mathcal{H}_+, \mathcal{H}_-, \kappa)$ on it. A polarized Hilbert space is denoted by $(\mathcal{H}, \kappa)$. Given a polarized Hilbert space $(\mathcal{H}, \kappa)$, denote the orthogonal projections from $\mathcal{H}$ onto $\mathcal{H}_\pm$ by $\pi_\pm$ respectively.

The Sato Grassmannian[2] $\mathrm{Gr}(\mathcal{H})$ associated with a polarized Hilbert space $(\mathcal{H}, \kappa)$ is the set of all closed subspaces W of $\mathcal{H}$ such that the orthogonal projection $\pi_-|_W : W \to \mathcal{H}_-$ is a Fredholm operator and the orthogonal projection $\pi_+|_W : W \to \mathcal{H}_+$ is a compact operator [3].

For each $W \in \mathrm{Gr}(\mathcal{H})$, let U_W be the set of all closed subspaces which are graphs of compact operators from W into $W^\perp$. In other words, $V \in U_W$ if and only if V consists of points of the form $w + Kw, w \in W$ for some compact operator $K : W \to W^\perp$. Define $\varphi_W(V) = K$, where $V = W + KW$ in U_W. Then $\varphi_W : U_W \to \mathcal{K}(W, W^\perp)$ is a bijection for each $W \in \mathrm{Gr}(\mathcal{H})$. (Here $\mathcal{K}(V, V')$ stands for the space of compact operators from a Banach space V to a Banach space V'). The family $\{(U_W, \varphi_W)\}$ gives $\mathrm{Gr}(\mathcal{H})$ a Banach manifold structure modelled on $\mathcal{K}(\mathcal{H}_+, \mathcal{H}_-)$. A typical example of a polarized Hilbert space is $L^2(S^1)$ together with the standard semi-infinite structure defined as follows. The subspaces $L^2(S^1)_+$ and $L^2(S^1)_-$ of $L^2(S^1)$ are the closed subspaces of $L^2(S^1)$ spanned by $\{z^i : i \geq 0\}$ and $\{z^j : j < 0\}$ respectively and form an orthogonal direct sum of $L^2(S^1)$. The standard semi-infinite structure on $L^2(S^1)$ is the map $\kappa(f)(z) = \frac{1}{z} f\left(\frac{1}{z}\right)$. One can see that κ maps $L^2(S^1)_\pm$ into $L^2(S^1)_\mp$. The standard Sato Grassmannian $\mathrm{Gr}(L^2(S^1))$ is the Sato Grassmannian associated with the standard polarized Hilbert space $(L^2(S^1), \kappa)$. From now on, we will assume $\mathcal{H} = L^2(S^1)$. A point W in $\mathrm{Gr}(\mathcal{H})$ is said to have virtual index d if the Fredholm operator $\pi_-|_W$ has index d. The set of all points of $\mathrm{Gr}(\mathcal{H})$ consisting of virtual index d forms a submanifold of $\mathrm{Gr}(\mathcal{H})$; it is denoted by $\mathrm{Gr}_d(\mathcal{H})$. The manifolds $\{\mathrm{Gr}_d(\mathcal{H})\}_{d \in \mathbb{Z}}$ are connected components of $\mathrm{Gr}(\mathcal{H})$.

2.2. Moduli Spaces and the Krichever Map. We denote by $\mathcal{M}_g$ the moduli space of complex curves of genus g (of Riemann surfaces of genus g). As a set this is a set of all equivalence classes of compact smooth complex curves of genus g. A rigorous definition of

[1]The statement that a submanifold of oriented manifold specifies a cohomology class of dimension equal to the codimension of submanifold is well known in finite-dimensional case, but its precise formulation and proof in infinite-dimensional case are non-trivial. However, in the situations we consider it is possible to justify our considerations representing infinite-dimensional manifolds as limits of finite-dimensional ones.

[2]We use the version of Sato Grassmannian defined by Segal and Wilson.

[3]Sometimes it is convenient to use Hilbert-Schmidt operators instead of compact operators in the definition of Grassmannian. Our calculations can be applied to this modification of Grassmannian.

moduli space $\mathcal{M}_g$ is complicated, because complex curves can have non-trivial automorphisms. This means that the moduli space should be regarded as an orbifold or as a stack. To avoid these complications we can work instead with families of curves . [4] A complex curve of genus g with n marked points is a collection $(C, p_1, \ldots, p_n)$, where C is a compact complex curve of genus g and $(p_1, \ldots, p_n)$ is an n-tuple of distinct points on C. A morphism from $(C, p_1, \ldots, p_n)$ to $(C', p'_1, \ldots, p'_n)$ is a holomorphic map $\varphi : C \to C'$ such that $\varphi(p_j) = p'_j$ for $1 \leq j \leq n$. The moduli space of complex curves with n marked points $\mathcal{M}_{g,n}$ consists of all isomorphism classes of compact complex curves of genus g with n marked points. It is obvious that $\mathcal{M}_{g,0} = \mathcal{M}_g$. Again we can work with families instead of moduli spaces. Similarly we can define other moduli spaces.

The moduli space $\widehat{\mathcal{M}}_g$ is defined as a space of triples (C, p, z), where C is a compact complex curve of genus g with a point p and a map $z : D \to \mathbb{D}$ is an isomorphism from a closed set D into the closed unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ obeying $z(p) = 0$. The moduli space of quintuples (C, p, z, L, ϕ), where (C, p, z) specifies a point of $\widehat{\mathcal{M}}_g$ and L is a line bundle over C together with a local trivialization ϕ over D will be denoted by $\widehat{\mathcal{F}}_{g,h}$. We also denote by $\mathcal{F}_{g,h}$ the moduli space of triples (C, p, L), where $(C, p) \in \mathcal{M}_{g,1}$ and L is a line bundle over C of degree h.The moduli space of pairs (C, L), where $C \in \mathcal{M}_g$ and L is a line bundle over C of degree h will be denoted by $\mathcal{P}_{g,h}$.

The moduli spaces $\widehat{\mathcal{M}}_g$ and $\widehat{\mathcal{F}}_{g,h}$ can be embedded into the standard Sato Grassmannian $\mathrm{Gr}(\mathcal{H})$.

Let (C, p, z, L, φ) be a point in $\widehat{\mathcal{F}}_{g,h}$. Identify D with the closed unit disk $\mathbb{D}$ and its boundary ∂D with S^1 via z. Let $H^0(C \setminus D, L)$ be the space of holomorphic sections of L over $C \setminus D$. Let $k(x)$ be the closed subspace of $L^2(S^1)$ consisting of functions f with the property that there exists $s \in H^0(C \setminus D, L)$ such that $f = s|_{S^1}$. One can show that $k(x) \in \mathrm{Gr}(\mathcal{H})$ ([14]). Moreover, $\ker \pi_-|_{k(x)}$ and $\operatorname{coker} \pi_-|_{k(x)}$ can be identified with $\ker \overline{\partial}_L = H^0(C, L)$ and $\operatorname{coker} \overline{\partial}_L = H^1(C, L)$ respectively. By the Riemann-Roch theorem,

$$\operatorname{ind} k(x) = h^0(C, L) - h^1(C, L) = h - g + 1,$$

where $h^i(C, L) = \dim H^i(C, L)$ for any line bundle L. The map

$$k : \widehat{\mathcal{F}}_{g,h} \to \mathrm{Gr}_d(\mathcal{H}) \tag{2.1}$$

where $d = h - g + 1$ is called the Krichever map . It is a continuous embedding. Similarly, one can construct continuous embeddings

$$k_q : \widehat{\mathcal{M}}_g \to \mathrm{Gr}_{d_q}(\mathcal{H}) \tag{2.2}$$

by defining $k_q(C, p, z) = k(C, p, z, K_C^{\otimes q}, dz^{\otimes q})$, where K_C is the canonical line bundle over C, $h_q = q(2g - 2)$ for $q \geq 1$, and $d_q = h_q - g + 1$. (One can say that these embeddings are obtained as compositions of Krichever map and natural embeddings of $\widehat{\mathcal{M}}_g$ into $\widehat{\mathcal{F}}_{g,h}$.)

2.3. **The Equivariant Cohomology.** Let G be a topological group and X be a G-space. The equivariant cohomology of X is defined to be

$$H^*_G(X) = H^*(EG \times_G X),$$

where EG is a contractible G-space such that G acts freely on EG and $EG \times_G X$ denotes the quotient space of $EG \times X$ modulo the relation $(h \cdot g, x) \sim (h, g \cdot x)$. (This definition works for any group of coefficients, but we always consider the cohomology with coefficients

[4] Recall that a family of curves of genus g with base B is a holomorphic map $p : E \to B$ that can be considered as locally trivial fibration with curves of genus g as fibers.

in $\mathbb{C}$.) When X is a point, $H^*_G(pt)$ is the cohomology $H^*(BG)$ of the classiying space $BG = EG/G$. When G acts freely on X, $H^*_G(X)$ is simply $H^*(X/G)$. The equivariant cohomology $H^*_G(X)$ is an algebra over $H^*_G(pt)$.

If G-space X is an orientable manifold then for every G- invariant cycle Z of codimension r in X one can construct an r-dimensional equivariant cohomology class $[Z]$. We will say that this class is dual to Z (the construction generalizes Poincare duality). If Y is a G-invariant submanifold of X and G-invariant cycle Z in X is in general position with respect to Y (the codimension of $Z \cap Y$ in Y is equal to intersection of Z in X) then

$$i^*[Z] = [Z \cap Y] \tag{2.3}$$

where i^* denotes the homomorphism $H^*_G(X) \to H^*_G(Y)$ induced by the embedding $i : Y \to X$.

Example 2.1. *Let S^∞ be an infinite-dimensional sphere (understood as the direct limit of finite-dimensional spheres with respect to maps $S^{2n-1} \to S^{2n+1}$ induced by natural embeddings $\mathbb{C}^n \subset \mathbb{C}^{n+1}$ or as the unit sphere in an infinite dimensional complex Hilbert space $\mathcal{H}$). It is a contractible space with a free S^1 action. Hence the classifying space BS^1 of S^1 is the infinite dimensional projective space $\mathbb{P}^\infty$. The equivariant cohomology ring $H_{S^1}(pt) = H^*(\mathbb{P}^\infty)$ of a point is the polynomial ring $\mathbb{C}[u]$, where u is a degree 2 element in $H^*(\mathbb{P}^\infty)$. Using this this statement one obtains that the equivariant cohomology ring $H_{\mathbb{T}}(pt)$ where $\mathbb{T} = (S^1)^n$ is an n-dimensional torus is a polynomial ring $\mathbb{C}[u_1, \dots, u_n]$.*

One says that a G-space X is equivariantly formal if its equivariant cohomology is a free module over $H^*_G(pt)$. There exist numerous conditions that guarantee equivariant formality (see [5]); for us it is sufficient to know that among these conditions is vanishing of odd-dimensional cohomology.

Let us suppose that $G = \mathbb{T}$ is a torus and the action of $\mathbb{T}$ on X is equivariantly formal. Then the restriction map $H_{\mathbb{T}}(X) \to H_{\mathbb{T}}(F)$ where F is the set of fixed points of torus action is injective. Hence to calculate the cohomology ring $H_{\mathbb{T}}(X)$ one should describe the image of this map. This can be done [5]. We will formulate the answer in the case when X is a non-singular algebraic variety, F is finite, the action of $\mathbb{T}$ can extended to an algebraic action of algebraic torus $\mathbb{T}^{al}$ and $\mathbb{T}^{al}$ has only a finite number of orbits of complex dimension 1. We define X_1 as a union of these orbits and F (a union of orbits of dimension ≤ 1). Then the image of restriction map can be characterized as the kernel of the homomorphism [5] $H_{\mathbb{T}}(F) \to H_{\mathbb{T}}(X_1, F))$ (GKM-theorem, [5]). Notice, that in the conditions of GKM theorem we can calculate not only equivariant cohomology with respect to the torus $\mathbb{T}$, but also equivariant cohomology with respect to any subtorus $\mathbb{T}' \subset \mathbb{T}$.

The moduli space $\widehat{\mathcal{M}}_g$ has a natural free S^1- action $S^1 \times \widehat{\mathcal{M}}_g \to \widehat{\mathcal{M}}_g$ defined by $(\lambda, (C, p, z)) \mapsto (C, p, \lambda z)$. Let us consider the forgetful map

$$F : \widehat{\mathcal{M}}_g \to \mathcal{M}_{g,1}$$

defined by $F(C, p, z) = (C, p)$. The moduli space $\widehat{\mathcal{M}}_g$ is homotopy equivalent to the moduli space $\mathcal{M}'_{g,1}$, where $\mathcal{M}'_{g,1}$ is the moduli space of triples (C, p, v), where $(C, p) \in \mathcal{M}_{g,1}$ and v is a nonzero tangent vector to C at p. Hence F is homotopy equivalent to the map:

$$F' : \mathcal{M}'_{g,1} \to \mathcal{M}_{g,1},$$

[5]This homomorphism is defined as the boundary homomorphism in the exact sequence of the pair (X_1, F).

where $F'(C,p,v) = (C,p)$ is the forgetful map. Since the fiber of F' is homotopy equivalent to S^1, we have the following identification:

$$\overline{F}' : \mathcal{M}'_{g,1}/S^1 \to \mathcal{M}_{g,1}. \tag{2.4}$$

Lemma 2.1. *We have natural isomorphisms:*

$$H^*_{S^1}(\widehat{\mathcal{M}}_g) = H^*_{S^1}(\mathcal{M}'_{g,1}) = H^*(\mathcal{M}_{g,1}). \tag{2.5}$$

The generator of the algebra $H_{S^1}(pt) = \mathbb{C}[u]$ acts on $H^*(\mathcal{M}_{g,1})$ as multiplication by $-\psi$ where ψ is the first Chern class of the complex line bundle K_π over $\mathcal{M}_{g,1}$ that over the point $(C,p) \in \mathcal{M}_{g,1}$ has a fiber defined as the cotangent space to C at the point p. (This bundle can be interpreted as relative dualizing sheaf of fibration $\mathcal{M}_{g,1} \to \mathcal{M}_g$.)

Proof. Since S^1 acts on both $\mathcal{M}'_{g,1}$ and $\widehat{\mathcal{M}}_g$ freely and $\mathcal{M}'_{g,1}$ is homotopy equivalent to $\widehat{\mathcal{M}}_g$, we have the natural identifications

$$H^*_{S^1}(\mathcal{M}'_{g,1}) = H^*(\mathcal{M}'_{g,1}/S^1) \cong H^*(\widehat{\mathcal{M}}_g/S^1) = H^*_{S^1}(\widehat{\mathcal{M}}_g). \tag{2.6}$$

The map $\overline{F}'$ defined in (2.4) is a homeomorphism which identifies the cohomology:

$$H^*(\mathcal{M}'_{g,1}/S^1) \cong H^*(\mathcal{M}_{g,1}). \tag{2.7}$$

By (2.6) and (2.7), we proved (2.5).

To find the action of the generator of the algebra $H_{S^1}(pt)$ we apply the general statement that in the case of free action of S^1 on X the action of the generator on $H^*_{S^1}(X) = H^*(X/S^1)$ can be described as multiplication on the first Chern class of the circle bundle $X \to X/S^1$. □

The moduli space $\widehat{\mathcal{F}}_{g,h}$ of quintuples (C,p,z,L,ϕ) is homotopy equivalent to the moduli space $\mathcal{F}'_{g,h}$ of quaduples (C,p,v,L), where (C,p,L) specifies an element of $\mathcal{F}_{g,h}$ and v is a tangent vector to C at p. Similarly, we have the following result:

Lemma 2.2. *There exists a natural isomorphism:*

$$H^*_{S^1}(\widehat{\mathcal{F}}_{g,h}) \cong H^*(\mathcal{F}_{g,h}).$$

The generator of the algebra $H_{S^1}(pt) = \mathbb{C}[u]$ acts on $H^*(\mathcal{F}_{g,h})$ as multiplication by $-\omega$ where ω denotes the first Chern class of line bundle over $\mathcal{F}_{g,h}$ having the cotangent space T^*_p to the curve C at p as a fiber over $(C,p,L) \in \mathcal{F}_{g,h}$. (This bundle can be regarded as the relative dualizing sheaf of the forgetful map $\pi' : \mathcal{F}_{g,h} \to \mathcal{P}_{g,h}$.)

Notice that the group S^1 acts on $\mathcal{H} = L^2(S^1)$ as the group of rotations of S^1; this action induces an action on $\mathrm{Gr}(\mathcal{H})$. It is easy to check that the Krichever map commutes with the S^1-action on moduli spaces and on Grassmannian, hence it induces a homomorphism on equivariant cohomology. Our goal is to study this homomorphism.

2.4. **Topology of Sato Grassmannian.** Let us remind some basic facts about topology of finite-dimensional Grassmannian $\mathrm{Gr}_{n,l}$ (of the space of l-dimensional complex vector subspaces of $\mathbb{C}^n$.) The torus $T = (S^1)^n$ (as well as the algebraic torus $\mathbb{T}^{al} = (\mathbb{C}^*)^n$) acts in natural way on $\mathbb{C}^n$ and therefore on $\mathrm{Gr}_{n,l}$. (The torus acts on $\mathbb{C}^n$ by means of linear transformations having vectors of the standard basis $\{e_1, \ldots, e_n\}$ as eigenvectors.) Fixed points of the torus action on $\mathrm{Gr}_{n,l}$ are vector subspaces H_S spanned by subsets S of the the set $\{e_1, \ldots, e_n\}$ consisting of l vectors. There exists a cell decomposition into even-dimensional cells invariant with respect to torus action (Schubert cells); these cells are in one-to-one correspondence with fixed points. It follows that the Grassmannian is equivariantly formal. This means that equivariant cohomology is a free module over the cohomology of one-point set; cocycles dual to Schubert cells (Schubert cocycles) constitute a basis of this module. The two-dimensional orbits of the torus action (orbits of $\mathbb{T}^{al}$ having complex dimension 1) can be described in the following way. Let us consider two fixed points of torus action corresponding to subsets S_1, S_2 having $l-1$ common vectors. Denote the subspace spanned by vector $e_i + \lambda e_j$ and vectors from $S_1 \bigcap S_2$ by V_λ (here $\lambda \in \mathbb{C}$, $e_i \in S_1 \setminus S_1 \bigcap S_2, e_j \in S_2 \setminus S_1 \bigcap S_2$). These subspaces form a two-dimensional orbit of the torus action. Applying GKM theorem one obtains the equivariant cohomology ring of Grassmannian as a subring of the ring of functions on the set of fixed points taking values in the polynomial ring $\mathbb{C}[u_1, ..., u_n]$.

The situation with Sato Grassmannian is similar. An infinite-dimensional torus $\mathbb{T}$ acts on $\mathcal{H}$ [6] and therefore on $\mathrm{Gr}(\mathcal{H})$. The fixed points of this action are subspaces $\mathcal{H}_S$ spanned by vectors z^j where $j \in S$. Such a subspace belongs to $\mathrm{Gr}(\mathcal{H})$ iff $S \in \mathcal{S}$ where $\mathcal{S}$ consists of subsets of $\mathbb{Z}$ that differ from $\mathbb{Z}_-$ only by finite number of points, i.e. the symmetric difference $S\Delta\mathbb{Z}_-$ is a finite set . Two-dimensional orbits of torus action correspond to pairs of subsets S_1, S_2 such that one can go from one subset to another deleting and adding one vector. The construction of such an orbit is similar to the construction in finite-dimensional case. We can describe equivariant cohomology classes of Grassmannian $\mathrm{Gr}_d(\mathcal{H})$ in terms of their restriction to fixed points (to the points of the form $\mathcal{H}_S$). We prove that the GKM theorem can be applied to Sato Grassmannian. This allows us to describe the ring $H_\mathbb{T}(\mathrm{Gr}_d(\mathcal{H}))$ as a subring of functions ϕ on $\mathcal{S}$ taking values in the ring $H^*_\mathbb{T}(pt)$ (we consider only functions with finite support). Namely, if $S_1 = (S_1 \cap S_2) \cup \{e_i\}, S_2 = (S_1 \cap S_2) \cup \{e_j\}$ the difference $\phi(S_1) - \phi(S_2)$ should be divisible by $u_i - u_j$. We are mostly interested in cohomology $H_{S^1}(\mathrm{Gr}_d(\mathcal{H}))$; it can be described as a ring of $\mathbb{C}[u]$-valued functions ϕ on $\mathcal{S}$ such that the difference $\phi(S_1) - \phi(S_2)$ is divisible by u. (We embed S^1 into $\mathbb{T}$ by the formula $t_i = \lambda^i$ and therefore we should substitute iu instead of u_i.)

There exists a stratification of $\mathrm{Gr}(\mathcal{H})$ in terms of Schubert cells: the Grassmannian can be represented as a disjoint union of Schubert cells Σ_S; again these cells are in one-to-one correspondence with fixed points (the fixed point $\mathcal{H}_S$ belongs to the cell Σ_S). Instead of a set S one can consider a decreasing sequence $(s_i)_{i \geq 1}$ of elements of this set; it is easy to check that for $n >> 0$ we have $s_n = -n + d$ where d stands for the index of $\mathcal{H}_S$. The complex codimension of Schubert cell Σ_S is given by the formula

$$l(S) = \sum_{i=1}^{\infty} (s_i + i - d).$$

To construct a stratification of Grassmannian $\mathrm{Gr}(\mathcal{H})$ we notice that every subspace $V \subset \mathcal{H}$ that specifies a point of Grassmannian has a canonical basis of the form $e_n = z^{s_n} + \sum_{l \geq s_n + 1} k_{nl} z^l$ where $(s_i)_{i \geq 1}$ is a decreasing sequence and $k_{ns_j} = 0$ for all $1 \leq j < n$. The Grassmannian is a union of sets labelled by sequences (s_i) that appear in the definition of canonical basis; these sets are called Schubert cells, they will be denoted by Σ_S. [7] Notice

[6] If $(\ldots, t_n, \ldots) \in \mathbb{T}$, $n \in \mathbb{Z}$, the corresponding map of $\mathcal{H}$ transforms a point $\sum a_n e_n$ into the point $\sum t_n a_n e_n$. Here $e_n = z^n$.The topology of $\mathbb{T}$ is specified by the operator norm. The cohomology ring $H^*_\mathbb{T}(pt)$ can be considered as a subring of the ring of functions of infinite number of variables u_n where $n \in \mathbb{Z}$ (this follows from the fact that every homomorphism of S^1 into $\mathbb{T}$ induces a map $H^*_\mathbb{T}(pt) \to H^*_{S^1}(pt)$.) One can give a precise description of this subring, but we do not need this description.

[7] An equivalent definition of Schubert cell can be given in the following way. For every $S \in \mathcal{S}$ we construct a set U_S consisting of such elements $V \in \mathrm{Gr}(\mathcal{H})$ that the projection $V \to \mathcal{H}_S$ is an isomorphism. Then we can find a basis of V having the form $e_n = z^n + \sum_l k_{nl} z^l$ where $n \in S, l \notin S$. To define $\Sigma_S \subset U_S$ we impose an additional condition $k_{nl} = 0$ for $l < n$.

that instead of sequences $S = (s_n)$ one can use partitions $\lambda = (\lambda_n)$ where $\lambda_n = s_n + n - d$ vanishes for $n >> 0$. Given a sequence S with its corresponding partition λ, we also denote Σ_S by $\Sigma_{\lambda,d}$ or simply Σ_λ when the index d is specified.

The closure $\overline{\Sigma}_S$ of Σ_S is called the Schubert cycle with the characteristic sequence $S = (s_i)$. It defines a cohomology class in $H^{2l(S)}(\mathrm{Gr}(\mathcal{H}))$ (Schubert class). The Schubert cycle is $\mathbb{T}$-invariant, hence it specifies an element of equivariant cohomology group $H^{2l(S)}_{\mathbb{T}}(\mathrm{Gr}(\mathcal{H}))$. This element (also called Schubert class) will be denoted by the symbol $[\overline{\Sigma}_S]$. [8] We will be mostly interested in equivariant cohomology $H^{2l(S)}_{S^1}(\mathrm{Gr}(\mathcal{H}))$, where S^1 stands for the subgroup of $\mathbb{T}$ corresponding to rotation $z \to \lambda z$, but the statements of the next paragraph can be generalized to any subtorus of $\mathbb{T}$.

One can prove that the Schubert cocycles specify a basis of cohomology of Grassmannian. Similarly equivariant cohomology is a free module over $H_{S^1}(pt)$ generated by equivariant Schubert classes. The multiplication of Schubert classes can be expressed in terms of Schur functions , as in finite-dimensional case. More precisely, if Schubert classes are labeled by partitions, the multiplication formula for Schubert classes in the case of Sato Grassmannian is the same as for finite-dimensional Grassmannian. The multiplication of equivariant Schubert classes can be expressed in terms of shifted Schur functions introduced in [12] ; see [8].

It is easy to check that (equivariant) cohomology of $\mathrm{Gr}_d(\mathcal{H})$ is a polynomial algebra generated by Schubert classes c_r corresponding to sequences $S = (s_j)$, where $s_j = 1 - j + d$ for $1 \leq j \leq r$ and $s_j = -j + d$ for $j \geq r+1$.[9] If we are working with equivariant cohomology we will use notations C_r for these Schubert classes.[10]

The proof of these statements can be based on the results of [13]. Following [13] we can consider the sequence $G_k \subset \mathrm{Gr}(\mathcal{H})$ where the subspace $V \in \mathrm{Gr}(\mathcal{H})$ belongs to G_k iff $\mathcal{H}_{-k} \subset V \subset \mathcal{H}_k$. (We use the notation $\mathcal{H}_k$ for the subspace of $\mathcal{H}$ spanned by z^n with $n < k$.) It follows from [13] that the homology and cohomology of $\mathrm{Gr}(\mathcal{H})$ coincide with homology (cohomology) of the union Gr_0 of sets G_k. It is easy to derive from this fact that the same is true for equivariant homology and cohomology. The space Gr_0 admits a cell decomposition consisting of invariant even-dimensional cells [13]; this decomposition can be used to calculate (equivariant) (co)homology and justify the above statements. A little bit different proof is based on the remark that the homology of Gr_0 can be represented as direct limit of homology groups of G_k; the same is true for equivariant homology. [11] The remark that G_k is homeomorphic to disjoint union of finite-dimensional Grassmannians permits us to finish the proof.

3. Cohomological properties of Krichever map

The action of the Krichever map on the cohomology of Grassmanian $\mathrm{Gr}_d(\mathcal{H})$ can be expressed in terms of lambda-classes. If we are working with equivariant cohomology, we can get analogous results by introducing the notion of equivariant lambda-classes.

Recall that the Hodge bundle $\mathbb{E}$ over moduli space $\mathcal{M}_g$ is defined as a bundle having as a fiber over a curve $C \in \mathcal{M}_g$ the space of all holomorphic differentials on C. Replacing

[8] More precisely, if the sequence S has index d the class $[\overline{\Sigma}_S]$ belongs to $H^{2l(S)}_{\mathbb{T}}(\mathrm{Gr}_d(\mathcal{H}))$.

[9] The corresponding partitions are $(1, \ldots, 1, 0, \ldots) = (1^r)$ for $r \geq 1$.

[10] The classes c_r can be interpreted as Chern classes of (infinite-dimensional) tautological vector bundle over $\mathrm{Gr}_d(\mathcal{H})$ (up to a factor $(-1)^r$). We do not use this interpretation, because it does not work in equivariant case: equivariant Chern classes are not well defined for tautological bundle.

[11] More generally, one can consider spaces G_{kl} consisting of subspaces obeying $\mathcal{H}_l \subset V \subset \mathcal{H}_k$) and take the limit $k \to \infty, l \to -\infty$.

the space of holomorphic differentials by the space of holomorphic q-differentials in the definition of the Hodge bundle, we obtain a more general notion of Hodge bundle $\mathbb{E}_q$. (For $q > 1$ this is a bundle of dimension $d_q = (2q-1)(g-1)$.) More rigorously we can define Hodge bundle $\mathbb{E}_q$ as the pushforward of q-th power of relative dualizing sheaf K_π of forgetful map $\pi : \mathcal{M}_{g,1} \to \mathcal{M}_g$. (As we noticed this sheaf can be identified with the complex line bundle over $\mathcal{M}_{g,1}$ that over the point $(C,p) \in \mathcal{M}_{g,1}$ has a fiber defined as cotangent space to C at the point p. The first Chern class of this bundle was denoted by ψ.)

Chern classes of Hodge bundle are called lambda-classes; they are denoted by λ_r (if we would like to emphasize that we are working with q-differentials we use the notation λ_r^q). For all other moduli spaces, we have a natural map onto $\mathcal{M}_g$; taking pullback with respect to this map, we construct Hodge bundles and lambda-classes on these spaces.

Hodge bundles over $\widehat{\mathcal{M}}_g$ are S^1-equivariant bundles, hence we can define corresponding equivariant Chern classes. They are called equivariant lambda-classes and denoted by Λ_r. The equivariant Chern classes of $\mathbb{E}_q$ are denoted by Λ_r^q. Equivariant lambda classes are equivariant cohomology classes of $\widehat{\mathcal{M}}_g$, or equivalently, cohomology classes of $\mathcal{M}_{g,1}$. If π is the natural projection $\mathcal{M}_{g,1} \to \mathcal{M}_g$ we can say that $\Lambda_r^q = \pi^*\lambda_r^q$. To prove this fact we notice that $\mathcal{M}_g$ can be considered as S^1-space and Hodge bundle as an equivariant bundle over it if we assume that S^1 acts trivially. The equivariant Hodge bundle over $\widehat{\mathcal{M}}_g$ can be regarded as pullback of Hodge bundle over $\mathcal{M}_g$ with respect to natural projection $\widehat{\mathcal{M}}_g \to \mathcal{M}_g$; this allows us to obtain Λ_r^q from λ_r^q considered as equivariant Chern class of a bundle with trivial action of S^1.

The following theorem describes the behavior of equivariant cohomology classes C_r with respect to the Krichever map. (Notice, that $k_q^*(u) = -\psi$ as follows from Lemma 2.1.) It is sufficient to calculate Krichever map on the equivariant cohomology ring of $\mathrm{Gr}_d(\mathcal{H})$ (classes C_r generate this ring), however, it is possible to calculate directly the image of Schubert classes $[\overline{\Sigma}_S]$ (The result can be expressed in terms of shifted Schur functions defined in [12]. This calculation will be published separately.)

Theorem 3.1. *In the case $q > 1$,*

$$k_q^* C_r = (-1)^r \sum_{j+m=r} (-1)^m h_m(q, q+1, \ldots, q+d_q-r)\psi^m \Lambda_j^q \tag{3.1}$$

for all $1 \le r \le d_q$ and

$$k_q^* C_r = (-1)^r \sum_{m+j=r} e_m(q-1, q-2, \cdots, q-r+d_q+1)\psi^m \Lambda_j^q, \tag{3.2}$$

if $r > d_q$.

Here $h_m(x_1, x_2, \ldots)$ is the m-th complete symmetric function in variables $\{x_1, x_2, \ldots\}$ and $e_m(x_1, x_2, \cdots)$ denotes the m-th elementary symmetric function in variables $\{x_1, x_2, \cdots\}$.

Proof. In order to calculate the homomorphism k_q^* we introduce the space Gr_d^l as the submanifold of $\mathrm{Gr}_d(\mathcal{H})$ consisting of all W such that the orthogonal projection $\pi_l : W \to z^{-l}\mathcal{H}_-$ is surjective. It follows from this requirement that there is an equivariant $(d+l)$-dimensional vector bundle $\mathcal{E}_l$ over Gr_d^l whose fiber over W is the kernel of the projection $\pi_l : W \to z^{-l}\mathcal{H}_-$.

The intersections of Schubert cells with Gr_d^l form a stratification of Gr_d^l; the strata are also called Schubert cells. For every Schubert cell in $\mathrm{Gr}_d(\mathcal{H})$ and sufficiently large l this cell is in general position with respect to $\mathrm{Gr}_d^l(\mathcal{H})$; in other words the corresponding cell in $\mathrm{Gr}_d^l(\mathcal{H})$ has the same codimension. (Recall that the codimension of the Schubert cell Σ_S is determined by the length $l(S)$). Denote the intersection of $\overline{\Sigma}_{(1^r)}$ and Gr_d^l by $\overline{\Sigma}_{(1^r),l}$. The equivariant cohomology class corresponding to $\overline{\Sigma}_{(1^r),l}$ is denoted by $C_{r,l}$. Since $\overline{\Sigma}_{(1^r),l}$ is in general position with respect to Gr_d^l for $l >> 0$ applying the formula (2.3) we obtain

$$f_l^* C_r = C_{r,l} \tag{3.3}$$

where $f_l^* : H_T^*(\mathrm{Gr}_d) \to H_T^*(\mathrm{Gr}_d^l)$ is the homomorphism induced by the inclusion map $f_l : \mathrm{Gr}_d^l \to \mathrm{Gr}_d$.

Denote $d = d_q$. (Here $d_q = (2q-1)(g-1)$ stands for the dimension of Hodge bundle $\mathbb{E}_q$, $q > 1$.). The Krichever locus $k_q(\widehat{\mathcal{M}}_g)$ lies in Gr_d^l for all $l \geq 0$. We obtain modified Krichever maps ${}^l k_q : \widehat{\mathcal{M}}_g \to \mathrm{Gr}_d^l$ for all $l \geq 0$. Then $k_q = f_l \circ {}^l k_q$, where $f_l : \mathrm{Gr}_d^l \to \mathrm{Gr}_d$ is the inclusion map. Hence we can compute $k_q^* : H_T^*(\mathrm{Gr}_d) \to H_T^*(\widehat{\mathcal{M}}_g)$ composing homomorphisms in the sequence

$$H_{S^1}^*(\mathrm{Gr}_d) \xrightarrow{f_l^*} H_{S^1}^*(\mathrm{Gr}_d^l) \xrightarrow{{}^l k_q^*} H_{S^1}^*(\widehat{\mathcal{M}}_g).$$

Due to (3.3) it is sufficient to calculate ${}^l k_q^* C_{r,l}$. To do this we express $C_{r,l}$ in terms of equivariant Chern classes. The expression we need can be obtained from general Kempf-Laksov formula (see [6] or [3], Lecture 8), but we can use also simpler Porteous formula.

Let us construct a vector bundle $\underline{\mathcal{H}}_{i,j}$ over Gr_d^l as a bundle with total space $\mathcal{H}_{i,j} \times \mathrm{Gr}_d^l$. Here $\mathcal{H}_{i,j}$ is the subspace of $\mathcal{H}$ spanned by $\{z^m : i \leq m \leq j\}$. We define the action of S^1 on this bundle by

$$(\lambda, f, W) \mapsto (\lambda^{-q} f(\lambda^{-1} z), \lambda(W)) \tag{3.4}$$

where $\lambda \in S^1$, $f \in \mathcal{H}_{i,j}$ and $W \in \mathrm{Gr}_d^l$. (We define $\lambda(W)$ as a space of functions $f(\lambda^{-1}z)$ where $f(z) \in W$.) We also define an S^1-action on $\mathcal{E}_l$ by (3.4) on the fiber of $\mathcal{E}_l$. Then the bundles $\underline{\mathcal{H}}_{i,j}$ and $\mathcal{E}_l$ are non-trivial equivariant bundles. The total equivariant Chern class c^T(the sum of all equivariant Chern classes) of $\underline{\mathcal{H}}_{i,j}$ is given by the formula

$$c^T(\underline{\mathcal{H}}_{i,j}) = \prod_{m=i}^{j} (1 - (q+m)u).$$

Lemma 3.1.

$$C_{r,l} = (-1)^r c_r^T(\mathcal{E}_l - \underline{\mathcal{H}}_{-l,d-r}). \tag{3.5}$$

Note that the class $c_r^T(\mathcal{E}_l - \underline{\mathcal{H}}_{-l,d-r})$ is well-defined because $\mathcal{E}_l$ and $\underline{\mathcal{H}}_{-l,d-r}$ are equivariant complex vector bundles of finite rank over Gr_d^l.

To prove this lemma we consider an equivariant bundle map $\mathcal{E}_l \to \underline{\mathcal{H}}_{-l,d-r}$ defined by means of orthogonal projection of fibers. The cycle $\overline{\Sigma}_{(1^r),l}$ can be considered as degeneracy locus of this bundle map (this is the locus where the rank of the map of fibers is $\leq l+d-r$). This allows us to apply the Porteous formula [4] to calculate the dual cohomology class.

Let us denote by $\mathcal{E}_l^0$ the restriction of the bundle $\mathcal{E}_l$ to Gr_d^0. The restriction of $\underline{\mathcal{H}}_{-l,-1}$ to Gr_d^0 will be denoted by $\underline{\mathcal{H}}_{-l,-1}^0$. There exists an exact sequence of equivariant bundles

$$0 \to \mathcal{E}_0 \to \mathcal{E}_l^0 \to \underline{\mathcal{H}}_{-l,-1}^0 \to 0.$$

This means that

$$c^T(\mathcal{E}_l^0) = c^T(\mathcal{E}_0) c^T(\underline{\mathcal{H}}_{-l,-1}^0).$$

Let us consider the case $1 \leq r \leq d$. Using the relation $\underline{\mathcal{H}}_{-l,d-r} = \underline{\mathcal{H}}_{-l,-1} \oplus \underline{\mathcal{H}}_{0,d-r}$ we obtain that

$$c^T(\underline{\mathcal{H}}_{-l,d-r}) = c^T(\underline{\mathcal{H}}_{-l,-1}) c^T(\underline{\mathcal{H}}_{0,d-r}).$$

We have

$$(-1)^r k_q^* C_r = (-1)^r\ {}^l k_q^* C_{r,l} = {}^l k_q^* c_r^T(\mathcal{E}_l - \underline{\mathcal{H}}_{-l,d-r}) = {}^0 k_q^* \iota_l^* c_r^T(\mathcal{E}_l - \underline{\mathcal{H}}_{-l,d-r})$$
$$= {}^0 k_q^* c_r^T(\mathcal{E}_l^0 - \underline{\mathcal{H}}^0_{-l,d-r}) = {}^0 k_q^* c_r^T(\mathcal{E}_0 - \underline{\mathcal{H}}^0_{0,d-r}).$$

Here ι_l stands for the embedding $\iota_l : \mathrm{Gr}_d^0 \to \mathrm{Gr}_d^l$, ${}^l k_q = \iota_l\ {}^0 k_q$. It remains to notice that $\mathbb{E}_q = {}^0 k_q^* \mathcal{E}_0$ (the bundle $\mathbb{E}_q$ is the pullback of $\mathcal{E}_0$ as an S^1-equivariant bundle with respect to the action (3.4)). Hence $\Lambda_r^q = {}^0 k_q^* c_r^T(\mathcal{E}_0)$. We obtain

$$(-1)^r k_q^* C_r = \left(\frac{c^T(\mathbb{E}_q)}{\prod_{j=0}^{d-r}(1+(q+j)\psi)} \right)_{[r]}.$$

where $(x)_{[r]}$ stand for $2r$-dimensional component of cohomology class x.

Using the Cauchy's identity,

$$\prod_{j=0}^{d-r}(1+(q+j)x)^{-1} = \sum_{m=0}^{\infty}(-1)^m h_m(q, q+1, \ldots, q+d-r)x^m, \tag{3.6}$$

we obtain

$${}^l k_q^* c^T(\mathcal{E}_l - \underline{\mathcal{H}}_{-l,d-r}) = \sum_{j=0}^{d_q}\sum_{m=0}^{\infty}(-1)^m h_m(q, q+1, \ldots, q+d_q-r)\psi^m \Lambda_j^q$$

which implies (3.1). In the case $r > d$, very similar arguments lead to the relation:

$$(-1)^r k_q^* C_r = {}^0 k_q^* c_r^T(\mathcal{E}_0 \oplus \underline{\mathcal{H}}^0_{-(r-d)+1,-1}).$$

This relation implies (3.2). □

Applying the forgetful map $H^*_{S^1}(\mathrm{Gr}_d(\mathcal{H})) \to H^*(\mathrm{Gr}_d(\mathcal{H}))$, we obtain:

Corollary 3.1.

$$k_q^* c_r = (-1)^r \lambda_r^q$$

if $r \le d_q$ *and*

$$k_q^* c_r = 0,$$

if $r > d_q$.

Of course, it is easy to give an independent proof of these formulas (for example, interpreting c_r as Chern classes of infinite-dimensional tautological vector bundle).

Mumford [9] has shown how to relate lambda classes λ_r to kappa classes $\kappa_r = \pi_* \psi^{r+1}$. The same method, based on Grothendieck-Riemann-Roch theorem, can be used to calculate λ_r^q in terms of kappa classes.

Theorem 3.2. *The r-th component of the Chern character of $\mathbb{E}_q$ is given by*

$$\mathrm{ch}_r\, \mathbb{E}_q = \frac{B_{r+1}(q)}{(r+1)!}\kappa_r,$$

where $B_n(q)$ is the n-th Bernoulli polynomial in q. (The Bernoulli polynomials $\{B_n(x)\}$ are defined by the generating function $te^{xt}/(e^t-1) = \sum_{n=0}^{\infty} B_n(x)t^n/n!$).

This formula was given in [2].

The expression of Chern classes in terms of Chern character is well known (see for example [11]).

The behavior of the equivariant cohomology with respect to the Krichever map k_1 is described as follows:

Theorem 3.3.

$$k_1^* C_r = (-1)^r \sum_{j+m=r} (-1)^m h_m(1,2,\dots,g-r)\psi^m \Lambda_j \tag{3.7}$$

for $r \le g-1$ and

$$k_1^* C_g = (-1)^g \Lambda_g \tag{3.8}$$

and

$$k_1^* C_{g+1} = 0$$

and if $r \ge g+2$, we have

$$k_1^* C_r = (-1)^r \sum_{m+j=r} (-1)^m e_m(1,2,\cdots,r-g-1)\psi^m \Lambda_j. \tag{3.9}$$

Proof. The Krichever locus $k_1(\widehat{\mathcal{M}}_g)$ lies in Gr_d^l for all $l \ge 1$. Consider the modified Krichever maps ${}^l k_1 : \widehat{\mathcal{M}}_g \to \mathrm{Gr}_d^l$ for $l \ge 2$. We compute k_1^* via

$$H^*_{S^1}(\mathrm{Gr}_d) \xrightarrow{f_l^*} H^*_{S^1}(\mathrm{Gr}_d^1) \xrightarrow{{}^l k_1^*} H^*_{S^1}(\widehat{\mathcal{M}}_g).$$

Denote $\mathcal{E}_l^1$ the restriction of the bundle $\mathcal{E}$ to Gr_d^1. The restriction of $\underline{\mathcal{H}}_{i,j}$ to Gr_d^1 is denoted by $\underline{\mathcal{H}}_{i,j}^1$ for each i,j. Then there exists an exact sequence of equivariant vector bundles:

$$0 \to \mathcal{E}_1 \to \mathcal{E}_l^1 \to \underline{\mathcal{H}}^1_{-l,-2} \to 0 \tag{3.10}$$

which gives

$$c^T(\mathcal{E}_l^1) = c^T(\mathcal{E}_1) c^T(\underline{\mathcal{H}}^1_{-l,-2}).$$

For $1 \le r \le g$, using the relation $\underline{\mathcal{H}}_{-l,g-1-r} = \underline{\mathcal{H}}_{-l,-2} \oplus \underline{\mathcal{H}}_{-1,g-1-r}$, we obtain

$$c^T(\underline{\mathcal{H}}_{-l,g-1-r}) = c^T(\underline{\mathcal{H}}_{-l,-2}) c^T(\underline{\mathcal{H}}_{-1,g-1-r}).$$

We have

$$\begin{aligned}(-1)^r k_1^* C_r &= (-1)^r \; {}^l k_1^* C_{r,l} = \; {}^l k_1^* c_r^T(\mathcal{E}_l - \underline{\mathcal{H}}_{-l,g-1-r}) = \; {}^1 k_1^* \; \iota_{1,l}^* c_r^T(\mathcal{E}_l - \underline{\mathcal{H}}_{-l,g-1-r}) \\ &= \; {}^1 k_1^* c_r^T(\mathcal{E}_l^1 - \underline{\mathcal{H}}^1_{-l,g-1-r}) = (-1)^r \; {}^1 k_1^* c_r^T(\mathcal{E}_1 - \underline{\mathcal{H}}_{-1,g-1-r}).\end{aligned}$$

Here $\iota_{1,l}$ stands for the embedding $\iota_{1,l} : \mathrm{Gr}_d^1 \to \mathrm{Gr}_d^l$, ${}^l k_1 = \iota_{1,l} \; {}^1 k_1$. Notice that $\mathbb{E} = \; {}^1 k_1^* \mathcal{E}_1$ and hence $\Lambda_r = \; {}^1 k_1^* c_t^T(\mathcal{E}_1)$. We obtain

$$(-1)^r k_1^* C_r = \left(\frac{c^T(\mathbb{E})}{\prod_{j=-1}^{g-1-r}(1+(j+1)\psi)} \right)_{[r]}$$

which implies (3.7) and (3.8) by the Cauchy's identity. By the exact sequence (3.10), we have

$$(-1)^{g+1} k_1^* C_{g+1} = \; {}^1 k_1^* c_{g+1}^T(\mathcal{E}_l^1 - \underline{\mathcal{H}}^1_{-l,-2}) c = \; {}^1 k_1^* c_{g+1}^T(\mathcal{E}_1) = c_{g+1}^T(\mathbb{E}) = 0.$$

In the case $r > g+1$, very similar arguments give us

$$(-1)^r k_1^* C_r = \; {}^1 k_1^* c_r^T(\mathcal{E}_1 \oplus \underline{\mathcal{H}}^1_{-(r-g),-2}).$$

This result gives us (3.9). □

Similarly, forgetting about the equivariant structure, we obtain

Corollary 3.2.

$$k_1^* c_r = (-1)^r \lambda_r$$

for $r \leq g$,

$$k_1^* c_r = 0,$$

for $r > g$.

The moduli space $\mathcal{M}_g$ can be embedded in the moduli space $\overline{\mathcal{M}}_g$ (Deligne-Mumford compactification). Similar embeddings exist for other moduli spaces we considered (we allow curves with simple double points, but the marked point should be non-singular). The Krichever map k_1 can be extended to the moduli space $\widehat{\overline{\mathcal{M}}}_g$, but this extension is not continuous (however, the extension is continuous on the subspace consisting of irreducible curves). More generally, the map k_1 can be extended to the moduli space of irreducible Cohen-Macaulay curves with a disk around a non-singular point; this extension is continuous in appropriate topology. (This follows from the results of [14] and from the remark that the dualizing sheaf of Cohen-Macaulay curve is torsion-free.) Our methods can be applied to the analysis of cohomological properties of the extended Krichever map.

Let us define a vector bundle $\mathbb{P}$ on the moduli space $\mathcal{P}_{g,h}$ (on the moduli space of pairs (C, L)) as a bundle having a fiber over a point (C, L) that can be identified with the space of holomorphic sections of L.[12] (To guarantee the existence of such a vector bundle we impose the condition $h > 2g - 2$; then one of the terms in Riemann-Roch theorem vanishes and $\mathbb{P}$ is a bundle of rank $d = h - g + 1$.) The Chern classes of $\mathbb{P}$ are denoted by the symbol p_r; we will use the same notation for their images in cohomology of other moduli spaces that can be mapped in $\mathbb{P}$ in natural way.

The classes p_r are analogous to lambda-classes λ_r^q. The methods that were applied to calculate lambda-classes can be used to compute p_r. It is easy to check that the bundle $\mathbb{P}$ is a pushforward of a bundle on $\mathcal{F}_{g,h}$. This bundle, denoted by $\mathcal{L}$, has a fiber L_p over the point $(C, p, L) \in \mathcal{F}_{g,h}$. (Here L_p stands for the fiber of L over the point p.) We denote its first Chern class by γ.

Recall that we denoted by ω be the first Chern class of line bundle over $\mathcal{F}_{g,h}$ having the cotangent space T_p^* to the curve C at p as a fiber over $(C, p, L) \in \mathcal{F}_{g,h}$ (see Lemma 2.2). Following [7] we define the generalized Mumford-Morita classes $m_{i,j}$ by

$$m_{i,j} = \pi_*(\gamma^i \omega^j) \in H^{i+j-1}(\mathcal{F}_{g,h}).$$

The Chern classes of $\mathbb{P}$ can be expressed in terms of the generalized Mumford-Morita classes $m_{i,j}$.

Theorem 3.4. *The k-th Chern character of* $\mathbb{P}$ *is given by*

$$\operatorname{ch}_k \mathbb{P} = \frac{1}{(k+1)!} m_{k+1,0} - \frac{1}{2(k!)} m_{k,1} + \sum_{j=1}^{[k/2]} \frac{B_{2j}}{(2j)!(k-2j)!} m_{k+1-2j,2j}, \tag{3.11}$$

where $\{B_n\}$ *are Bernoulli numbers* $\dfrac{x}{e^x - 1} = \displaystyle\sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$.

[12] The restriction of $\mathbb{P}$ to moduli spaces of curves embedded into $\mathcal{P}_{g,h}$ by means of q-differentials coincides with Hodge bundle.

Proof. The proof is based on the Grothendieck-Riemann-Roch theorem applied to the forgetful map $\pi' : \mathcal{F}_{g,h} \to \mathcal{P}_{g,h}$ and the bundle $\mathcal{L}$ on $\mathcal{F}_{g,h}$; it is similar to Mumford's calculation of lambda classes. The Todd class of the relative tangent sheaf $\mathcal{T}_{\pi'}$ of π' is

$$\mathrm{Td}\,\mathcal{T}_{\pi'} = \frac{-\omega}{1-e^{\omega}} = 1 - \frac{1}{2}\omega + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!}\omega^{2j}.$$

Since $\mathrm{ch}\,\mathcal{L} = e^{\gamma}$, we find

$$\mathrm{ch}\,\mathcal{L}\cdot \mathrm{Td}\,\mathcal{T}_{\pi'} = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{\gamma^k\omega}{k!} + \sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\frac{B_{2j}}{k!(2j)!}\gamma^k\omega^{2j}.$$

The Grothendieck-Riemann-Roch theorem states that $\mathrm{ch}\,\mathbb{P} = \pi'_*(\mathrm{ch}\,\mathcal{L}\,\mathrm{Td}\,\mathcal{T}_{\pi'})$. This gives us

$$\mathrm{ch}\,\mathbb{P} = \sum_{k=0}^{\infty}\frac{m_{k,0}}{k!} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{m_{k,1}}{k!} + \sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\frac{B_{2j}}{(2j)!k!}m_{k,2j}$$

which implies (3.11). □

The bundle $\mathbb{P}$ can be considered as S^1-equivariant bundle with respect to trivial action of S^1. Let us denote by $\mathcal{P}$ an equivariant bundle constructed as a pullback of $\mathbb{P}$ with respect to the forgetful map $\widehat{\mathcal{F}}_{g,h} \to \mathcal{P}_{g,h}$. The equivariant Chern classes P_r transforms into p_r under the identification $H^*_{S^1}(\widehat{\mathcal{F}}_{g,h}) = H^*(\mathcal{F}_{g,h})$. (Notice that the non-equivariant Chern classes of $\mathcal{P}$ are the classes p_r in $H^*(\widehat{\mathcal{F}}_{g,h})$.)

Now let us consider the Krichever map

$$k : \widehat{\mathcal{F}}_{g,h} \to \mathrm{Gr}_d(\mathcal{H}), \tag{3.12}$$

where $d = h - g + 1$ and $h > 2g - 2$. The Krichever locus $k(\widehat{\mathcal{F}}_{g,h})$ lies in $\mathrm{Gr}^l_d(\mathcal{H})$ for all $l \geq 0$.

To study (3.12), we consider the action of S^1 on the vector bundle $\underline{\mathcal{H}}_{i,j}$ and $\mathcal{E}_l$ defined by

$$\lambda\cdot(f, W) \mapsto (f(\lambda^{-1}z), \lambda(W)). \tag{3.13}$$

Then both $\underline{\mathcal{H}}_{i,j}$ and $\mathcal{E}_l$ are nontrivial equivariant vector bundles. The total equivariant Chern class of $\underline{\mathcal{H}}_{i,j}$ is

$$c^T(\underline{\mathcal{H}}_{i,j}) = \prod_{m=i}^{j}(1 - mu).$$

Moreover, the equivariant vector bundle $\mathcal{P}$ of rank d over $\widehat{\mathcal{F}}_{g,h}$ is the pullback of the equivariant d-dimensional bundle $\mathcal{E}_0$ over $\mathrm{Gr}^0_d(\mathcal{H})$ via k with respect to (3.13). It follows from 2.2 that $k^*u = -\omega$.

To calculate the homomorphism induced by (3.12) on the (equivariant) cohomology we repeat the arguments used in the proof of (3.1). We obtain

Theorem 3.5. *For the equivariant case, we have*

$$k^*C_r = (-1)^r \sum_{j+m=r} (-1)^m h_m(1, 2, \ldots, d-r)\omega^m P_j,$$

if $r \leq d$,

$$k^*C_r = (-1)^r \sum_{m+j=r} (-1)^m e_m(1, 2, \cdots, r-d-1)\omega^m P_j,$$

if $r > d$. For the nonequivariant case, we have

$$k^* c_r = (-1)^r p_r$$

if $r \leq d$,

$$k^* c_r = 0$$

if $r > d$.

Acknowledments We are indebted to Yu. Manin, M. Movshev, M. Mulase, A.Okounkov, B. Osserman, F. Plaza-Martin , A. Polishchuk and V. Vologodsky for very useful comments.

References

1. Arbarello, E., de Concini, C., Kac, V.G., Procesi, C.,: Moduli Spaces of Curves and Representation Theory. Comm. Math. Phys. 117, no. 1, 1-36 (1988).
2. Bini, G.,: Generalized Hodge Classes on the Moduli Space of Curves. Beiträge Algebra Geom. 44, no.2, 559-565 (2003).
3. Fulton, W.,: Equivariant Cohomology in Algebraic Geometry. Lecture Notes by D. Anderson (2007).
4. Griffiths, P., Harris, J.,: Principles of Algebraic Geometry. Wiley-Interscience, New York (1978).
5. Goresky, M., Kottwitz, R., MacPherson, R.,: Equivariant Cohomology, Koszul Duality, and the Localization Theorem. Invent. Math. 131, no. 1, 25-83 (1998).
6. Kempf, G., Laksov, D.,: The Determinantal Formula of Schubert Calculus. Acta Math. 132, 153-162 (1974).
7. Kawazumi, N,: A Generalization of the Morita-Mumford Classes to Extended Mapping Class Groups for Surfaces. Invent. Math. 131, no. 1, 137-149 (1998).
8. Liou Jia-Ming, Schwarz, A.: Equivariant cohomology of infinite-dimensional Grassmannian and shifted Schur functions, arXiv:1201.2554, to be published in Mathematical Research Letters
9. Mumford, D.,: Towards an Enumerative Geometry of the Moduli Space of Curves. Arithmetic and Geometry, vol. II, 271-328, Progr. Math., 36, Birkhäuser Boston, Boston, MA (1983).
10. Mulase, M.,: Algebraic theory of the KP equations. Perspective in mathematical physics, 151-217 (1994).
11. Macdonald, I.G.,: Symmetric functions and Hall polynomials. 2nd edition Clarendon Press, Oxford (1995).
12. Okounkov, A., Olshanski, G.,: Shifted Schur Functions, St. Petersburg Math. J. 9, 239-300 (1998),Shifted Schur Functions II. The Binomial Formula for Characters of Classical Groups and its Applications. Kirillov's Seminar on Representation Theory, 245-271, Amer. Math. Soc. Transl. Ser. 2, 181, Amer. Math. Soc., Providence, RI (1998).
13. Pressley, A., Segal, G.,: Loop groups. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1986).
14. Segal, G., Wilson, G.,: Loop groups and equations of KdV type. Inst. Hautes Etudes Sci. Publ Math. no. 61, 5-65 (1985).
15. Sen, A., Zwiebach, B.,: Quantum background independence of closed string field theory. Nuclear Phys. B 423, no 2-3, 580-630 (1994).
16. Schwarz, A.,: Grassmannian and String theory. Comm. Math. Phys. 199, no. 1, 1-24 (1998).

Max Planck Institut für Mathematik, Vivatsgasse 7, Bonn, 53111, Germany, fjmliou@gmail.com

Department of Mathematics, University of California, Davis, CA 95616, USA, schwarz@math.ucdavis.edu