Remarks on surfaces with $c_1^2 = 2\chi - 1$ having non-trivial 2-torsion

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On the occasion of 60-th birthday of Prof. Fabrizio Catanese

Abstract

We shall show that any complex minimal surface of general type with $c_1^2 = 2\chi - 1$ having non-trivial 2-torsion divisors, where c_1^2 and χ are the first Chern number of a surface and the Euler characteristic of the structure sheaf respectively, has the Euler characteristic χ not exceeding 4. Moreover, we shall give a complete description for the surfaces of the case $\chi = 4$, and prove that the coarse moduli space for surfaces of this case is a unirational variety of dimension 29. Using the description, we shall also prove that our surfaces of the case $\chi = 4$ have non-birational bicanonical maps and no pencil of curves of genus 2, hence being of so called non-standard case for the non-birationality of the bicanonical maps.

1 Introduction

In classification of regular surfaces of general type, the torsion parts of the Picard groups (the torsion groups for short) sometimes play an important role. One of the reasons for this lies in variety of topological types under single values of numerical invariants, which is common especially in cases of small geometric genus; the torsion group of a regular surface, isomorphic to the first homology group with integral coefficients, carries information that the numerical invariants c_1^2 and χ do not.

Studies on surfaces of general type done using the torsion groups are well-known for cases of vanishing geometric genus (see, e.g., Barth-Peters-Van de Ven [3, p. 237]). In those studies, they tried to determine the structures of surfaces with given isomorphism classes of the torsion groups. There are,

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however, some other cases of numerical invariants for which similar studies have been successfully developed. Consider the case $c_1^2 = 2\chi - 2$. In this case, by Ciliberto-Mendes Lopes [7], the orders of the torsion groups do not exceed 2, and the Euler characteristics χ's for the cases of non-trivial torsion do not exceed 5. Complete descriptions for the surfaces with non-trivial torsion with $\chi = 2$, 3, 4, and 5 are given in Catanese-Debarre [5], Ciliberto-Mendes Lopes [7], Bartalesi-Catanese [2], and Ciliberto-Mendes Lopes [7] respectively. We remark that even in cases of vanishing geometric genus, complete descriptions are known only for a small number of classes.

In the present paper, we study minimal surfaces with $c_1^2 = 2\chi - 1$ having non-trivial 2-torsion divisors. Note that if X is a minimal surface with $c_1^2 = 2\chi - 1$, then X has vanishing irregularity, hence geometric genus $p_g = \chi - 1$. We shall prove the bound $\chi \leq 4$ for the Euler characteristics χ's (Theorem 1), describe the surfaces of the case $\chi = 4$ (Theorem 2, Remark 2), and study the moduli space for surfaces of this case (Theorem 3). By the main theorem of [17], the order of the torsion group of a minimal surface with $c_1^2 = 2\chi - 1$ is at most 3 if $\chi = 2$, and at most 2 if $\chi \geq 3$. Thus for our surfaces with $\chi \geq 2$, two conditions $\mathbb{Z}/2 \subset \mathrm{Tors}$ and $\mathrm{Tors} \simeq \mathbb{Z}/2$ are equivalent, where Tors denotes the torsion group. The case $\chi = 1$ on this line is that of the numerical Godeaux surfaces (i.e., minimal surfaces of general type with $c_1^2 = 1$ and $p_g = 0$).

Surfaces with $c_1^2 = 2\chi - 1$, $\chi = 4$, and $\mathrm{Tors} \simeq \mathbb{Z}/2$ are known to exist and can be found in [8]. In [8], Ciliberto and Mendes Lopes completely classified regular surfaces with $p_g = 3$ having non-birational bicanonical maps and without genus 2 pencils, i.e., regular surfaces with $p_g = 3$ and of non-standard case for the non-birationality of the bicanonical maps. Among their results, they showed that any regular surface of non-standard case with $c_1^2 = 7$ and $p_g = 3$ is obtained by performing a certain operation on what is known as Du Val's ancestor with $c_1^2 = 8$ and $p_g = 4$. Since these surfaces have non-trivial 2-torsion divisors, as has been shown in [8], these are examples of our surfaces for the case $\chi = 4$. In fact, our structure theorem for surfaces with $c_1^2 = 2\chi - 1$, $\chi = 4$, and $\mathrm{Tors} \simeq \mathbb{Z}/2$ shows that although we start from the different assumption, the resulting surfaces are exactly those seen in the paper [8].

Our complete description for the surfaces with $\chi = 4$ asserts that any such surface X is obtained roughly as a free quotient by $\mathbb{Z}/2$ of a double cover of the Hirzebruch surface $\Sigma_d = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$ ($d = 0$ or 2). We shall describe the branch divisor of the double cover, and determine the free action by $\mathbb{Z}/2$ (Theorem 2, Remark 1). The branch divisor of the double cover turns out to be a member of the quadruple anticanonical system having exactly two $[3, 3]$-points. The action by $\mathbb{Z}/2$ turns out to be a lifting of that on the

Hirzebruch surface Σ_d.

This description induces another description of our surfaces of the case $\chi = 4$ (Proposition 14), which is almost the same as a description appearing in Ciliberto-Mendes Lopes [8]. Using our descriptions, we shall show that our surfaces of the case $\chi = 4$ has non-birational bicanonical maps and no pencil of curves of genus 2 (Proposition 15), hence completely coinciding with those seen in [8] (see also Remark 6).

The coincidence of the resulting surfaces certainly implies possibility of another proof of our complete description, i.e., of a proof, like one for the case $c_1^2 = 2\chi - 2$ in Ciliberto-Mendes Lopes [7], by showing that our surfaces with $\chi = 4$ are of non-standard case for the non-birationality of the bicanonical maps. We however do not chose this way. We remark that our method has an advantage in the sense that we can show the irreducibility of the moduli space in a very explicit and elementary way.

The present paper is organized as follows. In order to show our main theorem, we follow Miyaoka [14] and Reid [19], and take the unramified double cover $Y \to X$ corresponding to a torsion divisor. We study its canonical map Φ_{K_Y} using the action by the Galois group of Y over X. In Section 2, we state our main results and show, on the assumption $\chi \geq 4$, that we have $\deg \Phi_{K_Y} = 1$ or 2, and that $\deg \Phi_{K_Y} = 1$ implies $\chi = 4$. Note here that to obtain our main theorem, we only need to study the case $\chi \geq 4$. In Section 3, we study the case $\deg \Phi_{K_Y} = 2$. We divide this case into three according to the degree of the canonical image $Z = \Phi_{K_Y}(Y) \subset \mathbb{P}^n$: the case $\deg Z = n+1$, the case $\deg Z = n$, and the case $\deg Z = n-1$. We shall classify non-degenerate surfaces in $\mathbb{P}^n$ of degree $n+1$ of which minimal desingularizations have vanishing irregularities (Proposition 3), and use this classification to study the case $\deg Z = n+1$. In Section 4, we study the case $\deg \Phi_{K_Y} = 1$ and $\chi = 4$, and then prove Theorems 1 and 2. In the case $\deg \Phi_{K_Y} = 1$ and $\chi = 4$, the surface Y has the first Chern number 14, geometric genus 7, and irregularity 0. Hence the surface Y in this case is a canonical surface whose invariant lies on the Castelnuovo line. We use results given in Ashikaga-Konno [1] to exclude this case. Finally in Section 5, we study the coarse moduli space for the surfaces of the case $\chi = 4$, and prove Theorem 3. To prove the unirationality of the moduli space and the uniqueness of the deformation type, we describe our surfaces of the case $\chi = 4$ as double planes, which is almost the same as the description in Ciliberto-Mendes Lopes [8] for the surfaces of the non-standard case (see also Ciliberto-Francia-Mendes Lopes [6]). Using the two descriptions of our surfaces, we show that our surfaces of the case $\chi = 4$ in fact are of the non-standard case for the non-birationality of bicanonical maps.

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Notation and Terminology

Let S be a compact complex manifold of dimension 2. We denote by $c_1(S)$, $p_g(S)$, and $q(S)$ the first Chern class, the geometric genus, and the irregularity of S respectively. The torsion group of S, denoted by $\operatorname{Tors}(S)$, is the torsion part of the Picard group of S. If V is a complex manifold, K_V is a canonical divisor of V. For a coherent sheaf $\mathcal{F}$ on V, we denote by $H^i(\mathcal{F})$, $h^i(\mathcal{F})$, and $\chi(\mathcal{F})$ the i-th cohomology group, its dimension $\dim_{\mathbb{C}} H^i(\mathcal{F})$, and the Euler characteristic $\sum(-1)^i h^i(\mathcal{F})$ respectively. Let $f : V \to W$ be a morphism to a complex manifold W, and D, a divisor on W. Then $f^*(D)$ and $f_*^{-1}(D)$ denote the total transform and the strict transform respectively of D. The symbol $\sim$ means the linear equivalence of divisors. We denote by $\Sigma_d \to \mathbb{P}^1$ the Hirzebruch surface of degree d. The divisors Δ_0 and Γ are its minimal section and its fiber respectively. Let C be a curve on S. We denote by $\operatorname{mult}_x C$ the multiplicity of C at a point $x \in S$. Let x be a triple point of a reduced curve C on S, and $S' \to S$, the blowing-up at x. Assume that the strict transform C' of C has an infinitely near triple point x'. Then the point x is called a $[3, 3]$-point of C, if the strict transform C'' to S'', where $S'' \to S'$ is the blowing-up at x', has at most negligible singularities on the exceptional locus of $S'' \to S$.

2 Statement of the main theorem

In [17], we obtained a bound for the orders of the torsion groups of minimal surfaces with $c_1^2 = 2\chi - 1$ and $\chi \geq 2$. In the present paper, we study the case of 2-torsion divisors, and sharpen the bound. Our goals are a bound for the Euler characteristic χ, a complete description for the surfaces of the case of maximal χ, and the unirationality of the moduli space for surfaces of this case. The following three are the main theorems:

Theorem 1. *Let X be a minimal surface of general type with $c_1^2 = 2\chi - 1$ and torsion group* $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$. *Then the Euler characteristic χ of the structure sheaf does not exceed* 4.

Theorem 2. *Let X be a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and torsion group* $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$. *Then the unramified double cover Y of X admits a generically two-to-one morphism f onto the Hirzebruch surface Σ_d of degree $d = 0$ or 2 satisfying the following conditions*:

i) *the action by the Galois group $G = \mathrm{Gal}(Y/X) \simeq \mathbb{Z}/2$ of Y over X induces one on Σ_d, of which fixed locus is a set of four points on Σ_d*;

ii) *the branch divisor B of f is a member of the linear system $|-4K_{\Sigma_d}|$ passing no fixed point of the action by G*;

iii) *the branch divisor $B \in |-4K_{\Sigma_d}|$ has exactly two $[3,3]$-points, and all other singularities, if any, are negligible ones.*

Theorem 3. *Any two minimal surfaces with $c_1^2 = 2\chi - 1$, $\chi = 4$, and* $\mathrm{Tors} \simeq \mathbb{Z}/2$ *are equivalent under deformation of complex structures. The coarse moduli space for minimal surfaces with these invariants is a unirational variety of dimension* 29.

Theorem 1 sharpens the bound given in [17] into the following:

Theorem 4. *Let X be a minimal algebraic surface with $c_1^2 = 2\chi - 1$. Then the following hold*:

i) *if $\chi = 2$, then $\sharp\mathrm{Tors}(X) \leq 3$*;

ii) *if $\chi \geq 3$, then $\sharp\mathrm{Tors}(X) \leq 2$*;

iii) *if $\chi \geq 5$, then $\sharp\mathrm{Tors}(X) = 1$.*

Remark 1. In Theorem 2, we can describe the action by G on Σ_d more concretely: if an involution of the Hirzebruch surface Σ_d has exactly four fixed points (d: even), then there exists an open cover $\{U_i\}_{i=0,1}$ of Σ_d satisfying $U_i = \{(u_i, (t_i : 1))\} = \mathbb{C} \times \mathbb{P}^1$, $u_0 = 1/u_1$, and $t_0 = u_1^d t_1$, such that this involution is given by

$$(u_0, t_0) \mapsto (-u_0, -t_0). \tag{1}$$

Remark 2. Theorem 2 asserts that any minimal surface X with $c_1^2 = 2\chi - 1$, $\chi = 4$, and $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$ is obtained by the following procedure: 1) set $d = 0$ or 2; the involution (1) defines an action by $G = \mathbb{Z}/2$ on the Hirzebruch surface Σ_d; 2) take a reduced member $B \in |-4K_{\Sigma_d}|$ stable under this action that satisfies the conditions ii) and iii) in Theorem 2; 3) take the double cover of Σ_d branched along B, and denote by Y its minimal desingulraization; there exists a unique free lifting to Y of the action by G on Σ_d; 4) take the quotient of Y by this free action.

It is not difficult to check that this procedure in fact gives surfaces of the case $\chi = 4$ for sufficiently general B.

Remark 3. Let Σ_d be the Hirzebruch surface which appears in Theorem 2. It is obvious from Rmarks 1 and 2 that the fibration $\Sigma_d \to \mathbb{P}^1$ induces a hyperelliptic fibration $Y \to \mathbb{P}^1$ of genus 3 and that the divisor class of a fiber of this fibration is stable under the action by the Galois group $G = \mathrm{Gal}(Y/X)$. So we obtain a hyperelliptic fibration $X = Y/G \to \mathbb{P}^1/G$ of genus 3 with two multiple fibers $2A_1$ and $2A_2$ corresponding to the fixed points of the action by G on $\mathbb{P}^1$. As is explained also in [8, p. 85], the difference $A_1 - A_2$ gives a non-trivial 2-torsion divisor of our surface X.

In what follows, X is a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = \lambda \geq 4$, and $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$. We denote by $\pi : Y \to X$ the unramimfied double cover corresponding to the torsion group $\mathrm{Tors}(X)$. Note that we have assumed $\lambda \geq 4$. The following lemma follows from the unbranched covering trick.

Lemma 2.1. $K_Y^2 = 2(2\lambda - 1)$, $p_g(Y) = 2\lambda - 1$, *and* $q(Y) = 0$.

In order to show Theorems 1 and 2, we study the canonical map $\Phi_{K_Y} : Y \to \mathbb{P}^n$ of Y, where $n = 2\lambda - 2$. We denote by $Z = \Phi_{K_Y}(Y)$ the canonical image of the surface Y.

Proposition 1. *The canonical image Z is a surface. The equality* $\deg \Phi_{K_Y} = 1$ *or* 2 *holds. Moreover, if* $\deg \Phi_{K_Y} = 1$, *then* $\lambda = 4$.

Proof. Since we have assumed $\lambda \geq 4$, we have

$$K_Y^2 - 3p_g(Y) = -(2\lambda - 1) \leq -7.$$

By this together with $q(Y) = 0$ and [11, Theorem 1.1], we see that $|K_Y|$ is not composite with a pencil. Thus we have

$$\deg \Phi_{K_Y} \leq \frac{K_Y^2}{\deg Z} \leq \frac{2(n+1)}{n-1} = 2 + \frac{4}{n-1} \leq 2 + \frac{4}{5},$$

hence $\deg \Phi_{K_Y} \leq 2$. The second assertion follows from Castelnuovo's inequality. □

If $\lambda = 4$, then the Chern invariant of Y is on the Castelnuovo line. Thus we can use results given in [1] to study the case $\deg \Phi_{K_Y} = 1$.

3 The case $\deg \Phi_{K_Y} = 2$

In this section, we study the case $\deg \Phi_{K_Y} = 2$. We begin with the study of the base locus of the canonical system $|K_Y|$. Let $|M|$ and F be the variable part and the fixed part of the linear system $|K_Y|$. We take the shortest composite $p : \tilde{Y} \to Y$ of quadric transformations such that the variable part $|L|$ of $p^*|M|$ is free from base points, and denote by E the fixed part of $p^*|M|$. Then we have $p^*|K_Y| = |L| + E + p^*F$ and

$$K_Y^2 = L^2 + LE + MF + K_Y F, \tag{2}$$

where each term of the right hand is a non-negative integer. Note that the eigenvectors of the natural action by $G = \mathrm{Gal}(Y/X)$ span the the space of global section $H^0(\mathcal{O}_Y(K_Y))$. This implies that the linear systems $|K_Y|$, $|M|$, and F are spanned by the pull-backs of divisors on X. Hence, for example, we have $MF \equiv 0 \mod 2$, since $\pi : Y \to X$ is of mapping degree 2. In the same way, we obtain

$$L^2 \equiv LE = -E^2 \equiv MF \equiv K_Y F \equiv 0 \mod 2 \tag{3}$$

(for the detail, see [16, Section 3]).

Proposition 2. *Let M, F, L, and E be divisors as above. Then one of the following holds*:
1) $|K_Y| = |L|$: *the canonical system $|K_Y|$ is free from base points*;
2) $L^2 = K_Y^2 - 2$, $F = 0$, *and* $LE = 2$;
3-1) $L^2 = K_Y^2 - 4$, $F = 0$, *and* $LE = 4$;
3-2) $L^2 = K_Y^2 - 4$, $|L| = |M|$, $K_Y F = 0$, *and* $F^2 = -4$.

Proof. First, note that we have $L^2 = K_Y^2$, $K_Y^2 - 2$, or $K_Y^2 - 4$. This follows from (3) and [9, Lemma 2]. Second, note that

$$MF \equiv 0 \mod 4. \tag{4}$$

This follows from the Riemann–Roch theorem, since we have $MF = M(M + K_Y) - 2M^2 = M(M + \pi^* K_X) - 2M^2$, $\deg \pi = 2$, and $M \sim \pi^* M'$ for a certain divisor M' on X. Then the assertion follows from (2), (4), (3), and Hodge's index theorem. □

In case 3-1), the number of the base points of $|M|$ cannot be 1, since the action by G on Y has no fixed point. Thus in this case, the morphism $p : \tilde{Y} \to Y$ is a composite of four quadric transformations. In the same way, we see that, in case 2), the morphism $p : \tilde{Y} \to Y$ is a blowing-up of Y at two distinct points. In case 3-2), the divisor F is a sum of two fundamental cycles of rational double points.

We denote by $\Phi_L : \tilde{Y} \to Z \subset \mathbb{P}^n$ the morphism associated with the linear system $|L|$. The action by G on Y induces one on $\tilde{Y}$. We study the morphism Φ_L using this action.

3.1 The case $|K_Y| = |L|$

Let us first exclude case 1) in Proposition 2. In what follows, we assume $|K_Y| = |L|$. Thus we have $\deg Z = n+1$. We shall prove the following proposition in Appendix.

Proposition 3. *Let $n \geq 4$ be an integer, Z, a non-degenerate surface in $\mathbb{P}^n$ of degree $n+1$, and $Z' \to Z$, its minimal desingularization. Assume that the morphism $Z' \to Z$ is given by a complete linear system $|D'|$ and that $q(Z') = 0$ holds. Then n does not exceed 11. Further, there exist an integer $0 \leq d \leq 3$ and a blowing-up $r : Z' \to \Sigma_d$ at (possibly infinitely near) $11-n$ points such that the equivalence $D' \sim -K_{Z'} + r^*\Gamma$ holds. Here, the divisor Γ is a fiber of the Hirzebruch surface $\Sigma_d \to \mathbb{P}^1$.*

In our case, we have $n = 2\lambda - 2$, $\lambda \geq 4$, and $q(Y) = 0$. Moreover Z is the canonical image of Y. Thus our surface $Z = \Phi_{K_Y}(Y)$ satisfies all the conditions in the proposition above. It follows that there exist an integer $0 \leq d \leq 3$ and a blowing-up $r : Z' \to \Sigma_d$ at $11-n$ points such that the morphism $\Phi_{D'} : Z' \to Z$, where $\Phi_{D'}$ is a morphism corresponding to the complete linear system $|D'| = |-K_{Z'} + r^*\Gamma|$, gives the minimal desingularization of Z.

Proposition 4. *The canonical map $\Phi_{K_Y} : Y \to Z$ lifts to a morphism $f' : Y \to Z'$. The branch divisor B' of f' is a member of the linear system $|2(2D' - r^*\Gamma)|$ having at most negligible singularities.*

Proof. Let us first show the liftability of the canonical map Φ_{K_Y}. Let $p' : Y' \to Y$ be the shortest composite of quadric transformations such that the morphism $\Phi_{K_Y} \circ p'$ factors through $\Phi_{D'} : Z' \to Z$. We denote by $f' : Y' \to Z'$ the unique morphism satisfying $\Phi_{K_Y} \circ p' = \Phi_{D'} \circ f'$. Then we have $K_{Y'} \sim p'^*K_Y + \eta$ for a certain effective divisor η on Y'. If $f'_*\eta = 0$, then $p' : Y' \to Y$ is an isomorphism . Thus we only need to show $f'_*\eta = 0$.

So we prove the equality above. Let R' be the ramification divisor of f', and $B' = f'_*R'$, its direct image. Then from $R' \sim K_{Y'} - f'^*K_{Z'} \sim f'^*(2D' - r^*\Gamma) + \eta$, we infer

$$B' \sim 2(2D' - r^*\Gamma + \alpha), \tag{5}$$

where α is a divisor satisfying $2\alpha \sim f'_*\eta$. We denote by $Y'' \to Z'$ the double cover branched along B', and by $Y^\sharp \to Y''$ its canonical resolution. To show the equality $f'_*\eta = 0$, we compute the Euler characteristic $\chi(\mathcal{O}_{Y^\sharp})$ in two ways and compare them. Note that $\dim(\Phi_{K_Y} \circ p')(\eta) = 0$, and that any general member of $|r^*\Gamma|$ is a 0-curve. It follows $D'\alpha = D'f'_*\eta/2 = 0$ and $D'(r^*\Gamma) = -K_{Z'}(r^*\Gamma) = 2$. Thus by (5) and [9, Lemma 6], we obtain

$$\begin{aligned}\chi(\mathcal{O}_{Y_\sharp}) &= 2 + \frac{1}{2}(2D' - r^*\Gamma + \alpha)((2D' - r^*\Gamma + \alpha) + K_{Z'}) - \beta \\ &= 2 + \frac{1}{2}(2D' - r^*\Gamma + \alpha)(D' + \alpha) - \beta \\ &= D'^2 + 1 - \frac{1}{4}(r^*\Gamma)(f'_*\eta) + \frac{1}{8}(f'_*\eta)^2 - \beta, \qquad (6)\end{aligned}$$

where β is a term coming from essential singularities of the branch divisor B'. Here, we have three inequalities

$$-\frac{1}{4}(r^*\Gamma)(f'_*\eta) \leq 0, \quad \frac{1}{8}(f'_*\eta)^2 \leq 0, \quad \text{and} \quad -\beta \leq 0. \qquad (7)$$

The first one follows from the absence of base points of $|r^*\Gamma|$, the second one from $D'^2 > 0$ and $D'f'_*\eta = 0$, and the last one from the definition of β. Meanwhile we have $\chi(\mathcal{O}_{Y^\sharp}) = \chi(\mathcal{O}_Y) = n + 2 = D'^2 + 1$. Thus by (6) and (7), we obtain $(f'_*\eta)^2 = 0$, from which together with Hodge's index theorem, we infer $f'_*\eta = 0$. Hence the canonical map Φ_{K_Y} lifts.

The remaining assertion easily follows from the proof above. □

Note that the action by $G = \mathrm{Gal}(Y/X)$ on Y induces one on Z'. We can verify it as follows. Since Z is the canonical image of our surface Y, the action on Y induces one on Z. Meanwhile the surface Z' is the minimal desingularization of our surface Z. Thus this action on Z induces one on Z'.

Lemma 3.1. *The induced action by G on Z' is non-trivial. The fixed locus of this action has a one-dimensional irreducible component C'_0 satisfying $C_0'^2 \equiv 1 \bmod 2$.*

Proof. The first assertion is trivial, since the action on Y has no fixed point. Let us show the second assertion. Let $\{z_1, \ldots, z_b\}$ be the set of isolated fixed points of the action on Z', and $r'' : Z'' \to Z'$ the blowing-up at these b points. We denote by C''_i the (-1)-curve lying over z_i. Let $\{C'_1, \ldots, C'_a\}$ be the set of 1-dimensional irreducible components of the fixed locus of the action on Z'. We use the same symbol C'_i for the total transform to Z'' of the divisor C'_i. Note that the divisor $\sum_{i=1}^a C'_i + \sum_{i=1}^b C''_i$ has no singularity, since we have $G \simeq \mathbb{Z}/2$. It follows that the quotient Z''/G is smooth, where the action by G is the lifting of that on Z'. We denote by $\bar{C}'_i$ and $\bar{C}''_i$ the image to Z''/G of the divisor C'_i and that of the divisor C''_i, respectively. Then since the branch divisor $\sum_{i=1}^{a} \bar{C}'_i + \sum_{i=1}^{b} \bar{C}''_i$ is linearly equivalent to twice a divisor on Z''/G, we have

$$\sum_{i=1}^{a} C_i'^2 - b = (\sum_{i=1}^{a} C'_i + \sum_{i=1}^{b} C''_i)^2 = (\sum_{i=1}^{a} \bar{C}'_i + \sum_{i=1}^{b} \bar{C}''_i)^2/2 \equiv 0 \mod 2.$$

Meanwhile, since $K_{Z''}$ is linearly equivalent to a pull-back of a divisor on Z''/G, we have $K_{Z''}^2 = K_{Z'}^2 - b = n - 3 - b \equiv 0 \bmod 2$, hence $b \equiv 1 \bmod 2$. Thus we infer $\sum_{i=1}^{a} C_i'^2 \equiv 1 \bmod 2$, which implies the second assertion. □

Lemma 3.2. *Let C'_0 be an irreducible curve as in Lemma 3.1. Then $B'C'_0 \neq 0$ holds.*

Proof. We derive a contradiction by assuming $B'C'_0 = 0$. Assume that $B'C'_0 = 0$ holds. Then by Proposition 4, we have

$$(2D' - r^*\Gamma)C'_0 = (-2K_{Z'} + r^*\Gamma)C'_0 = 0.$$

If $(r^*\Gamma)C'_0 = 0$, then by the equality above, we obtain $K_{Z'}C'_0 = 0$, which contradicts $C_0'^2 \equiv 1 \bmod 2$. Thus we have $(r^*\Gamma)C'_0 > 0$, hence $-2K_{Z'}C'_0 = -(r^*\Gamma)C'_0 < 0$. It follows C'_0 is a fixed component of the anti-canonical system $|-K_{Z'}|$. Then since $-K_{\Sigma_d} \sim 2\Delta_0 + (2+d)\Gamma$, we obtain $(r^*\Gamma)C'_0 \leq 2$, hence $(r^*\Gamma)C'_0 = 2K_{Z'}C'_0 = 2$. Thus $r_*C'_0 \sim 2\Delta_0 + c\Gamma$ holds for a certain integer $c \geq 1$. Meanwhile since $0 \leq d \leq 3$, we have $h^0(\mathcal{O}_{Z'}(-K_{Z'})) \geq h^0(\mathcal{O}_{\Sigma_d}(-K_{\Sigma_d})) - (11-n) = n-2$. Thus we obtain

$$\begin{aligned} n-2 \leq h^0(\mathcal{O}_{Z'}(-K_{Z'})) &= h^0(\mathcal{O}_{Z'}(-K_{Z'} - C'_0)) \\ &\leq h^0(\mathcal{O}_{\Sigma_d}(-K_{\Sigma_d} - r_*C'_0)) = 3 + d - c, \end{aligned}$$

hence $c - d \leq 5 - n < 0$. It follows $(r_*C'_0)\Delta_0 = (c-d) - d < 0$, which contradicts the irreducibility of C'_0. Hence $B'C'_0 \neq 0$ holds. □

Now let us exclude case 1) in Proposition 2.

Proposition 5. *Case* 1) *in Proposition* 2 *does not occur.*

Take an irreducible curve C'_0 as in Lemma 3.1. Then by Lemma 3.2, we have $B' \cap C'_0 \neq \emptyset$. So let us take a point $x \in B' \cap C'_0$. Then the preimage $f'^{-1}(x) \subset Y$ is stable under the action by G on Y. By Proposition 4, however, the set $f'^{-1}(x)$ is either a point or a base space of the fundamental cycle of a rational double point. This implies that the action by G on $f'^{-1}(x)$ has a fixed point, which contradicts the definition of $\pi : Y \to X$. Thus we have the assertion. □

3.2 The case $L^2 = K_Y^2 - 4$

Next we exclude cases 3-1) and 3-2) in Proposition 2. In these two cases, we have $L^2 = 2(n-1)$; hence the canonical image Z is a non-degenerate surface in $\mathbb{P}^n$ of minimal degree $n-1$. Thus from the well-known classification, it follows that our Z is a image of the Hirzebruch surface $Z' = \Sigma_d$ by the morphism associated with the complete linear system $|D'| = |\Delta_0 + \frac{n-1+d}{2}\Gamma|$, where $0 \leq d \leq n-1$ and $d \equiv n-1 \bmod 2$ (see [18]or [10, Lemma 1.2]). Let us denote this morphism by $\Phi_{D'} : Z' \to Z \subset \mathbb{P}^n$. Then $\Phi_{D'}$ is an embedding if $d < n-1$, and is the contraction of Δ_0 if $d = n-1$. Note that in the later case, our Z is a cone over a rational curve embedded in $\mathbb{P}^{n-1}$ by $\mathcal{O}_{\mathbb{P}^1}(n-1)$.

For the case $d < n-1$, the lemma below is trivial. For the case $d = n-1$, we can give a proof by the same method as in [10, Lemma 1.5].

Lemma 3.3. *The morphism $\Phi_L : \tilde{Y} \to Z$ lifts to a morphism $f' : \tilde{Y} \to Z'$.*

By the same argument as in the exclusion of case 1), we see that the action by G on Y induces one on Z'.

Let us recall the morphism $p : \tilde{Y} \to Y$ and the base locus of $|K_Y|$. In case 3-1) in Proposition 2, the morphism p is the blowing-up at (possibly infinitely near) four points, which we shall call $y_1, \ldots, y_4$. Let E_i denote the total transform to $\tilde{Y}$ of the (-1)-curve corresponding to y_i. Then we have $E = \sum_{i=1}^4 E_i$ and $LE_i = 1$ $(1 \leq i \leq 4)$. Since the action by G on the set of base points of $|M|$ has no fixed point, we have only two cases: i) the case where $y_1, \ldots, y_4$ are four distinct points on $\tilde{Y}$, and ii) the case where y_1 and y_2 are distinct points on $\tilde{Y}$, and y_{i+2} is infinitely near to y_i for $i = 1, 2$. In the later case, the divisor $E_i' = E_i - E_{i+2}$ is a (-2)-curve satisfying $LE_i' = 0$.

Meanwhile in case 3-2), the morphism $p : \tilde{Y} \to Y$ is an isomorphism. Hence we may assume $\tilde{Y} = Y$. We have $|M| = |L|$ and $F = \sum_{i=1,2} F_i$, where F_i is a fundamental cycle of a rational double point. Since the action on Y has no fixed point, we have $F_1 \cap F_2 = \emptyset$; hence the generator of G maps F_1 onto F_2. It follows $LF_1 = LF_2 = 2$.

In what follows, we put $T = 2E$ for case 3-1), and $T = F$ for case 3-2). Then we have

$$K_{\tilde{Y}} \sim L + T.$$

Lemma 3.4. *Let T be the divisor above. Then $\Gamma(f'_*T) \equiv 2 \bmod 4$ holds.*

Proof. Since $d \equiv n-1 \equiv 1 \bmod 2$, we have $d \neq 0$. Thus the action by G on $Z' = \Sigma_d$ induces one on $\mathbb{P}^1$ via the natural fibration $\Sigma_d \to \mathbb{P}^1$ of the Hirzebruch surface. It follows there exists a member $\Gamma_0 \in |\Gamma|$ stable under the action by G. Let us take a blowing-up $\tilde{X} \to X$ such that $\tilde{Y} = \tilde{X} \times_X Y$ holds. The base change $\tilde{\pi} : \tilde{Y} \to \tilde{X}$ is an unramified double cover satisfying

$\mathrm{Gal}(\tilde{Y}/\tilde{X}) \simeq \mathrm{Gal}(Y/X)$. Then since $f'^*\Gamma_0$ is stable under the action by G on $\tilde{Y}$, the divisor $f'^*\Gamma_0$ is a pull-back by $\tilde{\pi}$ of a certain divisor on $\tilde{X}$. Thus from $\tilde{\pi}^*K_{\tilde{X}} \sim K_{\tilde{Y}}$ and the Riemann–Roch theorem, we infer

$$(f'^*\Gamma_0)^2 + (f'^*\Gamma_0)K_{\tilde{Y}} = (f'^*\Gamma_0)(L+T) = 2 + (f'^*\Gamma_0)T \equiv 0 \mod 4.$$

Hence we have the assertion. □

Lemma 3.5. *The morphism $f' : \tilde{Y} \to Z'$ contracts no (-1)-curve on $\tilde{Y}$. Further, the following hold*:

i) *if C is a (-1)-curve on $\tilde{Y}$ satisfying $LC = 1$, then $f'_*C \sim \Gamma$*;

ii) *if C is a (-2)-curve on $\tilde{Y}$ satisfying $LC = 0$, then f' contracts C.*

Proof. The first assertion trivially follows from the definition of $p : \tilde{Y} \to Y$. In order to prove i) and ii), we put $f'_*C \sim a\Delta_0 + b\Gamma$. We denote by θ the involution of $\tilde{Y}$ over Z'. This involution exists, since f' contracts no (-1)-curve.

First, let us prove the assertion i). Assume that C is a (-1)-curve on $\tilde{Y}$ satisfying $LC = 1$. Then since $L \sim f'^*D'$, we have

$$(\Delta_0 + d\Gamma)f'_*C + \frac{n-1-d}{2}\Gamma f'_*C = 1, \tag{8}$$

where each term of the left hand is a non-negative integer. Thus we obtain $(\Delta_0 + d\Gamma)f'_*C = 0$ or 1. Assume that $(\Delta_0 + d\Gamma)f'_*C = 1$. Then we have $f'_*C \sim a\Delta_0 + \Gamma$ and $(\frac{n-1-d}{2})a = 0$. Thus, in this case, we only have to show $a = 0$, which is trivial if $n-1-d \neq 0$. If $n-1-d = 0$, then by the irreducibility of C, we have $\Delta_0 f'_*C = 1 - a(n-1) \geq 0$, hence $a = 0$. Assume next that $(\Delta_0 + d\Gamma)f'_*C = 0$. Then by (8), we obtain $f'_*C = \Delta_0$ and $d = n-3$. We exclude this case as follows. We have $f'^*\Delta_0 = C + \theta(C) + \xi$ for a certain effective divisor ξ exceptional with respect to f'. It follows

$$(f'^*\Delta_0)^2 = (C + \theta(C) + \xi)(C + \theta(C)) \geq C^2 + \theta(C)^2 + 2C\theta(C) \geq -4,$$

hence $-2(n-3) \geq -4$. This contradicts $\lambda \geq 4$. Thus we have $(\Delta_0 + d\Gamma)f'_*C \neq 0$, which completes the proof of the assertion i).

Next, let us prove the assertion ii). Assume that $f'(C)$ is a curve. Then since $\Phi_{D'}$ contracts $f'(C)$, we have $d = n-1$ and $f'(C) = \Delta_0$. Note that we have $f'_*C = \Delta_0$ or $2\Delta_0$, since $\deg f' = 2$. Assume that $f'_*C = \Delta_0$. Then we have $f'^*\Delta_0 = C + \theta(C) + \xi$ for a certain effective divisor ξ exceptional with respect to f'. Then by the same method as in the proof of i), we obtain $-2(n-1) = (f'^*\Delta_0)^2 \geq -8$, which contradicts $\lambda \geq 4$. Assume next that $f'_*C = 2\Delta_0$. Then we have $f'^*\Delta_0 = C + \xi$ for a certain effective divisor ξ

exceptional with respect to f'. Then again by the same method, we obtain $-2(n-1) \geq -2$, which contradicts $\lambda \geq 4$. Thus we have the assertion ii). □

If our Y is of case 3-1) in Proposition 2, then by the lemma above we have $f'_*T = 2f'_*E \sim 8\Gamma$, which contradicts Lemma 3.4. Thus we have the following:

Proposition 6. *Case* 3-1) *in Proposition* 2 *does not occur.*

So in what follows, we assume that our Y is of case 3-2) in Proposition 2.

Lemma 3.6. *Let C be an irreducible component of F_1 satisfying $D'f'_*C > 0$. Then one of the following holds*:
i) $D'f'_*C = 1$ *and* $f'_*C \sim \Gamma$;
ii) $D'f'_*C = 2$ *and* $f'_*C \sim 2\Gamma$;
iii) $D'f'_*C = 2$, $f'_*C = \Delta_0$, *and* $d = n - 5 = 1$.

Proof. First, note that if $f'(C) = \Delta_0$, then we have $C \neq \theta(C)$, where θ is the involution of $\tilde{Y} = Y$ over Z'. We can verify this as follows. Let ι be the generator of the Galois group G, and $\iota|_{Z'}$, the corresponding automorphism of Z'. Then since $d \neq 0$, we have $f'(\iota(C)) = \iota|_{Z'}(f'(C)) = \Delta_0 = f'(C)$. This means $C \neq \theta(C) = \iota(C)$, since we have $\iota(C) \subset F_2$ and $F_1 \cap F_2 = \emptyset$. Next, note that C is a (-2) curve satisfying $0 < D'f'_*C \leq D'f'_*F_1 = 2$. Then we can prove the assertion by the same method as in the proof of Lemma 3.5. □

By $D'f'_*F_1 = 2$ together with Lemmas 3.5 and 3.6, we see that either of the following holds:
a) $f'_*F_1 = f'_*(\iota(F_2)) \sim 2\Gamma$;
b) $f'_*F_1 = f'_*(\iota(F_2)) = \Delta_0$, and $d = n - 5 = 1$,
where ι is the generator of the Galois group of G. Case a) above, however, contradicts the assertion in Lemma 3.4. Thus we have the following:

Lemma 3.7. $f'_*F_1 = f'_*F_2 = \Delta_0$ *and* $d = n - 5 = 1$.

Now let us study the morphism $f' : \tilde{Y} = Y \to Z' = \Sigma_1$. Let R' be the ramification divisor of f', and $B' = f'_*R'$, the branch divisor. Then by the lemma above we obtain

$$R' \sim f'^*(3\Delta_0 + 6\Gamma) + \sum_{i=1,2} F_i \qquad \text{and} \qquad B' \sim 2(4\Delta_0 + 6\Gamma). \tag{9}$$

We take the double cover of Z' with branch divisor B', and denote by $Y^\sharp$ its canonical resolution. Let us recall how to obtain the canonical resolution. Set $Z'_0 = Z'$ and $B'_0 = B'$. We define Z'_i and B'_i inductively as follows. Choose a singularity z_i, if any, of B'_{i-1}, and take the blowing-up $q'_i : Z'_i \to Z'_{i-1}$ at this point. We denote by ε_i the (-1)-curve corresponding to z_i. Let m_i be the multiplicity of B'_{i-1} at z_i, and $[\frac{m_i}{2}]$, the largest integer not exceeding $\frac{m_i}{2}$. Then we define B'_i by $B'_i = q'^*_i B'_{i-1} - 2[\frac{m_i}{2}]\varepsilon_i$. For a certain $s \geq 0$, the divisor B'_s is non-singular. So take the double cover $f^\sharp : \tilde{Y}_s \to Z^\sharp = Z'_s$ with branch divisor $B^\sharp = B'_s$. Then this $\tilde{Y}_s$ is our canonical resolution $Y^\sharp$. Put $q' = (q'_1 \circ q'_2 \circ \cdots \circ q'_s) : Z^\sharp \to Z'$. There exists a natural birational morphism $p^\sharp : Y^\sharp \to \tilde{Y}$ satisfying $q' \circ f^\sharp = f' \circ p^\sharp$. We use the same symbol ε_i for the total transform to $Z^\sharp$ of the (-1)-curve $\varepsilon_i \subset Z'_i$. Note, for our case, the action by the Galois group $G = \mathrm{Gal}(Y/X)$ on $\tilde{Y}$ induces one on $Z^\sharp$ and one on $Y^\sharp$. This action on $Y^\sharp$ is free.

By the same method as in [9, Section 2], we obtain the following:

Proposition 7. *There exist i_1 and i_2 $(i_1 < i_2)$ satisfying $[\frac{m_{i_1}}{2}] = [\frac{m_{i_2}}{2}] = 2$. For any $i \neq i_1, i_2$, the equality $[\frac{m_i}{2}] = 1$ holds. The morphism $p^\sharp : Y^\sharp \to \tilde{Y} = Y$ is a composite of two quadric transformations.*

Thus the branch divisor B' has an essential singularity. By the proposition above, we obtain

$$K_{Y^\sharp} \sim f^{\sharp *}(q'^*(2\Delta_0 + 3\Gamma) - \varepsilon_{i_1} - \varepsilon_{i_2}). \tag{10}$$

Lemma 3.8. *Every essential singularity of B' lies on Δ_0.*

Proof. Since f' contracts no (-1)-curve, $f'^*B' - 2R' = 2\zeta'$ holds for a certain effective divisor ζ' on $\tilde{Y}$. This ζ' satisfies

$$2\zeta' \sim 2(f'^*(\Delta_0) - \sum_{i=1,2} F_i), \tag{11}$$

since we have (9). Let $\zeta' = \sum \zeta'_i$ be the decomposition into connected components. Note that f' maps each ζ'_i to a point on Z'. Then, for any i satisfying $f'(\zeta'_i) \notin \Delta_0$, we infer from (11) that $\zeta'^2_i = \zeta'_i\zeta' = 0$, hence $\zeta'_i = 0$, which implies the assertion. □

Lemma 3.9. *Let $\eta^\sharp \sim K_{Y^\sharp} - p^{\sharp *}K_{\tilde{Y}}$ be the exceptional divisor corresponding to $p^\sharp : Y^\sharp \to \tilde{Y}$. Then the fixed part of $|K_{Y^\sharp}|$ is given by $\sum_{i=1,2} p^{\sharp *}F_i + \eta^\sharp$. Further, the linear equivalence $\sum_{i=1,2} p^{\sharp *}F_i + \eta^\sharp \sim f^{\sharp *}(q'^*\Delta_0 - \varepsilon_{i_1} - \varepsilon_{i_2})$ holds, where i_1 and i_2 are integers given in Proposition 7.*

Proof. The first assertion follows from $|K_{Y^\sharp}| = |K_{\tilde{Y}}| + \eta^\sharp$, since $|L|$ has no base point. The second assertion follows from (10) and $\sum p^{\sharp *}F_i + \eta^\sharp \sim K_{Y^\sharp} - p^{\sharp *}L \sim K_{Y^\sharp} - p^{\sharp *}f'^*D'$. □

Lemma 3.10. *There exists a member $\varGamma_1 \in |\varGamma|$ contained in the fixed locus of the action by G on $Z' = \varSigma_1$.*

Proof. The action by G on $Z' = \varSigma_1$ induces one on $\mathbb{P}^1$ via the natural fibration $Z' = \varSigma_1 \to \mathbb{P}^1$ of the Hirzebruch surface. Let us show that this induced action on $\mathbb{P}^1$ is non-trivial. There exists a member $\varDelta_1 \in |\varDelta_0 + \varGamma|$ stable under the action by G satisfying $\varDelta_1 \cap \varDelta_0 = \emptyset$. Assume that the induced action on $\mathbb{P}^1$ is trivial. Then this $\varDelta_1$ is contained in the fixed locus of the action by G on Z'. From this together with $B'\varDelta_1 = 12$ and Lemma 3.8, it follows that B' has a smooth point or a negligible singularity that is stable under the action by G. This, however, leads us to a contradiction by the same argument as in the proof of Proposition 5. Thus the induced action on $\mathbb{P}^1$ is non-trivial. Now take two fibers of $Z' \to \mathbb{P}^1$ that lie over the fixed points of the action on $\mathbb{P}^1$. Since $Z' = \varSigma_1$, one of these two fibers are contained in the fixed locus of the action by G. □

Let us exclude case 3-2) in Proposition 2.

Proposition 8. *Case* 3-2) *in Proposition* 2 *does not occur.*

Proof. Let $\varGamma_1 \in |\varGamma|$ be the member as in Lemma 3.10. By (9), we have $B'\varGamma_1 = 8$, hence $B' \cap \varGamma_1 \neq \emptyset$. If a smooth point or a negligible singularity of B' lies on $B' \cap \varGamma_1$, we can derive a contradiction by the same argument as in the proof of Proposition 5. Thus by Lemma 3.8, we see that $B' \cap \varGamma_1 = \varDelta_0 \cap \varGamma_1$ and that this point is an essential singularity of B'. So we put $\varDelta_0 \cap \varGamma_1 = \{z_1\}$, where the point z_1 is the center of the first blowing-up $q'_1 : Z'_1 \to Z'_0 = Z'$ in the procedure to obtain the canonical resolution $Y^\sharp$. Then, by Proposition 7, we have $3 \leq m_1 \leq 5$. If m_1 is odd, then the strict transform $\varepsilon_1^\sharp \simeq \mathbb{P}^1 \subset Z^\sharp$ of the exceptional curve $\varepsilon_1 \subset Z'_1$ is a component of $B^\sharp$ stable under the action by G. This, however, leads us to a contradiction, since the action by G on $Y^\sharp$ is free. It follows $m_1 \equiv 0 \mod 2$, hence $m_1 = 4$. Thus we have $B'_1 = {q'_1}^* B' - 4\varepsilon_1$ and $B'_1 {q'_{1*}}^{-1}(\varGamma_1) = 4$, where the divisor ${q'_{1*}}^{-1}(\varGamma_1)$ is the strict transform of $\varGamma_1$ by $q'_1 : Z'_1 \to Z'$. Note that the action by G on Z' induces one on Z'_1, and that the strict transform ${q'_{1*}}^{-1}(\varGamma_1)$ is contained in the fixed locus of this induced action. By the same argument as that on $\varGamma_1$ above, we see that the point $B'_1 \cap {q'_{1*}}^{-1}(\varGamma_1) = \varepsilon_1 \cap {q'_{1*}}^{-1}(\varGamma_1)$ is an essential singularity of B'_1, that we can set $\varepsilon_1 \cap {q'_{1*}}^{-1}(\varGamma_1) = \{z_2\}$, where the point z_2 is the center of the second blowing-up $q'_2 : Z'_2 \to Z'_1$, and that $m_2 = 4$, where m_2 is the multiplicity of B'_1 at z_2. Thus we have $i_1 = 1$ and $i_2 = 2$, where i_1 and i_2 are the integers given in Proposition 7.

Now we derive a contradiction. Let $\varGamma_1^\sharp$ be the strict transform to $Z^\sharp$ of of the divisor $\varGamma_1$. Note that we have $z_1 \in \varGamma_1$ and $z_2 \in {q'_{1*}}^{-1}(\varGamma_1)$. Thus by

Lemma 3.9, we obtain

$$f^{\sharp}_{*}(\sum p^{\sharp *}F_i + \eta^{\sharp})\mathit{\Gamma}^{\sharp}_1 = 2(\mathit{\Delta}_0\mathit{\Gamma} + {\varepsilon_1}^2 + {\varepsilon_2}^2) = -2 < 0.$$

From this together with Lemma 3.7, we infer that the divisor $\mathit{\Gamma}^{\sharp}_1$ is the image by $f^{\sharp}$ of an irreducible component of $\eta^{\sharp}$, which contradicts the equality $\dim(q' \circ f^{\sharp})(\eta^{\sharp}) = \dim(f' \circ p^{\sharp})(\eta^{\sharp}) = 0$. Hence we have the assertion. □

3.3 The case $L^2 = K_Y^2 - 2$

Finally, we study case 2) in Proposition 2. It will turn out that $\lambda = 4$ in this case, and that the surfaces of this case have the structure as in the statement of Theorem 2. In what follows, we assume that our Y is of case 2) in Proposition 2, hence $\deg Z = L^2/2 = n$. Note that in this case, the morphism $p : \tilde{Y} \to Y$ is a blowing-up at two distinct points on Y. Let E_1 and E_2 denote the (-1)-curves corresponding to the centers of this blowing-up. Then we have $p^*|K_Y| = |L| + \sum_{i=1,2} E_i$ and $LE_1 = LE_2 = 1$. The Galois group $G = \mathrm{Gal}(Y/X)$ acts transitively on the set $\{E_1, E_2\}$. We denote by Z' the minimal desingularization of Z.

Lemma 3.11. *There exists a blowing-up $r : Z' \to \mathbb{P}^2$ at (possibly infinitely near) $9-n$ points such that the anticanonical morphism $Z' \to Z \subset \mathbb{P}^n$ of Z' gives the minimal desingularization of Z.*

Proof. Note that our $Z = \mathit{\Phi}_{K_Y}(Y)$ is a non-degenerate surface in $\mathbb{P}^n$ of degree n. Hence our Z is one of the following (see [18] or [12, Section 3]):

i) a projection of a surface of degree n in $\mathbb{P}^{n+1}$ from a point outside the surface;

ii) the Veronese embedding into $\mathbb{P}^8$ of a quadric in $\mathbb{P}^3$ ($n = 8$);

iii) the anticanonical image of $\mathbb{P}^2$ blown up at $9 - n$ points;

iv) a cone over an elliptic curve in $\mathbb{P}^{n-1}$ of degree n.

Since $Z' \to Z$ is given by a complete linear system, case i) above is impossible for our case. Since $q(Y) = 0$, case iv) also is impossible. Thus it suffices to exclude case ii). In case ii), however, the divisor L is linearly equivalent to twice a divisor on $\tilde{Y}$, which contradicts the equality $LE_i = 1$. Hence we have the assertion. □

In what follows, we put $D' = -K_{Z'}$ and denote by $\mathit{\Phi}_{D'} : Z' \to Z \subset \mathbb{P}^n$ the anticanonical map of Z'. Note that the action by $G = \mathrm{Gal}(Y/X)$ on $\tilde{Y}$ induces one on Z'.

Lemma 3.12. *If the surface Z' has no (-2)-curve, or if every (-2)-curve on Z' is stable under the action by G on Z', then $\mathit{\Phi}_L : \tilde{Y} \to Z \subset \mathbb{P}^n$ lifts to a morphism $f' : \tilde{Y} \to Z'$.*

Proof. Take the shortest composite $p' : Y' \to \tilde{Y}$ of quadric transformations such that Y' admits a morphism $f' : Y' \to Z'$ satisfying $\Phi_L \circ p' = \Phi_{D'} \circ f'$. Then the action by G on $\tilde{Y}$ induces one on Y'. Note that f' contracts no (-1)-curve. This follows from $LE_i = 1$ and the definition of p', since the surface Y is of general type. To obtain the assertion, we only need to show that $p' : Y' \to \tilde{Y}$ is an isomorphism. Assume that $p' : Y' \to \tilde{Y}$ is not an isomorphism. Then there exists a (-1)-curve C on Y' exceptional with respect to p'. Since the anticanonical map $\Phi_{D'} : Z' \to Z \subset \mathbb{P}^n$ contracts $f'(C)$ to a point, the curve $f'(C)$ is a (-2)-curve on Z', hence, by the assumption in the statement, stable under the action by G on Z'. Meanwhile by the same method as in Lemma 3.5, we see that $f'_*C = f'(C)$ or $2f'(C)$, and that if $f'_*C = f'(C)$, then C is a component of the ramification divisor of f'. It follows that $C \simeq \mathbb{P}^1$ is stable under the action by G on Y', which implies the existence of fixed points of this action. This, however, contradicts the definition of $\pi : Y \to X$. Thus we have the assertion. □

Lemma 3.13. *Assume that $\Phi_L : \tilde{Y} \to Z$ lifts to a morphism $f' : \tilde{Y} \to Z'$. Then $f'(E_1)$ and $f'(E_2)$ are (-1)-curves on Z'. Further, the following hold:*

i) *$f'_*E_i = f'(E_i)$ for $i = 1, 2$;*

ii) *the ramification divisor R' of f' satisfies $R' \sim f'^*(-2K_{Z'}) + 2\sum_{i=1,2} E_i$;*

iii) *the branch divisor B' of f' satisfies $B' \sim -4K_{Z'} + 2\sum_{i=1,2} f'(E_i)$;*

iv) *$f'(E_1)$ and $f'(E_2)$ are distinct components of the branch divisor B'.*

Proof. The first assertion and the assertion i) follow from $E_iL = E_i f'^* D' = 1$, which implies $\Phi_L(E_i)$ is a line in $\mathbb{P}^n$. The assertions ii) and iii) follow from $D' \sim -K_{Z'}$ and the assertion i). So it suffices to prove the assertion iv). Let us prove the assertion iv). Let θ be the involution of $\tilde{Y}$ over Z'. Since Y is of general type, the divisors E_1 and E_2 are the only (-1)-curves on $\tilde{Y}$. It follows that if $f'(E_1) \neq f'(E_2)$, then $\theta(E_i) = E_i$ holds for $i = 1, 2$. Thus we only need to show $f'(E_1) \neq f'(E_2)$. Assume that $f'(E_1) = f'(E_2)$. Then $f'^*(f'(E_1)) = f'^*(f'(E_2)) = E_1 + E_2 + \xi$ holds for a certain effective divisor ξ exceptional with respect to f'. Since we have $E_1 \cap E_2 = \emptyset$, we see, by the same method as in the proof of Lemma 3.5, that $\xi^2 = -(E_1 + E_2)\xi = 0$, hence $\xi = 0$. It follows $f'^*(f'(E_1)) = f'^*(f'(E_2)) = E_1 + E_2$. From this together with the assertions ii) and iii), we infer $f'^*B' - 2R' = 0$, which implies that the branch divisor B' has at most negligible singularities. Thus by [9, Lemma 6], we obtain

$$\chi(\mathcal{O}_{\tilde{Y}}) = 2\chi(\mathcal{O}'_Z) + \frac{1}{2}(-2K_{Z'} + \sum f'(E_i))(-K_{Z'} + \sum f'(E_i)) = n + 3,$$

which contradicts $\chi(\mathcal{O}_Y) = n + 2$. Thus we have $f'(E_1) \neq f'(E_2)$, which completes the proof of the assertion iv). □

Lemma 3.14. *If the surface Y is of case* 2) *in Proposition* 2*, then* $\lambda = 4$.

Proof. By Lemma 3.11, we have $n = 2\lambda - 2 \leq 9$, hence $\lambda \leq 5$. Thus we only need to exclude the case $\lambda = 5$. Assume $\lambda = 5$. Then $r : Z' \to \mathbb{P}^2$ is a blowing-up at one pint, hence $Z' = \Sigma_1$. Thus by Lemmas 3.12 and 3.13, we see that $\Phi_L : \tilde{Y} \to Z$ lifts to a morphism $f' : \tilde{Y} \to Z'$, and that $f'(E_i)$'s are (-1)-curves. The minimal section Δ_0, however, is the unique (-1)-curve on the Hirzebruch surface Σ_1. Thus we have $f'(E_1) = f'(E_2) = \Delta_0$, which contradicts Lemma 3.13. Hence we have the assertion. □

Thus we only need to study the case $\lambda = 4$. In what follows we assume $\lambda = 4$, hence $n = 6$. In this case, the morphism $r : Z' \to \mathbb{P}^2$ is a blowing-up at three points.

Lemma 3.15. *Assume that $\Phi_L : \tilde{Y} \to Z$ lifts to a morphism $f' : \tilde{Y} \to Z'$, and that $f'(E_1) \cap f'(E_2) = \emptyset$ holds. Let $r' : Z' \to W$ denote the blowing-down of the two (-1)-curves $f'(E_1)$ and $f'(E_2)$. Then the branch divisor B of the morphism $r' \circ f' : \tilde{Y} \to W$ is a member of the linear system $|-4K_W|$ having $[3,3]$-points at $r'(f'(E_1))$ and $r'(f'(E_2))$. Except for these two $[3,3]$-points, the branch divisor B has at most negligible singularities. Further, the surface Y gives the minimal desingularization of the double cover* (*of the surface W*) *with branch divisor B.*

Proof. Note that f' contracts no (-1)-curve. Thus the divisor $f'^*B' - 2R'$, linearly equivalent to $2(\sum f'^*(f'(E_i)) - 2\sum E_i)$ by Lemma 3.13, is twice a certain effective divisor ζ on $\tilde{Y}$. We have $\zeta E_j = (\sum f'^*(f'(E_i)) - 2\sum E_i)E_j = 1$, hence $\sharp(\zeta \cap E_j) = 1$ for $j = 1$, 2. So we put $\{z_j\} = f'(\zeta \cap E_j)$. Then the point $z_j \in f'(E_j)$, where $1 \leq j \leq 2$, is an essential singularity of the branch divisor B'. Meanwhile, by Lemma 3.13, we see that the divisor $B' - \sum f'(E_i)$ is effective, and that $(B' - \sum f'(E_i))f'(E_j) = 3$ for each $j = 1$, 2, from which we infer $\mathrm{mult}_{z_j}(B' - \sum f'(E_i)) \leq 3$. If, moreover, we have $\mathrm{mult}_{z_j}(B' - \sum f'(E_i)) \leq 2$, then z_j is a negligible singularity of the branch divisor B'; the singularity z_j of B' decomposes into a sum of points of multiplicity at most 2 by the blowing-up at z_j. Thus we obtain $\mathrm{mult}_{z_j}(B' - \sum f'(E_i)) = 3$, hence $(B' - \sum f'(E_i)) \cap f'(E_j) = \{z_j\}$ and $\mathrm{mult}_{z_j} B' = 4$. Let $q_1 \circ q_2 : Z'_2 \to Z'$ be the blowing-up at z_1 and z_2, and $\varepsilon_j = (q_1 \circ q_2)^{-1}(z_j)$, the (-1)-curve corresponding to z_j. Then by the same method as in [9, Section 2], we infer that the divisor $B'_2 = (q_1 \circ q_2)^* B' - 4\sum_{i=1,2} \varepsilon_i$ has at most negligible singularities, and that the surface $\tilde{Y}$ gives the canonical resolution of the double cover with branch divisor B'. It follows that $B = r'_* B'$ has $[3,3]$-points at $r'(f'(E_1))$ and $r'(f'(E_2))$, that the divisor B has no essential singularity except for these two $[3,3]$-points, and that the surface Y gives the minimal desingularization of the double cover with branch divisor B. Now all we have left is the linear equivalence $B \sim -4K_W$, which, however, is trivial by iii) in Lemma 3.13. □

Lemma 3.16. *Let $r' : Z' \to W$ be the blowing-down given in Lemma 3.15. Then the surface W is the Hirzebruch surface Σ_d of degree $d = 0$ or 2. The action by $G = \mathrm{Gal}(Y/X)$ on Z' induces one on W, of which fixed locus is a set of four isolated points. Further, none of these four fixed points lies on the branch divisor B.*

Proof. The action by G on Z' induces one on W, since the divisor $f'(E_1)+f'(E_2)$ is stable under the action by G. Note that the anticanonical system $|-K_{Z'}|$ has no fixed component. From this together with $K_W^2 = K_Z^2+2 = 8$, we see that $W = \Sigma_d$ for a certain integer $0 \le d \le 2$.

Let us show that the class of Γ, a fiber of the Hirzebruch surface $W = \Sigma_d \to \mathbb{P}^1$, is stable under the action by G on W. If the class of Γ is not stable, then we see that $d = 0$ and that the generator of G maps Γ to a member of the linear system $|\Delta_0|$. It follows that there exists an irreducible member $\Delta \in |\Delta_0 + \Gamma|$ contained in the fixed locus of the action by G on W. We have $\Delta \cap B \neq \emptyset$, since Δ is a 2-curve. Meanwhile since the Galois group G acts transitively on the set $\{r'(f'(E_1)), r'(f'(E_2))\}$, neither $r'(f'(E_1))$ nor $r'(f'(E_2))$ lies on Δ. Thus by Lemma 3.15, every point in $\Delta \cap B$ is at most a negligible singularity of B. Then the same argument as in the proof of Proposition 5 leads us to a contradiction. Hence the class of Γ is stable.

Now let us show the assertions. The argument above shows that the action by G on W induces one on $\mathbb{P}^1$ via the natural fibration of the Hirzebruch surface $W = \Sigma_d \to \mathbb{P}^1$. Note that if this induced action on $\mathbb{P}^1$ is trivial, then there exists an irreducible member $\Delta_1 \in |\Delta_0 + d\Gamma|$ contained in the fixed locus of the action by G on W, which, together with the same argument as in the case of Δ above, leads us to a contradiction. Thus the induced action on $\mathbb{P}^1$ is non-trivial. It follows that $|\Gamma|$ has exactly two members stable under the action on W, which we shall call Γ_1 and Γ_2. The same argument as in the case of Δ above shows that the induced action on Γ_i is non-trivial for each $i = 1, 2$. Thus we see that $d \neq 1$, and that if $d = 0$ or 2, then the fixed locus of the induced action on W is a set of four isolated points. The absence of the fixed points lying on B follows from the same argument as in the case of Δ above. □

By Lemmas 3.15 and 3.16, we see that if Φ_L lifts to $f' : \tilde{Y} \to Z'$, and if $f'(E_1) \cap f'(E_2) = \emptyset$, then our surface X has the structure as in Theorem 2. Let us check that these two conditions are in fact satisfied. To do this, we study the arrangement of (-1)-curves and (-2)-curves on Z', and use Lemmas 3.12 and 3.13.

Let $r_i : Z'_i \to Z'_{i-1}$, where $-2 \leq i \leq 0$, be the blowing-up such that $r = (r_{-2} \circ r_{-1} \circ r_0) : Z'_0 = Z' \to Z'_{-3} = \mathbb{P}^2$ holds. We denote by $z_i \in Z'_{i-1}$ and $\varepsilon_i = r_i^{-1}(z_i)$ the center of the blowing-up r_i and its corresponding (-1)-curve, respectively. For each $-2 \leq i \leq 0$, we denote by ε'_i the strict transform to Z' of the exceptional curve ε_i. For the total transform to Z' of ε_i, we use the same symbol ε_i.

Lemma 3.17. *Let $m \leq 2$ be a non-negative integer, and C, a $(-m)$-curve on Z' not exceptional with respect to $r : Z' \to \mathbb{P}^2$. Then C is a strict transform to Z' of a line on $\mathbb{P}^2$ passing exactly $m+1$ of the tree points z_i's $(-2 \leq i \leq 0)$.*

Proof. Let l be a line on $\mathbb{P}^2$. Then we have $C \sim m_0 r^*(l) - \sum_{i=-2}^{0} n_i \varepsilon_i$ for certain integers $m_0 \geq 1$ and $n_i \geq 0$'s. Note that $C^2 = -m$ and $-K_{Z'}C = 2 - m$, since C is a $(-m)$-curve. Thus we have

$$m_0^2 - \sum_{i=-2}^{0} n_i^2 = -m, \qquad 3m_0 - \sum_{i=-2}^{0} n_i = -m + 2. \tag{12}$$

From these equalities, we infer

$$5\sum_{i=-2}^{0} n_i^2 + \sum_{-2 \leq i < j \leq 0} (n_i - n_j)^2 + \sum_{i=-2}^{0} (n_i + m - 2)^2 = 9m + 4(m-2)^2 \leq 18,$$

hence $\sum_{i=-2}^{0} n_i^2 \leq 3$. Thus we have $n_i^2 = n_i$ for any $-2 \leq i \leq 0$. By this together with the equalities (12), we obtain $m_0 = 1$ and $\sum_{i=-2}^{0} n_i = m + 1$. Thus we have the assertion. □

We study the arrangement of (-1)-curves and (-2)-curves on Z' according to the configuration of the centers z_i's of the blowing-up $r : Z' \to \mathbb{P}^2$. First, we study the case where no two of the centers z_{-2}, z_{-1}, and z_0 are infinitely near. This case is divided into two cases: case 2-1-1) and case 2-1-2).

Case 2-1-1): the case where the centers z_{-2}, z_{-1}, and z_0 are not collinear. In this case, the surface Z' has no (-2)-curve. Thus Φ_L lifts to a morphism $f' : \tilde{Y} \to Z'$. There exist exactly six (-1)-curves: ε_{-2}, ε_{-1}, ε_0, $r_*^{-1}(l_{-2,-1})$, $r_*^{-1}(l_{-1,0})$, and $r_*^{-1}(l_{-2,0})$, where $l_{i,j}$ denotes the line on $\mathbb{P}^2$ passing z_i and z_j. Let $(X_0 : X_1 : X_2)$ be homogeneous coordinates of $\mathbb{P}^2$ satisfying $z_{-2} = (1 : 0 : 0)$, $z_{-1} = (0 : 1 : 0)$, and $z_0 = (0 : 0 : 1)$. For each $(a, b) \in \mathbb{C}^\times \times \mathbb{C}^\times$, we denote by $\varphi_{(a,b)}$ the automorphism of Z' corresponding to the projective transformation $(X_0 : X_1 : X_2) \mapsto (X_0 : aX_1 : bX_2)$.

Let us study the induced action by G on Z'. Let $\mathrm{Aut}(Z')$ be the group of analytic automorphisms of the surface Z', and D_6, the dihedral group of degree 6. Then we have a short exact sequence

$$0 \to \mathbb{C}^\times \times \mathbb{C}^\times \to \mathrm{Aut}(Z') \to D_6 \to 0,$$

where the morphism $\mathbb{C}^\times \times \mathbb{C}^\times \to \mathrm{Aut}(Z')$ is given by $(a,b) \mapsto \varphi_{(a,b)}$, and the morphism $\mathrm{Aut}(Z') \to D_6$ corresponds to the transitions of six (-1)-curves on Z'. Let φ_σ and φ_τ be the automorphisms of Z' corresponding to the Cremona transformation $(X_0 : X_1 : X_2) \mapsto (X_2X_0 : X_0X_1 : X_1X_2)$ and the morphism $(X_0 : X_1 : X_2) \mapsto (X_0 : X_2 : X_1)$, respectively. Then we have

$$(\varphi_\sigma)^6 = \mathrm{id}_{Z'} \qquad (\varphi_\tau)^2 = \mathrm{id}_{Z'} \qquad \varphi_\sigma \circ \varphi_\tau \circ \varphi_\sigma \circ \varphi_\tau = \mathrm{id}_{Z'}.$$

Thus the short exact sequence above splits. We denote by σ and τ the image by $\mathrm{Aut}(Z') \to D_6$ of φ_σ and φ_τ, respectively. We have a group homomorphism $G \to \mathrm{Aut}(Z')$ corresponding to the action by G on Z'. Composing this homomorphism with $\mathrm{Aut}(Z') \to D_6$, we obtain a group homomorphism $\alpha : G \to D_6$. Note that by Lemma 3.13, the morphism α is an injection of G into D_6. Hence the image $\alpha(G)$ is conjugate to $\langle \tau \rangle$, $\langle \sigma^3\tau \rangle$, or $\langle \sigma^3 \rangle$ in D_6.

Assume that the image $\alpha(G)$ is conjugate to $\langle \tau \rangle$ in D_6. Replacing the morphism $r : Z' \to \mathbb{P}^2$ if necessary, we may assume that $\alpha(G) = \langle \tau \rangle$. Then since the Galois group G acts transitively on the set $\{f'(E_1), f'(E_2)\}$, we have $\{f'(E_1), f'(E_2)\} = \{r_*^{-1}(l_{-2,-1}), r_*^{-1}(l_{-2,0})\}$ or $\{\varepsilon_{-1}, \varepsilon_0\}$, hence $f'(E_1) \cap f'(E_2) = \emptyset$. It follows that the surface W, where $r' : Z' \to W$ is the blowing-down of the two (-1)-curves $f'(E_1)$ and $f'(E_2)$, is isomorphic to the Hirzebruch surface $\varSigma_1$, which contradicts Lemma 3.16. Thus $\alpha(G)$ is not conjugate to $\langle \tau \rangle$.

Assume that the image $\alpha(G)$ is conjugate to $\langle \sigma^3\tau \rangle$ in D_6. Replacing the morphism $r : Z' \to \mathbb{P}^2$ if necessary, we may assume that $\alpha(G) = \langle \sigma^3\tau \rangle$. Then the blowing-down of the two (-1)-curves ε_{-2} and $r_*^{-1}(l_{-1,0})$ gives a birational morphism $r'' : Z' \to \varSigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ satisfying $r''_*(\varepsilon_0) \sim r''_*(r_*^{-1}(l_{-2,-1})) \sim \varDelta_0$ and $r''_*(\varepsilon_{-1}) \sim r''_*(r_*^{-1}(l_{-2,0})) \sim \varGamma$. Note that the action by G on Z' induces one on $\varSigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$. We take homogeneous coordinates $((\xi_0 : \xi_1), (\eta_0 : \eta_1))$ of $\varSigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ in such a way that $r''(\varepsilon_{-2}) = ((1 : 0), (1 : 0))$ and $r''(r_*^{-1}(l_{-1,0})) = ((0 : 1), (0 : 1))$ hold, and that the automorphism of $\varSigma_0$ corresponding to the generator ι of G is given by $((\xi_0 : \xi_1), (\eta_0 : \eta_1)) \mapsto ((\eta_1 : \eta_0), (\xi_1 : \xi_0))$. Since we have $-K_{Z'} \sim r''^*(-K_{\varSigma_0}) - \varepsilon_{-2} - r_*^{-1}(l_{-1,0})$, the space $H^0(\mathcal{O}_{Z'}(-2K_{Z'}))$ corresponds to a certain subspace V of $H^0(\mathcal{O}_{\varSigma_0}(-2K_{\varSigma_0}))$. Every element in V is a homogeneous polynomial $\psi(\xi_0, \xi_1, \eta_0, \eta_1)$ of bidegree $(4, 4)$ vanishing with multiplicity at least 2 at $((1 : 0), (1 : 0))$ and $((0 : 1), (0 : 1))$.

Note that we have a natural inclusion $V \hookrightarrow H^0(\mathcal{O}_Y(2K_Y))$, since we have $L \sim f'^*D'$. We denote by ϕ the restriction to V of the natural action by G on

$H^0_Y(\mathcal{O}(2K_Y))$. Let $\phi'(\iota)$ be the automorphism of V given by $\psi(\xi_0, \xi_1, \eta_0, \eta_1) \mapsto \psi(\eta_1, \eta_0, \xi_1, \xi_0)$. Then $\iota \mapsto \phi'(\iota)$, where ι is the generator of the Galois group G, gives another action ϕ' by G on V. Note that for any $g \in G$ and $\psi \in V$, the two elements $\phi(g)\psi$ and $\phi'(g)\psi$ defines the same divisor on Σ_0. From this we infer that $\phi = c\phi'$ for a certain character $c \in \mathrm{Char}(G)$.

Now let V^+ be the set of all elements in V stable under the action ϕ'. Then by $\phi = c\phi'$, we see that $V^+ \subset H^0(\mathcal{O}_X(2K_X - T_c))$ for a torsion divisor $T_c \in \mathrm{Pic}(X)$ corresponding to the character c. Meanwhile, by the Riemann–Roch theorem, we have $h^0(\mathcal{O}_X(2K_X - T_c)) = \chi + K_X^2 = 11$. The space V^+, however, has a base consisting of twelve elements:

$$\xi_0^i \xi_1^{4-i} \eta_0^j \eta_1^{4-j} + \xi_0^{4-j} \xi_1^j \eta_0^{4-i} \eta_1^i \qquad (0 \le i, \quad 0 \le j, \quad 2 \le i+j \le 4).$$

This contradicts the inequality $\dim V^+ \le h^0(\mathcal{O}_X(2K_X - T_c))$. Hence, the image $\alpha(G)$ is not conjugate to $\langle \sigma^3 \tau \rangle$ in D_6.

Thus we have $\alpha(G) = \langle \sigma^3 \rangle$. Hence, replacing $r : Z' \to \mathbb{P}^2$ if necessary, we may assume that $\{f'(E_1), f'(E_2)\} = \{\varepsilon_{-2}, r_*^{-1}(l_{-1,0})\}$. Then the surface W as in Lemma 3.15, obtained by blowing down the two (-1)-curves $f'(E_1)$ and $f'(E_2)$ of Z', is isomorphic to the Hirzebruch surface Σ_0. Thus by Lemmas 3.15 and 3.16, our surface X, in case 2-1-1), has the structure as in the case $d = 0$ in Theorem 2.

Case 2-1-2): the case where three points z_{-2}, z_{-1}, and z_0 are collinear. Let $l_{-2,-1}$ be the line on $\mathbb{P}^2$ passing the tree points z_i's above. Then the strict transform $r_*^{-1}(l_{-2,-1})$ is the unique (-2)-curve on Z'. Hence by Lemma 3.12, the morphism $\Phi_L : \tilde{Y} \to Z$ lifts to $f' : \tilde{Y} \to Z'$. Meanwhile the surface Z' has exactly three (-1)-curves: ε_{-2}, ε_{-1}, and ε_0. Replacing $r : Z' \to \mathbb{P}^2$ if necessary, we may assume $\{f'(E_1), f'(E_2)\} = \{\varepsilon_{-2}, \varepsilon_{-1}\}$ by Lemma 3.13. Let $r' : Z' \to W$ be the blowing-down as in Lemma 3.15 of the two (-1)-curves $f'(E_1)$ and $f'(E_2)$. Then we have $W = \Sigma_1$, $r'_*(\varepsilon_0) = \Delta_0$, and $r'_*(r_*^{-1}(l_{-2,-1})) \sim \Gamma$, which contradicts Lemma 3.16. Thus case 2-1-2) does not occur.

Next, we study the case where z_{-2} and z_{-1} are distinct points on $\mathbb{P}^2$, and z_0 is infinitely near to z_{-1}. We denote by $l_{-2,-1}$ the unique line on $\mathbb{P}^2$ passing z_{-2} and z_{-1}. This case is divided into two cases: case 2-2-1) and case 2-2-2).

Case 2-2-1): the case where z_0 does not lie on the strict transform $(r_{-2} \circ r_{-1})_*^{-1}(l_{-2,-1})$ of $l_{-2,-1}$ by $r_{-2} \circ r_{-1}$. Let $l_{-1,0}$ be the unique line on $\mathbb{P}^2$ whose strict transform $(r_{-2} \circ r_{-1})_*^{-1}(l_{-1,0})$ by $r_{-2} \circ r_{-1}$ passes z_0. Then the surface Z' has a unique (-2)-curve ε'_{-1}, and exactly four (-1)-curves ε_{-2}, ε_0, $r_*^{-1}(l_{-2,-1})$, and $r_*^{-1}(l_{-1,0})$. Hence, by Lemma 3.12, the morphism $\Phi_L : \tilde{Y} \to Z$ lifts to $f' : \tilde{Y} \to Z'$. Note that $\{\varepsilon_0, r_*^{-1}(l_{-2,-1})\}$ is the set of all (-1)-curves intersecting the unique (-2)-curve ε'_{-1}. Thus we have $\{f'(E_1), f'(E_2)\} =$

$\{\varepsilon_0, r_*^{-1}(l_{-2,-1})\}$ or $\{\varepsilon_{-2}, r_*^{-1}(l_{-1,0})\}$, hence, in particular, $f'(E_1) \cap f'(E_2) = \emptyset$. We denote by $r' : Z' \to W$ the blowing-down as in Lemma 3.15 of the two (-1)-curves $f'(E_1)$ and $f'(E_2)$. If $\{f'(E_1), f'(E_2)\} = \{\varepsilon_0, r_*^{-1}(l_{-2,-1})\}$, then we have $W = \Sigma_0$, $r'_*(\varepsilon'_{-1}) \sim \Delta_0$ and $r'_*(\varepsilon_{-2}) \sim r'_*(r_*^{-1}(l_{-1,0})) \sim \Gamma$. If on the other hand $\{f'(E_1), f'(E_2)\} = \{\varepsilon_{-2}, r_*^{-1}(l_{-1,0})\}$, then we have $W = \Sigma_2$, $r'_*(\varepsilon'_{-1}) = \Delta_0$, and $r'_*(\varepsilon_0) \sim r'_*(r_*^{-1}(l_{-2,-1})) \sim \Gamma$. Thus by lemmas 3.15 and 3.16, our surface X, in case 2-2-1), has the structure as in the case $d = 0$ or the case $d = 2$ in Theorem 2, according as $\{f'(E_1), f'(E_2)\} = \{\varepsilon_0, r_*^{-1}(l_{-2,-1})\}$ or $\{f'(E_1), f'(E_2)\} = \{\varepsilon_{-2}, r_*^{-1}(l_{-1,0})\}$ respectively.

Case 2-2-2): the case where z_0 lies on the strict transform $(r_{-2} \circ r_{-1})_*^{-1}(l_{-2,-1})$. In this case, the surface Z' has exactly two (-2)-curves ε'_{-1} and $r_*^{-1}(l_{-2,-1})$, and exactly two (-1)-curves ε_{-2} and ε_0. Note that every (-2)-curve on Z' is stable under the action by G on Z'; the divisor $r_*^{-1}(l_{-2,-1})$ is the unique (-2)-curve intersecting all (-1)-curves on Z'. Thus by Lemma 3.12, the morphism $\Phi_L : \tilde{Y} \to Z$ lifts to $f' : \tilde{Y} \to Z'$. Then it follows from Lemma 3.13 that $\{f'(E_1), f'(E_2)\} = \{\varepsilon_{-2}, \varepsilon_0\}$. This, however, contradicts the transitivity of the action by G on $\{f'(E_1), f'(E_2)\}$, since ε_0 is the unique (-1)-curve intersecting all (-2)-curves on Z'. Thus case 2-2-2) does not occur.

Finally, we study the case where z_{-1} is infinitely near to z_{-2}, and z_0 is infinitely near to z_{-1}. We denote by $l_{-2,-1}$ the unique line on $\mathbb{P}^2$ whose strict transform $(r_{-2})_*^{-1}(l_{-2,-1})$ passes z_{-1}. Note that Z' has no (-3)-curve, since the linear system $|-K_{Z'}|$ has no fixed component. Thus the point z_0 does not lie on the strict transform $(r_{-1})_*^{-1}(\varepsilon_{-2})$. This case is divided into two cases: case 2-3-1) and case 2-3-2).

Case 2-3-1): the case where z_0 does not lie on the strict transform $(r_{-2} \circ r_{-1})_*^{-1}(l_{-2,-1})$. In this case, the surface Z' has exactly two (-2)-curves ε'_{-2} and ε'_{-1}, and exactly two (-1)-curves ε_0 and $r_*^{-1}(l_{-2,-1})$. Since ε'_{-2} is the unique (-2)-curve intersecting no (-1)-curve on Z', every (-2)-curve is stable under the action by G on Z'. Thus by Lemma 3.12, the morphism $\Phi_L : \tilde{Y} \to Z$ lifts to $f' : \tilde{Y} \to Z'$. Then it follows from Lemma 3.13 that $\{f'(E_1), f'(E_2)\} = \{\varepsilon_0, r_*^{-1}(l_{-2,-1})\}$, hence $f'(E_1) \cap f'(E_2) = \emptyset$. Let $r' : Z' \to W$ be the blowing-down as in Lemma 3.15 of the two (-1)-curves $f'(E_1)$ and $f'(E_2)$. Then we have $W = \Sigma_2$, $r'_*(\varepsilon'_{-2}) = \Delta_0$, and $r'_*(\varepsilon'_{-1}) \sim \Gamma$. Thus by Lemmas 3.15 and 3.16, our surface X, in case 2-3-1), has the structure as in the case $d = 2$ in Theorem 2.

Case 2-3-2): the case where z_0 lies on the strict transform $(r_{-2} \circ r_{-1})_*^{-1}(l_{-2,-1})$. In this case, the surface Z' has exactly three (-2)-curves ε'_{-2}, ε'_{-1}, and $r_*^{-1}(l_{-2,-1})$, and a unique (-1)-curve ε_0. Note that ε'_{-2} is the unique (-2)-curve intersecting no (-1)-curve on Z', and that ε'_{-1} is the unique (-2)-curve intersecting ε'_{-2}. Thus every (-2)-curve on Z' is stable under the action by G

on Z'. Thus by Lemma 3.12, the morphism $\Phi_L : \tilde{Y} \to Z$ lifts to $f' : \tilde{Y} \to Z'$. This, however, contradicts Lemma 3.13, since ε_0 is the unique (-1)-curve on Z'. Hence, case 2-3-2) does not occur.

Thus we have the following:

Proposition 9. *Assume that the surface Y is of case* 2) *in Proposition* 2. *Then $\lambda = 4$. Further, the surface X in this case has the structure as in Theorem* 2.

4 The case $\deg \Phi_{K_Y} = 1$

In this section, we exclude the case $\deg \Phi_{K_Y} = 1$ and give a proof for Theorems 1 and 2. In what follows, we assume that $\deg \Phi_{K_Y} = 1$. Note that by Proposition 1, we have $\lambda = 4$, hence $K_Y^2 = 14$, $p_g(Y) = 7$, and $q(Y) = 0$. Thus our Y is a canonical surface whose invariant lies on the Castelnuovo line. By [1, Lemma 1.1], the canonical system $|K_Y|$ is free from base points; hence the canonical map $\Phi_{K_Y} : Y \to \mathbb{P}^n$ is a morphism, where $n = 2\lambda - 2 = 6$. In what follows, we frequently use results given in [1].

Let $\mathcal{Q} \subset \mathbb{P}^n$ be the intersection of all quadrics containing the canonical image $Z = \Phi_{K_Y}(Y)$. By [1, Section 1], we obtain the following:

Proposition 10. *Let $\mathcal{Q}$ be the variety defined above. Then either of the following holds*:

1) *$\mathcal{Q}$ is the image by Φ_T of the variety $\mathcal{Q}' = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$, where Φ_T is the morphism associated with a tautological divisor T of the $\mathbb{P}^1$-bundle $\mathrm{pr}_{\mathcal{Q}'} : \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)) \to \mathbb{P}^2$*;

2) *$\mathcal{Q}$ is the image by Φ_T of the variety $\mathcal{Q}' = \mathbb{P}(\bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}^1}(a_i))$, where Φ_T is the morphism associated with a tautological divisor T of the $\mathbb{P}^2$-bundle $\mathrm{pr}_{\mathcal{Q}'} : \mathbb{P}(\bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}^1}(a_i)) \to \mathbb{P}^1$, and $0 \le a_0 \le a_1 \le a_2$ and $\sum_{i=0}^{2} a_i = n - 2$.*

First, we exclude case 1) in the proposition above.

Proposition 11. *Case* 1) *in Proposition* 10 *does not occur.*

Proof. Assume that our $\mathcal{Q}$ is as in case 1) in Proposition 10. Then $\mathcal{Q}$ is a cone over the Veronese surface. Let p_0 be the vertex of $\mathcal{Q}$, and Λ, the linear system consisting of pull-backs by Φ_{K_Y} of all hyperplanes in $\mathbb{P}^n$ passing p_0. We denote by Λ_0 and G_0 its variable part and fixed part respectively. By [1, Proof of Claim I], the linear system Λ_0 is free from base points and induces $\Phi_{\Lambda_0} : Y \to \mathbb{P}^{n-1}$, a morpshism of mapping degree 3 onto its image. The image $\Phi_{\Lambda_0}(Y)$ is the Veronese surface, i.e., the projective plane $\mathbb{P}^2$ embedded in $\mathbb{P}^5$ by $\mathcal{O}_{\mathbb{P}^2}(2)$. Note that by the definition of $\mathcal{Q}$, the varity $\mathcal{Q}$ and its vertex

p_0 are stable under the action by $G = \mathrm{Gal}(Y/X)$ on $\mathbb{P}^n$. This implies that the subspace of $H^0(\mathcal{O}_Y(K_Y))$ corresponding to Λ is stable under the action by G on $H^0(\mathcal{O}_Y(K_Y))$. Thus the action by G on Y induces one on $\Phi_{\Lambda_0}(Y) = \mathbb{P}^2$. Now let us derrive a contradiction. Since $G \simeq \mathbb{Z}/2$, the fixed locus of this induced action contains a line l_0 on $\mathbb{P}^2$. Then the divsor $\Phi^*_{\Lambda_0}(l_0)$, stable under the action by G, is a pull-back by $\pi : Y \to X$ of that on X. We however have $\Phi^*_{\Lambda_0}(l_0)^2 = \deg \Phi_{\Lambda_0} = 3$, which contradicts $\deg \pi = 2$. Thus we have the assertion. □

Next, we exclude case 2) in Proposition 10.

Lemma 4.1. *If the variety $\mathcal{Q}$ is as in case* 2) *of Proposition 10, then $a_0 = 0$.*

Proof. Assume that our variety $\mathcal{Q}$ is as in case 2) in Proposintion 10 and that $a_0 > 0$. Then $\Phi_T : \mathcal{Q}' \to \mathbb{P}^n$ is an embedding. We identify $\mathcal{Q}$ and $\mathcal{Q}'$ by Φ_T. By the same arguement as in the proof of Proposintion 11, we see that the variety $\mathcal{Q}$ is stable under the action by G on $\mathbb{P}^n$. Let P be a fiber of the $\mathbb{P}^2$-bundle $\mathrm{pr}_{\mathcal{Q}'} : \mathcal{Q} = \mathcal{Q}' \to \mathbb{P}^1$. Then P and T generate the Picard group of $\mathcal{Q}$. Using this, we see easily that if a divisor P' on $\mathcal{Q}$ satisfies $P'^3 = K_{\mathcal{Q}} P'^2 = 0$ and $h^0(\mathcal{O}_{\mathcal{Q}}(P')) = 2$, then $P' \sim P$. Thus the class of P is stable under the action by G on $\mathcal{Q}$. It follows that this action induces one on $\mathbb{P}^1$ via the projection $\mathrm{pr}_{\mathcal{Q}'} : \mathcal{Q} = \mathcal{Q}' \to \mathbb{P}^1$, and that there exsits a member $P_0 \in |P|$ stable under the action on $\mathcal{Q}$. Now let us derrive a contradiction. Since $G \simeq \mathbb{Z}/2$, the fixed locus of the action by G on $P_0 = \mathbb{P}^2$ contains a line l_0. Hence the action on Z has a fixed points. By [1, Theorem 1.5], however, the surface Z has at most rational double points as its singularities. Thus, by the same arguement as in the proof of Proposition 5, we infer that the action on Y has fixed points, which contradicts the definition of $\pi : Y \to X$. □

Proposition 12. *Case* 2) *in Proposition* 10 *does not occur.*

Proof. Assume that our $\mathcal{Q}$ is as in case 2) in Proposition 10. Then by Lemma 4.1 and [1, Claim II], we have $a_0 = 0$ and $a_1 > 0$. It follows that our $\mathcal{Q}$ is a cone over the Hirzebruch surface $\Sigma_{a_2 - a_1}$ embedded in $\mathbb{P}^{n-1}$ by $|\Delta_0 + a_2 \Gamma|$. Let p_0 be the vertex of $\mathcal{Q}$, and Λ, the linear system consisting of the pull-backs by Φ_{K_Y} of all hyperplanes in $\mathbb{P}^n$ passing p_0. We denote by Λ_0 and G_0 the variable part and the fixed part of Λ respectively. By [1, Proof of Claim II], the linear system Λ_0 is free from base points and induces $\Phi_{\Lambda_0} : Y \to \mathbb{P}^{n-1}$, a morphism of degree 3 onto its image. The image $\Phi_{\Lambda_0}(Y)$ is the Hirzebruch surface $\Sigma_{a_2 - a_1}$ embedded in $\mathbb{P}^{n-1}$ by $|\Delta_0 + a_2 \Gamma|$. By the same arguement as in the proof of Proposition 11, we see that the action by G on Y induces one on $\Sigma_{a_2 - a_1}$. The class of $\Delta_0 + a_2 \Gamma$ and that of $-K_{\Sigma_{a_2 - a_1}}$ are stable under this induced action on $\Sigma_{a_2 - a_1}$; hence so are the class of Δ_0

and that of Γ. Thus there exist members $\Delta_1 \in |\Delta_0|$ and $\Gamma_1 \in |\Gamma|$ stable under the action on $\Sigma_{a_2-a_1}$. Then from $\Phi^*_{\Lambda_0}(\Delta_1)\Phi^*_{\Lambda_0}(\Gamma_1) = \deg \Phi_{\Lambda_0} = 3$, we derrive a contradiction by the same arguement as in the proof of Proposition 11. Thus we have the assertion. $\square$

Now we are ready to prove Theorems 1 and 2.

Proof of Theorems 1 and 2.

By Propositions 1, 10, 11, and 12, we have $\deg \Phi_{K_Y} = 2$. Thus Theorems 1 and 2 follow from Propsitions 2, 5, 6, 8, and 9. $\square$

Remark 4. Let $X_{(1)}$ and $X_{(2)}$ be two minimal complex surfaces as in Theorem 2, $\pi_{(i)} : Y_{(i)} \to X_{(i)}$ ($i = 1, 2$), the unramified double cover corresponding to the torsion group, $f_{(i)} : Y_{(i)} \to W_{(i)} = \Sigma_{d_{(i)}}$, the generically two-to-one morphism as in Theorem 2, and $B_{(i)}$, the branch divisor of $f_{(i)}$. Then if $X_{(1)}$ and $X_{(2)}$ are isomorphic to each other, so are the triplets $(W_{(1)}, \iota|_{W_{(1)}}, B_{(1)})$ and $(W_{(2)}, \iota|_{W_{(2)}}, B_{(2)})$, where $\iota|_{W_{(i)}}$ denotes the involution of $W_{(i)}$ corresponding to the generator of the Galoirs group of $\pi_{(i)}$. This is verified as follows. Let $p_{(i)} : \tilde{Y}_{(i)} \to Y_{(i)}$ be the shortest composite of quadric transformations such that the variable part of $p^*_{(i)}|K_{Y_{(i)}}|$ is free from base points, and $r'_{(i)} : Z'_{(i)} \to W_{(i)} = \Sigma_{d_{(i)}}$, the blowing-up at two $[3,3]$-points of the branch divisor $B_{(i)}$. Then $f_{(i)}$ induces a morphims $\tilde{f}_{(i)} : \tilde{Y}_{(i)} \to Z'_{(i)}$. The projection $r'_{(i)}$ is the blowing-down of the image by $\tilde{f}_{(i)}$ of the exceptional divisor of $p_{(i)} : \tilde{Y}_{(i)} \to Y_{(i)}$. Since $Z'_{(i)}$ is the minimal desingularization of the canonical image of $Y_{(i)}$, we have the assertion.

5 The moduli space for the case $\chi = 4$

In this section, we shall study the moduli space for surfaces as in Theorem 2, and give a proof for Theorem 3. For this purpose, we shall first study the explicit description of our surfaces in more detail.

Let X be a minimal algebraic surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and $\mathrm{Tors} \simeq \mathbb{Z}/2$. We denote by $\pi : Y \to X$ the unramified double cover corresponding to the torsion group, and by $p : \tilde{Y} \to Y$, the shortest composite of quadric transformations such that the variable part of $p^*|K_Y|$ is free from base points. Then there exist an even integer $0 \leq d \leq 2$ and a generically two-to-one morphism $f : Y \to W = \Sigma_d$ satisfying the three conditions given in Theorem 2. In what follows, we denote by $\iota|_W$ the involution of W corresponding to the generator of the Galois group $G = \mathrm{Gal}(Y/X)$.

Let $r' : Z' \to W$ be the blowing-up at two $[3,3]$-points, which we shall call w_1 and w_2, of the branch divisor B of f. Then $f_W = f \circ p : \tilde{Y} \to W$ lifts to a morphism $f' : \tilde{Y} \to Z'$. We denote by $e_i = {r'}^{-1}(w_i)$ the exceptional divisor of r' lying over w_i. Let $\tilde{r} : \tilde{Z} \to Z'$ be the blowing-up at two quadraple points, which we shall call $w'_1 \in e_1$ and $w'_2 \in e_2$, of the branch divisor of f'. Then f' lifts to a morphism $\tilde{f} : \tilde{Y} \to \tilde{Z}$. We denote by $e'_i = \tilde{r}^{-1}(w'_i)$ the exceptional divisor of $\tilde{r}$ lying over w'_i. Let us use the same symbol e_i for the total transform to $\tilde{Z}$ of the divisor $e_i \subset Z'$. Then there exists a reduced member $\tilde{B}_0 \in |(r' \circ \tilde{r})^*(-4K_W) - 3\sum e_i - 3\sum e'_i|$ satisfying $\tilde{B}_0 \cap \tilde{r}_*^{-1}(e_1) = \tilde{B}_0 \cap \tilde{r}_*^{-1}(e_2) = \emptyset$ such that the branch divisor of $\tilde{f}$ is given by $\tilde{B}_0 + \sum \tilde{r}_*^{-1}(e_i)$. Note that the divisor $\tilde{B}_0$ has at most negligible singularities. In what follows, Δ_0 and Γ denote the minimal section and a fiber respectively of the Hirzebruch surface $W = \Sigma_d \to \mathbb{P}^1$.

Lemma 5.1. *Let $\iota|_{Z'}$ be the involution of Z' induced by the involution $\iota|_W$ of W. Then the configuration of the four points w_1, $w_2 = \iota|_W(w_1)$, w'_1, and $w'_2 = \iota|_{Z'}(w'_1)$ satisfies the following three conditions*:

i) if $d = 2$, then $w_1 \notin \Delta_0$;

ii) if the two points w_1 and w_2 lie on one and the same member of the linear system $|\Gamma|$, then for each $i = 1, 2$, the point w'_i does not lie on the strict transform to Z' of this member;

iii) if d equals 0, and the two points w_1 and w_2 lie on one and the same member of the linear system $|\Delta_0|$, then for each $i = 1, 2$, the point w'_i does not lie on the strict transform to Z' of this member.

Proof. i). Assume that $d = 2$ and $w_1 \in \Delta_0$. Then since Δ_0 is stable under the action by G on W, we have $w_2 \in \Delta_0$. Thus ${r'}_*^{-1}(\Delta_0)$ is a (-4)-curve on Z', hence ${r'}_*^{-1}(\Delta_0)(-K_{Z'}) < 0$. It follows that ${r'}_*^{-1}(\Delta_0)$ is a fixed component of the linear systme $|-K_{Z'}|$. This is impossible, since by the proof of our complete descripiton the pull-back $f'^*|-K_{Z'}|$ is the variable part of $|K_{\tilde{Y}}|$. Thus we have $w_1 \notin \Delta_0$ for the case $d = 2$.

ii). Assume that w_1 and w_2 lie on one and the same member $\Gamma_0 \in |\Gamma|$. Then since $w_2 = \iota|_W(w_1)$, the member Γ_0 is stable under the action by G. It follwos that Γ_0 passes exactly two of the fixed points of the involution $\iota|_W$. Moreover if $w'_1 \in {r'}_*^{-1}(\Gamma_0)$, then we obtain $w'_2 \in {r'}_*^{-1}(\Gamma_0)$, $(r' \circ \tilde{r})_*^{-1}(\Gamma_0) \sim (r' \circ \tilde{r})^*(\Gamma) - \sum e_i - \sum e'_i$, and $\tilde{B}_0((r' \circ \tilde{r})_*^{-1}(\Gamma_0)) = -4 < 0$. The last inequality implies that Γ_0 is an irreducible component of the branch divisor B. This however is impossible, since, by the condition in Theorem 2, the branch divisor B cannot pass any fixed points of the involution $\iota|_W$. Thus we have $w'_1 \notin {r'}_*^{-1}(\Gamma_0)$.

iii). By the same arguement as in the proof of ii), we can prove iii). □

Remark 5. As shown in the proof of Lemma above, if the two points w_1 and w_2 lie on one and the same member $\Gamma_0 \in |\Gamma|$, then this Γ_0 is stable under the action by G on W. There exist exactly two members of $|\Gamma|$ stable under the action by G. In what follows, we denote by Γ_1 and Γ_2 these two members. For each $i = 1, 2$, exactly two fixed points of the action by G lie on Γ_i.

Next let us show that if conversely the configuration of four points w_i's and w'_i's satisfies the three conditions in Lemma 5.1, then the proceedure implied by our structure theorem in fact produces a minimal surface with the desired invariants. Some of the results below will be used later, in our proof of the uniqueness of the deformation type. Let $W = \Sigma_d$ be the Hirzebruch surface of degree $d = 0$ or 2, and $\iota|_W$, the involution (1) given in Remark 1. Take a point $w_1 \in W$ outside the fixed locus of $\iota|_W$. We denote by $r' : Z' \to W$ the blowing-up at two points w_1 and $w_2 = \iota|_W(w_1)$, and by $e_i = {r'}^{-1}(w_i)$, the exceptional curve lying over w_i. Let $\iota|_{Z'}$ be the involution of Z' induced by $\iota|_W$. Take a point $w'_1 \in e_1 \subset Z'$. We donote by $\tilde{r} : \tilde{Z} \to Z'$ the blowing-up at two points w'_1 and $w'_2 = \iota|_{Z'}(w'_1)$, and by $e'_i = \tilde{r}^{-1}(w'_i)$, the exceptional curve lying over w'_i. We use the same symbol e_i for the total transform to $\tilde{Z}$ of the divisor e_i on Z'. We assume that the configuration of w_i's and w'_i's satisfies the three conditions i), ii), and iii) in Lemma 5.1.

Let Γ_1 and Γ_2 be two distinct members of $|\Gamma|$ stable under the natural action by $G = \langle \iota|_W \rangle$ on W (see Remark 5). We take the minimal section Δ_0 and an irreducible member $\Delta_\infty \in |\Delta_0 + d\Gamma|$ such that both are stable under the action by G, and $\Delta_0 \cap \Delta_\infty = \emptyset$ holds. Let m be a positive integer. Since the divisor $m(\Delta_0 + \frac{d+2}{2}\Gamma_1)$ is stable under the action by G, we obtain a natural action on $H^0(\mathcal{O}_W(m(\Delta_0 + \frac{d+2}{2}\Gamma)))$ by identifying this space with that of meromorphic functions with poles at most $m(\Delta_0 + \frac{d+2}{2}\Gamma_1)$.

We put $\Lambda_m = |m(\Delta_0 + \frac{d+2}{2}\Gamma)|$, and denote by Λ_m^+ and Λ_m^- the subsystems of Λ_m corresponding to the eigenspaces of eigenvalues $+1$ and -1 repectively with respect to ${\iota|_W}^*$. Moreover, for an effective divisor C on $\tilde{Z}$, we put

$$\Lambda_m(C) = \{D \in \Lambda_m; (r' \circ \tilde{r})^*(D) - C \succeq 0\} \qquad \tilde{\Lambda}_m(C) = (r' \circ \tilde{r})^*\Lambda_m(C) - C$$
$$\Lambda_m^+(C) = \{D \in \Lambda_m^+; (r' \circ \tilde{r})^*(D) - C \succeq 0\} \qquad \tilde{\Lambda}_m^+(C) = (r' \circ \tilde{r})^*\Lambda_m^+(C) - C$$
$$\Lambda_m^-(C) = \{D \in \Lambda_m^-; (r' \circ \tilde{r})^*(D) - C \succeq 0\} \qquad \tilde{\Lambda}_m^-(C) = (r' \circ \tilde{r})^*\Lambda_m^-(C) - C,$$

where the symbol $\succeq 0$ means effectiveness of a divisor. We abbreviate $\tilde{\Lambda}_m(0)$, $\tilde{\Lambda}_m^+(0)$, and $\tilde{\Lambda}_m^-(0)$ to $\tilde{\Lambda}_m$, $\tilde{\Lambda}_m^+$, and $\tilde{\Lambda}_m^-$ respectively. Note that if $\tilde{f} : \tilde{Y} \to \tilde{Z}$ is the generically two-to-one morphism obtained as in the begining of this section from our structure theorem, then we have $\tilde{B}_0 \in \tilde{\Lambda}_8^+(3\sum e_i + 3\sum e'_i)$, where $\tilde{B}_0 + \sum \tilde{r}_*^{-1}(e_i)$ gives the branch divisor of $\tilde{f} : \tilde{Y} \to \tilde{Z}$.

Lemma 5.2. *1) The linear system $\tilde{\Lambda}_2^+$ has no base point.*

2) The linear system $\tilde{\Lambda}_2^+(\sum e_i + \sum e'_i)$ has no base point.

3) The linear system $\tilde{\Lambda}_8^+(3\sum e_i + 3\sum e'_i)$ has no base point.

4) The linear systems $|-K_{Z'}|$ and $|-K_{\tilde{Z}}|$ have no base point.

Proof. Since we have $\tilde{\Lambda}_2^+ + 3\tilde{\Lambda}_2^+(\sum e_i + \sum e'_i) \subset \tilde{\Lambda}_8^+(3\sum e_i + 3\sum e'_i)$, the assertion 3) follows from the assertions 1) and 2).

Assume that we have the assertions 1) and 2). Then by $\tilde{\Lambda}_2^+(\sum e_i + \sum e'_i) \subset |-K_{\tilde{Z}}|$, we see that the linear system $|-K_{\tilde{Z}}|$ has no base point. Moreover, by this together with the Riemann–Roch theorem and the vanishing theorem, we obtain $h^0(\mathcal{O}_{\tilde{Z}}(-K_{\tilde{Z}})) = \chi(\mathcal{O}_{\tilde{Z}}) + K_{\tilde{Z}}^2 = 5$. Meanwhile, since $r' : Z' \to W$ is the blowing-up at two points w'_i's, we have $h^0(\mathcal{O}_{Z'}(-K_{Z'})) \geq h^0(\mathcal{O}_W(-K_W)) - 2 = 7$. Thus we obtain $h^0(\mathcal{O}_{Z'}(-K_{Z'})) - h^0(\mathcal{O}_{\tilde{Z}}(-K_{\tilde{Z}})) \geq 2$, which implies that neither of the two points w'_i's is a base point of $|-K_{Z'}|$. From this, we infer that $|-K_{Z'}|$ has no base point. So the assertion 4) also follows from the assertions 1) and 2).

Thus we only need to show the assertions 1) and 2). First, let us show the assertion 1). Let C_0 be a general member of $|\Gamma|$. Then since the divisor $2\Delta_0 + \frac{d+2}{2}(C_0 + \iota|_W(C_0)) \in \Lambda_2$ is stable under the action by G, and the divisor

$$(2\Delta_0 + \frac{d+2}{2}(C_0 + \iota|_W(C_0))) - 2(\Delta_0 + \frac{d+2}{2}\Gamma_1)$$

has no support at $\Delta_\infty \cap \Gamma_2$, the divisor $2\Delta_0 + \frac{d+2}{2}(C_0 + \iota|_W(C_0))$ is a member of Λ_2^+. Thus the base locus of Λ_2^+ is contained in Δ_0. Using a similar argument, we can show that $2\Delta_\infty + \frac{2-d}{2}(C_0 + \iota|_W(C_0)) \in \Lambda_2^+$, so that the base locus of Λ_2^+ is contained in Δ_∞. Thus since $\Delta_0 \cap \Delta_\infty = \emptyset$, the linear system $\tilde{\Lambda}_2^+$ has no base point. Hence we have the assertion 1).

Next, let us show the assertion 2). We shall show it by dividing our situation into several cases. In what follows, for each $i = 1, 2$, we denote by $\Gamma_{(i)}$ the unique member of $|\Gamma|$ passing w_i.

Case 1-1: the case where $d = 0$ holds, and the two points w_1 and w_2 lie neither on one and the same member of $|\Gamma|$ nor on that of $|\Delta_0|$. In this case, for each $i = 1, 2$, we denote by $\Delta_{(i)}$ the unique member of $|\Delta_0|$ passing w_i. This case is divided into two subcases: case 1-1-1 and case 1-1-2.

Case 1-1-1; the subcase of case 1-1 where $w'_1 \notin {r'}_*^{-1}(\Gamma_{(1)})$ and $w'_1 \notin {r'}_*^{-1}(\Delta_{(1)})$. In this case, take global coordinates (s'_1, ξ'_1) of $W \setminus (\Gamma_{(2)} \cup \Delta_{(2)}) \simeq \mathbb{A}^2$ such that $\Gamma_{(1)}$ is given by $s'_1 = 0$, $\Gamma_{(2)}$ by $s'_1 = \infty$, $\Delta_{(1)}$ by $\xi'_1 = 0$, and $\Delta_{(2)}$ by $\xi'_1 = \infty$. Then the involution $\iota|_W$ is given by $(s'_1, \xi'_1) \mapsto (1/s'_1, 1/\xi'_1)$, and the linear system Λ_2^+ is spanned by the five elements ${s'_1}^l {\xi'_1}^m + {s'_1}^{2-l}{\xi'_1}^{2-m}$ $(0 \leq l \leq 2,\ 0 \leq m \leq 2)$. Thus the linear system $\Lambda_2^+(\sum e_i + \sum e'_i)$ is spanned by the three elements

$$a_0(s'_1 + s'_1{\xi'_1}^2) + b_0(\xi'_1 + {s'_1}^2\xi'_1), \qquad s'_1\xi'_1, \qquad {s'_1}^2 + {\xi'_1}^2,$$

where $a_0 \neq 0$ and $b_0 \neq 0$ are certain non-zero complex numbers. From this, we infer that the set $\{w_1, w_2\}$ forms the base locus of $\Lambda_2^+(\sum e_i + \sum e'_i)$, and that any general member of this linear system is smooth. By this together with $\sum \mathit{\Gamma}_{(i)} + \sum \mathit{\Delta}_{(i)} \in \mathit{\Lambda}_2^+(\sum e_i + \sum e'_i)$, we see that the linear system $\tilde{\mathit{\Lambda}}_2^+(\sum e_i + \sum e'_i)$ has no base point.

Case 1-1-2: the subcase of case 1-1 where $w'_1 \in {r'}_*^{-1}(\mathit{\Gamma}_{(1)})$ or $w'_1 \in {r'}_*^{-1}(\mathit{\Delta}_{(1)})$. Since the proof is the same, we only give a proof for the case $w'_1 \in {r'}_*^{-1}(\mathit{\Gamma}_{(1)})$. Assume that $w'_1 \in {r'}_*^{-1}(\mathit{\Gamma}_{(1)})$. Since we have $C_0 + \iota|_W(C_0) + \sum \mathit{\Gamma}_{(i)} \in \mathit{\Lambda}_2^+(\sum e_i + \sum e'_i)$ for any general member C_0 of $|\mathit{\Delta}_0|$, the base locus of $\tilde{\mathit{\Lambda}}_2^+(\sum e_i + \sum e'_i)$ is contained in $\sum (r' \circ \tilde{r})_*^{-1}(\mathit{\Gamma}_{(i)})$. Meanwhile, since we have $C_1 + \iota|_W(C_1) \in \mathit{\Lambda}_2^+(\sum e_i + \sum e'_i)$ for any general member C_1 of $\mathit{\Lambda}_1(\sum e_i)$, the base locus of $\tilde{\mathit{\Lambda}}_2^+(\sum e_i + \sum e'_i)$ is contained in $\sum \tilde{r}_*^{-1}(e_i)$. Since we have $(\sum (r' \circ \tilde{r})_*^{-1}(\mathit{\Gamma}_{(i)})) \cap (\sum \tilde{r}_*^{-1}(e_i)) = \emptyset$, we see that the linear system $\tilde{\mathit{\Lambda}}_2^+(\sum e_i + \sum e'_i)$ has no base point.

Case 1-2: the case where $d = 0$ holds, and the two points w_1 and w_2 lie on one and the same member of $|\mathit{\Gamma}|$ or $|\mathit{\Delta}_0|$. In this case, for each $i = 1, 2$, we denote by $\mathit{\Delta}_{(i)}$ the unique member of $|\mathit{\Delta}_0|$ passing w_i. By exchanging $\mathit{\Delta}_0$ and $\mathit{\Gamma}$ if necessary, we may assume that the two points w_1 and w_2 lie on one and the member $\mathit{\Gamma}_0 \in |\mathit{\Gamma}|$. Moreover, by Remark 5, by exchanging $\mathit{\Gamma}_1$ and $\mathit{\Gamma}_2$ if necessary, we may assume that $\mathit{\Gamma}_0 = \mathit{\Gamma}_{(1)} = \mathit{\Gamma}_{(2)} = \mathit{\Gamma}_1$. Then this case is divided into two subcases: case 1-2-1 and case 1-2-2.

Case 1-2-1: the subcase of case 1-2 where $w'_1 \notin {r'}_*^{-1}(\mathit{\Delta}_{(1)})$. Note that we have assumed the condition ii) of Lemma 5.1 for our configuration, so that we have $w'_1, w'_2 \notin {r'}_*^{-1}(\mathit{\Gamma}_1)$. For any general member $C_0 \in |\mathit{\Delta}_0|$, we have $C_0 + \iota|_W(C_0) + 2\mathit{\Gamma}_1 \in \mathit{\Lambda}_2^+(\sum e_i + \sum e'_i)$. Thus the base locus of $\tilde{\mathit{\Lambda}}_2^+(\sum e_i + \sum e'_i)$ is contained in $2(r' \circ \tilde{r})_*^{-1}(\mathit{\Gamma}_1) + \sum \tilde{r}_*^{-1}(e_i)$. Meanwhile for any general C_1 $(\neq \mathit{\Delta}_{(1)} + \mathit{\Gamma}_1) \in \mathit{\Lambda}_1(e_1 + e'_1)$, we have $C_1 + \iota|_W(C_1) \in \mathit{\Lambda}_2^+(\sum e_i + \sum e'_i)$ and the irreducibility and smoothness at w_1 of C_1. Thus the base locus of $\tilde{\mathit{\Lambda}}_2^+(\sum e_i + \sum e'_i)$ is contained in $(r' \circ \tilde{r})_*^{-1}(C_1 + \iota|_W(C_1))$. Since $2(r' \circ \tilde{r})_*^{-1}(\mathit{\Gamma}_1) + \sum \tilde{r}_*^{-1}(e_i)$ and $(r' \circ \tilde{r})_*^{-1}(C_1 + \iota|_W(C_1))$ do not intersect each other, we see that the linear system $\tilde{\mathit{\Lambda}}_2^+(\sum e_i + \sum e'_i)$ has no base point.

Case 1-2-2: the subcase of case 1-2 where $w_1 \in {r'}_*^{-1}(\mathit{\Delta}_{(1)})$. By the same argument as one given in the proof for case 1-2-1, we see that the base locus of $\tilde{\mathit{\Lambda}}_2^+(\sum e_i + \sum e'_i)$ is contained in $2(r' \circ \tilde{r})_*^{-1}(\mathit{\Gamma}_1) + \sum \tilde{r}_*^{-1}(e_i)$. Meanwhile, for any general member $C_1 \in |\mathit{\Gamma}|$, we have $C_1 + \iota|_W(C_1) + \sum \mathit{\Delta}_{(i)} \in \mathit{\Lambda}_2^+(\sum e_i + \sum e'_i)$. Thus the base locus of $\tilde{\mathit{\Lambda}}_2^+(\sum e_i + \sum e'_i)$ is contained in $\sum (r' \circ \tilde{r})_*^{-1}(\mathit{\Delta}_{(i)})$. Since $2(r' \circ \tilde{r})_*^{-1}(\mathit{\Gamma}_1) + \sum \tilde{r}_*^{-1}(e_i)$ and $\sum (r' \circ \tilde{r})_*^{-1}(\mathit{\Delta}_{(i)})$ do not intersect each other, we see that the linear system $\tilde{\mathit{\Lambda}}_2^+(\sum e_i + \sum e'_i)$ has no base point.

Case 2-1: the case where $d = 2$ holds, and the two points w_1 and w_2 do not lie on one and the same member of $|\mathit{\Gamma}|$. Note that since we have assumed the condition i) in Lemma 5.1, we have $w_1 \notin \mathit{\Delta}_0$. Note also that for this case, or more generally for case $d = 2$, we have $\dim \mathit{\Lambda}_1^- = 1$, and any general member of this linear system is an irreducible curve stable under the action by G that passes two points $\Delta_\infty \cap \Gamma_1$ and $\Delta_\infty \cap \Gamma_2$. We denote by Δ_1 the unique member of Λ_1^- that passes the two points w_1 and w_2. Then this case is divided into three subcases: case 2-1-1, case 2-1-2, and case 2-1-3.

Case 2-1-1: the subcase of case 2-1 where $w_1' \notin {r'}_*^{-1}(\Gamma_{(1)})$ and $w_1' \notin {r'}_*^{-1}(\Delta_1)$. Since the divisor $\Delta_0 + 2\Gamma_{(1)}$ is the unique reducible member of $\Lambda_1(e_1 + e_1')$, and we have $h^0(\mathcal{O}_W(\Delta_0 + 2\Gamma)) = 4$, any general member of $\Lambda_1(e_1+e_1')$ is irreducible and non-singular. By this together with $\Delta_0+2\Gamma_{(1)} \in \Lambda_1(e_1 + e_1')$, we see that $\tilde{\Lambda}_1(e_1 + e_1')$ has no base point, and $(r' \circ \tilde{r})_*^{-1}(C_0 + \iota|_W(C_0)) \in \tilde{\Lambda}_2^+(\sum e_i + \sum e_i')$ for any general member $C_0 \in \Lambda_1(e_1 + e_1')$. Thus we see that $\tilde{\Lambda}_2^+(\sum e_i + \sum e_i')$ has no base point.

Case 2-1-2: the subcase of case 2-1 where $w_1' \in {r'}_*^{-1}(\Gamma_{(1)})$. Since we have $2\Delta_0 + \sum \Gamma_{(i)} + C_0 + \iota|_W(C_0) \in \Lambda_2^+(\sum e_i + \sum e_i')$ for any general member $C_0 \in |\Gamma|$, the base locus of $\tilde{\Lambda}_2^+(\sum e_i + \sum e_i')$ is contained in $2(r' \circ \tilde{r})^*(\Delta_0) + \sum(r' \circ \tilde{r})_*^{-1}(\Gamma_{(i)})$. By this together with $2\Delta_1 \in \Lambda_2^+(\sum e_i + \sum e_i')$, we see that the linear system $\tilde{\Lambda}_2^+(\sum e_i + \sum e_i')$ has no base point.

Case 2-1-3: the subcase of case 2-1 where $w_1' \in {r'}_*^{-1}(\Delta_1)$. Since we have $C_0 + \Delta_1 \in \Lambda_2^+(\sum e_i + \sum e_i')$ for any general member $C_0 \in \Lambda_1^-$, the base locus of $\tilde{\Lambda}_2^+(\sum e_i + \sum e_i')$ is contained in $(r' \circ \tilde{r})_*^{-1}(\Delta_1)$. By this together with $2(\Delta_0 + \sum \Gamma_{(i)}) \in \Lambda_2^+(\sum e_i + \sum e_i')$, we see that the linear system $\tilde{\Lambda}_2^+(\sum e_i + \sum e_i')$ has no base point.

Case 2-2: the case where $d = 2$ holds, and the two points w_1 and w_2 lie on one and the same member of $|\Gamma|$. By Remark 5, we may assume w_1, $w_2 \in \Gamma_1$. Note that we have assumed the conditions i) and ii) of Lemma 5.1 for our configuration, so that we have $w_1 \notin \Delta_0$ and $w_1' \notin {r'}_*^{-1}(\Gamma_1)$. Since we have $2\Delta_0 + 2\Gamma_1 + C_0 + \iota|_W(C_0) \in \Lambda_2^+(\sum e_i + \sum e_i')$ for any general member $C_0 \in |\Gamma|$, the base locus of $\tilde{\Lambda}_2^+(\sum e_i+\sum e_i')$ is contained in $2(r'\circ\tilde{r})^*(\Delta_0)+2(r'\circ \tilde{r})_*^{-1}(\Gamma_1) + \sum \tilde{r}_*^{-1}(e_i)$. Meanwhile since we have $h^0(\mathcal{O}_W(\Delta_0 + 2\Gamma)) = 4$, we have $C_1+\iota|_W(C_1) \in \Lambda_2^+(\sum e_i+\sum e_i')$ for an irreducible member $C_1 \in \Lambda_1(e_1+e_1')$. Since $2(r'\circ\tilde{r})^*(\Delta_0)+2(r'\circ\tilde{r})_*^{-1}(\Gamma_1)+\sum \tilde{r}_*^{-1}(e_i)$ and $(r'\circ\tilde{r})_*^{-1}(C_1+\iota|_W(C_1))$ do not intersect each other, we see that the linear system $\tilde{\Lambda}_2^+(\sum e_i + \sum e_i')$ has no base point.

Now that we have shown the absence of base points of $\tilde{\Lambda}_2^+(\sum e_i + \sum e_i')$ for all the eight cases 1-1-1, . . . , 2-2, we have the assertion 2). □

Let $\tilde{B}_0$ be a reduced member of $\tilde{\Lambda}_8^+(3\sum e_i + 3\sum e_i')$ that has at most negligible singularities, satisfies $\tilde{B}_0 \cap \sum \tilde{r}_*^{-1}(e_i) = \emptyset$, and passes no fixed point of the action by G on $\tilde{Z}$. Existence of such $\tilde{B}_0$ is ensured by Lemma 5.2. Let $\tilde{Y}$ be the canonical resolution of the double cover of $\tilde{Z}$ branched along $\tilde{B}_0 + \sum \tilde{r}_*^{-1}(e_i)$, and $\tilde{f} : \tilde{Y} \to \tilde{Z}$, the natural projection. We have $\tilde{f}^*(\tilde{r}_*^{-1}(e_i)) = 2E_i$ for a (-1)-curve E_i on $\tilde{Y}$. Let $p : \tilde{Y} \to Y$ be the blowing-down of E_1 and E_2. Then we have $|K_{\tilde{Y}}| = (\tilde{r} \circ \tilde{f})^*| - K_{Z'}| + 2\sum E_i$. Since $|-K_{Z'}|$ has no base point by Lemma 5.2, we see that Y is a minimal surface with $c_1^2 = 14$ and $\chi = 8$. By [15, Lemma 3.1] and

$$(r' \circ \tilde{r})(\tilde{B}_0)\Gamma_1 \equiv (r' \circ \tilde{r})(\tilde{B}_0)\Gamma_2 \equiv (r' \circ \tilde{r})(\tilde{B}_0)\Delta_0 \equiv 0 \mod 4,$$

there exists a unique free lifting to $\tilde{Y}$ of the action by G on W. Let $X = Y/G$ be the quotient of Y by the induced free action by G on Y. Then by [17, Theorem 1] or [7, (ii) in Theorem A], the surface X is a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$. Thus we have the following:

Proposition 13. *Let $W = \Sigma_d$ be the Hirzebruch surface of degree $d = 0$ or 2. Let $r' : Z' \to W$ be the blowing-up at two points w_1 and $w_2 = \iota|_W(w_1)$, where $\iota|_W$ is the involution of W given in Remark 1, and w_1, a point outside the fixed locus of $\iota|_W$. Let $\tilde{r} : \tilde{Z} \to Z'$ be the blowing-up at two points w'_1 and $w'_2 = \iota|_{Z'}(w'_1)$, where $\iota|_{Z'}$ is the induced involution of Z', and w'_1, a point infinitely near to w_1. Put $e_i = r'^{-1}(w_i)$ and $e'_i = \tilde{r}^{-1}(w'_i)$ for each $i = 1$, 2, and assume that the configuration of w_i's and w'_i's satisfies all the three conditions in Lemma 5.1. Let $\tilde{B}_0$ be a reduced member of $\tilde{\Lambda}_8^+(3\sum e_i + 3\sum e'_i)$ that has at most negligible singularities, satisfies $\tilde{B}_0 \cap \sum \tilde{r}_*^{-1}(e_i) = \emptyset$, and passes no fixed point of the induced action on $\tilde{Z}$ by $G = \langle \iota|_W \rangle$. Let $\tilde{Y}$ be the canonical resolution of the double cover of $\tilde{Z}$ branched along $\tilde{B}_0 + \sum \tilde{r}_*^{-1}(e_i)$, and $\tilde{f} : \tilde{Y} \to \tilde{Z}$, the natural projection. Let $p : \tilde{Y} \to Y$ be the blowing-down of two (-1)-curves $E_1 = \tilde{f}^{-1}(\tilde{r}_*^{-1}(e_1))$ and $E_2 = \tilde{f}^{-1}(\tilde{r}_*^{-1}(e_2))$. Then there exists a unique free lifting to $\tilde{Y}$ of the action by G on $\tilde{Z}$, and the quotient Y/G of Y by the induced free action is a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and* Tors $\simeq \mathbb{Z}/2$.

Our Theorem 2 together with Remark 1 and Lemma 5.1 says that all minimal surfaces with $c_1^2 = 2\chi - 1$, $\chi = 4$, and Tors $\simeq \mathbb{Z}/2$ are obtained by the procedure as in the proposition above. We use the following lemma in order to show the uniqueness of the deformation type.

Lemma 5.3. *Let $r' : Z' \to W$ and $\tilde{r} : \tilde{Z} \to Z'$, $w_i \in W$ and $w'_i \in Z'$ for $i = 1$, 2, and $e_i = r'^{-1}(w_i)$ and $e'_i = \tilde{r}^{-1}(w'_i)$ for $i = 1$, 2 be the morphisms, points, and divisors respectively as in Proposition 13. Then any general member $\tilde{B}_0$ of $\tilde{\Lambda}_8^+(3\sum e_i + 3\sum e'_i)$ is non-singular and reduced. Further $h^i(\mathcal{O}_{\tilde{Z}}(\tilde{B}_0)) = 0$ holds for any positive integer $i > 0$.*

Proof. The first assertion follows from 3) in Lemma 5.2. The second assertion follows from 3) and 4) in Lemma 5.2 and the vanishing theorem. □

Now let us show the uniqueness of the deformation type and the unirationality of the moduli space. For this purpose, we shall give another description of our surface X. Let $r' : Z' \to W$, $\tilde{r} : \tilde{Z} \to Z'$, w_i, w_i', e_i, and e_i' be as in Proposition 13. Let Γ_1 and Γ_2 be as in Remark 5. We take the minimal section Δ_0 and an irreducible member $\Delta_\infty \in |\Delta_0 + d\Gamma|$ satisfying $\Delta_0 \cap \Delta_\infty = \emptyset$ such that both are stable under the action by G. Note that if $d = 2$, such Δ_∞'s form a one-dimensional family.

The fixed locus of the action by G on $\tilde{Z}$ is a set of four isolated points: $\{(r' \circ \tilde{r})^{-1}(\Gamma_i \cap \Delta_j)\}_{i=1,2,\ j=0,\infty}$. Let $\bar{r} : \bar{Z} \to \tilde{Z}$ be the blowing-up at these four points. For $i = 1, 2$ and $j = 0, \infty$, we define the divisors J_{ij} on $\bar{Z}$ as follows:

if $d = 0$, then $J_{ij} = (r' \circ \tilde{r} \circ \bar{r})^{-1}(\Gamma_i \cap \Delta_j)$ for any $i = 1, 2$ and $j = 0, \infty$;

if $d = 2$, then $J_{10} = (r' \circ \tilde{r} \circ \bar{r})^{-1}(\Gamma_1 \cap \Delta_\infty)$, $J_{1\infty} = (r' \circ \tilde{r} \circ \bar{r})^{-1}(\Gamma_1 \cap \Delta_0)$, and $J_{2j} = (r' \circ \tilde{r} \circ \bar{r})^{-1}(\Gamma_2 \cap \Delta_j)$ for any $j = 0, \infty$.

Moreover for $i = 1, 2$ and $j = 0, \infty$, we define the divisors $\bar{\Gamma}_i$, $\bar{\Delta}_j$, $\bar{e}_i$, and $\bar{e}_i'$ on $\bar{Z}$ by $\bar{\Gamma}_i = (r' \circ \tilde{r} \circ \bar{r})_*^{-1}(\Gamma_i)$, $\bar{\Delta}_j = (r' \circ \tilde{r} \circ \bar{r})_*^{-1}(\Delta_j)$, $\bar{e}_i = \bar{r}^*(\tilde{r}_*^{-1}(e_i))$, and $\bar{e}_i' = \bar{r}^*(e_i')$. The four divisors J_{ij}'s form the set of all irreducible exceptional curves of $\bar{r} : \bar{Z} \to \tilde{Z}$. The action by G on $\tilde{Z}$ lifts to one on $\bar{Z}$. Note that $\sum J_{ij}$ gives the fixed locus of the induced action by G on $\bar{Z}$.

Now let $\bar{V} = \bar{Z}/G$ be the quotient of $\bar{Z}$ by the induced action by G, and $\wp : \bar{Z} \to \bar{V}$, the natural projection. Then $\bar{V}$ is smooth, and $\sum J_{ij}$ gives the ramification divisor of $\wp : \bar{Z} \to \bar{V}$. For $i = 1, 2$ and $j = 0, \infty$, we define the divisors $\bar{I}_{ij}$, $\bar{G}_i$, and $\bar{D}_j$ on $\bar{V}$ by $\bar{I}_{ij} = \wp(J_{ij})$, $\bar{G}_i = \wp(\bar{\Gamma}_i)$, and $\bar{D}_j = \wp(\bar{\Delta}_j)$. Moreover we define the divisors $\bar{\lambda}$ and $\bar{\lambda}'$ on $\bar{V}$ by $\bar{\lambda} = \wp(\bar{e}_1) = \wp(\bar{e}_2)$ and $\bar{\lambda}' = \wp(\bar{e}_1') = \wp(\bar{e}_2')$. The divisors $\bar{I}_{ij}$'s are non-singular rational curves with selfintersection $\bar{I}_{ij}^2 = -2$. Note that $\sum \bar{I}_{ij}$ gives the branch divisor of $\wp : \bar{Z} \to \bar{V}$.

Let $\bar{\nu} : \bar{V} \to \tilde{V}$ be the blowing-down of the (-1)-curve $\bar{\lambda}'$. For $i = 1, 2$ and $j = 0, \infty$, we define the divisor $\tilde{I}_{ij}$ on $\tilde{V}$ by $\tilde{I}_{ij} = \bar{\nu}(\bar{I}_{ij})$. Moreover we define the divisor $\tilde{\lambda}$ on $\tilde{V}$ by $\tilde{\lambda} = \bar{\nu}(\bar{\lambda})$. The divisors $\tilde{I}_{ij}$ and $\tilde{\lambda}$ are non-singular rational curves with $\tilde{I}_{ij}^2 = -2$ and $\tilde{\lambda}^2 = -1$ respectively.

Let $\tilde{\nu} : \tilde{V} \to V'$ be the blowing-down of the (-1)-curve $\tilde{\lambda}$. For $i = 1, 2$ and $j = 0, \infty$, we define the divisors I_{ij}', G_i', and D_j' on V' by $I_{ij}' = (\tilde{\nu} \circ \bar{\nu})_*(\bar{I}_{ij})$, $G_i' = (\tilde{\nu} \circ \bar{\nu})_*(\bar{G}_i)$, and $D_j' = (\tilde{\nu} \circ \bar{\nu})_*(\bar{D}_j)$. The divisors I_{ij}', G_i', D_0', and D_∞' are non-singular rational curves with ${I_{ij}'}^2 = -2$, ${G_i'}^2 = -1$, ${D_0'}^2 = -(d+2)/2$, and ${D_\infty'}^2 = (d-2)/2$ respectively.

Let $\nu' : V' \to V''$ be the blowing-down of the two (-1)-curves G_1' and G_2'. For $i = 1, 2$ and $j = 0, \infty$, we define the divisors I_{ij}'' and D_j'' on V'' by $I_{ij}'' = \nu'(I_{ij}')$ and $D_j'' = \nu'(D_j')$. The divisors I_{ij}'', D_0'', and D_∞'' are non-singular rational curves with ${I_{ij}''}^2 = -1$, ${D_0''}^2 = -(d+2)/2$ and ${D_\infty''}^2 = (d-2)/2$ respectively.

Let $\nu'' : V'' \to V'''$ be the blowing-down of the two (-1)-curves $I''_{1\infty}$ and $I''_{2\infty}$. For $i = 1, 2$ and $j = 0, \infty$, we define the divisors I'''_{i0} and D'''_j on V''' by $I'''_{i0} = \nu''(I''_{i0})$ and $D'''_j = \nu''(D''_j)$. The divisors I'''_{i0}, D'''_0, and D'''_∞ are non-singular rational curves with $I'''^2_{i0} = 0$, $D'''^2_0 = -1$, and $D'''^2_\infty = 1$. By $K^2_{V'''} = 8$, we see easily that V''' is isomorphic to the Hirzebruch surface Σ_1 of degree 1, where D'''_0 and $I'''_{10} \sim I'''_{20}$ give the minimal section and the fiber class respectively. We define the point $v'''_1 \in V'''$ by $v'''_1 = \nu''(I''_{1\infty})$. Note that we have $v'''_1 \notin D'''_0$ if $d = 0$, and $v'''_1 \in D'''_0$ if $d = 2$.

We put $\nu = (\nu'' \circ \nu' \circ \tilde{\nu} \circ \bar{\nu}) : \bar{V} \to V''' \simeq \Sigma_1$, and use the same symbols $I''_{i\infty}$, G'_i, and $\tilde{\lambda}$ for the total transforms to $\bar{V}$ of the divisors $I''_{i\infty} \subset V''$, $G'_i \subset V'$, and $\tilde{\lambda} \subset \tilde{V}$ respectively. Note that the morphism $\nu : \bar{V} \to V''' \simeq \Sigma_1$ is a blowing-up at six points some of which are infinitely near.

Proposition 14. *The linear system* $|-4K_{\bar{V}} + \tilde{\lambda} + \bar{\lambda}'| = |\nu^*(-4K_{V'''}) - 4\sum I''_{i\infty} - 4\sum G'_i - 3\tilde{\lambda} - 3\bar{\lambda}'|$ *has no base point. Let* $\bar{A}_0$ *be a reduced member of* $|-4K_{\bar{V}} + \tilde{\lambda} + \bar{\lambda}'|$ *that has at most negligible singularities, and satisfies* $\bar{A}_0 \cap \bar{\lambda} = \emptyset$ *and* $\bar{A}_0 \cap \sum \bar{I}_{ij} = \emptyset$. *Let* $\bar{X}$ *be the canonical resolution of the double cover of* $\bar{V}$ *branched along* $\bar{A}_0 + \bar{\lambda} + \sum \bar{I}_{ij}$, *and* $\bar{h} : \bar{X} \to \bar{V}$, *the natural projection. Let* $\bar{X} \to X$ *be the blowing-down of five* (-1)*-curves* $\bar{h}^{-1}(\bar{\lambda})$, $\bar{h}^{-1}(\bar{I}_{10})$, $\bar{h}^{-1}(\bar{I}_{20})$, $\bar{h}^{-1}(\bar{I}_{1\infty})$, *and* $\bar{h}^{-1}(\bar{I}_{2\infty})$. *Then* X *is a minimal surface with* $c_1^2 = 2\chi - 1$, $\chi = 4$, *and* Tors $\simeq \mathbb{Z}/2$. *Conversely, for any minimal surface* $X_{(1)}$ *with these invariants, there exist configuration of* w_1 *and* w'_1 *and a reduced member* $\bar{A}_0$ *as above such that the surface* X *constructed by this procedure is isomorphic to* $X_{(1)}$.

Proof. Since $K_{\bar{Z}} \sim \bar{\wp}^*(K_{\bar{V}}) + \sum J_{ij}$, $2J_{ij} = \bar{\wp}^*(\bar{I}_{ij})$, $\bar{\wp}^*(\bar{\lambda}) = \sum \bar{e}_i$, $\bar{\wp}^*(\bar{\lambda}') = \sum \bar{e}'_i$, and $\tilde{\lambda} \sim \bar{\lambda} + \bar{\lambda}'$, we have

$$\bar{\wp}^*(-4K_{\bar{V}} + \tilde{\lambda} + \bar{\lambda}') \sim \bar{r}^*((r' \circ \tilde{r})^*(-4K_W) - 3\sum e_i - 3\sum e'_i).$$

By this together with

$$\sum \bar{I}_{ij} = \nu^*(\sum I'''_{i0}) - 2\sum G'_i \sim 2(I'''_{10} - \sum G'_i), \tag{13}$$

we obtain

$$\bar{r}^*\tilde{\Lambda}^+_8(3\sum e_i + 3\sum e'_i) = \bar{\wp}^*|-4K_{\bar{V}} + \tilde{\lambda} + \bar{\lambda}'|,$$
$$\bar{r}^*\tilde{\Lambda}^-_8(3\sum e_i + 3\sum e'_i) = \bar{\wp}^*|-4K_{\bar{V}} + \tilde{\lambda} + \bar{\lambda}' - (I'''_{10} - \sum G'_i)| + \sum J_{ij}.$$

Thus the absence of base points of $|-4K_{\bar{V}} + \tilde{\lambda} + \bar{\lambda}'|$ follows from 3) of Lemma 5.2. Let $\bar{A}_0 \in |-4K_{\bar{V}} + \tilde{\lambda} + \bar{\lambda}'|$, $\bar{h} : \bar{X} \to \bar{V}$, and X be a reduced member, the induced morphism, and the obtained surface respectively as in the statement. Then $\tilde{B}_0 = \bar{r}(\bar{\wp}^*(\bar{A}_0))$ satisfies all the conditions given in Proposition 13. Thus for this $\tilde{B}_0$, we obtain morphisms $\tilde{f} : \tilde{Y} \to \tilde{Z}$ and $p : \tilde{Y} \to Y$ as in Proposition 13 and a minimal surface Y/G with $c_1^2 = 2\chi - 1$, $\chi = 4$, and Tors $\simeq \mathbb{Z}/2$. Note that the preimage by $\tilde{f}$ of the set $\{(r' \circ \tilde{r})^{-1}(\varGamma_i \cap \varDelta_j)\}_{i=1,2,\ j=0,\infty}$ is composed of exactly eight points. We denote by $\bar{Y} \to \tilde{Y}$ the blowing-up at these eight points. Then the morphism $\tilde{f} : \tilde{Y} \to \tilde{Z}$ induces a generically two-to-one morphism $\bar{f} : \bar{Y} \to \bar{Z}$. Moreover, the natural free action by $G = \langle \iota|_W \rangle$ on $\tilde{Y}$ lifts to one on $\bar{Y}$ that is compatible with the induced action by G on $\bar{Z}$. Thus $\bar{f} : \bar{Y} \to \bar{Z}$ induces a natural morphism $\bar{Y}/G \to \bar{V} = \bar{Z}/G$. Since the branch divisor of $\bar{Y}/G \to \bar{V} = \bar{Z}/G$ is $\bar{A}_0 + \bar{\lambda} + \sum \bar{I}_{ij}$, and $\bar{V}$ has no non-trivial torsion divisor, the morphism $\bar{Y}/G \to \bar{V}$ coincides with $\bar{h} : \bar{X} \to \bar{V}$. Thus by Proposition 13, $X \simeq Y/G$ is a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and Tors $\simeq \mathbb{Z}/2$. The final assertion follows from Theorem 2 and Lemma 5.1. □

Remark 6. The description above for our surfaces of the case $\chi = 4$ is almost the same as the description in Ciliberto-Mendes Lopes [8, Section 1] of the surfaces of the non-standard case for the non-birationality of bicanonical maps (see also [6, (b) in Theorem 3.1]). We emphasize here that in the present paper we have put neither the assumption of the non-birationality of bicanonical maps nor the assumption of the absence of pencils of curves of genus 2. By the description above, it is almost clear that our surfaces coincide with those found in [8]. To be precise, however, by showing the following proposition, we shall prove that they in fact coincide.

Proposition 15. *Any minimal surface X with $c_1^2 = 2\chi - 1$, $\chi = 4$, and* Tors $\simeq \mathbb{Z}/2$ *has non-birational bicanonical map. Moreover, it has no pencil of curves of genus* 2.

Proof. Let X be a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and Tors $\simeq \mathbb{Z}/2$, and assume that X has a pencil of curves of genus 2. This pencil is rational, since $c_1^2 \geq 2$. Let ϑ be a non-trivial 2-torsion divisor, and C, a general member of the pencil. Then by the same method as in the proof of (i), Lemma 1.2 of [7], we see that $|K_X + \vartheta| = |(\chi - 1)C| + D$ for a certain effective divisor D satisfying $K_X D = 1$, $CD = 2$, $D^2 = 3 - 2\chi$, and $\mathcal{O}_C(D) \not\simeq \mathcal{O}_C(K_C)$. Since $K_X D = 1$, the divisor D contains an irreducible curve D_1 satisfying $K_X D_1 = 1$, and all other components of D are (-2)-curves. Then since $-3 \leq D_1^2 + D_1(D - D_1) = D_1 D = K_X D_1 - (\chi - 1)CD_1 = 1 - (\chi - 1)CD_1$, we obtain $0 \leq CD_1 \leq 1$. Assume that $CD_1 = 1$. Then $D - D_1$ contains a (-2)-curve D_2 such that $CD_2 = 1$, and so $-2 \leq D_2^2 + D_2(D - D_2) = DD_2 \leq -(\chi - 1)CD_2$, hence a contradiction. Thus we have $CD_1 = 0$. In this case, if we let $D = D' + D''$ be the decomposition of D such that D' is the sum of all the irreducible components meeting C with positive intersection number, and D'' is the sum of all the irreducible components meeting C with intersection number 0, then we have $D_1 \subset D''$ and $CD' = 2$. Since D' has at most two irreducible components and any irreducible component of D' is a (-2)-curve, by using exactly the same argument as in the proof of Lemma 2.1 of [7], we obtain a contradiction. Hence the surface X has no pencil of curves of genus 2.

Thus in order to prove Proposition 15, it only remains to prove that X has non-birational bicanonical map. For this purpose, let us use the notation in Proposition 14. We have $H^0(\bar{h}_* \mathcal{O}_{\bar{X}}(2K_{\bar{X}})) = H^0(\mathcal{O}_{\bar{V}}(2(K_{\bar{V}} + \varrho))) \oplus H^0(\mathcal{O}_{\bar{V}}(2K_{\bar{V}} + \varrho))$ for a certain divisor ϱ with $\bar{A}_0 + \bar{\lambda} + \sum \bar{I}_{i,j} \sim 2\varrho$. We however have

$$2K_{\bar{V}} + \varrho \quad \sim \quad \nu^*(I_{10}''') - \sum G_i' + \tilde{\lambda},$$

hence $h^0(\mathcal{O}_{\bar{V}}(2K_{\bar{V}} + \varrho)) = 0$. This implies that the bicanonical map of $\bar{X}$ factors through the rational map of $\bar{V}$ associated to the linear system $|2(K_{\bar{V}} + \varrho)|$. Hence we have the assertion. □

To give a proof for Theorem 3, we also need the following lemma:

Lemma 5.4. *Let $\bar{V}$ be the smooth surface as in Proposition 14. Then for a member $\bar{A}_0 \in |-4K_{\bar{V}} + \tilde{\lambda} + \tilde{\lambda}'|$, the following equalities hold*:

$$h^0(\mathcal{O}_{\bar{V}}(\bar{A}_0)) = 29, \qquad h^1(\mathcal{O}_{\bar{V}}(\bar{A}_0)) = 0, \qquad h^2(\mathcal{O}_{\bar{V}}(\bar{A}_0)) = 0.$$

Proof. Since we have $\wp^*(\bar{A}_0) \in \bar{r}^* \tilde{\Lambda}_8^+(3\sum e_i + 3\sum e_i')$, the equality $h^i(\mathcal{O}_{\bar{V}}(\bar{A}_0)) = 0$ for any $i > 0$ follows from Lemma 5.3. From this together with the Riemann–Roch theorem, we infer $h^0(\mathcal{O}_{\bar{V}}(\bar{A}_0)) = 29$. □

As the first part of our proof for Theorem 3, we shall show the following:

Lemma 5.5. *Any two minimal algebraic surfaces with $c_1^2 = 2\chi - 1$, $\chi = 4$, and* Tors $\simeq \mathbb{Z}/2$ *are equivalent under deformation of complex structures. The coarse moduli space $\mathcal{M}$ for surfaces with these invariants is irreducible.*

Proof. Let $\mathcal{M}$ be the coarse moduli space for minimal surfaces X's with $c_1^2 = 2\chi - 1$, $\chi = 4$, and Tors $\simeq \mathbb{Z}/2$. In what follows, for a surface X with these invariants, we denote by $[X]$ the point in $\mathcal{M}$ corresponding to the isomorphism class of X. As a reference point of $\mathcal{M}$, let us fix a surface $X_{(1)}$ with these invariants for which $d = 0$ holds, the two points w_1 and w_2 lie neither on one and the same member of $|\Gamma|$ nor on that of $|\Delta_0|$, and $\bar{A}_0$ is smooth. Below, we shall give an irreducible component $\mathcal{M}_{(1)}$ of $\mathcal{M}$

containing $[X_{(1)}]$, and show that for any X with these invariants we have $[X] \in \mathcal{M}_{(1)}$ and X has the same deformation type as that of the reference surface $X_{(1)}$. We divide our situation into several cases according to d, the configuration of w_i's and w_i''s, and smoothness of $\bar{A}_0$ for our X. In what follows, ϵ and ϵ_0 will denote positive real numbers small enough. We shall replace these numbers with smaller ones without mentioning it explicitly.

Case 1-1: the case where $d = 0$ holds, the two points w_1 and w_2 lie neither on one and the same member of $|\Gamma|$ nor on that of $|\Delta_0|$. This case splits into two subcases: case 1-1-1 and case 1-1-2.

Case 1-1-1: the subcace of case 1-1 where $\bar{A}_0$ is smooth. From the point of view of description as in Proposition 14, this case corresponds to the case where $v_1''' \notin D_0'''$, $v_0' \notin \sum I_{ij}' + \sum G_i' + \sum D_j'$, and moreover $\bar{A}_0$ is smooth, where we put $v_1''' = \nu''(I_{1\infty}'')$ and $v_0' = \tilde{\nu}(\tilde{\lambda})$. Note that for all X's of this case $\tilde{V}$'s have one and the same isomorphism class. Let $\mathrm{pr}_{\tilde{V}\times\tilde{\lambda}} : \tilde{V} \times \tilde{\lambda} \to \tilde{\lambda} \simeq \mathbb{P}^1$ be the trivial family. Let $\mathrm{pr}_{\tilde{V}\times\tilde{\lambda},\tilde{V}} : \tilde{V} \times \tilde{\lambda} \to \tilde{V}$ be the first projection.

Then we can easily construct an analytic family $\mathrm{pr}_{\bar{\mathcal{V}}} : \bar{\mathcal{V}} \to \tilde{\lambda} \simeq \mathbb{P}^1$ together with a projection $\mathrm{pr}_{\bar{\mathcal{V}},\tilde{V}\times\tilde{\lambda}} : \bar{\mathcal{V}} \to \tilde{V} \times \tilde{\lambda}$ satisfying the following condition: for each $t \in \tilde{\lambda}$, the natural projection $\bar{V}_t = \mathrm{pr}_{\bar{\mathcal{V}}}{}^{-1}(t) \to \tilde{V} = \mathrm{pr}_{\tilde{V}\times\tilde{\lambda}}{}^{-1}(t)$ is the blowing-up at $t \in \tilde{\lambda} \subset \tilde{V}$ with exceptional divisor $\bar{\lambda}_t'$. Let us denote by $\bar{\lambda}_t$ and $\tilde{\lambda}_t$ the strict transform and the total transform by $\bar{V}_t = \mathrm{pr}_{\bar{\mathcal{V}}}{}^{-1}(t) \to \tilde{V} = \mathrm{pr}_{\tilde{V}\times\tilde{\lambda}}{}^{-1}(t)$ of the divisor $\tilde{\lambda}$, respectively. We denote by $\mathrm{pr}_{\bar{\mathcal{V}},\tilde{V}} : \bar{\mathcal{V}} \to \tilde{V}$ the composite of two projections $\mathrm{pr}_{\bar{\mathcal{V}},\tilde{V}\times\tilde{\lambda}}$ and $\mathrm{pr}_{\tilde{V}\times\tilde{\lambda},\tilde{V}}$.

Consider the divisor $-4K_{\bar{\mathcal{V}}} + \mathrm{pr}_{\bar{\mathcal{V}},\tilde{V}}{}^*(\tilde{\lambda}) + \cup_t \bar{\lambda}_t'$ on $\bar{\mathcal{V}}$. The restriction to $\bar{V}_t$ of this divisor is linearly equivalent to $-4K_{\bar{V}_t} + \tilde{\lambda}_t + \bar{\lambda}_t'$. Since we have $h^1(\mathcal{O}_{\bar{V}_t}(-4K_{\bar{V}_t} + \tilde{\lambda}_t + \bar{\lambda}_t')) = 0$ and $h^0(\mathcal{O}_{\bar{V}_t}(-4K_{\bar{V}_t} + \tilde{\lambda}_t + \bar{\lambda}_t')) = 29$ by Lemma 5.4, it follows that the direct image

$$\mathcal{F}_0 = \mathrm{pr}_{\bar{\mathcal{V}}*}\mathcal{O}_{\bar{\mathcal{V}}}(-4K_{\bar{\mathcal{V}}} + \mathrm{pr}_{\bar{\mathcal{V}},\tilde{V}}{}^*(\tilde{\lambda}) + \cup_t \bar{\lambda}_t')$$

is a locally free sheaf on $\tilde{\lambda} \simeq \mathbb{P}^1$ of rank 29. We denote by $\mathcal{F}_0^\vee$ the dual sheaf of $\mathcal{F}_0$ on $\tilde{\lambda}$.

Let $\mathrm{pr}_{\mathbb{P}} : \mathbb{P} = \mathbb{P}(\mathcal{F}_0^\vee) \to \tilde{\lambda}$ be the $\mathbb{P}^{28}$-bundle over $\tilde{\lambda}$ associated with $\mathcal{F}_0^\vee$. Then $\mathbb{P}$ is the projectivised total space of vector bundle $\mathcal{F}_0$. We consider the Cartesian diagram

$$\begin{array}{ccc} \bar{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P} & \longrightarrow & \mathbb{P} \\ \downarrow & & \downarrow \mathrm{pr}_{\mathbb{P}} \\ \bar{\mathcal{V}} & \xrightarrow{\mathrm{pr}_{\bar{\mathcal{V}}}} & \tilde{\lambda}, \end{array}$$

and denote by $\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P},\bar{\mathcal{V}}} : \bar{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P} \to \bar{\mathcal{V}}$, $\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P},\mathbb{P}} : \bar{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P} \to \mathbb{P}$, and $\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P}} :$

$\bar{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P} \to \tilde{\lambda}$ the first projection, the second projection, and the induced natural projection respectively.

Let $\mathcal{O}_{\mathbb{P}}(1)$ be the tautological bundle of $\mathrm{pr}_{\mathbb{P}} : \mathbb{P} = \mathbb{P}(\mathcal{F}_0^\vee) \to \tilde{\lambda}$. Then there exists a natural non-zero global section

$$\Psi_0 \in H^0(\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P},\mathbb{P}}{}^*\mathcal{O}_{\mathbb{P}}(1) \otimes \mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P},\bar{\mathcal{V}}}{}^*\mathcal{O}_{\bar{\mathcal{V}}}(-4K_{\bar{\mathcal{V}}} + \mathrm{pr}_{\bar{\mathcal{V}},\tilde{V}}{}^*(\tilde{\lambda}) + \cup_t \bar{\lambda}'_t))$$

on $\bar{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P}$ satisfying the following condition: for each open set $U \subset \tilde{\lambda}$ such that the restriction $\mathcal{F}_0|_U$ is trivial, the restriction $\Psi_0|_{\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P}}{}^{-1}(U)}$ of Ψ_0 to $\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P}}{}^{-1}(U)$ is given by $\Psi_0|_{\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P}}{}^{-1}(U)} = \sum_{i=1}^{29} a_i\psi_i$, where $\{\psi_1, \ldots, \psi_{29}\}$ and $\{a_1, \ldots, a_{29}\}$ are a base of $H^0(\mathcal{F}_0|_U)$ and its dual base respectively (note here that we have the natural isomorphism $\mathrm{pr}_{\mathbb{P}*}\mathcal{O}_{\mathbb{P}}(1) \simeq \mathcal{F}_0^\vee$). We denote by $\bar{\alpha}_0 = (\Psi_0)$ the divisor on $\bar{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P}$ defined by the section Ψ_0.

For each $u \in \mathbb{P}$, we put $t(u) = \mathrm{pr}_{\mathbb{P}}(u) \in \tilde{\lambda}$. Then we have the natural isomorphism $\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P},\mathbb{P}}{}^{-1}(u) \simeq \bar{V}_{t(u)}$. Moreover, via this identification, the restriction $\bar{A}_{0\,u} = \bar{\alpha}_0|_{\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P},\mathbb{P}}{}^{-1}(u)} \in |-4K_{\bar{V}_{t(u)}} + \tilde{\lambda}_{t(u)} + \bar{\lambda}'_{t(u)}|$ is a divisor on $\bar{V}_{t(u)}$ given by the local defining function $\sum_{i=1}^{29} a_i(u)\psi_i|_{\bar{V}_{t(u)}}$. Let $\mathbb{P}_0 \subset \mathbb{P}$ be the set of all u's such that $\bar{A}_{0\,u}$ is a reduced smooth divisor satisfying $\bar{A}_{0\,u} \cap (\bar{\lambda}_{t(u)} + \sum_{i=1,2,\ j=0,\infty} \bar{I}_{ij\,t(u)}) = \emptyset$, where $\bar{I}_{ij\,t}$ $(t \in \tilde{\lambda})$ is the restriction to $\bar{V}_t$ of the divisor $\mathrm{pr}_{\bar{\mathcal{V}},\tilde{V}}{}^*(\tilde{I}_{ij})$. Then $\mathbb{P}_0$ is a non-empty Zariski open subset of $\mathbb{P} = \mathbb{P}(\mathcal{F}_0^\vee)$. Since $\mathbb{P} \to \tilde{\lambda}$ is a $\mathbb{P}^{28}$-bundle over a non-singular rational curve $\tilde{\lambda} \simeq \mathbb{P}^1$, there exists a covering $\{U_\mu^\vee\}_\mu$ of $\mathbb{P}$ by a finite number of Zariski open subsets $U_\mu^\vee$'s satisfying the following condition: for any μ, the restriction $\mathcal{O}_{\mathbb{P}}(1)|_{U_\mu^\vee}$ is trivial, and $U_\mu^\vee$ is isomorphic to the 29-dimensional linear space $\mathbb{A}^{29}$. We fix one such cover $\{U_\mu^\vee\}_\mu$, and put $U_\mu^0 = U_\mu^\vee \cap \mathbb{P}_0$ for each μ.

Let $\mathrm{pr}_{\bar{\mathcal{V}},V'''} : \bar{\mathcal{V}} \to V'''$ and $\mathrm{pr}_{\bar{\mathcal{V}},V'} : \bar{\mathcal{V}} \to V'$ be the natural projections $\nu'' \circ \nu' \circ \tilde{\nu} \circ \mathrm{pr}_{\bar{\mathcal{V}},\tilde{V}}$ and $\tilde{\nu} \circ \mathrm{pr}_{\bar{\mathcal{V}},\tilde{V}}$ respectively. Then since the restriction to $\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P},\mathbb{P}}{}^{-1}(U_\mu^0)$ of $\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P},\mathbb{P}}{}^*\mathcal{O}_{\mathbb{P}}(1)$ is trivial, it follows from (13) that the restriction to $\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P},\mathbb{P}}{}^{-1}(U_\mu^0)$ of the divisor

$$\bar{\alpha}_0 + \mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P},\bar{\mathcal{V}}}{}^*(\cup_t \bar{\lambda}_t + \mathrm{pr}_{\bar{\mathcal{V}},\tilde{V}}{}^*(\sum \tilde{I}_{ij})) \tag{14}$$

is linearly equivalent to twice the restriction to $\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P},\mathbb{P}}{}^{-1}(U_\mu^0)$ of the divisor

$$\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P},\bar{\mathcal{V}}}{}^*(-2K_{\bar{\mathcal{V}}} + \mathrm{pr}_{\bar{\mathcal{V}},\tilde{V}}{}^*(\tilde{\lambda}) + \mathrm{pr}_{\bar{\mathcal{V}},V'''}{}^*(I'''_{10}) - \mathrm{pr}_{\bar{\mathcal{V}},V'}{}^*(\sum G'_i)).$$

So let $\bar{\mathcal{X}}^{(\mu)} \to \mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P},\mathbb{P}}{}^{-1}(U_\mu^0)$ be the double cover branched along the restriction to $\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P},\mathbb{P}}{}^{-1}(U_\mu^0)$ of the divisor (14). Composing this morphism with the projection $\mathrm{pr}_{\bar{\mathcal{V}}\times_{\tilde{\lambda}}\mathbb{P},\mathbb{P}} : \bar{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P} \to \mathbb{P}$, we obtain an analytic family

$\mathrm{pr}_{\bar{\mathcal{X}}^{(\mu)}} : \bar{\mathcal{X}}^{(\mu)} \to U_\mu^0$. For each $u \in U_\mu^0$, we put $\bar{X}_u^{(\mu)} = {\mathrm{pr}_{\bar{\mathcal{X}}^{(\mu)}}}^{-1}(u)$. The inverse image by $\bar{\mathcal{X}}^{(\mu)} \to {\mathrm{pr}_{\bar{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P}, \mathbb{P}}}^{-1}(U_\mu^0)$ of ${\mathrm{pr}_{\bar{\mathcal{V}} \times_{\tilde{\lambda}} \mathbb{P}, \bar{\mathcal{V}}}}^*(\cup_t \bar{\lambda}_t + {\mathrm{pr}_{\bar{\mathcal{V}}, \tilde{V}}}^*(\sum \tilde{I}_{ij}))$ gives a family over U_μ^0 whose fiber over each $u \in U_\mu^0$ is a sum of five disjoint (-1)-curves on $\bar{X}_u^{(\mu)}$. Blowing down this family of disjoint five (-1)-curves relatively to $\mathrm{pr}_{\bar{\mathcal{X}}^{(\mu)}} : \bar{\mathcal{X}}^{(\mu)} \to U_\mu^0$, we obtain an analytic family $\mathrm{pr}_{\mathcal{X}^{(\mu)}} : \mathcal{X}^{(\mu)} \to U_\mu^0$. For each $u \in U_\mu^0$, we put $X_u^{(\mu)} = {\mathrm{pr}_{\mathcal{X}^{(\mu)}}}^{-1}(u)$. Then by Proposition 14, for each $u \in U_\mu^0$, the fiber $X_u^{(\mu)}$ is a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and $\mathrm{Tors} \simeq \mathbb{Z}/2$.

Let $U_\mu^0 \to \mathcal{M}$ be the natural morphism induced from the family $\mathrm{pr}_{\mathcal{X}^{(\mu)}} : \mathcal{X}^{(\mu)} \to U_\mu^0$, i.e., the morphism given by $u \mapsto [X_u^{(\mu)}]$, where $[X_u^{(\mu)}]$ is a point in $\mathcal{M}$ corresponding to the isomorphism class of the fiber $X_u^{(\mu)}$. The two morphisms $U_{\mu_1}^0 \to \mathcal{M}$ and $U_{\mu_2}^0 \to \mathcal{M}$ coincide on $U_{\mu_1}^0 \cap U_{\mu_2}^0$. Thus gluing $U_\mu^0 \to \mathcal{M}$'s, we obtain a morphism $\mathbb{P}_0 \to \mathcal{M}$ given locally by $u \mapsto [X_u^{(\mu)}]$. Since $\mathbb{P}_0$ is irreducible, the image of this $\mathbb{P}_0 \to \mathcal{M}$ lies on an irreducible component of the moduli space $\mathcal{M}$. We fix one such irreducible component and denote it by $\mathcal{M}_{(1)}$.

Now let X be a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and $\mathrm{Tors} \simeq \mathbb{Z}/2$ for which $d = 0$ holds, the two points w_1 and w_2 lie neither on one and the same member of $|\mathit{\Delta}_0|$ nor on that of $|\mathit{\Gamma}|$, and $\bar{A}_0$ is smooth. Then by Proposition 14 and the construction of $\mathrm{pr}_{\mathcal{X}^{(\mu)}} : \mathcal{X}^{(\mu)} \to U_\mu^0$ above, there exist a μ and a $u \in U_\mu^0$ such that $X \simeq X_u^{(\mu)}$ holds. Since $\mathbb{P}_0$ is connected, we infer that X has the same deformation type as that of the reference surface $X_{(1)}$. Moreover, we infer that the corresponding point $[X]$ lies on the irreducible component $\mathcal{M}_{(1)}$.

Case 1-1-2: the subcase of case 1-1 where $\bar{A}_0$ is singular. Let X be a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and $\mathrm{Tors} \simeq \mathbb{Z}/2$ of this case. In this case, $\bar{A}_0$ has at most negligible singularities, and by Proposition 14, the linear system $|\bar{A}_0|$ has no base point. Thus by the same method as in [9, Proof of Theorem 4], we obtain an analytic family $\mathrm{pr}_{\mathcal{X}} : \mathcal{X} \to N = \{u \in \mathbb{C} : |u| < \epsilon\}$ of minimal surfaces with $c_1^2 = 2\chi - 1$, $\chi = 4$, and $\mathrm{Tors} \simeq \mathbb{Z}/2$ such that $X_u = {\mathrm{pr}_{\mathcal{X}}}^{-1}(u)$ is of case 1-1-1 for each $u \neq 0 \in N$, and $X_0 = {\mathrm{pr}_{\mathcal{X}}}^{-1}(0) \simeq X$. From this together with the results for case 1-1-1, we infer that X has the same deformation type as that of the reference surface $X_{(1)}$, and that the point $[X]$ lies on the irreducible component $\mathcal{M}_{(1)}$ in the proof for case 1-1-1.

Case 1-2: the case where $d = 0$ holds, and the two points w_1 and w_2 lie on one and the same member of $|\mathit{\Gamma}|$ or $|\mathit{\Delta}_0|$. In this case, by Remark 5, we may assume that the two points w_1 and w_2 lie on the member $\mathit{\Gamma}_1 \in |\mathit{\Gamma}|$. Then by Lemma 5.1, we have $w_1' \notin {r'}_*^{-1}(\mathit{\Gamma}_1)$. This case is divided into two subcases: case 1-2-1 and case 1-2-2.

Case 1-2-1: the subcase of case 1-2 where $\bar{A}_0$ is smooth. From the point of view of description as in Proposition 14, this case corresponds to the case where $v_1''' \notin D_0'''$, $v_0' \in G_1' \backslash (\sum_j I_{1j}')$, and moreover $\bar{A}_0$ is smooth, where we put $v_1''' = \nu''(I_{1\infty}'')$ and $v_0' = \tilde{\nu}(\tilde{\lambda})$. Let X be a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and Tors $\simeq \mathbb{Z}/2$ of this case. Let ϵ be a positive real number small enough. We put $N = \{t \in \mathbb{C} : |t| < \epsilon\}$, and denote by $\mathrm{pr}_{V' \times N} : V' \times N \to N$ the trivial family. Let $\mathrm{pr}_{V' \times N, V'} : V' \times N \to V'$ be the first projection.

Then we can easily construct anatitic families $\mathrm{pr}_{\tilde{\mathcal{V}}} : \tilde{\mathcal{V}} \to N$, $\mathrm{pr}_{\bar{\mathcal{V}}} : \bar{\mathcal{V}} \to N$ together with projections $\mathrm{pr}_{\tilde{\mathcal{V}}, V' \times N} : \tilde{\mathcal{V}} \to V' \times N$, $\mathrm{pr}_{\bar{\mathcal{V}}, \tilde{\mathcal{V}}} : \bar{\mathcal{V}} \to \tilde{\mathcal{V}}$ satisfying the following conditions: for each $t \in N$, the projection $\tilde{V}_t = {\mathrm{pr}_{\tilde{\mathcal{V}}}}^{-1}(t) \to V' = {\mathrm{pr}_{V' \times N}}^{-1}(t)$ is the blowing-up at $v'(t)$ with exceptional divisor $\tilde{\lambda}_t$, where $v' : N \to V' \times N$ is a holomorphic section of the analytic family $\mathrm{pr}_{V' \times N} : V' \times N \to N$; for each $t \in N$, the projection $\bar{V}_t = {\mathrm{pr}_{\bar{\mathcal{V}}}}^{-1}(t) \to \tilde{V}_t = {\mathrm{pr}_{\tilde{\mathcal{V}}}}^{-1}(t)$ is the blowing-up at $\tilde{v}(t)$ with exceptional divisor $\bar{\lambda}_t'$, where $\tilde{v} : N \to \tilde{\mathcal{V}}$ is a holomorphic section of the analytic family $\mathrm{pr}_{\tilde{\mathcal{V}}} : \tilde{\mathcal{V}} \to N$; $v'(0) = v_0' (= \tilde{\nu}(\tilde{\lambda}))$ holds, and $\mathrm{pr}_{V' \times N, V'}(v'(t)) \in \sum G_i' + \sum D_j' + \sum I_{ij}'$ if and only if $t = 0$; $\tilde{v}(0) = \tilde{v}_0 (= \bar{\nu}(\bar{\lambda}'))$ holds, and $\tilde{v}(t) \in \tilde{\lambda}_t$ for any $t \in N$. Note that from the conditions above, we have in particular $\tilde{V}_0 = \tilde{V}$ and $\bar{V}_0 = \bar{V}$. Let us denote by $\bar{\lambda}_t$ the strict transform of $\tilde{\lambda}_t$ by $\bar{V}_t = {\mathrm{pr}_{\bar{\mathcal{V}}}}^{-1}(t) \to \tilde{V}_t = {\mathrm{pr}_{\tilde{\mathcal{V}}}}^{-1}(t)$.

Consider the divisor $-4K_{\bar{\mathcal{V}}} + {\mathrm{pr}_{\bar{\mathcal{V}}, \tilde{\mathcal{V}}}}^*(\cup_t \tilde{\lambda}_t) + \cup_t \bar{\lambda}_t'$ on $\bar{\mathcal{V}}$. The restriction to $\bar{V}_0 = \bar{V}$ of this divisor is linearly equivalent to $-4K_{\bar{V}} + \tilde{\lambda} + \bar{\lambda}'$. Since we have $h^1(\mathcal{O}_{\bar{V}}(-4K_{\bar{V}} + \tilde{\lambda} + \bar{\lambda}')) = 0$ by Lemma 5.4, there exists a non-zero global section $\Psi \in H^0(\mathcal{O}_{\bar{\mathcal{V}}}(-4K_{\bar{\mathcal{V}}} + {\mathrm{pr}_{\bar{\mathcal{V}}, \tilde{\mathcal{V}}}}^*(\cup_t \tilde{\lambda}_t) + \cup_t \bar{\lambda}_t'))$ on $\bar{\mathcal{V}}$ satisfying the following conditions: the restriction $(\Psi)|_{\bar{V}_0}$ to $\bar{V}_0$ coincides with $\bar{A}_0$, where (Ψ) denotes the divisor on $\bar{\mathcal{V}}$ defined by the global section Ψ; for any $t \in N$, the restriction $(\Psi)|_{\bar{V}_t}$ to $\bar{V}_t$ is a reduced non-singular divisor on $\bar{V}_t$; the divisor (Ψ) does not intersects $\cup_t \bar{\lambda}_t + {\mathrm{pr}_{\bar{\mathcal{V}}, V'}}^*(\sum I_{ij}')$, where $\mathrm{pr}_{\bar{\mathcal{V}}, V'} : \bar{\mathcal{V}} \to V'$ is the composite of three projections $\mathrm{pr}_{\bar{\mathcal{V}}, \tilde{\mathcal{V}}}$, $\mathrm{pr}_{\tilde{\mathcal{V}}, V' \times N}$, and $\mathrm{pr}_{V' \times N, V'}$.

Let $\bar{\mathcal{X}} \to \bar{\mathcal{V}}$ be the double cover branched along $(\Psi) + \cup_t \bar{\lambda}_t + {\mathrm{pr}_{\bar{\mathcal{V}}, V'}}^*(\sum I_{ij}')$. Then composing this morphism with the projection $\mathrm{pr}_{\bar{\mathcal{V}}} : \bar{\mathcal{V}} \to N$, we obtain an analytic family $\mathrm{pr}_{\bar{\mathcal{X}}} : \bar{\mathcal{X}} \to N$. For each $t \in N$, we put $\bar{X}_t = {\mathrm{pr}_{\bar{\mathcal{X}}}}^{-1}(t)$. The inverse image by $\bar{\mathcal{X}} \to \bar{\mathcal{V}}$ of $\cup_t \bar{\lambda}_t + {\mathrm{pr}_{\bar{\mathcal{V}}, V'}}^*(\sum I_{ij}')$ gives a family over N whose fiber over each $t \in N$ is a disjoint union of five (-1)-curves on $\bar{X}_t$. Blowing down this family of five (-1)-curves relatively to $\mathrm{pr}_{\bar{\mathcal{X}}} : \bar{\mathcal{X}} \to N$, we obtain an analytic family $\mathrm{pr}_{\mathcal{X}} : \mathcal{X} \to N$. Then by the construction of $\mathrm{pr}_{\mathcal{X}} : \mathcal{X} \to N$ above, we have $X_0 = {\mathrm{pr}_{\mathcal{X}}}^{-1}(0) \simeq X$, and for each $t \neq 0 \in N$, the fiber $X_t = {\mathrm{pr}_{\mathcal{X}}}^{-1}(t)$ is a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and Tors $\simeq \mathbb{Z}/2$ of case 1-1-1. From this together with the results for case 1-1-1, we infer that X has the same deformation type as that of the reference surface $X_{(1)}$, and that the point $[X]$ lies on the irreducible component $\mathcal{M}_{(1)}$ in the proof for case 1-1-1.

Case 1-2-2: the subcase of case 1-2 where $\bar{A}_0$ is singular. Let X be a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and Tors $\simeq \mathbb{Z}/2$ of this case. Then using the same argument as in case 1-1-2, we infer from the results for case 1-2-1 that X has the same deformation type as that of the reference surface $X_{(1)}$, and that the point $[X]$ lies on the irreducible component $\mathcal{M}_{(1)}$ in the proof for case 1-1-1.

Case 2-1: the case where $d = 2$ holds, and the two points w_1 and w_2 do not lie on one and the same member of $|\Gamma|$. Note that in this case we have $v_0' \notin D_0'$ by Lemma 5.1, where we put $v_0' = \tilde{\nu}(\tilde{\lambda})$. This case splits into two subcases: case 2-1-1 and case 2-1-2.

Case 2-1-1: the subcase of case 2-1 where $\bar{A}_0$ is smooth. From the point of view of description as in Proposition 14, this case corresponds to the case where $v_1''' \in D_0'''$, $v_0' \notin D_0' + \sum G_i' + \sum I_{ij}'$, and moreover $\bar{A}_0$ is smooth, where we put $v_1''' = \nu''(I_{1\infty}'')$ and $v_0' = \tilde{\nu}(\tilde{\lambda})$. Let X be a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and $\mathbb{Z}/2$ of this case. Let ϵ be a positive real number small enough. We put $N = \{t \in \mathbb{C} : |t| < \epsilon\}$, and denote by $\mathrm{pr}_{V''' \times N} : V''' \times N \to N$ the trivial family. Let $\mathrm{pr}_{V''' \times N, V'''} : V''' \times N \to V'''$ be the first projection. Let us take a holomorphic section $v_{(1)}''' : N \to V''' \times N$ satisfying the following conditions: $\mathrm{pr}_{V''' \times N, V'''}(v_{(1)}'''(0)) = v_1'''$ $(= \nu''(I_{1\infty}''))$ holds; $\mathrm{pr}_{V''' \times N, V'''}(v_{(1)}'''(t)) \in I_{10}'''$ for any $t \in N$; $\mathrm{pr}_{V''' \times N, V'''}(v_{(1)}'''(t)) = v_1'''$ if and only if $t = 0$.

Recall that the configuration corresponding to Case 1-1-1 was $v_1''' \notin D_0'''$, $v_0' \notin \sum I_{ij}' + \sum G_i' + \sum D_j'$. Thus using the holomorphic section $v_{(1)}''' : N \to V''' \times N$ above, and by the same arguement as in the proof for case 1-2-1, we obtain an analytic family $\mathrm{pr}_{\mathcal{X}} : \mathcal{X} \to N = \{t \in \mathbb{C} : |t| < \epsilon\}$ satisfying the following conditions: $X_0 = {\mathrm{pr}_{\mathcal{X}}}^{-1}(0) \simeq X$; and for each $t \neq 0 \in N$, the fiber $X_t = {\mathrm{pr}_{\mathcal{X}}}^{-1}(t)$ is a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and Tors $\simeq \mathbb{Z}/2$ of case 1-1-1. From this together with the results for case 1-1-1, we infer that X has the same deformation type as that of the reference surface $X_{(1)}$, and that the point $[X]$ lies on the irreducible component $\mathcal{M}_{(1)}$ in the proof for case 1-1-1.

Case 2-1-2: the subcase of case 2-1 where $\bar{A}_0$ is singular. Let X be a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and Tors $\simeq \mathbb{Z}/2$ of this case. Then using the same argument as in case 1-1-2, we infer from the results for case 2-1-1 that X has the same deformation type as that of the reference surface $X_{(1)}$, and that the point $[X]$ lies on the irreducible component $\mathcal{M}_{(1)}$ in the proof for case 1-1-1.

Case 2-2: the case where $d = 2$ holds, the two points w_1 and w_2 lie on one and the same member of $|\varGamma|$. Note that in this case we have $v_0' \notin D_0'$ by Lemma 5.1, where we put $v_0' = \tilde{\nu}(\tilde{\lambda})$. In this case, by Remark 5, we may assume that the two points w_1 and w_2 lie on the member $\varGamma_1 \in |\varGamma|$. Then by Lemma 5.1, we have $w_1' \notin {r'}_*^{-1}(\varGamma_1)$. This case is divided into two subcases: case 2-2-1 and case 2-2-2.

Case 2-2-1: the subcase of case 2-2 where $\bar{A}_0$ is smooth. From the point of view of description as in Proposition 14, this case corresponds to the case where $v_1''' \in D_0'''$, $v_0' \in G_1' \setminus (\sum_j I_{1j}')$, and moreover $\bar{A}_0$ is smooth, where we put $v_1''' = \nu''(I_{1\infty}'')$ and $v_0' = \tilde{\nu}(\tilde{\lambda})$. Note that by Lemma 5.1, we have $\tilde{v}_0 \notin \tilde{\nu}_*^{-1}(G_1')$, where we put $\tilde{v}_0 = \bar{\nu}(\bar{\lambda}')$. Let X be a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and Tors $\simeq \mathbb{Z}/2$ of this case. Let ϵ be a positive real number small enough. We put $N = \{t \in \mathbb{C} : |t| < \epsilon\}$, and denote by $\mathrm{pr}_{V' \times N} : V' \times N \to N$ the trivial family. Let $\mathrm{pr}_{V' \times N, V'} : V' \times N \to V'$ be the first projection. Let us take a holomorphic section $v' : N \to V' \times N$ satisfying the following conditions: $\mathrm{pr}_{V' \times N, V'}(v'(0)) = v_0' (= \tilde{\nu}(\tilde{\lambda}))$ holds; $\mathrm{pr}_{V' \times N, V'}(v'(t)) \in D_0' + \sum G_i' + \sum I_{ij}'$ if and only if $t = 0$.

Recall that the configuration corresponding to case 2-1-1 was $v_1''' \in D_0'''$, $v_0' \notin D_0' + \sum G_i' + \sum I_{ij}'$. Thus using the holomorphic section $v' : N \to V' \times N$ above, and by the same arguement as in the proof for case 1-2-1, we obtain an analytic family $\mathrm{pr}_{\mathcal{X}} : \mathcal{X} \to N = \{t \in \mathbb{C} : |t| < \epsilon\}$ satisfying the following conditions: $X_0 = {\mathrm{pr}_{\mathcal{X}}}^{-1}(0) \simeq X$; for each $t \neq 0 \in N$, the fiber $X_t = {\mathrm{pr}_{\mathcal{X}}}^{-1}(t)$ is a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and Tors $\simeq \mathbb{Z}/2$ of case 2-1-1. From this together with the results for case 2-1-1, we infer that X has the same deformation type as that of the reference surface $X_{(1)}$, and that the point $[X]$ lies on the irreducible component $\mathcal{M}_{(1)}$ in the proof for case 1-1-1.

Case 2-2-2: the subcase of case 2-2 where $\bar{A}_0$ is singular. Let X be a minimal surface with $c_1^2 = 2\chi - 1$, $\chi = 4$, and Tors $\simeq \mathbb{Z}/2$ of this case. Then using the same argument as in case 1-1-2, we infer from the results for case 2-2-1 that X has the same deformation type as that of the reference surface $X_{(1)}$, and that the point $[X]$ lies on the irreducible component $\mathcal{M}_{(1)}$ in the proof for case 1-1-1.

Now that we have the results for all the eight cases 1-1-1, …, 2-2-2, we have the assertion. □

Note that from the proof above, we see that the morphism $\mathbb{P}_0 \to \mathcal{M}$ in the proof for case 1-1-1 is dominant.

Now let us prove Theorem 3.

Proof of Theorem 3

Let $\mathbb{P}_0 \to \mathcal{M}$ be the morphism $u \mapsto [X_u^{(\mu)}]$ given in the proof (for case 1-1-1) of Lemma 5.5. Recall that we have $X_u^{(\mu_1)} \simeq X_u^{(\mu_2)}$ if $u \in U_{\mu_1}^0 \cap U_{\mu_2}^0$. So in what follows, we abbreviate $X_u^{(\mu)}$ to X_u. Since $\mathbb{P}_0 \to \mathcal{M}$ is a dominant morphism from the 29-dimensional variety $\mathbb{P}_0$, we only need to show that for each $u_0 \in \mathbb{P}_0$, there exist at most eight $u \in \mathbb{P}_0$'s satisfying $[X_u] = [X_{u_0}]$. Recall also that for all X's of case 1-1-1 in the proof of Lemma 5.5, $\tilde{V}$'s have one and the same isomorphism class. In what follows, we assume that W, Z', $\tilde{V}$, and the configuration of w_i's are those for X's of case 1-1-1.

Let $\mathrm{Aut}(W)$ be the group of analytic automorphisms of $W \simeq \Sigma_0$, and $\iota|_W$, the involution of W as in Proposition 13. Let $\mathrm{Aut}(W, \iota|_W, \{w_i\})$ be the subgroup of $\mathrm{Aut}(W)$ consisting of all $\sigma \in \mathrm{Aut}(W)$'s satisfying $(\iota|_W)\circ\sigma = \sigma\circ(\iota|_W)$ and $\sigma(\{w_i\}_{i=1,2}) = \{w_i\}_{i=1,2}$. Since $\mathrm{Aut}(W, \iota|_W, \{w_i\})$ acts naturally on the sets $\{w_i\}_{i=1,2}$ and $\{\Delta_0, \Delta_\infty, \Gamma_1, \Gamma_2\}$, we have corresponding group homomorphisms $\mathrm{Aut}(W, \iota|_W, \{w_i\}) \to \mathfrak{S}_2$ and $\mathrm{Aut}(W, \iota|_W, \{w_i\}) \to D_4$, where $\mathfrak{S}_2$ and D_4 denote the symmetric group of degree 2 and the dihedral group of degree 4 respectively. It is easy to see that the product $\mathrm{Aut}(W, \iota|_W, \{w_i\}) \to \mathfrak{S}_2 \times D_4$ of these two morphisms is an isomorphism.

Let Z'/G be the quotient of the surface Z' by the natural action by the group $G = \langle \iota|_W \rangle$. Then the quotient Z'/G has four nodes, and the natural morphism $\tilde{V} \to Z'/G$ gives the minimal desingularization of Z'/G. Thus via the diagram $\tilde{V} \to Z'/G \leftarrow Z' \to W$, the action by $\mathrm{Aut}(W, \iota|_W, \{w_i\})$ on the surface W induces one on the pair $(\tilde{V}, \tilde{\lambda})$. Let $\mathfrak{S}_2 = \langle \iota|_W \rangle \to \mathrm{Aut}(W, \iota|_W, \{w_i\})$ be the natural inclusion. Since $\mathfrak{S}_2$ acts trivially on $\tilde{V}$ via this inclusion, we obtain a natural action by $\mathrm{Aut}(W, \iota|_W, \{w_i\})/\mathfrak{S}_2 \simeq D_4$ on the pair $(\tilde{V}, \tilde{\lambda})$. This action on $(\tilde{V}, \tilde{\lambda})$ induces one on $\mathbb{P}_0$. Remark 4 however implies that we have $X_{u_1} \simeq X_{u_2}$ if and only if two points $u_1 \in \mathbb{P}_0$ and $u_2 \in \mathbb{P}_0$ belong to the same orbit of the action by $\mathrm{Aut}(W, \iota|_W, \{w_i\})/\mathfrak{S}_2$ on $\mathbb{P}_0$. Thus by $\sharp D_4 = 8$, we see that for any $u_0 \in \mathbb{P}_0$ there exist at most eight $u \in \mathbb{P}_0$'s satisfying $X_u \simeq X_{u_0}$. Hence we have the assertion. □

6 Appendix

Proof of Proposition 3. Let us prove Proposition 3. The method we employ here is the same as the one used in [13, Proof of Lemma 4.5], to which we refer the readers for details of the following argument. Let $Z \subset \mathbb{P}^n$, where $n \geq 4$, be a non-degenerate surface satisfying the assumptions in Proposition 3, and $Z' \to Z$, its minimal desingularization. Since we have $\deg Z < 2n-2$, and $Z' \to Z$ is given by a complete linear system $|D'|$, the surface Z' is a rational surface not isomorphic to $\mathbb{P}^2$. Thus, for an integer d, the surface Z' admits a birational morphism $r : Z' \to \Sigma_d = Z'_0$ onto the Hirzebruch surface Σ_d of degree d. Let D'_0 be a general member of the linear system $r_*|D'|$, and ε'_i's, the total transforms to Z' of the (-1)-curves appearing at the blowings up in $Z' \to Z$. Then we have $D' \sim r^*D'_0 - \sum_{i=1}^{s} m_i\varepsilon'_i$, where m_i's, $s \in \mathbb{Z}$.

Lemma 6.1. *There exists an $r : Z' \to Z'_0$ as above such that for any i's, the equality $m_i = 1$ holds.*

Proof of Lemma. Note that the general member D' is a non-singular irreducible curve on Z'. If $h^1(\mathcal{O}_{D'}(D')) > 0$, then by Clifford's theorem on special divisors, we have $D'^2 \geq 2(h^0(\mathcal{O}_{D'}(D')) - 1)$, which contradicts $n \geq 4$. Thus we have $h^1(\mathcal{O}_{D'}(D')) = 0$. From this together with the natural short exact sequence $0 \to \mathcal{O}_{Z'} \to \mathcal{O}_{Z'}(D') \to \mathcal{O}_{D'}(D') \to 0$ and the Riemann–Roch theorem, we infer

$$\chi(\mathcal{O}_{Z'}(D')) = n + 1,$$

$D'K_{Z'} = D'^2 + 2(1 - \chi(\mathcal{O}_{Z'}(D'))) = 1 - n$, and $(K_{Z'} + D'/2)D' = (3-n)/2 < 0$. Thus by Cone Theorem, we find that if Z' is not the Hirzebruch surface, then there exists a (-1)-curve ε' on Z' satisfying $(K_{Z'} + D'/2)\varepsilon' < 0$. Since $Z' \to Z$ contracts no (-1)-curve, we obtain $D'\varepsilon' = 1$. Let $r' : Z' \to Z''$ be the blowing-down of ε'. We put $D'' = r'_*(D')$. If Z'' is not the Hirzebruch surface, then the same argument as above ensures the existence of a (-1)-curve ε'' on Z'' satisfying $D''\varepsilon'' = 1$ (for the detail, see [13, Lemma 4.4]). We can repeat the same steps until we obtain the Hirzebruch surface. □

In what follows, we assume our r satisfies the condition in the lemma above, hence $D' \sim r^*D'_0 - \sum_{i=1}^{s} \varepsilon'_i$. We put $D'_0 \sim a\Delta_0 + b\Gamma$, where if $d = 0$, we chose Δ_0 and Γ in such a way that $b \geq a$. Then by $\chi(\mathcal{O}_{Z'}(D')) = n + 1$ and $D'^2 = n + 1$, we obtain the following three equalities:

$$n + 1 = D'(D' - K_{Z'})/2 + 1 = (a + 1)(b - ad/2) + a - s + 1,$$

$$n + 1 = 2a(b - ad/2) - s, \tag{15}$$

$$0 = D'^2 - \chi(\mathcal{O}_{Z'}(D')) = (a - 1)(b - ad/2) - (a + 1). \tag{16}$$

Note that we have $b - ad/2 \geq a$ if $d \neq 1$, and that $b - ad/2 \geq a/2$ if $d = 1$. Thus by (15) and (16), we find $a = 2$, $b = d + 3$, and $s = 11 - n$, hence $D' \sim -K_{Z'} + r^*\Gamma$. Since $|D'_0|$ has no fixed component, we obtain $d \leq 3$. □

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