Derived equivalence classification of the cluster-tilted algebras of Dynkin type E

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Abstract

We obtain a complete derived equivalence classification of the cluster-tilted algebras of Dynkin type E. There are 67, 416, 1574 algebras in types E_6, E_7 and E_8 which turn out to fall into 6, 14, 15 derived equivalence classes, respectively. This classification can be achieved computationally and we outline an algorithm which has been implemented to carry out this task. We also make the classification explicit by giving standard forms for each derived equivalence class as well as complete lists of the algebras contained in each class; as these lists are quite long they are provided as supplementary material to this paper. From a structural point of view the remarkable outcome of our classification is that two cluster-tilted algebras of Dynkin type E are derived equivalent if and only if their Cartan matrices represent equivalent bilinear forms over the integers which in turn happens if and only if the two algebras are connected by a sequence of "good" mutations. This is reminiscent of the derived equivalence classification of cluster-tilted algebras of Dynkin type A, but quite different from the situation in Dynkin type D where a far-reaching classification has been obtained using similar methods as in the present paper but some very subtle questions are still open.

1 Introduction

1.1 The problem

Cluster categories have been introduced in [9] (see also [13] for Dynkin type A) as a representation-theoretic approach to Fomin and Zelevinsky's cluster algebras without coefficients having skew-symmetric exchange matrices (so that matrix mutation becomes the combinatorial recipe of mutation of quivers). This highly successful approach allows to use deep algebraic and representation-theoretic methods in the context of cluster algebras. A crucial role is played by the so-called cluster tilting objects in the cluster category which model the clusters in the cluster algebra. The endomorphism algebras of these cluster tilting objects are called cluster-tilted algebras.

Cluster-tilted algebras are particularly well-understood if the quiver underlying the cluster algebra, and hence the cluster category, is of Dynkin type. Cluster-tilted algebras of Dynkin type can be described as quivers with relations where the possible quivers are precisely the quivers in the mutation class of the Dynkin quiver, and the relations are uniquely determined by the quiver in an explicit way [10]. By a result of Fomin and Zelevinsky [15], the mutation class of a Dynkin quiver is finite. Moreover, the quivers in the mutation classes of Dynkin quivers are explicitly known; for type A they can be found in [12], for type D in [24] and for type E they can be enumerated using a computer, for example by the Java applet [17].

However, despite knowing the cluster-tilted algebras of Dynkin type as quivers with relations, many structural properties are not understood yet. One important structural aspect is to understand the derived module categories of the cluster-tilted algebras. In particular, one would want to know when two cluster-tilted algebras have equivalent derived categories. A complete derived equivalence classification has been achieved so far for cluster-tilted algebras of Dynkin type A by Buan and Vatne [12] and for those of extended Dynkin type $\tilde{A}$ by

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the first-named author [2]. For Dynkin type D, a far-reaching classification has been presented by the authors in [3]. In the present paper we address this problem for cluster-tilted algebras of Dynkin type E and obtain a complete derived equivalence classification of these algebras.

1.2 Main results

There are two natural approaches to address derived equivalence classification problems of a given collection of algebras arising from some combinatorial data. The top-to-bottom approach is to divide the algebras into equivalence classes according to some invariants of derived equivalence, so that algebras belonging to different classes are not derived equivalent. The bottom-to-top approach is to systematically construct, based on the combinatorial data, derived equivalences between pairs of these algebras and then to arrange these algebras into groups where any two algebras are related by a sequence of such derived equivalences. To obtain a complete derived equivalence classification one has to combine these approaches and hope that the two resulting partitions of the entire collection of algebras coincide.

The invariant of derived equivalence we use in this paper is the integer equivalence class of the bilinear form represented by the Cartan matrix of an algebra A. As this invariant is sometimes arithmetically subtle to compute directly, we instead compute the determinant of the Cartan matrix C_A and the characteristic polynomial of its asymmetry matrix $S_A = C_A C_A^{-T}$, defined whenever C_A is invertible over $\mathbb{Q}$, and encode them conveniently in a single polynomial that we call the *associated polynomial* of A. This quantity is generally a weaker invariant of derived equivalence, but in our case it will turn out to be enough for the classification.

The constructions we use are the following. Since any two quivers in a mutation class are connected by a sequence of mutations, it is natural to ask when a single mutation of quivers is accompanied by derived equivalence of their corresponding cluster-tilted algebras. The third author [20] has presented a procedure to determine when two cluster-tilted algebras whose quivers are related by a single mutation are also related by Brenner-Butler (co-)tilting, which is a particular kind of derived equivalence. We call such quiver mutation *good mutation*. In other words, a mutation at some vertex is good if the corresponding Brenner-Butler tilting module is defined and its endomorphism algebra is isomorphic to the cluster-tilted algebra of the mutated quiver. Obviously, the cluster-tilted algebras of quivers connected by a sequence of good mutations are derived equivalent. The explicit knowledge of the relations for cluster-tilted algebras of Dynkin type together with the procedure in [20] imply that for these algebras there is an algorithm to decide if a mutation is good or not.

It turns out that for cluster-tilted algebras of Dynkin type E the two approaches can be successfully combined to give a complete derived equivalence classification.

Theorem. *The following conditions are equivalent for two cluster-tilted algebras Λ and Λ' of Dynkin type E:*

(a) *Λ and Λ' have the same associated polynomial;*

(b) *The Cartan matrices of Λ and Λ' represent equivalent bilinear forms over $\mathbb{Z}$;*

(c) *Λ and Λ' are derived equivalent;*

(d) *The quivers of Λ and Λ' are connected by a sequence of good mutations.*

Note that the implication (c) $\Rightarrow$ (b) holds in general for any two (finite-dimensional) algebras Λ and Λ', and the implication (b) $\Rightarrow$ (a) holds whenever the associated polynomials are defined, i.e. when the Cartan matrices are invertible over $\mathbb{Q}$. Moreover, for cluster-tilted algebras the implication (d) $\Rightarrow$ (c) is evident from the definition.

Let us mention a few consequences from the derived equivalence classification. Recall that a good mutation involves a particular kind of derived equivalence, namely Brenner-Butler tilting. However, it turns out that whenever two cluster-tilted algebras of Dynkin type E whose quivers are related by a single mutation are derived equivalent, they are also related by Brenner-Butler tilting. In other words:

Corollary 1. *A single mutation of quivers in the mutation class of Dynkin type E is good if and only if the corresponding cluster-tilted algebras are derived equivalent.*

In view of this corollary, condition (d) in the theorem can thus be rephrased as follows:

(d') *The quivers of Λ and Λ' are connected by a sequence of mutations such that all the intermediate cluster-tilted algebras are derived equivalent.*

Another consequence of the classification is the following.

Corollary 2. *Let Λ be a cluster-tilted algebra of Dynkin type E. Then Λ and its opposite algebra Λ^{op} are derived equivalent.*

In addition to the above general statement we make the derived equivalence classification explicit by giving standard forms for each derived equivalence class as well as providing complete lists of the algebras contained in each class. As these lists are quite long, we will refrain from reproducing them here. Rather, they can be found in the supplementary material to this paper, which is freely and permanently accessible on the arXiv as part of the text in earlier versions of [4]. It can also be accessed via the third author's personal web page [21], in a version generated by using a computer program.

In the following tables we list the associated polynomials of the cluster-tilted algebras, and also the total number of algebras in each derived equivalence class.

For type E_6 the mutation class consists of 67 quivers. The corresponding cluster-tilted algebras turn out to fall into six derived equivalence classes as follows.

Derived equivalence classes for type E_6			
Associated polynomial	#	Associated polynomial	#
$x^6 - x^5 + x^3 - x + 1$	20	$3(x^6 + x^3 + 1)$	19
$2(x^6 - x^4 + 2x^3 - x^2 + 1)$	16	$4(x^6 + x^4 + x^2 + 1)$	7
$2(x^6 - 2x^4 + 4x^3 - 2x^2 + 1)$	3	$4(x^6 + x^5 - x^4 + 2x^3 - x^2 + x + 1)$	2

For type E_7 the mutation class consists of 416 quivers. The derived equivalence classes of the corresponding cluster-tilted algebras are again characterized by the associated polynomials; there are 14 classes in total, given as follows.

Derived equivalence classes for type E_7			
Associated polynomial	#	Associated polynomial	#
$x^7 - x^6 + x^4 - x^3 + x - 1$	64	$4(x^7 + x^6 - 2x^5 + 2x^4 - 2x^3 + 2x^2 - x - 1)$	2
$2(x^7 - x^5 + 2x^4 - 2x^3 + x^2 - 1)$	32	$4(x^7 + x^5 - x^4 + x^3 - x^2 - 1)$	56
$2(x^7 - x^5 + x^4 - x^3 + x^2 - 1)$	72	$4(x^7 + x^5 - 2x^4 + 2x^3 - x^2 - 1)$	8
$2(x^7 - 2x^5 + 4x^4 - 4x^3 + 2x^2 - 1)$	8	$5(x^7 + x^5 - x^4 + x^3 - x^2 - 1)$	17
$3(x^7 - 1)$	124	$6(x^7 + x^6 - x^4 + x^3 - x - 1)$	11
$4(x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x - 1)$	16	$6(x^7 + x^5 - x^2 - 1)$	1
$4(x^7 + x^6 - x^5 - x^4 + x^3 + x^2 - x - 1)$	4	$8(x^7 + x^6 + x^5 - x^4 + x^3 - x^2 - x - 1)$	1

For type E_8 the mutation class consists of 1574 quivers. The corresponding cluster-tilted algebras turn out to fall into 15 different derived equivalence classes which are characterized as follows.

Derived equivalence classes for type E_8			
Associated polynomial	#	Associated polynomial	#
$x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$	128	$4(x^8 + x^6 - x^5 + 2x^4 - x^3 + x^2 + 1)$	221
$2(x^8 - x^6 + 2x^5 - 2x^4 + 2x^3 - x^2 + 1)$	64	$4(x^8 + x^6 - 2x^5 + 4x^4 - 2x^3 + x^2 + 1)$	22
$2(x^8 - x^6 + x^5 + x^3 - x^2 + 1)$	256	$5(x^8 + x^6 + x^4 + x^2 + 1)$	167
$2(x^8 - 2x^6 + 4x^5 - 4x^4 + 4x^3 - 2x^2 + 1)$	16	$6(x^8 + x^6 + x^5 + x^3 + x^2 + 1)$	38
$3(x^8 + x^4 + 1)$	384	$6(x^8 + x^7 + 2x^4 + x + 1)$	118
$4(x^8 + x^7 - x^6 + x^5 + x^3 - x^2 + x + 1)$	72	$8(x^8 + 2x^7 + 2x^4 + 2x + 1)$	4
$4(x^8 + x^7 - x^6 + 2x^4 - x^2 + x + 1)$	48	$8(x^8 + x^7 + x^6 + 2x^4 + x^2 + x + 1)$	24
$4(x^8 + x^7 - 2x^6 + 2x^5 + 2x^3 - 2x^2 + x + 1)$	12		

1.3 Comparison with the other Dynkin types

We now put our results in perspective by comparing them to the derived equivalence classifications of the cluster-tilted algebras of the other Dynkin types. In type A, a complete classification has been achieved by Buan and Vatne [12], whereas in type D, a far-reaching classification has been presented by the authors in [3].

It turns out that two cluster-tilted algebras of type A_n are derived equivalent if and only if their quivers have the same number of 3-cycles. For distinguishing such algebras up to derived equivalence one uses the determinants of the Cartan matrices; these have been determined explicitly for arbitrary gentle algebras by the second author in [16].

A quick look at the above tables reveals that in Dynkin type E, unlike as in type A, the determinant itself is not sufficient for distinguishing the algebras up to derived equivalence. However, it is interesting to note that two cluster-tilted algebras of type E_n ($n = 6, 7, 8$) with the same *odd* Cartan determinant are derived equivalent.

For this reason one needs to use further derived invariants, such as the characteristic polynomial of the asymmetry matrix of the Cartan matrix, in order to distinguish the derived equivalence classes. Looking again at the above tables we see that this polynomial alone is not enough to distinguish the derived classes: there are two derived equivalence classes of cluster-tilted algebras of type E_7 with the same characteristic polynomial $x^7 + x^5 - x^4 + x^3 - x^2 - 1$. Our main theorem claims that by combining this polynomial with the Cartan determinant to form the associated polynomial we get a complete invariant of derived equivalence for cluster-tilted algebras of type E.

There are five non-trivial assertions which are satisfied by the cluster-tilted algebras of type E. These include the three implications (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) in the main theorem, together with the statements of Corollaries 1 and 2. In the table below we specify, for each of these assertions and each of the Dynkin types A, D and E, whether the assertion holds ($\surd$), does not hold ($\times$) or is unknown (?) for cluster-tilted algebras of the given type. In the two latter cases, we also give the smallest numbers of vertices for which there is a counterexample (in the case '$\times$') or an example that could not be settled (in the case '?').

	Type A	Type D		Type E
(a) $\Rightarrow$ (b)	$\surd$	$\times$	(D_{15})	$\surd$
(b) $\Rightarrow$ (c)	$\surd$	?	(D_{15}, D_{19})	$\surd$
(c) $\Rightarrow$ (d)	$\surd$	$\times$	(D_6, D_8)	$\surd$
Corollary 1	$\surd$	$\surd$		$\surd$
Corollary 2	$\surd$	?	(D_{15})	$\surd$

We see that as far as these assertions are concerned, cluster-tilted algebras of Dynkin type E behave similarly to that of type A. For type D, however, the picture is quite different; already in types D_6 and D_8 there are pairs of derived equivalent cluster-tilted algebras whose quivers cannot be connected by sequences of good mutations, making it necessary to use further constructions of derived equivalence. Thus, the derived equivalence classification of cluster-tilted algebras of type E is simpler than that of type D, even for small numbers of vertices.

Furthermore, starting at type D_{15} there are pairs of cluster-tilted algebras which we could not decide whether they are derived equivalent, or not; there does not seem to be an accessible tilting complex for these algebras but on the other hand all computable derived invariants available to us are the same for both algebras. It is therefore a remarkable coincidence that the two effectively decidable conditions (a) and (d) are equivalent for cluster-tilted algebras of type E, thus enabling a complete derived equivalence classification of these algebras.

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2 Assembling the proof

Throughout this paper let K be an algebraically closed field. All algebras are assumed to be finite-dimensional K-algebras. For an algebra A, we denote the bounded derived category of right A-modules by $\mathcal{D}^b(A)$. Two algebras A and B are called *derived equivalent* if $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.

2.1 The equivalence class of the Euler form as derived invariant

Let A be an algebra and let $P_1, \ldots, P_n$ be a complete collection of non-isomorphic indecomposable projective right A-modules (finite-dimensional over K). The *Cartan matrix* of A is then the $n \times n$ matrix C_A defined by $(C_A)_{ij} = \dim_K \operatorname{Hom}_A(P_j, P_i)$.

Denote by $\operatorname{per} A$ the triangulated category of *perfect* complexes of A-modules inside the derived category of A, that is, the complexes which are quasi-isomorphic to finite complexes of finitely generated projective A-modules. The Grothendieck group $K_0(\operatorname{per} A)$ is a free abelian group on the generators $[P_1], \dots, [P_n]$, and the expression

$$\langle X, Y \rangle = \sum_{r \in \mathbb{Z}} (-1)^r \dim_K \operatorname{Hom}_{\operatorname{per} A}(X, Y[r])$$

is well defined for any $X, Y \in \operatorname{per} A$ and induces a bilinear form on $K_0(\operatorname{per} A)$, known as the *Euler form*, whose matrix with respect to the basis of projectives is C_A^T.

The following proposition is well known. For the convenience of the reader, we give the short proof, see also the *proof* of Proposition 1.5 in [5].

Proposition 2.1. *Let A and B be two finite-dimensional, derived equivalent algebras. Then the matrices C_A and C_B represent equivalent bilinear forms over $\mathbb{Z}$, that is, there exists $P \in \mathrm{GL}_n(\mathbb{Z})$ such that $PC_AP^T = C_B$, where n denotes the number of non-isomorphic indecomposable projective modules of A (and B).*

Proof. Indeed, by [23], if A and B are derived equivalent, then $\operatorname{per} A$ and $\operatorname{per} B$ are equivalent as triangulated categories. Now any triangulated functor $F : \operatorname{per} A \to \operatorname{per} B$ induces a linear map from $K_0(\operatorname{per} A)$ to $K_0(\operatorname{per} B)$. When F is also an equivalence, this map is an isomorphism of the Grothendieck groups preserving the Euler forms. Thus, if $[F]$ denotes the matrix of this map with respect to the bases of indecomposable projectives, then $[F]^T C_B [F] = C_A$. □

In general, to decide whether two integral bilinear forms are equivalent is a very subtle arithmetical problem. Therefore, it is useful to introduce somewhat weaker invariants that are computationally easier to handle. In order to do this, assume further that C_A is invertible over $\mathbb{Q}$, that is, $\det C_A \neq 0$. In this case one can consider the rational matrix $S_A = C_A C_A^{-T}$ (here C_A^{-T} denotes the inverse of the transpose of C_A), known in the theory of non-symmetric bilinear forms as the *asymmetry* of C_A.

Proposition 2.2. *Let A and B be two finite-dimensional, derived equivalent algebras with invertible (over $\mathbb{Q}$) Cartan matrices. Then we have the following assertions, each implied by the preceding one:*

(a) *There exists $P \in \mathrm{GL}_n(\mathbb{Z})$ such that $PC_AP^T = C_B$.*

(b) *There exists $P \in \mathrm{GL}_n(\mathbb{Z})$ such that $PS_AP^{-1} = S_B$.*

(c) *There exists $P \in \mathrm{GL}_n(\mathbb{Q})$ such that $PS_AP^{-1} = S_B$.*

(d) *The matrices S_A and S_B have the same characteristic polynomial.*

For proofs and discussion, see for example [19, Section 3.3]. Since the determinant of an integral bilinear form is also invariant under equivalence, we obtain the following discrete invariant of derived equivalence.

Definition 2.3. For an algebra A with invertible Cartan matrix C_A over $\mathbb{Q}$, we define its *associated polynomial* as $(\det C_A) \cdot \chi_{S_A}(x)$, where $\chi_{S_A}(x)$ is the characteristic polynomial of the asymmetry matrix $S_A = C_A C_A^{-T}$.

Remark 2.4. The matrix $S_A = C_A C_A^{-T}$ (or better, minus its transpose $-C_A^{-1} C_A^T$) is related to the *Coxeter transformation* which has been widely studied in the case when A has finite global dimension (so that C_A is invertible over $\mathbb{Z}$), see [22]. It is then the K-theoretic shadow of the Serre functor and the related Auslander-Reiten translation in the derived category. The characteristic polynomial is then known as the *Coxeter polynomial* of the algebra.

Remark 2.5. In general, S_A might have non-integral entries. However, when the algebra A is *Gorenstein*, the matrix S_A is integral, which is an incarnation of the fact that the injective modules have finite projective resolutions. By a result of Keller and Reiten [18], this is the case for cluster-tilted algebras.

2.2 Mutations of algebras

We recall the notion of mutations of algebras from [20]. These are local operations on an algebra A producing new algebras derived equivalent to A.

Let $A = KQ/I$ be an algebra given as a quiver with relations. For any vertex i of Q, there is a trivial path e_i of length 0; the corresponding indecomposable projective $P_i = e_iA$ is spanned by the images of the paths starting at i. Thus an arrow $i \xrightarrow{\alpha} j$ gives rise to a map $P_j \to P_i$ given by left multiplication with α.

Let k be a vertex of Q without loops. Consider the following two complexes of projective A-modules

$$T_k^-(A) = \big(P_k \xrightarrow{f} \bigoplus_{j\to k} P_j\big) \oplus \big(\bigoplus_{i\neq k} P_i\big), \qquad T_k^+(A) = \big(\bigoplus_{k\to j} P_j \xrightarrow{g} P_k\big) \oplus \big(\bigoplus_{i\neq k} P_i\big)$$

where the map f is induced by all the maps $P_k \to P_j$ corresponding to the arrows $j \to k$ ending at k, the map g is induced by the maps $P_j \to P_k$ corresponding to the arrows $k \to j$ starting at k, the term P_k lies in degree -1 in $T_k^-(A)$ and in degree 1 in $T_k^+(A)$, and all other terms are in degree 0.

Definition 2.6. Let A be an algebra given as a quiver with relations and k a vertex without loops.

(a) We say that the negative mutation of A at k is *defined* if $T_k^-(A)$ is a tilting complex over A. In this case, we call the algebra $\mu_k^-(A) = \operatorname{End}_{\mathcal{D}^b(A)} T_k^-(A)$ the *negative mutation* of A at the vertex k.

(b) We say that the positive mutation of A at k is *defined* if $T_k^+(A)$ is a tilting complex over A. In this case, we call the algebra $\mu_k^+(A) = \operatorname{End}_{\mathcal{D}^b(A)} T_k^+(A)$ the *positive mutation* of A at the vertex k.

Remark 2.7. By Rickard's Morita theory for derived categories [23], the negative and the positive mutations of an algebra A at a vertex, when defined, are always derived equivalent to A.

Example 2.8. When k is a sink in the quiver of A, then $\mu_k^-(A)$ is defined and moreover $T_k^-(A)$ is isomorphic in $\mathcal{D}^b(A)$ to the APR-tilting module [1] corresponding to k. Similarly, when k is a source then $\mu_k^+(A)$ is defined.

There is a combinatorial criterion to determine whether a mutation at a vertex is defined, see [20, Prop. 2.3]. Since the algebras we will be dealing with in this paper are schurian, we state here the criterion only for this case, as it takes a particularly simple form. Recall that an algebra is *schurian* if the entries of its Cartan matrix are only 0 or 1.

Proposition 2.9. *Let A be a schurian algebra and let k be a vertex in the quiver of A.*

(a) *The negative mutation $\mu_k^-(A)$ is defined if and only if for any non-zero path $k \rightsquigarrow i$ starting at k and ending at some vertex i, there exists an arrow $j \to k$ such that the composition $j \to k \rightsquigarrow i$ is non-zero in A.*

(b) *The positive mutation $\mu_k^+(A)$ is defined if and only if for any non-zero path $i \rightsquigarrow k$ starting at some vertex i and ending at k, there exists an arrow $k \to j$ such that the composition $i \rightsquigarrow k \to j$ is non-zero in A.*

Remark 2.10. It follows from [20, Remark 2.10] that in many cases, and in particular when A is schurian, the negative mutation of A at k is defined if and only if one can associate with k the corresponding Brenner-Butler tilting module [7]. Moreover, $T_k^-(A)$ is then isomorphic in $\mathcal{D}^b(A)$ to that Brenner-Butler tilting module.

Remark 2.11. It is enough to introduce only one of the notions of negative and positive mutation, since the other can then be defined using the notion of the opposite algebra. Indeed, if Q is the quiver of an algebra A, then the quiver of the opposite algebra A^{op} is the opposite quiver Q^{op} which has the same vertices as Q and is obtained from it by inverting all the arrows. The positive mutation $\mu_k^+(A)$ is then defined if and only if the negative mutation $\mu_k^-(A^{op})$ is defined. Moreover, in that case, we have $\mu_k^+(A) \simeq \big(\mu_k^-(A^{op})\big)^{op}$.

2.3 Cluster-tilted algebras

In this section we assume that all quivers are without loops and 2-cycles. Given such a quiver Q and a vertex k, we denote by $\mu_k(Q)$ the Fomin-Zelevinsky quiver mutation [14] of Q at k. Two quivers are called *mutation equivalent* if one can be reached from the other by a finite sequence of quiver mutations. The *mutation class* of a quiver Q is the set of all quivers which are mutation equivalent to Q.

For a quiver Q' without oriented cycles, the corresponding cluster category $\mathcal{C}_{Q'}$ was introduced in [9]. A *cluster-tilted algebra* of *type* Q' is an endomorphism algebra of a cluster-tilting object in $\mathcal{C}_{Q'}$, see [11]. It is known by [11] that for any quiver Q mutation equivalent to Q', there is a cluster-tilted algebra whose quiver is Q. Moreover, by [8], it is unique up to isomorphism. Hence, there is a bijection between the quivers in the mutation class of an acyclic quiver Q' and the isomorphism classes of cluster-tilted algebras of type Q'. This justifies the following notation.

Notation 2.12. Throughout the paper, for a quiver Q which is mutation equivalent to an acyclic quiver, we denote by Λ_Q the corresponding cluster-tilted algebra.

When Q' is a Dynkin quiver of types A, D or E, the corresponding cluster-tilted algebras are said to be of Dynkin type. These algebras have been investigated in [10], where it is shown that they are schurian and moreover they can be defined by using only zero and commutativity relations that can be extracted from their quivers in an algorithmic way.

2.4 Good quiver mutations

For simplicity, we assume in this section that all the quivers we deal with are mutation equivalent to Dynkin quivers. A more general treatment can be found in [20].

Let Q be such a quiver and let k be a vertex of Q. Starting with the cluster-tilted algebra Λ_Q, there are two notions of mutation that one may consider. The first is mutation of quivers, leading to the cluster-tilted algebra $\Lambda_{\mu_k(Q)}$. The second is mutation of algebras, leading (when defined) to the algebra $\mu_k^-(\Lambda_Q)$ or $\mu_k^+(\Lambda_Q)$. It is interesting to ask when these two notions are compatible. This motivates the following definition.

Definition 2.13. Let Q be a quiver which is mutation equivalent to a Dynkin quiver and let k be a vertex of Q. The mutation of Q at k is *good* if $\Lambda_{\mu_k(Q)} \simeq \mu_k^-(\Lambda_Q)$ or $\Lambda_{\mu_k(Q)} \simeq \mu_k^+(\Lambda_Q)$ (or both).

Obviously, a good mutation implies the derived equivalence of the corresponding cluster-tilted algebras.

Remark 2.14. Observe that the condition $\Lambda_{\mu_k(Q)} \simeq \mu_k^+(\Lambda_Q)$ is equivalent to the condition $\Lambda_Q \simeq \mu_k^-(\Lambda_{\mu_k(Q)})$, so that the mutation of Q at k is good if and only if that of $\mu_k(Q)$ at k is good. Hence being "good" is a property of the mutation, independently of which of the two quivers we start with.

A-priori, checking a condition like $\Lambda_{\mu_k(Q)} \simeq \mu_k^-(\Lambda_Q)$ involves two steps: the first is to check that the algebra mutation $\mu_k^-(\Lambda_Q)$ is defined, and the second is to check that it is isomorphic to $\Lambda_{\mu_k(Q)}$. This last step is not so easily adapted for a computer since it involves the computation of an endomorphism algebra of a tilting complex.

For cluster-tilted algebras of Dynkin type, the statement of Theorem 5.3 in [20], linking more generally mutation of cluster-tilting objects in 2-Calabi-Yau categories with mutations of their endomorphism algebras, takes the following form.

Proposition 2.15. *Let Q be mutation equivalent to a Dynkin quiver and let k be a vertex of Q.*

(a) $\Lambda_{\mu_k(Q)} \simeq \mu_k^-(\Lambda_Q)$ *if and only if the two algebra mutations* $\mu_k^-(\Lambda_Q)$ *and* $\mu_k^+(\Lambda_{\mu_k(Q)})$ *are defined.*

(b) $\Lambda_{\mu_k(Q)} \simeq \mu_k^+(\Lambda_Q)$ *if and only if the two algebra mutations* $\mu_k^+(\Lambda_Q)$ *and* $\mu_k^-(\Lambda_{\mu_k(Q)})$ *are defined.*

Thus, in order to check the conditions in the definition of good mutation it is enough to check that certain algebra mutations are defined.

Remark 2.16. In view of Propositions 2.9 and 2.15, there is an algorithm which decides, given a quiver which is mutation equivalent to a Dynkin quiver, whether a mutation at a vertex is good or not.

We recollect a few basic observations on good mutations.

Lemma 2.17. *A mutation of Q at a sink or a source is always good.*

Proof. Assume that k is a sink in Q. Then the negative mutation $\mu_k^-(\Lambda_Q)$ is defined. Since k becomes a source in the quiver $\mu_k(Q)$, the positive mutation $\mu_k^+(\Lambda_{\mu_k(Q)})$ is defined. Now apply Proposition 2.15. □

Hence, for the purpose of derived equivalence classification it is enough to consider quivers only up to *sink/source equivalence*: two quivers are called sink/source equivalent if one can be obtained from the other by performing mutations only at vertices which are sinks or sources.

Lemma 2.18. *The mutation of Q at a vertex k is good if and only if the mutation of Q^{op} at k is good.*

Proof. Observe that $\mu_k(Q^{op}) \simeq (\mu_k(Q))^{op}$ and $\Lambda_{Q^{op}} \simeq \Lambda_Q^{op}$. Now use Remark 2.11 and Proposition 2.15. $\square$

2.5 Putting everything together

We now have all the ingredients needed for the proof of the main theorem. It is enough to show the equivalence of conditions (a) and (d) in that theorem, i.e. that the quivers of any two cluster-tilted algebras of Dynkin type E with the same associated polynomial can be connected by a sequence of good mutations.

The proof is computational; we give below an outline of an algorithm to carry out this task. We start with a Dynkin quiver Q of type E_n where $n \in \{6, 7, 8\}$.

Step 1 (Mutation data)**.** Compute the mutation class $Q_1, Q_2, \dots, Q_N$ of Q together with functions

$$s : \{1, \dots, N\} \times \{1, \dots, n\} \to \{1, \dots, N\}, \qquad \pi : \{1, \dots, N\} \times \{1, \dots, n\} \to S_n$$

such that for each index $1 \leq t \leq N$ and vertex $1 \leq k \leq n$ we have $\mu_k(Q_t) \simeq Q_{s(t,k)}$ with the isomorphism given by the permutation $\pi(t, k)$ on the set of vertices. This is done as follows:

1. Set $Q_1 \leftarrow Q$, $t \leftarrow 1$, $m \leftarrow 1$.
2. At the t-th stage, we have a list $Q_1, \dots, Q_m$ for some $m \geq t$. For each $1 \leq k \leq n$, compute $\mu_k(Q_t)$. If it is isomorphic to some member in the list, set $s(t, k)$ and $\pi(t, k)$ accordingly; otherwise, append it to the list and set $m \leftarrow m + 1$.
3. If $m = t$, set $N \leftarrow m$, stop; otherwise, set $t \leftarrow t + 1$ and go to stage 2.

For $1 \leq t \leq N$, denote by Λ_t the cluster-tilted algebra of Q_t and by C_t its Cartan matrix. Perform Steps 2, 3 and 4 below for each $1 \leq t \leq N$:

Step 2 (Algebras and non-zero paths)**.** Compute the zero-relations and the commutativity-relations of Λ_t from the quiver Q_t according to the algorithm in [10]. Then, for each $1 \leq i, j \leq n$, store the list $L_t(i, j)$ of paths starting at i and ending at j whose image in Λ_t is non-zero. Note that since Λ_t is schurian, in computing these lists it is enough to consider paths traversing through each vertex at most once.

Step 3 (Cartan matrix and associated polynomial)**.** For each $1 \leq i, j \leq n$, set $(C_t)_{ij} = 1$ if $L_t(i, j)$ is not empty, otherwise set $(C_t)_{ij} = 0$. Then compute the asymmetry $S_t = C_t C_t^{-T}$ and the associated polynomial $p_t(x) = (\det C_t) \cdot \chi_{S_t}(x)$.

Step 4 (Tilting data)**.** For each vertex $1 \leq k \leq n$, check which of the algebra mutations $\mu_k^-(\Lambda_t)$ and $\mu_k^+(\Lambda_t)$ is defined; to this end use the lists $L_t(i, j)$ and apply the criteria of Proposition 2.9.

We encode the good mutations in an undirected graph G whose vertices are the indices $1, 2, \dots, N$ and for any $1 \leq t, t' \leq N$ there is an edge $t - t'$ if and only if the quivers Q_t and $Q_{t'}$ are related by a good mutation at some vertex. The graph G is computed in the next step as follows:

Step 5 (Good mutations graph)**.** For each $1 \leq t \leq N$ and $1 \leq k \leq n$:

- Set $t' \leftarrow s(t, k)$ and $\ell \leftarrow \pi(t, k)^{-1}(k)$;
- If $\mu_k^-(\Lambda_t)$ and $\mu_\ell^+(\Lambda_{t'})$ are defined, set an edge $t - t'$ in G;
- If $\mu_k^+(\Lambda_t)$ and $\mu_\ell^-(\Lambda_{t'})$ are defined, set an edge $t - t'$ in G.

Observe that two quivers Q_t and $Q_{t'}$ are connected by a sequence of good mutations if and only if the indices t and t' are in the same connected component of the graph G.

Step 6 (End of proof)**.** Compute the connected components of the graph G and then take one representative from each connected component (e.g. the smallest one in that component) to get a sequence $t_1, t_2, \dots, t_M$. Finally verify that the associated polynomials $p_{t_1}(x), \dots, p_{t_M}(x)$ are all distinct.

We have made two implementations of this algorithm. The first is a fully automatic one developed by the third author by writing computer code for the Magma computational algebra system [6]. The program takes a Dynkin quiver as input and produces all the relevant information (quivers in the mutation class, defining relations for the cluster-tilted algebras, Cartan matrices, lists of good mutations and the partition according to the associated polynomials). Its output can be found in the supplementary material to this paper, available on the third author's personal web page [21]. Since all the mutations are examined by the program, it can also verify Corollary 1 in addition to the main theorem.

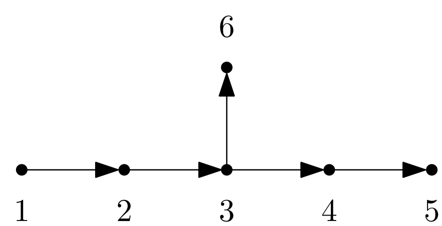
The other implementation is more "traditional". The mutation class is computed using Keller's Java applet [17], then Steps 2 and 3 are performed and the quivers are divided into groups according to the associated polynomials of their corresponding cluster-tilted algebras. Then one finds enough good mutations to show that in each such group, any two quivers can be connected by good mutations.

In this approach one does not build the graph G completely, but rather finds enough of its edges to construct a spanning tree for each set of vertices with the same associated polynomial. Shortcuts to reduce the number of calculations are provided by Lemma 2.17 (sink/source equivalence) and Lemma 2.18 (opposites). We shall demonstrate this approach for the quiver E_6 in the next section. For the quivers E_7 and E_8, lists of good mutations can be found in a previous version of this paper [4].

3 Cluster-tilted algebras of type E_6

In this section we demonstrate the techniques of the previous sections and describe in detail the derived equivalence classification of cluster-tilted algebras of type E_6. Since the number of these algebras is not large, we can present the full classification and give enough details so that the reader may reproduce all our findings.

We start by computing the mutation class of the E_6 quiver given by

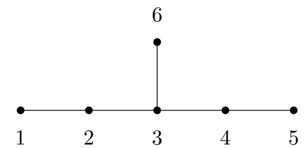


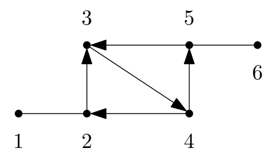
This can be done, for example, by using the Java applet of Keller [17]. The mutation class of E_6 consists of 67 quivers. To reduce the number of calculations, observe that for the purpose of derived equivalence classification of the corresponding cluster-tilted algebras it suffices to consider the quivers only up to sink/source equivalence, see Lemma 2.17. There are 21 quivers up to sink/source equivalence, and we number them $1, 2, \dots, 21$ according to the output of the program [17].

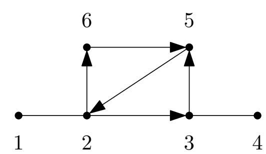
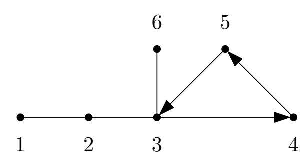
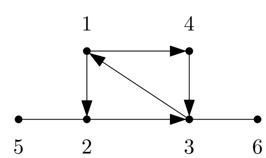
For each representative quiver in a sink/source equivalence class we compute the relations of the corresponding cluster-tilted algebra according to [10]. From this we deduce the Cartan matrix and the associated polynomial, obtained by multiplying the determinant of the Cartan matrix by the characteristic polynomial of its asymmetry matrix.

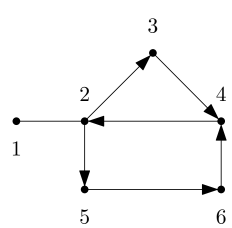
As there are six distinct such polynomials, the quivers are divided into six groups according to the associated polynomial of their corresponding cluster-tilted algebras. We claim that these groups form the six derived equivalence classes of the cluster-tilted algebras of type E_6.

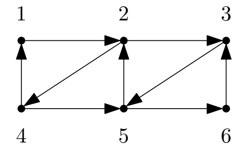
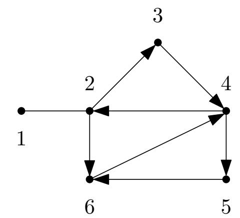
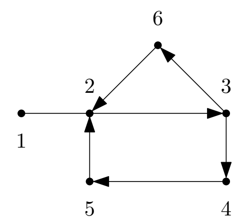
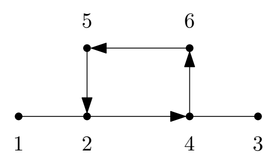
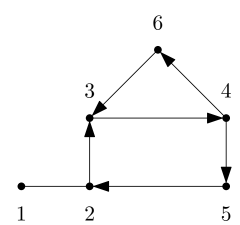
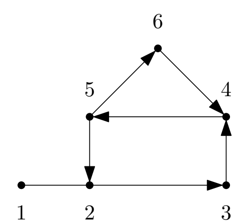
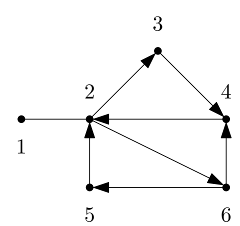
In the table below we list, for each of these six polynomials, all the quivers in the mutation class of type E_6 whose corresponding cluster-tilted algebras have this polynomial as associated polynomial. For each set of sink/source equivalent quivers we give only one picture where certain arrows are replaced by undirected lines; this has to be read that these lines can take any orientation. We also give the relations of the corresponding cluster-tilted algebras. Since there is at most one arrow between any two vertices, we indicate a path by the sequence of vertices it traverses. A zero-relation is then indicated by a sequence of the form $(a, b, c, \dots)$ and a commutativity relation has the form $(a, b, c, \dots) - (a', b', c', \dots)$.

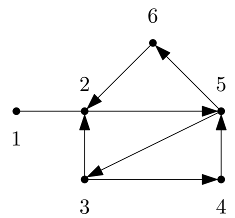
$\boldsymbol{x^6 - x^5 + x^3 - x + 1}$		
no.	quiver	relations
1		None

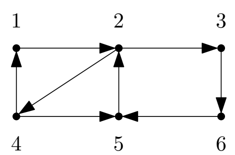
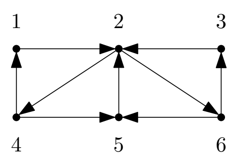
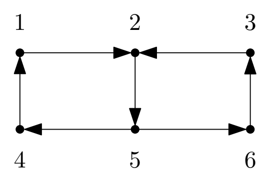
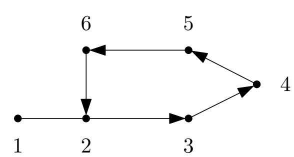
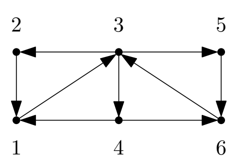
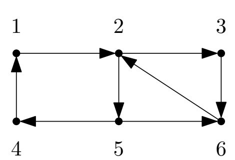
$\boldsymbol{2(x^6 - 2x^4 + 4x^3 - 2x^2 + 1)}$		
no.	quiver	relations
11		$(2,3,4),\ (3,4,2),\ (5,3,4),\ (3,4,5),$ $(4,2,3)-(4,5,3)$

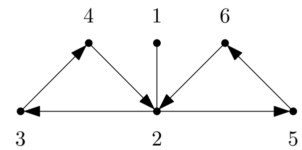
$\boldsymbol{2(x^6 - x^4 + 2x^3 - x^2 + 1)}$		
no.	quiver	relations
2		$(3,5,2),\ (5,2,3),\ (6,5,2),\ (5,2,6),$ $(2,6,5)-(2,3,5)$
7		$(3,4,5),\ (4,5,3),\ (5,3,4)$
12		$(2,3,1),\ (3,1,2),\ (4,3,1),\ (3,1,4),$ $(1,2,3)-(1,4,3)$

$\boldsymbol{3(x^6 + x^3 + 1)}$		
no.	quiver	relations
3		$(3,4,2),\ (4,2,3),\ (5,6,4,2),\ (6,4,2,5),$ $(4,2,5,6),\ (2,3,4)-(2,5,6,4)$

no.	quiver	relations
4		$(1,2,4)$, $(2,4,1)$, $(5,2,4)$, $(3,5,2)$, $(3,5,6)$, $(6,3,5)$, $(2,4,5)-(2,3,5)$, $(4,1,2)-(4,5,2)$, $(5,2,3)-(5,6,3)$
6		$(3,4,2)$, $(4,2,3)$, $(6,4,2)$, $(5,6,4)$, $(6,4,5)$, $(2,3,4)-(2,6,4)$, $(4,2,6)-(4,5,6)$
10		$(2,3,6)$, $(6,2,3)$, $(4,5,2,3)$, $(5,2,3,4)$, $(2,3,4,5)$, $(3,6,2)-(3,4,5,2)$
14		$(2,4,6,5)$, $(4,6,5,2)$, $(6,5,2,4)$, $(5,2,4,6)$
16		$(3,4,6)$, $(6,3,4)$, $(5,2,3,4)$, $(2,3,4,5)$, $(3,4,5,2)$, $(4,6,3)-(4,5,2,3)$
19		$(4,5,6)$, $(6,4,5)$, $(2,3,4,5)$, $(3,4,5,2)$, $(4,5,2,3)$, $(5,6,4)-(5,2,3,4)$
20		$(3,4,2)$, $(4,2,3)$, $(4,2,6)$, $(2,6,5)$, $(5,2,6)$, $(2,3,4)-(2,6,4)$, $(6,4,2)-(6,5,2)$

no.	quiver	relations
21		$(2,5,6)$, $(6,2,5)$, $(2,5,3)$, $(4,5,3)$, $(5,3,4)$, $(5,6,2)-(5,3,2)$, $(3,2,5)-(3,4,5)$

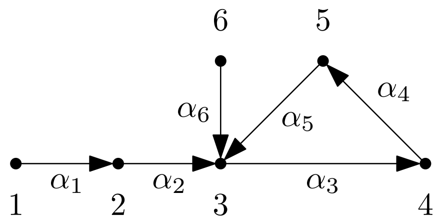
$\mathbf{4(x^6+x^4+x^2+1)}$		
no.	quiver	relations
5		$(1,2,4)$, $(2,4,1)$, $(5,2,4)$, $(3,6,5,2)$, $(6,5,2,3)$, $(5,2,3,6)$, $(4,1,2)-(4,5,2)$, $(2,4,5)-(2,3,6,5)$
9		$(1,2,4)$, $(2,4,1)$, $(5,2,4)$, $(5,2,6)$, $(2,6,3)$, $(3,2,6)$, $(4,1,2)-(4,5,2)$, $(2,4,5)-(2,6,5)$, $(6,3,2)-(6,5,2)$
13		$(1,2,5,4)$, $(2,5,4,1)$, $(4,1,2,5)$, $(3,2,5,6)$, $(2,5,6,3)$, $(6,3,2,5)$, $(5,4,1,2)-(5,6,3,2)$
15		$(2,3,4,5,6)$, $(3,4,5,6,2)$, $(4,5,6,2,3)$, $(5,6,2,3,4)$, $(6,2,3,4,5)$
17		$(1,3,2)$, $(2,1,3)$, $(1,3,4)$, $(6,3,4)$, $(6,3,5)$, $(5,6,3)$, $(3,2,1)-(3,4,1)$, $(3,4,6)-(3,5,6)$, $(4,1,3)-(4,6,3)$
18		$(3,6,2)$, $(6,2,3)$, $(6,2,5)$, $(1,2,5,4)$, $(2,5,4,1)$, $(4,1,2,5)$, $(2,3,6)-(2,5,6)$, $(5,6,2)-(5,4,1,2)$

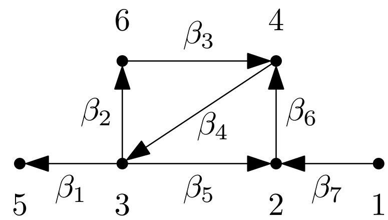
$\mathbf{4(x^6+x^5-x^4+2x^3-x^2+x+1)}$		
no.	quiver	relations
8		$(2,3,4)$, $(3,4,2)$, $(4,2,3)$, $(2,5,6)$, $(5,6,2)$, $(6,2,5)$

The rest of this section is devoted to proving our main theorem (as stated in the introduction) for type E_6. In order to do this, it is enough to show the equivalence of conditions (a) and (d) in that theorem, namely, that any two cluster-tilted algebras of type E_6 with the same associated polynomial are connected by a sequence of good mutations.

First, we give a detailed example. Then we list the other good mutations in a shorter form which is explained after the example.

Example 3.1. Let A_7 be the cluster-tilted algebra corresponding to one representative of the quiver number 7 (as depicted in the left picture). After mutation at vertex 3 we get a representative of the quiver number 2 as in the right picture. We denote the corresponding cluster-tilted algebra by A_2.





Invoking the algorithm in [10], we see that the relations of A_7 are given by the three zero-relations $\alpha_3\alpha_4$, $\alpha_4\alpha_5$ and $\alpha_5\alpha_3$. Similarly, the cluster-tilted algebra A_2 has the zero-relations $\beta_3\beta_4$, $\beta_4\beta_2$, $\beta_4\beta_5$, $\beta_6\beta_4$ and the commutativity-relation $\beta_2\beta_3 = \beta_5\beta_6$. The corresponding Cartan matrices are therefore

$$C_{A_7} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_{A_2} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

and we compute the associated polynomial to be $2(x^6 - x^4 + 2x^3 - x^2 + 1)$.

We examine now the algebra mutations at the vertex 3. Since the arrow $\alpha_6 : 6 \to 3$ does not appear in any relation of A_7, its composition with any non-zero path starting at vertex 3 is non-zero. Thus the negative mutation $\mu_3^-(A_7)$ is defined. Since the arrow $\beta_1 : 3 \to 5$ does not appear in any relation of A_2, its composition with any non-zero path ending at vertex 3 is non-zero. Thus the positive mutation $\mu_3^+(A_2)$ is defined.

Since the quiver of A_2 is obtained by mutating the quiver of A_7 at the vertex 3, condition (a) of Proposition 2.15 holds, hence the quiver mutation at vertex 3 is good and the corresponding cluster-tilted algebras A_7 and A_2 are derived equivalent.

We now write enough good mutations so that the reader can easily verify that the quivers of any two cluster-tilted algebras with the same associated polynomial are indeed connected by a sequence of these mutations.

We present these good mutations in a concise form in the tables below. The algebras are numbered according to the numbering of their quivers, i.e. A_i denotes the cluster tilted algebra corresponding to the quiver no. i in the above tables. We list the cluster-tilted algebra together with an orientation of the arrows incident to the mutated vertex (if they can be oriented arbitrarily), the vertex we mutate at and the resulting cluster-tilted algebra. Note that up to a permutation of the vertices and sink/source equivalence, it is one of the 21 in our list, and the necessary permutations (written as a product of disjoint cycles) are given in each case. By Lemma 2.18, the mutation of the opposite quivers at the same vertex is also good, so we also indicate this under "opposite case" when the corresponding algebras are not isomorphic to the original ones. The verification that all these mutations are good is done as in the previous example.

$\mathbf{2(x^6 - x^4 + 2x^3 - x^2 + 1)}$			
algebra (orientation)	vertex	mutated algebra	permutation
A_7 ($2 \to 3$, $6 \to 3$)	3	A_2	(145)(23)
A_2 ($1 \to 2$)	2	A_{12}	(23)(456)

$\mathbf{3(x^6 + x^3 + 1)}$				
algebra (orientation)	vertex	mutated algebra	permutation	opposite case
A_3	5	A_{20}	id	$A_{10} \sim A_{20}$
A_3 ($1 \to 2$)	2	A_4	(143)	$A_{10} \sim A_4$
A_{20}	4	A_{14}	id	
A_6	5	A_3	(56)	$A_{21} \sim A_{10}$
A_{16}	5	A_6	(365)	$A_{19} \sim A_{21}$

$\mathbf{4(x^6 + x^4 + x^2 + 1)}$				
algebra (orientation)	vertex	mutated algebra	permutation	opposite case
A_5	3	A_9	id	$A_{18} \sim A_{17}$
A_{15} ($1 \to 2$)	2	A_5	(243)(56)	$A_{15} \sim A_{18}$
A_{13}	4	A_5	(14)(25)(36)	

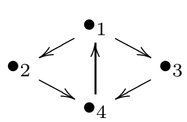
4 Standard forms for the derived equivalence classes

In this section we provide standard forms for the derived equivalence classes of cluster-tilted algebras of Dynkin type E. For type E_6 the derived equivalence classification has been carried out in Section 3. For more details on the actual classification in types E_7 and E_8 we refer the reader to the supplementary material [21] as well as to earlier versions of the paper [4] on the arXiv.

Given such derived equivalence class, we calculate its standard form in the following way. We start by considering the cluster-tilted algebras within that class whose quivers have minimal number of arrows. It turns out that except for one derived class in type E_7 consisting of a single algebra, the quivers with relations of all these "minimal" algebras can be described as iterated gluing of quivers with relations of three possible kinds:

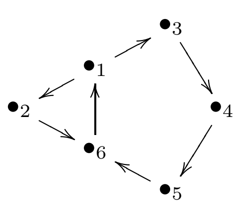
1. *Cycles*, consisting of n vertices $a_0, a_1, \dots, a_{n-1}$ and n arrows $a_i \to a_{i+1}$ (where indices are taken modulo n). The relations are given by the n paths $a_i, a_{i+1}, \dots, a_{i+n-1}$ of length $n-1$ (again indices are taken modulo n) for $0 \le i < n$. Note that these are actually cluster-tilted algebras of type D_n [3, 24]. The values of n which occur are $3, 4, 5, 6, 7$.

2. The two cluster-tilted algebras of types D_4 and D_6 given by the quivers with relations



$(1,2,4) - (1,3,4), (2,4,1), (3,4,1), (4,1,2), (4,1,3)$

and



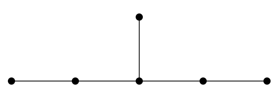
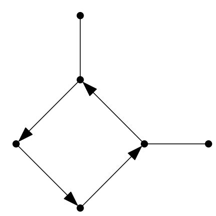
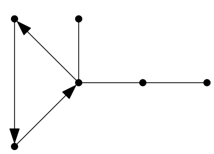
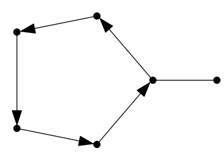
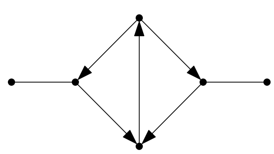
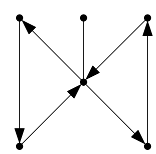
$(1,2,6) - (1,3,4,5,6), (2,6,1), (6,1,2),$
$(3,4,5,6,1), (4,5,6,1,3), (5,6,1,3,4), (6,1,3,4,5).$

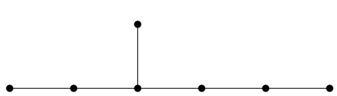
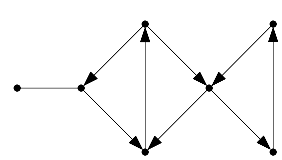
3. *Tails*, which are orientations of a Dynkin diagram of type A_n with $n \ge 2$.

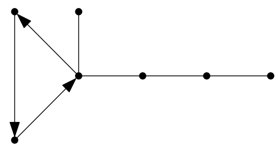
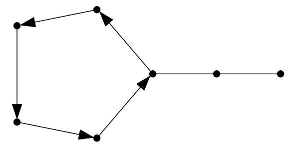
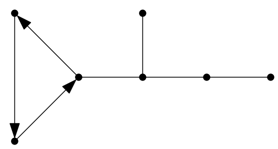
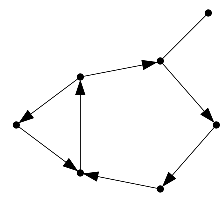
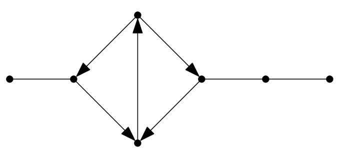
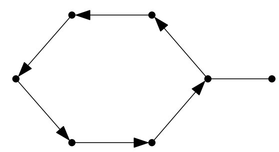
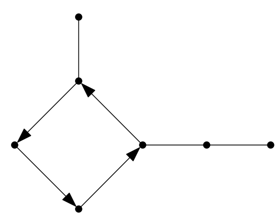
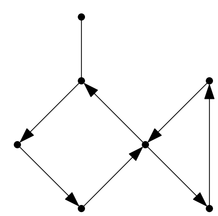
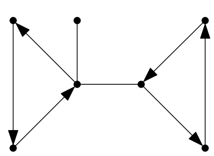
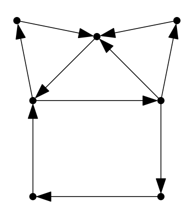
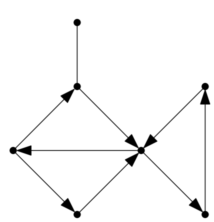
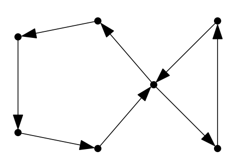
Recall that a *gluing* of two quivers with relations Q_1 and Q_2 with respect to some chosen vertices v_1 in Q_1 and v_2 in Q_2 is obtained by taking the disjoint union $Q_1 \sqcup Q_2$ and identifying the vertices v_1 and v_2. The relations are induced from those of Q_1 and Q_2 (i.e. there are no additional relations).

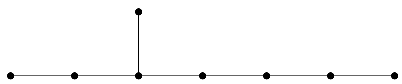
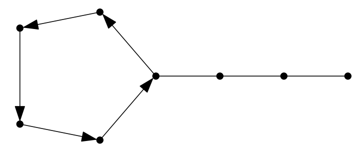
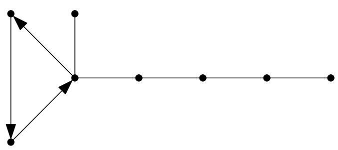
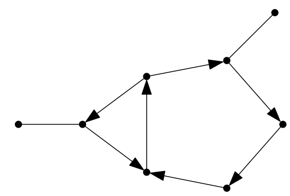
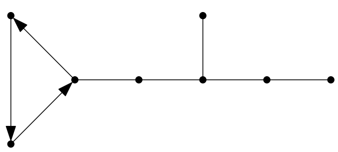
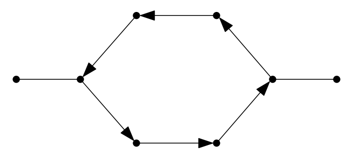
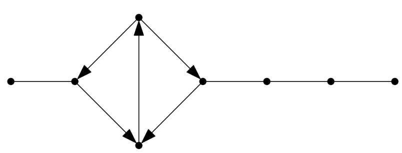
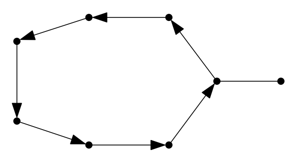
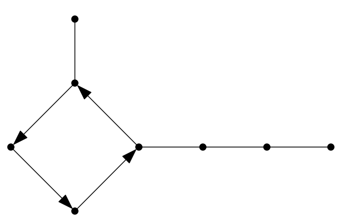
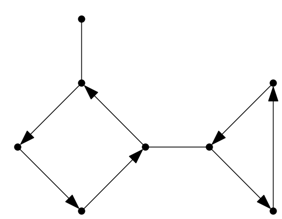
We can consider the minimal algebras up to sink/source equivalence and taking opposites, as these operations do not change the derived equivalence class (cf. Lemma 2.17 and Corollary 2). For many derived equivalence classes this determines uniquely a standard form within the class. For the classes where this is not the case, imposing the additional condition that the quiver should have minimal number of "free" tails (i.e. tails where no quiver of the above kinds 1 and 2 is glued to one of their ends) determines uniquely a standard form.

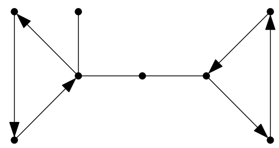
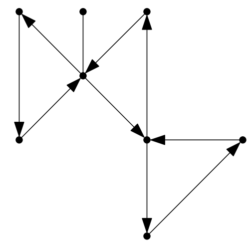
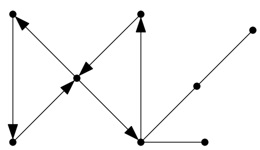
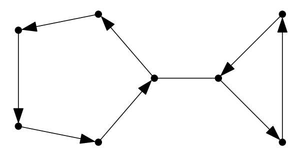
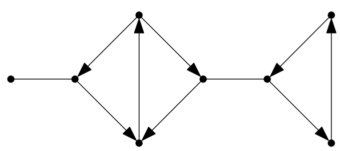
In the tables below we provide lists of these standard forms along with their associated polynomials. Arrows which are displayed without orientation can be oriented arbitrarily, without changing the derived equivalence class.

Representatives of derived equivalence classes for type E_6			
Quiver and associated polynomial	#	Quiver and associated polynomial	#
 $x^6 - x^5 + x^3 - x + 1$	20	 $3(x^6 + x^3 + 1)$	19
 $2(x^6 - x^4 + 2x^3 - x^2 + 1)$	16	 $4(x^6 + x^4 + x^2 + 1)$	7
 $2(x^6 - 2x^4 + 4x^3 - 2x^2 + 1)$	3	 $4(x^6 + x^5 - x^4 + 2x^3 - x^2 + x + 1)$	2

Representatives of derived equivalence classes for type E_7			
Quiver and associated polynomial	#	Quiver and associated polynomial	#
 $x^7 - x^6 + x^4 - x^3 + x - 1$	64	 $4(x^7 + x^6 - 2x^5 + 2x^4 - 2x^3 + 2x^2 - x - 1)$	2

Quiver and associated polynomial	#	Quiver and associated polynomial	#
 $2(x^7 - x^5 + 2x^4 - 2x^3 + x^2 - 1)$	32	 $4(x^7 + x^5 - x^4 + x^3 - x^2 - 1)$	56
 $2(x^7 - x^5 + x^4 - x^3 + x^2 - 1)$	72	 $4(x^7 + x^5 - 2x^4 + 2x^3 - x^2 - 1)$	8
 $2(x^7 - 2x^5 + 4x^4 - 4x^3 + 2x^2 - 1)$	8	 $5(x^7 + x^5 - x^4 + x^3 - x^2 - 1)$	17
 $3(x^7 - 1)$	124	 $6(x^7 + x^6 - x^4 + x^3 - x - 1)$	11
 $4(x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x - 1)$	16	 $6(x^7 + x^5 - x^2 - 1)$	1
 $4(x^7 + x^6 - x^5 - x^4 + x^3 + x^2 - x - 1)$	4	 $8(x^7 + x^6 + x^5 - x^4 + x^3 - x^2 - x - 1)$	1

Representatives of derived equivalence classes for type E_8			
Quiver and associated polynomial	#	Quiver and associated polynomial	#
 $x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$	128	 $4(x^8 + x^6 - x^5 + 2x^4 - x^3 + x^2 + 1)$	221
 $2(x^8 - x^6 + 2x^5 - 2x^4 + 2x^3 - x^2 + 1)$	64	 $4(x^8 + x^6 - 2x^5 + 4x^4 - 2x^3 + x^2 + 1)$	22
 $2(x^8 - x^6 + x^5 + x^3 - x^2 + 1)$	256	 $5(x^8 + x^6 + x^4 + x^2 + 1)$	167
 $2(x^8 - 2x^6 + 4x^5 - 4x^4 + 4x^3 - 2x^2 + 1)$	16	 $6(x^8 + x^6 + x^5 + x^3 + x^2 + 1)$	38
 $3(x^8 + x^4 + 1)$	384	 $6(x^8 + x^7 + 2x^4 + x + 1)$	118

Quiver and associated polynomial	#	Quiver and associated polynomial	#
 $4(x^8+x^7-x^6+x^5+x^3-x^2+x+1)$	72	 $8(x^8+2x^7+2x^4+2x+1)$	4
 $4(x^8+x^7-x^6+2x^4-x^2+x+1)$	48	 $8(x^8+x^7+x^6+2x^4+x^2+x+1)$	24
 $4(x^8+x^7-2x^6+2x^5+2x^3-2x^2+x+1)$	12		

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