Welschinger invariants of real Del Pezzo surfaces of degree ≥ 3

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Abstract

We give a recursive formula for purely real Welschinger invariants of real Del Pezzo surfaces of degree $K^2 \geq 3$, where in the case of surfaces of degree 3 with two real components we introduce a certain modification of Welschinger invariants and enumerate exclusively the curves traced on the non-orientable component. As an application, we prove the positivity of the invariants under consideration and their logarithmic asymptotic equivalence, as well as congruence modulo 4, to genus zero Gromov-Witten invariants.

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From a dictionary for mathematicians:

Recursion – see *recursion*.

"Mathematicians are also joking"
(compiled by S. N. Fedin)

1 Introduction

In this paper we continue the study of purely real Welschinger invariants of Del Pezzo surfaces. A particular interest of this class of surfaces is related to the fact that the Welschinger invariants of an unnodal real Del Pezzo surface are enumerative; in particular, purely real Welschinger invariants of such a surface count with certain signs the real rational curves that belong to a given linear system and interpolate a suitable amount of real points.

As we proved in [8, 9, 11, 13], if Σ is a real Del Pezzo surface of degree ≥ 4 with nonempty real part (except the case of surfaces containing four disjoint (-1)-curves which form two complex conjugate pairs) and D is a nef and big real divisor class, then the purely real Welschinger invariant $W(\Sigma, D)$ is positive (which implies the existence of interpolating real rational curves). Furthermore, for these surfaces the purely real Welschinger invariants and the corresponding genus zero Gromov-Witten invariants are asymptotically equivalent in the logarithmic scale, *i.e.*,

$$\lim_{n\to+\infty}\frac{\log W(\Sigma,nD)}{n\log n}=\lim_{n\to+\infty}\frac{\log GW(\Sigma,nD)}{n\log n}=-DK_\Sigma \tag{1}$$

(which implies a supexponential growth of the number of interpolating real rational curves provided that Σ is unnodal).

The main result of the present paper is a new recursive formula of Caporaso-Harris type that applies to all purely real Welschinger invariants of real Del Pezzo surfaces of degree ≥ 3 with nonempty real part (Corollary 14, section 3.5). Using this formula, we extend the previous positivity and asymptotic results to the plane blown up at a real points and b pairs of complex conjugate points, where $a+2b\leq 6$, $b\leq 2$, as well as to minimal two-component real conic bundles over $\mathbb{P}^1$ and two-component real cubic surfaces (see Theorems 2, 3, and Remark 20, section 4.1). Additionally, for the surface Σ which is the plane blown up at $a\leq 6$ real points, we prove the monotone dependence of $W(\Sigma,D)$ on the divisor class D and the Mikhalkin-type congruence

$$W(\Sigma,D)=GW(\Sigma,D)\mod 4$$

(both claims were known before for $a\leq 5$, *cf.* [2, 13]).

The present paper contains the first treatment of the positivity and asymptotic relations of Welschinger invariants for surfaces having at least two connected components of the real point set. The original purely real Welschinger invariants are no more unconditionally positive in such a case (see Remark 20). We introduce some variations in the definition of Welschinger signs that give us modified invariants (see details in section 2), which are positive and do satisfy logarithmic equivalence with genus zero Gromov-Witten invariants (Theorem 3, section 4.1.2).

Unlike our previous works [12, 13], here we do not use tropical geometry to derive the recursive formula. Instead, we convert to a real form a complex Caporaso-Harris type formula obtained in [16] for the plane blown up at 6 points. The latter formula is in the spirit of [3, 18]; it differs from the similar formula obtained in [18] by the fact that the 6 blown up points are in general position.

A tropical calculation of purely real Welschinger invariants of the plane blown up at 6 real points was recently proposed by E. Brugallé [1].

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2 Welschinger invariants

Recall the original definition of Welschinger invariants in a form adapted to the case of Del Pezzo surfaces. Let Σ be a real *unnodal* (*i.e.*, not containing any rational $(-n)$-curve, $n \ge 2$) Del Pezzo surface, and let $D \subset \Sigma$ be a real effective divisor class. Consider a connected component F of the real point set $\mathbb{R}\Sigma$ of Σ and a generic set $\boldsymbol{p} \subset F$ of $c_1(\Sigma) \cdot D - 1$ points. The set $\mathcal{R}(\Sigma, D, \boldsymbol{p})$ of real rational curves $C \in |D|$ passing through the points of $\boldsymbol{p}$ is finite, and all these curves are nodal and irreducible. Due to the Welschinger theorem [19] (and the genericity of the complex structure on Σ), the number

$$W(\Sigma, D, \boldsymbol{p}) = \sum_{C \in \mathcal{R}(\Sigma, D, \boldsymbol{p})} (-1)^{s(C)} , \tag{2}$$

where $s(C)$ is the number of *solitary nodes* of C (*i.e.*, real points, where a local equation of the curve can be written over $\mathbb{R}$ in the form $x^2 + y^2 = 0$), does not depend on the choice of a generic set $\boldsymbol{p} \subset F$. We denote this (original) Welschinger invariant by $W(\Sigma, D, F)$.

If $\mathbb{R}\Sigma$ has more than one connected component (for example, if Σ is a cubic surface and $\mathbb{R}\Sigma$ has two connected components), we modify the above construction of invariants in the following way. Let, as above, F be one of these components. For a real nodal curve $C \subset \Sigma$, we introduce its *Welschinger weight reduced to* F by putting $w_F(C) = (-1)^{s(C,F)}$, where $s(C, F)$ is the number of real solitary nodes of C belonging to F. Then, given a real effective divisor class D on Σ, and a generic set $\boldsymbol{z}$ of $c_1(\Sigma)D - 1$ points in F, we define the *Welschinger number of D reduced to F* by the formula

$$W_F(\Sigma, D, \boldsymbol{z}) = \sum_{C \in \mathcal{R}(\Sigma, D, \boldsymbol{p})} w_F(C) . \tag{3}$$

Such a *twisting* of the Welschinger construction can be reformulated and slightly generalized. In addition to choosing one of the real components, F, let us pick a homology class $\phi \in H_2(\Sigma \setminus F; \mathbb{Z}/2\mathbb{Z})$ invariant under the action of complex conjugation, $\mathrm{conj}_*\phi = \phi$. Given a real effective divisor class D on Σ, and a generic set $\boldsymbol{z}$ of $c_1(\Sigma)D - 1$ points in F, we define the *twisted Welschinger number of D* by the formula

$$W_\phi(\Sigma, D, \boldsymbol{z}) = \sum_{C \in \mathcal{R}(\Sigma, D, \boldsymbol{z})} w_\phi(C), \quad w_\phi(C) = (-1)^{s(C) + C_\pm \circ \phi} , \tag{4}$$

where $C_\pm$ denotes any of the two halves of C (any of the two discs cut from C by $\mathbb{R}C$) and $C_\pm \circ \phi \in \mathbb{Z}/2\mathbb{Z}$. Clearly, when ϕ_F is the homology class realized in $H_2(\Sigma \setminus F; \mathbb{Z}/2\mathbb{Z})$ by the union of the components of $\mathbb{R}\Sigma \setminus F$, we get

$$W_{\phi_F}(\Sigma, D, \boldsymbol{z}) = W_F(\Sigma, D, \boldsymbol{z}) \ .$$

Proposition 1 *The number $W_\phi(\Sigma, D, \boldsymbol{z})$ does not depend on the choice of a generic set $\boldsymbol{z}$ of $c_1(\Sigma)D - 1$ points in F.*

Proof. The statement is an immediate consequence of the invariance of $W(\Sigma, D, F)$ due to the following observation.

In a one-parametric family of curves $C(t)$ of class D interpolating $c_1(\Sigma) \circ D - 2$ fixed generic points of F and one additional point of F moving generically, the homology classes of the discs $C_\pm(t)$ are jumping only at those moments $t = t_0$ when the curve $C(t_0)$ splits into two irreducible components $C'(t_0)$ and $C''(t_0)$. When such a jump happens, the number $(-1)^{s(C(t))+C_+(t)\circ\phi}$ does not change, since $C_+(t < t_0) \circ \phi = C'_+(t_0) \circ \phi + C''_+(t_0) \circ \phi = C'_+(t_0) \circ \phi + \mathrm{conj}_* C''_+(t_0) \circ \mathrm{conj}_*\phi = C'_+(t_0) \circ \phi + C''_-(t_0) \circ \phi = C_+(t > t_0) \circ \phi$. $\square$

The above proposition implies existence of modified Welschinger invariants $W_\phi(\Sigma, D) = W_\phi(\Sigma, D, \boldsymbol{z})$. As a particular example, we may take $\phi = \phi_F$, the fundamental class of the union of real components of Σ different from F. This is the invariant which we use below in the case of two-component real cubic surfaces (and conic bundles); we denote it, in accordance with our previous notation, $W_F(\Sigma, D)$.

One may also choose as ϕ any combination of the fundamental classes of real components different from F, and more generally, combine them with vanishing classes between a pair of real components different from F. In fact, one may prove, that for multi-component real structures there do exist twists such that some of the curves C in $\mathcal{R}(\Sigma, D, \boldsymbol{p})$ (with certain D depending on the twist) change and some do not change the sign with respect to the original Welschinger definition. Note also that the number of independent possible twists is preserved under real blow-ups (that is, a blow-up at a real point or at a pair of complex conjugate points), so that interesting twists exist only for surfaces with a disconnected real part.

3 Recursive formula for Welschinger invariants

3.1 Preliminaries

In this chapter, we consider unnodal Del Pezzo surfaces of degree 3. From the complex point of view, such a surface, denoted by $\mathbb{P}^2_6$, is the complex projective plane blown up at 6 points in general position. Denote by $L \subset \mathbb{P}^2_6$ the strict transform of a generic line, and by $E_1, ..., E_6$ the exceptional divisors of the blow up.

We equip $\Sigma = \mathbb{P}^2_6$ with a real structure, *i.e.*, an anti-holomorphic involution $\text{conj} : \Sigma \to \Sigma$. Then, as is well known, the surface becomes isomorphic either to $\mathbb{P}^2_{a,b}$, $a + 2b = 6$, that is, the plane $\mathbb{P}^2$ (equipped with its standard real structure) blown up at a real and b pairs of complex conjugate points, all in general position, or to $\mathbb{B}_1$, a real cubic surface with the two-component real part. By F we denote the non-orientable connected component of $\mathbb{R}\Sigma$ (which is the only one if $\Sigma = \mathbb{P}^2_{a,b}$), and we choose a class $\phi \in H_2(\Sigma \setminus F; \mathbb{Z}/2\mathbb{Z})$ invariant under the action of complex conjugation, $\text{conj}_*\phi = \phi$.

We pick a real smooth (-1)-curve E on Σ with $\mathbb{R}E \subset F$: if $\Sigma = \mathbb{P}^2_{a,b}$, then choose $E \in |L - E_1 - E_2|$, where E_1 and E_2 are assumed to be either both real, or complex conjugate; if $\Sigma = \mathbb{B}_1$, then choose for E any of the three lines whose real parts are contained in F (see, for example, [15]).

By $\text{Pic}(\Sigma, E)$ we denote the subsemigroup of $\text{Pic}(\Sigma)$ generated by (complex) irreducible curves, crossing E non-negatively. The involution of complex conjugation conj acts on $\text{Pic}(\Sigma, E)$. By $\text{Pic}^{\mathbb{R}}(\Sigma, E)$ we denote the disjoint union of the sets

$$\{D \in \text{Pic}(\Sigma, E) \ : \ \text{conj} D = D\}$$

and

$$\big\{\{D_1, D_2\} \in \text{Sym}^2(\text{Pic}(\Sigma, E)) \ : \ \text{conj} D_1 = D_2\big\} \ .$$

For an element $\mathcal{D} \in \text{Pic}^{\mathbb{R}}(\Sigma, E)$, define $[\mathcal{D}] \in \text{Pic}(\Sigma, E)$ by

$$[\mathcal{D}] = \begin{cases} D, & \text{if } \mathcal{D} = D, \ \text{a divisor class,} \\ D_1 + D_2, & \text{if } \mathcal{D} = \{D_1, D_2\}, \ \text{a pair of divisor classes.} \end{cases}$$

Let $\mathbb{Z}^\infty_+$ be the direct sum of countably many additive semigroups $\mathbb{Z}_+ = \{m \in \mathbb{Z} \mid m \ge 0\}$ with the standard basis

$$\theta_i = (\alpha_1, \alpha_2, ...), \quad \alpha_i = 1, \quad \alpha_j = 0, \ j \ne i \ .$$

For $\alpha, \alpha' \in \mathbb{Z}^\infty_+$, the relation $\alpha \ge \alpha'$ means that $\alpha - \alpha' \in \mathbb{Z}^\infty_+$. For $\alpha = (\alpha_1, \alpha_2, ...) \in \mathbb{Z}^\infty_+$ put

$$\|\alpha\| = \sum_{k=1}^\infty \alpha_k, \quad I\alpha = \sum_{k=1}^\infty k\alpha_k, \quad I^\alpha = \prod_{k=1}^\infty k^{\alpha_k}, \quad \alpha! = \prod_{k=1}^\infty \alpha_k! \ ,$$

and for $\alpha^{(0)}, ..., \alpha^{(m)}, \alpha \in \mathbb{Z}^\infty_+$ such that $\alpha^{(0)} + ... + \alpha^{(m)} \le \alpha$, put

$$\binom{\alpha}{\alpha^{(0)}, ..., \alpha^{(m)}} = \frac{\alpha!}{\alpha^{(0)}!...\alpha^{(m)}!(\alpha - \alpha^{(0)} - ... - \alpha^{(m)})!} \ .$$

Introduce also the semigroup

$$\mathbb{Z}^{\infty,\text{odd}}_+ = \{\alpha \in \mathbb{Z}^\infty_+ \ : \ \alpha_{2i} = 0, \ i \ge 1\} \ .$$

For an element $\mathcal{D} \in \text{Pic}^{\mathbb{R}}(\Sigma, E)$ and a vector $\beta \in \mathbb{Z}^\infty_+$, put

$$R_\Sigma(\mathcal{D}, \beta) = -[\mathcal{D}](K_\Sigma + E) + \|\beta\| - \begin{cases} 1, & \text{if } \mathcal{D} = D, \ \text{a divisor class,} \\ 2, & \text{if } \mathcal{D} = \{D_1, D_2\}, \ \text{a pair of divisor classes.} \end{cases}$$

3.2 Families of real curves on Σ

Let $\mathcal{D} \in \mathrm{Pic}^{\mathbb{R}}(\Sigma, E)$, and vectors $\alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}} \in \mathbb{Z}_+^\infty$ satisfy $I(\alpha + \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}}) = [\mathcal{D}]E$. Let $\boldsymbol{p}^\flat = \{p_{i,j} \ : \ i \ge 1,\ 1 \le j \le \alpha_i\}$ be a sequence of $\|\alpha\|$ distinct real generic points on E. Such tuples $(\mathcal{D}, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$ are called *admissible*.

By $V_\Sigma^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$ we denote the closure of the family of real reduced curves C belonging to the linear system defined by $[\mathcal{D}]$ and such that

(i) if $\mathcal{D} = D$ is a divisor class, then $C \in |D|$ is an irreducible (over $\mathbb{C}$) rational curve,

(ii) if $\mathcal{D} = \{D_1, D_2\}$ is a pair of divisors classes, then $C = C_1 \cup C_2$, where $C_1 \in |D_1|$, $C_2 \in |D_2|$ are distinct, irreducible, rational, conjugate imaginary curves;

(iii) $C \cap E$ consists of the points $\boldsymbol{p}^\flat$ and $\|\beta^{\mathrm{re}} + 2\beta^{\mathrm{im}}\|$ other points; $\|\beta^{\mathrm{re}}\|$ of the latter points are real, and the remaining points form $\|\beta^{\mathrm{im}}\|$ pairs of complex conjugate points;

(iv) C has one local branch at each of the points of $C \cap E$, and the intersection multiplicities of C with E are as follows:

- $(C \cdot E)_{p_{i,j}} = i$ for all $i \ge 1$, $1 \le j \le \alpha_i$,
- for each $i \ge 1$, there are precisely β_i^{re} real points $q \in (C \cap E) \setminus \boldsymbol{p}^\flat$ such that $(C \cdot E)_q = i$;
- for each $i \ge 1$, there are precisely β_i^{im} pairs q, q' of complex conjugate points of $C \cap E$ such that $(C \cdot E)_q = (C \cdot E)_{q'} = i$.

Lemma 2 *Let $(\mathcal{D}, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$ be an admissible tuple. If $V_\Sigma^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$ is nonempty, then $R_\Sigma(\mathcal{D}, \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}}) \ge 0$, and each component of $V_\Sigma^{\mathbb{R}}(\mathcal{D}, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$ has dimension $\le R_\Sigma(\mathcal{D}, \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}})$. Moreover, a generic element of any component of $V_\Sigma^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$ of dimension $R_\Sigma(\mathcal{D}, \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}})$ is a nodal curve, nonsingular along E.*

Proof. Follows from [16, Propositions 2.1 and 2.2]. We notice only that the (-1)-curve $E \in |2L - E_1 - ... - E_5|$, chosen in [16], can be replaced by any other (-1)-curve. $\square$

Let $(\mathcal{D}, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$ be an admissible tuple such that $R_\Sigma(\mathcal{D}, \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}}) \ge 0$. Pick a set $\boldsymbol{p}^\sharp$ of $R_\Sigma(\mathcal{D}, \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}})$ generic points of $F \setminus E$ and denote by $V_\Sigma^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$ the set of curves belonging to $V_\Sigma^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$ and passing through the points of $\boldsymbol{p}^\sharp$.

Lemma 3 *Let $(\mathcal{D}, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$ be an admissible tuple.*

(1) if $\mathcal{D} = D$ is a divisor class, then $V_\Sigma^{\mathbb{R}}(D, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$ is a finite set of real nodal irreducible rational curves, nonsingular along E;

(2) if $\mathcal{D} = \{D_1, D_2\}$ is a pair of divisor classes, then $V_\Sigma^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$ is nonempty only if $\alpha = \beta^{\rm re} = 0$, $R_\Sigma(\mathcal{D}, 2\beta^{\rm im}) = 0$, and $\boldsymbol{p}^\flat = \boldsymbol{p}^\sharp = \emptyset$; furthermore, in this case the set $V_\Sigma^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \boldsymbol{p}^\sharp)$ is finite.

Proof. By Lemma 2 we have to show only that $R_\Sigma(\mathcal{D}, 2\beta^{\rm im}) = 0$ is necessary for the nonemptyness of $V_\Sigma^{\mathbb{R}}(\mathcal{D}, 0, 0, \beta^{\rm im}, \emptyset, \boldsymbol{p}^\sharp)$ with $\mathcal{D} = \{D_1, D_2\}$. A curve $C \in V_\Sigma^{\mathbb{R}}(\mathcal{D}, 0, 0, \beta^{\rm im}, \emptyset, \boldsymbol{p}^\sharp)$ splits in the following way:

$$C = C_1 \cup C_2, \quad C_1 \in |D_1|, \quad C_2 \in |D_2|, \quad {\rm conj} C_1 = C_2 \ ,$$

and, by [16, Proposition 2.1], the component C_1 varies in a family of complex dimension

$$-D_1(K_\Sigma + E) + \|\beta^{\rm im}\| - 1 = \frac{1}{2} R_\Sigma(\mathcal{D}, 2\beta^{\rm im}) \ .$$

Hence, a curve C can match at most $\frac{1}{2} R_\Sigma(\mathcal{D}, 2\beta^{\rm im})$ generic points in $F \setminus E$, and the claim follows. □

Lemma 4 (see, for example, [15]) *(1) The linear system $|-(K_\Sigma + E)|$ is of dimension 1 and contains precisely two nonsingular curves Q', Q'' tangent to E, and five reducible curves; each of the latter curves consists of two distinct smooth (-1)-curves intersecting at one point.*

(2) If $\Sigma = \mathbb{B}_1$, then

- *(i) one of the five reducible curves in the linear system $|-(K_\Sigma + E)|$ is formed by two real lines; each of the other four reducible curves is formed by two complex conjugate lines which intersect in one real point.*
- *(ii) Σ has exactly three real lines, and these lines generate the semigroup of real effective divisor classes on Σ.*

The three real lines of $\Sigma = \mathbb{B}_1$ are denoted by L_1, L_2, and L_3 (if the contrary is not explicitly stated, we always assume that $E = L_1$). The lines forming the four pairs of complex conjugate lines are denoted by L_j, $j = 4, \ldots, 11$, in a way that, for any $i = 2, 3, 4, 5$, the lines L_{2i} and L_{2i+1} are complex conjugate.

The next two lemmas follow from [16, Proposition 2.3] and Lemma 4.

Lemma 5 *Let $\Sigma = \mathbb{P}^2_{a,b}$, $a + 2b = 6$, and $E \in |L - E_1 - E_2|$. Then, among the sets $V_\Sigma^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat)$, where $(\mathcal{D}, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat)$ is an admissible tuple such that*

$$[\mathcal{D}]E > 0, \quad R_\Sigma(\mathcal{D}, \beta^{\rm re} + 2\beta^{\rm im}) = 0,$$

the only nonempty sets are as follows:

(1) in the case of a divisor class $\mathcal{D} = D$,

(1i) $V^{\mathbb{R}}_\Sigma(E_i,0,\theta_1,0,\emptyset)$, *where* $i=1,2$, *consists of one element, provided that* E_1 *and* E_2 *are real;*

(1ii) $V^{\mathbb{R}}_\Sigma(L-E_i-E_j,0,\theta_1,0,\emptyset)$, *where* $3\le i<j\le 6$, *consists of one element, provided that* E_i *and* E_j *are either both real or are complex conjugate;*

(1iii) $V^{\mathbb{R}}_\Sigma(-(K_\Sigma+E)-E_i,0,\theta_1,0,\emptyset)$, *where* $i=1,2$, *consists of one element, provided that* E_1 *and* E_2 *are real;*

(1iv) $V^{\mathbb{R}}_\Sigma(-(K_\Sigma+E),0,\theta_2,0,\emptyset)$ *consists of two elements* Q',Q'', *provided that* Q' *and* Q'' *are both real;*

(1v) $V^{\mathbb{R}}_\Sigma(-(K_\Sigma+E),\theta_1,\theta_1,0,\boldsymbol{p}^\flat)$ *consists of one element;*

(1vi) $V^{\mathbb{R}}_\Sigma(-s(K_\Sigma+E)+L-s_1E_1-s_2E_2-E_i,\alpha,0,0,\boldsymbol{p}^\flat)$ *consists of one element, if* $s\ge 0$, $0\le s_1,s_2\le 1$, $s_1+s_2\le 2s$, $3\le i\le 6$, *the divisor* E_i *is real, and the divisors* E_1,E_2 *are real whenever* $s_1\ne s_2$*;*

(1vii) $V^{\mathbb{R}}_\Sigma(-s(K_\Sigma+E)-s_1E_1-s_2E_2+E_i,\alpha,0,0,\boldsymbol{p}^\flat)$ *consists of one element, if* $s\ge 1$, $0\le s_1,s_2\le 1$, $s_1+s_2<2s$, $3\le i\le 6$, *the divisor* E_i *is real, and the divisors* E_1,E_2 *are real whenever* $s_1\ne s_2$*;*

(2) in the case of a pair $\mathcal{D}=\{D_1,D_2\}$ *of divisor classes,*

(2i) $V^{\mathbb{R}}_\Sigma(\{E_1,E_2\},0,0,\theta_1,\emptyset)$ *consists of one element, provided that* E_1 *and* E_2 *are complex conjugate;*

(2ii) $V^{\mathbb{R}}_\Sigma(\{L-E_i-E_j,L-E_i-E_k\},0,0,\theta_1,\emptyset)$, *where* $\{i,j,k\}\subset\{3,4,5,6\}$, *consists of one element, if* E_i *is real, and* E_j,E_k *are complex conjugate;*

(2iii) $V^{\mathbb{R}}_\Sigma(\{-(K_\Sigma+E)-E_1,-(K_\Sigma+E)-E_2\},0,0,\theta_1,\emptyset)$ *consists of one element, provided that* E_1 *and* E_2 *are complex conjugate;*

(2iv) $V^{\mathbb{R}}_\Sigma(\{L-E_i-E_j,L-E_k-E_l\},0,0,\theta_1,\emptyset)$, *where* $\{i,j,k,l\}=\{3,4,5,6\}$, *consists of one element, if* E_i,E_j *and* E_k,E_l *are two pairs of complex conjugate exceptional divisors;*

(2v) $V^{\mathbb{R}}_\Sigma(\{-(K_\Sigma+E),-(K_\Sigma+E)\},0,0,\theta_2,\emptyset)$ *consists of one element* $\{Q',Q''\}$, *provided that* Q' *and* Q'' *are complex conjugate.*

Lemma 6 *Let* $\Sigma=\mathbb{B}_1$ *and* $E=L_1$. *Then, among the sets* $V^{\mathbb{R}}_\Sigma(\mathcal{D},\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat)$, *where* $(\mathcal{D},\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat)$ *is an admissible tuple such that*

$$[\mathcal{D}]E>0,\quad R_\Sigma(\mathcal{D},\beta^{\mathrm{re}}+2\beta^{\mathrm{im}})=0,$$

the only nonempty sets are as follows:

(1) in the case of a divisor class $\mathcal{D}=D$,

(1i) $V^{\mathbb{R}}_\Sigma(L_i,0,\theta_1,0,\emptyset)$, *where* $i=2,3$, *consists of one element* L_i*;*

(1ii) $V^{\mathbb{R}}_\Sigma(-(K_\Sigma+E),0,\theta_2,0,\emptyset)$ *consists of two elements* Q' *and* Q''*;*

(1iii) $V^{\mathbb{R}}_\Sigma(-(K_\Sigma+E),\theta_1,\theta_1,0,\boldsymbol{p}^\flat)$, *consists of one element;*

(2) *in the case of a pair* $\mathcal{D}=\{D_1,D_2\}$ *of divisor classes,*

$V^{\mathbb{R}}_{\Sigma}(\{L_{2i},L_{2i+1}\},0,0,\theta_1,\emptyset)$, *where* $i=2,3,4,5$, *consists of one element* $\{L_{2i},L_{2i+1}\}$. □

3.3 Deformation diagrams

Let $(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat)$ be an admissible tuple, where $D\in\mathrm{Pic}^{\mathbb{R}}(\Sigma,E)$ is a divisor class and $R_\Sigma(D,\beta^{\rm re}+2\beta^{\rm im})>0$. Pick a set $\widetilde{\boldsymbol{p}}^\sharp$ of $R_\Sigma(D,\beta^{\rm re}+2\beta^{\rm im})-1$ generic real points of $F\setminus E$, a generic real point $p\in E\setminus\boldsymbol{p}^\flat$, and a smooth real algebraic curve germ Λ crossing E transversally at p. Denote by $\Lambda^+=\{p(t)\ :\ t\in(0,\varepsilon)\}$ a parameterized connected component of $\Lambda\setminus\{p\}$ with $\lim_{t\to0}p(t)=p$. There exists $\varepsilon_0>0$ such that, for all $t\in(0,\varepsilon_0]$, the sets $V_\Sigma(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\widetilde{\boldsymbol{p}}^\sharp\cup\{p(t)\})$ are finite, their elements remain nodal, nonsingular along E as t runs over the interval $(0,\varepsilon_0]$, and the closure in $V_\Sigma(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat)$ of the family

$$V=\bigcup_{t\in(0,\varepsilon_0]}V_\Sigma(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\widetilde{\boldsymbol{p}}^\sharp\cup\{p(t)\})\qquad(5)$$

is a union of real algebraic arcs which are disjoint for $t>0$. This closure is called a *deformation diagram* of $(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\widetilde{\boldsymbol{p}}^\sharp,p)$. The elements of $V_\Sigma(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\widetilde{\boldsymbol{p}}^\sharp\cup\{p(\varepsilon_0)\}$ are called *leaves* of the deformation diagram, and the elements of $\overline{V}\setminus V$ are called *roots* of the deformation diagram.

Lemma 7 *Each connected component of a deformation diagram* $\overline{V}$ *defined by (5) contains exactly one root. The roots are curves* $C\in|D|$ *of the following two types.*

(I) The curve C *is a generic member of an* $(R_\Sigma(D,\beta^{\rm re}+2\beta^{\rm im})-1)$*-dimensional component of one of the families*

$$V^{\mathbb{R}}_\Sigma(D,\alpha+\theta_j,\beta^{\rm re}-\theta_j,\beta^{\rm im},\boldsymbol{p}^\flat\cup\{p\},\widetilde{\boldsymbol{p}}^\sharp\})\ ,$$

where j *is a natural number such that* $\beta^{\rm re}_j>0$.

(R) The curve C *decomposes into* E *and curves of the following four types (for each type, the collection of curves can be empty):*

(R1) *distinct reduced irreducible over* $\mathbb{R}$ *curves* $C^{(i)}$, $1\le i\le m$, *which are generic members in some* $R_\Sigma(\mathcal{D}^{(i)},(\beta^{\rm re})^{(i)}+2(\beta^{\rm im})^{(i)})$*-dimensional components of families*

$$V^{\mathbb{R}}_\Sigma(\mathcal{D}^{(i)},\alpha^{(i)},(\beta^{\rm re})^{(i)},(\beta^{\rm im})^{(i)},(\boldsymbol{p}^\flat)^{(i)},(\boldsymbol{p}^\sharp)^{(i)})\ ,$$

respectively, where $\mathcal{D}^{(i)}$ *is a divisor class if* $C^{(i)}$ *is irreducible over* $\mathbb{C}$, *and is a pair of divisor classes if* $C^{(i)}$ *is the union of two complex conjugate components, and, in addition,* $\mathcal{D}^{(i)}$ *is neither* $-(K_\Sigma+E)$, *nor* $\{-(K_\Sigma+E),-(K_\Sigma+E)\}$, $1\le i\le m$, *(R2)* *distinct curves* $jQ(p_{j,s})$*, where* $p_{j,s}$ *runs over some subset* $(\boldsymbol{p}^\flat)^{(0)}$ *of* $\boldsymbol{p}^\flat$*, and* $Q(p_{j,s}) \in |-(K_\Sigma + E)|$ *is the (real) curve passing through* $p_{j,s}$*,*

(R3) *distinct curves* $j(z)Q(z)$*, where* z *runs over some subset* $(\boldsymbol{p}^\sharp)^{(0)}$ *of* $\widetilde{\boldsymbol{p}}^\sharp$*,* $j(z) \ge 1$*, and* $Q(z) \in |-(K_\Sigma + E)|$ *is the (real) curve passing through* z*,*

(R4) *curves* $l'Q'$ *and* $l''Q''$*, where* $Q', Q'' \in |-(K_\Sigma + E)|$ *are the two curves tangent to* E *(cf. Lemma 4(1)), and* $l' = l'' \ge 0$ *if* Q', Q'' *are complex conjugate, and* $l', l'' \ge 0$ *if* Q', Q'' *are real.*

Furthermore, the parameters of the above decomposition are subject to the following restrictions:

- $\sum_{i=1}^m \alpha^{(i)} \le \alpha - \alpha^{(0)}$*, where* $\alpha^{(0)}$ *encodes the sequence of multiplicities* j *over all* $p_{j,k} \in (\boldsymbol{p}^\flat)^{(0)}$*,*
- $\sum_{i\in S} (\beta^{\rm im})^{(i)} = \beta^{\rm im}$*, where* $S = \{i \in [1, m] \ :\ \mathcal{D}^{(i)}$ *is a divisor class*$\}$*,*
- $\beta^{(0)} \le \beta^{\rm re}$*, where* $\beta^{(0)}$ *encodes the sequence of multiplicities* $j(z)$ *over all* $z \in (\boldsymbol{p}^\sharp)^{(0)}$*,*
- *there is a sequence of vectors* $\gamma^{(i)} \in \mathbb{Z}_+^\infty$*,* $i \in S$*, such that* $\|\gamma^{(i)}\| = 1$*,* $\gamma^{(i)} \le (\beta^{\rm re})^{(i)}$*,* $i \in S$*, and* $\sum_{i\in S}((\beta^{\rm re})^{(i)} - \gamma^{(i)}) = \beta^{\rm re} - \beta^{(0)}$*,*
- $\sum_{i=1}^m [\mathcal{D}^{(i)}] = D - E + (I\alpha^{(0)} + I\beta^{(0)} + l' + l'')(K_\Sigma + E)$*.*

Proof. All claims follow from [16, Proposition 3.1]. We only make two comments. In the case (I), an imaginary moving intersection point with E cannot merge to p, since otherwise the conjugate moving intersection point must merge to p too. In the case (R), in view of [16, Lemma 4.2(2)] and due to the rationality of curves in $V_\Sigma^{\mathbb{R}}(D, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \widetilde{\boldsymbol{p}}^\sharp \cup \{p(\varepsilon_0)\}$, *cf.* [16, Corollary 5.2], in the deformation of C induced by $\overline{V}$, for each irreducible over $\mathbb{C}$ component $C' \ne E$ of C, precisely one of the intersection points of C' with $E \setminus \boldsymbol{p}^\flat$ is smoothed out, whereas the remaining intersection points of C' with E turn into smooth points of the deformed curve, where it crosses E with the same multiplicity as C' (these intersection points stay fixed if they were in $\boldsymbol{p}^\flat$ or move along E otherwise). To get restrictions on the parameters of the decomposition, we notice also that, by Lemmas 3(2), 5(2), and 6(2), if $C^{(i)}$ has a pair of complex conjugate components, then each component crosses E at a unique point, and this point is imaginary. □

Lemma 8 *Let* C *be the root of a connected component* δ *of the deformation diagram* $\overline{V}$*. Assume that* $C \in V_\Sigma^{\mathbb{R}}(D, \alpha + \theta_j, \beta^{\rm re} - \theta_j, \beta^{\rm im}, \boldsymbol{p}^\flat \cup \{p\}, \widetilde{\boldsymbol{p}}^\sharp)$ *is of type (I). If* j *is odd, then* δ *has a unique leaf; if* j *is even, then* δ *has two leaves. In both cases, each leaf has the same real topology as the root* C*.*

Proof. Straightforward from [16, Formulas (18) and (19)]. □

Let C be the root of a connected component δ of the deformation diagram $\overline{V}$. Assume that C is of type (R). To describe the leaves of δ, we introduce *deformation labels*, certain nodal curves specified below. Each deformation label is given by a polynomial equation $\psi(x,y)=0$ in the toric surface $\mathrm{Tor}(\psi)$ defined by the Newton polygon of ψ. The following list contains the deformation labels and, in the case of real deformation labels, specifies the number of their solitary nodes.

(DL1) Deformation label defined by the equation

$$\psi(x,y)=y^2+1+y\cdot\mathrm{cheb}_j(x)=0\ ,$$

where j is a positive odd number, and $\mathrm{cheb}_j(x)=\cos(j\arccos x)$; this curve has $j-1$ solitary nodes.

(DL2) Two deformation labels defined by the equations

$$\psi_1(x,y)=y^2+1+y\cdot\mathrm{cheb}_j(x)=0\quad\text{and}\quad\psi_2(x,y)=\psi_1(\sqrt{-1}\ x,y)=0\ ,$$

where j is a positive even number; the former curve has $j-1$ solitary nodes, the latter curve has no solitary nodes.

(DL3) Deformation label defined by the equation

$$\psi(x,y)=(x-1)(y^j-x)=0\ ;$$

this curve has no solitary nodes.

(DL4) Deformation label defined by the equation

$$\psi(x,y)=(x-1)(1+x((y+2^{1/j})^j-1))=0\ ,$$

where j is a positive odd number; this curve has no solitary nodes.

(DL5) Two deformation labels defined by the equations

$$\psi(x,y)=(x-1)(1+x((y\pm 2^{1/j})^j-1))=0\ ,$$

where j is a positive even number; these curves have no solitary nodes.

(DL6) Deformation label defined by the equation

$$\psi(x,y)=1+\frac{y+x^2}{2y}\left(\mathrm{cheb}_{l'+1}\left(\frac{\pm y}{2^{(l'-1)/(l'+1)}}+y'\right)-1\right)=0\ ,$$

where l' is a positive even number, and y' is the only positive simple root of $\mathrm{cheb}_{l'+1}(y)-1$; this curve has either l' solitary nodes, or no solitary nodes at all.

(DL7) Two deformation labels given by the equations

$$\psi_1(x,y) = 1 + \frac{y+x^2}{2y}\left(\mathrm{cheb}_{l'+1}\left(\frac{y}{2^{(l'-1)/(l'+1)}} + y'\right) - 1\right) = 0\ ,$$
$$\psi_2(x,y) = 1 + \frac{y+x^2}{2y}\left(\mathrm{cheb}_{l'+1}\left(\frac{y}{2^{(l'-1)/(l'+1)}} - y'\right) - 1\right) = 0,$$

where l' is a positive odd number, and y' is the only positive simple root of $\mathrm{cheb}_{l'+1}(y) - 1$; the former curve has l' solitary nodes, the latter curve has no solitary nodes.

(DL8) $l'+1$ deformation labels defined by the equations

$$\psi(x,y) = 1 + \frac{y+\sqrt{-1}\,x^2}{2y}\left(\mathrm{cheb}_{l'+1}\left(\frac{y\varepsilon}{2^{(l'-1)/(l'+1)}} + y'\right) - 1\right) = 0, \quad \varepsilon^{l'+1} = 1\ ,$$

where l' is a positive integer.

(DL9) $l''+1$ deformation labels defined by the equations

$$\psi(x,y) = 1 + \frac{y-\sqrt{-1}\,x^2}{2y}\left(\mathrm{cheb}_{l''+1}\left(\frac{y\varepsilon}{2^{(l''-1)/(l''+1)}} + y'\right) - 1\right) = 0, \quad \varepsilon^{l''+1} = 1\ ,$$

where l'' is a positive integer.

Consider now the following data:

(C1) choose a sequence of vectors $\gamma^{(i)} \in \mathbb{Z}_+^\infty$, $i = 1, \ldots, m$, such that

- $\|\gamma^{(i)}\| = 1$ for $i \in S = \{i \ : \ 1 \le i \le m,\ \mathcal{D}^{(i)}$ is a divisor class$\}$,
- $\gamma^{(i)} = 0$ for $1 \le i \le m$ and $i \notin S$,
- $\gamma^{(i)} \le (\beta^{\mathrm{re}})^{(i)}$, $i = 1, \ldots, m$,
- $\sum_{i=1}^m ((\beta^{\mathrm{re}})^{(i)} - \gamma^{(i)}) = \beta^{\mathrm{re}} - \beta^{(0)}$;

(C2) for each $i \in S$, choose a real point $q_i \in (C^{(i)} \cap E) \setminus \boldsymbol{p}^\flat$ such that $(C^{(i)} \cdot E)_{q_i} = j$, where $\gamma^{(i)} = \theta_j$;

(C3) for each $z \in (\boldsymbol{p}^\sharp)^{(0)}$, choose a point $q(z)$ which is of one of the two real points of $Q(z) \cap E$.

Denote by $\mathcal{C}(C)$ the set of all possible choices of data (C1)-(C3). For any element of $\mathcal{C}(C)$, a *suitable deformation label collection* is a sequence of deformation labels as follows:

- one deformation label for each point q_i chosen in (C2); this deformation label is of type (DL1) or (DL2) depending on the parity of $j = (C^{(i)} \cdot E)_{q_i}$;
- one deformation label of type (DL3) for each component $jQ(p_{j,s})$ of C,

- one deformation label for each component $j(z)Q(z)$ of C and each point $q(z)$ chosen in (C3); this deformation label is of type (DL4) or (DL5) depending on the parity of $j = j(z)$;
- if Q' and Q'' are real, one deformation label for $l'Q'$ and one deformation label for $l''Q''$, the former (respectively, the latter) deformation label is of type (DL6) or (DL7) depending on the parity of l' (respectively, l'');
- if Q' and Q'' are complex conjugate (in this case $l' = l''$), one deformation label of type (DL8) for $l'Q'$ and the complex conjugate deformation label of type (DL9) for $l''Q''$.

Denote by $\mathcal{D}ef(C, \sigma)$ the set of all suitable deformation label collections of a given element $\sigma \in \mathcal{C}$.

Lemma 9 *Let C be the root of a connected component δ of the deformation diagram $\overline{V}$. Assume that C is of type (R).*

(1) Suppose that $\beta^{(0)} \in \mathbb{Z}_+^{\infty,\text{odd}}$, $\sum_{i=1}^m (\beta^{\text{re}})^{(i)} - \beta^{\text{re}} \in \mathbb{Z}_+^{\infty,\text{odd}}$, and either Q', Q'' are complex conjugate, or Q', Q'' are real and l', l'' are both even. Then, there is a one-to-one correspondence between the set of leaves of δ and the disjoint union of the sets $\mathcal{D}ef(C, \sigma)$ over all $\sigma \in \mathcal{C}(C)$.

Suppose that either $\beta^{(0)} \notin \mathbb{Z}_+^{\infty,\text{odd}}$, or $\sum_{i=1}^m (\beta^{\text{re}})^{(i)} - \beta^{\text{re}} \notin \mathbb{Z}_+^{\infty,\text{odd}}$, or both Q', Q'' are real and at least one of l', l'' is odd. Then, the set of leaves of δ is in one-to-one correspondence with the disjoint union of sets $\mathcal{D}ef(C, \sigma)$, where σ runs over some nonempty subset of $\mathcal{C}(C)$.

(2) The set of solitary nodes of each leaf of δ bijectively corresponds to the disjoint union of the sets of solitary nodes of the corresponding deformation labels and the sets of solitary nodes of the components $C^{(i)}$, $i = 1, \ldots, m$, of C. Moreover, the solitary nodes coming from the deformation labels all belong to the connected component $F \subset \mathbb{R}\Sigma$ which contains the line $\mathbb{R}E$.

Proof. Statement (1) follows from [16, Lemma 4.13] (one-to-one correspondence) and [16, Lemma 4.2] (geometry of deformation), which describe all complex deformations of C via so-called deformation patterns. Our deformation labels can be viewed as normalized versions of these deformation patterns. The restricted correspondence in the second case of assertion (1) comes from the fact that, for some elements of $\mathcal{C}(C)$, all complex deformations of C are non-real. Statement (2) follows from [16, Lemmas 4.4, 4.6, 4.8, and 4.9], where one can find a complete description of complex deformation patterns and formulas for them. $\square$

3.4 Welschinger numbers

For any admissible tuple $(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \boldsymbol{p}^\flat)$ such that $R_\Sigma(\mathcal{D}, \beta^{\text{re}} + 2\beta^{\text{im}}) \geq 0$, and for any set $\boldsymbol{p}^\sharp$ of $R_\Sigma(\mathcal{D}, \beta^{\text{re}} + 2\beta^{\text{im}})$ generic points of $F \setminus E$, consider the set

$V^{\mathbb{R}}_\Sigma(\mathcal{D},\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$, see section 3.2. According to Lemma 3, this set is finite and consists of real nodal irreducible rational curves. Put

$$W_{\Sigma,\phi}(\mathcal{D},\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)=\sum_{C\in V^{\mathbb{R}}_\Sigma(\mathcal{D},\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)}(-1)^{s(C)+C_\pm\circ\phi}\ . \qquad (6)$$

In view of Proposition 1, for any divisor class $D\in\mathrm{Pic}^{\mathbb{R}}(\Sigma,E)$ and a set $\boldsymbol{p}^\sharp$ of $c_1(\Sigma)D-1$ distinct generic points of $F\setminus E$, one has

$$W_\phi(\Sigma,D)=\sum_{k+2l=DE}W_{\Sigma,\phi}(D,0,k\theta_1,l\theta_1,\emptyset,\boldsymbol{p}^\sharp)\ . \qquad (7)$$

Pick a divisor class $D_0\in\mathrm{Pic}^{\mathbb{R}}(\Sigma,E)$, and put $N=\dim|D_0|$. Note that the set

$$\mathrm{Prec}(D_0)=\{D\in\mathrm{Pic}^{\mathbb{R}}(\Sigma,E)\ :\ D_0\ge D\}$$

is finite, and we have $\dim|D|\le N$ for each $D\in\mathrm{Prec}(D_0)$. Furthermore, for each nonempty variety $V^{\mathbb{R}}_\Sigma(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat)$ with $D\in\mathrm{Prec}(D_0)$, we have

$$\|\alpha\|+R_\Sigma(D,\beta^{\mathrm{re}}+2\beta^{\mathrm{im}})\le N\ .$$

Lemma 10 *Let $D_0\in\mathrm{Pic}^{\mathbb{R}}(\Sigma,E)$ be a divisor class with $N=\dim|D_0|>0$. Then, there exists a sequence $\Lambda(D_0)=(\Lambda_i)_{i=1,\dots,N}$ of N disjoint smooth real algebraic arcs in Σ, which are parameterized by $t\in[0,1]\mapsto p_i(t)\in\Lambda_i$, such that $p_i(0)\in E$, $i=1$, $\dots$, N, the arcs Λ_i are transverse to E at $p_i(0)$, $i=1,\dots,N$, and the following condition holds:*

for an arbitrary admissible tuple $(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat)$, disjoint subsets $J^\flat,J^\sharp\subset\{1,...,N\}$, and a positive integer $k\le N$ such that

(i) $D\le D_0$,

(ii) $R_\Sigma(D,\beta^{\mathrm{re}}+2\beta^{\mathrm{im}})>0$,

(iii) $i<k<j$ for all $i\in J^\flat$, $j\in J^\sharp$,

(iv) the number of elements in $J^\sharp$ is equal to $R_\Sigma(D,\beta^{\mathrm{re}}+2\beta^{\mathrm{im}})-1$,

(v) $\boldsymbol{p}^\flat=\{p_i(0)\ :\ i\in J^\flat\}$,

the closure of the family

$$\bigcup_{t\in(0,1]}V_\Sigma(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,\widetilde{\boldsymbol{p}}^\sharp\cup\{p_k(t)\})\ ,$$

where $\widetilde{\boldsymbol{p}}^\sharp=\{p_j(1)\}_{j\in J^\sharp}$, is a deformation diagram of $(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,\widetilde{\boldsymbol{p}}^\sharp,p_k(0))$.

Proof. Take a sequence $\widehat{\Lambda}_i$, $i=1,...,N$, of disjoint smooth real algebraic arcs in Σ, which are parameterized by $t\in[0,1]\mapsto p_i(t)\in\widehat{\Lambda}_i$, such that $(p_i(0))_{i=1,\dots,N}$ is a generic sequence of points in E, and the arcs $\widehat{\Lambda}_i$ are transverse to E at $p_i(0)$, $i=1,...,N$. We will inductively shorten and reparameterize these arcs in order to satisfy the diagrammatic condition.

Suppose that we have constructed the arcs $\Lambda_1,...,\Lambda_{k-1}$, $1\le k\le N$. There are finitely many admissible tuples $(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat)$ and subsets $J^\flat,J^\sharp\subset\{1,...,N\}$ satisfying restrictions (i)-(v) in Lemma. Given such data $D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,J^\flat,J^\sharp$, we take a closed neighborhood $\Lambda_k(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,J^\flat,J^\sharp)$ of $p_k(0)$ in $\widehat{\Lambda}_k$, parameterized by $[0,\varepsilon_k]$, such that the closure of the family

$$\bigcup_{p'\in\Lambda_k(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,J^\flat,J^\sharp),p'\ne p_k(0)} V_\Sigma(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,\widetilde{\boldsymbol{p}}^\sharp\cup\{p'\})\ ,$$

where $\widetilde{\boldsymbol{p}}^\sharp=\{p_i(\varepsilon_i)\}_{1\le i<k}$, is a deformation diagram of $(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,\widetilde{\boldsymbol{p}}^\sharp,p_k(0))$. Then we define

$$\Lambda_k=\bigcap_{(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,J^\flat,J^\sharp)}\Lambda_k(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,J^\flat,J^\sharp)$$

and reparameterize this arc by $[0,1]$. □

Take a divisor class $D_0\in\mathrm{Pic}^{\mathbb{R}}(\Sigma,E)$ such that $N=\dim|D_0|>0$ and a sequence of arcs $(\Lambda_i)_{i=1,\dots,N}$ as in Lemma 10. Given two subsets $J^\flat,J^\sharp\subset\{1,...,N\}$ such that $i<j$ for all $i\in J^\flat$, $j\in J^\sharp$, we say that the pair of point configurations

$$\boldsymbol{p}^\flat=\{p_i(0)\ :\ i\in J^\flat\},\quad \boldsymbol{p}^\sharp=\{p_j(1)\ :\ j\in J^\sharp\}$$

is *in D_0-CH position.*

Proposition 11 *Fix a tuple $(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}})$, where $D\in\mathrm{Pic}^{\mathbb{R}}(\Sigma,E)$ is a divisor class, $\alpha,\beta^{\mathrm{re}}\in\mathbb{Z}_+^{\infty,\mathrm{odd}}$, and $\beta^{\mathrm{im}}\in\mathbb{Z}_+^\infty$ such that $R_\Sigma(D,\beta^{\mathrm{re}}+2\beta^{\mathrm{im}})>0$. Choose two point sequences $\boldsymbol{p}^\flat$ and $\boldsymbol{p}^\sharp$ satisfying the following restrictions:*

- *the tuple $(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat)$ is admissible,*
- *the number of points in $\boldsymbol{p}^\sharp$ is equal to $R_\Sigma(D,\beta^{\mathrm{re}}+2\beta^{\mathrm{im}})$,*
- *the pair $(\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ is in D_0-CH position for some divisor class $D_0\in\mathrm{Pic}^{\mathbb{R}}(\Sigma,E)$, $D_0\ge D$.*

Then, the number $W_{\Sigma,\phi}(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ does not depend on the choice of sequences $\boldsymbol{p}^\flat$ and $\boldsymbol{p}^\sharp$.

The proof is presented in section 3.6.

We skip $\boldsymbol{p}^\flat$ and $\boldsymbol{p}^\sharp$ in the notation of the above numbers and simply write $W_{\Sigma,\phi}(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}})$ calling them **Welschinger numbers**.

Proposition 12 *Let* $\Sigma = \mathbb{P}^2_{a,b}$*,* $a + 2b = 6$*, and* $E \in |L - E_1 - E_2|$*. If* $\phi = 0 \in H_2(\Sigma \setminus F; \mathbb{Z}/2\mathbb{Z})$*, then among the Welschinger numbers* $W_{\Sigma,\phi}(\mathcal{D}, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, \emptyset)$*, where* $(\mathcal{D}, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$ *is in admissible tuple such that*

$$\alpha, \beta^{\mathrm{re}} \in \mathbb{Z}_+^{\infty,\mathrm{odd}}, \quad [\mathcal{D}]E > 0, \text{ and } \quad R_\Sigma(\mathcal{D}, \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}}) = 0,$$

the only non-zero numbers are as follows:

(1) in the case of a divisor class $\mathcal{D} = D$*,*

- *(1i)* $W_{\Sigma,\phi}(E_i, 0, \theta_1, 0, \emptyset, \emptyset) = 1$*, where* $i = 1, 2$*, and* E_1, E_2 *are real;*
- *(1ii)* $W_{\Sigma,\phi}(L - E_i - E_j, 0, \theta_1, 0, \emptyset, \emptyset) = 1$*, where* $3 \le i < j \le 6$*, and* E_i, E_j *are both real or are complex conjugate;*
- *(1iii)* $W_{\Sigma,\phi}(-(K_\Sigma + E) - E_i, 0, \theta_1, 0, \emptyset, \emptyset) = 1$*, where* $i = 1, 2$*, and* E_1, E_2 *are real;*
- *(1iv)* $W_{\Sigma,\phi}(-(K_\Sigma + E), \theta_1, \theta_1, 0, \boldsymbol{p}^\flat, \emptyset) = 1$*;*
- *(1v)* $W_{\Sigma,\phi}(-s(K_\Sigma + E) + L - s_1E_1 - s_2E_2 - E_i, \alpha, 0, 0, \boldsymbol{p}^\flat, \emptyset) = 1$*, where* $s \ge 0$*,* $0 \le s_1, s_2 \le 1$*,* $s_1 + s_2 \le 2s$*,* $3 \le i \le 6$*, the sequence* $\alpha \in \mathbb{Z}^{\infty,\mathrm{odd}}$ *verifies* $I\alpha = 2s + 1 - s_1 - s_2$*, the divisor* E_i *is real, and the divisors* E_1 *and* E_2 *are both real if* $s_1 \ne s_2$*;*
- *(1vi)* $W_{\Sigma,\phi}(-s(K_\Sigma + E) - s_1E_1 - s_2E_2 + E_i, \alpha, 0, 0, \boldsymbol{p}^\flat, \emptyset) = 1$*, where* $s \ge 1$*,* $0 \le s_1, s_2 \le 1$*,* $s_1 + s_2 < 2s$*,* $3 \le i \le 6$*, the sequence* $\alpha \in \mathbb{Z}^{\infty,\mathrm{odd}}$ *verifies* $I\alpha = 2s - s_1 - s_2$*, the divisor* E_i *is real, and the divisors* E_1 *and* E_2 *are both real if* $s_1 \ne s_2$*;*

(2) in the case of a pair $\mathcal{D} = (D_1, D_2)$ *of divisor classes,*

- *(2i)* $W_{\Sigma,\phi}(\{E_1, E_2\}, 0, 0, \theta_1, \emptyset, \emptyset) = 1$*, where* E_1 *and* E_2 *are complex conjugate;*
- *(2ii)* $W_{\Sigma,\phi}(\{L - E_i - E_j, L - E_i - E_k\}, 0, 0, \theta_1, \emptyset, \emptyset) = 1$*, where* $\{i, j, k\} \subset \{3, 4, 5, 6\}$*, the divisor* E_i *is real, and the divisors* E_j, E_k *are complex conjugate;*
- *(2iii)* $W_{\Sigma,\phi}(\{-(K_\Sigma + E) - E_1, -(K_\Sigma + E) - E_2\}, 0, 0, \theta_1, \emptyset, \emptyset) = 1$*, where* E_1 *and* E_2 *are complex conjugate;*
- *(2iv)* $W_{\Sigma,\phi}(\{L - E_i - E_j, L - E_k - E_l\}, 0, 0, \theta_1, \emptyset, \emptyset) = -1$*, where* $\{i, j, k, l\} = \{3, 4, 5, 6\}$*,* E_i, E_k *and* E_j, E_l *are two pairs of complex conjugate divisors.*
- *(2v)* $W_{\Sigma,\phi}(\{-(K_\Sigma + E), -(K_\Sigma + E)\}, 0, 0, \theta_2, \emptyset, \emptyset) = 1$*, if* Q', Q'' *are complex conjugate.*

Proposition 13 *Let* $\Sigma = \mathbb{B}_1$*. (Recall that* F *is the non-orientable component of* $\mathbb{R}\Sigma$*, and* $E = L_1$*.) If* ϕ *is either* 0*, or* ϕ_F *(cf. section 2), then among the Welschinger numbers* $W_{\Sigma,\phi}(\mathcal{D}, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat, \emptyset)$*, where* $(\mathcal{D}, \alpha, \beta^{\mathrm{re}}, \beta^{\mathrm{im}}, \boldsymbol{p}^\flat)$ *is an admissible tuple such that*

$$\alpha, \beta^{\mathrm{re}} \in \mathbb{Z}_+^{\infty,\mathrm{odd}}, \quad [\mathcal{D}]E > 0, \text{ and } \quad R_\Sigma(\mathcal{D}, \beta^{\mathrm{re}} + 2\beta^{\mathrm{im}}) = 0,$$

the only non-zero numbers are as follows:

(1) *in the case of a divisor class* $\mathcal{D} = D$,

(1i) $W_{\Sigma,\phi}(L_i, 0, \theta_1, 0, \emptyset, \emptyset) = 1$, *where* $i = 2, 3$;

(1ii) $W_{\Sigma,\phi}(-(K_\Sigma + E), \theta_1, \theta_1, 0, \boldsymbol{p}^\flat, \emptyset) = 1$;

(2) *in the case of a pair* $\mathcal{D} = (D_1, D_2)$ *of divisor classes,*

$$W_{\Sigma,\phi}(\{L_{2i}, L_{2i+1}\}, 0, 0, \theta_1, \emptyset, \emptyset) = \begin{cases} 1, & \text{if } \phi = \phi_F \text{ and } L_{2i} \cap L_{2i+1} \cap F = \varnothing, \\ -1, & \text{otherwise,} \end{cases}$$

where $i = 2, 3, 4, 5$.

Proof of Propositions 12 and 13. Both propositions can be easily derived from Lemmas 5 and 6. We only make a comment concerning the statement of Proposition 13(2). By Lemma 4(2), there are four pairs (L_{2i}, L_{2i+1}), $i = 2, 3, 4, 5$, of complex conjugate lines crossing E, and the lines of each pair intersect at a real point, which is a solitary node of the corresponding curve $L_{2i}L_{2i+1}$. If $\phi = 0$, then the contribution of such a curve to the Welschinger number is -1, whereas if $\phi = \phi_F$, then by formula (6) the contribution is -1 when the solitary node occurs on the component F and it is 1 otherwise. □

The numbers $W_{\Sigma,\phi}(D, \alpha, \beta^{\rm re}, \beta^{\rm im}, \boldsymbol{p}^\flat, \emptyset)$ in Propositions 12 and 13 do not depend on the choice of $\boldsymbol{p}^\flat$. We skip $\boldsymbol{p}^\flat$ and $\emptyset$ in the notation of these numbers and simply write $W_{\Sigma,\phi}(D, \alpha, \beta^{\rm re}, \beta^{\rm im})$.

3.5 Recursive formula

Theorem 1 *Let* $\Sigma = \mathbb{P}^2_{a,b}$, $a + 2b = 6$, *or* $\mathbb{B}_1$, *let* E *be a real smooth* (-1)*-curve with* $\mathbb{R}E \subset F$, *and let* $\phi \in H_2(\Sigma \setminus F; \mathbb{Z}/2\mathbb{Z})$ *be a class invariant under the action of complex conjugation,* $\text{conj}_*\phi = \phi$.

Let $D \in \text{Pic}^{\mathbb{R}}(\Sigma, E)$ *be a divisor class, and let* $\alpha, \beta^{\rm re} \in \mathbb{Z}_+^{\infty,\text{odd}}$, $\beta^{\rm im} \in \mathbb{Z}_+^\infty$ *satisfy the following conditions:*

$$I(\alpha + \beta^{\rm re} + 2\beta^{\rm im}) = DE, \quad R_\Sigma(D, \beta^{\rm re} + 2\beta^{\rm im}) > 0 \ .$$

Then,

$$W_{\Sigma,\phi}(D, \alpha, \beta^{\rm re}, \beta^{\rm im}) = \sum_{k \geq 1,\ \beta^{\rm re}_k > 0} W_{\Sigma,\phi}(D, \alpha + \theta_k, \beta^{\rm re} - \theta_k, \beta^{\rm im})$$
$$+ \sum \frac{2^{\|\beta^{(0)}\|}}{\beta^{(0)}!}(l+1)\binom{\alpha}{\alpha^{(0)}\alpha^{(1)}...\alpha^{(m)}}\frac{(n-1)!}{n_1!...n_m!}$$
$$\times \prod_{i=1}^m \left(\binom{(\beta^{\rm re})^{(i)}}{\gamma^{(i)}} W_{\Sigma,\phi}(\mathcal{D}^{(i)}, \alpha^{(i)}, (\beta^{\rm re})^{(i)}, (\beta^{\rm im})^{(i)})\right) \ , \tag{8}$$

where

$$n = R_\Sigma(D, \beta^{\rm re} + 2\beta^{\rm im}), \quad n_i = R_\Sigma(\mathcal{D}^{(i)}, (\beta^{\rm re})^{(i)} + 2(\beta^{\rm im})^{(i)}),\ i = 1, ..., m \ ,$$

and the second sum in (8) is taken

- *over all integers* $l \ge 0$ *and vectors* $\alpha^{(0)} \le \alpha$, $\beta^{(0)} \le \beta^{\rm re}$;
- *over all sequences*

$$(\mathcal{D}^{(i)}, \alpha^{(i)}, (\beta^{\rm re})^{(i)}, (\beta^{\rm im})^{(i)}),\ 1 \le i \le m\ , \tag{9}$$

such that, for all $i = 1, ..., m$,

(1a) $\mathcal{D}^{(i)} \in {\rm Pic}^{\mathbb{R}}(\Sigma, E)$, *and* $\mathcal{D}^{(i)}$ *is neither the divisor class* $-(K_\Sigma + E)$, *nor the pair* $\{-(K_\Sigma + E), -(K_\Sigma + E)\}$,

(1b) $I(\alpha^{(i)} + (\beta^{\rm re})^{(i)} + 2(\beta^{\rm im})^{(i)}) = [\mathcal{D}^{(i)}]E$, $(\beta^{\rm re})^{(i)} + (\beta^{(\rm im)})^{(i)} \ne 0$, *and* $R_\Sigma(\mathcal{D}^{(i)}, (\beta^{\rm re})^{(i)} + 2(\beta^{\rm im})^{(i)}) \ge 0$,

and

(1c) $D - E = \sum_{i=1}^m [\mathcal{D}^{(i)}] - (2l + I\alpha^{(0)} + I\beta^{(0)})(K_\Sigma + E)$,

(1d) $\sum_{i=1}^m \alpha^{(i)} \le \alpha - \alpha^{(0)}$,

(1e) $\sum_{i=1}^m (\beta^{\rm re})^{(i)} \ge \beta^{\rm re}$ *and* $\sum_{i\in S} (\beta^{\rm im})^{(i)} = \beta^{\rm im}$, *where*

$$S = \{i\ :\ 1 \le i \le m,\ \mathcal{D}^{(i)}\ \textit{is a divisor class}\}\ ,$$

(1f) each tuple $(\mathcal{D}^{(i)}, 0, (\beta^{\rm re})^{(i)}, (\beta^{\rm im})^{(i)})$ *with* $n_i = 0$ *appears in (9) at most once,*

- *over all sequences*

$$\gamma^{(i)} \in \mathbb{Z}_+^{\infty,{\rm odd}}, \quad \|\gamma^{(i)}\| = \begin{cases} 1, & i \in S, \\ 0, & i \notin S, \end{cases} \qquad i = 1, ..., m\ , \tag{10}$$

satisfying

(2a) $(\beta^{\rm re})^{(i)} \ge \gamma^{(i)}$, $i = 1, \ldots, m$, *and* $\sum_{i=1}^m \left((\beta^{\rm re})^{(i)} - \gamma^{(i)}\right) = \beta^{\rm re} - \beta^{(0)}$,

and the second sum in (8) is factorized by simultaneous permutations in the sequences (9) and (10).

The proof is presented in section 3.6.

Corollary 14 *Let* $\Sigma = \mathbb{P}^2_{a,b}$, $a + 2b = 6$, *or* $\mathbb{B}_1$, *let* E *be a real smooth* (-1)*-curve with* $\mathbb{R}E \subset F$, *and let* $\phi \in H_2(\Sigma \setminus F; \mathbb{Z}/2\mathbb{Z})$ *be a class invariant under the action of complex conjugation,* ${\rm conj}_*\phi = \phi$.

(1) For any divisor class $D \in {\rm Pic}^{\mathbb{R}}(\Sigma, E)$ *and vectors* $\alpha, \beta^{\rm re} \in \mathbb{Z}_+^{\infty,{\rm odd}}$, $\beta^{\rm im} \in \mathbb{Z}_+^\infty$ *such that* $I(\alpha + \beta^{\rm re} + 2\beta^{\rm im}) = DE$, $R_\Sigma(D, \beta^{\rm re} + 2\beta^{\rm im}) \ge 0$, *and* $\beta^{\rm im} \ne 0$, *one has*

$$W_{\Sigma,\phi}(D, \alpha, \beta^{\rm re}, \beta^{\rm im}) = 0\ . \tag{11}$$

(2) For any divisor class $D \in \mathrm{Pic}^{\mathbb{R}}(\Sigma, E)$ and vectors $\alpha, \beta \in \mathbb{Z}_+^{\infty,\mathrm{odd}}$ such that $I(\alpha+\beta) = DE$ and $R_\Sigma(D, \beta) > 0$, one has

$$W_{\Sigma,\phi}(D,\alpha,\beta,0) = \sum_{k\ge1,\ \beta_k>0} W_{\Sigma,\phi}(D,\alpha+\theta_k,\beta-\theta_k,0)$$

$$+ \sum \frac{2^{\|\beta^{(0)}\|}}{\beta^{(0)}!}(l+1)\binom{\alpha}{\alpha^{(0)}\alpha^{(1)}...\alpha^{(m)}}\frac{(n-1)!}{n_1!...n_m!}$$

$$\times \prod_{i=1}^m \left(\binom{(\beta^{\mathrm{re}})^{(i)}}{\gamma^{(i)}} W_{\Sigma,\phi}(\mathcal{D}^{(i)},\alpha^{(i)},(\beta^{\mathrm{re}})^{(i)},(\beta^{\mathrm{im}})^{(i)})\right), \tag{12}$$

where

$$n = R_\Sigma(D,\beta), \quad n_i = R_\Sigma(\mathcal{D}^{(i)}, (\beta^{\mathrm{re}})^{(i)} + 2(\beta^{\mathrm{im}})^{(i)}),\ i = 1, ..., m\ ,$$

and the second sum in (12) is taken

- *over all integers $l \ge 0$ and vectors $\alpha^{(0)} \le \alpha$, $\beta^{(0)} \le \beta$;*
- *over all sequences*

$$(\mathcal{D}^{(i)}, \alpha^{(i)}, (\beta^{\mathrm{re}})^{(i)}, (\beta^{\mathrm{im}})^{(i)}),\ 1 \le i \le m\ , \tag{13}$$

  such that, for all $i = 1, ..., m$,

  (1a) $\mathcal{D}^{(i)} \in \mathrm{Pic}^{\mathbb{R}}(\Sigma, E)$, and $\mathcal{D}^{(i)}$ is neither the divisor class $-(K_\Sigma + E)$, nor the pair $(-(K_\Sigma+E), -(K_\Sigma+E))$,

  (1b) $I(\alpha^{(i)} + (\beta^{\mathrm{re}})^{(i)} + 2(\beta^{\mathrm{im}})^{(i)}) = [\mathcal{D}^{(i)}]E$, and $R_\Sigma(\mathcal{D}^{(i)}, (\beta^{\mathrm{re}})^{(i)} + 2(\beta^{\mathrm{im}})^{(i)}) \ge 0$,

  (1c) $(\beta^{\mathrm{im}})^{(i)} \ne 0$ if and only if $\mathcal{D}^{(i)}$ is a pair of divisor classes, and, in such a case, $n_i = 0$, $\alpha^{(i)} = (\beta^{\mathrm{re}})^{(i)} = 0$,

  and

  (1d) $D - E = \sum_{i=1}^m [\mathcal{D}^{(i)}] - (2l + I\alpha^{(0)} + I\beta^{(0)})(K_\Sigma + E)$,

  (1e) $\sum_{i=0}^m \alpha^{(i)} \le \alpha$, $\sum_{i=1}^m (\beta^{\mathrm{re}})^{(i)} \ge \beta - \beta^{(0)}$,

  (1f) each tuple $(\mathcal{D}^{(i)}, 0, (\beta^{\mathrm{re}})^{(i)}, (\beta^{\mathrm{im}})^{(i)})$ with $n_i = 0$ appears in (13) at most once,

- *over all sequences*

$$\gamma^{(i)} \in \mathbb{Z}_+^{\infty,\mathrm{odd}}, \quad \|\gamma^{(i)}\| = \begin{cases} 1, & \mathcal{D}^{(i)} \text{ is a divisor class,} \\ 0, & \mathcal{D}^{(i)} \text{ is a pair of divisor classes,}\end{cases} \quad i = 1, ..., m\ , \tag{14}$$

  satisfying

  (2a) $(\beta^{\mathrm{re}})^{(i)} \ge \gamma^{(i)}$, $i = 1, ..., m$, and $\sum_{i=1}^m \left((\beta^{\mathrm{re}})^{(i)} - \gamma^{(i)}\right) = \beta^{\mathrm{re}} - \beta^{(0)}$,

and the second sum in (12) is factorized by simultaneous permutations in the sequences (13) and (14).

(3) Assume that $\phi = 0$ *or* $\phi = \phi_F$. *Then, all Welschinger numbers* $W_{\Sigma,\phi}(D,\alpha,\beta,0)$, *where* $D \in \mathrm{Pic}^{\mathbb{R}}(\Sigma, E)$ *is a divisor class and* $\alpha, \beta \in \mathbb{Z}^{\infty,\mathrm{odd}}$ *are vectors such that* $I(\alpha+\beta) = DE$ *and* $R_\Sigma(D,\beta) > 0$, *are recursively determined by the formula (12) and the initial conditions in Propositions 12, 13.*

(4) For any divisor class $D \in \mathrm{Pic}^{\mathbb{R}}(\Sigma, E)$, *one has*

$$W_\phi(\Sigma, D) = W_{\Sigma,\phi}(D, 0, (DE)\theta_1, 0) \ . \tag{15}$$

Proof. The condition (1e) in Theorem 1 and the vanishing of the Welschinger numbers $W_{\Sigma,\phi}(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}})$ such that D is a divisor class, $R_\Sigma(D,\beta^{\mathrm{re}}+2\beta^{\mathrm{im}}) = 0$, and $\beta^{\mathrm{im}} \ne 0$ (see Propositions 12, 13) directly imply the statements (1) and (2). The claim (3) is straightforward. Formula (15) comes from (7) and (11). □

Remark 15 *(1) Formula (12) holds for any surface* Σ' *obtained from* Σ *by successive blowing down of real* (-1)*-curves or pairs of disjoint imaginary conjugate* (-1)*-curves: one simply has to reduce* $\mathrm{Pic}^{\mathbb{R}}(\Sigma, E)$ *to the divisor classes/pairs of divisor classes which do not cross the blown down curves.*

(2) If one blows down L_3 *on* $\mathbb{B}_1$, *the resulting surface* $\mathbb{B}$ *is a conic bundle and has two real spherical components; these components give rise to the same collections of Welschinger invariants.*

Corollary 16 *Let* E *be one of the three real lines of* $\mathbb{B}_1$.

(1) For any divisor class $D \in \mathrm{Pic}^{\mathbb{R}}(\mathbb{B}_1, E)$ *and vectors* $\alpha, \beta \in \mathbb{Z}_+^{\infty,\mathrm{odd}}$ *such that* $I(\alpha+\beta) = DE$ *and* $R_{\mathbb{B}_1}(D,\beta) > 0$, *one has*

$$\begin{aligned} W_{\mathbb{B}_1,\phi_F}(D,\alpha,\beta,0) = \sum_{k\ge 1,\ \beta_k>0} W_{\mathbb{B}_1,\phi_F}(D,\alpha+\theta_k,\beta-\theta_k,0) \\ + \sum \frac{2^{\|\beta^{(0)}\|}}{\beta^{(0)}!} \binom{\alpha}{\alpha^{(0)}\alpha^{(1)}...\alpha^{(m)}} \frac{(n-1)!}{n_1!...n_m!} \\ \times \prod_{i=1}^m \left(\binom{(\beta^{\mathrm{re}})^{(i)}}{\gamma^{(i)}} W_{\mathbb{B}_1,\phi_F}(D^{(i)},\alpha^{(i)},\beta^{(i)},0) \right) , \end{aligned} \tag{16}$$

where

$$n = R_{\mathbb{B}_1}(D,\beta), \quad n_i = R_{\mathbb{B}_1}(D^{(i)},\beta^{(i)}),\ i = 1,...,m \ ,$$

and the second sum in (16) is taken

- *over all vectors* $\alpha^{(0)} \le \alpha$, $\beta^{(0)} \le \beta$;
- *over all sequences*

$$(D^{(i)},\alpha^{(i)},\beta^{(i)},0),\ 1 \le i \le m \ , \tag{17}$$

  such that, for all $i = 1,...,m$,

(i) $D^{(i)} \in \mathrm{Pic}^{\mathbb{R}}(\mathbb{B}_1, E)$ *is a divisor class different from* $-(K_{\mathbb{B}_1}+E)$,

(ii) $I(\alpha^{(i)}+\beta^{(i)}) = D^{(i)}E$, *and* $R_{\mathbb{B}_1}(D^{(i)}, \beta^{(i)}) \ge 0$,

and

(iii) $D - E = \sum_{i=1}^m D^{(i)} - (I\alpha^{(0)} + I\beta^{(0)})(K_{\mathbb{B}_1}+E)$,

(iv) $\sum_{i=0}^m \alpha^{(i)} \le \alpha$, $\sum_{i=1}^m \beta^{(i)} \ge \beta - \beta^{(0)}$,

(v) each tuple $(D^{(i)}, 0, \beta^{(i)}, 0)$ *with* $n_i = 0$ *appears in (17) at most once, and coincides with* $(L_j, 0, \theta_1, 0)$, *where* $L_j \ne E$ *is a real line on* $\mathbb{B}_1$,

- *over all sequences*

$$\gamma^{(i)} \in \mathbb{Z}_+^{\infty,\mathrm{odd}}, \quad \|\gamma^{(i)}\| = 1, \quad i = 1, ..., m \ , \tag{18}$$

satisfying

$$\beta^{(i)} \ge \gamma^{(i)}, \ i = 1, \ldots, m, \quad \text{and} \quad \sum_{i=1}^m \left(\beta^{(i)} - \gamma^{(i)}\right) = \beta - \beta^{(0)},$$

and the second sum in (17) is factorized by simultaneous permutations in the sequences (17) and (18).

(2) All Welschinger numbers $W_{\mathbb{B}_1,\phi_F}(D, \alpha, \beta, 0)$, *where* $D \in \mathrm{Pic}^{\mathbb{R}}(\mathbb{B}_1, E)$ *is a divisor class and* $\alpha, \beta \in \mathbb{Z}^{\infty,\mathrm{odd}}$ *are vectors such that* $I(\alpha+\beta) = DE$ *and* $R_{\mathbb{B}_1}(D, \beta) > 0$, *are recursively determined by the formula (16) and the initial conditions in Proposition 13(1).*

Proof. Let L_1, L_2, and L_3 be the three real lines of $\mathbb{B}_1$, and let L_j, $j = 4, \ldots, 11$, be the eight non-real lines of $\mathbb{B}_1$ which intersect L_1 and are numbered in such a way that L_{2i} and L_{2i+1} are complex conjugate for any $i = 2, 3, 4, 5$ (*cf.* Lemma 4). Assume that $E = L_1$.

For precisely two pairs (L_{2i}, L_{2i+1}), $2 \le i \le 5$, say, for $i = 2, 3$, the intersection point belongs to F, and for the other two pairs (L_{2i}, L_{2i+1}), $i = 4, 5$, the intersection point belongs to $\mathbb{R}\mathbb{B}_1 \setminus F$, and hence (*cf.* Proposition 13(2))

$$W_{\mathbb{B}_1,\phi_F}(\{L_{2i}, L_{2i+1}\}, 0, 0, \theta_1) = \begin{cases} -1, & i = 2, 3, \\ 1, & i = 4, 5. \end{cases} \tag{19}$$

Combining the terms of the second sum in the righthand side of the formula (12) applied to $W_{\mathbb{B}_1,\phi_F}(D, \alpha, \beta, 0)$, we obtain the expression

$$\sum_{k=0}^{4} \sum_{l \ge 0} x_k (l+1) \lambda_{k,l},$$

where k is the number of non-divisorial factors in a summand, and x_k is the sum of products of k distinct non-divisorial terms. Any two coefficients $\lambda_{k,l}$ and $\lambda_{k',l'}$ such that $k+2l=k'+2l'$ coincide, so we put $\lambda_{k+2l}=\lambda_{k,l}$. One has

$$x_0=x_4=1,\quad x_1=x_3=0,\quad x_2=-2.$$

Thus, the second sum in the righthand side of the formula is equal to

$$\sum_{l\ge 0}(l+1)(\lambda_{2l}-2\lambda_{2+2l}+\lambda_{4+2l})=\lambda_0.$$

□

Example 17 *We present here some values of Welschinger invariants computed by means of formulas (12) and (16). In the case of* $\mathbb{P}^2_{a,b}$ *these are the usual Welschinger invariants; for the conic bundle* $\mathbb{B}$ *(see Remark 15) we take for* F *one of the components of* $\mathbb{R}\mathbb{B}$*; for* $\mathbb{B}_1$*, as always,* F *is the non-orientable component of* $\mathbb{R}\mathbb{B}_1$*.*

$D\setminus\Sigma,\phi$	$\mathbb{P}^2_{6,0}$	$\mathbb{P}^2_{4,1}$	$\mathbb{P}^2_{2,2}$	$\mathbb{P}^2_{0,3}$	$\mathbb{B},0$	$\mathbb{B},\phi_F$	$\mathbb{B}_1,0$	$\mathbb{B}_1,\phi_F$
-K	8	6	4	2	0	4	0	4
-2K	1000	522	236	78	0	512	0	160

3.6 Proof of Proposition 11 and Theorem 1

We simultaneously prove Proposition 11 and Theorem 1 by induction on $R_\Sigma(D,\beta^{\mathrm{re}}+2\beta^{\mathrm{im}})$.

The claim of Proposition 11 for $R_\Sigma(D,\beta^{\mathrm{re}}+2\beta^{\mathrm{im}})=0$ (the base of induction) follows from Propositions 12 and 13.

For induction step, we fix a tuple $(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}})$ satisfying the hypotheses of Theorem 1 and choose a divisor class $D_0\in\mathrm{Pic}^{\mathbb{R}}(\Sigma,E)$, $D_0\ge D$, and a sequence of arcs $(\Lambda_s)_{s=1,\dots,N}$ as in Lemma 10. Pick two point sequences $\boldsymbol{p}^\flat=(p_s(0))_{s\in J^\flat}$ and $\boldsymbol{p}^\sharp=(p_s(1))_{s\in J^\sharp}$, $J^\flat,J^\sharp\subset\{1,...,N\}$, such that

- $s_1<s_2$ for all $s_1\in J^\flat$, $s_2\in J^\sharp$,
- the tuple $(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat)$ is admissible,
- the number of points in $\boldsymbol{p}^\sharp$ is equal to $R_\Sigma(D,\beta^{\mathrm{re}}+2\beta^{\mathrm{im}})$,

and prove that $W_{\Sigma,\phi}(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ equals the right-hand side of formula (8).

Consider the deformation diagram Δ of $(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,\widetilde{\boldsymbol{p}}^\sharp,p_k(0))$ provided by Λ_k, where $k=\min J^\sharp$ and $\widetilde{\boldsymbol{p}}^\sharp=\boldsymbol{p}^\sharp\setminus\{p_k(1)\}$. We intend to compute $W_{\Sigma,\phi}(D,\alpha,\beta^{\mathrm{re}},\beta^{\mathrm{im}},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ by summing up the contributions of all connected components of Δ.

The connected components of Δ are enumerated by their roots described in Lemma 7.

Since $\beta^{\rm re} \in \mathbb{Z}_+^{\infty,{\rm odd}}$, Lemma 8 implies that each connected component $\delta \subset \Delta$ with a root of type (I) has a unique leaf, and this leaf has the same Welschinger sign as the root. The contribution of these components of Δ gives the first summand in the right-hand side of formula (8).

Let δ be a connected component of Δ with a root C of type (R). Using the description of leaves of δ given in Lemma 9, we immediately conclude that the contribution of the leaves of δ into $W_{\Sigma,\phi}(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ is as follows:

- 0, if either there is $(\beta^{\rm re})^{(i)} \notin \mathbb{Z}_+^{\infty,{\rm odd}}$, or Q',Q'' are real and at least one of l',l'' is odd;
- $(l+1)\prod_{i=1}^m w_\phi(C^{(i)})$, if $(\beta^{\rm re})^{(i)} \in \mathbb{Z}_+^{\infty,{\rm odd}}$ for all $i=1,\ldots,m$, and Q',Q'' are complex conjugate, $l'=l''=l$;
- $\prod_{i=1}^m w_\phi(C^{(i)})$, if $(\beta^{\rm re})^{(i)} \in \mathbb{Z}_+^{\infty,{\rm odd}}$ for all $i=1,\ldots,m$, and Q',Q'' are real, l',l'' are even.

Finally, taking into account the induction assumption of the independence of the Welschinger numbers

$$W_{\Sigma,\phi}(\mathcal{D}^{(i)},\alpha^{(i)},(\beta^{\rm re})^{(i)},(\beta^{\rm im})^{(i)},(\boldsymbol{p}^\flat)^{(i)},(\boldsymbol{p}^\sharp)^{(i)}),\quad i=1,\ldots,m\ ,$$

on the choice of pairs of point sequences $(\boldsymbol{p}^\flat)^{(i)},(\boldsymbol{p}^\sharp)^{(i)}$ in D_0-CH position, $i=1,\ldots,m$, and summing up over connected components of Δ, we immediately obtain formula (8) and, in virtue of this formula, also the independence of $W_{\Sigma,\phi}(D,\alpha,\beta^{\rm re},\beta^{\rm im},\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp)$ on the choice of $\boldsymbol{p}^\flat,\boldsymbol{p}^\sharp$. □

4 Applications

4.1 Positivity and asymptotics

A divisor class D on a surface Σ is called *nef* if D non-negatively intersects any algebraic curve on Σ. A nef divisor class D is big if $D^2>0$.

4.1.1 The case of $\Sigma=\mathbb{P}^2_{a,b}$ with $a+2b\le 6$, $b\le 2$

In this case, the real part $\mathbb{R}\Sigma$ is nonempty and connected, and hence we can speak only of the usual Welschinger invariants, which we simply denote by $W(\Sigma,D)$ omitting ϕ in the notation.

Theorem 2 *Let $\Sigma=\mathbb{P}^2_{a,b}$, where $a+2b\le 6$ and $b\le 2$. Then, for any nef and big real divisor class D on Σ,*

- *the invariant* $W(\Sigma,D)$ *is positive; in particular, through any generic collection of* $c_1(\Sigma)D-1$ *real points in* Σ *one can trace a real rational curve* $C\in|D|$*,*

- *the following asymptotic relation holds:*

$$\log W(\Sigma,nD)=c_1(\Sigma)D\cdot n\log n+O(n),\quad n\to+\infty\ , \tag{20}$$

and in particular,

$$\lim_{n\to+\infty}\frac{\log W(\Sigma,nD)}{\log GW(\Sigma,nD)}=1\ . \tag{21}$$

Remark 18 *(1) Theorem 2 covers all the cases studied in [9, 11, 13, 17]. The statement holds true for the surface* $\mathbb{P}^2_{0,3}$ *as well, but the proof requires another approach and will be presented in a forthcoming paper.*

(2) If D *is not nef or not big, then* $W(\Sigma,D)=1$ *or* 0 *depending on whether or not the linear system* $|D|$ *contains an irreducible curve (for the existence of rational irreducible representatives see, for instance, [7]).*

(3) The positivity statement and the existence of real rational curves do not extend in the same form to all real unnodal Del Pezzo surfaces, for example, $W(\mathbb{P}^2_{0,4},-K_{\mathbb{P}^2_{0,4}})=0$*, and there are generic configurations of four pairs of imaginary conjugate blown up points such that the linear system* $|-K_{\mathbb{P}^2_{0,4}}|$ *does not contain any real rational curve (cf. [9, Section 3.1(1)]).*

Lemma 19 *Let* Σ *be an unnodal Del Pezzo surface, and* $D\in\mathrm{Pic}(\Sigma)$*.*

(i) The divisor class D *is nef and nonzero if and only if its intersection with any* (-1)*-curve on* Σ *is non-negative and* $K_\Sigma D<0$*.*

(ii) Assume that $\Sigma=\mathbb{P}^2_{a,b}$*, where* $a+2b\le 6$*. If* D *is nef and nonzero, then* $D^2\ge 0$*, and* D *can be represented by a union of rational curves, which are real if* D *is real. More precisely, if* $D^2>0$*, then* D *can be represented by an irreducible rational curve; if* $D^2=0$*, then* $D=kD'$*, where* D' *is primitive* (*not divisible by a natural number* >1)*,* D' *can be represented by an irreducible nonsingular rational curve, and* D *can be represented by a union of* k *disjoint irreducible nonsingular rational curves.*

Proof. The statements follow, for instance, from [7, Theorems 5.1, 5.2, and Remark 5.3]. In particular, if D is real, the construction in [7, Section 5.2] gives real representatives. □

Proof of Theorem 2. First, decreasing the parameter $-K_\Sigma D$, we prove that

$$W(\Sigma,D)=W_\Sigma(D,0,(DE)\theta_1,0)>0 \tag{22}$$

for all nef and big real divisor classes D on Σ.

If $DE' = 0$ for a real (-1)-curve E', we blown down E'. If $DE' = 0$ for an imaginary (-1)-curve E', then $D\overline{E'} = 0$ and $E'\overline{E'} = 0$ (otherwise, one would have $\dim|E' + \overline{E'}| = 1$ yielding $D^2 \le 0$), and thus, we can blown down E' and $\overline{E'}$. In the both cases, we obtain another unnodal Del Pezzo surface and keep the same values of D^2 and $K_\Sigma D$. Continuing in this way, we either come to a surface treated in [8, 17] ($\mathbb{P}^2$, $\mathbb{P}^2_1$, $\mathbb{P}^2_2$, or $\mathbb{P}^1 \times \mathbb{P}^1$) or end up with a nef and big real divisor class $D \subset \Sigma \ne \mathbb{P}^2, \mathbb{P}^2_1, \mathbb{P}^2_2, \mathbb{P}^1 \times \mathbb{P}^1$ crossing each (-1)-curve of Σ.

In the latter case, in a suitable conjugation invariant basis $L, E_1, ..., E_k$ ($k \ge 3$) of $\mathrm{Pic}(\Sigma)$ we can choose $E = L - E_1 - E_2$ so that $\{E_i\}_{3 \le i \le k}$ contain at most one pair of imaginary (-1)-curves. Then, by Proposition 12, all the initial Welschinger numbers are non-negative, and hence, due to the positivity of the coefficients in formula (12), we get that

$$W_\Sigma(D', \alpha, \beta, 0) \ge 0 \quad \text{for any} \quad D' \in \mathrm{Pic}^{\mathbb{R}}(\Sigma, E),\ \alpha, \beta \in \mathbb{Z}_+^{\infty,\mathrm{odd}},\ I(\alpha + \beta) = D'E \ .$$

Putting $W_\Sigma(D, (DE + 1)\theta_1, -\theta_1, 0) = 0$ and applying $(DE + 1)$ times the formula (12), we get

$$W_\Sigma(D, k\theta_1, (DE-k)\theta_1, 0) = W_\Sigma(D, (k+1)\theta_1, (DE-k-1)\theta_1, 0) + \tau_k, \quad k = 0, \ldots, DE \ ,$$

where τ_k stands for the second sum in the right-hand side of (12). One has

$$W_\Sigma(D, 0, (DE)\theta_1, 0) = \tau_0 + \ldots + \tau_{DE} \ , \tag{23}$$

and to prove the positivity of $W_\Sigma(D, 0, (DE)\theta_1, 0)$, it is sufficient to find at least one positive term in $\tau_0 + \ldots + \tau_{DE}$.

Since E crosses any other (-1)-curve in at most one point, the divisor class $D - E$ non-negatively intersects with each (-1)-curve, and thus, is nef and nonzero (see Lemma 19(i)).

In addition, $-K_\Sigma(D - E) < -K_\Sigma D$ and

$$\tau_0 \ge (DE + 1)W_\Sigma(D - E, 0, (DE + 1)\theta_1, 0).$$

Hence, if $D - E$ is big then, we can replace D with $D - E$ in our procedure.

Otherwise, according to Lemma 19, one has $D - E = kD'$ with a nef primitive D' such that $(D')^2 = 0$ and $D'E > 0$. Since $R_\Sigma(D', (D'E)\theta_1) = \dim|D'| = 1$, we have $R_\Sigma(D', (D'E - 1)\theta_1) = 0$, and then from Proposition 12, we can derive that $W_\Sigma(D', \theta_1, (D'E - 1)\theta_1, 0) = 1$. Formula (12) gives then

$$W_\Sigma(D', 0, (D'E)\theta_1, 0) \ge W_\Sigma(D', \theta_1, (D'E - 1)\theta_1, 0) > 0 \ .$$

This implies that the term τ_{k-1} in (23) is positive, since this term contains the summand $(W_\Sigma(D', 0, (D'E)\theta_1, 0))^k > 0$; hence (22).

Due to the upper bound $W(\Sigma, D) \le GW(\Sigma, D)$ and the asymptotics $\log GW(\Sigma, nD) = c_1(\Sigma)D \cdot n \log n + O(n)$ (see [10]), to prove the asymptotic relations (20) and (21), it is enough to establish for any nef and big real divisor class D the inequality

$$\log W_\Sigma(nD, 0, n(DE)\theta_1, 0) \ge c_1(\Sigma)D \cdot n \log n + O(n), \quad n \to +\infty \ . \tag{24}$$

As above, if $DE' = 0$ for a real (-1)-curve E' (respectively, for an imaginary (-1)-curve E'), we blown down E' (respectively, E' and $\overline{E'}$). Continuing in this way, we either come to a surface treated in [9, 11] ($\mathbb{P}^2$, $\mathbb{P}^2_1$, $\mathbb{P}^2_2$, or $\mathbb{P}^1 \times \mathbb{P}^1$) or end up with a nef and big real divisor class $D \subset \Sigma \ne \mathbb{P}^2, \mathbb{P}^2_1, \mathbb{P}^2_2, \mathbb{P}^1 \times \mathbb{P}^1$ crossing each (-1)-curve of Σ.

In the latter case, there exist positive numbers ξ, η, ζ (depending on D) such that

$$W(\Sigma, nD) = W_\Sigma(nD, 0, n(DE)\theta_1, 0) \ge \xi\eta^{n\zeta}(-nK_\Sigma D - 1)! \ , \tag{25}$$

for any $n \ge 1$, which clearly implies (24). Indeed, if $D - E$ is big, then, applying formula (23) to $W_\Sigma((n{+}1)D, 0, (n{+}1)(DE)\theta_1, 0)$, we obtain that the term τ_2 contains the sum

$$\frac{(-(n+1)K_\Sigma D - 2)!}{(-K_\Sigma(D-E) - 1)!}(DE+1) \cdot W_\Sigma(D - E, 0, (DE+1)\theta_1, 0)$$

$$\times \sum_{i=1}^{n-1} \frac{i(n-i)(DE)^2 \cdot W(iD, 0, i(DE)\theta_1, 0) \cdot W_\Sigma((n-i)D, 0, (n-i)(DE)\theta_1, 0)}{(-iK_\Sigma D - 1)!(-(n-i)K_\Sigma D - 1)!} \ ,$$

which means that the sequence

$$u_n = \frac{nW_\Sigma(nD, 0, n(DE)\theta_1, 0)}{(-nK_\Sigma D - 1)!}$$

satisfies the condition

$$u_n \ge \sum_{i=1}^{n-1} cu_iu_{n-i}$$

for a certain positive constant c. As is well known (*cf.* [6]), this implies the inequality (25).

If $D - E$ is not big, then, according to Lemma 19, one has $D - E = kD'$, where D' is a nef divisor class with $D'E > 0$, $(D')^2 = 0$, and $W_\Sigma(D', 0, (D'E)\theta_1, 0) = 1$. Again applying formula (12) to $W_\Sigma((n+1)D, 0, (n+1)(DE)\theta_1, 0)$, we obtain that the term τ_{k+1} contains the sum

$$(-(n+1)K_\Sigma D - 2)!(D'E)^k$$

$$\times \sum_{i=1}^{n-1} \frac{i(n-i)(DE)^2 \cdot W(iD, 0, i(DE)\theta_1, 0) \cdot W_\Sigma((n-i)D, 0, (n-i)(DE)\theta_1, 0)}{(-iK_\Sigma D - 1)!(-(n-i)K_\Sigma D - 1)!} \ ,$$

which, as above, implies the inequality (25). □

4.1.2 The case of $\Sigma = \mathbb{B}_1$

Recall that F denotes the non-orientable component of $\mathbb{R}\mathbb{B}_1$.

Theorem 3 *For any nef and big real divisor class D on $\mathbb{B}_1$, the Welschinger invariant $W_F(\mathbb{B}_1, D)$ is positive. In particular, through any collection of $c_1(\mathbb{B}_1)D - 1$ generic points of F one can trace a real rational curve $C \in |D|$ passing through the given points. Furthermore, one has*

$$\log W_F(\mathbb{B}_1, nD) = c_1(\mathbb{B}_1)D \cdot n \log n + O(n), \quad n \to \infty \ .$$

In particular,

$$\lim_{n\to\infty} \frac{\log W_F(\mathbb{B}_1, nD)}{\log GW(\mathbb{P}^2_6, nD)} = 1 \ .$$

Remark 20 *(1) The positivity of the usual Welschinger invariants $W(\mathbb{B}_1, D)$ does not hold: $W(\mathbb{B}_1, -K_{\mathbb{B}_1}) = 0$, whereas $W_F(\mathbb{B}_1, -K_{\mathbb{B}_1}) = 4$ (cf. Example 17). Indeed, for any $2 = K^2_{\mathbb{B}_1} - 1$ generic points of F, there are exactly two planes passing through these points and tangent to the spherical component of $\mathbb{R}\mathbb{B}_1$; each of these two planes intersects $\mathbb{B}_1$ along a cubic with a solitary node belonging to the spherical component. The planes passing through the chosen two points and tangent to F give rise to rational cubics whose total contribution to $W(\mathbb{B}_1, -K_{\mathbb{B}_1})$ (as well as to $W_F(\mathbb{B}_1, -K_{\mathbb{B}_1})$) is equal to the negative Euler characteristics of F blown up at 3 points.*

(2) For any real Del Pezzo surface Σ with disconnected real part and for any connected component F of $\mathbb{R}\Sigma$, one has $W(\Sigma, -K_\Sigma, F) \le 0$. Indeed, $W(\Sigma, -K_\Sigma, F) = -\chi(\mathbb{R}\Sigma) + K^2_\Sigma$, where $\chi(\mathbb{R}\Sigma)$ is the Euler characteristics of $\mathbb{R}\Sigma$, and the inequality $-\chi(\mathbb{R}\Sigma) + K^2_\Sigma \le 0$ follows, for example, from Comessatti's classification of real rational surfaces [4] (see also [5]).

(3) Theorem 3 implies a similar statement for the invariants $W_F(\mathbb{B}, D)$, where F is any of the two connected components of $\mathbb{R}\mathbb{B}$: the nef and big real divisor classes on $\mathbb{B}$ can be viewed as the divisor classes $D = d_1L_1 + d_2L_2 + d_3L_3$ on $\mathbb{B}_1$, where L_1, L_2, and L_3 are the three real lines on $\mathbb{R}\mathbb{B}_1$ and $d_1 = d_2 + d_3$, $d_2, d_3 > 0$.

Proof of Theorem 3. Recall that a real divisor class D on $\mathbb{B}_1$ is nef and big if and only if $D = d_1L_1 + d_2L_2 + d_3L_3$, where

$$d_1, d_2, d_3 > 0 \quad \text{and} \quad d_i + d_j \ge d_k, \ \{i, j, k\} = \{1, 2, 3\} \ .$$

Let $D = d_1L_1 + d_2L_2 + d_3L_3$, and $d_i = \max\{d_1, d_2, d_3\} > 1$. Applying formula (16) with $E = L_i$, we obtain

$$\begin{gathered} W_F(\mathbb{B}_1, D) = W_{\mathbb{B}_1,\phi_F}(D, 0, (d_1 + d_2 + d_3 - 2d_i)\theta_1, 0) \\ \ge (d_1 + d_2 + d_3 - 2d_i + 1)W_{\mathbb{B}_1,\phi_F}(D - E, 0, (d_1 + d_2 + d_3 - 2d_i + 1)\theta_1, 0) \\ \ge W_F(\mathbb{B}_1, D - E) \end{gathered}$$

with a nef and big real divisor class $D - E$. After $d_1 + d_2 + d_3 - 3$ steps we get

$$W_F(\mathbb{B}_1, D) \ge W_F(\mathbb{B}_1, L_1 + L_2 + L_3) \ .$$

Since $W_F(\mathbb{B}_1, L_1+L_2+L_3) = 4$ (see Remark 20(1)), one has $W_F(\mathbb{B}_1, D) > 0$ for any nef and big real divisor D on $\mathbb{B}_1$.

Assume now that $d_1 \ge d_2 \ge d_3 > 0$, and pick a number $n \in \mathbb{N}$. Putting $E = L_1$ and applying $n(d_1 - d_2)$ times the formula (16), we get

$$W_F(\mathbb{B}_1, nD) \ge \frac{(nd_3)!}{(n(d_2+d_3-d_1))!} W_F(\mathbb{B}_1, nD'),$$

where $D' = d_2(L_1+L_2) + d_3L_3$. Then, putting alternatively $E = L_1$ and $E = L_2$, we apply $2n(d_2 - d_3)$ times the formula (16) and get

$$W_F(\mathbb{B}_1, nD') \ge (nd_3(nd_3+1))^{n(d_2-d_3)}\, W_F(\mathbb{B}_1, nD''),$$

where $D'' = d_3(L_1 + L_2 + L_3)$. Finally, putting cyclically $E = L_1$, $E = L_2$, and $E = L_3$, we apply $3nd_3 - 3$ times the formula (16) and get

$$W_F(\mathbb{B}_1, nD'') \ge \frac{(nd_3+1)!(nd_3)!(nd_3-1)!}{2} W_F(\mathbb{B}_1, L_1+L_2+L_3).$$

The above inequalities give

$$\log W_F(\mathbb{B}_1, nD) \ge (d_1+d_2+d_3)n\log n + O(n) = c_1(\mathbb{B}_1)D \cdot n\log n + O(n)$$

which implies the desired asymptotics. □

4.2 Monotonicity

Lemma 21 *(1) Let Σ be an unnodal Del Pezzo surface, and D, D' be nef and big divisor classes on Σ. If $D-D'$ is effective, then $D-D'$ can be decomposed into a sum $E^{(1)}+...+E^{(k)}$ of smooth rational (-1)-curves such that each of $D^{(i)} = D' + \sum_{j\le i} E^{(j)}$ is nef and big, and satisfies $D^{(i)}E^{(i+1)} > 0$, $i = 0, \dots, k-1$.*

(2) Let D and D' be nef and big real divisor classes on $\mathbb{B}_1$. If $D - D'$ is effective, then $D - D' = E^{(1)} + \ldots + E^{(k)}$, where $E^{(1)}, \dots, E^{(k)} \in \{L_1, L_2, L_3\}$, and the following properties hold for any $i = 0, \dots, k-1$: the divisor class $D^{(i)} = D' + \sum_{j\le i} E^{(j)}$ is nef and big, and $D^{(i)}E^{(i+1)} > 0$.

Proof. The proof of the first claim literally coincides with the proof of [13, Lemma 30]. To prove the second statement, write $D = d_1L_1 + d_2L_2 + d_3L_3$ and $D' = d'_1L_1 + d'_2L_2 + d'_3L_3$, where d_j, $j = 1, 2, 3$, and d'_j, $j = 1, 2, 3$, are positive integers such that

$$d_1 + d_2 \ge d_3, \quad d_1 + d_3 \ge d_2, \quad d_2 + d_3 \ge d_1,$$
$$d'_1 + d'_2 \ge d'_3, \quad d'_1 + d'_3 \ge d'_2, \quad d'_2 + d'_3 \ge d'_1.$$

Put $k = (d_1 + d_2 + d_3) - (d'_1 + d'_2 + d'_3)$ and $D^{(k)} = D$. Define inductively $E^{(i+1)}$ and $D^{(i)}$, $i = k-1, \dots, 0$, in such a way that each $E^{(i+1)}$ is a real line of $\mathbb{B}_1$, each divisor class $D^{(i)}$ is nef and big, $D^{(0)} = D'$, and $D^{(i)}E^{(i+1)} > 0$, for any $i = k-1, \ldots, 0$. This can be done as follows. Write $D^{(i+1)}$ in the form $d_1^{(i+1)}L_1 + d_2^{(i+1)}L_2 + d_3^{(i+1)}L_3$, where $d_1^{(i+1)}$, $d_2^{(i+1)}$, and $d_3^{(i+1)}$ are positive integers, and choose among the coefficients $d_j^{(i+1)}$ such that $d_j^{(i+1)} > d'_j$ a maximal one, $d_{j^{(i+1)}}^{(i+1)}$. Put $E^{(i+1)} = L_{j^{(i+1)}}$ and $D^{(i)} = D^{(i+1)} - E^{(i+1)}$. □

Theorem 4 *(1) Let D and D' be nef and big divisor classes on $\mathbb{P}^2_{6,0}$ such that $D - D'$ is effective. Then $W(\mathbb{P}^2_{6,0}, D) \ge W(\mathbb{P}^2_{6,0}, D')$. Moreover, in the notation of Lemma 21(1), one has*

$$W(\mathbb{P}^2_{6,0}, D) \ge \prod_{i=1}^{k}(D^{(i-1)}E^{(i)}) \cdot W(\mathbb{P}^2_{6,0}, D') \ .$$

(2) Let D and D_2 be nef and big real divisor classes on $\mathbb{B}_1$ such that $D - D'$ is effective. Then $W_F(\mathbb{B}_1, D) \ge W_F(\mathbb{B}_1, D')$. Moreover, in the notation of Lemma 21(2), one has

$$W_F(\mathbb{B}_1, D) \ge \prod_{i=1}^{k}(D^{(i-1)}E^{(i)}) \cdot W_F(\mathbb{B}_1, D') \ .$$

Proof. The statements immediately follow from formulas (12) and (16). □

4.3 Mikhalkin's congruence

Theorem 5 *Let $\Sigma = \mathbb{P}^2_{6,0}$. Then, for any nef and big divisor class D on Σ, one has*

$$W(\Sigma, D) = GW(\Sigma, D) \mod 4 \ . \tag{26}$$

Proof. Let E be a (-1)-curve on Σ. For any big and nef divisor class D on Σ and any sequences $\alpha, \beta \in \mathbb{Z}_+^\infty$ such that $I(\alpha+\beta) = DE$, consider a generic collection $\boldsymbol{z}^\flat = \{z_{i,j} \ : \ i \ge 1, \ 1 \le j \le \alpha_i\}$ of $\|\alpha\|$ points on E, and the variety $V_\Sigma^{\mathbb{C}}(D, \alpha, \beta, \boldsymbol{z}^\flat)$ which is the union of $R_\Sigma(D, \beta)$-dimensional components of the family of complex reduced irreducible rational curves $C \in |D|$ which intersect E in the following way:

- C has one local branch at each of the points of $C \cap E$,
- $(C \cdot E)_{z_{i,j}} = i$ for all $i \ge 1$, $1 \le j \le \alpha_i$,
- for each $i \ge 1$, there are precisely β_i points $q \in (C \cap E) \setminus \boldsymbol{z}^\flat$ such that $(C \cdot E)_q = i$

(*cf.* Section 3.2 and [16, Definition 2.4]). Denote by $N_\Sigma(D, \alpha, \beta)$ the degree of $V_\Sigma^{\mathbb{C}}(D, \alpha, \beta, \boldsymbol{z}^\flat)$. In particular, $N_\Sigma(D, 0, (DE)\theta_1) = GW(\Sigma, D)$ for any nef and big divisor class D.

We prove the following statement:

$$W_\Sigma(D,\alpha,\beta,0) = I^\beta N_\Sigma(D,\alpha,\beta) \mod 4\ , \tag{27}$$

for any divisor class $D \in \mathrm{Pic}^{\mathbb{R}}(\Sigma, E)$ and any sequences $\alpha, \beta \in \mathbb{Z}_+^{\infty,\mathrm{odd}}$ such that $I(\alpha+\beta) = DE$. This statement immediately implies the statement of the theorem.

Using induction on $R_\Sigma(D,\beta)$ and the recursive formula (65) from [16], we easily derive that the numbers $N_\Sigma(D,\alpha,\beta)$ are even if $\beta \notin \mathbb{Z}_+^{\infty,\mathrm{odd}}$, and hence

$$I^\beta \cdot N_\Sigma(D,\alpha,\beta) = 0 \mod 4 \quad \text{if } \beta \notin \mathbb{Z}_+^{\infty,\mathrm{odd}}\ . \tag{28}$$

Then, using Proposition 12, we check the congruence (27) in the case $R_\Sigma(D,\beta) = 0$. Finally, we proceed by induction on $R_\Sigma(D,\beta)$, comparing term by term [16, Formula (65)] and the formula (12) and using the following observations:

- $\mathrm{Pic}^{\mathbb{R}}(\Sigma, E)$ contains only divisor classes, and hence the parameters $(\beta^{\mathrm{im}})^{(i)}$ in (12) always vanish,
- for any integer j, one has
$$j^2 = \begin{cases} 0 \mod 4, & \text{if } j = 0 \mod 2, \\ 1 \mod 4, & \text{if } j = 1 \mod 2, \end{cases}$$
- for any non-negative integer k, one has
$$\binom{k+3}{3} = \begin{cases} 0 \mod 4, & \text{if } k = 1 \mod 2, \\ l+1 \mod 4, & \text{if } k = 2l, \end{cases}$$
- for any sequence $\beta^{(0)} \in \mathbb{Z}_+^{\infty,\mathrm{odd}}$, one has
$$2^{\|\beta^{(0)}\|} I^{\beta^{(0)}} = 2^{\|\beta^{(0)}\|} \mod 4,$$

□

The congruence (26) was established by G. Mikhalkin ([14], *cf.* [2]) for $\Sigma = \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{P}^2_{a,0}$, $a = 1,2,3$, and then extended to the cases of $\mathbb{P}^2_{4,0}$ and $\mathbb{P}^2_{5,0}$ in [13].

References

[1] Brugallé, E., private communication.

[2] Brugallé, E., Mikhalkin, G.: Enumeration of curves via floor diagrams. C. R. Math. Acad. Sci., Paris **345**, no. 6, (2007), 329–334.

[3] Caporaso, L., and Harris, J.: Counting plane curves of any genus. *Invent. Math.* **131** (1998), no. 2, 345–392.

[4] Comessatti, A;: Fondamenti per la geometria sopra le superficie razionali dal punto di vista reale. *Math. Ann.* **73** (1912), 1–72.

[5] Degtyarev, A., Kharlamov, V.: Around real Enriques surfaces. *Revista Matematica de la Universidad Complutense de Madrid* **10** (1997), 93–109.

[6] Di Francesco, P., Itzykson, C.: Quantum intersection rings. *in "The moduli space of curves"* (Texel Island, 1994), 81–148, Progr. Math., 129, Birkhäuser Boston, Boston, MA, 1995.

[7] Greuel, G.-M., Lossen, C., and Shustin, E.: Geometry of families of nodal curves on the blown up projective plane. *Trans. Amer. Math. Soc.* **350** (1998), no. 1, 251–274.

[8] Itenberg, I., Kharlamov, V., and Shustin, E.: Welschinger invariant and enumeration of real rational curves. *International Math. Research Notices* **49** (2003), 2639–2653.

[9] Itenberg, I., Kharlamov, V., and Shustin, E.: Logarithmic equivalence of Welschinger and Gromov-Witten invariants. *Russian Math. Surveys* **59** (2004), no. 6, 1093–1116.

[10] Itenberg, I., Kharlamov, V., and Shustin, E.: Logarithmic asymptotics of the genus zero Gromov-Witten invariants of the blown up plane. *Geometry and Topology* **9** (2005), paper no. 14, 483–491.

[11] Itenberg, I., Kharlamov, V., and Shustin, E.: New cases of logarithmic equivalence of Welschinger and Gromov-Witten invariants. *Proc. Steklov Math. Inst.* **258** (2007), 65–73.

[12] Itenberg, I., Kharlamov, V., and Shustin, E.: A Caporaso-Harris type formula for Welschinger invariants of real toric Del Pezzo surfaces. *Comment. Math. Helv.* **84** (2009), 87–126.

[13] Itenberg, I., Kharlamov, V., and Shustin, E.: *Welschinger invariants of small non-toric Del Pezzo surfaces*. Preprint at arXiv:1002.1399 (to appear in J. of the European Math. Society).

[14] Mikhalkin, G.: Private communication.

[15] Segre, B.: *The nonsingular cubic surfaces*. Clarendon Press, Oxford, 1942.

[16] Shoval, M., and Shustin, E.: *Counting curves of any genus on* $\mathbb{P}^2_6$. Preprint at arXiv:1106.6155.

[17] Shustin, E.: Welschinger invariants of toric Del Pezzo surfaces with non-standard real structures. *Proc. Steklov Math. Inst.* **258** (2007), 218–247.

[18] Vakil, R.: Counting curves on rational surfaces. *Manuscripta Math.* **102** (2000), 53–84.

[19] Welschinger, J.-Y.: Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry. *Invent. Math.* **162** (2005), no. 1, 195–234.

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