THE STICKELBERGER SPLITTING MAP AND EULER SYSTEMS IN THE K–THEORY OF NUMBER FIELDS

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Abstract. For a CM abelian extension F/K of an arbitrary totally real number field K, we construct the Stickelberger splitting maps (in the sense of [1]) for both the étale and the Quillen K–theory of F and we use these maps to construct Euler systems in the even Quillen K–theory of F. The Stickelberger splitting maps give an immediate proof of the annihilation of the groups of divisible elements $divK_{2n}(F)_l$ of the even K–theory of the top field by higher Stickelberger elements, for all odd primes l. This generalizes the results of [1], which only deals with CM abelian extensions of $\mathbb{Q}$. The techniques involved in constructing our Euler systems at this level of generality are quite different from those used in [3], where an Euler system in the odd K–theory with finite coefficients of abelian CM extensions of $\mathbb{Q}$ was given. We work under the assumption that the Iwasawa μ–invariant conjecture holds. This permits us to make use of the recent results of Greither-Popescu [16] on the étale Coates-Sinnott conjecture for arbitrary abelian extensions of totally real number fields, which are conditional upon this assumption. In upcoming work, we will use the Euler systems constructed in this paper to obtain information on the groups of divisible elements $divK_{2n}(F)_l$, for all $n > 0$ and odd l. It is known that the structure of these groups is intimately related to some of the deepest unsolved problems in algebraic number theory, e.g. the Kummer-Vandiver and Iwasawa conjectures on class groups of cyclotomic fields. We make these connections explicit in the introduction.

1. Introduction

Let F/K be an abelian CM extension of a totally real number field K. Let $\mathbf{f}$ be the conductor of F/K and let $K_{\mathbf{f}}/K$ be the ray–class field extension with conductor $\mathbf{f}$. Let $G_{\mathbf{f}} := G(K_{\mathbf{f}}/K)$. For all $n \in \mathbb{Z}_{\geq 0}$, Coates [10] defined higher Stickelberger elements $\Theta_n(\mathbf{b}, \mathbf{f}) \in \mathbb{Q}[G(F/K)]$, for integral ideals $\mathbf{b}$ of K coprime to $\mathbf{f}$. Deligne and Ribet [12] proved that $\Theta_n(\mathbf{b}, \mathbf{f}) \in \mathbb{Z}[G(F/K)]$, if $\mathbf{b}$ is also coprime to $w_{n+1}(F) := \operatorname{card} H^0(F, \mathbb{Q}/\mathbb{Z}(n+1))$. A detailed discussion of the Stickelberger elements and their basic properties is given in §2 below. In 1974, Coates and Sinnott [11] formulated the following conjecture.

Conjecture 1.1 (Coates-Sinnott)**.** *For all $n \geq 1$ and all $\mathbf{b}$ coprime to $w_{n+1}(F)$, $\Theta_n(\mathbf{b}, \mathbf{f})$ annihilates $K_{2n}(\mathcal{O}_F)$.*

This should be viewed as a higher analogue of the classical conjecture of Brumer.

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Conjecture 1.2 (Brumer)**.** *For all* $\mathbf{b}$ *coprime to* $w_1(F)$*,* $\Theta_0(\mathbf{b},\mathbf{f})$ *annihilates* $K_0(\mathcal{O}_F)_{\mathrm{tors}} = Cl(O_F)$*.*

Coates and Sinnott [11] proved that for the base field $K = \mathbb{Q}$ the element $\Theta_1(\mathbf{b},\mathbf{f})$ annihilates $K_2(\mathcal{O}_F)$ for $F/\mathbb{Q}$ abelian and $\mathbf{b}$ coprime to the order of $K_2(\mathcal{O}_F)$. Moreover, in the case $K = \mathbb{Q}$, they proved that $\Theta_n(\mathbf{b},\mathbf{f})$ annihilates the l–adic étale cohomology groups $H^2(\mathcal{O}_F[1/l], \mathbb{Z}_l(n+1)) \simeq K_{2n}^{et}(\mathcal{O}_F[1/l])$ for any odd prime l, and any odd $n \geq 1$. One of the ingredients used in the proof is the fact that Brumer's conjecture holds true if $K = \mathbb{Q}$. This is the classical theorem of Stickelberger. The passage from annihilation of étale cohomology to that of K–theory in the case $n = 1$ was possible due to the following theorem (see [28], [8] and [9].)

Theorem 1.3 (Tate)**.** *The* l*–adic Chern map gives a canonical isomorphism*

$$K_2(\mathcal{O}_L) \otimes \mathbb{Z}_l \xrightarrow{\cong} K_2^{et}(\mathcal{O}_L[1/l]),$$

for any number field L *and any odd prime* l*.*

The following deep conjecture aims at generalizing Tate's theorem.

Conjecture 1.4 (Quillen-Lichtenbaum)**.** *For any number field* L*, any* $m \geq 1$ *and any odd prime* l *there is a natural* l*–adic Chern map isomorphism*

$$K_m(\mathcal{O}_L) \otimes \mathbb{Z}_l \xrightarrow{\cong} K_m^{et}(\mathcal{O}_L[1/l]) \tag{1}$$

Very recently, Greither and the second author used Iwasawa theoretic techniques to prove the following results for a general abelian CM extension F/K of an arbitrary totally real field K (see [16].)

Theorem 1.5 (Greither-Popescu)**.** *Assume that* l *is odd and the Iwasawa* μ*–invariant* $\mu_{F,l}$ *associated to* F *and* l *vanishes. Then, we have the following.*

(1) $\prod_{\mathbf{l}}(1 - (\mathbf{l}, F/K)^{-1} \cdot N\mathbf{l}) \cdot \Theta_0(\mathbf{b},\mathbf{f})$ *annihilates* $Cl(O_F)_l$*, for all* $\mathbf{b}$ *coprime to* $w_1(K)l$*, where the product is taken over primes* $\mathbf{l}$ *of* K *which divide* l *and are coprime to* $\mathbf{f}$*.*

(2) $\Theta_n(\mathbf{b},\mathbf{f})$ *annihilates* $K_{2n}^{et}(\mathcal{O}_F[1/l])$*, for all* $n \geq 1$ *and all* $\mathbf{b}$ *coprime to* $w_{n+1}(F)l$*.*

In fact, stronger results are proved in [16], involving Fitting ideals rather than annihilators and, in the case $n = 0$, a refinement of Brumer's conjecture, known as the Brumer-Stark conjecture (see Theorems 6.5 and 6.11 in loc.cit.)

Results similar to the Fitting ideal version of part (2) of Theorem 1.5 were also obtained with different methods by Burns–Greither in [5] and by Nguyen Quang Do in [19], under some extra hypotheses.

Note that a well known conjecture of Iwasawa states that $\mu_{F,l} = 0$, for all l and F as above. This conjecture is known to hold if F is an abelian extension of $\mathbb{Q}$, due independently to Ferrero-Washington and Sinnott. Consequently, if the Quillen-Lichtenbaum conjecture is proved, then, for all odd primes l, the l–primary part of the Coates-Sinnott conjecture is established unconditionally for all abelian extensions $F/\mathbb{Q}$ and for general extensions F/K, under the assumption that $\mu_{F,l} = 0$. It is hoped that recent work of Suslin, Voyevodsky, Rost, Friedlander, Morel, Levine, Weibel and others will lead to a proof of the Quillen-Lichtenbaum conjecture.

In 1992, a different approach towards the Coates-Sinnott conjecture was used in [1], in the case $K = \mathbb{Q}$. Namely, for all $n \geq 1$, all $\mathbf{b}$ coprime to $w_{n+1}(F)$, and $l > 2$, the first author constructed in Ch. IV of loc.cit. the Stickelberger splitting map $\Lambda := \Lambda_n$ of the boundary map ∂_F in the Quillen localization sequence

$$0 \longrightarrow K_{2n}(\mathcal{O}_F)_l \longrightarrow K_{2n}(F)_l \underset{\Lambda}{\overset{\partial_F}{\rightleftarrows}} \bigoplus_v K_{2n-1}(k_v)_l \longrightarrow 0.$$

By definition, Λ is a homomorphism such that $\partial_F \circ \Lambda$ is the multiplication by $\Theta_n(\mathbf{b}, \mathbf{f})$. Above, k_v denotes the residue field of a prime v in O_F.

The existence of such a map Λ implies that $\Theta_n(\mathbf{b}, \mathbf{f})$ annihilates the group $div(K_{2n}(F)_l)$ of divisible elements in $K_{2n}(F)_l$ (see loc.cit. as well as Theorem 4.23 below.) This group is contained in $K_{2n}(\mathcal{O}_F)_l$, which is obvious from the exact sequence above and the finiteness of $K_{2n-1}(k_v)_l$, for all v.

The construction of Λ in loc.cit. was done without appealing to étale cohomology and the Quillen-Lichtenbaum conjecture. However, it was based on the fact that Brumer's Conjecture is known to hold for abelian extensions of $\mathbb{Q}$ (Stickelberger's theorem). Since Brumer's conjecture was not yet proved over arbitrary totally real base fields (and it is still not proved unconditionally at that level of generality), the construction of Λ in loc.cit. could not be generalized. Also, it should be mentioned that in loc.cit. various technical difficulties arose at primes $l|n$ and the map Λ was constructed only up to a certain power $l^{v_l(n)}$ in those cases.

In 1996, in joint work with Gajda [3], the first author discovered a new, perhaps deeper and farther reaching application of the existence of Λ for abelian extensions $F/\mathbb{Q}$. Namely, Λ was used in [3] to construct special elements which give rise to Euler systems in the K–theory with finite coefficients $\{K_{2n+1}(L, \mathbb{Z}/l^k)\}_L$, where L runs over all abelian extensions of $\mathbb{Q}$, such that $F \subseteq L$ and L/F has a square-free conductor coprime to $\mathbf{f}l$. Now, it is hoped that these Euler systems can be used to study the structure of the group of divisible elements $divK_{2n}(F)_l$, for all $n \geq 1$. This is a goal truly worth pursuing, as this group structure is linked to some of the deepest unsolved problems in algebraic number theory, as shown at the end of this introduction.

The main goal of this paper is to generalize the results obtained in [1] and [3] to the case of CM abelian extensions F/K of arbitrary totally real number fields K. Moreover, in terms of constructing Euler systems, we go far beyond [3] in that we construct Euler systems in Quillen K–theory rather than K–theory with finite coefficients only. Roughly speaking, our strategy is as follows.

Step 1. We fix an integer $m > 0$ and assume that the m–th Stickelberger elements $\Theta_m(\mathbf{b}, \mathbf{f}_k)$ annihilate $K_{2m}(\mathcal{O}_{F_k})_l$ (respectively $K_{2m}^{et}(\mathcal{O}_{F_k})_l$) for each k, where $F_k := F(\mu_{l^k})$ and $\mathbf{f}_k$ is the conductor of F_k/K. Under this assumption, we construct the Stickelberger splitting maps Λ_m (respectively Λ_m^{et}) for the K–theory (respectively étale K–theory) of F_k, for all $k \geq 1$. (See Lemma 4.5 and the constructions which lead to it.) Note that, if combined with Theorem 1.3, Theorem 1.5 shows that $\Theta_m(\mathbf{b}, \mathbf{f}_k)$ annihilates $K_{2m}(\mathcal{O}_{F_k})_l$, for $m = 1$ and l odd, under the assumption that $\mu_{F,l} = 0$ (and unconditionally if $F/\mathbb{Q}$ is abelian.) Also, in [20], the first author constructs an infinite class of abelian CM extensions F/K of an arbitrary totally real number field K for which the annihilation of $K_{2m}(\mathcal{O}_{F_k})_l$ by $\Theta_m(\mathbf{b}, \mathbf{f}_k)$, for $m = 1$ and l odd is proved unconditionally.

Step 2. We use the map Λ_m (respectively Λ_m^{et})) of Step 1 to construct special elements $\boldsymbol{\lambda}_{v,l^k}$ (respectively $\boldsymbol{\lambda}_{v,l^k}^{et}$) in the K–theory with coefficients $K_{2n}(O_{F,S_v};\mathbb{Z}/l^k)$ (respectively étale K–theory with coefficients $K_{2n}^{et}(O_{F,S_v};\mathbb{Z}/l^k)$), for all $n>0$, all $k\geq 0$ and all primes v in O_F, where S_v is a sufficiently large finite set of primes in F. (See Definition 4.7 .)

Step 3. We use the special elements of Step 3 and a projective limit process with respect to k to construct the Stickelberger splitting maps Λ_n and Λ_n^{et} taking values in $K_{2n}(F)_l$ and $K_{2n}^{et}(F)$, respectively, for all $n\geq 1$. (See Definition 4.16 and Theorem 4.17.) This step generalizes the constructions in [1] to abelian CM extensions of arbitrary totally real fields. It also eliminates the extra-factor $l^{v_l(n)}$ which appeared in loc.cit. in the case $l|n$, for abelian CM extensions of $\mathbb{Q}$.

Step 4. We use the special elements of Step 2 as well as the maps Λ_n of Step 3 to construct Euler Systems $\{\Lambda_n(\xi_{v(\mathbf{L})})\}_{\mathbf{L}}$ in the K-theory without coefficients $\{K_{2n}(F_{\mathbf{L}})_l\}_{\mathbf{L}}$, for every $n>0$, where $\mathbf{L}$ runs through the squarefree ideals of O_F which are coprime to $\mathbf{f}l$, $F_{\mathbf{L}}$ is the ray class field of F corresponding to $\mathbf{L}$ and S is a sufficiently large finite set of primes in O_F. (See Definitions 5.4 and 5.5 as well as Theorem 5.7.) A similar construction of Euler systems in étale K–theory can be done without difficulty. This step generalizes the constructions of [3] to the case of abelian CM extensions of totally real number fields. It is also worth noting that while [3] contains a construction of Euler systems only in the case of K–theory with coefficients, we deal with both the K–theory with and without coefficients in the more general setting discussed in this paper.

In the process, as a consequence of the construction of Λ_n (Step 3), we obtain a direct proof that $\Theta_n(\mathbf{b},\mathbf{f})$ annihilates the group $div(K_{2n}(F)_l)$, for arbitrary CM abelian extensions F/K of totally real base field K and all $n>0$, under the assumption that $l>2$ and $\mu_{F,l}=0$ (see Theorem 4.26.)

In our upcoming work, we are planning on using the Euler systems described in Step 4 above to study the structure of the groups of divisible elements $divK_{2n}(F)_l$, for all $n>0$ and all $l>2$.

We conclude this introduction with a few paragraphs showing that the groups of divisible elements in the K–theory of number fields lie at the heart of several important conjectures in number theory, which justifies the effort to understand their structure in terms of special values of global L–functions. In 1988, Warren Sinnott pointed out to the first author that Stickelberger's Theorem for an abelian extension $F/\mathbb{Q}$ or, more generally, Brumer's conjecture for a CM extension F/K of a totally real number field K is equivalent to the existence of a Stickelberger splitting map Λ in the following basic exact sequence

$$0\longrightarrow \mathcal{O}_F^{\times}\longrightarrow F^{\times}\underset{\xleftarrow{\Lambda}}{\xrightarrow{\partial_F}}\bigoplus_v \mathbb{Z}\longrightarrow Cl(O_F)\longrightarrow 0.$$

This means that Λ is a group homomorphism, such that $\partial_F\circ\Lambda$ is the multiplication by $\Theta_0(\mathbf{b},\mathbf{f})$. Obviously, the above exact sequence is the lower part of the Quillen localization sequence in K–theory, since $K_1(\mathcal{O}_F)=\mathcal{O}_F^{\times}$, $K_1(F)=F^{\times}$, $K_0(k_v)=\mathbb{Z}$, $K_0(\mathcal{O}_F)_{\text{tors}}=Cl(O_F)$ and Quillen's ∂_F is the direct sum of the valuation maps in this case.

Further, by [2] p. 292 we observe that for any prime $l>2$, the annihilation of $div(K_{2n}(F)_l)$ by $\Theta_n(\mathbf{b},\mathbf{f})$ is equivalent to the existence of a "splitting" map Λ in the following exact sequence

$$0 \longrightarrow K_{2n}(\mathcal{O}_F)[l^k] \longrightarrow K_{2n}(F)[l^k] \underset{\Lambda}{\overset{\partial_F}{\rightleftarrows}} \bigoplus_v K_{2n-1}(k_v)[l^k] \longrightarrow div(K_{2n}(F)_l) \longrightarrow 0$$

such that $\partial_F \circ \Lambda$ is the multiplication by $\Theta_n(\mathbf{b}, \mathbf{f})$, for any $k \gg 0$. Hence, the group of divisible elements $div(K_{2n}(F)_l)$ is a direct analogue of the l–primary part $Cl(O_F)_l$ of the class group. Any two such "splittings" Λ differ by a homomorphism in $\mathrm{Hom}(\bigoplus_v K_{2n-1}(k_v)[l^k],\ K_{2n}(\mathcal{O}_F)[l^k])$. Moreover, the Coates-Sinnott conjecture is equivalent to the existence of a "splitting" Λ, such that $\Lambda \circ \partial_F$ is the multiplication by $\Theta_n(\mathbf{b}, \mathbf{f})$. If the Coates-Sinnott conjecture holds, then such a "splitting" Λ is unique and satisfies the property that $\partial_F \circ \Lambda$ is equal to the multiplication by $\Theta_n(\mathbf{b}, \mathbf{f})$. This is due to the fact that $div(K_{2n}(F)_l) \subset K_{2n}(\mathcal{O}_F)_l$. Clearly, in the case $div(K_{2n}(F)_l) = K_{2n}(\mathcal{O}_F)_l$, our map Λ also has the property that $\Lambda \circ \partial_F$ equals multiplication by $\Theta_n(\mathbf{b}, \mathbf{f})$. Observe that if the Quillen-Lichtenbaum conjecture holds, then by Theorem 4 in [2], we have

$$div(K_{2n}(F)_l) = K_{2n}(\mathcal{O}_F)_l \ \Leftrightarrow\ \left| \frac{\prod_{v|l} w_n(F_v)}{w_n(F)} \right|_l^{-1} = 1.$$

In particular, for $F = \mathbb{Q}$ and n odd, we have $w_n(\mathbb{Q}) = w_n(\mathbb{Q}_l) = 2$. Hence, according to the Quillen-Lichtenbaum conjecture, for any $l > 2$ we should have $div(K_{2n}(\mathbb{Q})_l) = K_{2n}(\mathbb{Z})_l$.

Now, let $A := Cl(\mathbb{Z}[\mu_l])_l$ and let $A^{[i]}$ denote the eigenspace corresponding to the i–th power of the Teichmuller character $\omega : G(\mathbb{Q}(\mu_l)/\mathbb{Q}) \to (\mathbb{Z}/l\mathbb{Z})^\times$. Consider the following classical conjectures in cyclotomic field theory.

Conjecture 1.6 (Kummer-Vandiver)**.**

$$A^{[l-1-n]} = 0 \quad \textit{for all } n \textit{ even and} \quad 0 \leq n \leq l-1$$

Conjecture 1.7 (Iwasawa)**.**

$$A^{[l-1-n]} \quad \textit{is cyclic for all } n \textit{ odd, such that} \quad 1 \leq n \leq l-2$$

We can state the Kummer-Vandiver and Iwasawa conjectures in terms of divisible elements in K–theory of $\mathbb{Q}$ (see [3] and [4]):

(1) $A^{[l-1-n]} = 0 \Leftrightarrow div(K_{2n}(\mathbb{Q})_l) = 0,$ for all n even, with $1 \leq n \leq (l-1)$.
(2) $A^{[l-1-n]}$ is cyclic $\Leftrightarrow div(K_{2n}(\mathbb{Q})_l)$ is cyclic, for all n odd, with $n \leq (l-2)$.

Finally, we would like to point out that the groups of divisible elements discussed in this paper are also related to the Quillen-Lichtenbaum conjecture. Namely, by comparing the exact sequence of [24], Satz 8 with the exact sequence of [2], Theorem 2 we conclude that the Quillen-Lichtenbaum conjecture for the K-group $K_{2n}(F)$ (for any number field F and any prime $l > 2$) holds if and only if

$$div(K_{2n}(F)_l) = K^w_{2n}(\mathcal{O}_F)_l$$

where $K^w_{2n}(\mathcal{O}_F)_l$ is the wild kernel defined in [2].

2. Basic facts about the Stickelberger ideals

Let F/K be an abelian CM extension of a totally real number field K. Let $\mathbf{f}$ be the conductor of F/K and let $K_{\mathbf{f}}/K$ be the ray class field extension corresponding to $\mathbf{f}$. Let $G_{\mathbf{f}} := G(K_{\mathbf{f}}/K)$. Every element of $G_{\mathbf{f}}$ is the Frobenius morphism $\sigma_{\mathbf{a}}$, for some ideal $\mathbf{a}$ of $\mathcal{O}_K$, coprime to the conductor $\mathbf{f}$. Let $(\mathbf{a}, F)$ denote the image of $\sigma_{\mathbf{a}}$ in $G(F/K)$ via the natural surjection $G_{\mathbf{f}} \to G(F/K)$. Choose a prime number l.

With the usual notations, we let $I(\mathbf{f})/P_1(\mathbf{f})$ be the ray class group of fractional ideals in K coprime to $\mathbf{f}$. Let $\mathbf{a}$ and $\mathbf{a}'$ be two fractional ideals in $I(\mathbf{f})$. The symbol $\mathbf{a} \equiv \mathbf{a}' \mod \mathbf{f}$ will mean that $\mathbf{a}$ and $\mathbf{a}'$ are in the same class modulo $P_1(\mathbf{f})$. For every $\mathbf{a} \in I(\mathbf{f})$ we consider the partial zeta function of [10], p. 291, given by

$$\zeta_{\mathbf{f}}(\mathbf{a}, s) := \sum_{\mathbf{c} \equiv \mathbf{a} \bmod \mathbf{f}} \frac{1}{N\mathbf{c}^s}\,, \qquad \mathrm{Re}(s) > 1, \tag{2}$$

where the sum is taken over the integral ideals $\mathbf{c} \in \mathbf{I}(\mathbf{f})$ and $N\mathbf{c}$ denotes the usual norm of the integral ideal $\mathbf{c}$. The partial zeta $\zeta_{\mathbf{f}}(\mathbf{a}, s)$ can be meromorphically continued to the complex plane with a single pole at $s = 1$. For $s \in \mathbb{C} \setminus \{1\}$, consider the Sickelberger element of [C], p. 297,

$$\Theta_s(\mathbf{b}, \mathbf{f}) := (N\mathbf{b}^{s+1} - (\mathbf{b},\, F)) \sum_{\mathbf{a}} \zeta_{\mathbf{f}}(\mathbf{a}, -s)(\mathbf{a},\, F)^{-1} \in \mathbb{C}[G(F/K)] \tag{3}$$

where $\mathbf{b}$ is an integral ideal in $I(\mathbf{f})$ and the summation is over a finite set $\mathcal{S}$ of ideals $\mathbf{a}$ of $\mathcal{O}_K$ coprime to $\mathbf{f}$, chosen such that the Artin map

$$\mathcal{S} \longrightarrow G(K_{\mathbf{f}}/K)\,, \quad \mathbf{a} \longrightarrow \sigma_{\mathbf{a}}$$

is bijective. The element $\Theta_s(\mathbf{b}, \mathbf{f})$ can be written in the following way

$$\Theta_s(\mathbf{b}, \mathbf{f}) := \sum_{\mathbf{a}} \Delta_{s+1}(\mathbf{a}, \mathbf{b}, \mathbf{f})(\mathbf{a},\, F)^{-1}, \tag{4}$$

where

$$\Delta_{s+1}(\mathbf{a}, \mathbf{b}, \mathbf{f}) := N\mathbf{b}^{s+1}\zeta_{\mathbf{f}}(\mathbf{a}, -s) - \zeta_{\mathbf{f}}(\mathbf{ab}, -s). \tag{5}$$

Arithmetically, the Stickelberger elements $\Theta_s(\mathbf{b}, \mathbf{f})$ are most interesting for values $s = n$, with $n \in \mathbb{N} \cup \{0\}$. If $\mathbf{a}, \mathbf{b}, \mathbf{f}$ are integral ideals, such that $\mathbf{ab}$ is coprime to $\mathbf{f}$, then Deligne and Ribet [12] proved that $\Delta_{n+1}(\mathbf{a}, \mathbf{b}, \mathbf{f})$ are l-adic integers for all primes $l \nmid N\mathbf{b}$ and all $n \geq 0$. Moreover, in loc.cit. it is proved that

$$\Delta_{n+1}(\mathbf{a}, \mathbf{b}, \mathbf{f}) \equiv N(\mathbf{ab})^n \Delta_1(\mathbf{a}, \mathbf{b}, \mathbf{f}) \mod w_n(K_{\mathbf{f}}). \tag{6}$$

As usual, if L is a number field, then $w_n(L)$ is the largest number $m \in \mathbb{N}$ such that the Galois group $G(L(\mu_m)/L)$ has exponent dividing n. Note that

$$w_n(L) = |H^0(G(\overline{L}/L),\, \mathbb{Q}/\mathbb{Z}(n))|\,,$$

where $\mathbb{Q}/\mathbb{Z}(n) := \oplus_l \mathbb{Q}_l/\mathbb{Z}_l(n)$. By Theorem 2.4 of [C], the results in [12] lead to

$$\Theta_n(\mathbf{b}, \mathbf{f}) \in \mathbb{Z}[G(F/K)],$$

whenever $\mathbf{b}$ is coprime to $w_{n+1}(F)$. The ideal of $\mathbb{Z}[G(F/K)]$ generated by the elements $\Theta_n(\mathbf{b}, \mathbf{f})$, for all integral ideals $\mathbf{b}$ coprime to $w_{n+1}(F)$ is called the n-th Stickelberger ideal for F/K.

When $K \subset F \subset E$ is a tower of finite abelian extensions then

$$Res_{E/F} \,:\, G(E/K) \to G(F/K), \qquad Res_{E/F} : \mathbb{C}[G(E/K)] \to \mathbb{C}[G(F/K)]$$

denote the restriction map and its $\mathbb{C}$–linear extension at the level of group rings, respectively. If $\mathbf{f} \,|\, \mathbf{f}'$ and $\mathbf{f}$ and $\mathbf{f}'$ are divisible by the same prime ideals of $\mathcal{O}_K$ then, for all $\mathbf{b}$ coprime to $\mathbf{f}$, we have the following equality (see [10] Lemma 2.1, p. 292).

$$Res_{K_{\mathbf{f}'}/K_{\mathbf{f}}}\, \Theta_s(\mathbf{b}, \mathbf{f}') = \Theta_s(\mathbf{b}, \mathbf{f}). \tag{7}$$

Let $\mathbf{l}$ is a prime ideal of O_K coprime to $\mathbf{f}$. Then, we have

$$\zeta_{\mathbf{f}}(\mathbf{a}, s) := \sum_{\substack{\mathbf{c} \equiv \mathbf{a} \bmod \mathbf{f} \\ \mathbf{l} \nmid \mathbf{c}}} \frac{1}{N\mathbf{c}^s} + \sum_{\substack{\mathbf{c} \equiv \mathbf{a} \bmod \mathbf{f} \\ \mathbf{l} \,|\, \mathbf{c}}} \frac{1}{N\mathbf{c}^s}. \tag{8}$$

Observe that we also have

$$\sum_{\substack{\mathbf{c} \equiv \mathbf{a} \bmod \mathbf{f} \\ \mathbf{l} \nmid \mathbf{c}}} \frac{1}{N\mathbf{c}^s} = \sum_{\substack{\mathbf{a}' \bmod \mathbf{lf} \\ \mathbf{a}' \equiv \mathbf{a} \bmod \mathbf{f}}} \sum_{\mathbf{c} \equiv \mathbf{a}' \bmod \mathbf{lf}} \frac{1}{N\mathbf{c}^s} = \sum_{\substack{\mathbf{a}' \bmod \mathbf{lf} \\ \mathbf{a}' \equiv \mathbf{a} \bmod \mathbf{f}}} \zeta_{\mathbf{lf}}(\mathbf{a}', s) \tag{9}$$

Let us fix a finite $\mathcal{S}$ of integral ideals $\mathbf{a}$ in $I(\mathbf{f})$ as above. Observe that every class corresponding to an integral ideal $\mathbf{a}$ modulo $P_1(\mathbf{f})$ can be written uniquely as a class $\mathbf{la}''$ modulo $P_1(\mathbf{f})$, for some $\mathbf{a}''$ from our set $\mathcal{S}$ of chosen integral ideals. This establishes a one–to–one correspondence between classes $\mathbf{a}$ modulo $P_1(\mathbf{f})$ and $\mathbf{a}''$ modulo $P_1(\mathbf{f})$. If $\mathbf{l} \,|\, \mathbf{c}$, we put $\mathbf{c} = \mathbf{lc}'$. Hence, we have the following equality.

$$\sum_{\substack{\mathbf{c} \equiv \mathbf{a} \bmod \mathbf{f} \\ \mathbf{l} \,|\, \mathbf{c}}} \frac{1}{N\mathbf{c}^s} \;=\; \frac{1}{N\mathbf{l}^s} \sum_{\mathbf{c}' \equiv \mathbf{a}'' \bmod \mathbf{f}} \frac{1}{N\mathbf{c}'^{\,s}} = \frac{1}{N\mathbf{l}^s}\, \zeta_{\mathbf{f}}(\mathbf{a}'', s) \tag{10}$$

Formulas (8), (9) and (10) lead to the following equality:

$$\zeta_{\mathbf{f}}(\mathbf{a}, s) - \frac{1}{N\mathbf{l}^s} \zeta_{\mathbf{f}}(\mathbf{l}^{-1}\mathbf{a}, s) = \sum_{\substack{\mathbf{a}' \bmod \mathbf{lf} \\ \mathbf{a}' \equiv \mathbf{a} \bmod \mathbf{f}}} \zeta_{\mathbf{lf}}(\mathbf{a}', s). \tag{11}$$

For all $\mathbf{f}$ coprime to $\mathbf{l}$ and for all $\mathbf{b}$ coprime to $\mathbf{lf}$, equality (11) gives:

$$Res_{K_{\mathbf{lf}}/K_{\mathbf{f}}}\; \Theta_s(\mathbf{b}, \mathbf{lf}) = (1 - (\mathbf{l},\, F)^{-1} N\mathbf{l}^s)\, \Theta_s(\mathbf{b}, \mathbf{f}) \tag{12}$$

Indeed we easily check that:

$$Res_{K_{\mathbf{lf}}/K_{\mathbf{f}}}\, (N\mathbf{b}^{s+1} - (\mathbf{b},\, F)) \sum_{\mathbf{a}' \bmod \mathbf{lf}} \zeta_{\mathbf{lf}}(\mathbf{a}', -s)(\mathbf{a}',\, F)^{-1} =$$

$$(N\mathbf{b}^{s+1} - (\mathbf{b},\, F)) \sum_{\mathbf{a} \bmod \mathbf{f}} \sum_{\substack{\mathbf{a}' \bmod \mathbf{lf} \\ \mathbf{a}' \equiv \mathbf{a} \bmod \mathbf{f}}} \zeta_{\mathbf{lf}}(\mathbf{a}', -s)(\mathbf{a},\, F)^{-1} =$$

$$(N\mathbf{b}^{s+1} - (\mathbf{b},\, F)) \sum_{\mathbf{a} \bmod \mathbf{f}} (\zeta_{\mathbf{f}}(\mathbf{a}, -s) - N\mathbf{l}^s \zeta_{\mathbf{f}}(\mathbf{l}^{-1}\mathbf{a}, -s))(\mathbf{a},\, F)^{-1} =$$

$$(N\mathbf{b}^{s+1} - (\mathbf{b},\, F))(\sum_{\mathbf{a} \bmod \mathbf{f}} \zeta_{\mathbf{f}}(\mathbf{a}, -s)(\mathbf{a},\, F)^{-1} - (\mathbf{l},\, F)^{-1} N\mathbf{l}^s\, \zeta_{\mathbf{f}}(\mathbf{l}^{-1}\mathbf{a}, -s)(\mathbf{l}^{-1}\mathbf{a},\, F)^{-1}) =$$

$$(1-(\mathbf{l},\,F)^{-1}N\mathbf{l}^s)(N\mathbf{b}^{s+1}-(\mathbf{b},\,F))\sum_{\mathbf{a}\bmod\mathbf{f}}\zeta_{\mathbf{f}}(\mathbf{a},-s)(\mathbf{a},\,F)^{-1}$$

Lemma 2.1. *Let $\mathbf{f}\,|\,\mathbf{f}'$ be ideals of $\mathcal{O}_K$ coprime to $\mathbf{b}$. Then, we have the following.*

$$Res_{K_{\mathbf{f}'}/K_{\mathbf{f}}}\;\Theta_s(\mathbf{b},\mathbf{f}')=\Big(\prod_{\substack{\mathbf{l}\nmid\mathbf{f}\\ \mathbf{l}\,|\,\mathbf{f}'}}(1-(\mathbf{l},\,F)^{-1}N\mathbf{l}^s)\Big)\;\Theta_s(\mathbf{b},\mathbf{f}) \tag{13}$$

Proof. The lemma follows from (7) and (12). □

Remark 2.2. The property of higher Stickelberger elements given by the above Lemma will translate naturally into the Euler System property of the special elements in Quillen K–theory constructed in §5 below.

In what follows, for any given abelian extension F/K of conductor $\mathbf{f}$, we consider the field extensions $F(\mu_{l^k})/K$, for all $k\geq 0$ and a fixed prime l, where μ_{l^k} denotes the group of roots of unity of order dividing l^k. We let $\mathbf{f}_k$ denote the conductor of the abelian extension $F(\mu_{l^k})/K$. We suppress from the notation the explicit dependence of $\mathbf{f}_k$ on l, since the prime l will be chosen and fixed once and for all in this paper.

3. Basic facts about algebraic K-theory

3.1. **The Bockstein sequence and the Bott element.** Let us fix a prime number l. For a ring R we consider the Quillen K-groups

$$K_m(R):=\pi_m(\Omega BQP(R)):=[S^m,\,\Omega BQP(R)]$$

(see [21]) and the K-groups with coefficients

$$K_m(R,\,\mathbb{Z}/l^k):=\pi_m(\Omega BQP(R),\,\mathbb{Z}/l^k):=[M^m_{l^k},\,\Omega BQP(R)]$$

defined by Browder and Karoubi in [6]. Quillen's K–groups can also be computed using Quillen's plus construction as $K_n(R):=\pi_n(BGL(R)^+)$. Any unital homomorphism of rings $\phi\,:\,R\to R'$ induces natural homomorphisms

$$\phi_{R|R'}\,:\,K_m(R,\,\diamondsuit)\longrightarrow K_m(R',\,\diamondsuit)$$

where $K_m(R,\,\diamondsuit)$ denotes either $K_m(R)$ or $K_m(R,\,\mathbb{Z}/l^k)$.

Quillen K-theory and K-theory with coefficients admit product structures:

$$K_n(R,\,\diamondsuit)\times K_m(R,\,\diamondsuit)\xrightarrow{\;*\;}K_{m+n}(R,\,\diamondsuit)$$

(see [21] and [6].) These induce graded ring structures on the groups $\bigoplus_{n\geq 0}K_n(R,\,\diamondsuit)$.

For a topological space X, there is a Bockstein exact sequence

$$\longrightarrow\pi_{m+1}(X,\,\mathbb{Z}/l^k)\xrightarrow{\;b\;}\pi_m(X)\xrightarrow{\;l^k\;}\pi_m(X)\longrightarrow\pi_m(X,\,\mathbb{Z}/l^k)\longrightarrow$$

In particular, if we take $X:=\Omega BQP(R))$, we get the Bockstein exact sequence in K-theory given by

$$\longrightarrow K_{m+1}(R,\,\mathbb{Z}/l^k)\xrightarrow{\;b\;}K_m(R)\xrightarrow{\;l^k\;}K_m(R)\longrightarrow K_m(R,\,\mathbb{Z}/l^k)\longrightarrow \tag{14}$$

For any discrete group G, we have:

$$\pi_n(BG) \;=\; \begin{cases} G & \text{if} \quad n=1 \\ 0 & \text{if} \quad n>1. \end{cases}$$

Consequently, for a commutative group G and $X := BG$ the Bockstein map b gives an isomorphism $b\,:\,\pi_2(BG,\,\mathbb{Z}/l^k) \xrightarrow{\cong} G[l^k]$. Here, $G[m]$ denotes the m–torsion subgroup of the commutative group G, for all $m \in \mathbb{N}$.

For a commutative ring with identity R we have $GL_1(R) = R^\times$. Assume that $\mu_{l^k} \subset R^\times$. Then $R^\times[l^k] = \mu_{l^k}$. Let β denote the natural composition of maps:

$$\begin{tikzcd} \mu_{l^k} \arrow[r, "b^{-1}"] & \pi_2(BGL_1(R);\mathbb{Z}/l^k) \arrow[r] & \pi_2(BGL(R);\mathbb{Z}/l^k) \arrow[d] & \\ & & \pi_2(BGL(R)^+;\mathbb{Z}/l^k) \arrow[r, "="] & K_2(R,\,\mathbb{Z}/l^k) \end{tikzcd}$$

We fix a generator ξ_{l^k} of μ_{l^k}. We define the Bott element

$$\beta_k := \beta(\xi_{l^k}), \qquad \beta_k \in K_2(R;\,\mathbb{Z}/l^k) \tag{15}$$

as the image of ξ_{l^k} via β. Further, we let

$$\beta_k^{*\,n} := \beta_k * \cdots * \beta_k \in K_{2n}(R;\,\mathbb{Z}/l^k).$$

The Bott element β_k depends of course on the ring R. However, we suppress this dependence from the notation since it will be always clear where a given Bott element lives. For example, if $\phi\,:\,R \to R'$ is a homomorphism of commutative rings containing μ_{l^k}, then it is clear from the definitions that the map

$$\phi_{R|R'}\;:\;K_2(R;\,\mathbb{Z}/l^k) \longrightarrow K_2(R',\,\mathbb{Z}/l^k)$$

transports the Bott element for R into the Bott element for R'. By a slight abuse of notation, this will be written as $\phi_{R|R'}(\beta_k) = \beta_k$.

Dwyer and Fiedlander [13] constructed the étale K-theory $K_*^{et}(R)$ and étale K-theory with coefficients $K_*^{et}(R,\mathbb{Z}/l^k)$ for any commutative, Noetherian $\mathbb{Z}[1/l]$–algebra R. Moreover, they proved that if $l>2$ then there are natural graded ring homomorphisms, called the Dwyer-Friedlander maps:

$$K_*(R) \longrightarrow K_*^{et}(R) \tag{16}$$

$$K_*(R;\mathbb{Z}/l^k) \longrightarrow K_*^{et}(R;\mathbb{Z}/l^k). \tag{17}$$

If R has finite $\mathbb{Z}/l$-cohomological dimension then there are Atiyah-Hirzebruch type spectral sequences (see [13], Propositions 5.1, 5.2):

$$E_2^{p,-q} = H^p(R;\mathbb{Z}_l(q/2)) \Rightarrow K_{q-p}^{et}(R). \tag{18}$$

$$E_2^{p,-q} = H^p(R;\mathbb{Z}/l^k(q/2)) \Rightarrow K_{q-p}^{et}(R;\mathbb{Z}/l^k). \tag{19}$$

Throughout, we will denote by $r_{k'/k}$ the reduction maps at the level of coefficients

$$r_{k'/k} : K_*(R;\mathbb{Z}/l^{k'}) \to K_*(R;\mathbb{Z}/l^k),$$

$$r_{k'/k} : K_*^{et}(R;\mathbb{Z}/l^{k'}) \to K_*^{et}(R;\mathbb{Z}/l^k),$$

for any R as above and $k' \geq k$.

3.2. **K-theory of finite fields.** Let $\mathbb{F}_q$ be the finite field with q elements. In [Q3], Quillen proved that:

$$K_n(\mathbb{F}_q) \simeq \begin{cases} \mathbb{Z} & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n = 2m \quad \text{and} \quad m > 0 \\ \mathbb{Z}/(q^m - 1)\mathbb{Z} & \text{if} \quad n = 2m - 1 \quad \text{and} \quad m > 0 \end{cases}$$

Moreover, in loc.cit, pp. 583-585, it is also showed that for an inclusion $i : \mathbb{F}_q \to \mathbb{F}_{q^f}$ of finite fields and all $n \geq 1$ the natural map

$$i : K_{2n-1}(\mathbb{F}_q) \to K_{2n-1}(\mathbb{F}_{q^f})$$

is injective and the transfer map

$$N : K_{2n-1}(\mathbb{F}_{q^f}) \to K_{2n-1}(\mathbb{F}_q)$$

is surjective, where we simply write i instead of $i_{\mathbb{F}_q|\mathbb{F}_{q^f}}$ and N instead of $Tr_{\mathbb{F}_{q^f}/\mathbb{F}_q}$. Further (see loc.cit., pp. 583-585), i induces an isomorphism

$$K_{2n-1}(\mathbb{F}_q) \cong K_{2n-1}(\mathbb{F}_{q^f})^{G(\mathbb{F}_{q^f}/\mathbb{F}_q)}$$

and the q–power Frobenius automorphism Fr_q (the canonical generator of $G(\mathbb{F}_{q^f}/\mathbb{F}_q)$) acts on $K_{2n-1}(\mathbb{F}_{q^f})$ via multiplication by q^n. Observe that

$$i \circ N = \sum_{i=0}^{f-1} Fr_q^i.$$

Hence, we have the equalities

$$\operatorname{Ker} N = K_{2n-1}(\mathbb{F}_{q^f})^{Fr_q - Id} = K_{2n-1}(\mathbb{F}_{q^f})^{q^n-1}$$

since $\operatorname{Ker} N$ is the kernel of multiplication by $\sum_{i=0}^{f-1} q^{ni} = \frac{q^{nf}-1}{q^n-1}$ in the cyclic group $K_{2n-1}(\mathbb{F}_{q^f})$. In particular, this shows that the norm map N induces the following group isomorphism

$$K_{2n-1}(\mathbb{F}_{q^f})_{G(\mathbb{F}_{q^f}/\mathbb{F}_q)} \cong K_{2n-1}(\mathbb{F}_q).$$

By the Bockstein exact sequence (14) and Quillen's results above, we observe that

$$K_{2n}(\mathbb{F}_q, \mathbb{Z}/l^k) \xrightarrow{b} K_{2n-1}(\mathbb{F}_q)[l^k]$$

is an isomorphism. Hence, $K_{2n}(\mathbb{F}_q, \mathbb{Z}/l^k)$ is a cyclic group.

Let us assume that $\mu_{l^k} \subset \mathbb{F}_q^\times$ (i.e. $l^k \mid q - 1$.) In this case, Browder [6] proved that the element $\beta_k^{*\,n}$ is a generator of $K_{2n}(\mathbb{F}_q, \mathbb{Z}/l^k)$. Dwyer and Friedlander [13] proved that there is a natural isomorphism of graded rings:

$$K_*(\mathbb{F}_q, \mathbb{Z}/l^k) \xrightarrow{\cong} K_*^{et}(\mathbb{F}_q, \mathbb{Z}/l^k).$$

By abuse of notation, let β_k denote the image of the Bott element defined in (15) via the natural isomorphism:

$$K_2(\mathbb{F}_q, \mathbb{Z}/l^k) \xrightarrow{\cong} K_2^{et}(\mathbb{F}_q, \mathbb{Z}/l^k).$$

Then by Theorem 5.6 in [13] multiplication with β_k induces isomorphisms:

$$\times\, \beta_k : K_i(\mathbb{F}_q, \mathbb{Z}/l^k) \xrightarrow{\cong} K_{i+2}(\mathbb{F}_q, \mathbb{Z}/l^k),$$

$$\times\, \beta_k : K_i^{et}(\mathbb{F}_q, \mathbb{Z}/l^k) \xrightarrow{\cong} K_{i+2}^{et}(\mathbb{F}_q, \mathbb{Z}/l^k).$$

In particular, if $l^k \,|\, q - 1$ and α is a generator of $K_1(\mathbb{F}_q, \mathbb{Z}/l^k) = K_1(\mathbb{F}_q)/l^k$, then the element $\alpha * \beta_k^{*\,n-1}$ is a generator of the cyclic group $K_{2n-1}(\mathbb{F}_q, \mathbb{Z}/l^k)$.

3.3. **K-theory of number fields and their rings of integers.** Let F be a number field. As usual, $\mathcal{O}_F$ denotes the ring of integers in F and k_v is the residue field for a prime v of $\mathcal{O}_F$. For a finite set of primes S of $\mathcal{O}_F$ the ring of S-integers of F is denoted $\mathcal{O}_{F,S}$.

Quillen [22] proved that $K_n(\mathcal{O}_F)$ is a finitely generated group for every $n \geq 0$. Borel computed the ranks of the groups $K_n(\mathcal{O}_F)$ as follows:

$$K_n(\mathcal{O}_F)\otimes_{\mathbb{Z}}\mathbb{Q} \;\simeq\; \begin{cases} \mathbb{Q} & \text{if} \quad n=0 \\ \mathbb{Q}^{r_1+r_2-1} & \text{if} \quad n=1 \\ 0 & \text{if} \quad n=2m \quad \text{and} \quad n>0 \\ \mathbb{Q}^{r_1+r_2} & \text{if} \quad n\equiv 1 \mod 4 \quad \text{and} \quad n\neq 1 \\ \mathbb{Q}^{r_2} & \text{if} \quad n\equiv 3 \mod 4 \end{cases}$$

We have the following localization exact sequences in Quillen K-theory and K-theory with coefficients [21].

$$\longrightarrow K_m(\mathcal{O}_F,\,\diamondsuit) \longrightarrow K_m(F,\,\diamondsuit) \xrightarrow{\partial_F} \bigoplus_v K_{m-1}(k_v,\,\diamondsuit) \longrightarrow K_{m-1}(\mathcal{O}_F,\,\diamondsuit) \longrightarrow$$

Let E/F be a finite extension. The natural maps in K-theory induced by the embedding $i\,:\,F\to E$ and $\sigma\,:\,E\to E$, for $\sigma\in G(E/F)$, will be denoted for simplicity by $i\,:\,K_m(F,\,\diamondsuit)\longrightarrow K_m(E,\,\diamondsuit)$ and $\sigma\,:\,K_m(E,\,\diamondsuit)\longrightarrow K_m(E,\,\diamondsuit)$. Observe that $i := i_{F|E}$ and $\sigma := \sigma_{E|E}$, according to the notation in section 3.1.

In addition to the natural maps i, σ, ∂_F, ∂_E, and product structures $*$ for K-theory of F and E introduced above, we have (see [21]) the transfer map

$$Tr_{E/F}\,:\,K_m(E,\,\diamondsuit)\longrightarrow K_m(F,\,\diamondsuit)$$

and the reduction map

$$r_v\,:\,K_m(\mathcal{O}_{F,S},\,\diamondsuit)\longrightarrow K_m(k_v,\,\diamondsuit)$$

for any prime $v\notin S$.

The maps discussed above enjoy many compatibility properties. For example, σ is naturally compatible with i, ∂_F, ∂_E, the product structure $*$, $Tr_{E/F}$ and r_w and r_v. See e.g. [1] for explanations of some of these compatibility properties. Let us mention below two nontrivial such compatibility properties which will be used in what follows. By a result of Gillet [17], we have the following commutative diagrams in Quillen K-theory and K-theory with coefficients:

$$\begin{tikzcd} K_m(F,\,\diamondsuit)\times K_n(\mathcal{O}_F,\,\diamondsuit) \arrow[r, "*"] \arrow[d, "\partial_F\times id"] & K_{m+n}(F,\,\diamondsuit) \arrow[d, "\partial_F"] \\ \bigoplus_v K_{m-1}(k_v,\,\diamondsuit)\times K_n(\mathcal{O}_F,\,\diamondsuit) \arrow[r, "*"] & \bigoplus_v K_{m+n-1}(k_v,\,\diamondsuit) \end{tikzcd} \tag{20}$$

Let E/F be a finite extension unramified over a prime v of $\mathcal{O}_F$. Let w be a prime of $\mathcal{O}_E$ over v. From now on, we will write $N_{w/v} := Tr_{k_w/k_v}$. The following diagram shows the compatibility of transfer with the boundary map in localization sequences for Quillen K-theory and K-theory with coefficients.

$$
\begin{tikzcd}
K_m(E,\,\diamondsuit) \arrow[r, "\partial_E"] \arrow[d, "Tr_{E/F}"'] & \bigoplus_v \bigoplus_{w\,|\,v} K_{m-1}(k_w,\,\diamondsuit) \arrow[d, "\bigoplus_v \bigoplus_{w\,|\,v} N_{w/v}"] \\
K_m(F,\,\diamondsuit) \arrow[r, "\partial_F"] & \bigoplus_v K_{m-1}(k_v,\,\diamondsuit)
\end{tikzcd} \tag{21}
$$

where the direct sums are taken with respect primes v in F and w in E, respectively.

4. Construction of Λ and Λ^{et} and first applications

In this section, we construct special elements in K-theory and étale K-theory with coefficients, under the assumption that for some fixed $m > 0$ the Stickelberger elements $\Theta_m(\mathbf{b}, \mathbf{f}_k)$ annihilate $K_{2m}(\mathcal{O}_{F_{l^k}})$ for all $k \geq 0$. This will produce special elements in K-theory and étale K-theory without coefficients which are of primary importance in our construction of Euler systems given in §5. These constructions will also give us the Stickelberger splitting maps $\Lambda := \Lambda_n$ and $\Lambda^{et} := \Lambda_n^{et}$ announced in the introduction. As a byproduct, we obtain a direct proof of the annihilation of the groups divisible elements $div(K_{2n}(F)_l)$, for all $n > 0$, generalizing the results of [1].

All the results in this section are stated for both K-theory and étale K-theory. However, detailed proofs will be given only in the case of K-theory since the proofs in the case of étale K-theory are very similar. The key idea in transferring the $K-$theoretic constructions to étale $K-$theory is the following. Replace Gillet's result [17] for K-theory (commutative diagram (20) of §3) with the compatibility of the Dwyer-Friedlander spectral sequence with the product structure ([13], Proposition 5.4) combined with Soulé's observation (see [25], p. 275) that the localization sequence in étale cohomology (see [25], p. 268) is compatible with the product by the étale cohomology of $\mathcal{O}_{F,S}$.

4.1. **Constructing special elements in $K-$theory with coefficients.** Let L be a number field, such that $\mu_{l^k} \subset L$. Let S be a finite set of prime ideals of $\mathcal{O}_L$ containing all primes over l. Let $i \in \mathbb{N}$ and let $m \in \mathbb{Z}$, such that $i + 2m > 0$. Then, for $R = L$ or $R = \mathcal{O}_{L,S}$ there is a natural group isomorphism (see [13] Theorem 5.6):

$$K_i^{et}(R;\mathbb{Z}/l^k) \xrightarrow{\cong} K_{i+2m}^{et}(R;\mathbb{Z}/l^k) \tag{22}$$

which sends η to $\eta * \beta_k^{*\,m}$ for any $\eta \in K_i^{et}(R;\mathbb{Z}/l^k)$. If $m \geq 0$ this isomorphism is just the multiplication by $\beta_k^{*\,m}$. If $m < 0$ and $i + 2m > 0$, then the isomorphism (22) is the inverse of the multiplication by $\beta_k^{*\,-m}$ isomorphism:

$$*\ \beta_k^{*\,-m}\ :\ K_{i+2m}^{et}(R;\mathbb{Z}/l^k) \xrightarrow{\cong} K_i^{et}(R;\mathbb{Z}/l^k). \tag{23}$$

Now, let us consider Quillen K-theory. If $m \geq 0$, there is a natural homomorphism

$$*\ \beta^{*\,m}\ :\ K_i(R;\mathbb{Z}/l^k) \to K_{i+2m}(R;\mathbb{Z}/l^k) \tag{24}$$

which is just multiplication by $\beta_k^{*\,m}$. The homomorphism (24) is compatible with the isomorphism (22) via the Dwyer-Friedlander map. If $m < 0$ and $i + 2m > 0$, then take the homomorphism

$$t(m)\,:\,K_i(R;\mathbb{Z}/l^k)\to K_{i+2m}(R;\mathbb{Z}/l^k) \tag{25}$$

to be the unique homomorphism which makes the following diagram commutative.

$$\begin{tikzcd}
K_i(R;\mathbb{Z}/l^k) \arrow[r, "t(m)"] \arrow[d] & K_{i+2m}(R;\mathbb{Z}/l^k) \\
K_i^{et}(R;\mathbb{Z}/l^k) \arrow[r, "(*\,\beta_k^{*\,-m})^{-1}"] & K_{i+2m}^{et}(R;\mathbb{Z}/l^k) \arrow[u]
\end{tikzcd}$$

The left vertical arrow is the Dwyer-Friedlander map, while the right vertical arrow is the Dwyer-Friedlander splitting map (see [13], Proposition 8.4.) The latter map is obtained as the multiplication of the inverse of the isomorphism $K_{i'}(R;\mathbb{Z}/l^k)\xrightarrow{\cong}K_{i'}^{et}(R;\mathbb{Z}/l^k)$, for $i'=1$ or $i'=2$, by a nonnegative power of the Bott element $\beta_k^{*\,m'}$, with $m'\geq 0$ (see the proof of Proposition 8.4 in [13].)

Remark 4.1. It is clear that the Dwyer-Friedlander splitting from [13], Proposition 8.4 is compatible with the maps $\mathbb{Z}/l^j\to\mathbb{Z}/l^{j-1}$ at the level of coefficients, for all $1\leq j\leq k$. Consequently, the map $t(m)$ is naturally compatible with these maps. In addition, $t(m)$ is naturally compatible with the ring embedding $R\to R'$, where $R'=L'$ or $R'=\mathcal{O}_{L',S}$ for a number field extension L'/L. Let

$$t^{et}(m):=(*\,\beta_k^{*\,-m})^{-1}.$$

It is clear from the above diagram that $t(m)$ and $t^{et}(m)$ are naturally compatible with the Dwyer-Friedlander maps.

Lemma 4.2. *Let $L=F(\mu_{l^k})$ and let $i>0$ and $m<0$, such that $i+2m>0$. Then, for $R=L$ or $R=\mathcal{O}_{L,S}$, the natural group homomorphisms $t^{et}(m)$ and $t(m)$ have the following properties:*

$$t^{et}(m)(\alpha)^{\sigma_{\mathbf{a}}}=t^{et}(m)(\alpha^{N\mathbf{a}^m\sigma_{\mathbf{a}}}) \tag{26}$$

$$t(m)(\alpha)^{\sigma_{\mathbf{a}}}=t(m)(\alpha^{N\mathbf{a}^m\sigma_{\mathbf{a}}}) \tag{27}$$

for any ideal $\mathbf{a}$ of O_F coprime to $\mathbf{f}_k$ and for $\alpha\in K_i^{et}(R;\mathbb{Z}/l^k)$ and $\alpha\in K_i(R;\mathbb{Z}/l^k)$, respectively.

Lemma 4.3. *If $i\in\{1,2\}$, $\alpha\in K_i(R;\mathbb{Z}/l^k)$ and $n+m>0$ then*

$$t^{et}(m)(\alpha*\beta_k^{*\,n})=\alpha*\beta_k^{*\,n+m}. \tag{28}$$

$$t(m)(\alpha*\beta_k^{*\,n})=\alpha*\beta_k^{*\,n+m}. \tag{29}$$

Proof of Lemmas 4.2 and 4.3. The properties in Lemmas 4.2 and 4.3 follow directly from the definition of the maps $t^{et}(m)$ and $t(m)$. □

If v is a prime of $\mathcal{O}_{L,S}$, $m<0$ and $i+2m>0$, then we construct the morphism

$$t_v(m)\,:\,K_i(k_v;\mathbb{Z}/l^k)\to K_{i+2m}(k_v;\mathbb{Z}/l^k) \tag{30}$$

in the same way as we have done for $\mathcal{O}_{L,S}$ or L. Namely, $t_v(m)$ is the homomorphism which makes the following diagram commute.

$$\begin{array}{ccc} K_i(k_v;\mathbb{Z}/l^k) & \xrightarrow{t_v(m)} & K_{i+2m}(k_v;\mathbb{Z}/l^k) \\ \downarrow \cong & & \cong \uparrow \\ K_i^{et}(k_v;\mathbb{Z}/l^k) & \xrightarrow{(*\,\beta_k^{*\,-m})^{-1}} & K_{i+2m}^{et}(k_v;\mathbb{Z}/l^k) \end{array}$$

The right vertical arrow is the inverse of the Dwyer-Friedlander map which, in the case of a finite field, is clearly seen to be equal to the Dwyer-Friedlander splitting map described above.

Similarly to $t^{et}(m)$ we can construct $t_v^{et}(m) := (*\,\beta_k^{*\,-m})^{-1}$. We observe that the maps $t(m)$ and $t_v(m)$ are compatible with the reduction maps and the boundary maps. In other words, we have the following commutative diagrams.

$$\begin{array}{ccc} K_i(\mathcal{O}_{L,S};\,\mathbb{Z}/l^k) & \xrightarrow{r_v} & K_i(k_v;\,\mathbb{Z}/l^k) \\ \downarrow t(m) & & \downarrow t_v(m) \\ K_{i+2m}(\mathcal{O}_{L,S};\,\mathbb{Z}/l^k) & \xrightarrow{r_v} & K_{i+2m}(k_v;\,\mathbb{Z}/l^k) \end{array}$$

$$\begin{array}{ccc} K_i(\mathcal{O}_{L,S},\,\mathbb{Z}/l^k) & \xrightarrow{\partial} & \bigoplus_{v\in S}\; K_{i-1}(k_v;\,\mathbb{Z}/l^k) \\ \downarrow t(m) & & \downarrow t_v(m) \\ K_{i+2m}(\mathcal{O}_{L,S};\,\mathbb{Z}/l^k) & \xrightarrow{\partial} & \bigoplus_{v\in S}\; K_{i-1+2m}(k_v;\,\mathbb{Z}/l^k) \end{array}$$

Let us point out that we have similar commutative diagrams for étale K-theory and the maps $t^{et}(m)$ and $t_v^{et}(m)$.

As observed above, the map $t(m)$ for $m < 0$ has the same properties as the multiplication by $\beta^{*\,m}$ for $m \geq 0$. So, we make the following.

Definition 4.4. For $m < 0$, we define the symbols

$$\alpha * \beta^{*\,m} := t(m)(\alpha), \qquad \alpha_v * \beta^{*\,m} := t_v(m)(\alpha_v),$$

for all $\alpha \in K_i(\mathcal{O}_L;\,\mathbb{Z}/l^k)$ and $\alpha_v \in K_i(k_v;\mathbb{Z}/l^k))$, respectively. For $m \geq 0$, the symbols $\alpha * \beta^{*\,m}$ and $\alpha_v * \beta^{*\,m}$ denote the usual products.

Let $m > 0$ be a natural number. Throughout the rest of this section we assume that $\Theta_m(\mathbf{b}, \mathbf{f}_k)$ annihilates $K_{2m}(\mathcal{O}_{F_{l^k}})$ for all $k \geq 0$. For a prime v of $\mathcal{O}_F$, let k_v be its residue field and q_v the cardinality of k_v . Similarly, for any prime w of $\mathcal{O}_{F_{l^k}}$, we let k_w be its residue field. We put $E := F_{l^k}$. If $v \not| l$, we observe that $k_w = k_v(\xi_{l^k})$, since the corresponding local field extension E_w/F_v is unramified. For any finite set S of primes in O_F and any $k \geq 0$, there is an exact sequence (see [22]):

$$0 \longrightarrow K_{2m}(\mathcal{O}_{F_{l^k}}) \longrightarrow K_{2m}(\mathcal{O}_{F_{l^k},S}) \xrightarrow{\partial} \bigoplus_{v\in S}\bigoplus_{w|v} K_{2m-1}(k_w) \longrightarrow 0$$

Let $\xi_{w,k} \in K_{2m-1}(k_w)_l$ be a generator of the l-torsion part of $K_{2m-1}(k_w)$. Pick an element $x_{w,k} \in K_{2m}(\mathcal{O}_{F_{l^k},S})_l$ such that $\partial(x_{w,k}) = \xi_{w,k}$. Obviously, $x_{w,k}^{\Theta_m(\mathbf{b},\mathbf{f}_k)}$ does not depend on the choice of $x_{w,k}$ since $\Theta_m(\mathbf{b}, \mathbf{f}_k)$ annihilates $K_{2m}(\mathcal{O}_{F_{l^k}})$. If

$\mathrm{ord}(\xi_{w,k}) = l^a$, then $x_{w,k}^{l^a} \in K_{2m}(\mathcal{O}_{F_{l^k}})$. Hence, $(x_{w,k}^{\Theta_m(\mathbf{b},\mathbf{f}_k)})^{l^a} = (x_{w,k}^{l^a})^{\Theta_m(\mathbf{b},\mathbf{f}_k)} = 0$. Consequently, there is a well defined map:

$$\Lambda_m \,:\, \bigoplus_{v\in S}\bigoplus_{w|v} K_{2m-1}(k_w)_l \longrightarrow K_{2m}(\mathcal{O}_{F_{l^k},S})_l,$$

$$\Lambda_m(\xi_{w,k}) \,:=\, x_{w,k}^{\Theta_m(\mathbf{b},\mathbf{f}_k)}. \tag{31}$$

If R is either a number field L or its ring of (S,l)–integers $\mathcal{O}_{L,S}[1/l]$, for some finite set $S \subseteq \mathrm{Spec}(\mathcal{O}_L)$, Tate proved in [28] that there is a natural isomorphism:

$$K_2(R)_l \xrightarrow{\cong} K_2^{et}(R).$$

Dwyer and Friedlander [13] proved that the natural maps:

$$K_j(R;\mathbb{Z}/l^k) \longrightarrow K_j^{et}(R;\mathbb{Z}/l^k),$$

are surjections for $j \geq 1$ and isomorphisms for $j = 1, 2$. As explained in [2], for any number field L, any finite set $S \subset \mathrm{Spec}(\mathcal{O}_L)$ and any $j \geq 1$, we have the following commutative diagrams with exact rows and (surjective) Dwyer-Friedlander maps as vertical arrows.

$$\begin{tikzcd}
0 \arrow[r] & K_{2j}(\mathcal{O}_L)_l \arrow[r] \arrow[d] & K_{2j}(\mathcal{O}_{L,S})_l \arrow[r, "\partial"] \arrow[d] & \bigoplus_{v\in S} K_{2j-1}(k_v)_l \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & K_{2j}^{et}(\mathcal{O}_L[1/l]) \arrow[r] & K_{2j}^{et}(\mathcal{O}_{L,S}[1/l]) \arrow[r, "\partial^{et}"] & \bigoplus_{v\in S} K_{2j-1}^{et}(k_v) \arrow[r] & 0
\end{tikzcd}$$

For $j = 1$, the left and the middle vertical arrows in the above diagram are also isomorphisms, according to Tate's theorem. If the Quillen-Lichtenbaum conjecture holds, then these are isomorphisms for all $j > 0$.

Our assumption that $\Theta_m(\mathbf{b},\mathbf{f}_k)$ annihilates $K_{2m}(\mathcal{O}_{F_{l^k}})$ for all $k \geq 0$ implies that $\Theta_m(\mathbf{b},\mathbf{f}_k)$ annihilates $K_{2m}^{et}(\mathcal{O}_{F_{l^k}}[1/l])$, for all $k \geq 0$. In the diagram above, let $y_{w,k}$ and $\zeta_{w,k}$ denote the images of $x_{w,k}$ and $\xi_{w,k}$ via the middle vertical and right vertical arrows, respectively. Then, we define

$$\Lambda_m^{et}(\zeta_{w,k}) \,:=\, y_{w,k}^{\Theta_m(\mathbf{b},\mathbf{f}_k)}.$$

Clearly, the following diagram is commutative

$$\begin{tikzcd}
K_{2m}(\mathcal{O}_{F_{l^k},S})_l \arrow[d] & \bigoplus_{v\in S}\bigoplus_{w|v} K_{2m-1}(k_w)_l \arrow[l, "\Lambda_m"'] \arrow[d, "\cong"] \\
K_{2m}^{et}(\mathcal{O}_{F_{l^k},S}[1/l]) & \bigoplus_{v\in S}\bigoplus_{w|v} K_{2m-1}^{et}(k_w)_l \arrow[l, "\Lambda_m^{et}"']
\end{tikzcd}$$

where the vertical maps are the Dwyer-Friedlander maps.

Lemma 4.5. *The maps Λ_m and Λ_m^{et} satisfy the following properties*

$$\partial\Lambda_m(\xi_{w,k}) \,:=\, \xi_{w,k}^{\Theta_m(\mathbf{b},\mathbf{f}_k)}, \qquad \partial^{et}\Lambda_m^{et}(\zeta_{w,k}) \,:=\, \zeta_{w,k}^{\Theta_m(\mathbf{b},\mathbf{f}_k)}.$$

Proof. The lemma follows immediately by compatibility of ∂ and ∂^{et} with the $G(E/F)$ action. $\square$

Let us fix an $n \in \mathbb{N}$. Let v be a prime in $\mathcal{O}_F$ sitting above $p \neq l$ in $\mathbb{Z}$. Let $S := S_v$ be the finite set primes of $\mathcal{O}_F$ consisting of all the primes over p and all the primes over l. Let $k(v)$ be the natural number for which $l^{k(v)} \,||\, q_v^n - 1$. Observe that if $l \,|\, q_v - 1$ then $k(v) = v_l(q_v - 1) + v_l(n)$ (see e.g. [1, p. 336].)

Definition 4.6. As in loc.cit. p. 335, let us define:

$$\gamma_l := \prod_{\substack{\mathbf{l} \nmid \mathbf{f} \\ \mathbf{l} \,|\, l}} (1 - (\mathbf{l},\, F)^{-1} N\mathbf{l}^n)^{-1} = \prod_{\substack{\mathbf{l} \nmid \mathbf{f} \\ \mathbf{l} \,|\, l}} (1 + (\mathbf{l},\, F)^{-1} N\mathbf{l}^n + (\mathbf{l},\, F)^{-2} N\mathbf{l}^{2n} + \cdots).$$

If $\mathbf{l} | \, \mathbf{f}$ for every $\mathbf{l} \,|\, l$ then naturally we let $\gamma_l := 1$. Observe that γ_l is a well defined operator on any $\mathbb{Z}_l[G(F/K)]$-module which is a torsion abelian group with a finite exponent.

Definition 4.7. For all $k \geq 0$ and $E := F(\mu_{l^k})$, let us define elements:

$$\boldsymbol{\lambda}_{v,l^k} := Tr_{E/F}(\Lambda_m(\xi_{w,k}) * \beta_k^{*\,n-m})^{N\mathbf{b}^{n-m}\gamma_l} \in K_{2n}(\mathcal{O}_{F,S};\, \mathbb{Z}/l^k)).$$

Similarly, define elements:

$$\boldsymbol{\lambda}^{et}_{v,l^k} := Tr_{E/F}(\Lambda^{et}_m(\zeta_{w,k}) * \beta_k^{*\,n-m})^{N\mathbf{b}^{n-m}\gamma_l} \in K^{et}_{2n}(\mathcal{O}_{F,S};\, \mathbb{Z}/l^k)).$$

Obviously, $\boldsymbol{\lambda}^{et}_{v,l^k}$ is the image of $\boldsymbol{\lambda}_{v,l^k}$ via the Dwyer-Friedlander map.

Let us fix a prime sitting above v in each of the fields $F(\mu_{l^k})$, such that if $k \leq k'$ and w and w' are the fixed primes in $E = F(\mu_{l^k})$ and $E' := F(\mu_{l^{k'}})$, respectively, then w' sits above w. By the surjectivity of the transfer maps for K-theory of finite fields (see the end of §3), we can associate to each k and the chosen prime w in $E = F(\mu_{l^k})$ a generator $\xi_{w,k}$ of $K_{2m-1}(k_w)_l$ and a generator $\zeta_{w,k}$ of $K^{et}_{2m-1}(k_w)_l$, such that

$$N_{w'/w}(\xi_{w',k'}) = \xi_{w,k}, \qquad N_{w'/w}(\zeta_{w',k'}) = \zeta_{w,k},$$

for all $k \leq k'$, where w and w' are the fixed primes in $E = F(\mu_{l^k})$ and $E' = F(\mu_{l^{k'}})$, respectively.

Lemma 4.8. *With notations as above, for every $k \leq k'$ we have*

$$r_{k'/k}(N_{w'/v}(\xi_{w',k'} * \beta_{k'}^{*\,n-m})) = N_{w/v}(\xi_{w,k} * \beta_k^{*\,n-m}),$$

$$r_{k'/k}(N_{w'/v}(\zeta_{w',k'} * \beta_{k'}^{*\,n-m})) = N_{w/v}(\zeta_{w,k} * \beta_k^{*\,n-m}).$$

Proof. First, let us consider the case $n-m \geq 0$. The formula follows by the compatibility of the elements $(\xi_{w,k})_w$ with respect to the norm maps, by the compatibility of Bott elements with respect to the coefficient reduction map $r_{k'/k}(\beta_{k'}) = \beta_k$, and by the projection formula. More precisely, we have the following equalities:

$$\begin{aligned} r_{k'/k}(N_{w'/v}(\xi_{w',k'} \, * \, \beta_{k'}^{*\,n-m})) &= N_{w'/v}(r_{k'/k}(\xi_{w',k'} \, * \, \beta_{k'}^{*\,n-m})) = \\ N_{w'/v}(\xi_{w',k'} \, * \, \beta_{k}^{*\,n-m})) &= N_{w/v}(N_{w'/w}(\xi_{w',k'}) * \beta_k^{*\,n-m})) = \\ &= N_{w/v}(\xi_{w,k} * \beta_k^{*\,n-m}). \end{aligned}$$

Next, let us consider the case $n - m < 0$. Observe that the Dwyer-Friedlander maps commute with $N_{w/v}$ and $N_{w'/v}$. Hence we can argue in the same way as in the case $n - m \geq 0$ by using the projection formula for the negative twist in étale cohomology, since for any finite field $\mathbb{F}_q$ with $l \not| q$, we have natural isomorphisms coming from the Dwyer-Friedlander spectral sequence (cf. the end of §3.1):

$$K^{et}_{2j-1}(\mathbb{F}_q) \cong H^1(\mathbb{F}_q;\, \mathbb{Z}_l(j)) \tag{32}$$

$$K^{et}_{2j-1}(\mathbb{F}_q;\, \mathbb{Z}/l^k) \cong H^1(\mathbb{F}_q;\, \mathbb{Z}/l^k(j)). \tag{33}$$

□

Lemma 4.9. *For all $0 \leq k \leq k'$, we have*

$$r_{k'/k}(\boldsymbol{\lambda}_{v,l^{k'}}) = \boldsymbol{\lambda}_{v,l^k}$$

$$r_{k'/k}(\boldsymbol{\lambda}^{et}_{v,l^{k'}}) = \boldsymbol{\lambda}_{v,l^k}.$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} K_{2m}(\mathcal{O}_{E',S}) & \xrightarrow{\bigoplus_{w'\in S}\partial_{w'}} & \bigoplus_{w'\in S} K_{2m-1}(k_{w'}) \\ \downarrow{\scriptstyle Tr_{E'/E}} & & \downarrow{\scriptstyle \bigoplus_{w\in S}\bigoplus_{w'|w} N_{w'/w}} \\ K_{2m}(\mathcal{O}_{E,S}) & \xrightarrow{\bigoplus_{w\in S}\partial_{w}} & \bigoplus_{w\in S} K_{2m-1}(k_{w}) \end{array}$$

It follows that we have $Tr_{E'/E}(x_{w',k'})^{\Theta_m(\mathbf{b},\mathbf{f}_k)} = x_{w,k}^{\Theta_m(\mathbf{b},\mathbf{f}_k)}$. Hence the case $n-m \geq 0$ follows by the projection formula:

$$r_{k'/k}(Tr_{E'/F}(x_{w',k'}^{\Theta_m(\mathbf{b},\mathbf{f}_{k'})} * \beta_{k'}^{*\,n-m})^{N\mathbf{b}^{n-m}\gamma_l}) =$$

$$= Tr_{E/F}(Tr_{E'/E}(x_{w',k'}^{\Theta_m(\mathbf{b},\mathbf{f}_{k'})} * \beta_{k}^{*\,n-m}))^{N\mathbf{b}^{n-m}\gamma_l} =$$

$$= Tr_{E/F}(x_{w,k}^{\Theta_m(\mathbf{b},\mathbf{f}_{k})} * \beta_{k}^{*\,n-m})^{N\mathbf{b}^{n-m}\gamma_l}.$$

Now, consider the case $n-m < 0$. We observe that $Tr_{E'/E}$ commutes with the Dwyer-Friedlander map. Hence $Tr_{E'/E}$ also commutes with the splitting of the Dwyer-Friedlander map since the splitting is a monomorphism. By the Dwyer-Friedlander spectral sequence for any number field L and any finite set S of prime ideals of $\mathcal{O}_L$ containing all primes over l, we have the following isomorphism

$$K^{et}_{2j}(\mathcal{O}_{L,S}) \cong H^2(\mathcal{O}_{L,S};\, \mathbb{Z}_l(j+1)) \tag{34}$$

and the following exact sequence

$$0 \to H^2(\mathcal{O}_{L,S};\, \mathbb{Z}/l^k(j+1)) \to K^{et}_{2j}(\mathcal{O}_{L,S};\, \mathbb{Z}/l^k) \to H^0(\mathcal{O}_{L,S};\, \mathbb{Z}/l^k(j)) \to 0. \tag{35}$$

Since $x_{w,k}^{\Theta_m(\mathbf{b},\mathbf{f}_k)} \in K_{2m}(\mathcal{O}_{F_k,S})$, its image in $K^{et}_{2m}(\mathcal{O}_{F_k,S};\, \mathbb{Z}/l^k)$ lies in fact in in the ételae cohomology group $H^2(\mathcal{O}_{F_k,S};\, \mathbb{Z}/l^k(m+1))$. Hence, one can settle the case $n-m<0$ as well by using the projection formula for the étale cohomology with negative twists. □

Theorem 4.10. *For every $k \geq 0$, we have*

$$\partial_F(\boldsymbol{\lambda}_{v,l^k}) = N(\xi_{w,k} * \beta_k^{*\,n-m})^{\Theta_m(\mathbf{b},\mathbf{f})}\ ,$$

$$\partial^{et}_F(\boldsymbol{\lambda}^{et}_{v,l^k}) = N(\zeta_{w,k} * \beta_k^{*\,n-m})^{\Theta_m(\mathbf{b},\mathbf{f})}\ .$$

Proof. The proof is similar to the proofs of Theorem 1, pp. 336-340 of [1] and Proposition 2, pp. 221-222 of [3]. The diagram at the end of §3 gives the following commutative diagram of K–groups with coefficients

$$\begin{array}{ccc} K_{2n}(\mathcal{O}_{E,S};\, \mathbb{Z}/l^k) & \xrightarrow{\partial_E} & \bigoplus_{v \in S} \bigoplus_{w \,|\, v} K_{2n-1}(k_w;\, \mathbb{Z}/l^k) \\ \downarrow{\scriptstyle Tr_{E/F}} & & \downarrow{\scriptstyle N} \\ K_{2n}(\mathcal{O}_{F,S};\, \mathbb{Z}/l^k) & \xrightarrow{\partial_F} & \bigoplus_{v \in S} K_{2n-1}(k_v,;\, \mathbb{Z}/l^k) \end{array},$$

where $N := \bigoplus_v \bigoplus_{w \,|\, v} N_{w/v}$. Hence we have $\partial_F \circ Tr_{E/F} = N \circ \partial_E$. The compatibilities of some of the natural maps mentioned in §3 which will be used next can be expressed via the following commutative diagrams, explaining the action of the groups $G(E/K)$ and $G(F/K)$ on the K–groups with coefficients in the diagram above. For $j > 0$ we use the following commutative diagram.

$$\begin{array}{ccc} K_{2j}(\mathcal{O}_{E,S};\, \mathbb{Z}/l^k) & \xrightarrow{r_w} & K_{2j}(k_w;\, \mathbb{Z}/l^k) \\ \downarrow{\scriptstyle \sigma_{\mathbf{a}}^{-1}} & & \downarrow{\scriptstyle \sigma_{\mathbf{a}}^{-1}} \\ K_{2j}(\mathcal{O}_{E,S};\, \mathbb{Z}/l^k) & \xrightarrow{r_{w^{\sigma_{\mathbf{a}}^{-1}}}} & K_{2j}(k_{w^{\sigma_{\mathbf{a}}^{-1}}};\, \mathbb{Z}/l^k) \end{array}$$

The above diagram gives the following equality:

$$r_{w^{\sigma_{\mathbf{a}}^{-1}}}(\beta_k^{*\, n-m}) = r_{w^{\sigma_{\mathbf{a}}^{-1}}}((\beta_k^{*\, n-m})^{N\mathbf{a}^{n-m}\sigma_{\mathbf{a}}^{-1}}) = (r_w(\beta_k^{*\, n-m}))^{N\mathbf{a}^{n-m}\sigma_{\mathbf{a}}^{-1}}. \tag{36}$$

For any $j \in \mathbb{Z}$, we have the following commutative diagram:

$$\begin{array}{ccc} H^0(\mathcal{O}_{E,S};\, \mathbb{Z}/l^k(j)) & \xrightarrow{r_w} & H^0(k_w;\, \mathbb{Z}/l^k(j)) \\ \downarrow{\scriptstyle \sigma_{\mathbf{a}}^{-1}} & & \downarrow{\scriptstyle \sigma_{\mathbf{a}}^{-1}} \\ H^0(\mathcal{O}_{E,S};\, \mathbb{Z}/l^k(j)) & \xrightarrow{r_{w^{\sigma_{\mathbf{a}}^{-1}}}} & H^0(k_{w^{\sigma_{\mathbf{a}}^{-1}}};\, \mathbb{Z}/l^k(j)) \end{array}$$

If $\xi_{l^k} := exp(\frac{2\,\pi\, i}{l^k})$ is the generator of μ_{l^k} then the above diagram gives

$$r_{w^{\sigma_{\mathbf{a}}^{-1}}}(\xi_{l^k}^{\otimes\, n-m}) = r_{w^{\sigma_{\mathbf{a}}^{-1}}}((\xi_{l^k}^{\otimes\, n-m})^{N\mathbf{a}^{n-m}\sigma_{\mathbf{a}}^{-1}}) = (r_w(\xi_{l^k}^{\otimes\, n-m}))^{N\mathbf{a}^{n-m}\sigma_{\mathbf{a}}^{-1}}. \tag{37}$$

We can write the m–th Stickelberger element as follows

$$\Theta_m(\mathbf{b}, \mathbf{f}_k) = {\sum_{\mathbf{a} \bmod \mathbf{f}_k}}' \left(\sum_{\mathbf{c} \bmod \mathbf{f}_k,\, w^{\sigma_{\mathbf{c}^{-1}}} = w} \Delta_{m+1}(\mathbf{ac}, \mathbf{b}, \mathbf{f}) \sigma_{\mathbf{c}^{-1}} \right) \cdot \sigma_{\mathbf{a}^{-1}}\,, \tag{38}$$

where $\sum'_{\mathbf{a} \bmod \mathbf{f_k}}$ denotes the sum over a maximal set $\mathcal{S}$ of ideal classes $\mathbf{a} \mod \mathbf{f_k}$, such that the primes $w^{\sigma_{\mathbf{a}}^{-1}}$, for $\mathbf{a} \in \mathcal{S}$, are distinct. By formula (6), for every $m \geq 1$ and $n \geq 1$ we have

$$\Delta_{n+1}(\mathbf{a}, \mathbf{b}, \mathbf{f}) \equiv N\mathbf{a}^{n-m}\, N\mathbf{b}^{n-m}\, \Delta_{m+1}(\mathbf{ac}, \mathbf{b}, \mathbf{f}) \mod w_{\min\{m,n\}}(K_{\mathbf{f}}).$$

It is clear that for all $m, n \geq 1$ we get the following congruence $\mod w_{\min\{m,n\}}(K_{\mathbf{f}_k})$.

$$\Theta_n(\mathbf{b}, \mathbf{f}_k) \equiv$$
$${\sum_{\mathbf{a} \bmod \mathbf{f}_k}}' \Big(\sum_{\mathbf{c} \bmod \mathbf{f}_k,\, w^{\sigma_{\mathbf{c}^{-1}}} = w} N\mathbf{a}^{n-m}\, N\mathbf{c}^{n-m}\, N\mathbf{b}^{n-m}\, \Delta_{m+1}(\mathbf{ac}, \mathbf{b}, \mathbf{f}_k) \sigma_{\mathbf{c}^{-1}} \Big) \sigma_{\mathbf{a}^{-1}}$$

Equalities (36), (37), (38), Lemma 4.2, Gillet's result [17] (diagram (20)), the compatibility of $t(n-m)$ and $t_v(n-m)$ with ∂ and the above congruences satisfied by Stickelberger elements lead in both cases $n-m\geq 0$ and $n-m<0$ to the following equalities.

$$\partial_E(x_{w,k}^{\Theta_m(\mathbf{b},\mathbf{f}_k)} * \beta_k^{*\,n-m})^{N\mathbf{b}^{n-m}} =$$
$$= \sum_{\mathbf{a} \bmod \mathbf{f}_k}{}' \, \xi_{w,k}^{\sum_{\mathbf{c} \bmod \mathbf{f}_k,\, w^{\sigma_{\mathbf{c}}-1}=w} \Delta_{m+1}(\mathbf{ac},\mathbf{b},\mathbf{f}_k)\sigma_{(\mathbf{ac})^{-1}}} * (\beta_k^{*\,n-m})^{(N\mathbf{ac})^{n-m}N\mathbf{b}^{n-m}\sigma_{(ac)^{-1}}} =$$
$$= (\xi_{w,k} * \beta_k^{*\,n-m})^{\sum'_{\mathbf{a} \bmod \mathbf{f}_k} \sum_{\mathbf{c} \bmod \mathbf{f}_k,\, w^{\sigma_{\mathbf{c}}-1}=w} \Delta_{m+1}(\mathbf{ac},\mathbf{b},\mathbf{f}_k)(N\mathbf{ac})^{n-m}N\mathbf{b}^{n-m}\sigma_{(\mathbf{ac})^{-1}}} =$$
$$= (\xi_{w,k} * \beta_k^{*\,n-m})^{\Theta_n(\mathbf{b},\mathbf{f}_k)}.$$

By the first commutative diagram of this proof, the equalities above and Lemma 2.1, we obtain:

$$\partial_F(\boldsymbol{\lambda}_{v,l^k}) = N(\partial_E(x_{w,k}^{\Theta_m(\mathbf{b},\mathbf{f}_k)} * \beta_k^{*\,n-m})^{N\mathbf{b}^{n-m}})^{\gamma_l} = N((\xi_{w,k} * \beta_k^{*\,n-m})^{\Theta_n(\mathbf{b},\mathbf{f}_k)})^{\gamma_l} =$$
$$= (N(\xi_{w,k} * \beta_k^{*\,n-m}))^{\gamma_l{}^{-1}\Theta_n(\mathbf{b},\mathbf{f})\gamma_l} = (N(\xi_{w,k} * \beta_k^{*\,n-m}))^{\Theta_n(\mathbf{b},\mathbf{f})}.$$

□

Theorem 4.11. *For every v such that $l \mid q_v^n - 1$ and for all $k \geq k(v)$, there are homomorphisms*

$$\Lambda_{v,\,l^k} \;:\; K_{2n-1}(k_v;\, \mathbb{Z}/l^k) \to K_{2n}(\mathcal{O}_{F,S};\, \mathbb{Z}/l^k),$$
$$\Lambda_{v,\,l^k}^{et} \;:\; K_{2n-1}^{et}(k_v;\, \mathbb{Z}/l^k) \to K_{2n}^{et}(\mathcal{O}_{F,S};\, \mathbb{Z}/l^k) :$$

which satisfy the following equalities:

$$\Lambda_{v,\,l^k}\,(N(\xi_{w,k} * \beta_k^{*\,n-m})) \;=\; \boldsymbol{\lambda}_{v,l^k},$$
$$\Lambda_{v,\,l^k}^{et}\,(N(\zeta_{w,k} * \beta_k^{*\,n-m})) \;=\; \boldsymbol{\lambda}_{v,l^k}^{et}.$$

Proof. The definition of Λ_m (see (31)), combined with the natural isomorphism $K_{2m-1}(k_w)/l^k \cong K_{2m-1}(k_w;\mathbb{Z}/l^k\mathbb{Z})$ and the natural monomorphism

$$K_{2m}(\mathcal{O}_{E,S})/l^k \to K_{2m}(\mathcal{O}_{E,S};\mathbb{Z}/l^k\mathbb{Z}),$$

coming from the corresponding Bockstein exact sequences, leads to the following homomorphism:

$$\widetilde{\Lambda}_m : K_{2m-1}(k_w;\mathbb{Z}/l^k\mathbb{Z}) \to K_{2m}(\mathcal{O}_{E,S};\mathbb{Z}/l^k\mathbb{Z}).$$

Multiplying on the target and on the source of this homomorphism with the $n-m$ power of the Bott element if $n-m\geq 0$ (resp. applying the map $t_w(n-m)$ to the source and $t(n-m)$ to the target if $n-m<0$) under the observation that the following map is an isomorphism:

$$K_{2m-1}(k_w;\mathbb{Z}/l^k\mathbb{Z}) \xrightarrow[\sim]{*\beta_k^{*\,n-m}} K_{2n-1}(k_w;\mathbb{Z}/l^k\mathbb{Z})$$

(cf. the notation of $t(j)$ and $t_w(j)$) show that there exists a unique homomorphism

$$\widetilde{\Lambda}_m * \beta_k^{*\,n-m} \;:\; K_{2n-1}(k_w;\, \mathbb{Z}/l^k) \to K_{2n}(\mathcal{O}_{E,S};\, \mathbb{Z}/l^k),$$

sending $\xi_{w,k} * \beta_k^{*\, n-m} \to x_{w,k}^{\Theta_m(\mathbf{b},\mathbf{f}_k)} * \beta_k^{*\, n-m}$. Next, we compose the homomorphisms $\widetilde{\Lambda}_m * \beta_k^{*\, n-m}$ defined above and

$$Tr_{E/F} \,:\, K_{2n}(\mathcal{O}_{E,S};\, \mathbb{Z}/l^k) \to K_{2n}(\mathcal{O}_{F,S};\, \mathbb{Z}/l^k)$$

to obtain the following homomorphism:

$$Tr_{E/F} \circ (\widetilde{\Lambda}_m * \beta_k^{*\, n-m}) : K_{2n-1}(k_w;\, \mathbb{Z}/l^k) \to K_{2n}(\mathcal{O}_{F,S};\, \mathbb{Z}/l^k)\,.$$

We observe that this homomorphism factors through the quotient of $G(k_w/k_v)$–coinvariants

$$K_{2n-1}(k_w;\, \mathbb{Z}/l^k)_{G(k_w/k_v)} := K_{2n-1}(k_w;\, \mathbb{Z}/l^k)/K_{2n-1}(k_w;\, \mathbb{Z}/l^k)^{Fr_v - Id},$$

where $Fr_v \in G(k_w/k_v) \subseteq G(E/F)$ is the Frobenius element of the prime w over v. Since Fr_v acts via q_v^n–powers on $K_{2n-1}(k_w)$, the canonical isomorphism $K_{2n-1}(k_w;\, \mathbb{Z}/l^k) \cong K_{2n-1}(k_w)/l^k$ (see §3) and assumption $k \geq k(v)$ give

$$K_{2n-1}(k_w;\, \mathbb{Z}/l^k)_{G(k_w/k_v)} \cong K_{2n-1}(k_w;\, \mathbb{Z}/l^k)/l^{k(v)} \cong K_{2n-1}(k_w)/l^{k(v)}.$$

The obvious commutative diagram with surjective vertical morphisms (see §3)

$$\begin{array}{ccc} K_{2n-1}(k_w)/l^k & \xrightarrow{\cong} & K_{2n-1}(k_w;\, \mathbb{Z}/l^k) \\ \Big\downarrow{\scriptstyle N_{w/v}} & & \Big\downarrow{\scriptstyle N_{w/v}} \\ K_{2n-1}(k_v)/l^k & \xrightarrow{\cong} & K_{2n-1}(k_v;\, \mathbb{Z}/l^k) \end{array}$$

combined with the last isomorphism above, gives an isomorphism

$$K_{2n-1}(k_w;\, \mathbb{Z}/l^k)_{G(k_w/k_v)} \xrightarrow[\sim]{N_{w/v}} K_{2n-1}(k_v;\, \mathbb{Z}/l^k)$$

Now, the required homomorphism is:

$$\Lambda_{v,\, l^k} \,:\, K_{2n-1}(k_v;\, \mathbb{Z}/l^k) \longrightarrow K_{2n}(\mathcal{O}_{F,S};\, \mathbb{Z}/l^k) \tag{39}$$

defined by

$$\Lambda_{v,\, l^k}(x) := [Tr_{E/F} \circ (\widetilde{\Lambda}_m * \beta_k^{*\, n-m}) \circ {N_{w/v}}^{-1}(x)]^{N\mathbf{b}^{n-m}\gamma_l}\,,$$

for all $x \in K_{2n-1}(k_v;\, \mathbb{Z}/l^k)$. By definition, this map sends $N(\xi_{w,k} * \beta_k^{*\, n-m})$ onto the element $\boldsymbol{\lambda}_{v,l^k} := Tr_{E/F}(x_{w,k}^{\Theta_m(\mathbf{b},\mathbf{f}_k)} * \beta_k^{*\, n-m})^{N\mathbf{b}^{n-m}\gamma_l}$. □

4.2. **Constructing Λ and Λ^{et} for K–theory without coefficients.** Let us fix $n > 0$. In this section, we use the special elements and $\boldsymbol{\lambda}_{v,l^k}$ and $\boldsymbol{\lambda}_{v,l^k}^{et}$ defined above to construct the maps Λ_n and Λ_n^{et} for the K–theory (respectively étale K–theory) without coefficients. Since n is fixed throughout, we will denote $\Lambda := \Lambda_n$ and $\Lambda^{et} := \Lambda_n^{et}$.

Observe that for every $j > 0$ and every prime l, the Bockstein exact sequence (14) and results of Quillen [22], [23] give natural isomorphisms

$$K_j(\mathcal{O}_{F,S})_l \;\cong\; \varprojlim_k K_j(\mathcal{O}_{F,S};\, \mathbb{Z}/l^k), \tag{40}$$

$$K_j(k_v)_l \;\cong\; \varprojlim_k K_j(k_v;\, \mathbb{Z}/l^k)\,. \tag{41}$$

Similar isomorphisms hold for the étale K-theory.

Definition 4.12. We define $\boldsymbol{\lambda}_v \in K_{2n}(\mathcal{O}_{F,S})_l$ and $\boldsymbol{\lambda}_v^{et} \in K_{2n}^{et}(\mathcal{O}_{F,S})$ to be the elements corresponding to

$$(\boldsymbol{\lambda}_{v,l^k})_k \in \varprojlim_k K_{2n}(\mathcal{O}_{F,S};\, \mathbb{Z}/l^k), \qquad (\boldsymbol{\lambda}_{v,l^k}^{et})_k \in \varprojlim_k K_{2n}^{et}(\mathcal{O}_{F,S};\, \mathbb{Z}/l^k)$$

via the isomorphism (40) and its étale analogue, respectively.

Definition 4.13. We define $\xi_v \in K_{2n-1}(k_v)_l$ and $\zeta_v \in K_{2n-1}^{et}(k_v)$ to be the elements corresponding to

$$(N(\xi_{w,k} * \beta_k^{*\,n-m}))_k \in \varprojlim_k K_{2n-1}(k_v;\, \mathbb{Z}/l^k),$$

$$(N(\zeta_{w,k} * \beta_k^{*\,n-m}))_k \in \varprojlim_k K_{2n-1}^{et}(k_v;\, \mathbb{Z}/l^k),$$

via the isomorphism (41) and its étale analogue, respectively.

Definition 4.14. Assume that $l \mid q_v^n - 1$. Since the homomorphisms $\Lambda_{v,\,l^k}$, and $\Lambda_{v,\,l^k}^{et}$, from Theorem 4.11, are compatible with the coefficient reduction maps $r_{k'/k}$, for all $k' \geq k \geq k(v)$, we can define homomorphisms

$$\Lambda_v := \varprojlim_k \Lambda_{v,\,l^k} : K_{2n-1}(k_v)_l \to K_{2n}(\mathcal{O}_{F,S})_l \hookrightarrow K_{2n}(F)_l,$$

$$\Lambda_v^{et} := \varprojlim_k \Lambda_{v,\,l^k}^{et} : K_{2n-1}^{et}(k_v) \to K_{2n}^{et}(\mathcal{O}_{F,S}) \hookrightarrow K_{2n}^{et}(F)_l,$$

for all v. Here, the rightmost arrows are the inclusions $K_{2n}(\mathcal{O}_{F,S}) \subset K_{2n}(F)$ and $K_{2n}^{et}(\mathcal{O}_{F,S}) \subset K_{2n}^{et}(F)_l$, respectively. If $l \nmid q_v^n - 1$, then the morphisms Λ_v and Λ_v^{et} are trivial, by default.

Remark 4.15. It is clear from Theorem 4.11 that, for all v, we have

$$\Lambda_v(\xi_v) = \boldsymbol{\lambda}_v, \qquad \Lambda_v^{et}(\zeta_v) = \boldsymbol{\lambda}_v^{et}.$$

Definition 4.16. We define the maps Λ_n and Λ_n^{et} as follows:

$$\Lambda \,:\, \bigoplus_v K_{2n-1}(k_v)_l \,\to K_{2n}(F)_l, \qquad \Lambda \,:=\, \prod_v \Lambda_v.$$

$$\Lambda^{et} \,:\, \bigoplus_v K_{2n-1}^{et}(k_v) \,\to K_{2n}^{et}(F)_l, \qquad \Lambda^{et} \,:=\, \prod_v \Lambda_v^{et}.$$

Theorem 4.17. *The maps Λ and Λ^{et} satisfy the following properties:*

$$\partial_F \circ \Lambda(\xi_v) = \xi_v^{\Theta_n(\mathbf{b},\mathbf{f})},$$

$$\partial_F^{et} \circ \Lambda^{et}(\zeta_v) = \zeta_v^{\Theta_n(\mathbf{b},\mathbf{f})}.$$

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc} K_{2n}(\mathcal{O}_{F,S})/l^k & \xrightarrow{\bigoplus_{v\in S} \partial_v} & \bigoplus_{v\in S} K_{2n-1}(k_v)/l^k \\ \downarrow & & \downarrow \\ K_{2n}(\mathcal{O}_{F,S};\, \mathbb{Z}/l^k) & \xrightarrow{\bigoplus_{v\in S} \partial_v} & \bigoplus_{v\in S} K_{2n-1}(k_v;\, \mathbb{Z}/l^k) \end{array}$$

The vertical arrows in the diagram come from the Bockstein exact sequence. It is clear from the diagram that the inverse limit over k of the bottom horizontal arrow gives the boundary map $\partial_F = \bigoplus_{v \in S} \partial_v$:

$$\partial_F \;:\; K_{2n}(\mathcal{O}_{F,S})_l \to \bigoplus_v K_{2n-1}(k_v)_l.$$

Now, the theorem follows by Theorems 4.10 and 4.11. $\square$

In the next proposition we will construct a Stickelberger splitting map Γ which is complementary to the map Λ constructed above.

$$0 \longrightarrow K_{2n}(\mathcal{O}_F)_l \underset{\Gamma}{\overset{i}{\rightleftarrows}} K_{2n}(F)_l \underset{\Lambda}{\overset{\partial_F}{\rightleftarrows}} \bigoplus_v K_{2n-1}(k_v)_l \longrightarrow 0. \tag{42}$$

The existence of Γ was suggested in 1988 by Christophe Soulé in a letter to the first author and it is a direct consequence of the following module theoretic lemma.

Lemma 4.18. *Let R be a commutative ring with 1 and let $r \in R$ be fixed. Let*

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 0$$

be an exact sequence of R–modules. Then, the following are equivalent:

(1) *There exists an R–module morphism $\Lambda : C \to B$, such that $\pi \circ \Lambda = r \cdot id_C$.*
(2) *There exists an R–module morphism $\Gamma : B \to A$, such that $\Gamma \circ \iota = r \cdot id_A$.*

Moreover, if Λ and Γ exist, they can be chosen so that $\Gamma \circ \Lambda = 0$.

Proof of Lemma. Assume that (1) holds. By the defining property of Λ, we have

$$(\Lambda \circ \pi)(-b) + rb \in \mathrm{Im}(\iota), \qquad \forall\, b \in B. \tag{43}$$

We define $\Gamma(b) := \iota^{-1}((\Lambda \circ \pi)(-b) + rb)$, for all $b \in B$, where $\iota^{-1}(x)$ is the preimage of x via ι, for all $x \in \mathrm{Im}(\iota)$. One can check without difficulty that Γ is an R–module morphism which satisfies

$$\Gamma \circ \iota = r \cdot id_A, \qquad \Gamma \circ \Lambda = 0.$$

Now, assume that (2) holds. Let $c \in C$. Take $b \in B$, such that $\pi(b) = c$. Then, by the defining property of Γ, one can check that the element $(\iota \circ \Gamma)(-b) + rb \in B$ is independent on the chosen b. For all $c \in C$, we define $\Lambda(c) := (\iota \circ \Gamma)(-b) + rb$, where $b \in B$, such that $\pi(b) = c$. It is easily seen that the map Λ defined this way is an R–module morphism and it satisfies

$$\pi \circ \Lambda = r \cdot id_C, \qquad \Gamma \circ \Lambda = 0.$$

$\square$

Proposition 4.19. *The existence of a map Λ satisfying the property $(\partial_F \circ \Lambda)(\xi_v) = \xi_v^{\Theta_n(\mathbf{b},\mathbf{f})}$ is equivalent to the existence of a map $\Gamma : K_{2n}(F)_l \to K_{2n}(\mathcal{O}_F)_l$ with the property $(\Gamma \circ i)(\eta) = \eta^{\Theta_n(\mathbf{b},\mathbf{f})}$. Moreover, if they exist, the maps Λ and Γ can be chosen so that $\Gamma \circ \Lambda = 0$.*

Proof. The proof of the Proposition follows directly from the above Lemma applied to $R := \mathbb{Z}[G(F/K)]$, $r := \Theta_n(\mathbf{b}, \mathbf{f})$ and Quillen's localization exact sequence (42). $\square$

Remark 4.20. From the proof of Lemma 4.18 it is clear that if one of the maps Λ and Γ is given, then the other one can be chosen such that

$$r \cdot id_B = \Lambda \circ \pi + i \circ \Gamma.$$

Remark 4.21. Observe that the map Λ is defined in the same way for both cases $l \nmid n$ and $l \mid n$. If restricted to the particular case $K = \mathbb{Q}$, our construction improves upon that of [1]. In loc.cit., in the case $l \mid n$ the map Λ was constructed only up to a factor of $l^{v_l(n)}$.

Analogously, there is a Stickelberger splitting map Γ^{et} which is complementary to the map Λ^{et} such that the étale analogue of the Proposition 4.19 holds.

4.3. **Annihilating** $div\, K_{2n}(F)_l$**.** Now, let us give a set of immediate applications of our construction of the Stickelberger splitting maps Λ_n. In what follows, if A ia an abelian group, $div\, A$ denotes its subgroup of divisible elements. The applications which follow concern annihilation of the groups $div\, K_{2n}(F)_l$ by higher Stickelberger elements of the type proved in [1] in the case where the base field is $\mathbb{Q}$. The difference is that while [1] deals with abelian extensions $F/\mathbb{Q}$, under certain restrictions if $l \mid n$, we deal with abelian extensions F/K of an arbitrary totally real number field K under no restrictive conditions. The desired annihilation result follows from the following.

Lemma 4.22. *With notations as in Lemma 4.18, assume that a map Λ exists. Further, assume that C and A viewed as abelian groups satisfy $div\, C = 0$ and A has finite exponent. Then, we have the following.*

1. *$div\, B \subseteq \mathrm{Im}(\iota)$.*
2. *r annihilates $div\, B$.*

Proof. Since any morphism maps divisible elements to divisible elements, we have $\pi(div\, B) = 0$, by our assumption on C. This concludes the proof of (1).

Let $x \in div\, B$. Let m be the exponent of A and let $b \in B$, such that $m \cdot b = x$. Multiply (43) by m to conclude that

$$(\Lambda \circ \pi)(-x) + r \cdot x = 0.$$

Now, part (1) implies that $\pi(x) = 0$. Consequently, the last equality implies that $r \cdot x = 0$, which concludes the proof. $\square$

Theorem 4.23. *Let $m > 0$ be a natural number. Assume that the Stickelberger elements $\Theta_m(\mathbf{b}, \mathbf{f}_k)$ annihilate the groups $K_{2m}(\mathcal{O}_{F_k})_l$ for all $k \geq 1$. Then the Stickelberger's element $\Theta_n(\mathbf{b}, \mathbf{f})$ annihilates the group $div\, K_{2n}(F)_l$ for every $n \geq 1$.*

Proof. The proof is very similar to that of [Ba1, Cor. 1, p. 340]. Let us fix $n \geq 1$. Under our annihilation hypothesis, we have constructed a map $\Lambda := \Lambda_n$ satisfying the properties in Proposition 4.19 relative to the Quillen localization sequence (42). Note that $A := K_{2n}(O_F)_l$ is finite and therefore it has a finite exponent. Also, note that $C := \bigoplus_v K_{2n-1}(k_v)_l$ is a direct sum of finite abelian groups and therefore $div\, C = 0$. Consequently, the exact sequence (42) together with the map Λ and element $r := \Theta_n(\mathbf{b}, \mathbf{f})$ in the ring $R := \mathbb{Z}[G(F/K)]$ satisfy the hypotheses of Lemma 4.22. Therefore, we have

$$\Theta_n(\mathbf{b}, \mathbf{f}) \cdot div\, K_{2n}(F)_l = 0.$$

$\square$

Remark 4.24. Observe that we can restrict the map Λ to the l^k–torsion part, for any $k \geq 1$. For any $k \gg 0$, there is an exact sequence

$$0 \longrightarrow K_{2n}(\mathcal{O}_F)[l^k] \longrightarrow K_{2n}(F)[l^k] \underset{\overleftarrow{\ \Lambda\ }}{\xrightarrow{\partial_F}} \bigoplus_v K_{2n-1}(k_v)[l^k] \longrightarrow div(K_{2n}(F)_l) \longrightarrow 0$$

By Theorem 4.17, we know that $\partial_F \circ \Lambda$ is the multiplication by $\Theta_n(\mathbf{b}, \mathbf{f})$. As pointed out in the Introduction, this implies the annihilation of $div\,(K_{2n}(F)_l)$ and consequently gives a second proof for Theorem 4.23

Let us define $F_0 := F$ and:

$$\Theta_n(\mathbf{b}, \mathbf{f}_0) \ = \ \begin{cases} \left(\prod_{\substack{\mathbf{l}\nmid\mathbf{f} \\ \mathbf{l}\,|\,l}} (1-(\mathbf{l},\,F)^{-1}N\mathbf{l}^n)\right)\,\Theta_n(\mathbf{b}, \mathbf{f}) & \text{if} \quad l \nmid \mathbf{f} \\ \Theta_n(\mathbf{b}, \mathbf{f}) & \text{if} \quad l \mid \mathbf{f} \end{cases}$$

Hence by the formula (13) we get

$$Res_{F_{k+1}/F_k}\ \Theta_n(\mathbf{b}, \mathbf{f}_{k+1}) = \ \Theta_n(\mathbf{b}, \mathbf{f}_k) \tag{44}$$

Hence by formula (44) we can define the element

$$\Theta_n(\mathbf{b}, \mathbf{f}_\infty) := \varprojlim_k \Theta_n(\mathbf{b}, \mathbf{f}_k) \in \varprojlim_k\ \mathbb{Z}_l[G(F_k/F)]. \tag{45}$$

Corollary 4.25. *Let $m > 0$ be a natural number. Assume that the Stickelberger elements $\Theta_m(\mathbf{b}, \mathbf{f}_k)$ annihilate the groups $K_{2m}(\mathcal{O}_{F_k})_l$ for all $k \geq 1$. Then the Stickelberger element $\Theta_n(\mathbf{b}, \mathbf{f}_k)$ annihilates the group $div\, K_{2n}(F_k)_l$ for every $k \geq 0$ and every $n \geq 1$. In particular $\Theta_n(\mathbf{b}, \mathbf{f}_\infty)$ annihilates the group $\varinjlim_k div\, K_{2n}(F_k)_l$ for every $n \geq 1$.*

Proof. Follows immediately from Theorem 4.23. □

Theorem 4.26. *Let F/K be an abelian CM extension of an arbitrary totally real number field K and let l be an odd prime. If the Iwasawa μ–invariant $\mu_{F,l}$ associated to F and l vanishes, then $\Theta_n(\mathbf{b}, \mathbf{f})$ annihilates the group $div(K_{2n}(F)_l)$, for all $n \geq 1$ and all $\mathbf{b}$ coprime to $w_{n+1}(F)\mathbf{f}l$.*

Proof. In [16] (see Theorem 6.11), it is shown that if $\mu_{F,l} = 0$, then $\Theta_n(\mathbf{b}, \mathbf{f})$ annihilates $K_{2n}^{et}(O_F[1/l])$, for all $n \geq 1$ and all $\mathbf{b}$ as above. From the definition of Iwasawa's μ–invariant one concludes right away that if $\mu_{F,l} = 0$, then $\mu_{F_k,l} = 0$, for all k. Consequently, $\Theta_1(\mathbf{b}, \mathbf{f_k})$ annihilates $K_2^{et}(O_{F_k}[1/l])$, for all k. Now, one applies Tate's Theorem 1.3 to conclude that $\Theta_1(\mathbf{b}, \mathbf{f})$ annihilates $K_2(O_{F_k})_l$, for all k. Theorem 4.23 implies the desired result. □

Remark 4.27. It is a classical conjecture of Iwasawa that $\mu_{F,l} = 0$, for all number fields F and all primes l.

Corollary 4.28. *Let $F/\mathbb{Q}$ be an abelian extensions of conductor f. Then $\Theta_n(b, f)$ annihilates the group $div\, K_{2n}(F)_l$ for all $n \geq 1$ and all $\mathbf{b}$ coprime to $w_{n+1}(F)\mathbf{f}l$.*

Proof. By a well known theorem of Ferrero-Washington and Sinnott, $\mu_{F,l} = 0$ for all fields F which are abelian extensions of $\mathbb{Q}$ and all l. Now, the Corollary is an immediate consequence of the previous Theorem. □

Remark 4.29. Observe that the Corollary above strengthens Corollary 1, p. 340 of [1] in the case $l \,|\, n$.

5. Constructing Euler systems out of Λ–elements

As mentioned in §4, in this section we construct Euler systems for the even K-theory of CM abelian extensions of totally real number fields. We construct an Euler system in the K–theory with coefficients. Then, by passing to a projective limit, we obtain an Euler system in Quillen K–theory. Our constructions are quite different from those in [3], where an Euler systems in the odd K–theory with finite coefficients of CM abelian extensions of $\mathbb{Q}$ was described.

As above, we fix a finite abelian CM extension F/K of a totally real number field of conductor $\mathbf{f}$ and fix a prime number l. We let $\mathbf{L} = \mathbf{l}_1 \dots \mathbf{l}_t$ be a product of mutually distinct prime ideals of $\mathcal{O}_K$ coprime to $l\,\mathbf{f}$. We let $F_{\mathbf{L}} := FK_{\mathbf{L}}$, where $K_{\mathbf{L}}$ is the ray class field of K for the ideal $\mathbf{L}$. Since F/K has conductor $\mathbf{f}$ the CM-extension $F_{\mathbf{L}}/K$ has conductor dividing $\mathbf{Lf}$. As usual, we let $F_{\mathbf{L}l^k} := F_{\mathbf{L}}(\mu_{l^k})$, for every $k \geq 0$.

Let us fix a prime v in $\mathcal{O}_F$ sitting above a rational prime $p \neq l$. Let $S := S_v$ be the set consisting of all the primes of $\mathcal{O}_F$ sitting above p or above l. For all $\mathbf{L}$ as above and $k \geq 0$, we fix primes $w_k(\mathbf{L})$ of $O_{F_{\mathbf{L}l^k}}$ sitting above v, such that $w_{k'}(\mathbf{L}')$ sits above $w_k(\mathbf{L})$ whenever $l^k\mathbf{L} \mid l^{k'}\mathbf{L}'$. For simplicity, we let $v(\mathbf{L}) := w_0(\mathbf{L})$, for all $\mathbf{L}$ as above. Also, if k is fixed, we let $w(\mathbf{L}) := w_k(\mathbf{L})$, for all $\mathbf{L}$ as above.

Let us fix integers $m > 0$. For all $\mathbf{L}$ as above and all $k \geq 0$, let $\Theta_m(\mathbf{b}_{\mathbf{L}}, \mathbf{Lf}_k)$ denote the m-th Stickelberger element for the integral ideal $\mathbf{b}_{\mathbf{L}}$ of O_F, coprime to $\mathbf{Lf}l$, and the extension $F_{\mathbf{L}l^k}/K$. As usual, we assume throughout that $\Theta_m(\mathbf{b}_{\mathbf{L}}, \mathbf{Lf}_k)$ annihilates $K_{2m}(\mathcal{O}_{F_{\mathbf{L}l^k}})$, for all $\mathbf{L}$ as above and all $k \geq 0$.

By the surjectivity of the transfer maps for the K-theory of finite fields we can fix generators $\xi_{w_k(\mathbf{L}),k}$ of $K_{2m-1}(k_{w_k(\mathbf{L})})_l$, for all $k \geq 0$ and $\mathbf{L}$ as above, such that

$$N_{w_{k'}(\mathbf{L})/w_k(\mathbf{L})}(\xi_{w_{k'}(\mathbf{L}),k'}) = \xi_{w_k(\mathbf{L}),k}\,,$$

whenever we have $k \leq k'$.

Remark 5.1. Note that the cyclicity of the groups $K_{2m-1}(k_{w_k(\mathbf{L})})_l$ and the surjectivity of the appropriate transfer maps implies that the elements

$$(\xi_{w_k(\mathbf{L}),k})_k, \qquad (N_{w_k(\mathbf{L}')/w_k(\mathbf{L})}(\xi_{w_k(\mathbf{L}'),k}))_k,$$

viewed inside of $\mathbb{Z}_l$–module $\varprojlim_k K_{2m-1}(k_{w_k(\mathbf{L})})_l$, differ by a factor in $\mathbb{Z}_l^\times$, for all $\mathbf{L}$ and $\mathbf{L}'$ as above, such that $\mathbf{L}|\mathbf{L}'$. Above, the projective limit is taken with respect to the transfer maps.

Let us fix $k \geq 0$. For any $\mathbf{L}$ as above, we have the localization exact sequence:

$$0 \longrightarrow K_{2m}(\mathcal{O}_{F_{\mathbf{L}l^k}}) \longrightarrow K_{2m}(\mathcal{O}_{F_{\mathbf{L}l^k}, S}) \xrightarrow{\partial} \bigoplus_{v_0 \in S} \bigoplus_{w|v_0} K_{2m-1}(k_w) \longrightarrow 0,$$

where the direct sum is taken with respect to all the primes w in $\mathcal{O}_{F_{\mathbf{L}l^k}}$ which sit above primes v_0 in S. Pick an element $x_{w(\mathbf{L}),k} \in K_{2m}(\mathcal{O}_{F_{\mathbf{L}l^k},S})_l$, such that $\partial(x_{w(\mathbf{L}),k}) = \xi_{w(\mathbf{L}),k}$. The following element:

$$\Lambda_m(\xi_{w(\mathbf{L}),k}) := x_{w(\mathbf{L}),k}^{\Theta_m(\mathbf{b}_{\mathbf{L}},\mathbf{Lf}_k)} \tag{46}$$

does not depend on the choice of $x_{w(\mathbf{L}),k}$ since $\Theta_m(\mathbf{b}_{\mathbf{L}}, \mathbf{Lf}_k)$ annihilates $K_{2m}(\mathcal{O}_{F_{\mathbf{L}l^k}})$. Observe that by construction we have the following equalities:

$$(47) \qquad \partial_{F_{\mathbf{L}l^k}}(Tr_{F_{\mathbf{L}'l^k}/F_{\mathbf{L}l^k}}(x_{w(\mathbf{L}'),k})) = N_{w(\mathbf{L}')/w(\mathbf{L})}(\partial_{F_{\mathbf{L}'l^k}}(x_{w(\mathbf{L}'),k}) = $$
$$= N_{w(\mathbf{L}')/w(\mathbf{L})}(\xi_{w(\mathbf{L}'),k}),$$

$$(48) \qquad N_{w(\mathbf{L})/v(\mathbf{L})}(N_{w(\mathbf{L}')/w(\mathbf{L})}(\xi_{w(\mathbf{L}'),k})) = N_{v(\mathbf{L}')/v(\mathbf{L})}(N_{w(\mathbf{L}')/v(\mathbf{L}')}(\xi_{w(\mathbf{L}'),k})) =$$
$$= N_{v(\mathbf{L}')/v(\mathbf{L})}(\xi_{v(\mathbf{L}'),0}))$$

We choose the ideals $\mathbf{b_L}$ in such a way so that they are coprime to $l\,\mathbf{Lf}$ and

$$N\mathbf{b}_{\mathbf{L}'} \equiv N\mathbf{b}_{\mathbf{L}} \bmod l^k.$$

Then, the elements $\{\Lambda_m(\xi_{w(\mathbf{L}),k})\}_{\mathbf{L}}$ form an Euler system in K-theory without coefficients $\{K_{2m}(O_{\mathbf{L}l^k,S})_l\}_{\mathbf{L}}$. Namely, we have:

Proposition 5.2. *If $\mathbf{L}' = \mathbf{l}'\mathbf{L}$, then the following equality holds:*

$$(49) \quad Tr_{F_{\mathbf{L}'l^k}/F_{\mathbf{L}l^k}}(\Lambda_m(\xi_{w(\mathbf{L}'),k})) = \Lambda_m(N_{w(\mathbf{L}')/w(\mathbf{L})}(\xi_{w(\mathbf{L}'),k}))^{1-N(\mathbf{l}')^m(\mathbf{l}',\, F_{\mathbf{L}l^k})^{-1}}.$$

Proof. The Proposition follows by (47) and Lemma 2.1. □

Let us fix an arbitrary integer $n > 0$. Next, we use the Euler system above to construct Euler systems $\{\boldsymbol{\lambda}_{v(\mathbf{L})}\}_{\mathbf{L}}$ in the K–groups $\{K_{2n}(O_{\mathbf{L},S})_l\}_{\mathbf{L}}$. The general idea is as follows. First, one constructs Euler Systems $\{\boldsymbol{\lambda}_{v(\mathbf{L}),k}\}_{\mathbf{L}}$ in the K–theory with coefficients $\{K_{2n}(O_{\mathbf{L},S}, \mathbb{Z}/l^k)\}_{\mathbf{L}}$, for all $k > 0$. Then one passes to a projective limit with respect to k. The constructions, ideas and results developed in §4 play a key role in what follows.

For every $\mathbf{L}$ as above and every $k \geq 0$, we follow the ideas in §4 and define the elements $\boldsymbol{\lambda}_{v(\mathbf{L}),k} \in K_{2n}(\mathcal{O}_{F_{\mathbf{L}},S};\, \mathbb{Z}/l^k)$ by:

$$(50) \qquad \begin{aligned} \boldsymbol{\lambda}_{v(\mathbf{L}),k} : \;&=\; Tr_{F_{\mathbf{L}l^k}/F_{\mathbf{L}}}(x_{w_k(\mathbf{L}),k}^{\Theta_m(\mathbf{b_L},\,\mathbf{Lf}_k)} * \beta_k^{*\,n-m})^{N\mathbf{b}_{\mathbf{L}}^{n-m}\gamma_l} = \\ & \quad Tr_{F_{\mathbf{L}l^k}/F_{\mathbf{L}}}(\Lambda_m(\xi_{w_k(\mathbf{L}),k}) * \beta_k^{*\,n-m})^{N\mathbf{b}_{\mathbf{L}}^{n-m}\gamma_l}, \end{aligned}$$

where the operator $\gamma_l \in \mathbb{Z}_l[G(F/K)]$ is given in Definition 4.6. The following theorem lies at the heart of our construction of the Euler system for higher K-groups of CM abelian extensions of arbitrary totally real number fields.

Theorem 5.3. *For every $k' \geq k$ and every $\mathbf{L}$ and $\mathbf{L}' = \mathbf{L}\mathbf{l}'$ we have:*

$$\begin{aligned} r_{k'/k}(\boldsymbol{\lambda}_{v(\mathbf{L}),k'}) &= \boldsymbol{\lambda}_{v(\mathbf{L}),k} \\ \partial_{F_{\mathbf{L}}}(\boldsymbol{\lambda}_{v(\mathbf{L}),k}) &= (N_{\mathbf{L}}(\xi_{w(\mathbf{L}),k} * \beta_k^{*\,n-m}))^{\Theta_n(\mathbf{b_L},\mathbf{fL})} \\ Tr_{F_{\mathbf{L}'}/F_{\mathbf{L}}}(\boldsymbol{\lambda}_{v(\mathbf{L}'),k}) &= (\boldsymbol{\lambda}'_{v(\mathbf{L}),k})^{1-N(\mathbf{l}')^n(\mathbf{l}',\,F_{\mathbf{L}})^{-1}}, \end{aligned}$$

where $N_{\mathbf{L}} := Tr_{k_{w(\mathbf{L})}/k_{v(\mathbf{L})}}$ and $\boldsymbol{\lambda}'_{v(\mathbf{L}),k}$ is defined by

$$\boldsymbol{\lambda}'_{v(\mathbf{L}),k} := Tr_{F_{\mathbf{L}l^k}/F_{\mathbf{L}}}(\Lambda_m(N_{w_k(\mathbf{L}')/w_k(\mathbf{L})}(\xi_{w_k(\mathbf{L}'),k})) * \beta_k^{*\,n-m})^{N\mathbf{b}_{\mathbf{L}}^{n-m}\gamma_l}.$$

Proof. The first formula follows by Lemma 4.9. The second formula follows by Theorem 4.10. Let us prove the Euler System property (the third formula in the statement of the Theorem.) We apply Lemma 2.1 and definition (50):

$$Tr_{F_{\mathbf{L}'}/F_{\mathbf{L}}}(\boldsymbol{\lambda}_{v(\mathbf{L}'),k}) = Tr_{F_{\mathbf{L}'}/F_{\mathbf{L}}} Tr_{F_{\mathbf{L}'l^k}/F_{\mathbf{L}'}}(x_{w(\mathbf{L}'),k}^{\Theta_m(\mathbf{b}_{\mathbf{L}'},\,\mathbf{f}_k\mathbf{L}')} * \beta_k^{*\,n-m})^{N\mathbf{b}^{n-m}\gamma_l} =$$
$$Tr_{F_{\mathbf{L}l^k}/F_{\mathbf{L}}} Tr_{F_{\mathbf{L}'l^k}/F_{\mathbf{L}l^k}}(x_{w(\mathbf{L}'),k}^{\Theta_m(\mathbf{b}_{\mathbf{L}'},\,\mathbf{f}_k\mathbf{L}')} * \beta_k^{*\,n-m})^{N\mathbf{b}^{n-m}\gamma_l} =$$
$$Tr_{F_{\mathbf{L}l^k}/F_{\mathbf{L}}} (Tr_{F_{\mathbf{L}'l^k}/F_{\mathbf{L}l^k}} x_{w(\mathbf{L}'),k}^{\Theta_m(\mathbf{b}_{\mathbf{L}'},\,\mathbf{f}_k\mathbf{L}')} * \beta_k^{*\,n-m})^{N\mathbf{b}^{n-m}\gamma_l} =$$
$$Tr_{F_{\mathbf{L}l^k}/F_{\mathbf{L}}} (Tr_{F_{\mathbf{L}'l^k}/F_{\mathbf{L}l^k}} \left(x_{w(\mathbf{L}'),k}\right)^{\Theta_m(\mathbf{b}_{\mathbf{L}'},\,\mathbf{f}_k\mathbf{L}')} * \beta_k^{*\,n-m})^{N\mathbf{b}^{n-m}\gamma_l} =$$
$$Tr_{F_{\mathbf{L}l^k}/F_{\mathbf{L}}} (Tr_{F_{\mathbf{L}'l^k}/F_{\mathbf{L}l^k}} \left(x_{w(\mathbf{L}'),k}\right)^{Res_{K_{\mathbf{f}_k\mathbf{L}'}/K_{\mathbf{f}_k\mathbf{L}}}\Theta_m(\mathbf{b}_{\mathbf{L}'},\,\mathbf{f}_k\mathbf{L}')} * \beta_k^{*\,n-m})^{N\mathbf{b}^{n-m}\gamma_l} =$$
$$Tr_{F_{\mathbf{L}l^k}/F_{\mathbf{L}}} (Tr_{F_{\mathbf{L}'l^k}/F_{\mathbf{L}l^k}} \left(x_{w(\mathbf{L}'),k}\right)^{\left(1-(\mathbf{l}',\,F_{\mathbf{L}l^k})^{-1}N(\mathbf{l}')^m\right)\,\Theta_m(\mathbf{b}_{\mathbf{L}},\,\mathbf{f}_k\mathbf{L})} * \beta_k^{*\,n-m})^{N\mathbf{b}^{n-m}\gamma_l} =$$
$$Tr_{F_{\mathbf{L}l^k}/F_{\mathbf{L}}} (Tr_{F_{\mathbf{L}'l^k}/F_{\mathbf{L}l^k}} \left(x_{w(\mathbf{L}'),k}\right)^{\,\Theta_m(\mathbf{b}_{\mathbf{L}},\,\mathbf{f}_k\mathbf{L})} * \beta_k^{*\,n-m})^{N\mathbf{b}_{\mathbf{L}}{}^{n-m}\gamma_l\left(1-(\mathbf{l}',\,F_{\mathbf{L}})^{-1}N(\mathbf{l}')^n)\right)} =$$
$$(\boldsymbol{\lambda}'_{v(\mathbf{L}),k})^{1-N(\mathbf{l}')^n(\mathbf{l}',\,F_{\mathbf{L}})^{-1}}.$$

The last equality is a direct consequence of equalities (47) and (48). □

Now, let $\mathbf{b}$ be a fixed ideal in O_K, coprime to $\mathbf{f}\,l$. Consider all $\mathbf{L}$ as above which are coprime to $l\mathbf{bf}$. Naturally, we can choose $\mathbf{b}_{\mathbf{L}} := \mathbf{b}$, for all such $\mathbf{L}$. These choices and the results of §4 permit us to define the elements $\boldsymbol{\lambda}_{v(\mathbf{L})} \in K_{2n}(\mathcal{O}_{F_{\mathbf{L}},S})_l$ and $\xi_{v(\mathbf{L})} \in K_{2n-1}(k_{v(\mathbf{L})})_l$ as follows.

Definition 5.4. Let $\boldsymbol{\lambda}_{v(\mathbf{L})} \in K_{2n}(\mathcal{O}_{F_{\mathbf{L}},S})_l$ be the element corresponding to

$$(\boldsymbol{\lambda}_{v(\mathbf{L}),l^k})_k \in \varprojlim_k K_{2n}(\mathcal{O}_{F_{\mathbf{L}},S};\, \mathbb{Z}/l^k)$$

via the isomorphism (40) for the ring $\mathcal{O}_{F_{\mathbf{L}},S}$.

Definition 5.5. Let $\xi_{v(\mathbf{L})} \in K_{2n-1}(k_{v(\mathbf{L})})_l$ be the element corresponding to

$$(N_{\mathbf{L}}(\xi_{w(\mathbf{L}),k} * \beta_k^{*\,n-m}))_k \in \varprojlim_k K_{2n-1}(k_{v(\mathbf{L})};\, \mathbb{Z}/l^k)$$

via the isomorphism (41) for the finite field $k_{v(\mathbf{L})}$.

Remark 5.6. Note that $\Lambda(\xi_{v(\mathbf{L})}) = \boldsymbol{\lambda}_{v(\mathbf{L})}$ (see Remark 4.15.)

The next result shows that the elements $\{\boldsymbol{\lambda}_{v(\mathbf{L})}\}_{\mathbf{L}}$ provide an Euler System for the K-theory without coefficients $\{K_{2n}(O_{F_{\mathbf{L}},S})_l\}_{\mathbf{L}}$.

Theorem 5.7. *For every $\mathbf{L}$ and $\mathbf{L}'$ as above, such that $\mathbf{L}' = \mathbf{L}\mathbf{l}'$, we have the following equalities:*

$$\partial_{F_{\mathbf{L}}}(\Lambda(\xi_{v(\mathbf{L})})) = \xi_{v(\mathbf{L})}^{\Theta_n(\mathbf{b},\,\mathbf{Lf})} \tag{51}$$

$$Tr_{F_{\mathbf{L}'}/F_{\mathbf{L}}}(\Lambda(\xi_{v(\mathbf{L}')})) = \Lambda(N_{v(\mathbf{L}')/v(\mathbf{L})}(\xi_{v(\mathbf{L}')}))^{1-N(\mathbf{l}')^n(\mathbf{l}',\,F_{\mathbf{L}})} \tag{52}$$

Proof. This follows directly from Theorem 5.3. □

Remark 5.8. It is easy to see that one can construct Euler systems for étale K-theory in a similar manner.

Remark 5.9. In our upcoming work, we will use the Euler systems constructed above to investigate the structure of the group of divisible elements $divK_{2n}(F)_l$ inside $K_{2n}(F)_l$. The structure of $divK_{2n}(F)_l$ is of principal interest vis a vis some classical conjectures in algebraic number theory, as explained in the introduction.

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