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MOCK PERIOD FUNCTIONS, SESQUIHARMONIC MAASS FORMS, AND NON-CRITICAL VALUES OF L-FUNCTIONS

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Abstract. We introduce a new technique of completion for 1-cohomology which parallels the corresponding technique in the theory of mock modular forms. This technique is applied in the context of non-critical values of L-functions of GL(2) cusp forms. We prove that a generating series of non-critical values can be interpreted as a mock period function we define in analogy with period polynomials. Further, we prove that non-critical values can be encoded into a sesquiharmonic Maass form. Finally, we formulate and prove an Eichler-Shimura-type isomorphism for the space of mock period functions.

1. Introduction

In this work, we establish a connection between two seemingly disparate topics and techniques: mock modular forms (holomorphic parts of harmonic Maass forms) and non-critical values of L-functions of cusp forms. To describe this connection, we first outline each of these topics and some of the corresponding questions that arise.

A very fruitful technique that has recently emerged in the broader area of automorphic forms and its arithmetic applications is based on "completing" a holomorphic but not quite automorphic form into a harmonic Maass form by addition of a suitable non-holomorphic function. This method originates in its modern form in Zwegers' PhD thesis [36]. Zwegers completed all of Ramanujan's mock theta functions introduced by Ramanujan in his famous last letter to Hardy [33], including

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}.$$

To be more precise, Zwegers found a (purely) non-holomorphic function

$$N_f(z) := \int_{-\overline{z}}^{i\infty} \frac{\Theta_f(w)}{\sqrt{z+w}} dw, \tag{1.1}$$

where Θ_f is some explicit weight $\frac{3}{2}$ cuspidal theta function, so that

$$f(q) + N_f(z)$$

transforms like an automorphic form of weight "dual" to that of f, i.e., of weight $\frac{1}{2}$ in our case (throughout we write $q := e^{2\pi i z}$). Such completions proved to be useful in obtaining information for the original function (f in our context), including exact formulas for Fourier coefficients, made use of, e.g., in the proof in [8] of the Andrews-Dragonette Conjecture

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[1, 19]. On the other hand, one can also reverse the question and start with a modular form, define an integral N resembling the one in (1.1) and find a holomorphic function F such that $N+F$ transforms like a modular form. Such "lifts" were constructed for cusp forms of weight $\frac{1}{2}$ in terms of combinatorial series by the first author, Folsom, and Ono [6] and by the first author and Ono for general cusp forms [9]. Recently, also lifts for non-cusp forms were found [18]. Obstructions to modularity occuring from functions like f may also be viewed in terms of critical values of L-functions [7] in a way we will describe later.

We next introduce the second topic, non-critical values of L-functions. We will first outline the background concerning general values of L-functions and critical values. Let f be an element of S_k, the space of cusp forms of weight $k \in 2\mathbb{N}$ for $\mathrm{SL}_2(\mathbb{Z})$, and let $L_f(s)$ denote its L-function. Special values of L-functions have been the focus of intense research in arithmetic algebraic geometry and analytic number theory, because they provide deep insight to f and associated arithmetic and geometric objects. Several of the outstanding conjectures in number theory are related to special values of L-functions, e.g. the ones posed by Birch-Swinnerton-Dyer, Beilinson and Bloch-Kato (see, for example, [28]). In particular, they are commonly interpreted as regulators in K-theory [34].

Among the special values, more is known about the *critical* values which, for our purposes, are $L_f(1), L_f(2), \dots, L_f(k-1)$ (see [16, 28] for an intrinsic characterization). For instance, Manin's Periods Theorem [30] implies that, when f is an eigenform of the Hecke operators, its critical values are algebraic linear combinations of two constants depending only on f. This result was established by incorporating a "generating function" of the critical values into a cohomology which has a rational structure. The generating function is the *period polynomial*

$$r_f(X) := \int_0^{i\infty} f(w)(w-X)^{k-2}dw,$$

and each of its coefficients is an explicit multiple of a critical values of $L_f(s)$ (see Lemma 2.1 for the precise statement).

The period polynomial of f satisfies the *Eichler-Shimura relations*:

$$r_f|_{2-k}(1+S) = r_f\Big|_{2-k}\left(1+U+U^2\right) = 0 \qquad \text{with } S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right),\ U := \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

in terms of the action $|_m$ on $G : \mathfrak{H} \to \mathbb{C}$ defined for each $m \in 2\mathbb{Z}$ by

$$G|_m\gamma(X) := G(\gamma X)(cX+d)^{-m} \qquad \text{for } \gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R}).$$

Because of the importance of these Eichler-Shimura relations, the space V_{k-2} of all polynomials of degree at most $k-2$ satisfying them has been studied independently. It is called the *space of period polynomials* and is denoted by W_{k-2}.

Non-critical values are much less understood and there are even some "negative" results such as that of Koblitz [26], asserting that, in a strong sense, there can not be a Period Theorem for non-critical values. In any case, it is generally expected that the algebraic structure of such values is more complicated than that of critical values. Nevertheless, in [15] it is shown that it is possible to define "generating series" of non-critical values, which can further be incorporated into a cohomology similar to the Eichler cohomology. This fits into the philosophy of Manin's [31] and Goldfeld's [22] cohomological interpretation of values and derivatives of L-functions, respectively. The generating series is a function $r_{f,2}$ on the

Poincaré upper-half plane $\mathfrak{H}$ given by

$$r_{f,2}(z) := \int_0^{i\infty} \frac{F_f(w)}{(wz-1)^k} dw,$$

where F_f is the *Eichler integral* associated to f

$$F_f(z) := \int_z^{i\infty} f(w)(w-z)^{k-2} dw.$$

The function $r_{f,2}$ is the direct counterpart of the period polynomial r_f associated to critical values. The non-critical values are obtained from $r_{f,2}$ as "Taylor coefficients" of $r_{f,2}$ (see Lemma 2.2), just as critical values are retrieved as coefficients of the period polynomial r_f. The ambient space of functions consists of harmonic functions rather than polynomials and the action is $|_k$ instead of $|_{2-k}$.

The *first* link between the aforementioned two topics emerges as we use techniques from the theory of mock modular forms to intrinsically interpret the constructions that were associated to non-critical values in [15]. Those constructions were in some respects ad hoc and not as intrinsic as those relating to critical values. For example, whereas the period polynomial is expressed as a constant multiple of

$$F_f|_{2-k}(S-1),$$

the generating function $r_{f,2}(z)$ has an analogous expression only up to an explicit "correction term". That problem would seem to be insurmountable, because $r_{f,2}(z)$ is not invariant under S.

However, in this paper we show that it is exactly thanks to the "correction term" that our generating function $r_{f,2}$ can be completed into a function which belongs to a natural analogue of the space of period polynomials W_{k-2}. We show that an appropriate counterpart of

$$W_{k-2} := \{P \in V_{k-2}; P|_{2-k}(1+S) = P|_{2-k}\left(1+U+U^2\right) = 0\}$$

is

$$W_{k,2} := \left\{\mathcal{P} : \mathfrak{H} \to \mathbb{C};\ \xi_k(\mathcal{P}) \in V_{k-2};\ \mathcal{P}|_k(1+S) = \mathcal{P}|_k\left(1+U+U^2\right) = 0\right\}.$$

Here, ξ_k is a key operator in the theory of mock modular forms defined, for $y :=\mathrm{Im}(z)$ by

$$\xi_k := 2iy^k \overline{\frac{d}{d\overline{z}}}.$$

Our first main result then is

Theorem 1.1. *Let $k \in 2\mathbb{N}$ and f a weight k cusp form. Then the function*

$$\widehat{r}_{f,2}(z) := r_{f,2}(z) - \int_{-\overline{z}}^{i\infty} \frac{r_f(w)}{(w+z)^k} dw$$

belongs to the space $W_{k,2}$.

Theorem 1.1 suggests the name *mock period function* for $r_{f,2}$ (see Definition 3.3)

The completion of $r_{f,2}$ by a purely non-holomorphic term does not cause us to lose information about non-critical values, because it only introduces critical values (see Lemma 2.4), which from our viewpoint can be thought of as understood.

The *second* link between the two main subjects of the paper amounts to a technique that allows us to encode information about the mock period function of $f \in S_k$ into a certain "higher order" version of harmonic Maass forms. This is the direct analogue of a recent result proved for critical values by the first author, Guerzhoy, Kent, and Ono (Theorem 1.1 of [7]) and in a different guise earlier in [20]:

Theorem 1.2. *([20, 7]) For each $f \in S_k$, there is a harmonic Maass form M_f with holomorphic part M_f^+, such that*

$$r_f(-z) = M_f^+|_{2-k}(1-S).$$

The authors further use similar techniques to establish a structure theorem for W_{k-2} (Theorem 1.2 of [7]).

The first step of our approach towards establishing the counterpart of Theorem 1.2 for non-critical values is to identify the objects taking the role played by harmonic Maass forms in [7]. The class of these objects is formed by *sesquiharmonic Maass forms* (see Definition 4.1). Sesquiharmonic Maass form are natural higher order versions of harmonic Maass forms, the first example of which has appeared in a different context [17, 18]. (See also [12, 13, 14] for an earlier application of the underlying method). The main difference of sesquiharmonic to harmonic Maass forms is that the latter are annihilated by the weight k Laplace-operator

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right),$$

whereas sesquiharmonic Maass forms are annihilated by

$$\Delta_{k,2} := \Delta_{2-k}\circ\xi_k = -\xi_k\circ\xi_{2-k}\circ\xi_k = \xi_k\circ\Delta_k.$$

In Section 4, we will show that we can isolate a "harmonic" piece from each sesquiharmonic Maass, paralleling the way we can isolate a "holomorphic" piece from each harmonic Maass form. This construction allows us to formulate and prove the analogue of Theorem 1.2:

Theorem 1.3. *For each $f \in S_k$, there is a sesquiharmonic Maass form $M_{f,2}$ with harmonic part $M_{f,2}^{+-}$, such that*

$$\widehat{r}_{f,2}(z) = M_{f,2}^{+-}(z)\Big|_k(S-1).$$

The above two techniques we just described can be considered as a new version of the "completion" method, this time applied to the level of 1-cohomology.

The *third* main result and technique of this paper is a mock Eichler-Shimura isomorphism for $W_{k,2}$. The classical Eichler-Shimura isomorphism "parametrizes" W_{k-2} in terms of cusp forms. It can be summarized as:

Theorem 1.4. *(e.g., [27]) Every $P \in W_{k-2}$ can be written as*

$$P(X) = r_f(X) + r_g(-X) + a|_{2-k}(S-1)$$

for unique $f, g \in S_k$ and $a \in \mathbb{C}$.

In Section 5, we show that $W_{k,2}$ can be "parametrised" by cusp forms in a very similar fashion:

Theorem 1.5. *Every $P \in W_{k,2}$ can be written as*

$$P = \widehat{r}_{f,2} + \widehat{r}^*_{g,2} + aF|_k(S-1)$$

for unique $f, g \in S_k$ and an $a \in \mathbb{C}$. Here, F is an element of an appropriate space of functions on $\mathfrak{H}$ and $\widehat{r}^*_{g,2}$ is a period function associated $r_g(-X)$. (They will be defined precisely in Section 5).

The construction of $\widehat{r}^*_{g,2}$ is of independent interest and involves (regularized) integrals (see Section 5). Some of the techniques are related to the theory of periods of weakly holomorphic forms as studied by Fricke [21].

It is surprising that pairs of cusp forms suffice for this Mock Eichler-Shimura isomorphism just as they suffice for the classical Eichler-Shimura isomorphism. A priori, the spaces W_{k-2} and $W_{k,2}$ appear to be very different, especially since, as shown here, they are associated with critical and non-critical values respectively, which are expected to have completely different behaviour.

In the final section we interpret our two first main results cohomologically (Theorem 6.1) in order to highlight the essential similarity of the construction we associate here to non-critical values with the corresponding setting for critical values. Since we have an entirely analogous reformulation (see (6.1)) of the Eichler-Shimura theory and the results of [7], Theorem 6.1 justifies the claim that our constructions form the non-critical value counterpart of the corresponding results in the case of *critical* values of L-functions.

A suggestive comparison of this cohomological interpretation with Hida's evidence for a possible description of non-critical values in terms of non-top degree cohomology (cf. [24]) might also be made. We intend to return to possible explicit connections with Hida's construction in a future work.

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2. Cusp forms and periods associated to their L-values

Set $\Gamma := \mathrm{S}L_2(\mathbb{Z})$. Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ ($q = e^{2\pi i z}$) be a cusp form of weight k for Γ. Further let $L_f(s)$ be the entire function obtained by analytic continuation of the series $L_f(s) = \sum_{n=1}^{\infty} a(n)/n^s$ originally defined in an appropriate right half plane.

In the Eichler-Shimura-Manin theory one associates to f an Eichler integral $F_f : \mathfrak{H} \to \mathbb{C}$ and a period polynomial $r_f : \mathbb{C} \to \mathbb{C}$ as follows:

$$\begin{aligned} F_f(z) &:= \int_z^{i\infty} f(w)(w-z)^{k-2} dw, \\ r_f(z) &:= \int_0^{i\infty} f(w)(w-z)^{k-2} dw. \end{aligned}$$

These objects are connected to each other and intimately related to critical values of $L_f(s)$ (see e.g. [27], Section 1.1): $L_f(1), \ldots, L_f(k-1)$.

Lemma 2.1. *For every $f \in S_k$, we have*

$$\begin{aligned} F_f|_{2-k}(1-S) &= r_f, \\ r_f(z) &= -\frac{(k-2)!}{(2\pi i)^{k-1}} \sum_{n=0}^{k-2} \frac{L_f(n+1)}{(k-2-n)!} (2\pi i z)^{k-2-n}. \end{aligned}$$

We shall consider the analogues of F_f and r_f yielding non-critical values of $L_f(s)$. Set

$$\begin{aligned} F_{f,2}(z) &:= \int_{-\bar{z}}^{i\infty} \frac{F_f(w)}{(w+z)^k} dw, \\ r_{f,2}(z) &:= \left(\int_0^{i\infty} \frac{F_f(w)}{(w+z)^k} dw \right)\Big|_k S = \int_0^{i\infty} \frac{F_f(w)}{(wz-1)^k} dw. \end{aligned}$$

The function $r_{f,2}$ is not a polynomial, but the next lemma, proved in [15], shows that we can still retrieve values of L-functions of f as its "Taylor coefficients at 0". It also explains the reason for letting S act on the integral in the definition of $r_{f,2}$ in an apparent disanalogy to r_f:

Lemma 2.2. *For every $f \in S_k$ and $m \in \mathbb{N}$, we have*

$$\lim_{z \to 0^+} \frac{d^m}{dz^m} \left(r_{f,2}(z) \right) = i^{k+m} \frac{(m+k-1)!m!}{(k-1)(2\pi)^{m+k}} L_f(k+m).$$

In [15], it is also proved that $F_{f,2}$ and $r_{f,2}$ are linked in a way that parallels the link between F_f and r_f. For our purposes, we will need a reformulation of that result:

Proposition 2.3. *For every $f \in S_k$, we have*

$$F_{f,2}\big|_k (S-1) = r_{f,2} - \widetilde{r}_{f,2} \tag{2.1}$$

with

$$\widetilde{r}_{f,2}(z) := \int_{-\bar{z}}^{i\infty} \frac{r_f(w)}{(w+z)^k} dw.$$

Proof: ¿From the proof of Theorem 3 of [15], it follows that

$$F_{f,2}(z)\big|_k (S-1) = r_{f,2}(z) + \left(\int_{-\bar{z}}^{0} \frac{r_f(w)}{(w+z)^k} dw \right)\Big|_k S.$$

The last term may now easily be simplified using that $r_f \in W_{k-2}$. □

The correction term $\widetilde{r}_{f,2}$ may be explicitly expressed in terms of critical values, and it does not affect the analogy with the relation between F_f and r_f.

Lemma 2.4. *For all $f \in S_k$,*

$$\widetilde{r}_{f,2}(z) = -(k-2)! \sum_{n=0}^{k-2} \sum_{\ell=0}^{k-2-n} \frac{L_f(n+1)}{\ell!(k-2-n-\ell)!(1+n+\ell)} (-4\pi i z)^\ell (-4\pi y)^{-1-n-\ell}.$$

Remark 1. We note that all of the exponents of y are negative, thus $\widetilde{r}_{f,2}$ is a purely non-holomorphic function.

Proof: ¿From Lemma 2.1,

$$\int_{-\bar{z}}^{i\infty} \frac{r_f(w)}{(w+z)^k} dw = (k-2)! \sum_{n=0}^{k-2} i^{-n+1} \frac{L_f(n+1)}{(2\pi)^{n+1}(k-2-n)!} \int_{-\bar{z}}^{i\infty} \frac{w^{k-2-n}}{(w+z)^k} dw.$$

Making the change of variable $w \to w - z$ and then using the Binomial Theorem, we obtain that the integral equals

$$\sum_{\ell=0}^{k-2-n} \binom{k-2-n}{\ell} (-z)^\ell \frac{(2iy)^{-1-n-\ell}}{1+n+\ell}.$$

This implies the result. $\square$

Because of Lemma 2.4, it is natural to complete $r_{f,2}$ by substracting this "lower-order" non-holomorphic function to obtain

$$\widehat{r}_{f,2} := r_{f,2} - \widetilde{r}_{f,2}.$$

Lemma 2.2 and Proposition 2.3 suggest, by comparison with Lemma 2.1, that $\widehat{r}_{f,2}$ can be viewed as an analogue of the period polynomial associated to non-critical values. In the next section, we will show that this interpretation can be formalized in a way that justifies the name *mock period function* for $r_{f,2}$.

3. Mock period functions

One of the reasons that the theory of periods has been so successful in proving important results about the values of L-functions is that they satisfy relations that allow us to view them as elements of a space with a rational structure. This space is, in effect, the first cohomology group of Eichler cohomology. However, to make the relation with L-functions more immediate we will use the more concrete formulation and notation of [27]. In the last section, we will give a cohomological interpretation of our results.

For $n \in \mathbb{N}$, let V_n denote the space of polynomials of degree at most n acted upon by $|_{-n}$, and set

$$W_n := \left\{P \in V_n; P|_{-n}(1+S) = P|_{-n}\left(1+U+U^2\right) = 0\right\}.$$

The period polynomial r_f associated to $f \in S_k$ belongs to W_{k-2} (cf. [27]). According to the well-known Eichler-Shimura Isomorphism (cf. [27] and the references therein), the polynomials characterize the entire space.

Theorem 3.1. *(Eichler-Shimura Isomorphism) Let k be an even positive integer. Then for each $P \in W_{k-2}$ there exists a unique pair $(f,g) \in S_k \times S_k$ and $c \in \mathbb{C}$ such that*

$$P(z) = r_f(z) + r_g(-z) + c\left(z^{k-2}-1\right).$$

Remark 2. Usually, the second term is written as $\overline{r_g(\bar{z})}$, that is the polynomial obtained by replacing each coefficient of the polynomial r_g with its conjugate. However, this may be rewritten as

$$\overline{r_g(\overline{z})} = \int_0^{i\infty} \overline{g(w)}(\overline{w}-z)^{k-2}d\overline{w} = -\int_0^{i\infty} \overline{g(-\overline{w})}(-w-z)^{k-2}dw = -r_{g^c}(-z). \tag{3.1}$$

Recall that $g^c(z) := \overline{g(-\overline{z})} \in S_k$.

We will show that there is a space similar to W_{k-2} within which the completed period-like functions $\widehat{r}_{f,2}$ live. We first recall the operator $\xi_k := 2iy^k\overline{\frac{d}{d\overline{z}}}$ ($y :=\mathrm{Im}(z)$). This map satisfies $\xi_k(f|_k\gamma) = (\xi_k f)|_{2-k}\gamma$ for all $\gamma \in \Gamma$, and thus maps weight k automorphic objects to weight $2-k$ automorphic objects. We then set

$$W_{k,2} := \left\{\mathcal{P} : \mathfrak{H} \to \mathbb{C}; \xi_k(\mathcal{P}) \in V_{k-2}; \mathcal{P}|_k\left(1+S\right) = \mathcal{P}|_k\left(1+U+U^2\right) = 0\right\}.$$

This space consists not of polynomials but of functions which become polynomials only after application of the ξ_k-operator.

The next theorem explains in what sense $r_{k,2}$ can be considered a mock period function.

Theorem 3.2. *Let $k \in 2\mathbb{N}$ and $f \in S_k$. Then the function $\widehat{r}_{f,2}$ is an element of $W_{k,2}$.*

Proof: The first condition follows from the identity

$$\xi_k\Big(\widehat{r}_{f,2}(z)\Big) = -2iy^k \overline{\frac{d}{d\overline{z}} \int_{-\overline{z}}^{i\infty} \frac{r_f(w)}{(w+z)^k}\, dw} = (2i)^{1-k} r_{f^c}(z) \in V_{k-2}, \tag{3.2}$$

where for the last equality we used (3.1). The relation

$$\widehat{r}_{f,2}\big|_k (1+S) = 0$$

follows directly from the identity in Proposition 2.3.

To deduce the relation for U we first note that $F_{f,2}|_k T = F_{f,2}$, which follows directly from $f(w+1) = f(w)$. Thus

$$F_{f,2}\big|_k (1-S) = F_{f,2}\big|_k (1-TS) = F_{f,2}\big|_k (1-U)$$

and the claim follows from $U^3 = 1$. □

Remark 3. It is immediate that, if $\xi_k(\mathcal{P}) \in V_{k-2}$, then $\Delta_k(\mathcal{P}) = -\xi_{2-k} \circ \xi_k(\mathcal{P}) = 0$, and thus Theorem 3.2 implies that $\widehat{r}_{f,2}$ is harmonic.

This theorem suggests the name *mock period function* for $r_{f,2}$ as well as the more general

Definition 3.3. A holomorphic function $p_2 : \mathfrak{H} \to \mathbb{C}$ is called a *mock period function* if there exists a $\widetilde{p}_2 \in \oplus_{j=1}^{k-1} y^{-j} V_{k-2}$ such that

$$p_2 + \widetilde{p}_2 \in W_{k,2}.$$

The Eichler-Shimura relations for $\widehat{r}_{f,2}$ proved in Theorem 3.2 are reflected in *mock* Eichler-Shimura relations for $r_{f,2}$.

Theorem 3.4. *We have*

$$r_{f,2}(z)\Big|_k (1+S) = \int_0^{i\infty} \frac{r_f(w)}{(w+z)^k}\, dw,$$

$$r_{f,2}(z)\Big|_k \Big(1+U+U^2\Big) = \int_{-1}^{i\infty} \frac{r_f(w)}{(w+z)^k}\, dw + \int_{-1}^{0} \frac{r_f|_{2-k}\widetilde{U}(w)}{(w+z)^k}\, dw$$

with $\widetilde{U} := \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right) = SU^2S^{-1}$.

Proof: By (2.1) and Theorem 3.2 it suffices to consider the action of $1+S$ and $1+U+U^2$ on $\widetilde{r}_{f,2}$ only. Further, since $r_f \in W_{k-2}$, we have

$$r_f\Big|_{2-k}(1+S) = r_f\Big|_{2-k}\Big(1+U+U^2\Big) = 0. \tag{3.3}$$

For the first identity we have by (3.3)

$$\begin{aligned} \widetilde{r}_{f,2}(z)\Big|_k S &= z^{-k} \int_{\frac{1}{\overline{z}}}^{i\infty} \frac{r_f(w)}{\left(w-\frac{1}{z}\right)^k}\, dw \\ &= \left(\int_{-\overline{z}}^{i\infty} - \int_0^{i\infty}\right) \frac{r_f|_{2-k}S(w)}{(w+z)^k}\, dw = -\widetilde{r}_{f,2}(z) + \int_0^{i\infty} \frac{r_f(w)}{(w+z)^k}\, dw. \end{aligned}$$

To prove the second identity, we observe that (3.3) implies that

$$r_f\Big|_{2-k}\Big(1+\widetilde{U}+\widetilde{U}^2\Big) = 0. \tag{3.4}$$

The change of variables $w \to \widetilde{U}w$ gives

$$\widetilde{r}_{f,2}(z)\Big|_k U = \int_{-\overline{z}}^{0} \frac{r_f|_{2-k}\widetilde{U}(w)}{(z+w)^k}\, dw.$$

Likewise, the change of variables $w \to \widetilde{U}^2 w$ yields

$$\widetilde{r}_{f,2}(z)\Big|_k U^2 = \int_{-\overline{z}}^{-1} \frac{r_f|_{2-k}\widetilde{U}^2(w)}{(w+z)^k}\, dw.$$

Thus

$$\begin{aligned}\widetilde{r}_{f,2}(z)\Big|_k \left(1+U+U^2\right) = \int_{-\overline{z}}^{i\infty} &\frac{r_f|_{2-k}\left(1+\widetilde{U}+\widetilde{U}^2\right)(w)}{(w+z)^k}\, dw \\ &- \int_0^{i\infty} \frac{r_f|_{2-k}\widetilde{U}(w)}{(z+w)^k}\, dw - \int_{-1}^{i\infty} \frac{r_f|_{2-k}\widetilde{U}^2(w)}{(w+z)^k} dw.\end{aligned}$$

Applying (3.4) we obtain the claim. □

4. Sesquiharmonic Maass forms

In this section, we introduce new automorphic objects related to non-critical values of L-functions.

Definition 4.1. A real-analytic function $\mathcal{F} : \mathfrak{H} \to \mathbb{C}$ is called a *sesquiharmonic Maass form of weight k* if the following conditions are satisfied:

i) We have for all $\gamma \in \Gamma$ that $\mathcal{F}|_k\gamma = \mathcal{F}$.
ii) We have that $\Delta_{k,2}(\mathcal{F}) = 0$.
iii) The function $\mathcal{F}$ has at most linear exponential growth at infinity.

We denote the space of such functions by $H_{k,2}$. The subspace of harmonic weak Maass forms, i.e., these sesquiharmonic forms $\mathcal{F}$ that satisfy

$$\Delta_k(\mathcal{F}) = -\xi_{2-k} \circ \xi_k(\mathcal{F}) = 0$$

is denoted by H_k. Our definition in particular implies that

$$\xi_k\left(H_{k,2}\right) \subset H_{2-k}.$$

The holomorphic differential $D := \frac{1}{2\pi i}\frac{d}{dz}$ plays a role originating in Bol's identity. It is well-known that (see [10])

$$\xi_{2-k}\left(H_{2-k}\right) \subset M_k^!, \qquad D^{k-1}\left(H_{2-k}\right) \subset M_k^!.$$

Here, $M_k^!$ denotes the space of weakly holomorphic modular form, i.e., those meromorphic modular forms whose poles may only lie at the cusps. This suggests the following distinguished subspaces.

Definition 4.2. For $k \in 2\mathbb{N}$, set

i) $H_{2-k}^+ := \{f \in H_{2-k}; D^{k-1}(f) \in S_k\}$ and $H_{2-k}^- := \{f \in H_{2-k}; \xi_{2-k}(f) \in S_k\}$,
ii) $H_{k,2}^+ := \{f \in H_{k,2}; \xi_k(f) \in H_{2-k}^+\}$.

Employing the theory of Poincaré series, we will prove that the restriction of ξ_k on $H_{k,2}^+$ surjects onto H_{2-k}^+. In general, for functions φ that are translation invariant, we define the following Poincaré series

$$\mathcal{P}_k(\varphi; z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi\Big|_k \gamma(z) \tag{4.1}$$

whenever this series converges absolutely. Here, Γ_∞ is the set of translations in Γ. For $k > 2$, the classical Poincaré series, spanning S_k for $m > 0$, are in this notation

$$P_k(m; z) := \mathcal{P}_k\left(q^m; z\right).$$

For all $m \in \mathbb{Z}\setminus\{0\}$, the *Maass Poincaré series* are defined by [23]

$$\mathbb{P}_k(m, s; z) := \mathcal{P}_k\left(\varphi_{m,s}; z\right)$$

with

$$\varphi_{m,s}(z) := \mathcal{M}_s^k(4\pi my)e(mx),$$

Here, $e(x) := e^{2\pi i x}$ and

$$\mathcal{M}_s^k(u) := |u|^{-\frac{k}{2}} M_{\operatorname{sgn}(u)\frac{k}{2}, s-\frac{1}{2}}\big(|u|\big),$$

where $M_{\nu,\mu}$ is the usual M-Whittaker function with the integral representation

$$M_{\mu,\nu}(y) = y^{\nu+\frac{1}{2}} e^{\frac{y}{2}} \frac{\Gamma(1+2\nu)}{\Gamma\left(\nu+\mu+\frac{1}{2}\right)\Gamma\left(\nu-\mu+\frac{1}{2}\right)} \int_0^1 t^{\nu+\mu-\frac{1}{2}}(1-t)^{\nu-\mu-\frac{1}{2}} e^{-yt}\, dt \tag{4.2}$$

for $\operatorname{Re}\left(\nu \pm \mu + \frac{1}{2}\right) > 0$. Using that as $y \to 0$

$$\mathcal{M}_s^k(y) = O\left(y^{\operatorname{Re}(s)-\frac{k}{2}}\right), \tag{4.3}$$

we see that the series $\mathbb{P}_k(m, s; z)$ converges absolutely for $\operatorname{Re}(s) > 1$ and satisfies

$$\Delta_k\left(\mathbb{P}_k(m, s; z)\right) = \left(s(1-s) + \frac{1}{4}\left(k^2 - 2k\right)\right)\mathbb{P}_k(m, s; z). \tag{4.4}$$

In particular, the Poincaré series is annihilated for $s = \frac{k}{2}$ or $s = 1 - \frac{k}{2}$ (depending on the range of absolute convergence). Moreover, for $m > 0$ and $k \geq 2$, we have

$$D^{k-1}\left(\mathbb{P}_{2-k}\left(m, \frac{k}{2}; z\right)\right) = -(k-1)!m^{k-1}P_k(m; z) \tag{4.5}$$

(see, e.g. [5]) and

$$\xi_{2-k}\left(\mathbb{P}_{2-k}\left(-m, \frac{k}{2}; z\right)\right) = (k-1)(4\pi m)^{k-1}P_k(m; z) \tag{4.6}$$

(see, e.g. Theorem 1.1 (2) of [9]). This implies

$$\mathbb{P}_{2-k}\left(m, \frac{k}{2}; z\right) \in H_{2-k}^+, \qquad \mathbb{P}_{2-k}\left(-m, \frac{k}{2}; z\right) \in H_{2-k}^-.$$

In fact, the Poincaré series span the respective spaces H_{2-k}^+ and H_{2-k}^-. For the space H_k^- this follows from Remark 3.10 of [10]. For the space H_k^+ one may argue analogously by using the flipping operator [5], which gives a bijection between the two spaces.

For $k > 0$, we then set

$$\mathbb{P}_{k,2}(m; z) := \mathcal{P}_k\left(\psi_m; z\right)$$

with

$$\psi_m(z) := \frac{d}{ds}\left[\mathcal{M}_s^k(4\pi m y)\right]_{s=\frac{k}{2}} e(mx).$$

Differentiation in s only introduces logarithms and thus, using (4.3), we can easily see that, for $\mathrm{Re}(s) > 1$ and for every $\epsilon > 0$, the derivative is $O(y^{\mathrm{Re}(s)-\epsilon-k/2})$, and thus, as $y \to 0$, we find $\psi_m(z) = O(y^{-\epsilon})$. Thus for all nonzero integers m, and $k > 0$, $\mathbb{P}_{k,2}(m; z)$ is absolutely convergent.

One could further explicitly compute the Fourier expansion of $\mathbb{P}_{k,2}$ but for the purposes of this paper, this is not required.

Theorem 4.3. *For $m \in \mathbb{N}$, the function $\mathbb{P}_{k,2}(-m; z)$ is an element of $H_{k,2}^{+}$ and satisfies:*

$$\xi_k\left(\mathbb{P}_{k,2}(-m; z)\right) = (4\pi m)^{1-k}\mathbb{P}_{2-k}\left(m, \frac{k}{2}; z\right), \tag{4.7}$$

$$D^{k-1} \circ \xi_k\left(\mathbb{P}_{k,2}(-m; z)\right) = -(k-1)!(4\pi)^{k-1}P_k(m; z). \tag{4.8}$$

In particular, the map

$$\xi_k \colon H_{k,2}^{+} \to H_{2-k}^{+}$$

is surjective.

Proof: Due to the absolute convergence of the series, the transformation law is satisfied by construction.

To verify the (at most) linear exponential growth at infinity of $\mathbb{P}_{k,2}(m; z)$ we recall that $M_{\mu,\nu}$ has at most linear exponential growth as $y \to \infty$ (cf. [32], (13.14.20)). We further note that this also holds for its derivative in s and thus $\psi_m(z)$ too, because differentiation in s only introduces logarithms. Therefore, since $\mathrm{Im}(\gamma y) \to 0$ as $y \to \infty$ whenever $\gamma \neq 1$, we have

$$\mathbb{P}_{k,2}(m; z) \ll |\psi_m(z)| + y^{-\frac{k}{2}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma - \{1\}} \mathrm{Im}(\gamma z)^{-\epsilon + \frac{k}{2}}.$$

This together with the well-known polynomial growth of Eisenstein series at the cusps implies the claim.

To prove (4.7) and (4.8), and thus the annihilation under $\Delta_{k,2}$, we first note that ξ_k commutes with the group action of Γ and therefore we only have to compute

$$\begin{aligned} &\xi_k\left(\frac{d}{ds}\left[\mathcal{M}_s^k(-4\pi m y)e(-mx)\right]_{s=\frac{k}{2}}\right) \\ &= y^k(4\pi m)\overline{q}^{-m}\frac{d}{ds}\left[\frac{d}{dy}\left[\mathcal{M}_{s+\frac{k}{2}}^k(-y)e^{-\frac{y}{2}}\right]_{y=4\pi m y}\right]_{s=0}. \end{aligned} \tag{4.9}$$

Notice that we do not need to conjugate the internal function because upon differentiation at $s = 0$ we obtain a real function. The integral representation (4.2) implies for $y > 0$

$$\mathcal{M}_{s+\frac{k}{2}}^k(-y)e^{-\frac{y}{2}} = \frac{y^s\Gamma(2s+k)}{\Gamma(s)\Gamma(s+k)}\int_0^1 t^{s-1}(1-t)^{s+k-1}e^{-yt}\,dt$$

which, in turn, gives that

$$\frac{d}{dy}\left(\mathcal{M}^k_{s+\frac{k}{2}}(-y)e^{-\frac{y}{2}}\right)$$

$$= \frac{s}{y}\cdot y^{-\frac{k}{2}} M_{-\frac{k}{2},s+\frac{k}{2}-\frac{1}{2}}(y)e^{-\frac{y}{2}} - \frac{y^s\Gamma(2s+k)}{\Gamma(s)\Gamma(s+k)}\int_0^1 t^s(1-t)^{s+k-1}e^{-yt}\,dt$$

$$= sy^{-\frac{k}{2}-1}M_{-\frac{k}{2},s+\frac{k}{2}-\frac{1}{2}}(y)e^{-\frac{y}{2}} - \frac{s}{2s+k}y^{-\frac{k}{2}-\frac{1}{2}}M_{\frac{1}{2}-\frac{k}{2},s+\frac{k}{2}}(y)e^{-\frac{y}{2}}.$$

Differentiating with respect to s and setting $s=0$ gives ([35], (2.5.2))

$$y^{-\frac{k}{2}-1}e^{-\frac{y}{2}}\frac{1}{k}\left(kM_{-\frac{k}{2},\frac{k}{2}-\frac{1}{2}}(y) - \sqrt{y}M_{\frac{1}{2}-\frac{k}{2},\frac{k}{2}}(y)\right) = y^{-\frac{k}{2}-1}e^{-\frac{y}{2}}M_{1-\frac{k}{2},\frac{k}{2}-\frac{1}{2}}(y) = e^{-\frac{y}{2}}y^{-k}\mathcal{M}^{2-k}_{\frac{k}{2}}(y).$$

Thus

$$\xi_k\left(\frac{d}{ds}\left[\mathcal{M}^k_s(-4\pi my)e(-mx)\right]_{s=\frac{k}{2}}\right) = (4\pi m)^{1-k}\mathcal{M}^{2-k}_{\frac{k}{2}}(4\pi my)e(mx),$$

which implies (4.7). ¿From (4.7) we may also deduce that $\Delta_{k,2}\Big(\mathbb{P}_{k,2}(m;z)\Big) = 0$. Equality (4.5) implies (4.8). Since, as mentioned above the functions $\mathbb{P}_{2-k}(m,k/2;z)$ span H^+_{2-k}, (4.7) implies the last assertion. □

Since we have a basis of S_k consisting of Poincaré series, Theorem 4.3 implies

Corollary 4.4. *For $f \in S_k$ there exists $\mathcal{M}_{f,2} \in H^+_{k,2}$ such that*

$$D^{k-1}\circ\xi_k\left(\mathcal{M}_{f,2}\right) = f.$$

To state and prove our second main theorem we analyze the Fourier expansion of $\mathcal{F}$ in $H^+_{k,2}$. Since $F := \xi_k(\mathcal{F}) \in H^+_{2-k}$, it has a Fourier expansion of the form

$$F(z) = \sum_{n\geq 0}\widetilde{a}(n)q^n + \sum_{\substack{n\gg-\infty\\ n\neq 0}}\widetilde{b}(n)\Gamma(k-1,4\pi ny)q^{-n}$$

for some $\widetilde{a}(n), \widetilde{b}(n) \in \mathbb{C}$ and $\Gamma(s,y)$ the incomplete gamma function (see, for instance, [10]). The first summand is called the *holomorphic part* and the second the *non-holomorphic part* of F, and we denote them by F^+ and F^-, respectively. A direct calculation implies that for some $a(n), b(n), c(n), d(0) \in \mathbb{C}$

$$\mathcal{F}(z) = \sum_{n\gg-\infty}a(n)q^n + \sum_{n>0}b(n)\Gamma(1-k,4\pi ny)q^{-n} + \sum_{\substack{n\gg-\infty\\ n\neq 0}}c(n)\mathbf{\Gamma}_{k-1}(4\pi ny)q^n + d(0)y^{1-k}, \tag{4.10}$$

where for $y>0$, we define

$$\mathbf{\Gamma}_s(y) := \int_y^\infty \Gamma(s,t)t^{-s}e^t\frac{dt}{t}.$$

Similarly for $y<0$, we integrate from $-\infty$ instead of ∞. We call the first summand of the right hand side of (4.10) the *holomorphic part*, the second the *harmonic part*, and the third the *non-harmonic part* of $\mathcal{F}$ and we denote them by $\mathcal{F}^{++}$, $\mathcal{F}^{+-}$, and $\mathcal{F}^{--}$ respectively. We note that for $\mathcal{F}^{++}\neq 0, \mathcal{F}^{+-}\neq 0$, and $\mathcal{F}^{--}\neq 0$, we have

$$\xi_k\left(\mathcal{F}^{++}\right) = 0, \quad \xi_k\left(\mathcal{F}^{+-}\right)\neq 0 \quad \xi_k\left(\mathcal{F}^{--}\right)\neq 0, \quad \xi_k\left(y^{1-k}\right)\neq 0, \tag{4.11}$$

$$\xi_{2-k} \circ \xi_k \left(\mathcal{F}^{+-}\right) = 0, \quad \xi_{2-k} \circ \xi_k \left(\mathcal{F}^{--}\right) \neq 0, \quad \xi_{2-k} \circ \xi_k \left(y^{1-k}\right) = 0, \tag{4.12}$$

$$D^{k-1} \circ \xi_k \left(\mathcal{F}^{+-}\right) \neq 0, \quad D^{k-1} \circ \xi_k \left(\mathcal{F}^{--}\right) = 0, \quad D^{k-1} \circ \xi_k \left(y^{1-k}\right) = 0. \tag{4.13}$$

With this terminology and notation we have

Theorem 4.5. *For $f \in S_k$, there is a $\mathcal{M}_{f,2} \in H^+_{k,2}$ such that $D^{k-1} \circ \xi_k \left(\mathcal{M}_{f,2}\right) = -\frac{(k-2)!}{(4\pi)^{k-1}} f^c$ and*

$$\widehat{r}_{f,2}(z) = \mathcal{M}^{+-}_{f,2}(z)\Big|_k (S-1).$$

Proof: By equation (2.1),

$$\widehat{r}_{f,2} = F_{f,2}\Big|_k (S-1).$$

By Corollary 4.4, there is a $\mathcal{M}_{f,2} \in H^+_{k,2}$ such that

$$D^{k-1} \circ \xi_k \left(\mathcal{M}_{f,2}\right) = -\frac{(k-2)!}{(4\pi)^{k-1}} f^c. \tag{4.14}$$

We claim that

$$F_{f,2} = \mathcal{M}^{+,-}_{f,2}.$$

A direct computation inserting the Fourier expansion of f gives that $F_{f,2}(z)$ has a Fourier expansion of the form

$$\sum_n b(n)\Gamma(1-k, 4\pi n y) q^{-n}.$$

Next

$$\begin{aligned} \xi_k \left(F_{f,2}(z)\right) = (2i)^{1-k} F^c_f(z) &= (2i)^{1-k} \int_{-\overline{z}}^{i\infty} \overline{f(w)} (z+\overline{w})^{k-2}\, d\overline{w} \\ &= -(2i)^{1-k} \int_z^{i\infty} f^c(w)(z-w)^{k-2} dw. \end{aligned}$$

This implies that

$$D^{k-1} \circ \xi_k \Big(F_{f,2}\Big) = -\frac{(k-2)!}{(4\pi)^{k-1}} f^c.$$

Thus by (4.14),

$$D^{k-1} \circ \xi_k \Big(F_{f,2} - \mathcal{M}_{f,2}\Big) = 0.$$

By (4.11) and (4.13), non-zero expansions in incomplete gamma functions are not in the kernel of $D^{k-1} \circ \xi_k$. This implies that $F_{f,2} - \mathcal{M}^{+-}_{f,2}$=0. □

5. A Mock Eichler-Shimura isomorphism

In this section, we will show an Eichler-Shimura type theorem for harmonic period functions of positive weight. We first note that

$$\xi_k(W_{k,2}) \subset W_{k-2}, \tag{5.1}$$

because ξ_k is compatible with the group action of Γ.

Fix $P \in W_{k,2}$. Then (5.1) and Theorem 3.1 imply that there exist $f, g \in S_k$ and $a \in \mathbb{C}$ such that

$$\xi_k(P(z)) = r_f(z) + r_g(-z) + a\left(z^{k-2} - 1\right). \tag{5.2}$$

This can be viewed as a differential equation for P, and we will now describe the general solution in $W_{k,2}$. To find a preimage of the second summand we require regularized integrals as they are defined, for instance, by Fricke in his upcoming PhD thesis [21].

Consider a function $f : \mathbb{H} \to \mathbb{C}$ that is continuous. Assume that there is a $c \in \mathbb{R}^+$ such that

$$f(z) = O\left(e^{c\,\mathrm{Im}(z)}\right) \tag{5.3}$$

uniformly in $\mathrm{Re}(z)$ as $\mathrm{Im}(z) \to \infty$. Then, for each $z_0 \in \mathfrak{H}$, the integral

$$\int_{z_0}^{i\infty} e^{uw} f(w)\, dw$$

(where the path of integration lies within a vertical strip) is convergent for $u \in \mathbb{C}$ with $\mathrm{Im}(u) \gg 0$. If it has an analytic continuation to $u = 0$, we define the *regularized integral*

$$R.\int_{z_0}^{i\infty} f(w)\, dw := \left[\int_{z_0}^{i\infty} e^{uw} f(w)\, dw\right]_{u=0},$$

where the right hand side means the value at $u = 0$ of the analytic continuation of the integral. Similarly, we define integrals at other cusps $\mathfrak{a}$. Specifically, suppose that $\mathfrak{a} = \sigma_{\mathfrak{a}}(i\infty)$ for a scaling matrix $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{Z})$. If $f(\sigma_{\mathfrak{a}} z)$ satisfies (5.3), then we define

$$R.\int_{z_0}^{\mathfrak{a}} f(w)\, dw := R.\int_{\sigma_{\mathfrak{a}}^{-1} z_0}^{i\infty} f\big|_2 \gamma(w)\, dw.$$

For cusps $\mathfrak{a}, \mathfrak{b}$ we define:

$$R.\int_{\mathfrak{a}}^{\mathfrak{b}} f(w)\, dw := R.\int_{z_0}^{\mathfrak{b}} f(w)\, dw + R.\int_{\mathfrak{a}}^{z_0} f(w)\, dw \tag{5.4}$$

for any $z_0 \in \mathfrak{H}$. An easy calculation shows:

Lemma 5.1. *The integral $R.\int_{\mathfrak{a}}^{\mathfrak{b}} f(w)\, dw$ as defined in* (5.4) *is independent of $z_0 \in \mathfrak{H}$.*

By Theorem 1.2, there exists a harmonic Maass form M_f such that

$$r_f(-z) = M_f^+\Big|_k (1 - S)(z). \tag{5.5}$$

Set

$$\mathcal{F}_{f,2}^*(z) := R.\int_{-\overline{z}}^{i\infty} \frac{M_f^+(w)}{(w+z)^k}\, dw,$$

$$r_{f,2}^*(z) := R.\int_{0}^{i\infty} \frac{M_f^+(w)}{(w+z)^k}\, dw\,\Big|_k S,$$

$$\widetilde{r}_{f,2}^*(z) := \int_{-\overline{z}}^{i\infty} \frac{r_f(-w)}{(w+z)^k}\, dw,$$

$$\widehat{r}_{f,2}^*(z) := r_{f,2}^*(z) - \widetilde{r}_{f,2}^*(z).$$

We note that, by definition,

$$M_f^+(z) = \sum_{n=N}^{0} a_n e^{2\pi i n z} + O\left(e^{-2\pi y}\right) \quad \text{for some } N < 0, \text{ as } y \to \infty.$$

We insert the above Fourier expansion into $\mathcal{F}^*_{f,2}$ and integrate each of the terms separately. Terms with $n \geq 0$ do not require regularization. For terms with $n < 0$ we obtain a linear combination of incomplete gamma functions of the form $\Gamma(\ell, z)$ ($\ell \in \mathbb{Z}$, $z \neq 0$). These functions can be analytically continued, from which we may deduce that the integrals can be extended to $u = 0$. Therefore, the regularized integrals are well-defined. The integral $r^*_{f,2}$ ist treated analogously.

We also note that $\widetilde{r}^*_{f,2}$ does not require regularization, since $r_f(-z) \in V_{k-2}$. We easily compute, using (3.1), that

$$\xi_k\left(\widehat{r}^*_{f,2}(z)\right) = (2i)^{1-k} r_{f^c}(-z). \tag{5.6}$$

We claim that a special solution in $W_{k,2}$ to (5.1) is then given by

$$R^*_{f,2}(z) := -(2i)^{k-1}\widehat{r}_{f^c,2}(z) - (2i)^{k-1}\widehat{r}^*_{g^c,2}(z) + \overline{a}(2i)^{k-1}\left(\int_{-\bar{z}}^{i\infty} \frac{dw}{(w+z)^k}\right)\Big|_k (1-S). \tag{5.7}$$

It is clear by (3.2), (5.6) and the identity

$$\xi_k\left(\int_{-\bar{z}}^{i\infty} \frac{dw}{(w+z)^k}\right) = (2i)^{1-k} \tag{5.8}$$

that $R^*_{f,2}$ satisfies (5.2).

By Theorem 3.2, the function $\widehat{r}_{f^c,2}$ is an element of $W_{k,2}$. The same is true for $\widehat{r}^*_{f,2}$:

Lemma 5.2. *We have*

$$\mathcal{F}^*_{f,2}\Big|_k (S-1)(z) = \widehat{r}^*_{f,2}(z).$$

In particular, $\widehat{r}^*_{f,2} \in W_{k,2}$.

Proof: We first note, with Lemma 5.1 and the definition of regularized integrals, that

$$\begin{aligned} r^*_{f,2}|_k S &= \left[\int_{-\bar{z}}^{i\infty} \frac{e^{wu} M^+_f(w)\, dw}{(w+z)^k}\right]_{u=0} - \left[\int_{1/\bar{z}}^{i\infty} \frac{e^{wu} M^+_f(-1/w)\, d(-1/w)}{(-1/w+z)^k}\right]_{u=0} \\ &= \left[\int_{-\bar{z}}^{i\infty} \frac{e^{wu} M^+_f(w)\, dw}{(w+z)^k}\right]_{u=0} - \left[\int_{-\bar{z}}^{0} \frac{e^{-u/w} M^+_f(w)\, dw}{(w+z)^k}\right]_{u=0}. \end{aligned} \tag{5.9}$$

On the other hand, to compute $\mathcal{F}^*_{f,2}|_k(S-1)(z) = \mathcal{F}^*_{f,2}(-1/z)z^{-k} - \mathcal{F}^*_{f,2}(z)$ we recall that, by definition, this is the value of u at 0 of the analytic continuation of

$$\int_{1/\bar{z}}^{i\infty} \frac{e^{wu} M^+_f(w)\, dw}{(wz-1)^k} - \int_{-\bar{z}}^{i\infty} \frac{e^{wu} M^+_f(w)\, dw}{(w+z)^k}.$$

For $\mathrm{Im}(u) \gg 0$, with (5.5) this equals

$$\begin{aligned} &\int_{-\bar{z}}^{0} \frac{e^{-u/w} M^+_f(-1/w)\, d(-1/w)}{(-z/w-1)^k} - \int_{-\bar{z}}^{i\infty} \frac{e^{wu} M^+_f(w)\, dw}{(w+z)^k} \\ &\qquad = \int_{-\bar{z}}^{0} \frac{e^{-u/w} M^+_f(w)\, dw}{(z+w)^k} - \int_{-\bar{z}}^{0} \frac{e^{-u/w} r_f(-w)\, dw}{(z+w)^k} - \int_{-\bar{z}}^{i\infty} \frac{e^{wu} M^+_f(w)\, dw}{(w+z)^k}. \end{aligned} \tag{5.10}$$

Because of (3.3), the second integral of (5.10) equals

$$\int_{-1/\bar{z}}^{i\infty} \frac{e^{wu} r_f(-1/w) w^k \, dw}{(zw-1)^k} = -\int_{1/\bar{z}}^{i\infty} \frac{e^{wu} r_f(w) \, dw}{(zw-1)^k}$$

This is analytic at $u = 0$ with value $\tilde{r}^*_{f,2}|_k S(z)$. Therefore, with analytic continuation and (5.9), (5.10) gives

$$\mathcal{F}^*_{f,2}|_k(S-1) = -r^*_{f,2}|_k S + \widetilde{r}^*_{f,2}\big|_k S = -\widehat{r}^*_{f,2}\big|_k S,$$

which implies the result. □

That the third term of (5.7) is an element of $W_{k,2}$ follows directly from (5.8) and the invariance of the integral under T.

Therefore, the general solution of (5.2) is

$$-(2i)^{k-1}\left(\widehat{r}_{f^c,2}(z) + \widehat{r}^*_{g^c,2}(z) - \overline{a}\int_{-\overline{z}}^{i\infty} \frac{dw}{(w+z)^k}\bigg|_k (1-S) + G(z)\right),$$

where G is a holomorphic function on $\mathfrak{H}$. The last summand G must be annihilated by $1+S$ and $1+U+U^2$ in terms of $|_k$, because all the others satisfy the Eichler-Shimura relations. This implies that $G = H|_k(S-1)$ for some translation invariant holomorphic function H. Indeed, this follows from $H^1(\Gamma, \mathcal{A}) = 0$, where $\mathcal{A}$ is a the module of holomorphic functions on $\mathfrak{H}$ (see equation (5.3) of [25] citing [29]).

Set

$$U_{k,2} := \Big(\mathcal{O}(\mathfrak{H}) + \big\{f \in \oplus_{j=1}^{k-1} y^{-j} V_{k-2};\, \xi_k(f) \in V_{k-2}\big\}\Big) \cap \{f : \mathfrak{H} \to \mathbb{C}; f|_k T = f\},$$

where $\mathcal{O}(\mathfrak{H})$ is the space of holomorphic functions on $\mathfrak{H}$. We can then complete the proof of

Theorem 5.3. *The map $\phi : S_k \oplus S_k \to W_{k,2}$ defined by*

$$\phi(f,g) := \widehat{r}_{f^c,2} + \widehat{r}^*_{g^c,2}$$

induces an isomorphism

$$\overline{\phi}:\ S_k \oplus S_k \cong_{\mathbb{R}} W_{k,2}/V_{k,2},$$

where $V_{k,2} := U_{k,2}|_k(S-1)$.

Proof: We have already shown above that $\overline{\phi}$ is surjective. To show that it is injective, suppose that $P \in \ker(\overline{\phi})$. Then

$$\widehat{r}_{f^c,2} + \widehat{r}^*_{g^c,2} = A|_k(S-1) \tag{5.11}$$

for some $A \in U_{k,2}$. Applying ξ_k on both sides of (5.11), we deduce that $r_f(z) + r_g(-z)$ is an Eichler coboundary. The classical Eichler-Shimura isomorphism (Theorem 3.1) implies that f, g must vanish. □

Remark 4. Since $\big\{f \in \oplus_{j=1}^{k-1} y^{-j} V_{k-2};\, \xi_k(f) \in V_{k-2}\big\}$ does not contain any holomorphic elements, it is isomorphic to V_{k-2}. The corresponding isomorphism is ξ_k.

6. Cohomological interpretation

Theorem 4.5 has a cohomological interpretation which makes apparent the similarity of our construction with the one associated to critical values in [7]. We shall first give a cohomological interpretation of the period polynomials in the context of the results of [7].

We recall the definition of parabolic cohomology in our setting. For $m \in \mathbb{Z}$ and a Γ-submodule V of the space of functions $f : \mathfrak{H} \to \mathbb{C}$ we define

$$Z_p^1(\Gamma, V) := \big\{ g : \Gamma \to V; g(\gamma\delta) = g(\gamma)|_m \delta + g(\delta) \text{ and } \\ g(T) = h|_m(T-1) \text{ for some } h \in V \big\},$$

$$B_p^1(\Gamma, V) = B^1(\Gamma, V) := \big\{ g : \Gamma \to V; \text{for some } h \in V, \\ g(\gamma) = h|_m(\gamma - 1) \text{ for all } \gamma \in \Gamma \big\},$$

and

$$H_p^1(\Gamma, V) := Z_p^1(\Gamma, V)/B_p^1(\Gamma, V).$$

A basic map in the theory of period polynomials is

$$\rho : S_k \to H_p^1(\Gamma, V_{k-2}).$$

It assigns to $f \in S_k$ the class of a cocycle ϕ_f determined by $\phi_f(T) = 0$ and $\phi_f(S) = r_f(-z)$. We further consider the Γ-module $\mathcal{O}^*(\mathfrak{H})$ of holomorphic functions $F : \mathfrak{H} \to \mathbb{C}$ of at most linear exponential growth at the cusps. The group Γ acts on $\mathcal{O}^*(\mathfrak{H})$ via $|_{2-k}$. Then the natural injection i of V_{k-2} into $\mathcal{O}^*(\mathfrak{H})$ induces a map

$$i^* : H_p^1\left(\Gamma, V_{k-2}\right) \to H_p^1\left(\Gamma, \mathcal{O}^*(\mathfrak{H})\right).$$

Theorem 1.1 of [7] states that $r_f(-z)$ is a constant multiple of $F_f^+|_{2-k}(1-S)$ for the holomorphic part F_f^+ of some harmonic Maass form F_f that grows at most linear exponentially at the cusps. This can then be reformulated as:

$$i^* \circ \rho = 0. \tag{6.1}$$

To formulate the analogue of this result in our context and the setting of non-critical values we consider the following Γ-modules, all in terms of the action $|_k$,

i) $\mathcal{H}^*(\mathfrak{H})$ the Γ-module of harmonic functions on $\mathfrak{H}$ of at most linear exponential growth at the cusps.
ii) $\mathcal{V}_{k,2} := \{f : \mathfrak{H} \to \mathbb{C}$ of at most lin. exp. growth at the cusps, $\xi_k(f) \in V_{k-2}\}$.

Because of the compatibility of ξ_k with the slash action, these spaces are Γ-invariant.

According to Theorem 3.2, for each $f \in S_k$, the map ψ_f such that $\psi_f(T) = 0$ and $\psi_f(S) = \widehat{r}_{f,2}$ induces a cocycle with values in $\mathcal{V}_{k,2}$. Therefore, the assignment $f \to \psi_f$ induces a linear map

$$\rho' : S_k \to H_p^1\left(\Gamma, \mathcal{V}_{k,2}\right).$$

Because of Remark 3, there is a natural injection i' from $\mathcal{V}_{k,2}$ to $\mathcal{H}^*(\mathfrak{H})$, and this induces a map:

$$i'^* : H_p^1\left(\Gamma, \mathcal{V}_{k,2}\right) \to H_p^1\left(\Gamma, \mathcal{H}^*(\mathfrak{H})\right).$$

Theorem 4.5 then implies that

Theorem 6.1. *The composition $i'^* \circ \rho'$ is the zero map.*

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