MODULAR FORMS AND PERIOD POLYNOMIALS

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ABSTRACT. We study the space of period polynomials associated with modular forms of integral weight for finite index subgroups of the modular group. For the modular group, this space is endowed with a pairing, corresponding to the Petersson inner product on modular forms via a formula of Haberland, and with an action of Hecke operators, defined algebraically by Zagier. We generalize Haberland's formula to (not necessarily cuspidal) modular forms for finite index subgroups, and we show that it conceals two stronger formulas. We extend the action of Hecke operators to period polynomials of modular forms, we show that the pairing on period polynomials appearing in Haberland's formula is nondegenerate, and we determine the adjoints of Hecke operators with respect to it. We give a few applications for $\Gamma_1(N)$: an extension of the Eichler-Shimura isomorphism to the entire space of modular forms; the determination of the relations satisfied by the even and odd parts of period polynomials associated with cusp forms, which are independent of the period relations; and an explicit formula for Fourier coefficients of Hecke eigenforms in terms of their period polynomials, generalizing the Coefficients Theorem of Manin.

1. INTRODUCTION

Let Γ be a finite index subgroup of $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$, and let $S_k(\Gamma)$ be the space of cusp forms of integer weight $k \geqslant 2$ for Γ. Let V_w be the Γ-module of complex polynomials of degree at most $w = k - 2$. Each form $f \in S_k(\Gamma)$ determines a collection of polynomials $\rho_f : \Gamma \backslash \Gamma_1 \to V_w$ given by

$$\rho_f(A) = \int_0^{i\infty} f|_k A(t)(t - X)^w dt.$$

The object ρ_f belongs to the induced Γ_1-module $\mathrm{Ind}_\Gamma^{\Gamma_1} V_w$, and we call it the (multiple) period polynomial associated to f. The goal of this paper is to investigate the structure of the space of period polynomials, reflecting the Petersson inner product and the Hecke operators on modular forms. Working inside the subspace of period polynomials $W_w^\Gamma \subset \mathrm{Ind}_\Gamma^{\Gamma_1} V_w$, we show that the Petersson product, and the action of Hecke operators, can be stated in a simple way in terms of period polynomials. On an abstract level, this is explained by the fact that the parabolic cohomology class associated to f is completely determined by ρ_f, as reviewed in Section 2 where we restate the Eichler-Shimura isomorphism in terms of period polynomials. Our results can be interpreted as translating the cup product and the action of Hecke operators on cohomology, into a pairing and a Hecke action on the space of period polynomials.

An essential ingredient in our approach is a generalization of a formula of Haberland expressing the Petersson product of two cusp forms for the modular group in terms of a pairing on their period polynomials [Ha83, KZ84]. In Section 3 we show that Haberland's formula can be extended to finite index subgroups of Γ_1. More importantly, using an involution on period polynomials that corresponds to complex conjugation, we show that Haberland's formula splits in two simpler formulas, pairing the opposite sign parts (respectively the same sign parts) of the period polynomials

of the two forms when k is even (respectively when k is odd). For the full modular group, the stronger formulas were proved by different means in [Po11]. When $k = 2$, a generalization of Haberland's formula was given by Merel [Me09], and a proof for finite index subgroups and arbitrary weight was very recently given by Cohen [Co12]. Our proof is simplified by the use of Stokes' theorem on a fundamental domain for $\Gamma(2)$, which clarifies the appearence of the period polynomial pairing in the formula.

The action of Hecke operators on period polynomials was defined algebraically by Zagier for the full modular group [Za90, Za93, CZ93]. It was extended by Diamantis to operators of index coprime with the level for the congruence subgroups $\Gamma_0(N)$ [Di01]. We show in Section 5 that the same elements as in the full level case, which go back to work of Manin [Ma73], have actions on period polynomials that correspond to actions of a large class of double coset operators on modular forms, including Hecke and Atkin-Lehner operators for $\Gamma_1(N)$. We also determine the adjoint of the Hecke action with respect to the pairing on period polynomials appearing in Haberland's formula. The proof of Hecke equivariance given here relies on the generalization of Haberland's formula, and a completely algebraic proof is given in [PP12a].

As an application of the action of Hecke operators on period polynomials, we obtain a generalization of the Coefficients Theorem of Manin, giving the Fourier coefficients of a Hecke eigenform for $\Gamma_1(N)$ in terms of its even period polynomial. We also give a simple proof of the rationality of period polynomials of Hecke eigenforms for $\Gamma_1(N)$ in §5.3. We discuss period polynomials of cusp forms with nontrivial Nebentypus in §5.4. Our results can be used to efficiently compute period polynomials of Hecke eigenforms numerically, as well as Hecke eigenvalues and Petersson norms, and we give an example in §5.5.

We give two applications of the stronger form of Haberland's formula for cusp forms: in Section 6 we prove a decomposition of cusp forms in terms of Poincaré series generators; while in Section 7 we obtain the extra relations satisfied by the even and odd period polynomials of cusp forms, obtained by Kohnen and Zagier in the full level case [KZ84]. For $\Gamma = \Gamma_0(N)$, we characterize those N for which the extra relations involve only the even parts of period polynomials just like in the full level case, that is those N for which the map $\rho^- : S_k(\Gamma) \to (W_w^\Gamma)^-$ is an isomorphism (Prop. 4.4). The extra relations are explicit once the period polynomials of the generators with rational periods are computed, as partially done in [An92], [FY09]. For small N that is enough to give completely explicit relations, and we illustrate this for $\Gamma_0(2)$.

In the last section, we define the space $\widehat{W}_w^\Gamma$ of period polynomials of all modular forms, following the construction for the modular group in [Za91]. We generalize Haberland's formula and its refinement to this larger space, and we show that the pairing appearing in Haberland's formula is nondegenerate on $\widehat{W}_w^\Gamma$. In contrast, on W_w^Γ the radical of this pairing consists of the "coboundary polynomials", of dimension equal to the dimension of the Eisenstein subspace of $M_k(\Gamma)$, as shown in Section 4.

For $\Gamma = \Gamma_1(N)$ and $k > 2$, we show that the plus and minus period polynomial maps extend to isomorphisms $\rho^\pm : M_k(\Gamma) \to (\widehat{W}_w^\Gamma)^\pm$. This can be seen as an extension of the Eichler-Shimura isomorphism to the entire space of modular forms. Surprisingly, when $k = 2$ the two maps are not always isomorphisms: for $\Gamma = \Gamma_0(N)$ with N square free with at least two prime factors, precisely one of the two maps is an isomorphism (Proposition 8.4 and Remark 8.5). In this context we point out that Haberland's formula has been generalized to weakly holomorphic modular forms of full level in [BGKO], and it would be interesting to investigate if the results proved here for modular forms hold in that setting as well.

In §8.3 we extend the action of Hecke operators to the space of period polynomials of all modular forms, and we show that the adjoints of Hecke operators on the larger space $\widehat{W}_w^\Gamma$ are the same as on W_w^Γ. As an application, for $\Gamma = \Gamma_1(N)$ we show that for $(n, N) = 1$

$$\operatorname{Tr}(W_w^\Gamma|_\Delta \widetilde{T}_n) = \operatorname{Tr}(M_k(\Gamma)|T_n) + \operatorname{Tr}(S_k(\Gamma)|T_n)$$

where the $|_\Delta \widetilde{T}_n$ is the action on period polynomials corresponding to the action of the Hecke operator T_n on modular forms. A similar statement holds for modular forms with Nebentypus, and for traces of Atkin-Lehner operators on $\Gamma_0(N)$. This fact is used in upcoming joint work of the second author with Don Zagier to give a simple proof of the Eichler-Selberg trace formula for modular forms on $\Gamma_0(N)$ with Nebentypus.

A second method of obtaining explicit extra relations among periods of cusp forms is sketched in Section 8.4. It generalizes the Γ_1 approach in [KZ84], by using period polynomials of Eisenstein series and Haberland's formula for arbitrary modular forms.

We note that period polynomials are dual to modular symbols, in the sense that the coefficients of period polynomials are values of the integration pairing between modular forms and Manin symbols (the duality between cohomology and homology). The results of this paper are therefore parallel to the modular symbol formalism developed by Merel [Me94], and they also lead to efficient algorithms for modular form computations, as shown in §5.5. An additional structure that we introduce here is the extended pairing on $\widehat{W}_w^\Gamma$, whose nondegeneracy and Hecke equivariance properties are important even if one is interested only in period polynomials of cusp forms. For example, the properties of the extended pairing were used in the computation of the Hecke and Atkin-Lehner traces on W_w^Γ mentioned above, and in the proof of rationality of $\rho_f^\pm$ for newforms $f \in S_k(\Gamma_1(N))$ (Prop. 5.11).

The paper is self-contained, except for using the fact that the dimension of the parabolic cohomology group $H_P^1(\Gamma, V_w)$ equals twice the dimension of $S_k(\Gamma)$, a consequence of the Eichler-Shimura isomorphism.

2. Period polynomials and the Eichler-Shimura isomorphism

The theory of period polynomials for $\Gamma_0(N)$ has been treated in [Sk90, An92, Di01]. We review it here in a general setting, and interpret the Eichler-Shimura isomorphism in terms of period polynomials. We use the properties of the pairing on period polynomials introduced in Section 3 to prove injectivity of the Eichler-Shimura map. In this section we fix notations in use throughout the paper.

Let Γ be a finite index subgroup of $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$, and denote by $\overline{\Gamma} = \Gamma/(\Gamma \cap \{\pm 1\})$ the projectivisation of Γ. Throughout the paper, the weight $k \geqslant 2$ is an integer, and we set $w = k - 2$. Let V_w be the module of complex polynomials of degree at most w, with (right) Γ_1-action by the $|_{-w}$ operator: $P|_{-w}g(z) = P(gz)j(g, z)^w$ where $j(g, z) = cz + d$ for $g = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix} \right) \in \mathrm{GL}_2(\mathbb{R})$. Since this is the only action on polynomials, we will omit the subscript.

Viewing V_w as a Γ-module, let $\widetilde{V}_w^\Gamma$ be the induced Γ_1-module $\mathrm{Ind}_\Gamma^{\Gamma_1}(V_w)$. Since V_w is also a Γ_1-module, we can identify $\widetilde{V}_w^\Gamma$ with the space of maps $P : \Gamma \backslash \Gamma_1 \to V_w$ with Γ_1 action:

$$P|g(A) = P(Ag^{-1})|_{-w}g.$$

By the Shapiro isomorphism, we have $H_P^1(\Gamma, V_w) \simeq H_P^1(\Gamma_1, \widetilde{V}_w^\Gamma)$ (parabolic cohomology groups). For background on Shapiro's lemma and induced modules, see [NSW, p.59].

Letting $J = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$, for any cocycle $\sigma : \Gamma_1 \to V_w^\Gamma$ we have $\sigma(g)|(1 - J) = \sigma(J)|(1 - g)$ for all $g \in \Gamma_1$(which follows from $\sigma(Jg) = \sigma(gJ)$). It follows that the cocycle $\tilde{\sigma} = \frac{\sigma + \sigma|J}{2}$ is in the same cohomology class as σ, where $\sigma|J(g) := \sigma(g)|J$. Since the cocycle $\tilde{\sigma}$ takes values in the subspace

$$V_w^\Gamma := \{P \in \widetilde{V}_w^\Gamma : P|J = P, \text{ that is } P(A) = (-1)^w P(-A)\}$$

we have $H_P^1(\Gamma_1, \widetilde{V}_w^\Gamma) \simeq H_P^1(\Gamma_1, V_w^\Gamma)$ and from now on we will only work inside the space V_w^Γ. Note that when k is even, V_w is both a $\overline{\Gamma}$ and a $\overline{\Gamma}_1$ module, and the space V_w^Γ can be identified with $\mathrm{Ind}_{\overline{\Gamma}}^{\overline{\Gamma}_1}(V_w)$.

Let now $f \in S_k(\Gamma)$, and define a cocycle $\sigma_f : \Gamma_1 \to V_w^\Gamma$ by:

$$\sigma_f(g)(A) = \int_{g^{-1}i\infty}^{i\infty} f|A(t)(t - X)^w dt,$$

where the stroke operator acting on modular forms of weight k is $f|g = f|_k g$ for $g \in GL_2(\mathbb{R})^+$. The action of the coset A is defined by acting with any coset representative; this is independent of the representative chosen since $f|_k \gamma = f$ for $\gamma \in \Gamma$. We will show at the end of this section that σ_f satisfies the cocycle relation

$$\sigma_f(g_1 g_2) = \sigma_f(g_2) + \sigma_f(g_1)|g_2.$$

Let $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$, and let $U = TS$, so that $U^3 = J$. Clearly $\sigma_f(\pm T^n)$ vanishes for $n \in \mathbb{Z}$, and it is easy to see (by a change of variables) that it is a coboundary for other parabolic elements of Γ_1, hence σ_f defines an element $[\sigma_f] \in H_P^1(\Gamma_1, V_w^\Gamma)$. Since T, S and J generate Γ_1, it follows that the cohomology class $[\sigma_f]$ is completely determined by the value $\rho_f = \sigma_f(S) \in V_w^\Gamma$, which is the multiple period polynomial attached to f in the introduction. Using the fact that $\sigma_f(S^2) = \sigma_f(U^3) = \sigma_f(US) = 0$ and the cocycle relation, it follows that ρ_f satisfies the period polynomial relations:

$$\rho_f|(1 + S) = 0, \quad \rho_f|(1 + U + U^2) = 0.$$

We also have $\rho_f(-A) = (-1)^w \rho_f(A)$, so $\rho_f|J = \rho_f$. Therefore the image of the map $f \to \rho_f$ is contained in the subspace[1]

$$W_w^\Gamma = \{P \in V_w^\Gamma \; : \; P|(1 + S) = 0, \;\; P|(1 + U + U^2) = 0, \;\; P|J = P\}$$

whose elements we call *period polynomials* (each element is in fact a collection of $[\Gamma_1 : \Gamma]$ polynomials belonging to V_w).

In fact, setting $C_w^\Gamma = \{P|(1 - S) : P \in V_w^\Gamma, \;\; P|T = P\} \subset W_w^\Gamma$, we have an isomorphism

(2.1) $$H_P^1(\Gamma_1, V_w^\Gamma) \simeq W_w^\Gamma / C_w^\Gamma,$$

obtained by choosing representative cocycles σ such that $\sigma(T) = \sigma(J) = 0$, and sending $[\sigma]$ to $\sigma(S) \in W_w^\Gamma$. The space C_w^Γ is the image of coboundaries, and we show in Lemma 4.2 that its dimension equals the dimension of the Eisenstein subspace $\mathcal{E}_k(\Gamma)$ of $M_k(\Gamma)$.

Assume now that Γ is normalized by $\epsilon = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$. The matrix ϵ acts on $P \in V_w^\Gamma$ by

(2.2) $$P|\epsilon(A) = P(A')|_{-w}\epsilon,$$

where $A' = \epsilon A \epsilon$. This action is compatible with the action of Γ_1: $P|g|\epsilon = P|\epsilon|\epsilon g \epsilon$ for all $g \in \Gamma_1$. If $f^* \in S_k(\Gamma)$ denotes the form $f^*(z) = \overline{f(-\overline{z})}$, then

(2.3) $$\overline{\rho_{f^*}} = (-1)^{w+1} \rho_f|\epsilon$$

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[1]The condition $P|J = P$ is part of the definition of V_w^Γ, but we include it for clarity.

where $\overline{P}(A)$ is obtained by taking the complex conjugates of the coefficients of $P(A)$.

Under the action of ϵ, the space V_w^Γ breaks into ± 1-eigenspaces, denoted by $(V_w^\Gamma)^\pm$. For $P \in V_w^\Gamma$ we denote its $+1$ and -1-components by P^+ and P^- respectively:

$$(2.4) \qquad P^\pm = \frac{1}{2}(P \pm P|\epsilon) \in (V_w^\Gamma)^\pm.$$

We call P^+ the *even part* and P^- the *odd part* of P, which is justified by the fact that $P(I)^+$ is an even polynomial, and $P(I)^-$ is an odd polynomial (I is the identity coset). For $P \in W_w^\Gamma$, it is easily checked that $P|\epsilon \in W_w^\Gamma$ as well. Therefore $P^+, P^- \in W_w^\Gamma$, and the space W_w^Γ also decomposes into eigenspaces $(W_w^\Gamma)^\pm$.

Making use of the pairing on period polynomials introduced in Section 3, we restate the Eichler-Shimura isomorphism in terms of period polynomials as follows.

Theorem 2.1 (Eichler-Shimura). *The two maps $\rho^\pm : S_k(\Gamma) \to (W_w^\Gamma)^\pm$, $f \mapsto \rho_f^\pm$, give rise to isomorphisms, denoted by the same symbols:*

$$(2.5) \qquad \rho^\pm : S_k(\Gamma) \longrightarrow (W_w^\Gamma)^\pm / (C_w^\Gamma)^\pm.$$

Proof. By the stronger version of Haberland's formula (Theorem 3.3), the two maps $\rho^\pm : S_k(\Gamma) \to (W_w^\Gamma)^\pm$ are injective. Moreover, their images intersect trivially with C_w^Γ by Lemma 4.1, so the two maps in (2.5) are also injective. Using (2.1) and the Eichler-Shimura isomorphism [Sh71, Ch. 8] we have $\dim W_w^\Gamma = 2 \dim S_k(\Gamma) + \dim C_w^\Gamma$, and we conclude that $\rho^\pm$ in (2.5) are isomorphisms. $\square$

We now show that σ_f satisfies the cocycle relation, while also giving another construction of the associated period polynomial. In analogy with the Γ_1 case, the "Eichler integral" associated with $f \in S_k(\Gamma)$ is a function $\widetilde{f} : \Gamma \backslash \Gamma_1 \to \mathcal{A}$, where $\mathcal{A}$ is the space of holomorphic functions on the upper half plane, given by:

$$(2.6) \qquad \widetilde{f}(A)(z) = \int_z^{i\infty} f|A(t)(t-z)^w dt,$$

with Γ_1-action as on period polynomials: $\widetilde{f}|g(A) = \widetilde{f}(Ag^{-1})|_{-w}g$ for $g \in \Gamma_1$. By a change of variables we see that $\widetilde{f}|(1-g) = \sigma_f(g)$, which implies that σ_f satisfies the cocycle relation. Note that this provides another construction for the period polynomial ρ_f attached to f, which we record for further use:

$$(2.7) \qquad \rho_f = \widetilde{f}|(1-S).$$

Remark 2.2. A similar construction will be used in Section 8 to define period polynomials of arbitrary modular forms, by means of an Eichler integral $\widetilde{f}$ of $f \in M_k(\Gamma)$, which has the property that $\widetilde{f}|(1-T) = 0$ and $\widetilde{f}|(1-S)$ is the (extended) period polynomial attached to f. As pointed out in [DIT10], the construction of period polynomials of cusp forms using their higher order integrals goes back to Poincaré.

3. GENERALIZATION OF HABERLAND'S FORMULA

In [Ha83], Haberland proved a formula expressing the Petersson product of two cusp forms for the full modular group in terms of a pairing on their period polynomials. In this section we extend Haberland's formula to a finite index subgroup Γ of Γ_1, and we prove a stronger version for subgroups normalized by ϵ.

For $f, g \in S_k(\Gamma)$, define the Petersson scalar product:

$$(f, g) = \frac{1}{[\overline{\Gamma}_1 : \overline{\Gamma}]} \int_{\Gamma \backslash \mathcal{H}} f(z) \overline{g(z)} y^k \frac{dxdy}{y^2}.$$

On $V_w \times V_w$ we have a natural pairing $\langle \sum a_n x^n, \sum b_n x^n \rangle = \sum (-1)^{w-n} \binom{w}{n}^{-1} a_n b_{w-n}$, satisfying $\langle P, Q \rangle = (-1)^w \langle Q, P \rangle$. We will mostly use the equivalent formulation

$$(3.1) \qquad\qquad \langle (aX + b)^w, (cX + d)^w \rangle = (ad - bc)^w.$$

An easy consequence of (3.1) is that $\langle P|g, Q \rangle = \langle P, Q|g^\vee \rangle$ for $g \in \mathrm{GL}_2(\mathbb{R})$, where $g^\vee = g^{-1} \det g$; in particular the pairing is $\mathrm{SL}_2(\mathbb{R})$-invariant.

We define a similar pairing on $V_w^\Gamma \times V_w^\Gamma$:

$$(3.2) \qquad\qquad \langle\!\langle P, Q \rangle\!\rangle = \frac{1}{[\overline{\Gamma}_1 : \overline{\Gamma}]} \sum_{A \in \overline{\Gamma} \backslash \overline{\Gamma}_1} \langle P(A), Q(A) \rangle \quad \text{for } P, Q \in V_w^\Gamma.$$

Remark 3.1. For odd k there is a sign ambiguity in defining $P(A)$, $Q(A)$ for $P, Q \in V_w^\Gamma$ and $A \in \overline{\Gamma} \backslash \overline{\Gamma}_1$, but the pairing is well-defined since $P(-A) = (-1)^w P(A)$, $Q(-A) = (-1)^w Q(A)$. For the same reason, one can replace the range by $A \in \Gamma \backslash \Gamma_1$ and the normalizing factor by $\frac{1}{[\Gamma_1 : \Gamma]}$ without changing the pairing. A similar observation applies below, when $f|A$ always appears paired with $\overline{g}|A$, for $f, g \in S_k(\Gamma)$.

This pairing is Γ_1-invariant: $\langle\!\langle P|g, Q|g \rangle\!\rangle = \langle\!\langle P, Q \rangle\!\rangle$, for all $P, Q \in V_w^\Gamma$ and $g \in \Gamma_1$. It is normalized such that if $f, g \in S_k(\Gamma)$ and $\Gamma' \subset \Gamma$ then $\langle\!\langle \rho_f, \rho_g \rangle\!\rangle_\Gamma = \langle\!\langle \rho_f, \rho_g \rangle\!\rangle_{\Gamma'}$. Define also the modified pairing on $V_w^\Gamma \times V_w^\Gamma$:

$$(3.3) \qquad\qquad \{P, Q\} = \langle\!\langle P|T - T^{-1}, Q \rangle\!\rangle,$$

which satisfies $\{P, Q\} = (-1)^{w+1} \{Q, P\}$.

Part a) of the following theorem generalizes Haberland's formula [Ha83, KZ84]. Part b) follows easily from the proof of part a), although to our knowledge it has not appeared previously in the literature (except for Γ_1 in [Po11], but the proof there is more complicated).

Theorem 3.2. *(a) For $f, g \in S_k(\Gamma)$, we have*

$$6 C_k \cdot (f, g) = \{\rho_f, \overline{\rho_g}\},$$

where complex conjugation acts coefficientwise on polynomials and $C_k = -(2i)^{k-1}$.

(b) For $f, g \in S_k(\Gamma)$, we have $\{\rho_f, \rho_g\} = 0$.

Proof. (a) The proof is based on Stokes' theorem, as in [KZ84], except that we apply it to a fundamental domain for $\Gamma(2)$, namely the quadrilateral region $\mathcal{D}$ with vertices $i\infty, -1, 0, 1$ and with sides the geodesics connecting the points in that order. The region $\mathcal{D}$ consists of six copies of the fundamental domain for Γ_1, which explains the constant 6 appearing in the formula. Therefore

we have:

$$6C_k[\overline{\Gamma}_1 : \overline{\Gamma}] \cdot (f, g) = 6 \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)}(z - \overline{z})^w dz d\overline{z}$$

(3.4)
$$= \sum_{A\in\overline{\Gamma}\backslash\overline{\Gamma}_1} \int_{\mathcal{D}} f|A(z)\overline{g|A}(z)(z - \overline{z})^w dz d\overline{z}$$

$$= \sum_{A\in\overline{\Gamma}\backslash\overline{\Gamma}_1} \int_{\partial\mathcal{D}} F_A(z)\overline{g|A}(z) d\overline{z}$$

where $F_A(z) = \int_{i\infty}^{z} f|_k A(t)(t - \overline{z})^w dt$, so that $\frac{\partial F_A}{\partial z} = f|A(z)(z - \overline{z})^w$, and the last line follows from Stokes' theorem. For $B \in \mathrm{SL}_2(\mathbb{Z})$ a change of variables shows that

(3.5)
$$j(B, \overline{z})^w F_A(Bz) = F_{AB}(z) - \int_{i\infty}^{B^{-1}i\infty} f|AB(t)(t - \overline{z})^w dt.$$

We denote by $\int_a^b$ the integral over the geodesic arc from the cusp a to the cusp b. A change of variables $z = T^2\tau$ and (3.5) yields:

$$\int_1^{i\infty} F_A(z)\overline{g|A}(z) d\overline{z} = \int_{-1}^{i\infty} F_{AT^2}(\tau)\overline{g|AT^2}(\tau) d\overline{\tau},$$

and it follows that the sum of integrals over the vertical sides of $\mathcal{D}$ vanishes.

A change of variables $z = S\tau$ and (3.5) yields:

$$\int_{-1}^{0} F_A(z)\overline{g|A}(z) d\overline{z} = \int_1^{i\infty} F_{AS}(\tau)\overline{g|AS}(\tau) d\overline{\tau} + \int_1^{i\infty}\int_0^{i\infty} f|AS(t)(t - \overline{\tau})^w \overline{g|AS(\tau)} dt d\overline{\tau}.$$

$$\int_0^{1} F_A(z)\overline{g|A}(z) d\overline{z} = \int_{i\infty}^{-1} F_{AS}(\tau)\overline{g|AS}(\tau) d\overline{\tau} - \int_{-1}^{i\infty}\int_0^{i\infty} f|AS(t)(t - \overline{\tau})^w \overline{g|AS(\tau)} dt d\overline{\tau}.$$

When adding the last two equations and summing over $A \in \overline{\Gamma}\backslash\overline{\Gamma}_1$, the single integrals cancel as before and (3.4) becomes

$$6C_k[\overline{\Gamma}_1 : \overline{\Gamma}] \cdot (f, g) = \sum_{A\in\overline{\Gamma}\backslash\overline{\Gamma}_1} \int_1^{i\infty}\int_0^{i\infty} - \int_{-1}^{i\infty}\int_0^{i\infty} f|A(t)(t - \overline{\tau})^w \overline{g|A(\tau)} dt d\overline{\tau}.$$

To write the double integrals in terms of the period polynomial pairing, we use (3.1). After a change of variables $\tau = Tz$ the first double integral becomes

$$\int_0^{i\infty}\int_0^{i\infty} f|A(t) \left\langle (t - X)^w, (\overline{Tz} - X)^w \right\rangle \overline{g|AT(z)} \, dt d\overline{z} = \left\langle \rho_f(A), \overline{\rho_g}(AT)|T^{-1} \right\rangle.$$

The second integral yields the same result, with T replaced by T^{-1}, and the conclusion follows from the fact that the pairing $\langle\langle \, , \, \rangle\rangle$ is Γ_1 invariant.

(b) Going backwards in the proof of part (a) we have:

$$\{\rho_f, \rho_g\} = \frac{1}{[\overline{\Gamma}_1 : \overline{\Gamma}]} \sum_{A\in\overline{\Gamma}\backslash\overline{\Gamma}_1} \int_{\partial\mathcal{D}} H_A(z) g|A(z) dz$$

where $H_A(z) = \int_{i\infty}^{z} f|_k A(t)(t - z)^w dt = -\widetilde{f}(A)(z)$. Since the integrand is now holomorphic and vanishes exponentially at the cusps, it follows that each integral above vanishes. $\qquad\square$

Now let Γ be a congruence subgroup normalized by ϵ. The pairing $\{\cdot, \cdot\}$ satisfies

(3.6) $$\{P|\epsilon, Q|\epsilon\} = (-1)^{w+1}\{P, Q\},$$

hence $\{P, Q\} = 0$ if k is even and $P, Q \in V_w^\Gamma$ have the same parity, or if k is odd and P, Q have opposite parity. We have the following stronger version of Haberland's theorem, generalizing the result for the full modular group from [Po11].

Theorem 3.3. *Let Γ be a subgroup of finite index in Γ_1, normalized by ϵ. For $f, g \in S_k(\Gamma)$:*

$$3C_k \cdot (f, g) = \{\rho_f^{\kappa_1}, \overline{\rho_g^{\kappa_2}}\}$$

for any $\kappa_1, \kappa_2 \in \{+, -\}$ with $\kappa_1 \neq \kappa_2$ if k even and $\kappa_1 = \kappa_2$ if k odd.

Proof. We assume k even, the case k odd being entirely similar. In view of Theorem 3.2 (a) and (3.6), it is enough to show that $\{\rho_f^+, \overline{\rho_g^-}\} = \{\rho_f^-, \overline{\rho_g^+}\}$. By (2.3), we have $\overline{\rho_g^+} = (-1)^{w+1}\rho_{g*}^+$, $\overline{\rho_g^-} = (-1)^w \rho_{g*}^-$, and the previous equality reduces to $\{\rho_f, \rho_{g*}\} = 0$, which is Theorem 3.2 b). $\square$

4. COBOUNDARY POLYNOMIALS

In this section we show that the space of coboundary polynomials

$$C_w^\Gamma = \{P|(1-S) : P \in V_w^\Gamma \cap \ker(1-T)\}$$

is the radical of the bilinear form $\{\cdot, \cdot\}$ on W_w^Γ. The dimension of C_w^Γ equals the dimension of the Eisenstein subspace $\mathcal{E}_k(\Gamma) \subset M_k(\Gamma)$. For $\Gamma = \Gamma_0(N)$, we characterize those N for which $(C_w^\Gamma)^-$ is trivial, namely those N for which the map $\rho^- : S_k(\Gamma) \to (W_w^\Gamma)^-$ is an isomorphism, as in the full level case.

Lemma 4.1. *Let Γ be a finite index subgroup of Γ_1. The period polynomials W_w^Γ are orthogonal to the coboundary polynomials $C_w^\Gamma \subset W_w^\Gamma$ with respect to the pairing $\{\cdot, \cdot\}$.*

Proof. Let $P|(1-S) \in C_w^\Gamma$ with $P|1-T = 0$, and let $Q \in W_w^\Gamma$. Then $\langle\!\langle P|(1-S)(T-T^{-1}), Q \rangle\!\rangle = 0$ follows from the following relation in $\mathbb{Z}[\overline{\Gamma}_1]$

(4.1) $$(1-S)(T-T^{-1}) = (T-1)(2+T^{-1}+ST-S) + (1+TS+ST^{-1})(1-S)$$

using the Γ_1 invariance of the pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ (recall $U = TS$). $\square$

Let $e_\infty(\Gamma)$, $e_\infty^{\mathrm{reg}}(\Gamma)$ denote the number of inequivalent cusps, respectively regular cusps [DS05, Ch. 3]. The next lemma shows that $\dim C_w^\Gamma = \dim \mathcal{E}_k(\Gamma)$.

Lemma 4.2. *Let Γ be any finite index subgroup of Γ_1. The dimension of C_w^Γ equals: $e_\infty(\Gamma)$ if $k > 2$ is even; $e_\infty(\Gamma)$-1 if $k = 2$; $e_\infty^{\mathrm{reg}}(\Gamma)$ if $k > 2$ is odd.*

Proof. Let $P \in V_w^\Gamma \cap \ker(1-T)$. Then $P|T^n = P$ for every $n \in \mathbb{Z}$, that is $P(AT^{-n})|T^n = P(A)$ for $A \in \Gamma\backslash\Gamma_1$. Since $\Gamma\backslash\Gamma_1$ is finite, there is n such that $AT^{-n} = A$, and $P(A)(X+n) = P(A)(X)$. Since the only periodic polynomials are the constants, it follows that $P(A)(X) = c_A \in \mathbb{C}$, with $c_{AT} = c_A$. Since $P|J = P$ we also have $c_{AJ} = (-1)^w c_A$. Hence we have:

(4.2) $$C_w^\Gamma = \{(c_A - c_{AS^{-1}}X^w)_A \in V_w^\Gamma : c_A \in \mathbb{C}, \ c_{AT} = c_A, \ c_{AJ} = (-1)^w c_A\}.$$

If $k > 2$ is even it follows that $\dim C_w^\Gamma = |\Gamma\backslash\Gamma_1/\Gamma_{1\infty}|$, with $\Gamma_{1\infty} = \{\pm T^n : n \in \mathbb{Z}\}$ the stabilizer of ∞. Since the map

$$\Gamma\backslash\Gamma_1/\Gamma_{1\infty} \to \Gamma\backslash\mathbb{P}^1(\mathbb{Q}), \quad [\gamma] \to [\gamma\infty]$$

is a bijection and $|\Gamma\backslash\mathbb{P}^1(\mathbb{Q})| = e_\infty(\Gamma)$, the claim follows.

If $k = 2$, we identify $\mathbb{C}^{e_\infty(\Gamma)}$ with the vector space $\{(c_A)_{A \in \Gamma \backslash \Gamma_1} : c_{AT} = c_A = c_{AJ} \in \mathbb{C}\}$ and define the (surjective) map $\mathbb{C}^{e_\infty(\Gamma)} \to C_w^\Gamma$ by $(c_A)_A \to (c_A - c_{AS})_A$. Its kernel consists of those vectors $(c_A)_A$ with $c_{AT} = c_{AS} = c_{AJ} = c_A$. Since S, T, J generate Γ_1, it follows that $c_A = c$ for all $A \in \Gamma \backslash \Gamma_1$, so the kernel is isomorphic to $\mathbb{C}$. Therefore $\dim C_w^\Gamma = e_\infty(\Gamma) - 1$.

If $k > 2$ is odd (so $-1 \notin \Gamma$), we have $c_{AJ} = -c_A$. Therefore $\dim C_w^\Gamma$ equals the number of classes $[A] \in \Gamma \backslash \Gamma_1 / \Gamma_{1\infty}$, such that the two associated classes $[A]^+, [AJ]^+ \in \Gamma \backslash \Gamma_1 / \Gamma_{1\infty}^+$ are distinct, where $\Gamma_{1\infty}^+ = \{T^n : n \in \mathbb{Z}\}$ (when $[A]^+ = [AJ]^+$ then clearly $c_A = c_{AJ} = 0$). But $[A]^+ \neq [AJ]^+$ precisely when $[A]$ corresponds to a regular cusp of Γ since $[A]^+ = [AJ]^+$ means that $A^{-1}\gamma A = -T^n$ for some $\gamma \in \Gamma$, $n > 1$, so the cusp $A\infty$ is irregular. We conclude that $\dim C_w^\Gamma = e_\infty^{\mathrm{reg}}(\Gamma)$. $\qquad\square$

Definition 4.3. Following the proof of Lemma 4.2, we call *cusp* a double coset $\mathcal{C} \in \Gamma \backslash \Gamma_1 / \Gamma_{1\infty}$, which corresponds to the Γ-equivalence class of the usual cusp $A\infty$ for any representative $A \in \mathcal{C}$. We call *regular* those cusps $\mathcal{C} = [A]$ such that the double cosets $[A]^+, [AJ]^+ \in \Gamma \backslash \Gamma_1 / \Gamma_{1\infty}^+$ are distinct. The terminology agrees with the usual one for the cusp $A\infty$ by the last paragraph in the proof of the lemma.

For $\Gamma_0(N)$ it turns out that $(C_w^\Gamma)^-$ is often trivial, in which case $\rho^- : S_k(\Gamma) \to (W_w^\Gamma)^-$ is an isomorphism just like for Γ_1. The following proposition was discovered using SAGE [SG].

Proposition 4.4. *Let* $\Gamma = \Gamma_0(N)$. *Then* $(C_w^\Gamma)^- = \{0\}$ *if and only if* $N = 2^e N'$ *with* N' *odd square free and* $0 \leqslant e \leqslant 3$.

Proof. From the proof of Lemma 4.2 we identify $(C_w^\Gamma)^-$ with the space $(\mathbb{C}^{e_\infty(\Gamma)})^-$ of vectors $(c_A)_{A \in \overline{\Gamma} \backslash \overline{\Gamma}_1}$ with $c_A = c_{AT}$ and $c_A = -c_{A'}$ (including for $k = 2$).

Assume N does not satisfy the conditions, so there exists $t \geqslant 3$ with $t^2 | N$. We claim that $[A] \neq [A']$ for $A = \begin{pmatrix} x & y \\ t & z \end{pmatrix} \in \Gamma_1$, so $(C_w^\Gamma)^- \neq \{0\}$. Assuming by contradiction that $\gamma A T^s = A'$ for $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$, it follows that $cx + dt = -t, c(y + sx) + d(z + st) = z$. The first equation implies that $t | d + 1$ while the second that $t | d - 1$, a contradiction with $t \geqslant 3$.

Assuming N satisfies the conditions, let $(c_A)_{A \in \overline{\Gamma} \backslash \overline{\Gamma}_1}$ with $c_A = c_{AT}$ and $c_A = -c_{A'}$. Identifying the coset space $\Gamma_0(N) \backslash \Gamma_1$ with $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$, it follows that $c_{(a:b)} = c_{(a:a+b)}$, $c_{(a:b)} = -c_{(-a:b)}$. The second relation implies $c_{(0:1)} = 0$. Let $N = dd'$ and $k \in \mathbb{Z}$, $(k, d) = 1$. We will show that $c_{(d:k)} = 0$. We have:

$$c_{(d:k)} = c_{(d:k+ad)} = -c_{(-d:k)} = -c_{((bd'-1)d:k)}$$

and it is enough to find $a, b \in \mathbb{Z}$ with $(bd' - 1, N) = 1$ and $k \equiv (bd' - 1)(k + ad) \pmod{N}$. The latter equation can be written $k(bd' - 2) \equiv ad \pmod{dd'}$ and the hypothesis on N ensures that $(d, d') | 2$, so that we can find b, u such that $bd' - 2 = du$. Taking $a \equiv ku \pmod{d'}$, it follows that $(d : k + ad) = ((bd' - 1)d : k)$, which implies that $c_{(d:k)} = 0$. It follows that $c_A = 0$ for all $A \in \overline{\Gamma} \backslash \overline{\Gamma}_1$, finishing the proof. $\qquad\square$

5. HECKE OPERATORS

Following the Eichler integral method sketched in [Za93] for the modular group, we show that the same elements $\widetilde{T}_n$ as in the full level case define actions on period polynomials corresponding to a large class of double coset operators on modular forms, including the Hecke operators and Atkin-Lehner operators. On modular symbols, the action of the same type of double coset operators was determined by Merel, and the results of §5.1 parallel those of [Me94]. We find the adjoints of these operators with respect to the pairing $\{\cdot, \cdot\}$ in §5.2.[2] In §5.3 we give a new proof of the rationality of

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[2]For Γ_1, the Hecke equivariance of the pairing is mentioned without proof in [GKZ, p.96].

the plus and minus parts of period polynomials of newforms, while in §5.4 we discuss modular forms with Nebentypus. We end with a discussion of the numerical computation of period polynomials, Hecke eigenvalues, and Petersson norms of newforms.

5.1. **The universal Hecke operators.** Let M_n be the set of integer matrices of determinant n, set $\overline{M}_n = M_n/\{\pm I\}$, and let $R_n = \mathbb{Q}[\overline{M}_n]$. Thus $\overline{\Gamma}_1$ acts on R_n by left and right multiplication. Let

$$M_n^\infty = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right) : n = ad, 0 \leqslant b < d \right\}$$

be the usual system of representatives for $\Gamma_1 \backslash M_n$, and $T_n^\infty = \sum_{M \in M_n^\infty} M \in R_n$. Following [CZ93], let $\widetilde{T}_n, Y_n \in R_n$ be such that

$$(5.1) \qquad\qquad T_n^\infty(1 - S) = (1 - S)\widetilde{T}_n + (1 - T)Y_n.$$

We will show that for all n, the elements $\widetilde{T}_n$ define Hecke operators on period polynomials for a large class of congruence subgroups. Note that the elements $\widetilde{T}_n$ are universal, not depending on the weight or level.

Let Γ be a finite index subgroup of Γ_1 normalized by ϵ, which we will often specialize to be $\Gamma_1(N)$ or $\Gamma_0(N)$. Fix an integer $n \geqslant 1$, and let $\Sigma_n \subset M_n$ such that $\Sigma_n = \Gamma\Sigma_n = \Sigma_n\Gamma$ and Σ_n is a disjoint finite union of cosets $\Gamma\sigma$. Such double cosets Σ_n define operators $[\Sigma_n]$ on $f \in M_k(\Gamma)$

$$f|[\Sigma_n] = n^{w+1} \sum_{\sigma \in \Gamma \backslash \Sigma_n} f|_k \sigma.$$

Setting $g^\vee = g^{-1} \det g$, the adjoint of $[\Sigma_n]$ with respect to the Petersson product is given by $\Sigma_n^\vee = \{g^\vee : g \in \Sigma_n\}$, namely $(f|[\Sigma_n], g) = (f, g|[\Sigma_n^\vee])$.

We now define an action of M_n on V_w^Γ which depends on Σ_n, and which is based on the following property of the pair (Γ, Σ_n)

$$(H) \qquad\qquad \text{The map } \Gamma \backslash \Sigma_n \to \Gamma_1 \backslash \Gamma_1 \Sigma_n, \quad \Gamma\sigma \mapsto \Gamma_1\sigma \text{ is bijective.}$$

See [Sh71, Prop. 3.36] for a large class of congruence subgroups when this property holds.

For $M \in M_n$, $A \in \Gamma_1$ and $MA^{-1} \in \Gamma_1\Sigma_n$, there exists a decomposition $MA^{-1} = A_M^{-1}M_A$, with $M_A \in \Sigma_n$, $A_M \in \Gamma_1$, and by (H) the coset ΓA_M is independent of the decomposition. Moreover ΓA_M depends only on ΓA, since $\Gamma\Sigma_n = \Sigma_n\Gamma$. For $P \in V_w^\Gamma$ we define an element $P|_\Sigma M = P|_{\Sigma_n} M \in V_w^\Gamma$ by[3]

$$(5.2) \qquad\qquad P|_\Sigma M(A) = \begin{cases} P(A_M)|_{-w}M & \text{if } MA^{-1} \in \Gamma_1\Sigma_n \\ 0 & \text{otherwise.} \end{cases}$$

Note that both M and JM act in the same way (recall $J = -I$), so the action of $\overline{M}_n = M_n/\{\pm I\}$ is also well defined. This action extends linearly to an action of R_n on V_w^Γ. In the same way we define an action of $\overline{M}_n$ and R_n on the Eichler integrals $\widetilde{f}$ in (2.6).

We are interested in the following double cosets Σ_n satisfying (H), for $\Gamma = \Gamma_1(N)$ or $\Gamma_0(N)$.

(1) The double coset Δ_n consisting of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in M_n$ with $N|c$, and $a \equiv 1 \pmod{N}$ if $\Gamma = \Gamma_1(N)$, or $(a, N) = 1$ if $\Gamma = \Gamma_0(N)$. The operator $[\Delta_n]$ is the usual Hecke operator, denoted by T_n, and Property (H) follows from [Sh71, Prop. 3.36]. Note that for $(n, N) = 1$ we have $\Gamma_1\Delta_n = M_n$, so the second case in (5.2) does not occur.

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[3] With the notation $P|_\Sigma M$, the dependence on n is recorded in the fact that $M \in M_n$.

(2) The double coset $\Delta_n^\vee$ for $(n, N) = 1$. The operator $[\Delta_n^\vee]$ is denoted by T_n^*, the adjoint of T_n with respect to the Petersson inner product. Property (H) can be checked directly.

(3) The double coset $\Theta_n = \Gamma w_n \Gamma$ for $N = nn'$, $(n, n') = 1$, where $w_n = \left(\begin{smallmatrix} nx & y \\ Nz & nt \end{smallmatrix}\right) \in M_n$ with $x, y, z, t \in \mathbb{Z}$ for $\Gamma = \Gamma_0(N)$, while for $\Gamma = \Gamma_1(N)$ we impose the extra conditions $nx \equiv 1 \pmod{n'}$, $y \equiv 1 \pmod{n}$. Then $\Gamma \backslash \Theta_n$ has one element so Property (H) is trivially satisfied. The operator $[\Theta_n]$ is denoted by W_n, and if $W_n^* = [\Theta_n^\vee]$ denotes its adjoint we have $f|W_n|W_n^* = n^w f$. For $\Gamma = \Gamma_0(N)$ (so that $W_n = W_n^*$), the usual Atkin-Lehner involution is $W_n/n^{w/2}$ due to our choice of normalization.

(4) The double coset $\Theta_n^\vee$ with Θ_n as in (3). If $\Gamma = \Gamma_0(N)$ then $[\Theta_n^\vee] = [\Theta_n]$, and if $\Gamma = \Gamma_1(N)$ then $[\Theta_n^\vee] = \langle d \rangle [\Theta_n]$ for $d \equiv -1 \pmod{n}$, $nd \equiv 1 \pmod{n'}$, where $\langle d \rangle$ denotes the diamond operator.

Remark 5.1. Assume $\Gamma = \Gamma_1(N)$ and Σ_n is as in (1) or (3). The action of M_n on V_w^Γ can be determined as follows. Applying adjoint we have that $MA^{-1} \in \Gamma_1 \Sigma_n$ if and only if

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = AM^\vee = M_A^\vee A_M \in \Sigma_n^\vee \Gamma_1.$$

For $\Sigma_n = \Delta_n$, the inclusion is satisfied if and only if $\gcd(c, d, N) = 1$, and in that case $A_M = \left(\begin{smallmatrix} * & * \\ c' & d' \end{smallmatrix}\right) \in \Gamma_1$ has $c' \equiv c, d' \equiv d \pmod{N}$, which uniquely defines its class in $\Gamma \backslash \Gamma_1$.

For $\Sigma_n = \Theta_n$, let $N = nn'$ with $n \| N$. The inclusion is satisfied if and only if $n|c, n|d$, and the class of $A_M = \left(\begin{smallmatrix} * & * \\ c' & d' \end{smallmatrix}\right)$ in $\Gamma \backslash \Gamma_1$ is determined by

$$c' \equiv -a, \; d' \equiv -b \pmod{n}, \quad c' \equiv c/n, \; d' \equiv d/n \pmod{n'}.$$

When $\Gamma = \Gamma_0(N)$ and $\Sigma_n = \Theta_n$, the last congruences are replaced by $yc' \equiv a, yd' \equiv b \pmod{n}$ and $tc' \equiv c/n, td' \equiv d/n \pmod{n'}$ where t, y are any integers such that $nxt - n'yz = 1$ (taking $M_A^\vee = \left(\begin{smallmatrix} nx & y \\ Nz & nt \end{smallmatrix}\right) \in M_n$). The class of (c', d') in $P^1(\mathbb{Z}/N\mathbb{Z}) \sim \Gamma \backslash \Gamma_1$ is then independent of y, t.

While the operation $P|_\Sigma M$ is not a proper action, it is compatible with the action of Γ_1 and of ϵ. For $h \in \Gamma_1$, $M \in M_n$ a formal computation shows that

(5.3) $$P|_\Sigma M|h = P|_\Sigma Mh, \quad P|h|_\Sigma M = P|_\Sigma hM, \quad P|_\Sigma M|\epsilon = P|\epsilon|_{\Sigma'}\epsilon M\epsilon,$$

where $\Sigma_n' = \{g' = \epsilon g \epsilon : g \in \Sigma_n\}$. Using the compatibility with the Γ_1 action, we show that the universal operators $\widetilde{T}_n$ play on period polynomials the role of *all* double coset operators $[\Sigma_n]$ on modular forms, as long as the pair (Γ, Σ_n) satisfies (H).

Proposition 5.2. *Assume the pair (Γ, Σ_n) satisfies* (H)*. If $\widetilde{T}_n \in R_n$ satisfies* (5.1) *then for every $f \in S_k(\Gamma)$*

$$\rho_{f|[\Sigma_n]} = \rho_f|_\Sigma \widetilde{T}_n.$$

In particular for $\Gamma = \Gamma_1(N)$, we have $\rho_{f|T_n} = \rho_f|_\Delta \widetilde{T}_n$ and, if $n \| N$, $\rho_{f|W_n} = \rho_f|_\Theta \widetilde{T}_n$.

Proof. Using the fact that $\rho_f = \widetilde{f}|(1 - S)$ and $\widetilde{f}|(1 - T) = \sigma_f(T) = 0$, we have as in [Za93]

$$\rho_f|_\Sigma \widetilde{T}_n = \widetilde{f}|_\Sigma (1 - S)\widetilde{T}_n = \widetilde{f}|_\Sigma T_n^\infty (1 - S) = \widetilde{f|[\Sigma_n]}|(1 - S) = \rho_{f|[\Sigma_n]},$$

once we show that $\widetilde{f}|_\Sigma T_n^\infty = \widetilde{f|[\Sigma]}$.

For $M \in M_n^\infty$ and $A \in \Gamma_1$, let $MA^{-1} = A_M^{-1} M_A$ with $A_M \in \Gamma_1$, $M_A \in \Sigma_n$. By (5.2)

$$\widetilde{f}|_\Sigma T_n^\infty(A) = \sum_{M \in M_n^\infty \cap \Gamma_1 \Sigma_n A} \int_{Mz}^{i\infty} f|A_M(t)(t - Mz)^w j(M,z)^w dt$$

(5.4)

$$= n^{w+1} \sum_{M \in M_n^\infty \cap \Gamma_1 \Sigma_n A} \int_z^{i\infty} f|M_A A(u)(u-z)^w du$$

where we made a change of variables $t = Mu$. For fixed A, the map

$$M_n^\infty \cap \Gamma_1 \Sigma_n A \to \Gamma \backslash \Sigma_n, \quad M \mapsto M_A$$

is well defined by (H). It is easy to check that the map is bijective, hence the last expression equals $\widetilde{f|[\Sigma_n]}(A)$ finishing the proof. $\qquad\square$

Corollary 5.3. *If in addition to the hypotheses of Proposition 5.2 we assume that Γ is normalized by ϵ and that $\Sigma_n' = \Sigma_n$ then*

$$\rho_{f|[\Sigma_n]}^\pm = \rho_f^\pm |_\Sigma \widetilde{T}_n.$$

In particular if $\Gamma = \Gamma_1(N)$, then $\rho_{f|T_n}^\pm = \rho_f^\pm |_\Delta \widetilde{T}_n$, and if $\Gamma = \Gamma_0(N)$, then $\rho_{f|W_n}^\pm = \rho_f^\pm |_\Theta \widetilde{T}_n$.

Proof. By the last compatibility in (5.3), it is enough to show that $\epsilon \widetilde{T}_n \epsilon$ also satisfied (5.1) (for a different Y_n). This follows from conjugating (5.1) by ϵ, and using that $T_n^\infty - \epsilon T_n^\infty \epsilon \in (1-T)R_n$.

We have $\Delta_n' = \Delta_n$ and, for $\Gamma = \Gamma_0(N)$, $\Theta_n' = \Theta_n$. Note that if $\Gamma = \Gamma_1(N)$, then $[\Theta_n'] = \langle d \rangle [\Theta_n]$ with $d \equiv -1 \pmod{n}$, $d \equiv 1 \pmod{n'}$, so the action $|_\Theta \widetilde{T}_n$ on $(W_w^\Gamma)^\pm$ does not correspond to W_n in this situation. $\qquad\square$

Taking $\Sigma_n = \Delta_n$ in Corollary 5.3, we obtain explicit formulas for the Fourier coefficients of a Hecke eigenform $f \in S_k(\Gamma)$ in terms of the polynomials $\rho_f^\pm$, generalizing the Coefficients Theorem of Manin [Ma73]. The formulas can be used for fast computation of Hecke eigenvalues, as explained in §5.5, and they also yield an explicit inverse of the Eichler-Shimura maps $\rho^\pm$ of Theorem 2.1.

We state the formula corresponding to ρ_f^+. For $\Gamma = \Gamma_1(N)$ we identify a coset representative $A \in \Gamma \backslash \Gamma_1$ with the element $(c_A, d_A) \in E_N$, where c_A, d_A are the lower left, respectively lower right entries of A and

$$E_N = \left\{ (c,d) \in (\mathbb{Z}/N\mathbb{Z})^2 : (c,d,N) = 1 \right\}.$$

For $P \in W_w^\Gamma$, we write $P(A) = P(c_A, d_A) \in V_w$.

Proposition 5.4. *Let $f \in S_k(\Gamma_1(N))$ be an eigenform for T_n with eigenvalue λ_n, and let $\widetilde{T}_n = \sum_{M \in M_n} \alpha(M)M$ be any operator satisfying (5.1).*

(a) *Assume $r_{I,w}(f) \neq 0$ (which is satisfied if $k > 2$ and f is a newform), and let $P_f^+ = \rho_f^+ / r_{I,w}(f)$. Then*

$$\lambda_n = \sum_{\substack{M \in M_n \\ (c_M, a_M, N) = 1}} \alpha(M) P_f^+(-c_M, a_M)|M(0)$$

where $M = \begin{pmatrix} a_M & b_M \\ c_M & d_M \end{pmatrix}$. When $(n, N) = 1$ the congruence condition in the sum can be omitted.

(b) *If $k = 2$, let $(x, y) \in E_N$ represent a coset in $\Gamma \backslash \Gamma_1$ with $\rho_f^+(x, y) \neq 0$ and let $P_f^+ = \rho_f^+ / \rho_f^+(x, y)$. For each $M \in M_n$, let $x_M = x d_M - y c_M$, $y_M = -x b_M + y a_M$. Then*

$$\lambda_n = \sum_{\substack{M \in M_n \\ (x_M, y_M, N) = 1}} \alpha(M) P_f^+(x_M, y_M).$$

Remark 5.5. By Proposition 5.11, when f is a newform the polynomials P_f^+ have coefficients in the field K_f of coefficients of f. When f has known Nebentypus, we obtain simpler formulas using the results of §5.4.

Proof. By Corollary 5.3, P_f^+ is an eigenform of $\widetilde{T}_n$ with eigenvalue λ_n, and the conclusion follows by writing the action of $\widetilde{T}_n$ on P_f^+ explicitly, using Remark 5.1. $\qquad \square$

Elements $\widetilde{T}_n$ satisfying condition (5.1) go back to work of Manin [Ma73], and particular examples are given in [CZ93, Za90]. The element $\widetilde{T}_n$ is unique, up to addition of any element in the right Γ_1-module

$$(5.5) \qquad\qquad\qquad \mathcal{I} = (1 + S) R_n + (1 + U + U^2) R_n.$$

Remark 5.6. In [Me94] Merel gives several examples of elements $\widetilde{\mathfrak{T}}_n \in R_n$ acting on modular symbols, satisfying a condition that plays the same role as (5.1). It can be shown that the elements $\widetilde{\mathfrak{T}}_n^\vee$ (with the notation explained in the next paragraph) satisfy Property (5.1), reflecting the fact that the action of Γ_1 on modular symbols is on the left, and on period polynomials is on the right.

5.2. **Adjoints of Hecke operators.** Next we determine the adjoint of the action of Hecke and Atkin-Lehner operators for the pairing on W_w^Γ defined in (3.3). For $g \in M_n$ we denote by $g^\vee = g^{-1} \det g$ the adjoint of g, and we apply this notation to all elements of R_n by linearity. Recall that $\langle P | g, Q \rangle = \langle P, Q | g^\vee \rangle$ for $P, Q \in V_w$, $g \in GL_2(\mathbb{C})$.

Lemma 5.7. *Assume that both (Γ, Σ_n) and $(\Gamma, \Sigma_n^\vee)$ satisfy (H). For $P, Q \in V_w^\Gamma$, $M \in M_n$ we have*

$$(5.6) \qquad\qquad\qquad \langle\!\langle P |_\Sigma M, Q \rangle\!\rangle = \langle\!\langle P, Q |_{\Sigma^\vee} M^\vee \rangle\!\rangle.$$

In particular, (5.6) holds for $\Gamma = \Gamma_1(N)$ and $\Sigma_n = \Delta_n$ with $(n, N) = 1$, or $\Sigma_n = \Theta_n$ for $n \| N$.

Proof. For $A \in \Gamma_1$ and $M \in \Gamma_1 \Sigma_n A$, let $M A^{-1} = A_M^{-1} M_A$ with $A_M \in \Gamma_1$, $M_A \in \Sigma_n$. We have (see Remark 3.1)

$$[\Gamma_1 : \Gamma] \langle\!\langle P | M, Q \rangle\!\rangle = \sum_{\substack{A \in \Gamma \backslash \Gamma_1 \\ M A^{-1} \in \Gamma_1 \Sigma_n}} \langle P(A_M), Q(A) | M^\vee \rangle.$$

Taking adjoint we have $M^\vee A_M^{-1} = A^{-1} M_A^\vee$ with $M_A^\vee \in \Sigma_n^\vee$, hence $M A^{-1} \in \Gamma_1 \Sigma_n$ if and only if $M^\vee A_M^{-1} \in \Gamma_1 \Sigma_n^\vee$. Moreover the map $A \mapsto A_M$ is injective, by Property (H) applied to $\Sigma_n^\vee$. Summing over A_M instead of A finishes the proof. $\qquad \square$

Next we give two proofs of the Hecke equivariance of the period polynomial pairing, one of them requiring the following lemma.

Lemma 5.8. *The space C_w^Γ is preserved by the Hecke operators $\widetilde{T}_n$, whenever an action of M_n on V_w^Γ satisfying (5.3) can be defined.*

Proof. Let $P|(1-S) \in C_w^\Gamma$ with $P|1-T=0$. By (5.1), $P(1-S)|_\Sigma \widetilde{T}_n = P|_\Sigma T_n^\infty(1-S)$, and the latter is an element of C_w^Γ, since $T_n^\infty(1-T) \in (1-T)R_n$, so $P|_\Sigma T_n^\infty(1-T)=0$. $\qquad\square$

Theorem 5.9. *Assume that both (Γ, Σ_n) and $(\Gamma, \Sigma_n^\vee)$ satisfy property* (H). *For $P, Q \in W_w^\Gamma$ and any $\widetilde{T}_n$ as in* (5.1) *we have:*

$$\{P|_\Sigma \widetilde{T}_n, Q\} = \{P, Q|_{\Sigma^\vee} \widetilde{T}_n\}.$$

Proof. We give two proofs. For the first we assume that both $\Sigma_n, \Sigma_n^\vee$ satisfy (H), and in addition $\Sigma_n = \Sigma_n'$. From Theorem 3.3 and Corollary 5.3, it follows that the claim is true for $P = \rho_f^\pm$ and $Q = \rho_g^\mp$, for any $f, g \in S_k(\Gamma)$. By (2.5), any $P \in W_w^\Gamma$ can be written $P = \rho_f^+ + \rho_g^- + Q$ with $Q \in C_w^\Gamma$. Since $\widetilde{T}_n$ preserves $(W_w^\Gamma)^\pm$ and C_w^Γ, the claim follows taking into account Lemmas 4.1.

The second proof is purely algebraic and we only assume that $\Sigma_n, \Sigma_n^\vee$ satisfy (H). Via (5.6) the equality to prove is equivalent to

$$\langle\!\langle\, P|_\Sigma \big[\widetilde{T}_n(T-T^{-1}) + (T^{-1}-T)\widetilde{T}_n^\vee\big], Q \,\rangle\!\rangle = 0.$$

Since $P, Q \in W_w^\Gamma$, we are reduced to proving the next theorem. $\qquad\square$

Theorem 5.10. *For any element $\widetilde{T}_n \in R_n$ satisfying property* (5.1) *we have*

$$\widetilde{T}_n(T-T^{-1}) + (T^{-1}-T)\widetilde{T}_n^\vee \in \mathcal{I} + \mathcal{I}^\vee,$$

where $\mathcal{I}$ is defined in (5.5).

The proof is quite involved, and we give it in the short article [PP12a].

5.3. **Rationality of period polynomials of Hecke eigenforms.** Let $\Gamma = \Gamma_1(N)$ and for a character χ modulo N let $S_k(N, \chi) \subset S_k(\Gamma)$ be the subspace of forms of Nebentypus χ. The following proposition is well-known, although the precise statement is hard to find in the literature. For the trivial character, an equivalent statement to part (a) was given in terms of modular symbols in [GS94]. The proof requires the extension of the pairing $\{\cdot, \cdot\}$ to the whole space of modular forms and its properties proved in Section 8.

Proposition 5.11. *Assume that $f \in S_k(N, \chi)$ is a newform (a normalized eigenform for all Hecke operators which does not come from lower levels) and let K_f be the field of coefficients of f.*

(a) There exist nonzero complex numbers ω_f^+ and ω_f^- such that all the polynomial components of $\rho_f^\pm / \omega_f^\pm$ have coefficients in K_f.

(b) We have

$$\frac{\omega_f^+ \overline{\omega_f^-}}{i(2\pi)^{k-1}(f,f)} \in K_f \ \text{if } k \text{ is even}, \qquad \frac{|\omega_f^\pm|^2}{(2\pi)^{k-1}(f,f)} \in K_f \ \text{if } k \text{ is odd}.$$

In particular, if k is even one can choose $\omega_f^\pm$ such that $\omega_f^+ \overline{\omega_f^-} = i(2\pi)^{k-1}(f,f)$.

(c) If f has real Fourier coefficients at infinity, then $\omega_f^+ \in i^{k+1}\mathbb{R}$ and $\omega_f^- \in i^k\mathbb{R}$.

Proof. (a) We view $f \in S_k(\Gamma)$ for $\Gamma = \Gamma_1(N)$. By multiplicity one and Corollary 5.3, the polynomials $\rho_f^\pm$ are the unique (up to scalars) elements in $\rho^\pm(S_k(\Gamma)) \subset W_w^\Gamma$ with the same eigenvalues under $|_\Delta \widetilde{T}_n$ as those of f. Moreover on C_w^Γ there is no eigenvector for $|_\Delta \widetilde{T}_n$ with the same eigenvalues λ_n as those of f for every n coprime to N. Indeed, $|\lambda_n| \ll n^{(k-1)/2+\epsilon}$ by the Ramanujan bounds (even a nontrivial upper bound would suffice), and by Corollary 8.9 the eigenvalues of $|_\Delta \widetilde{T}_n$ on C_w^Γ are the same as the eigenvalues of an Eisenstein series, which are of order n^{k-1} if $(n, N) = 1$.

Therefore the common eigenspace of $|_\Delta \widetilde{T}_n$ for $(n, N) = 1$ acting on $(W_w^\Gamma)^\pm$ having the same eigenvalues as those of f is one dimensional, generated by $\rho_f^\pm$. Since the matrix of $\widetilde{T}_n$ has integer coefficients with respect to a basis of V_w^Γ, and $(W_w^\Gamma)^\pm$ are subspaces cut by relations with integer coefficients, it follows that there is a common eigenvector defined over K_f.

(b) The claim follows from (a) and Theorem 3.3.

(c) The conclusion follows from (2.3). $\qquad\square$

The coefficients of $\rho_f^\pm(I)$ are related to the critical values of the L-function of $L(s, f)$, so the proposition implies well-known rationality statements about critical values. With the notation of (6.1), we have

(5.7) $$r_{I,n}(f) = i^{n+1}\Lambda(n+1, f), \text{ with } \Lambda(s, f) := (2\pi)^{-s}\Gamma(s)L(s, f)$$

the completed L-function. This proves the following special case of Theorem 1 in [Sh77].

Corollary 5.12. *Assume that $f \in S_k(N, \chi)$ is a newform, and let K_f be the field of coefficients of f. Then with the periods $\omega_f^\pm$ defined in the Proposition 5.11 we have*

$$\frac{i^n \Lambda(n, f)}{\omega_f^\pm} \in K_f \text{ for } 0 < n < k, \quad (-1)^n = \pm(-1)^{k-1}.$$

When k is even, our convention regarding the signs of the transcendental factors differs from the usual one eg. in [Sh77], since ω_f^+ is the normalizing factor for *odd* critical values (and for *even* period polynomials).

Remark 5.13. When k is odd it follows from Prop. 5.11 (b) that $\frac{|\omega_f^+|^2}{|\omega_f^-|^2} \in K_f$. In fact, when K_f is a real field, it follows from the functional equation $N^{s/2}\Lambda(s, f) = \pm N^{(k-s)/2}\Lambda(k-s, f)$ that $\frac{\omega_f^+}{i\omega_f^-} \in K_f(\sqrt{N})$. From (5.7) and the functional equation, when K_f is real we can choose

$$\omega_f^- = -i\Lambda(1, f), \quad \omega_f^+ = \pm i^{w+1} N^{(k-1)/2}\Lambda(k-1, f)$$

so that $\frac{\omega_f^+}{i\omega_f^-} = \sqrt{N}$.

5.4. **Modular forms with Nebentypus.** Let $\Gamma = \Gamma_1(N)$, $\Gamma' = \Gamma_0(N)$ and χ a Dirichlet character modulo N. For $f \in S_k(N, \chi) \subset S_k(\Gamma)$ a cusp form with Nebentypus χ, we view the period polynomial ρ_f as an element of W_w^Γ. It clearly satisfies $\rho_f(BC) = \chi(B)\rho_f(C)$ where B runs through a fixed system of coset representatives for $\Gamma \backslash \Gamma'$ and C runs through a fixed system of coset representatives for $\Gamma' \backslash \Gamma_1$. Here $\chi(B) = \chi(d_B)$ where d_B is the lower right entry of B (d_B is well defined modulo N). Let therefore $W_w^{\Gamma,\chi}$, $C_w^{\Gamma,\chi}$ be the subspaces of W_w^Γ, C_w^Γ consisting of elements P with $P(BC) = \chi(B)P(C)$ with B, C as above. We have orthogonal decompositions with respect to the pairing $\{\cdot, \cdot\}$

$$W_w^\Gamma = \bigoplus_\chi W_w^{\Gamma,\chi}, \quad C_w^\Gamma = \bigoplus_\chi C_w^{\Gamma,\chi}$$

where the direct sums are over all characters modulo N. It is easily checked that the spaces $W_w^{\Gamma,\chi}$, $C_w^{\Gamma,\chi}$ are preserved by the action of Hecke operators $\widetilde{T}_n$, and the dimension of $C_w^{\Gamma,\chi}$ equals the dimension of the Eisenstein subspace $\mathcal{E}_k(N, \chi) \subset M_k(\Gamma)$. An indirect proof that the dimensions are equal is provided by Prop. 8.3, which shows that $C_w^{\Gamma,\chi}$ is dual to the space of period polynomials

of Eisenstein series $\widehat{E}_w^{\Gamma,\overline{\chi}}$ with respect to an extension of $\{\cdot,\cdot\}$ to the space $\widehat{W}_w^\Gamma$ of extended period polynomials (see also Prop. 8.10).

The action of ϵ also preserves these subspaces and we have the following generalization of the Eichler-Shimura isomorphism theorem 2.1.

Theorem 5.14 ((Eichler-Shimura)). *The two maps $\rho^\pm : S_k(N,\chi) \to (W_w^{\Gamma,\chi})^\pm$, $f \mapsto \rho_f^\pm$, give rise to isomorphisms, denoted by the same symbols:*

$$\rho^\pm : S_k(N,\chi) \longrightarrow (W_w^{\Gamma,\chi})^\pm/(C_w^{\Gamma,\chi})^\pm.$$

Proof. Injectivity follows from the refinement of Haberland's formula (Theorem 3.3) as in the proof of Theorem 2.1. Surjectivity follows by considering the maps above for all characters χ, and using the fact that the sum of the dimensions of their domains equals the sum of the dimensions of their ranges. $\square$

When computing spaces of modular forms, it is much more efficient to fix coset representatives $C_1,\ldots,C_d$ for $\Gamma'\backslash\Gamma_1$, with $d = [\Gamma_1 : \Gamma']$, and to view $P \in W_w^{\Gamma,\chi}$ as a vector $(P(C_1),\ldots,P(C_d)) \in V_w^d$, since the values at other cosets in $\Gamma\backslash\Gamma_1$ are determined by those above. Note however that this representation depends on the coset representatives chosen, which are fixed once for all.

The action of $\gamma \in \Gamma_1$ on $P \in V_w^d$ can be described as follows. Let $C_i\gamma^{-1} = B_i C_{\sigma(i)}$ where B_i are coset representatives for $\Gamma\backslash\Gamma'$ and $\sigma = \sigma_\gamma$ is a permutation of $\{1,\ldots,d\}$. Then

$$P|\gamma(C_i) = \chi(B_i)P(C_{\sigma(i)})|\gamma.$$

Similarly one can compute the actions $P|_\Sigma \widetilde{T}_n$ with Σ_n the double cosets giving the action of Hecke or Atkin-Lehner operators.

5.5. **Computing period polynomials numerically.** The period polynomials of a newform $f \in S_k(\Gamma_0(N),\chi)$ can be easily computed numerically using the results of this chapter. One finds the space of period polynomials as the kernel of the matrix of period relations, and then its plus and minus subspaces. Assuming the Hecke eigenvalues of f at primes $p \nmid N$ are known, then one can find a rational common eigenvector of enough operators $\widetilde{T}_p$ acting on $(W_w^\Gamma)^\pm$ such that the resulting eigenspace is one dimensional (often one operator $\widetilde{T}_p$ for the smallest prime $p \nmid N$ suffices). The period polynomials $\rho_f^\pm$ are thus determined up to scalars. The normalizing factors can be determined from the critical values $L(w+1,f)$ (nonzero if $k > 2$) and $L(w,f)$ (nonzero if $k > 2$, $k \neq 4$), unless they vanish and one has to determine $L(n, f|A)$ for some $A \in \Gamma\backslash\Gamma_1$. Once the polynomials $\rho_f^\pm$ are determined, Theorem 3.3 gives the Petersson product (f,f), and Proposition 5.4 gives the Hecke eigenvalue λ_n of f for every n.

We have implemented the procedure sketched above in MAGMA [Mgm] (available upon request). To give an idea of the running time, when $f = q + 12q^3 + 88q^7 + \ldots \in S_6(\Gamma_0(100))$ is a newform with rational coefficients, the computation of $\rho_f^\pm$ takes 1 sec (in this case $\dim(W_w^\Gamma)^+ = 78$, $\dim(W_w^\Gamma)^- = 72$, $\dim S_k(\Gamma) = 66$, and $[\Gamma_1 : \Gamma] = 180$). Once the polynomial ρ_f^+ is computed, the Hecke eigenvalue for $p = 10037$ is computed in less than 1 sec using Prop. 5.4. The 'Coefficient' command in MAGMA computes the same eigenvalue in 140 sec, while the 'coefficients' command in SAGE takes 3 sec for the same computation (on the same machine). We used the efficient implementation by W. Stein of the coset space $P^1(\mathbb{Z}/N\mathbb{Z}) \simeq \Gamma_0(N)\backslash\Gamma_1$ via 'P1Classes', and the Hecke elements given by 'HeilbronnCremona' in MAGMA (which give the action of Hecke operators on modular symbols, but see Remark 5.6).

An example. Let $\Gamma = \Gamma_0(5)$, $k = 4$, and $f = q - 4q^2 + 2q^3 + 8q^4 - 5q^5 \ldots \in S_k(\Gamma)$ the unique normalized cusp form. In the following table, the first row contains the elements of $P^1(\mathbb{Z}/5\mathbb{Z})$, namely the bottom rows of a system of representatives for $\Gamma\backslash\Gamma_1$, while the second and third rows list the components of rational generators $P_f^\pm$ of the Hecke eigenspace of $(W_w^\Gamma)^\pm$ that corresponds to the Hecke eigenform f (here P_f^- generates $(W_w^\Gamma)^-$, and only the eigenvalue of T_2 is needed for the computation of P_f^+).

	(0 1)	(1 1)	(1 3)	(1 2)	(1 4)	(1 0)
P_f^+	$-5X^2 + 1$	$-5X^2 + 5$	$8X^2 + 13X - 8$	$8X^2 - 13X - 8$	$-5X^2 + 5$	$-X^2 + 5$
P_f^-	X	$X^2 + 2X + 1$	$2X^2 - 3X - 2$	$-2X^2 - 3X + 2$	$-X^2 + 2X - 1$	X

We have $\rho_f^\pm = \omega_f^\pm P_f^\pm$ with

$$\omega_f^+ = r_{I,w}(f) = -0.0051365773i, \quad \omega_f^- = -wr_{I,w-1}(f) = 0.0208651386,$$

computed via (5.7) using the numerical computation of L-series implemented by Tim Dokchitser in MAGMA. Theorem 3.3 then gives $(f, f) = 0.00014513335$, in agreement with the value obtained analytically in [Co12].

Notice that $r_{A,n}^+(f)/r_{I,w}(f) \equiv 0 \pmod{13}$ for all n with $0 < n < w$, and all $A \in \Gamma\backslash\Gamma_1$ (see (6.1) for notation). As in Manin's proof of Ramanujan's congruence mod 691 [Ma73], this suggests that there should be a congruence between f and an Eisenstein series. Indeed by looking at the first few coefficients[4] we obtain that

$$f \equiv E_4(z) - E_4(5z) \pmod{13}$$

where E_4 is the weight 4 Eisenstein series of full level, normalized to have the coefficient of q equal to 1. A generalization of Ramanujan's congruence to Hecke eigenforms and Eisenstein series for $\Gamma_0(N)$ is in progress.

6. RATIONAL DECOMPOSITION OF MODULAR FORMS

For Γ a finite index subgroup of Γ_1 normalized by ϵ, we give an explicit decomposition of $f \in S_k(\Gamma)$ in terms of explicit generators, generalizing the result in the full level case [Po11]. For Γ_1, these are the generators with rational periods studied in [KZ84], where their periods were first computed. For $\Gamma_0(N)$, the periods of these generators were computed for N square free in [An92], and for arbitrary N in [FY09]. These generators have explicit formulas as Poincaré series when $k > 2$.

To define these generators, for $A \in \Gamma\backslash\Gamma_1$, $0 \leqslant n \leqslant w$, define the periods $r_{A,n}(f)$ by

$$(6.1) \qquad \rho_f(A)(X) = \sum_{n=0}^{w} (-1)^{w-n} \binom{w}{n} r_{A,n}(f) X^{w-n}$$

and similarly define $r_{A,n}^\pm(f)$ with ρ_f replaced by $\rho_f^\pm$. Let $R_{A,n} \in S_k(\Gamma)$ be the dual of the linear functional $f \to \frac{1}{[\overline{\Gamma_1}:\overline{\Gamma}]} r_{A,n}(f)$, with respect to the Petersson product:

$$(f, R_{A,n}) = \frac{r_{A,n}(f)}{[\overline{\Gamma}_1 : \overline{\Gamma}]}, \text{ for all } f \in S_k(\Gamma),$$

---

[4]By Sturm's result checking the first three coefficients of both sides is enough to prove the congruence.

and similarly define $R_{A,n}^+$, $R_{A,n}^-$. For $\Gamma = \Gamma_0(N)$, the polynomials $\rho^-(R_{A,n}^+)$, $\rho^+(R_{A,n}^-)$ have rational coefficients.

For $\kappa \in \{+,-\}$ and $0 \leqslant j \leqslant w$, define the linear combinations of periods:

$$s_{A,j}^\kappa(f) = \sum_{n=0}^{j} \binom{j}{n}(-1)^{j-n}r_{A,j}^\kappa(f).$$

We then have the following generalization of Theorem 1.1 in [Po11], which gives explicit inverses of the Eichler-Shimura maps (2.5).

Theorem 6.1. *Let Γ be a finite index subgroup of Γ_1 normalized by ϵ, and let $\kappa_1, \kappa_2 \in \{+,-\}$ with $\kappa_1 \neq \kappa_2$ if k even and $\kappa_1 = \kappa_2$ if k odd. For $f \in S_k(\Gamma)$*

$$\frac{-3C_k}{2} \cdot f = \sum_{A \in \overline{\Gamma}\backslash\overline{\Gamma}_1} \sum_{n=0}^{w} \binom{w}{n} s_{AU^{-1},n}^{\kappa_1}(f) R_{A,n}^{\kappa_2}.$$

Proof. Let $P \in (W_w^\Gamma)^{\kappa_1}$, $Q \in (W_w^\Gamma)^{\kappa_2}$. Then

$$\{P, \overline{Q}\} = \langle\!\langle P|US - SU^2, \overline{Q}\rangle\!\rangle = \langle\!\langle P|U^2 - U, \overline{Q}\rangle\!\rangle = -2\langle\!\langle P|U, \overline{Q}\rangle\!\rangle$$

where the second equality follows since $P|S = -P, Q|S = -Q$, while the third from $P|U^2 - U = P| - 1 - 2U$, together with (3.6).

If $R(X) = \sum_{n=0}^{w}(-1)^{w-n}\binom{w}{n}r_n X^{w-n} \in V_w$ then

$$R|U(X) = \sum_{j=0}^{w} \binom{w}{j}s_j X^j, \text{ with } s_j = \sum_{n=0}^{j}(-1)^{j-n}\binom{j}{n}r_n.$$

By Theorem 3.3 and the preceding computations it follows that for $f, g \in S_k(\Gamma)$

$$3C_k(f,g) = \{\rho_f^{\kappa_1}, \overline{\rho_g^{\kappa_2}}\} = -\frac{2}{[\overline{\Gamma}_1 : \overline{\Gamma}]} \sum_{A \in \overline{\Gamma}\backslash\overline{\Gamma}_1} \sum_{j=0}^{w} \binom{w}{j}s_{AU^{-1},j}^{\kappa_1}(f)\overline{r_{A,j}^{\kappa_2}(g)}.$$

Since $\overline{r_{A,j}^+(g)} = [\overline{\Gamma}_1 : \overline{\Gamma}] \cdot (R_{A,j}^+, g)$, the claim follows. □

7. Extra relations satisfied by period polynomials of cusp forms

In this section, we determine the image of the maps $\rho^\pm : S_k(\Gamma) \to (W_w^\Gamma)^\pm$, namely the extra relations satisfied by $\rho_f^\pm$ for $f \in S_k(\Gamma)$ which are independent of the period relations. To be explicit, the extra relations we obtain require the determination of the periods $r_{B,m}^\mp(R_{A,w}^\pm)$ of the generators defined in the previous section. For Γ_1, these periods were computed in [KZ84], and for $\Gamma_0(N)$, they were computed in [An92] (for N square free, and not quite in closed form), and in [FY09] (only the principal periods $r_{I,m}^\mp(R_{I,n}^\pm)$). For $\Gamma_0(N)$ with small N, the computations in [FY09] are sufficient to make completely explicit the extra relations, and we illustrate this for the case $\Gamma = \Gamma_0(2)$. The relations are similar to the relation found by Kohnen and Zagier in the full-level case [KZ84] (see also [Po11, Sec. 2]).

We first define bases of $(C_w^\Gamma)^\pm$, using the terminology in Definition 4.3. For each cusp $\mathcal{C} \in \Gamma\backslash\Gamma_1/\Gamma_{1\infty}$, which is regular if k is odd, define $P_\mathcal{C} \in C_w^\Gamma$ as in (4.2) by fixing $A_\mathcal{C} \in \mathcal{C}$ a representative, and setting $c_A = (-1)^w c_{AJ} = 1$ if $[A]^+ = [A_\mathcal{C}]^+$, and $c_A = 0$ if $[A] \neq [A_\mathcal{C}]$. Then $\{P_\mathcal{C}\}$ form a basis of C_w^Γ if $k > 2$, and if $k = 2$ there is only one relation $\sum_\mathcal{C} P_\mathcal{C} = 0$.

Assume that Γ is normalized by ϵ, and denote $A' = \epsilon A \epsilon$. Note that if $\mathcal{C} = [A]$ is a cusp, then $\mathcal{C}' = [A']$ is well-defined. Since $P_\mathcal{C}|\epsilon = P_{\mathcal{C}'}$, we have $P_\mathcal{C}^+ = P_{\mathcal{C}'}^+$ and $P_\mathcal{C}^- = -P_{\mathcal{C}'}^-$. Therefore a basis of $(C_w^\Gamma)^-$ consists of $P_\mathcal{C}^-$ for each unordered pair $(\mathcal{C}, \mathcal{C}')$ of cusps with $\mathcal{C} \neq \mathcal{C}'$, and a basis of $(C_w^\Gamma)^+$ consists of $P_\mathcal{C}^+$ for each unordered pair $(\mathcal{C}, \mathcal{C}')$ of cusps (when k is odd only pairs of regular cusps are considered; note that $\mathcal{C}$ is regular iff $\mathcal{C}'$ is regular).

We now fix $A \in \mathcal{C}$ for each (regular if k odd) cusp $\mathcal{C} \in \Gamma \backslash \Gamma_1 / \Gamma_{1\infty}$ and we write, by a slight abuse of notation, $R_{\mathcal{C},n} = R_{A,n}$, $r_{\mathcal{C},n} = r_{A,n}$ with the notation of the previous section ($R_{\mathcal{C},n}$ does depend on the choice of representative A, but we fix such a choice). For each unordered pair of cusps $(\mathcal{C}, \mathcal{C}')$ we take $g = R_{\mathcal{C},w}^+$ in Theorem 3.3, and if $\mathcal{C} \neq \mathcal{C}'$ we take also $g = R_{\mathcal{C},w}^-$ (again only for regular cusps if k is odd), obtaining the following linear relations satisfied by all $f \in S_k(\Gamma)$ [5]

$$(7.1) \qquad \frac{3C_k}{[\overline{\Gamma_1} : \overline{\Gamma}]} r_{\mathcal{C},w}^+(f) = \{\rho^+(f), \overline{\rho^-(R_{\mathcal{C},w}^+)}\}, \qquad \frac{3C_k}{[\overline{\Gamma_1} : \overline{\Gamma}]} r_{\mathcal{C},w}^-(f) = \{\rho^-(f), \overline{\rho^+(R_{\mathcal{C},w}^-)}\}.$$

The linear forms appearing in these relations can be applied to all $P \in (W_w^\Gamma)^\pm$, and putting together the relations involving $\rho^+(f)$ into a map λ_+, and the relations involving $\rho^-(f)$ (for pairs with $\mathcal{C} \neq \mathcal{C}'$) into a map λ_-, we obtain two linear maps $\lambda_\pm : (W_w^\Gamma)^\pm \to \mathbb{C}^{d_\pm}$ with $d_\pm = \dim(C_w^\Gamma)^\pm$. If $d_- = 0$ the map λ_- is trivial.

Proposition 7.1. *Assume $k \geqslant 3$ and let Γ be a finite index subgroup of Γ_1 normalized by ϵ. With $\lambda_\pm$ defined above, we have exact sequences:*

$$0 \to S_k(\Gamma) \xrightarrow{\rho^\pm} (W_w^\Gamma)^\pm \xrightarrow{\lambda_\pm} \mathbb{C}^{d_\pm} \to 0.$$

Proof. We have $\mathrm{Im}\rho^\pm \subset \ker \lambda_\pm$ by construction. Note that the first relation in (7.1) is not satisfied by $P_\mathcal{C}^+$, while the second is not satisfied by $P_\mathcal{C}^-$ if $\mathcal{C} \neq \mathcal{C}'$, since the LHS is nonzero, while the RHS vanishes by Lemma 4.1 a). Since $\{P_\mathcal{C}^\pm\}$ form a basis of $(C_w^\Gamma)^\pm$, the conclusion follows. $\square$

Remark 7.2. For $\Gamma = \Gamma_0(N)$, Proposition 4.4 characterizes those N for which the extra relations involve only the even parts of the period polynomials.

For example, if $\Gamma = \Gamma_0(p)$ with p prime, there are only two cusps $[I]$ and $[S]$, and for all $P \in W_w^\Gamma$ we have $P(S) = -P(I)|S$ by the period relations. In particular $r_{S,w}(f) = -r_{I,0}(f)$. Noting also that $P(I)^+$, $P(S)^+$ are the even parts of $P(I), P(S)$, so that $R_{I,n}^+ = R_{I,n}$ for n even, we have the following simpler version of Proposition 7.1.

Corollary 7.3. *Let $\Gamma = \Gamma_0(p)$ with p prime, and let $k > 2$ even. Then the two extra relations satisfied by all even period polynomials $\rho^+(f)$ for $f \in S_k(\Gamma)$ are*

$$\frac{3C_k}{[\overline{\Gamma_1} : \overline{\Gamma}]} r_{I,a}(f) = \{\rho^+(f), \overline{\rho^-(R_{I,a})}\}, \qquad for \ a = 0, w.$$

For small values of N (eg $N = 2, 3, 4, 5$), the polynomials ρ_f are completely determined by the principal parts $\rho_f(I)$, so that the relations above are completely explicit via the computation of $\rho^-(R_{I,a})(I)$ in [FY09]. In the remainder of this section, we discuss in detail the case of period polynomials for $\Gamma = \Gamma_0(2)$, which have been studied in [IK05],[FY09], [KT11].

We take as coset representatives for $\Gamma \backslash \Gamma_1$ the set $\{I, U, U^2\}$. Denoting by $\overline{A}$ the coset ΓA, we have $\overline{S} = \overline{U}$, $\overline{US} = \overline{I}$, $\overline{U^2 S} = \overline{U^2}$. For $P \in W_w^\Gamma$, the period relations $P|1 + S = 0$, $P|1 + U + U^2 = 0$

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[5]In this section we occasionally write $\rho(f)$ instead of ρ_f to simplify notation.

reduce to

(7.2) $P(U) + P(I)|S = 0,$ $P(U^2)|1 + S = 0,$ $P(U^2) + P(U)|U + P(I)|U^2 = 0.$

The polynomials $P(U), P(U^2)$ are therefore determined by $P(I)$ which satisfies the relation

(7.3) $P(I)|(ST - ST^{-1})(1 + S) = 0.$

Let $U_w \subset V_w$ denote the set of polynomials satisfying (7.3), so that we can identify W_w^Γ with U_w via $P \to P(I)$. Conjugation by ϵ leaves unchanged each coset $\overline{I}, \overline{U}, \overline{U^2}$, hence P^+, P^- correspond to the even and odd parts of the polynomial $P(I)$ in this identification.

   To express the formula in Theorem 3.3 in terms of $\rho_f(I), \rho_g(I)$ alone, let $P \in (W_w^\Gamma)^+, Q \in (W_w^\Gamma)^-$. We have $\lang\!\langle P|T - T^{-1}, Q \rangle\!\rangle = -2 \lang\!\langle P|U, Q \rangle\!\rangle$ as in the proof of Theorem 6.1, and using (7.2) we obtain

$$\lang\!\langle P|U, Q \rangle\!\rangle = \frac{1}{3} < P(I)|2T^{-1} - 2I - T, Q(I) >= -\frac{1}{2} < P(I)|T - T^{-1}, Q(I) >$$

where $\langle P(I), Q(I) \rangle = \langle P(I)|T + T^{-1}, Q(I) \rangle = 0$ since $P(I), Q(I)$ have opposite parity. We can therefore restate Theorem 3.3 as follows, setting $P_f = \rho_f(I) \in U_w$

(7.4) $3C_k \cdot (f, g) = \langle P_f^+|T - T^{-1}, \overline{P_g^-} \rangle .$

   By Proposition 4.4 and the Eichler-Shimura isomorphism (2.5), the map $f \to P_f^-$ gives an isomorphism $S_k(\Gamma_0(2)) \simeq U_w^-$, while the image of the map $f \to P_f^+$ is a codimension 2 subspace of U_w^+. [6] To simplify notation, let $r_n(f) = r_{I,n}(f)$, and $R_n = R_{I,n}$ for $0 \leqslant n \leqslant w$.

Corollary 7.4. *For f in $S_k(\Gamma_0(2))$, let $s_n(f) = \displaystyle\sum_{\substack{j=0 \\ n-j \ odd}}^{n} \binom{n}{j} r_j(f)$. The extra relations satisfied by the even periods of f are*

$$r_a(f) = \sum_{\substack{n=0 \\ n \ odd}}^{w} \binom{w}{n} s_{w-n}(f) \frac{2}{C_k} r_a(R_n)   for \ a = 0, w.$$

From [FY09], for $0 < w < n$, n odd, we have $r_w(R_n) = -\dfrac{r_0(R_{\widetilde{n}})}{N^n}$ and

$$\frac{2}{C_k} r_0(R_n) = -N^{\widetilde{n}} \frac{B_{\widetilde{n}+1}}{\widetilde{n}+1} + \frac{k}{B_k} \frac{B_{n+1}}{n+1} \frac{B_{\widetilde{n}+1}}{\widetilde{n}+1} \frac{\alpha_{N,k}(n)}{N} + \frac{\delta_{w,n+1}}{w},$$

where $\widetilde{n} = w - n$, $\alpha_{N,k}(n) = \frac{1 - N^{-n-1}}{1 - N^{-k}}$ (recall $N = 2$), and B_m are the Bernoulli numbers. Note that there is a minus sign missing in the normalization of the generators denoted by $R_{\Gamma,w,n}$ in [FY09, Def. 1.1], and with this correction we have $R_n = -\frac{C_k}{2} R_{\Gamma,w,n}$.

Proof. Since $P_f|T - T^{-1}(X) = -2\sum_{n=0}^{w}(-1)^n \binom{w}{n} s_n(f) X^{w-n}$, and $\overline{r_n(R_0)} = r_0(R_n)$, the claim follows from Corollary 7.3.

                                                                                    $\square$

   The periods $r_n(f)$ are related to the critical values of the L-series associated to f, and when f is a newform they can be readily computed using MAGMA [Mgm]. The relations in Corollary 7.4 have been checked numerically for $k = 8, 10, 14$.

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[6]A direct proof in this case is contained in [KT11, Theorem 4].

8. PERIOD POLYNOMIALS OF ARBITRARY MODULAR FORMS

In this section we define period polynomials for noncuspidal modular forms, and extend Haberland's formula and the action of Hecke operators to the larger space of period polynomials of all modular forms. An important feature of the larger space $\widehat{W}_w^\Gamma$ is that the the pairing $\{\cdot, \cdot\}$ has a natural nondegenerate extension to it, while on W_w^Γ it is degenerate (its radical is C_w^Γ). If $\Gamma = \Gamma_1(N)$ and $k > 2$, the period polynomial maps $\rho^\pm$ extend naturally to the larger space, and they give isomorphisms between $M_k(\Gamma)$ and $(\widehat{W}_w^\Gamma)^\pm$. Surprisingly, when $k = 2$, $\Gamma = \Gamma_0(N)$ and N is squarefree with at least two prime factors, only one of the two maps is an isomorphism.

For the full modular group, period polynomials of Eisenstein series were defined in [KZ84], using the description of periods as special values of the associated L-function, and the enlarged space of period polynomials was introduced in [Za91]. A different construction using an Eichler integral was given more recently in [BGKO], in the more general context of weakly holomorphic modular forms. We extend both the Eichler integral and the L-function approach to a finite index subgroup Γ of Γ_1.

For $f \in M_k(\Gamma)$, we define $\widehat{\rho}_f = \widetilde{f}|1 - S$ as in (2.7) (with the same action as on period polynomials), where the Eichler integral $\widetilde{f} : \Gamma\backslash\Gamma_1 \to \mathcal{A}$ is given by

$$(8.1) \qquad \widetilde{f}(A)(z) = \int_z^{i\infty} [f|A(t) - a_0(f|A)](t - z)^w dt$$

with $a_0(f|A)$ the constant term of the Fourier expansion of $f|A$. Note that $a_0(f|A) = a_0(f|AT)$, so $\widetilde{f}|1 - T = 0$. Let

$$\widehat{V}_w := \Big\{ \sum_{-1 \leqslant i \leqslant w+1} a_i X^i : a_i \in \mathbb{C} \Big\},$$

and let $\widehat{V}_w^\Gamma$ be the space of functions $P : \Gamma\backslash\Gamma_1 \to \widehat{V}_w$ with $P(-A) = (-1)^w P(A)$ for $A \in \Gamma\backslash\Gamma_1$. We define an action of $g \in \Gamma_1$ on $P \in \widehat{V}_w^\Gamma$ by $P|g(A) = P(Ag^{-1})|_{-w}g$ as before. Note that this is no longer well defined in general, as elements of Γ_1 do not preserve the space $\widehat{V}_w$. However one can still define the subspace

$$\widehat{W}_w^\Gamma = \{P \in \widehat{V}_w^\Gamma : P|1 + S = P|1 + U + U^2 = 0\}.$$

We will show below that $\widehat{\rho}_f \in \widehat{W}_w^\Gamma$. That it satisfies the period relations is immediate: we have $\widehat{\rho}_f|1 + S = 0$, and $\widehat{\rho}_f|1 - T = 0$ from the definition, while

$$\widehat{\rho}_f|1 + U + U^2 = \widetilde{f}|(1 - S)(1 + U + U^2) = \widetilde{f}|(1 - T^{-1})(1 + U + U^2) = 0.$$

It remains to show that $\widehat{\rho}_f \in \widehat{V}_w^\Gamma$, and we will do this by relating it with the polynomial $\rho_f \in V_w^\Gamma$ defined by (6.1), where

$$(8.2) \qquad r_{A,n}(f) = (-1)^{n+1} \frac{\Gamma(n + 1)}{(2\pi i)^{n+1}} L(n + 1, f|A).$$

The L-function $L(s, f) = \sum_{n=1}^\infty a_n(f) n^{-s}$ is given, if $\mathrm{Re}(s) > k$, by the Mellin transform

$$(-1)^s (2\pi i)^{-s} \Gamma(s) L(s, f) = \int_0^{i\infty} [f(t) - a_0(f)] t^{s-1} dt,$$

and it can be extended meromorphically to $\mathbb{C}$ by fixing $z_0 \in \mathcal{H}$, decomposing $\int_0^{i\infty} = \int_0^{z_0} + \int_{z_0}^{i\infty}$, and making a change of variables $t = Su$ in the first integral. We obtain a meromorphic function

with at most simple poles at $s = 0$ and $s = k$:

$$\frac{(-1)^s \Gamma(s)}{(2\pi i)^s} L(s, f) = \int_{z_0}^{i\infty} [f(t) - a_0(f)] t^{s-1} dt + (-1)^s \int_{Sz_0}^{i\infty} [f|S(t) - a_0(f|S)] t^{k-s-1} dt -$$
$$- a_0(f) \frac{z_0^s}{s} - (-1)^s a_0(f|S) \frac{(Sz_0)^{k-s}}{k-s}.$$

Introducing as in [KZ84] the function $H_{z_0} \in V_w^\Gamma$ defined for $A \in \Gamma \backslash \Gamma_1$ by

(8.3) $$H_{z_0}(A) = \int_{z_0}^{i\infty} [f|A(t) - a_0(f|A)](t - X)^w dt - a_0(f|A) \int_0^{z_0} (t - X)^w dt \in V_w$$

we obtain from (6.1) and the analytic continuation above

(8.4) $$\rho_f(A) = H_{z_0}(A) - H_{Sz_0}(AS^{-1})|S,$$

namely $\rho_f = H_{z_0} - H_{Sz_0}|S$.

We now determine the relation between $\widehat{\rho}_f$ and ρ_f, which also shows that $\widehat{\rho}_f \in \widehat{W}_w^\Gamma$.

Proposition 8.1. *For $f \in M_k(\Gamma)$, let $\rho_f^0 \in \widehat{V}_w^\Gamma$ be given by $\rho_f^0(A) = (-1)^w \frac{a_0(f|A)}{w+1} X^{w+1}$. We have*

$$\widehat{\rho}_f = \rho_f + \rho_f^0|(1 - S),$$

namely $\widehat{\rho}_f(A) = \rho_f(A) + (-1)^w \frac{a_0(f|A)}{w+1} X^{w+1} + \frac{a_0(f|AS^{-1})}{w+1} X^{-1}$.

Proof. Fixing $z_0 \in \mathcal{H}$ and decomposing the integral in (8.1) as $\int_{z_0}^{i\infty} + \int_z^{z_0}$ we have

$$\widetilde{f}(A)(z) = H_{z_0}(A)(z) + \int_z^{z_0} f|A(t)(t - z)^w dt + a_0(f|A) \int_0^z (t - z)^w dt.$$

Using the same relation for $\widetilde{f}(AS^{-1})(Sz)j(S, z)^w$ with Sz_0 in place of z_0 we obtain, after a change of variables $u = St$ in the first integral above

$$(\widetilde{f}|1 - S)(A)(z) = H_{z_0}(A)(z) - H_{Sz_0}(AS^{-1})|S(z) +$$
$$+ a_0(f|A) \int_0^z (t - z)^w dt - a_0(f|AS^{-1}) \int_0^{Sz} (t - Sz)^w j(S, z)^w dt.$$

Computing the last integrals and comparing with (8.4) yields the conclusion. □

We now determine the exact relationship between $\widehat{W}_w^\Gamma$ and W_w^Γ. For $\widehat{P} \in \widehat{W}_w^\Gamma$ write $\widehat{P} = P + P_0$ where $P \in V_w^\Gamma$ and $P_0(A) = c_A X^{w+1} + d_A X^{-1}$ for $A \in \Gamma \backslash \Gamma_1$. From $\widehat{P}|1 + S = 0$ we obtain $d_A = c_{AS}$. From $\widehat{P}|1 + U + U^2 = 0$, it follows that $P_0|1 + U + U^2 \in V_w^\Gamma$, which implies that $c_A = c_{AT}$ for all $A \in \Gamma_1$. Therefore we have $P_0 = P^0|1 - S$, where $P^0(A) = c_A X^{w+1}$, with $c_A = c_{AT}$. In conclusion, letting

(8.5) $$D_w^\Gamma = \{(c_A X^{w+1})_A|(1 - S) : c_A = c_{AT} = (-1)^w c_{AJ} \in \mathbb{C}\} \subset \widehat{V}_w^\Gamma$$

we have a unique decomposition of $\widehat{P} \in \widehat{W}_w^\Gamma$ as above

(8.6) $$\widehat{P} = P + P^0|(1 - S), \quad P \in V_w^\Gamma, \ P^0|(1 - S) \in D_w^\Gamma.$$

For $\widehat{\rho}_f$ this is the decomposition in Proposition 8.1, since $a_0(f|A) = a_0(f|AT) = (-1)^w a_0(f|AJ)$. As in the proof of Lemma 4.2, note that $\dim D_w^\Gamma$ equals $e_\infty(\Gamma)$ or $e_\infty^{\mathrm{reg}}(\Gamma)$, depending on whether k is even or odd respectively.

When $k = 2$ there is an extra relation satisfied by the coefficients of P^0 in (8.6). Letting $P(A) = d_A \in \mathbb{C}$, $P^0(A) = c_A X^{w+1}$, the period relations now imply that

$$d_A + d_{AU} + d_{AU^2} + 2(c_A + c_{AU} + c_{AU^2}) = 0, \text{ with } d_A + d_{AS} = 0, c_A = c_{AT}$$

for all $A \in \Gamma \backslash \Gamma_1$. The relation $d_A + d_{AS} = 0$ implies $\sum_A d_A = 0$, and then the first relation above implies that $\sum_A c_A = 0$ as well, where the sum is over a complete system of representatives for $\Gamma \backslash \Gamma_1$. From Proposition 8.1 it follows that $\sum_A a_0(f|A) = 0$ for all $f \in M_2(\Gamma)$.

Proposition 8.2. *a) If $k \geqslant 3$ there is an exact sequence*

$$0 \to W_w^\Gamma \to \widehat{W}_w^\Gamma \to D_w^\Gamma \to 0$$

where the first map is inclusion, and the second is the map $\widehat{P} \to P^0|1 - S$ defined above.
b) If $k = 2$ there is an exact sequence

$$0 \to W_w^\Gamma \to \widehat{W}_w^\Gamma \to D_w^\Gamma \to \mathbb{C} \to 0$$

where the last map takes $P^0|1 - S \in D_w^\Gamma$ with $P^0(A) = c_A X^{w+1}$ to $\sum_{A \in \Gamma \backslash \Gamma_1} c_A$.

Proof. Exactness at $\widehat{W}_w^\Gamma$ follows from the definition. If $k \geqslant 3$, surjectivity of the second map follows from Proposition 8.1, and the fact that there is a basis of Eisenstein series $E_k^\mathcal{C} \in M_k(\Gamma)$ for $\mathcal{C} = [A_\mathcal{C}]$ a complete system of representatives for the (regular if k is odd) cusps in $\Gamma \backslash \Gamma_1 / \Gamma_{1\infty}$, such that $a_0(E_k^\mathcal{C}|A) = (-1)^w a_0(E_k^\mathcal{C}|AJ)$ equals 1 if $[A]^+ = [A_\mathcal{C}]^+$, and 0 if $[A] \neq \mathcal{C}$ (see Definition 4.3 for notation). If $k = 2$ the Eisenstein subspace of $M_2(\Gamma)$ is spanned by modular forms f which are nonzero only at a fixed pair of nonequivalent cusps and are zero at other cusps, and such that $\sum_{A \in \Gamma \backslash \Gamma_1} a_0(f|A) = 0$. Their images in D_w^Γ span the kernel of the last map, proving exactness at D_w^Γ. $\qquad \square$

The previous proposition shows that $\dim \widehat{W}_w^\Gamma = 2 \dim M_k(\Gamma)$. From the proof we conclude that there is a direct sum decomposition

$$(8.7) \qquad \widehat{W}_w^\Gamma = W_w^\Gamma \oplus \widehat{E}_w^\Gamma$$

where $\widehat{E}_w^\Gamma$ is the image of the Eisenstein subspace $\mathcal{E}_k(\Gamma) \subset M_k(\Gamma)$ under the map $f \to \widehat{\rho}_f$.

The pairing $\{\cdot, \cdot\}$ extends to a pairing on $\widehat{W}_w^\Gamma \times \widehat{W}_w^\Gamma$, by decomposing $\widehat{P}, \widehat{Q} \in \widehat{W}_w^\Gamma$ as in (8.6) and setting:

$$(8.8) \quad \{\widehat{P}, \widehat{Q}\} = \langle\!\langle P|T - T^{-1}, Q \rangle\!\rangle + \langle\!\langle 2P^0|T - T^{-1}, Q \rangle\!\rangle + \langle\!\langle P, 2Q^0|T^{-1} - T \rangle\!\rangle + I_k(P^0, Q^0),$$

where $I_k(P^0, Q^0) = 0$ if k even and $I_k(P^0, Q^0) = \frac{6(k-1)}{k[\Gamma_1:\Gamma]} \sum_{A \in \Gamma \backslash \Gamma_1} c_A c_A'$ if k is odd, where $P^0(A) = c_A X^{w+1}$, $Q^0(A) = c_A' X^{w+1}$.

Since $P^0|T - T^{-1} \in V_w^\Gamma$ this pairing is well-defined, and it is easily checked that it behaves as in (3.6) under the action of ϵ defined as in (2.2). We will show below that this definition is natural for two reasons: Haberland's formula generalizes to arbitrary modular forms, if the Petersson product is extended in a natural way to all modular forms, and this pairing is Hecke equivariant for $\Gamma = \Gamma_1(N)$, with the same action of Hecke operators on $\widehat{W}_w^\Gamma$ as on W_w^Γ.

Recall that on W_w^Γ the pairing $\{\cdot, \cdot\}$ is degenerate, its radical being C_w^Γ (Lemma 4.1). We now show that the extended pairing is nondegenerate on $\widehat{W}_w^\Gamma$, more precisely that the dual of C_w^Γ inside $\widehat{W}_w^\Gamma$ is the space $\widehat{E}_w^\Gamma$.

Proposition 8.3. (a) *Let* $P = P'|1 - S \in C_w^\Gamma$ *and* $\widehat{Q} = Q + Q^0|1 - S \in \widehat{W}_w^\Gamma$, *and let* $P'(A) = c_A'$, $Q^0(A) = (-1)^w c_A \frac{X^{w+1}}{w+1}$ *for* $A \in \Gamma \backslash \Gamma_1$ *(so that* $c_A' = c_{AT}' = (-1)^w c_{AJ}', c_A = c_{AT} = (-1)^w c_{AJ}$*). Then*

$$\{P, \widehat{Q}\} = -\frac{6}{[\overline{\Gamma}_1 : \overline{\Gamma}]} \sum_{A \in \overline{\Gamma} \backslash \overline{\Gamma}_1} c_A' c_A.$$

(b) *The pairing* $\{\cdot, \cdot\}$ *is nondegenerate on* $\widehat{W}_w^\Gamma$, *and the dual of* C_w^Γ *is* $\widehat{E}_w^\Gamma$.

Proof. (a) As in the proof of Lemma 4.1, we use the formal relation (4.1), together with the relation $(1 - S)(1 + U + U^2) = (1 - T^{-1})(1 + U + U^2)$ and the Γ_1 invariance of the pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$:

$$\begin{aligned}
\{P, \widehat{Q}\} &= \langle\!\langle P'|(1 - S)(T - T^{-1}), Q \rangle\!\rangle + 2 \langle\!\langle P', Q^0|(T^{-1} - T)(1 - S) \rangle\!\rangle \\
&= 2 \langle\!\langle P', Q|1 + U + U^2 \rangle\!\rangle + 2 \langle\!\langle P', Q^0|(T^{-1} - T)(1 - S) \rangle\!\rangle \\
&= -2 \langle\!\langle P', Q^0|[(1 - T^{-1})(1 + U + U^2) + (T - T^{-1})(1 - S)] \rangle\!\rangle \\
&= -2 \langle\!\langle P', Q^0|[2(1 - T^{-1}) + T - 1] \rangle\!\rangle \\
&= -\frac{6}{[\overline{\Gamma}_1 : \overline{\Gamma}]} \sum_A c_A' c_A.
\end{aligned}$$

(b) We choose a basis of $\widehat{W}_w^\Gamma$ by concatenating bases for $C_w^\Gamma, \rho^-(S_k(\Gamma)) + \rho^+(S_k(\Gamma)), \widehat{E}_w^\Gamma$ in this order. The block form matrix of the pairing $\{\cdot, \cdot\}$ with respect to this basis is

$$\begin{pmatrix} 0 & 0 & \mathcal{A} \\ 0 & B & 0 \\ (-1)^{w+1} \mathcal{A}^t & 0 & C \end{pmatrix},$$

so it is enough to show that $\mathcal{A}$ is nonsingular (B is nonsingular by Theorem 3.3). When $k > 2$ this is obvious from part (a). For $k = 2$, we fix a cusp $\mathcal{C}_0$ in $\Gamma \backslash \Gamma_1 / \Gamma_{1\infty} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_n\}$ and we let a basis of C_w^Γ consist of P_i, $1 \leqslant i \leqslant n$, as in the statement of the lemma, with the constants $c_{iA}' = 1$ if $[A] \in \mathcal{C}_i$, and $c_{iA}' = 0$ otherwise (note that $P_0 = -\sum_{i=1}^n P_i$). Letting $l_i = \#\{A \in \overline{\Gamma} \backslash \overline{\Gamma}_1 : [A] \in \mathcal{C}_i\}$ (the width of the cusp $\mathcal{C}_i$), we take a basis of $\widehat{E}_w^\Gamma$ to consist of $\widehat{Q}_i$ as in the statement, $1 \leqslant i \leqslant n$, with $c_{iA} = 1$ if $[A] \in \mathcal{C}_i$, $c_{iA} = -\frac{l_i}{l_0}$ if $[A] \in \mathcal{C}_0$, and $c_{iA} = 0$ otherwise. With respect to this basis the matrix $\mathcal{A}$ is diagonal, so the pairing is nondegenerate. $\square$

Assume now that Γ is normalized by ϵ. Since the action of ϵ given by (2.2) preserves $\widehat{W}_w^\Gamma$ and D_w^Γ, passing to the ± 1 eigenspaces in Proposition 8.2 gives exact sequences

(8.9) $$0 \to (W_w^\Gamma)^\pm \to (\widehat{W}_w^\Gamma)^\pm \to (D_w^\Gamma)^\pm \to \mathbb{C} \to 0,$$

where the last map is nontrivial only if $k = 2$ and the sign is minus, when it is defined in Proposition 8.2 b) (when $k = 2$ and $P^0|1 - S \in (D_w^\Gamma)^+$ with $P^0(A) = c_A x^{w+1}$, then $c_A = -c_{A'}$ and $\sum_A c_A = 0$ automatically). From (4.2) and (8.5) we see that $\dim(D_w^\Gamma)^+ = \dim(C_w^\Gamma)^-$ for all k; $\dim(D_w^\Gamma)^- = \dim(C_w^\Gamma)^+$ for $k \geqslant 3$; and $\dim(C_w^\Gamma)^- - 1 = \dim(C_w^\Gamma)^+$ for $k = 2$. Combined with the Eichler-Shimura isomorphism (2.5) and Lemma 4.2, this implies that $\dim(\widehat{W}_w^\Gamma)^\pm = \dim M_k(\Gamma)$.

The next Proposition can be seen as a extension of the Eichler-Shimura isomorphism (2.5) to the entire space of modular forms.

Proposition 8.4. (a) *Assume that* k, Γ *are such that the extended Petersson scalar product on* $M_k(\Gamma)$ *defined in §8.2 is nondegenerate (see Remark 8.5). Then the maps* $\widehat{\rho}^\pm : M_k(\Gamma) \to (\widehat{W}_w^\Gamma)^\pm$, $f \to \widehat{\rho}_f^\pm$, *are isomorphisms.*

(b) *Assume that* $(C_w^\Gamma)^- = 0$ *(for example* $\Gamma = \Gamma_0(N)$ *with* N *as in Proposition 4.4). Then* $\widehat{\rho}^-$ *is an isomorphism.*

Remark 8.5. It is shown in [PP12] that the extended Petersson product is nondegenerate for $\Gamma_1(N)$ (and therefore also for $\Gamma_0(N)$) when $k > 2$. When $k = 2$ the extended Petersson product is nondegenerate for $\Gamma_1(p)$ or $\Gamma_0(p)$ with p prime, while it is degenerate for $\Gamma_0(N)$ with N squarefree with at least two prime factors. This implies that in the latter case the map $\widehat{\rho}^+$ is not an isomorphism; indeed, part b) shows that $\widehat{\rho}^-$ is an isomorphism, and if both $\widehat{\rho}^\pm$ were isomorphisms, then the Petersson product would be nondegenerate by Theorem 8.6 c) below, since the pairing $\{\cdot, \cdot\}$ is nondegenerate.

Proof. a) Since the dimensions of the spaces are equal, we only have to prove injectivity. If $\widehat{\rho}_f^\pm = 0$, it follows from Theorem 8.6 c) that $f = 0$, when the extended Petersson product on $M_k(\Gamma)$ is nondegenerate.

b) If $(C_w^\Gamma)^- = 0$, then $(D_w^\Gamma)^+ = 0$ as well. Assuming $\widehat{\rho_f} = 0$ for $f \in M_k(\Gamma)$, it follows that $\rho_f^0 = 0$ so f is a cusp form, hence $f = 0$ by Theorem 3.3. $\qquad\square$

8.1. An example.
As an example, we check directly that the map $\widehat{\rho}^+$ is not an isomorphism for $k = 2$ and $\Gamma = \Gamma_0(6)$. As explained in Remark 8.5, this gives an alternative proof that the extended Petersson product is degenerate in this case, in agreement with [PP12].

As representatives A_j for $\Gamma \backslash \Gamma_1$ we take the matrices

$$ST^{-i}S\{I, U^2, U\}, \quad i = 0, 1, 2, 3$$

in this order (namely $(A_1, \ldots, A_{12}) = (I, U^2, U, ST^{-1}S, ST^{-1}SU^2, \ldots, ST^{-3}SU)$), obtained from the set of representatives provided by the command 'CosetRepresentatives' in MAGMA. There are four cusps $\mathcal{C}_i \in \Gamma \backslash \Gamma_1 / \Gamma_{1\infty}$, and we have $\mathcal{C}_1 = [A_1]$, $\mathcal{C}_2 = [A_9] = [A_{12}]$, $\mathcal{C}_3 = [A_6] = [A_7] = [A_{11}]$, while the remaining six matrices are in the class $\mathcal{C}_4$.

Since there are no cusp forms of weight two on $\Gamma_0(6)$, we have $W_w^\Gamma = C_w^\Gamma$. The latter space is spanned by polynomials P_i supported at the class $\mathcal{C}_i$, $i = 1, 2, 3$, namely $P_i = (c_A)_A | 1 - S$ with $c_A = c_{AT}$ and $c_A = 1$ if $[A] = \mathcal{C}_i$, $c_A = 0$ otherwise. We identify a polynomial $P \in C_w^\Gamma$ with a vector $\mathbf{d} = (d_i) \in \mathbb{C}^{12}$ with $P(A_i) = d_i$. Let $\sigma \in \mathcal{S}_{12}$ be the permutation such that $A_j S = A_{\sigma j}$. We have $\sigma = (3, 4, 1, 2, 7, 10, 5, 12, 11, 6, 9, 8)$, and it follows that the vectors $\mathbf{d}$ corresponding to the polynomials P_1, P_2, P_3 are given respectively by (the entries not specified are equal to 0):

$$d_1 = 1, d_3 = -1; \quad d_9 = d_{12} = 1, d_{11} = d_8 = -1; \quad d_6 = d_7 = d_{11} = 1, d_{10} = d_5 = d_9 = -1.$$

Therefore in order to decompose a polynomial $P \in C_w^\Gamma$ with respect to the basis $\{P_1, P_2, P_3\}$ it is enough to know d_1, d_{12} and d_9.

The space $M_2(\Gamma_0(6))$ is spanned by the Eisenstein series $E_2^t(z) = E_2(z) - tE_2(tz)$, for $t = 2, 3, 6$, where $E_2(z) = -\frac{1}{24} + \sum_{n \geqslant 1} \sigma_1(n)e^{2\pi i n z}$. Since $(D_w^\Gamma)^+ = (C_w^\Gamma)^- = 0$, we have $\widehat{\rho}^+(E_2^t) = \rho^+(E_2^t) \in C_w^\Gamma$. Letting $\rho(E_2^t)(A_i) = e_i$, $\rho(E_2^t)(A_i') = e_i'$, we have $\rho^+(E_2^t)(A_i) = \frac{e_i + e_i'}{2} = d_i$, where $e_j' = e_{\tau j}$ with $\tau = (1, 4, 3, 2, 10, 7, 6, 8, 9, 5, 11, 12) \in \mathcal{S}_{12}$.

We now determine the constants d_i for each of the three Eisenstein series. Taking into account that $L(s, E_2) = \zeta(s)\zeta(s-1)$ and $L(s, E_2^t) = \zeta(s)\zeta(s-1)\big(1 - \frac{1}{t^{s-1}}\big)$, the constant d_1 is given by (8.2):

$$d_1 = C \ln(t), \quad \text{where } C = -\frac{\zeta(0)}{2\pi i}.$$

Since $E_2^2 \in M_2(\Gamma_0(2))$, and $A_6, A_7, A_{11} \in \Gamma_0(2)$ it follows that $e_1 = e_6 = e_7 = e_{11}$. We also have $e_6' = e_7$, $e_{11}' = e_{11}$, hence $d_1 = d_6 = d_7 = d_{11}$. We obtain $\rho^+(E_2^2) = C \ln(2)(P_1 + P_3)$.

Since $E_2^3 \in M_2(\Gamma_0(3))$, and $A_9, A_{12} \in \Gamma_0(3)$, we have $d_1 = d_9 = d_{12}$ and $\rho^+(E_2^3) = C\ln(3)(P_1 + P_2)$.

For E_2^6, in order to determine d_9, d_{12} we easily find

$$L(s, E_2^6|A_9) = \zeta(s)\zeta(s-1)\left(1 - \tfrac{2^{s-1}}{3^{s-1}}\right), \quad L(s, E_2^6|A_{12}) = \zeta(s)\zeta(s-1)\left(1 - 3^{2-s} + \tfrac{2^{s-1}}{3^{s-1}} + 6^{1-s}\right).$$

By (8.2) it follows $d_9 = C(\ln(3) - \ln(2))$, $d_{12} = C\ln 3$, so that

$$\rho^+(E_2^6) = C(P_1 \ln 6 + P_2 \ln 3 + P_3 \ln 2) = \rho^+(E_2^2) + \rho^+(E_2^3)$$

concluding that $\widehat{\rho}^+$ is not surjective.

8.2. **Haberland's formula for arbitrary modular forms.** The Petersson scalar product of two Eisenstein series of full level is defined by Zagier in [Za81]. Let $\mathcal{F}$ be the fundamental domain $\{z \in \mathcal{H} : |z| \geqslant 1, |\operatorname{Re} z| \leqslant 1/2\}$ for Γ_1, and for $T > 1$ let $\mathcal{F}_T$ be the truncated domain for which $\operatorname{Im} z < T$. Since $\sum_A f|A(z)\overline{g}|A(z)y^k$ is a Γ_1-invariant, renormalizable function in the sense of [Za81], we can define for $f, g \in M_k(\Gamma)$

$$(8.10) \qquad (f, g) = \frac{1}{[\overline{\Gamma}_1 : \overline{\Gamma}]} \lim_{T \to \infty} \sum_A \left[\int_{\mathcal{F}_T} f|A(z)\overline{g|A(z)} y^w dx dy - \frac{T^{k-1}}{k-1} a_0(f|A)\overline{a_0(g|A)} \right]$$

where the sum is over a complete system of representatives $A \in \overline{\Gamma}\backslash\overline{\Gamma}_1$. As in [Za81], it can be shown that the extended Petersson product equals $\operatorname{Res}_{s=k} L(s, f, \overline{g})$ up to a nonzero constant, where $L(s, f, \overline{g}) = \sum_{n \geqslant 1} \frac{a_n(f)\overline{a_n(g)}}{n^s}$. Using this fact, we show in [PP12] that for $\Gamma = \Gamma_1(N)$ the extended Petersson product is nondegenerate when $k > 2$, while for $k = 2$ it may or may not be degenerate.

We have the following generalization of Theorems 3.2 and 3.3.

Theorem 8.6. *Assume $k \geqslant 2$, and Γ is a finite index subgroup of Γ_1. Let $f, g \in M_k(\Gamma)$.*
a) We have: $6C_k \cdot (f, g) = \{\widehat{\rho}_f, \overline{\widehat{\rho}_g}\}, \quad$ *where $C_k = -(2i)^{k-1}$.*
b) We have: $\{\widehat{\rho}_f, \widehat{\rho}_g\} = 0$.
c) Assuming further that Γ is normalized by ϵ, and letting $\kappa_1, \kappa_2 \in \{+, -\}$ as in Theorem 3.3:

$$3C_k \cdot (f, g) = \{\widehat{\rho}_f^{\kappa_1}, \overline{\widehat{\rho}_g^{\kappa_2}}\}.$$

Proof. a) If one of f, g is a cusp form, then we can apply Stokes' theorem over the fundamental domain $\mathcal{D}$ for $\Gamma(2)$ as in the proof of Theorem 3.2, and easily obtain the desired identity. When both f, g have nonzero constant terms, this approach is complicated by the fact that both f and g blow up at the cusps $-1, 0, 1$, and we prefer to apply Stokes' theorem to the domain $\mathcal{F}$ as in [KZ84]. We use the following abbreviations: $f_A = f|A$, $g_A = g|A$, $a_A = a_0(f|A)$, $b_A = a_0(g|A)$, $C_\Gamma = [\overline{\Gamma}_1 : \overline{\Gamma}]$, $C_k = -(2i)^{k-1}$. Sums over A are over systems of representatives $A \in \overline{\Gamma}\backslash\overline{\Gamma}_1$. For all $T > 1$ we have

$$C_k C_\Gamma(f, g) = \sum_A \int_{\mathcal{F}} [f_A(z)\overline{g_A}(z) - a_A\overline{b_A}](z - \overline{z})^w dz d\overline{z} + a_A\overline{b_A}\left[\int_{\mathcal{F}_T} (z - \overline{z})^w dz d\overline{z} - C_k\frac{T^{k-1}}{k-1}\right]$$

By Stokes' theorem we find $\int_{\mathcal{F}_T} (z - \overline{z})^w dz d\overline{z} = C_k\frac{T^{w+1}}{w+1} + \frac{1}{w+1}\int_{\rho^2}^\rho (z - \overline{z})^{w+1} d\overline{z}$. In the first integral we apply Stokes' theorem after writing $f_A\overline{g_A} - a_A\overline{b_A} = (f_A - a_A)\overline{g_A} + a_A(\overline{g_A} - \overline{b_A})$ to get

$$C_k C_\Gamma(f, g) = \sum_A \int_{\partial\mathcal{F}} -F_A(z)\overline{g_A}(z) + a_A[\overline{g_A}(z) - \overline{b_A}]\frac{(z - \overline{z})^{w+1}}{w+1} d\overline{z} + \frac{a_A\overline{b_A}}{w+1}\int_{\rho^2}^\rho (z - \overline{z})^{w+1} d\overline{z}$$

where $F_A(z) = \int_z^{i\infty}[f_A(t) - a_A](t - \overline{z})^w dt$. Since $F_{AT}(z) = F_A(Tz)$, the integrals over the vertical sides of $\mathcal{F}$ cancel (after summing over A) and setting $\widetilde{F}_A(z) = F_A(z) - a_A \int_0^z (t - \overline{z})^w dt$ we obtain:

$$C_k C_\Gamma(f, g) = \sum_A \int_\rho^{\rho^2} \widetilde{F}_A(z)\overline{g_A}(z)d\overline{z} + (-1)^w \frac{a_A}{w+1} \int_\rho^{\rho^2} \overline{g_A}(z)\overline{z}^{w+1}d\overline{z}.$$

In the first integral we change variables $z \to Sz$, which reverses the order of integration. As in the proof of Proposition 8.1 we have $\widetilde{F}_A(z) - \widetilde{F}_{AS^{-1}}|_{-w}S(z) = \rho_f(A)(\overline{z})$ obtaining

$$(8.11) \qquad C_k C_\Gamma(f, g) = \sum_A \frac{1}{2} \int_\rho^{\rho^2} \rho_f(A)(\overline{z})\overline{g_A}(z)d\overline{z} + (-1)^w \frac{a_A}{w+1} \int_\rho^{\rho^2} \overline{g_A}(z)\overline{z}^{w+1}d\overline{z}.$$

We now proceed to write the result in terms of the pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on V_w^Γ. Define $H_{z_0} \in V_w^\Gamma$ as in (8.3), with g in place of f. Using $\int_\rho^{\rho^2} \overline{g_A}(z)(\overline{z} - X)^w d\overline{z} = \overline{H}_\rho(A) - \overline{H}_{\rho^2}(A)$ and (3.1) we get

$$\int_\rho^{\rho^2} \rho_f(A)(\overline{z})\overline{g_A}(z)d\overline{z} = \left\langle \rho_f(A), \int_\rho^{\rho^2} \overline{g_A}(z)(\overline{z} - X)^w d\overline{z} \right\rangle = \left\langle \rho_f(A), \overline{H}_\rho(A) - \overline{H}_{\rho^2}(A) \right\rangle.$$

The second integral in (8.11) can be written

$$\int_\rho^{\rho^2} \overline{g_A}(z)\overline{z}^{w+1}d\overline{z} = \overline{b_A} \int_\rho^{\rho^2} \overline{z}^{w+1}d\overline{z} + \int_\rho^{i\infty} - \int_{\rho^2}^{i\infty}(\overline{g_A}(z) - \overline{b_A})\overline{z}^{w+1}d\overline{z}$$

and changing variables $z = t - 1$ in the last integral, recalling that $\rho_f^0(A) = (-1)^w a_A \frac{X^{w+1}}{w+1}$ $(= \rho_f^0(AT))$, and using (3.1) we obtain

$$\sum_A (-1)^w \frac{a_A}{w+1} \int_\rho^{\rho^2} \overline{g_A}(z)\overline{z}^{w+1}d\overline{z} = \sum_A \langle \rho_f^0(A)|1 - T^{-1}, \overline{H}_\rho(A) \rangle +$$

$$\sum_A a_A \overline{b_A} \Big[\int_0^1 \int_0^\rho (t - \overline{z})^w d\overline{z}dt + (-1)^w \int_\rho^{\rho^2} \frac{\overline{z}^{w+1}}{w+1}d\overline{z} \Big].$$

The expression inside square brackets equals $\frac{1}{(k-1)k}$, and setting $I(f, g) = \frac{2}{k(k-1)C_\Gamma} \sum_A a_A \overline{b_A}$ we get

$$2C_k(f, g) = \langle\!\langle \rho_f, \overline{H}_\rho - \overline{H}_{\rho^2} \rangle\!\rangle + 2\langle\!\langle \rho_f^0|1 - T^{-1}, \overline{H}_\rho \rangle\!\rangle + I(f, g).$$

Now we use

$$(8.12) \qquad H_{\rho^2} = H_\rho|T - \rho_g^0|(1 - T), \quad \rho_g = H_\rho - H_{\rho^2}|S = H_\rho|(1 - TS) + \rho_g^0|(1 - T)S.$$

Taking into account the relation $(1 + U^2) = \frac{1}{3}(U^2 - U)(1 - U^{-1}) + \frac{2}{3}(1 + U + U^2)$ in $\mathbb{Q}[\overline{\Gamma}_1]$ we have

$$\langle\!\langle \rho_f, \overline{H}_\rho|1 - T \rangle\!\rangle = \langle\!\langle \rho_f|1 - T^{-1}, \overline{H}_\rho \rangle\!\rangle = \langle\!\langle \rho_f|1 + ST^{-1}, \overline{H}_\rho \rangle\!\rangle =$$

$$= \frac{1}{3}\langle\!\langle \rho_f|U^2 - U, \overline{H}_\rho|1 - U \rangle\!\rangle + \frac{2}{3}\langle\!\langle \rho_f|1 + U + U^2, \overline{H}_\rho \rangle\!\rangle$$

$$= \frac{1}{3}\langle\!\langle \rho_f| - T^{-1} - TS, \overline{\rho_g} - \overline{\rho_g^0}|(1 - T)S \rangle\!\rangle + \frac{2}{3}\langle\!\langle \rho_f|1 + U + U^2, \overline{H}_\rho \rangle\!\rangle$$

$$= \frac{1}{3}\langle\!\langle \rho_f|T - T^{-1}, \overline{\rho_g} \rangle\!\rangle + \frac{1}{3}\langle\!\langle \rho_f|T + T^{-1}S, \overline{\rho_g^0}|1 - T \rangle\!\rangle -$$

$$- \frac{2}{3}\langle\!\langle \rho_f^0|(1 - T^{-1})(1 + U + U^2), \overline{H}_\rho \rangle\!\rangle$$

(on the last line we used Proposition 8.1) and collecting terms we get

$$6C_k(f,g) = \langle\!\langle \rho_f | T - T^{-1}, \overline{\rho_g} \rangle\!\rangle + \langle\!\langle \rho_f | 3 + T^{-1}S + T, \overline{\rho}_g^0 | 1 - T \rangle\!\rangle +$$
$$+ 2 \langle\!\langle \rho_f^0 | (1 - T^{-1})(2 - U - U^2), \overline{H}_\rho \rangle\!\rangle + 3I(f,g)$$

In the second term we use the relation

$$(1 - T)(3 + ST + T^{-1}) = 2(T^{-1} - T) + (1 - T)S[1 + U + U^2 - U^2(1 + S)]S,$$

while in the third we use $2 - U - U^2 = (1 - U)(1 - U^2)$ and (8.12):

(8.13)
$$6C_k(f,g) = \langle\!\langle \rho_f | T - T^{-1}, \overline{\rho_g} \rangle\!\rangle + \langle\!\langle \rho_f, 2\overline{\rho}_g^0 | (T^{-1} - T) \rangle\!\rangle + \langle\!\langle 2\rho_f^0 | (T - T^{-1}), \overline{\rho_g} \rangle\!\rangle +$$
$$+ \langle\!\langle \rho_f^0 | (1 - T)(T^{-1}S - ST - 3), \overline{\rho}_g^0 | 1 - T \rangle\!\rangle + 3I(f,g)$$

Let $p(X) = \frac{X^{w+1}}{w+1} | 1 - T$, $q(X) = \frac{X^{w+1}}{w+1} | 1 - T^{-1}$. Then

$$\langle\!\langle \rho_f^0 | (1 - T)(T^{-1}S - ST), \overline{\rho}_g^0 | 1 - T \rangle\!\rangle = \frac{1}{C_\Gamma} \sum_A (a_A \overline{b_{AS^{-1}}} - (-1)^w a_{AS^{-1}} \overline{b_A}) \langle p, q | S \rangle = 0,$$

where we changed A to AS^{-1} in one of the sums and used that $a_{AJ} = (-1)^w a_A$. We also have

$$\langle\!\langle \rho_f^0 | (1 - T), \overline{\rho}_g^0 | 1 - T \rangle\!\rangle = \frac{1}{C_\Gamma} \sum_A a_A \overline{b_A} \langle p, p \rangle = \frac{1 + (-1)^w}{k(k-1)C_\Gamma} \sum_A a_A \overline{b_A},$$

which vanishes if w is odd, and equals $I(f,g)$ if w is even. Therefore the second line in (8.13) vanishes if k is even, and it equals $3I(f,g) = I_k(\rho_f^0, \overline{\rho}_g^0)$ if k is odd, finishing the proof.

b) Going backwards in the proof of part a) up to the first equation after applying Stokes' theorem, we obtain

$$C_\Gamma\{\widehat{\rho}_f, \widehat{\rho}_g\} = -6 \sum_A \int_{\partial\mathcal{F}} \widetilde{f}(A)(z)g_A(z)dz$$

with $\widetilde{f}$ defined in (8.1). Since the integrand is holomorphic and vanishes at $i\infty$, each term vanishes.

c) Since the extended pairing $\{\cdot,\cdot\}$ behaves as the original one under the action of ϵ, the claim follows from a) and b) as in the proof of Theorem 3.3. $\square$

8.3. **Hecke operators.** For a finite index subgroup Γ and a double coset Σ_n satisfying (H), we define the operation $|_\Sigma$ of the Hecke operators $\widetilde{T}_n$ on $\widehat{W}_w^\Gamma$ as in Section 5. Although matrices in the definition of $\widetilde{T}_n$ do not preserve $\widehat{V}_w^\Gamma$, we have the following generalization of Proposition 5.2 and of Corollary 5.3.

Proposition 8.7. *Assume the pair (Γ, Σ_n) satisfy (H), and for part (b) assume that $\Sigma_n = \Sigma_n'$ and Γ is normalized by ϵ. Let $\widetilde{T}_n \in R_n$ be any element satisying (5.1).*
(a) We have $\widehat{\rho}_{f|[\Sigma_n]} = \widehat{\rho}_f |_\Sigma \widetilde{T}_n$ for $f \in M_k(\Gamma)$.
(b) We have $\widehat{\rho}_{f|[\Sigma_n]}^\pm = \widehat{\rho}_f^\pm |_\Sigma \widetilde{T}_n$ for $f \in M_k(\Gamma)$.
(c) The operators $\widetilde{T}_n$ preserve the space $\widehat{W}_w^\Gamma$.

Proof. (a) The proof is the same as of Prop. 5.2, once we show that $\widetilde{f}|_\Sigma T_n^\infty = \widetilde{f|[\Sigma_n]}$. Equation (5.4) becomes

$$\widetilde{f}|_\Sigma T_n^\infty(A) = \sum_{M \in M_n^\infty \cap \Gamma_1 \Sigma_n A} \int_{Mz}^{i\infty} [f|A_M(t) - a_0(f|A_M)](t - Mz)^w j(M, z)^w dt$$

$$= n^{w+1} \sum_{M \in M_n^\infty \cap \Gamma_1 \Sigma_n A} \int_z^{i\infty} \left[f|M_A A(u) - a_0(f|A_M) j(M, u)^{-k} \right] (u - z)^w du$$

As in Proposition 5.2, we obtain $\widetilde{f}|_\Sigma T_n^\infty(A) = \int_z^{i\infty} \left[(f|[\Sigma_n])|A - c(n, f, A) \right] (u - z)^w du$ where $c(n, f, A)$ is the sum of the terms involving $a_0(f|A_M)$ (which is independent of u since $j(M, u) = d_M$ for $M \in M_n^\infty$). Since the integral converges, we must have $c(n, f, A) = a_0(f|[\Sigma_n]|A)$ (which can alsp be proved directly, using Prop. 8.1), hence the last expression equals $\widetilde{f|[\Sigma_n]}(A)$.

(b) The proof is the same as of Corollary 5.3.

(c) This follows from part (a) and the decomposition (8.7) □

Proposition 8.8. *Assume the hypotheses of Prop. 8.7 and furthermore that $\Sigma_n^\vee$ satisfies* (H). *We have for all $\widehat{P}, \widehat{Q} \in \widehat{W}_w^\Gamma$*

$$\{\widehat{P}|_\Sigma \widetilde{T}_n, \widehat{Q}\} = \{\widehat{P}, \widehat{Q}|_{\Sigma^\vee} \widetilde{T}_n\}.$$

Proof. As in the first proof of Theorem 5.9, we decompose $\widehat{P} = R + \rho_f^+ + \rho_g^- + \widehat{\rho}_e$ with $R \in C_w^\Gamma$, $f, g \in S_k(\Gamma)$, $e \in M_k(\Gamma)$. Taking into account Theorem 8.6, Proposition 8.7 and the fact that the adjoint of the operator $[\Sigma_n]$ is $[\Sigma_n^\vee]$ with respect to the extended Petersson inner product on $M_k(\Gamma)$ [PP12], it remains to show that

(8.14) $$\{R|_\Sigma \widetilde{T}_n, \widehat{\rho}_e\} = \{R, \widehat{\rho}_{e|[\Sigma_n^\vee]}\}.$$

We use Prop. 8.3. Let $R = R'|(1 - S)$, with $R'(A) = c(A) \in \mathbb{C}$ and $R'|(1 - T) = 0$. By (5.1) $R|_\Sigma \widetilde{T}_n = R'|_\Sigma T_n^\infty (1 - S)$, and for $A \in \Gamma_1$ we have $R'|_\Sigma T_n^\infty(A) = \sum_{M \in M_n^\infty} c(A_M) d_M^w$ where d_M is the lower right entry of M, and $MA^{-1} = A_M^{-1} M_A$ with $A_M \in \Gamma_1$, $M_A \in \Sigma_n$. The left side of (8.14) becomes (up to a constant which we ignore in the right side as well)

$$\text{LHS} = \sum_{M \in M_n^\infty} \sum_{\substack{A \in \Gamma \backslash \Gamma_1 \\ MA^{-1} \in A_M^{-1} \Sigma_n}} c(A_M) a_0(e|A) d_M^w.$$

Since $c(AT) = c(A)$, $a_0(e|AT) = a_0(e|A)$, we can replace M by $T^i M T^j$ in the interior sum without changing it. Therefore, if we write $M = M_{a,d,b} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and fix d, the interior sum depends only on b modulo $g_{a,d} = \gcd(a, d)$ and we have

$$\text{LHS} = \sum_{\substack{d|n \\ b \bmod g_{a,d}}} \sum_{\substack{A \in \Gamma \backslash \Gamma_1 \\ M_{a,d,b} A^{-1} \in A_M^{-1} \Sigma_n}} c(A_M) a_0(e|A) \frac{d^{w+1}}{g_{a,d}}.$$

For the right side of (8.14), from the proof of Prop. 8.7 we have for $B \in \Gamma_1$

$$a_0(e|[\Sigma_n^\vee]|B) = \sum_{\substack{M \in M_n^\infty \\ MB^{-1} \in B_M^{-1} \Sigma_n^\vee}} a_0(e|B_M) \frac{n^{w+1}}{d_M^k}.$$

Since $MB^{-1} \in B_M^{-1}\Sigma_n^\vee \iff M^\vee B_M^{-1} \in B^{-1}\Sigma_n$, the right side of (8.14) becomes, after interchanging B_M and B

$$\text{RHS} = \sum_{M \in M_n^\infty} \sum_{\substack{B \in \Gamma \backslash \Gamma_1 \\ M^\vee B^{-1} \in B_M^{-1}\Sigma_n}} c(B_M)a_0(e|B)\frac{n^{w+1}}{d_M^k}$$

$$= \sum_{d|n} \sum_{\substack{B \in \Gamma \backslash \Gamma_1 \\ b \bmod g_{a,d} \ M_{a,d,b}^\vee B^{-1} \in B_M^{-1}\Sigma_n}} c(B_M)a_0(e|B)\frac{a^{w+1}}{d}\frac{d}{g_{a,d}}$$

where the second line follows by writing $M = M_{a,d,b}$ as before. Comparing the expressions obtained for RHS and LHS finishes the proof of (8.14). □

From the duality in Proposition 8.3 and from Propositions 8.7, 8.8 we immediately obtain:

Corollary 8.9. *Assume that both Σ_n and $\Sigma_n^\vee$ satisfy property* (H). *There exist bases of $\mathcal{E}_k(\Gamma)$ and C_w^Γ such that the matrix of the operator $[\Sigma_n^\vee]$ acting on $\mathcal{E}_k(\Gamma)$ is the same as the matrix of the operator $|_\Sigma \widetilde{T}_n$ acting on C_w^Γ.*

As an application, we let $\Gamma = \Gamma_1(N)$, and χ a character modulo N, and we show that the trace of Hecke operators T_n on the Eisenstein subspace $\mathcal{E}_k(N,\chi) \subset M_k(N,\chi)$ is the same as the trace of $\widetilde{T}_n$ on $C_w^{\Gamma,\chi}$ (see §5.4 for the notation). For Γ_1, when $C_w^{\Gamma_1} = <X^w - 1>$, a direct proof is immediate, but for $\Gamma = \Gamma_1(N)$ it seems difficult to prove the statement without using the dual space $\widehat{E}_w^\Gamma$ and the pairing $\{\cdot,\cdot\}$.

Proposition 8.10. (a) *Let $\Gamma = \Gamma_1(N)$ and let $\widehat{E}_w^{\Gamma,\chi} \subset \widehat{W}_w^\Gamma$ be the image of the Eisenstein subspace $\mathcal{E}_k(N,\chi) \subset M_k(\Gamma)$ under the map $f \to \widehat{\rho}_f$. For $(n,N) = 1$ we have*

$$\text{Tr}(\mathcal{E}_k(N,\chi)|T_n) = \text{Tr}(\widehat{E}_w^{\Gamma,\chi}|_\Delta \widetilde{T}_n) = \text{Tr}(C_w^{\Gamma,\chi}|_\Delta \widetilde{T}_n).$$

(b) *For $\Gamma = \Gamma_0(N)$ and $n\|N$, let Θ_n be the double coset and let W_n be the Atkin-Lehner operator defined in Section 5.1. We have*

$$\text{Tr}(\mathcal{E}_k(\Gamma)|W_n) = \text{Tr}(\widehat{E}_w^\Gamma|_\Theta \widetilde{T}_n) = \text{Tr}(C_w^\Gamma|_\Theta \widetilde{T}_n).$$

Proof. (a) The duality in Prop. 8.3 between C_w^Γ and $\widehat{E}_w^\Gamma$ with respect to the pairing $\{\cdot,\cdot\}$ implies dualities between $C_w^{\Gamma,\chi}$ and $\widehat{E}_w^{\Gamma,\overline{\chi}}$. Therefore, taking $\Sigma_n = \Delta_n$ for $(n,N) = 1$ in Corollary 8.9, it follows that the eigenvalues of $|_\Delta \widetilde{T}_n$ on $C_w^{\Gamma,\chi}$ are the same as the eigenvalues of $T_n^* = [\Delta_n^\vee]$ on $\mathcal{E}_k(N,\overline{\chi})$, which are the same as the eigenvalues of T_n on $\mathcal{E}_k(N,\chi)$ (the latter space has a basis of eigenforms for T_n with $(n,N) = 1$).

(b) The claim follows from Corollary 8.9, using the fact that $\Theta_n = \Theta_n^\vee$. □

Remark 8.11. Prop 8.10 shows that for $\Gamma = \Gamma_1(N)$ and $(n,N) = 1$ we have

$$\text{Tr}(W_w^{\Gamma,\chi}|_\Delta \widetilde{T}_n) = \text{Tr}(M_k(N,\chi)|T_n) + \text{Tr}(S_k(N,\chi)|T_n),$$

and the same for Atkin-Lehner operators on $\Gamma_0(N)$. For $\Gamma = \Gamma_1$, this fact was an ingredient used by Zagier to sketch an elementary proof of the Eichler-Selberg trace formula, by computing directly the left side for an appropriately chosen $\widetilde{T}_n$ [Za93]. A generalization of this approach giving a simple trace formula for $M_k(N,\chi)$ is work in progress of the second author and Don Zagier.

8.4. **Extra relations revisited.** Theorem 8.6 gives another way of determining the extra relations satisfied by all period polynomials of cusp forms which are independent of the period relations. Assuming that $\widehat{\rho}^-$ is an isomorphism (see Proposition 8.4), it follows that there exist $g \in \mathcal{E}_k(\Gamma)$ such that $\widehat{\rho_g^-}$ form a basis for $(\widehat{E}_w^\Gamma)^-$. Since the pairing $\{\cdot,\cdot\}$ is nondegenerate, it follows that the linear relations $\{P, \overline{\widehat{\rho_g^-}}\} = 0$ are satisfied by $P = \rho_f^+$, for all $f \in S_k(\Gamma)$, but they are not satisfied by some $P \in (C_w^\Gamma)^+$. A similar argument applies to determine the relations satisfied by ρ_f^-, when $(C_w^\Gamma)^- \neq 0$ and $\widehat{\rho}^+$ is an isomorphism. These linear relations can be used to define other versions of the linear forms λ_+, λ_- in Proposition 7.1, which are entirely explicit once the period polynomials of Eisenstein series are determined.

As an example we take $\Gamma = \Gamma_1(N)$, and we assume $k \geqslant 3$. Then the Eisenstein subspace $\mathcal{E}_k(\Gamma)$ has a basis of Eisenstein series which are Hecke eigenforms for the Hecke operators of index coprime with the level [DS05, Ch. 5]. Their period polynomials for the identity coset can be determined in terms of special values of Dirichlet L-functions by Proposition 8.1. For other cosets $A \in \Gamma \backslash \Gamma_1$ the period polynomials of the Hecke eigenforms are harder to compute. Instead, consider a second basis, consisting of Eisenstein series which vanish at all but one cusp, so that the action $|A$ permutes the elements of this basis. The elements of the second basis can be decomposed in terms of Hecke eigenforms, so their period polynomials corresponding to all cosets A can be determined explicitly. For $\Gamma_0(N)$ we are planning to return to this question in a future work.

References

[An92] Antoniadis, J. *Modulformen auf $\Gamma_0(N)$ mit rationalen perioden*, Manuscripta Math. 74 (1992), 359–384

[BGKO] Bringmann, K., Guerzhoy, P., Kent, Z., Ono, K., *Eichler-Shimura theory for mock modular forms*, Math. Annalen, to appear

[Co12] Cohen, H., *Haberland's formula and numerical computation of Petersson scalar products.* ANTS-X Conference Proceedings, to appear

[CZ93] Choie, YJ., Zagier, D., *Rational period functions for PSL(2,Z).* Contemporary Math. 143 (1993), 89–108

[Di01] Diamantis, N., *Hecke operators and derivatives of L-Functions*, Compositio Mathematica 125 (2001), 39–54

[DIT10] Duke, W., Imamoḡlu, Ö., Tóth, Á., *Rational period functions and cycle integrals.* Abh. Math. Semin. Univ. Hambg. 80 (2010), 255–264

[DS05] Diamond, F., Shurman, J., *A First Course in Modular Forms.* Springer (2005)

[GKZ] Gangl, H., Kaneko, M., Zagier, D., *Double zeta values and modular forms.* Automorphic forms and zeta functions, (2006), 71–106

[FY09] Fukuhara, S., Yang, Y. *Period polynomials and explicit formulas for Hecke operators on $\Gamma_0(2)$.* Math.Proc. Cambridge Philos. Soc. 146 (2009), 321–350

[GS94] Greenberg, R., Stevens, G., *On the conjecture of Mazur, Tate, and Teitelbaum.* in: p-Adic Monodromy and the Birch and Swinnerton-Dyer conjecture, Boston, MA, 1991, in: Contemp. Math., vol. 165, American Mathematical Society, Providence, RI, 1994, pp. 123211.

[Ha83] Haberland, H., *Perioden von Modulformen einer Variablen und Gruppenkohomologie, I*, Math. Nachr. 112 (1983), 245–282

[IK05] Imamoglu, O., Kohnen, W., *Representations of integers as sums of an even number of squares.* Math. Ann. 333(4) (2005), 815–829.

[KT11] Kaneko, M., Tasaka, K., *Double zeta values, double Eisenstein series, and modular forms of level 2.* Preprint, arXiv:1112.5697v1 (2011)

[KZ84]   Kohnen, W., Zagier, D., *Modular forms with rational periods.* in *Modular forms,* R.A. Rankin editor, Ellis Horwood series in math. and its applications, 1984

[Mgm]   Bosma, W., Cannon, A., and Playoust, C., *The Magma algebra system. I. The user language.* J. Symbolic Comput., 24 (1997), 235–265.

[Ma73]   Manin, Ju. I., *Periods of parabolic forms and p-adic Hecke series.* Mat. Sb. 21 (1973), 371–393

[Me94]   Merel, L., *Universal Fourier expansions of modular forms.* On Artin's conjecture for odd 2-dimensional representations, Springer (1994), 59–94

[Me09]   Merel, L., *Symboles de Manin et valeurs de fonctions L.* in Algebra, Arithmetic, and Geometry in Honor of Yu. I. Manin vol. II, Progress in Math. 270, Birkhäuser (2009), 283–309

[NSW]   Neukirch, J., Schmidt, A., Wingberg, K., *Cohomology of number fields.* Springer-Verlag (2000)

[PP12]   Paşol, V., Popa, A.A., *On the Petersson scalar product of arbitrary modular forms.* Proc. Amer. Math. Soc., to appear. Arxiv:1204.0502

[PP12a]   Paşol, V., Popa, A.A., *An algebraic property of Hecke operators and two indefinite theta series.* Forum Math., to appear. Arxiv:1207.5766

[Po11]   Popa, A.A., *Rational decomposition of modular forms.* Ramanujan J., 26/3 (2011), 419–435

[SG]   Stein,W.A. et al., *Sage Mathematics Software (Version 4.4.1)*, The Sage Development Team, 2011, `http://www.sagemath.org`

[Sh71]   Shimura, G., *Introduction to the arithmetic theory of automorphic functions*, Princeton U. Press (1971)

[Sh77]   Shimura, G., *On the periods of modular forms.* Math. Ann. 229 (1977), 211-221

[Sk90]   Skoruppa, N.-P., *Binary quadratic forms and the Fourier coefficients of elliptic and Jacobi modular forms*, J. reine angew. Math. 411 (1990), 66–95

[Za81]   Zagier, D. *The Rankin-Selberg method for automorphic forms which are not of rapid decay.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1981), 415–437.

[Za90]   Zagier, D., *Hecke operators and periods of modular forms.* Israel Math. Conf. Proc. 3, (1990), 321–336

[Za91]   Zagier, D., *Periods of modular forms and Jacobi theta functions*, Invent. Math. 104 (1991), 449–465

[Za93]   Zagier, D., *Periods of modular forms, traces of Hecke operators, and multiple zeta values.* Research into automorphic forms and *L* functions. Sūrikaisekikenkyūsho Kōkyūroku No. 843 (1993), 162–170

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