Tetramodules over a bialgebra form a 2-fold monoidal category

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Abstract. Let B be an associative bialgebra over any field. A *module* over B in the sense of deformation theory is a *tetramodule* over B. All tetramodules form an abelian category. This category was studied by R.Taillefer [Tai1,2]. In particular, she proved that for any bialgebra B, the abelian category $\mathcal{T}etra(B)$ has enough injectives, and that $\mathrm{Ext}^{\bullet}(B,B)$ in this category coincides with the Gerstenhaber-Schack cohomology of B.

We prove that the category $\mathcal{T}etra(B)$ of tetramodules over any bialgebra B is a 2-fold-monoidal category, with B a unit object in it. Roughly, this means that the category $\mathcal{T}etra(B)$ admits two monoidal structures, with common unit B, which are compatible in some rather non-trivial way (the concept of an n-fold monoidal category is introduced in [BFSV]). Within (yet unproven) 2-fold monoidal analogue of the Deligne conjecture, our result would imply that $\mathrm{RHom}^{\bullet}(B,B)$ in the category of tetramodules is naturally a *homotopy 3-algebra*.

Introduction

This paper contains a part of results of our 2009 archive preprint [Sh1], presented at a glance suitable for a journal publication. It covers a construction of a 2-fold monoidal structure on the category of tetramodules, with all necessary definitions, and an overview of the results of R.Taillefer [Tai1,2] on tetramodules and the Gerstenhaber-Schack cohomology [GS] (formerly served as Appendix in [Sh1]), as well as a computation of the Gerstenhaber-Schack cohomology for the free commutative cocommutative bialgebra $S(V)$, for a V is a vector space.

Our approach to the n-fold monoidal Deligne conjecture from [Sh1] still remains unfinished, and is not considered here.

The paper is organized as follows.

In Section 1 we recall, in some detail, the definition of n-fold monoidal categories from [BFSV].

In Section 2 we define the abelian category of tetramodules over a bialgebra, and construct a 2-fold monoidal category structure on it. This Section is the core of the paper.

In Section 3 we recall (with proofs) some of results of R.Taillefer [Tai1,2], which give a relation between our results in Section 2, and deformation theory of associative bialgebras. We slightly modify the original proof in presentation (with the same main ideas), which, we hope, makes it more readable. As an application, we compute the Gerstenhaber-Schack cohomology of the free commutative cocommutative bialgebra $S(V)$ (the answer is well-known to specialists, but we were unable to find any published proof of it).

The main application of the results of Section 2 claims, modulo the Deligne conjecture for n-fold monoidal abelian categories, that $\mathrm{RHom}^\bullet_{\mathcal{T}etra(A)}(A, A)$ is a homotopy 3-algebra, when the bialgebra A is a Hopf algebra (over any field). However, the n-fold monoidal Deligne conjecture is unproven yet (for $n > 1$). In [Sh1], we tried to use some constructions of [Sch] to prove it, but (so far) we did not succeed in this direction.

We suggested a new approach to the n-fold monoidal Deligne conjecture, based on the Kock-Toën's simplicial Deligne conjecture [KT], and on the Leinster's definition [L] of weak Segal monoids. Our recent paper [Sh2] is a first step in realization of this approach. We hope to complete it in our next papers.

1 n-FOLD MONOIDAL CATEGORIES

1.1 INTRODUCTION

An n-fold monoidal category $\mathcal{C}$ is a category with n monoidal structures $\otimes_1, \ldots, \otimes_n \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which obey some compatibility relations, which may seem complicated when are written down explicitly. They were introduced in [BFSV], with the following motivation.

When $\mathcal{C}$ is a (1-)monoidal category, its nerve $N\mathcal{C}$ a simplicial H-space, up to a homotopy. In fact, all homotopies between (the higher) homotopies can be chosen in a coherent way, such that

the classifying space $B\mathcal{C}$ becomes an algebra over the topological Stasheff operad. It implies that the group completion of $B\mathcal{C}$ is a 1-fold loop space, $B\mathcal{C} = \Omega X$ for some connected space X.

The n-fold loop spaces $\Omega^n(X)$ are defined iteratively, $\Omega^n(X) = \Omega(\Omega^{n-1}(X))$. The n-fold monoidal categories are defined iteratively, in a similar way. The definition achieves the following goal: the group completion of the classifying space $B\mathcal{C}$, for an n-fold monoidal category $\mathcal{C}$, is an n-fold loop space.

1.2 Definition

Definition 1.1. A (strict) monoidal category is a category $\mathcal{C}$ together with a functor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object $\mathbb{I} \in \mathrm{Ob}(\mathcal{C})$ such that

1. $\otimes$ is strictly associative;

2. $\mathbb{I}$ is a strict two-sided unit for $\otimes$.

A monoidal functor $(F, \eta) \colon \mathcal{C} \to \mathcal{D}$ between monoidal categories is a functor F such that $F(\mathbb{I}_{\mathcal{C}}) = \mathbb{I}_{\mathcal{D}}$ with a natural transformation

$$\eta_{A,B} \colon F(A) \otimes F(B) \to F(A \otimes B) \tag{1.1}$$

which satisfies the following conditions:

1. Internal associativity: the following diagram commutes

$$\begin{array}{ccc}
F(A) \otimes F(B) \otimes F(C) & \xrightarrow{\eta_{A,B} \otimes id_{F(C)}} & F(A \otimes B) \otimes F(C) \\
\downarrow{\scriptstyle id_{F(A)} \otimes \eta_{B,C}} & & \downarrow{\scriptstyle \eta_{A \otimes B, C}} \\
F(A) \otimes F(B \otimes C) & \xrightarrow{\eta_{A, B \otimes C}} & F(A \otimes B \otimes C)
\end{array} \tag{1.2}$$

2. Internal unit conditions: $\eta_{A, \mathbb{I}} = \eta_{\mathbb{I}, A} = id_{F(A)}$.

The crucial in this definition is that *the map η is not required to be an isomorphism.*
Denote by $\mathcal{MC}at$ the category of (small) monoidal categories and monoidal functors.

1.2.1

Definition 1.2. A 2-fold monoidal category is a monoid in $\mathcal{MC}at$. This means, that we are given a monoidal category $(\mathcal{C}, \otimes_1, \mathbb{I})$, and a monoidal functor $(\otimes_2, \eta) \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which satisfies the following axioms:

1. External associativity: the following diagram commutes in $\mathcal{MC}at$

$$
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} \times \mathcal{C} & \xrightarrow{(\otimes_2, \eta) \times id_{\mathcal{C}}} & \mathcal{C} \times \mathcal{C} \\
{\scriptstyle id_{\mathcal{C}} \times (\otimes_2, \eta)} \big\downarrow & & \big\downarrow {\scriptstyle (\otimes_2, \eta)} \\
\mathcal{C} \times \mathcal{C} & \xrightarrow{(\otimes_2, \eta)} & \mathcal{C}
\end{array}
\tag{1.3}
$$

2. External unit conditions: the following diagram commutes in $\mathcal{MC}at$

$$
\begin{array}{ccccc}
\mathcal{C} \times \mathbb{I} & \xrightarrow{\subseteq} & \mathcal{C} \times \mathcal{C} & \xleftarrow{\supseteq} & \mathbb{I} \times \mathcal{C} \\
{\scriptstyle \cong} \big\downarrow & & {\scriptstyle (\otimes_2, \eta)} \big\downarrow & & \big\downarrow {\scriptstyle \cong} \\
\mathcal{C} & \xrightarrow{=} & \mathcal{C} & \xleftarrow{=} & \mathcal{C}
\end{array}
\tag{1.4}
$$

Note that the role of the monoidal structures $\otimes_1$ and $\otimes_2$ in this definition is not symmetric.

Explicitly the definition above means that we have an operation $\otimes_2$ with the two-sided unit $\mathbb{I}$ (the same that for $\otimes_1$) and a natural transformation

$$
\eta_{A,B,C,D} \colon (A \otimes_2 B) \otimes_1 (C \otimes_2 D) \to (A \otimes_1 C) \otimes_2 (B \otimes_1 D)
\tag{1.5}
$$

The internal unit conditions are: $\eta_{A,B,\mathbb{I},\mathbb{I}} = \eta_{\mathbb{I},\mathbb{I},A,B} = id_{A \otimes_2 B}$, and the external unit conditions are: $\eta_{A,\mathbb{I},B,\mathbb{I}} = \eta_{\mathbb{I},A,\mathbb{I},B} = id_{A \otimes_1 B}$. As well, one has the morphisms

$$
\eta_{A,\mathbb{I},\mathbb{I},B} \colon A \otimes_1 B \to A \otimes_2 B
\tag{1.6}
$$

and

$$
\eta_{\mathbb{I},A,B,\mathbb{I}} \colon A \otimes_1 B \to B \otimes_2 A
\tag{1.7}
$$

The internal associativity gives the commutative diagram:

$$
\begin{array}{ccc}
(U \otimes_2 V) \otimes_1 (W \otimes_2 X) \otimes_1 (Y \otimes_2 Z) & \xrightarrow{\eta_{U,V,W,X} \otimes_1 id_{Y \otimes_2 Z}} & \big((U \otimes_1 W) \otimes_2 (V \otimes_1 X)\big) \otimes_1 (Y \otimes_2 Z) \\
{\scriptstyle id_{U \otimes_2 V} \otimes_1 \eta_{W,X,Y,Z}} \big\downarrow & & \big\downarrow {\scriptstyle \eta_{U \otimes_1 W, V \otimes_1 X, Y, Z}} \\
(U \otimes_2 V) \otimes_1 \big((W \otimes_1 Y) \otimes_2 (X \otimes_1 Z)\big) & \xrightarrow{\eta_{U,V,W \otimes_1 Y, X \otimes_1 Z}} & (U \otimes_1 W \otimes_1 Y) \otimes_2 (V \otimes_1 X \otimes_1 Z)
\end{array}
\tag{1.8}
$$

The external associativity condition gives the commutative diagram:

$$(U \otimes_2 V \otimes_2 W) \otimes_1 (X \otimes_2 Y \otimes_2 Z) \xrightarrow{\eta_{U \otimes_2 V, W, X \otimes_2 Y, Z}} \left((U \otimes_2 V) \otimes_1 (X \otimes_2 Y)\right) \otimes_2 (W \otimes_1 Z)$$

with vertical maps $\eta_{U, V \otimes_2 W, X, Y \otimes_2 Z}$ on the left and $\eta_{U, V, X, Y} \otimes_2 id_{W \otimes_1 Z}$ on the right,
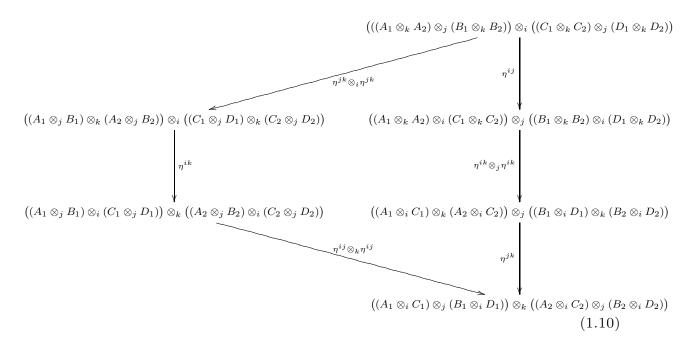
$$(U \otimes_1 X) \otimes_2 \left((V \otimes_2 W) \otimes_1 (Y \otimes_2 Z)\right) \xrightarrow{id_{U \otimes_1 X} \otimes_2 \eta_{V, W, Y, Z}} (U \otimes_1 X) \otimes_2 (V \otimes_1 Y) \otimes_2 (W \otimes_1 Z)$$

$$(1.9)$$

Finally, [BFSV] gives

DEFINITION 1.3. Denote by $\mathcal{MC}at_n$ the category of (small) n-fold monoidal categories. Then an $(n+1)$-fold monoidal category is a monoid in $\mathcal{MC}at_n$.

This gives the following compatibility axiom: for $1 \leq i < j < k \leq n$ the following diagram is commutative:

$$\left(\left((A_1 \otimes_k A_2) \otimes_j (B_1 \otimes_k B_2)\right) \otimes_i \left((C_1 \otimes_k C_2) \otimes_j (D_1 \otimes_k D_2)\right)\right)$$

with maps $\eta^{jk} \otimes_i \eta^{jk}$ (diagonal down-left) and η^{ij} (down-right),

$$\left((A_1 \otimes_j B_1) \otimes_k (A_2 \otimes_j B_2)\right) \otimes_i \left((C_1 \otimes_j D_1) \otimes_k (C_2 \otimes_j D_2)\right) \qquad \left((A_1 \otimes_k A_2) \otimes_i (C_1 \otimes_k C_2)\right) \otimes_j \left((B_1 \otimes_k B_2) \otimes_i (D_1 \otimes_k D_2)\right)$$

with maps η^{ik} (left) and $\eta^{ik} \otimes_j \eta^{ik}$ (right),

$$\left((A_1 \otimes_j B_1) \otimes_i (C_1 \otimes_j D_1)\right) \otimes_k \left((A_2 \otimes_j B_2) \otimes_i (C_2 \otimes_j D_2)\right) \qquad \left((A_1 \otimes_i C_1) \otimes_k (A_2 \otimes_i C_2)\right) \otimes_j \left((B_1 \otimes_i D_1) \otimes_k (B_2 \otimes_i D_2)\right)$$

with maps $\eta^{ij} \otimes_k \eta^{ij}$ (diagonal down-right) and η^{jk} (down),

$$\left((A_1 \otimes_i C_1) \otimes_j (B_1 \otimes_i D_1)\right) \otimes_k \left((A_2 \otimes_i C_2) \otimes_j (B_2 \otimes_i D_2)\right)$$

$$(1.10)$$

1.3 EXAMPLES

EXAMPLE 1.4. Let A be an associative algebra. The category $\mathcal{B}imod(A)$ of A-bimodules is monoidal (1-fold monoidal), with $\otimes_A$ as the monoidal product. The tautological bimodule A is the unit in this monoidal category.

EXAMPLE 1.5. Let B be an associative bialgebra over a field k (see Section 2.1 below for definition). The category of left B-modules is monoidal, with the following monoidal structure. Let M, N be two left B-modules, then $M \otimes_k N$ is naturally a $B \otimes_k B$-module. Now the coproduct $\Delta \colon B \to B \otimes_k B$ is a homomorphism of algebras; this makes $M \otimes_k N$ a B-module. Let $\varepsilon \colon B \to k$ be the counit in B, it endows k with a B-module structure. This left B-module k is the unit in this monoidal category.

EXAMPLE 1.6. According to a result of Joyal and Street [JS], there are just few examples for $n \geq 2$ when the map $\eta_{A,B,C,D}^{ij}$ are *isomorphisms* for any A, B, C, D and any $1 \leq i < j \leq n$ and when *there is a common unit object for all n monoidal structures*. For $n = 2$ any such category is equivalent as a 2-fold monoidal category to a category with $A \otimes_1 B = A \otimes_2 B$ with a *braiding* $c_{A,B} \colon A \otimes B \to B \otimes A$ defining a structure of a *braided category* (see, e.g., [ES]) on $\mathcal{C}$. Then we can construct a map $\eta_{A,B,C,D} \colon (A \otimes B) \otimes (C \otimes D) \to (A \otimes C) \otimes (B \otimes D)$ just as $\eta_{A,B,C,D} = id \otimes c_{23} \otimes id$. This construction gives a 2-fold monoidal category. For $n > 2$ and $\eta_{A,B,C,D}$ isomorphisms one necessarily has $A \otimes_i B = A \otimes_j B$ for any i, j and *all $\otimes_i$ are symmetric*. See the Fiedorowicz's Obervolfach talk [F] (page 4 and thereafter) for a clear overview of this result.

EXAMPLE 1.7. Let A be an associative bialgebra. We define a *tetramodule* over it as a k-vector space M such that there is a bialgebra structure on $A \oplus \epsilon M$, where $\epsilon^2 = 0$ and the restriction of the bialgebra structure to A is the initial one (see Section 2 for details). If we rephrase this definition replacing "bialgebra" by "associative algebra", we recover the concept of a bimodule over an associative algebra; thus, this Example is a generalization of Example 1.4. We construct a 2-fold monoidal structure on the abelian category $\mathcal{T}etra(A)$ of tetramodules over A in Section ???.

EXAMPLE 1.8. Examples 1.4 and 1.7 can be generalized as follows. Recall from Example 1.5 that the category of left modules over an associative bialgebra is a monoidal category, with the monoidal structure equal to the tensor product over k on the level of the underlying vector spaces. Define an *n-fold monoidal bialgebra* as an associative algebra with n coassociative coproducts $\Delta_1, \ldots, \Delta_n \colon A \to A \otimes_k A$ such that the corresponding n monoidal structures on the category of left A-modules form an n-fold monoidal category. Thus, 0-monoidal bialgebra is just an associative algebra, and 1-monoidal bialgebra is a bialgebra. One can define the category of tetramodules over an n-monoidal bialgebra analogously to the previous Example. We claim that this category is an $(n + 1)$-fold monoidal category; a proof is straightforward. We think this definition of n-monoidal bialgebra is a conceptually right n-categorical generalization of the concept of bialgebra, for higher n.

1.4   THE OPERAD OF CATEGORIES GOVERNING THE n-FOLD MONOIDAL CATEGORIES

Fix $n \geq 1$. For any $d \geq 0$ denote by $\mathcal{M}_n(d)$ the full subcategory of the free n-fold monoidal category generated by objects $x_1, \ldots, x_d$ consisting of objects which are monomials in x_i, where

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each x_i occurs exactly ones. For example, such monomials for $d = 3$ and $n = 2$ could be $(x_3 \otimes_1 x_1) \otimes_2 x_2$, or $(x_2 \otimes_2 x_3) \otimes_1 x_1$. For fixed n and d the category $\mathcal{M}_n(d)$ has a finite number of objects. The morphisms in $\mathcal{M}_n(d)$ are exactly those which can be obtained as compositions of the associativities for a fixed $\otimes_i$, and η_{ijkl}, with exactly the same commutative diagrams as in n-fold monoidal category.

When n is fixed and d is varied, the categories $\mathcal{M}_n(d)$ form an operad of categories. The following lemma follows from the definitions.

LEMMA 1.9. *A category is n-fold monoidal if and only if there is an action of the operad $\{\mathcal{M}_n(d)\}_{d \geq 0}$ of categories on it.*

The following very deep theorem is proven in [BFSV]:

THEOREM 1.10. *The classifying space of the operad of categories $\{\mathcal{M}_n(d)\}$ is an operad of topological space which is homotopically equivalent (as operad) to the n-dimensional little discs operad.*

2   THE CATEGORY OF TETRAMODULES AND A 2-FOLD MONOIDAL STRUCTURE ON IT

2.1

Recall that an *associative bialgebra* is a vector space A over a field k equipped with two operations, the product $* \colon A^{\otimes 2} \to A$ and the coproduct $\Delta \colon A \to A^{\otimes 2}$, which obey the axioms 1.-4. below:

1. Associativity: $a * (b * c) = (a * b) * c$;

2. Coassociativity: $(\Delta \otimes id)\Delta(a) = (id \otimes \Delta)\Delta(a)$;

3. Compatibility: $\Delta(a * b) = \Delta(a) * \Delta(b)$.

We use the classical notation
$$\Delta(a) = \Delta^1(a) \otimes \Delta^2(a)$$
which is just a simplified form of the equation
$$\Delta(a) = \sum_i \Delta_i^1(a) \otimes \Delta_i^2(a)$$

We always assume that our bialgebras have a unit and a counit. A unit is a map $i \colon k \to A$ and the counit is a map $\varepsilon \colon A \to k$. We always assume

4. $i(k_1 \cdot k_2) = i(k_1) * i(k_2)$, $\varepsilon(a * b) = \varepsilon(a) \cdot \varepsilon(b)$.

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We also denote the product $*$ by m.

A *Hopf algebra* is a bialgebra with antipode. An antipode is a k-linear map $S\colon A \to A$ which obeys

5. $m(1 \otimes S)\Delta(a) = m(S \otimes 1)\Delta(a) = i(\varepsilon(a))$

As we already mentioned above, an "operadic" definition of a bimodule over an associative algebra A reads: it is a k-vector space M such that $A \oplus \epsilon M$ is again an associative algebra, where $\epsilon^2 = 0$, and the restriction of the algebra structure to A coincides with the initial one. (The latter condition can be formulated equivalently that the associative algebra $A \oplus \epsilon M$ is defined *over* A). A vector space M obeying this definition is the same that an A-bimodule. We give an analogous definition in the case when A is an associative bialgebra.

DEFINITION 2.1. Let A be an associative bialgebra. A *Bernstein-Khovanova tetramodule* [BKh], [Kh] M over A is a vector space such that $A \oplus \epsilon M$ is an associative bialgebra, when $\epsilon^2 = 0$ and the restriction of the bialgebra structure to A is the initial one. The category of tetramodules over a bialgebra A is denoted $\mathcal{T}etra(A)$.

More precisely, one has maps $m_\ell\colon A \otimes M \to M$, $m_r\colon M \otimes A \to M$ (which make M an A-bimodule), and maps $\Delta_\ell\colon M \to A \otimes M$ and $\Delta_r\colon M \to M \otimes A$ (which make M an A-bicomodule), with some compatibility between these 4 maps. The compatibility written down explicitly is the following 4 equations:

$$\Delta_\ell(a * m) = (\Delta^1(a) * \Delta_\ell^1(m)) \otimes (\Delta^2(a) * \Delta_\ell^2(m)) \subset A \otimes_k M \qquad (2.1)$$

$$\Delta_\ell(m * a) = (\Delta_\ell^1(m) * \Delta^1(a)) \otimes (\Delta_\ell^2(m) * \Delta^2(a)) \subset A \otimes_k M \qquad (2.2)$$

$$\Delta_r(a * m) = (\Delta^1(a) * \Delta_r^1(m)) \otimes (\Delta^2(a) * \Delta_r^2(m)) \subset M \otimes_k A \qquad (2.3)$$

$$\Delta_r(m * a) = (\Delta_r^1(m) * \Delta^1(a)) \otimes (\Delta_r^2(m) * \Delta^2(a)) \subset M \otimes_k A \qquad (2.4)$$

Here we use the natural notation like $\Delta_\ell(m) = \Delta_\ell^1(m) \otimes \Delta_\ell^2(m)$ with $\Delta_\ell^1(m) \in A$, $\Delta_\ell^2(m) \in M$, etc. As well, we use the sign $*$ for the both product in A and the module products m_ℓ and m_r.

The tetramodules over a bialgebra A form an abelian category. The main (non-trivial) example of a tetramodule over A is A itself; it is called *the tautological tetramodule*.

When A is finite-dimensional over k, a tetramodule is the same that a left module over some associative algebra $H(A)$. This algebra $H(A)$ is, as a vector space, the tensor product $H(A) = A \otimes_k A \otimes_k A^* \otimes_k A^*$, and the product in it is defined to fulfill the equations (2.1)-(2.4) above. (An analogous associative algebra constructed from a bialgebra A, whose underlying vector space is $A \otimes_k A^*$, and the product is defined to fulfill only the single relation (2.1) among the 4 relations above, is known as *the Heisenberg double* of A.).

In particular, if A is finite-dimensional over k, the abelian category $\mathcal{T}etra(A)$ has enough projective and enough injective objects. For general A, R.Taillefer proved [Tai2] that the category $\mathcal{T}etra(A)$ has enough injectives.

Denote by $H^{\bullet}_{\mathrm{GS}}(A, A)$ the Gerstenhaber-Schack cohomology [GS] of a bialgebra A (we recall the definition in Section 3.1 below). The Gerstenhaber-Schack cohomology $H^{\bullet}_{\mathrm{GS}}(A, A)$ is known to control the infinitesimal deformations of the bialgebra A.

The interplay between the category of tetramodules and the deformation theory of associative bialgebras is given in the following result:

THEOREM 2.2 (Taillefer, [Tai1,2]). *For any bialgebra A one has:*

$$H^{\bullet}_{\mathrm{GS}}(A, A) = \mathrm{Ext}^{\bullet}_{\mathcal{T}etra(A)}(A, A) \tag{2.5}$$

We recall the original Taillefer's proof of Theorem 2.2 in Section 3 below.

EXAMPLE 2.3. Consider the case when $A = S(V)$ is a free (co)commutative bialgebra, for V a vector space over k. Suppose for simplicity that V is finite-dimensional. We prove in Section 4 that

$$H^k_{\mathrm{GS}}(A, A) = \oplus_{i+j=k} \Lambda^i V \otimes_k \Lambda^j V^* \tag{2.6}$$

The total graded space $F = \oplus_{i,j \geq 0} \Lambda^i V \otimes_k \Lambda^j V^*[-i-j]$ can be thought as the space of functions on the space $W = V[1] \oplus V^*[1]$. The space W is a Poisson space, with the Poisson bracket of degree -2 given by the contraction of $V[1]$ with $V^*[1]$. Altogether, F admits a product and an even Poisson-Lie bracket of degree -2, which are compatible by the Leibniz rule. It can be rephrased saying that F is a 3-algebra.

In fact, the Deligne conjecture for 2-fold monoidal categories (mentioned in Introduction), and our Theorem 2.6 below, would imply together that the Gerstenhaber-Schack cohomology of any Hopf algebra is naturally a 3-algebra structure. Presumably, it is not true for general bialgebra which is not a Hopf algebra.

2.2 THE STRUCTURE OF A 2-FOLD MONOIDAL CATEGORY ON $\mathcal{T}etra(A)$

2.2.1 TWO "EXTERNAL" TENSOR PRODUCTS

Before defining two "real" monoidal structures on the category of tetramodules, we start with two preliminary monoidal products. Our preliminary products are related to the real ones, as $M \otimes_k N$ is related to $M \otimes_A N$, for an associative algebra A over k, and two its bimodules.

Let A be an associative bialgebra, and let M_1, M_2 be two tetramodules over it. We define two their "external" tensor products $M_1 \boxtimes_1 M_2$ and $M_1 \boxtimes_2 M_2$ (which are A-tetramodules again). In the both cases the underlying vector space is $M_1 \otimes_k M_2$.

The case of $M_1 \boxtimes_1 M_2$:

1. $m_\ell(a \otimes m_1 \boxtimes m_2) = (am_1) \boxtimes m_2,$

2. $m_r(m_1 \boxtimes m_2 \otimes a) = m_1 \boxtimes (m_2 a)$,

3. $\Delta_\ell(m_1 \boxtimes m_2) = (\Delta_\ell^1(m_1) * \Delta_\ell^1(m_2)) \otimes (\Delta_\ell^2(m_1) \boxtimes \Delta_\ell^2(m_2))$,

4. $\Delta_r(m_1 \boxtimes m_2) = (\Delta_r^1(m_1) \boxtimes \Delta_r^1(m_2)) \otimes (\Delta_r^2(m_1) * \Delta_r^2(m_2))$.

The case of $M_1 \boxtimes_2 M_2$:

1. $m_\ell(a \otimes m_1 \boxtimes m_2) = (\Delta^1(a)m_1) \boxtimes (\Delta^2(a)m_2)$,

2. $m_r(m_1 \boxtimes m_2 \otimes a) = (m_1 \Delta^1(a)) \boxtimes (m_2 \Delta^2(a))$,

3. $\Delta_\ell(m_1 \boxtimes m_2) = \Delta_\ell^1(m_1) \otimes (\Delta_\ell^2(m_1) \boxtimes m_2)$,

4. $\Delta_r(m_1 \boxtimes m_2) = (m_1 \boxtimes \Delta_r^1(m_2)) \otimes \Delta_r^2(m_2)$.

The main lack of the two external products is that the tautological tetramodule A is not a unit object for them.

In the both definitions we use only "the half" of the tetramodule structures on M_1, M_2. In particular, in the first definition we do not use the right multiplication m_r for M_1 and the left multiplication m_ℓ for M_2. Similarly, in the second definition we do not use Δ_r for M_1 and Δ_ℓ for M_2.

This provides us some extra possibilities to modify our two products, which we use below to define the real "internal" tensor products $M_1 \otimes_1 M_2$ and $M_1 \otimes_2 M_2$. For the two internal products, the tautological tetramodule A is a unit object.

2.2.2 Two "internal" tensor products

DEFINITION 2.4. Let M_1, M_2 be two tetramodules over a bialgebra A. Their first tensor product $M_1 \otimes_1 M_1$ is defined as the quotient-tetramodule

$$M_1 \otimes_1 M_2 = M_1 \boxtimes_1 M_2 / ((m_1 a) \boxtimes_1 m_2 - m_1 \boxtimes_1 (am_2)) \tag{2.7}$$

One easily checks that this definition is correct. Analogously, the second tensor product $M_1 \otimes_2 M_2$ is defined as a sub-tetramodule

$$M_1 \otimes_2 M_2 = \left\{ \sum_i m_{1i} \boxtimes_2 m_{2i} \subset M_1 \boxtimes_2 M_2 | \sum_i \Delta_r(m_{1i}) \otimes_k m_{2i} = \sum_i m_{1i} \otimes_k \Delta_\ell(m_{2i}) \right\} \tag{2.8}$$

Again, one easily checks that this definition is correct.

LEMMA 2.5. *Let A be an associative bialgebra. Then the tautological tetramodule A is the unit object for both monoidal structures.*

Proof. Let M be a tetramodule. One can check that the maps

$$m_\ell \colon A \otimes_1 M \to M$$
$$m_r \colon M \otimes_1 A \to M$$

(2.9)

and

$$\Delta_\ell \colon M \to A \otimes_2 M$$
$$\Delta_r \colon M \to M \otimes_2 A$$

(2.10)

are morphisms of tetramodules. As A has a unit, the first two maps are bijective; as A has a counit, the last two maps are bijective too. $\qquad\square$

2.2.3 THE 2-FOLD MONOIDAL STRUCTURE

We construct the map $\eta_{M,N,P,Q} \colon (M \otimes_2 N) \otimes_1 (P \otimes_2 Q) \to (M \otimes_1 P) \otimes_2 (N \otimes_1 Q)$ in several steps.

The first step is to check that the map $\phi_0 \colon (M \boxtimes_2 N) \boxtimes_1 (P \boxtimes_2 Q) \to (M \boxtimes_1 P) \boxtimes_2 (N \boxtimes_1 Q)$, $\phi_0(m \otimes_k n \otimes_k p \otimes_k q) = m \otimes_k p \otimes_k n \otimes_k q$, is a map of tetramodules. We have:

$$a * \big((m \boxtimes_2 n) \boxtimes_1 (p \boxtimes_2 q) \big) = \big(a * (m \boxtimes_2 n) \big) \boxtimes_1 (p \boxtimes_2 q) = (\Delta^1(a) * m) \otimes (\Delta^2(a) * n) \otimes p \otimes q \quad (2.11)$$

and

$$a * \big((m \boxtimes_1 p) \boxtimes_2 (n \boxtimes_1 q) \big) = \big(\Delta^1(a) * (m \boxtimes_1 p) \big) \boxtimes_2 \big(\Delta^2(a) * (n \boxtimes_1 q) \big) = (\Delta^1(a) * m) \otimes p \otimes (\Delta^2(a) * n) \otimes q$$

(2.12)

We see that

$$\phi_0(\text{r.h.s. of } (2.11)) = (\text{r.h.s. of } (2.12)) \tag{2.13}$$

That is, ϕ_0 is a map of left modules; analogously it is a map of right modules.

Now prove that ϕ_0 is a map of left comodules. We have:

$$\Delta_\ell \big((m \boxtimes_2 n) \boxtimes_1 (p \boxtimes_2 q) \big) = \big(\Delta_\ell^1(m \boxtimes_2 n) * \Delta_\ell^1(p \boxtimes_2 q) \big) \otimes_k \big(\Delta_\ell^2(m \boxtimes_2 n) * \Delta_\ell^2(p \boxtimes_2 q) \big) =$$
$$\big(\Delta_\ell^1(m) * \Delta_\ell^1(p) \big) \otimes_k \big(\Delta_\ell^2(m) \otimes_k n \otimes_k \Delta_\ell^2(p) \otimes_k q \big)$$

(2.14)

and

$$\Delta_\ell \big((m \boxtimes_1 p) \boxtimes_2 (n \boxtimes_1 q) \big) = \Delta_\ell^1(m \boxtimes_1 p) \otimes_k \big(\Delta_\ell^2(m \boxtimes_1 p) \boxtimes_2 (n \boxtimes_1 q) \big) =$$
$$\big(\Delta_\ell^1(m) * \Delta_\ell^1(p) \big) \otimes_k \big(\Delta_\ell^2(m) \otimes_k \Delta_\ell^2(p) \otimes_k n \otimes_k q \big)$$

(2.15)

We see that

$$\Delta_\ell \circ \phi_0 = id \otimes_k (\phi_0 \circ \Delta_\ell) \tag{2.16}$$

that is, ϕ_0 is a map of left comodules. It is proven analogously that ϕ_0 is a map of right comodules.

At the second step we consider the natural projections of tetramodules $p_{M,P} \colon M \boxtimes_1 P \to M \otimes_1 P$ and $p_{N,Q} \colon N \boxtimes_1 Q \to N \otimes_1 Q$. We consider the composition

$$\phi_1 = (p_{M,P} \boxtimes_2 p_{N,Q}) \circ \phi_0 \colon (M \boxtimes_2 N) \boxtimes_1 (P \boxtimes_2 Q) \to (M \otimes_1 P) \boxtimes_2 (N \otimes_1 Q) \qquad (2.17)$$

We want to check that the map ϕ_1 defines naturally a map

$$\phi_2 = \overline{\phi}_1 \colon (M \boxtimes_2 N) \otimes_1 (P \boxtimes_2 Q) \to (M \otimes_1 P) \boxtimes_2 (N \otimes_1 Q) \qquad (2.18)$$

that is, the elements of the form

$$\big((m \boxtimes_2 n) * a\big) \otimes_k (p \boxtimes_2 q) - (m \boxtimes_2 n) \otimes_k \big(a * (p \boxtimes_2 q)\big) \qquad (2.19)$$

are mapped to 0 by ϕ_1.

Indeed,

$$(2.19) = (m * \Delta^1 a) \otimes_k (n * \Delta^2 a) \otimes_k (p \otimes_k q) - (m \otimes_k n) \otimes_k (\Delta^1 a * p) \otimes_k (\Delta^2 a * q) \qquad (2.20)$$

which, after the permutation ϕ_0 of the second and the third factors, is mapped to 0 in $(M \otimes_1 P) \boxtimes_2 (N \otimes_1 Q)$. Therefore, the map ϕ_2 is well-defined.

At the third step we restrict the map ϕ_2 to $(M \otimes_2 N) \otimes_1 (P \otimes_2 Q) \subset (M \boxtimes_2 N) \otimes_1 (P \boxtimes_2 Q)$, and we need to check that the image of this restricted map belongs to $(M \otimes_1 P) \otimes_2 (N \otimes_1 Q) \subset (M \otimes_1 P) \boxtimes_2 (N \otimes_1 Q)$.

Suppose $m \otimes_k n \in M \otimes_2 N$ and $p \otimes_k q \in P \otimes_2 Q$ (we assume the summation over several such monomials, but for simplicity we skip this summation). Then

$$\Delta_r^1(m) \otimes_k \Delta_r^2(m) \otimes_k n = m \otimes_k \Delta_\ell^1(n) \otimes_k \Delta_\ell^2(n) \qquad (2.21)$$

with the middle factors in A, and analogously

$$\Delta_r^1(p) \otimes_k \Delta_r^2(p) \otimes_k q = p \otimes_k \Delta_\ell^1(q) \otimes_k \Delta_\ell^2(q) \qquad (2.22)$$

again, with the middle factors in A.

One needs to prove that (2.21) and (2.22) together imply that

$$(m \boxtimes_1 p) \boxtimes_2 (n \boxtimes_1 q) \in (M \otimes_1 P) \otimes_2 (N \otimes_1 Q) \subset (M \otimes_1 P) \boxtimes_2 (N \otimes_1 Q) \qquad (2.23)$$

that is,

$$\Delta_r^1(m) \otimes_k \Delta_r^2(p) \otimes_k (\Delta_r^2(m) * \Delta_r^2(p)) \otimes_k n \otimes_k q = m \otimes_k p \otimes_k (\Delta_\ell^1(n) * \Delta_\ell^1(q)) \otimes_k \Delta_\ell^2(n) \otimes_k \Delta_\ell^2(q) \qquad (2.24)$$

To get (2.24) from (2.22) and (2.23) we permute (2.22) such that the factors in A are the most right, permute (2.22) such that the factors in A are the most left, then take the equation

(l.h.s. of (2.22)) $\otimes_k$ (l.h.s. of (2.23)) = (r.h.s. of (2.22)) $\otimes_k$ (r.h.s. of (2.23)) (after they are permuted). Then for the two middle factor (in A) we apply the product $*\colon A \otimes_k A \to A$, and then permute again.

The map $\eta_{M,N,P,Q}$ is constructed.

THEOREM 2.6. *The maps $\eta_{M,N,P,Q}$, constructed above, and the two tensor products $\otimes_1$ and $\otimes_2$, define a 2-fold monoidal structure on the category $\mathcal{T}etra(A)$ of tetramodules over a bialgebra A. The tautological tetramodule A is a unit object of this 2-fold monoidal category.*

Proof. First of all, the two tensor products $\boxtimes_1$ and $\boxtimes_2$ with $\widetilde{\eta}_{M,N,P,Q} = \phi_0$, clearly define a 2-fold monoidal structure (without a unit object) on the category $\mathcal{T}etra(A)$. In particular, the diagrams (1.8) and (1.9) are commutative for $\widetilde{\eta}_{M,N,P,Q}$ (because ϕ_0 is just the permutation which switches the second and the third factors). Now the same diagrams for the actual structure $\eta_{M,N,P,Q}$ are obtained from the above "external structure" diagrams just by passing to subquotients. Therefore, they are commutative as well. $\square$

REMARK 2.7. The "external" 2-fold monoidal structure does not obey properly the definition of n-fold monoidal categories given in Section 1, as it does not have a unit object (one easily sees that k is not a unit). Therefore, the Joyal-Street result from Example 1.6 is not applied to it.

3   TETRAMODULES AND THE GERSTENHABER-SCHACK COHOMOLOGY: AN OVERVIEW OF TAILLEFER'S RESULTS

In this Section we present some results of R.Taillefer [Tai1,2] on the category of tetramodules. Our presentation follows the same ideas as the original proofs of Taillefer, but the details are slightly different, and, we hope, more transparent. In particular, we prove Theorem 2.2 for any bialgebra, whereas Taillefer assumes it to be a Hopf algebra.

As an application of the general theory, we present a computation of the Gerstenhaber-Schack cohomology for $A = S(V)$, the free commutative cocommutative bialgebra generated by a vector space V.

3.1   THE GERSTENHABER-SCHACK COMPLEX

Let A be a (co)associative bialgebra. Note that the bar-differential in $\mathrm{Bar}^{\boxtimes_1}(A)$ is given by maps of tetramodules; analogously, the cobar-differential in $\mathrm{Cobar}^{\boxtimes_2}(A)$ is given by maps of tetramodules.

Let us recall, that originally the Gerstenhaber-Schack complex was defined in [GS] as

$$C^{\bullet}_{\mathrm{GS}}(A) = \mathrm{Hom}_{\mathcal{T}etra(A)}(\mathrm{Bar}^{\boxtimes_1}_{-}(A), \mathrm{Cobar}^{\boxtimes_2}_{+}(A)) \tag{3.1}$$

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Here $\text{Bar}_-(B)$ and $\text{Cobar}_+(C)$ are truncated complexes, which end (start) with $B\boxtimes_1 B$ ($C\boxtimes_2 C$) correspondingly.

For convenience of the reader let us write down here the Gerstenhaber-Schack differential in $C^\bullet_{\text{GS}}(A)$ explicitly:

First of all, as a graded vector space,

$$C^\bullet_{\text{GS}}(A) = \oplus_{m,n\geq 0}\text{Hom}_k(A^{\otimes m}, A^{\otimes n})[-m-n] \tag{3.2}$$

Now let $\Psi\colon A^{\otimes m} \to A^{\otimes n} \in C^{m+n}_{GS}(A)$. We are going to define the Gerstenhaber-Schack differential $d_{\text{GS}}(\Psi) \in \text{Hom}(A^{\otimes(m+1)}, A^{\otimes n}) \oplus \text{Hom}(A^{\otimes m}, A^{\otimes(n+1)})$. Denote the projection of d_{GS} to the first summand by $(d_{\text{GS}})_1$, and the projection to the second summand by $(d_{\text{GS}})_2$. The formulas for $(d_{\text{GS}})_1$ and $(d_{\text{GS}})_2$ are:

$$(d_{\text{GS}})_1(\Psi)(a_0 \otimes \cdots \otimes a_m) =$$
$$\Delta^{n-1}(a_0) * \Psi(a_1 \otimes \cdots \otimes a_m) + \sum_{i=0}^{m-1}(-1)^{i+1}\Psi(a_0 \otimes \cdots \otimes (a_i * a_{i+1}) \otimes \cdots \otimes a_m) + \tag{3.3}$$
$$(-1)^{m-1}\Psi(a_0 \otimes \cdots \otimes a_{m-1}) * \Delta^{n-1}(a_m)$$

and

$$(d_{\text{GS}})_2(\Psi)(a_1 \otimes \cdots \otimes a_m) =$$
$$(\Delta^{(1)}(a_1) * \Delta^{(1)}(a_2) * \cdots * \Delta^{(1)}(a_m)) \otimes \Psi(\Delta^{(2)}(a_1) \otimes \cdots \otimes \Delta^{(2)}(a_m)) +$$
$$\sum_{i=1}^{n}(-1)^i \Delta_i \Psi(a_1 \otimes \cdots \otimes a_m) +$$
$$(-1)^{n+1}\Psi(\Delta^{(1)}(a_1) \otimes \Delta^{(1)}(a_2) \otimes \cdots \otimes \Delta^{(1)}(a_m)) \otimes (\Delta^{(2)}(a_1) * \Delta^{(2)}(a_2) * \cdots * \Delta^{(2)}(a_m))$$
$$\tag{3.4}$$

The goal of this Section is to recall the original proof, with some minor improvements in presentation, of the following result due to R.Taillefer [Tai1,2]:

THEOREM 3.1 (Taillefer). *For any bialgebra A one has:*

$$\text{Ext}^\bullet_{\mathcal{T}etra(A)}(A, A) = H^\bullet\big(\text{Hom}_{\mathcal{T}etra(A)}(\text{Bar}^{\boxtimes_1}_-(A), \text{Cobar}^{\boxtimes_2}_+(A))\big)$$

3.2 TWO FORGETFUL FUNCTORS AND THEIR ADJOINTS

3.2.1

Let A be a (co)associative bialgebra. Besides the category $\mathcal{T}etra(A)$, we can consider the categories $\mathcal{B}imod(A)$ of A-bimodules (when we consider A as an algebra) and $\mathcal{B}icomod(A)$ of A-bicomodules (when we consider A as a coalgebra). Clearly there are two exact forgetful functors $F_1\colon \mathcal{T}etra(A) \to \mathcal{B}icomod(A)$ and $F_2\colon \mathcal{T}etra(A) \to \mathcal{B}imod(A)$. We have the following

LEMMA 3.2. *Let A be a bialgebra which has unit and counit. Then the functor F_1 admits a left adjoint L and the functor F_2 admits a right adjoint R. The functors L and R are exact.*

DEFINITION 3.3. Let A be a bialgebra, and let N be an A-bicomodule, and let M be an A-bimodule. The tetramodule $L(N)$ is called *an induced tetramodule* (from N), and the tetramodule $R(M)$ is called *a coinduced tetramodule* (from M). The induced and coinduced tetramodules form full additive subcategories in the abelian category $\mathcal{T}etra(A)$. We denote them $\mathcal{T}etra_{\mathrm{Ind}}(A)$ and $\mathcal{T}etra_{\mathrm{Coind}}(A)$, respectively.

Proof of Lemma: we set

$$L(N) = A \boxtimes_1 N \boxtimes_1 A \tag{3.5}$$

and

$$R(M) = A \boxtimes_2 M \boxtimes_2 A \tag{3.6}$$

(see Section 2.2.1 for the definitions of $\boxtimes_1$ and $\boxtimes_2$). Rigorously, to write down formulas like this, M and N should be tetramodules. However, the definition of $M_1 \boxtimes_1 M_2$ does not use the right A-module structure in M_1 and the left A-module structure in M_2. Analogously, in the definition of $M_1 \boxtimes_2 M_2$ does not use the right comodule structure in M_1 and the left comodule structure in M_2. Therefore, (3.5) and (3.6) make sense.

The adjunction properties of L and R are clear.

The exactness of L and R is clear from the constructions (3.5) and (3.6).

3.2.2

LEMMA 3.4. *Let A be an associative bialgebra. Then any tetramodule $M \in \mathcal{T}etra(A)$ can be imbedded into a coinduced tetramodule, and there is a surjection to M from an induced tetramodule.*

Proof. Let M be an A-tetramodule. Consider $P(M) = A \boxtimes_1 M \boxtimes_1 A$, it is induced from the bicomodule $F_1(M)$. The map $p \colon A \boxtimes_1 M \boxtimes_1 A \to M$, $a \boxtimes_1 m \boxtimes_1 b \mapsto a \cdot m \cdot b$ is clearly a map (and an epimorphism, because A contains a unit) of tetramodules. Analogously, the tetramodule $Q(M) = A \boxtimes_2 M \boxtimes_2 A$ is coinduced from the bimodule $F_2(M)$, and we have a monomorphism $j \colon M \to A \boxtimes_2 M \boxtimes_2 A$, $m \mapsto \Delta_\ell \circ \Delta_r(m)$. $\square$

COROLLARY 3.5 ([Tai2]). *For any bialgebra A the category $\mathcal{T}etra(A)$ has enough injectives.*

Proof. The functor R is a right adjoint to an exact functor F_2, and, therefore, maps injective objects to injective (see [W], Prop. 2.3.10). Moreover, R is left exact ([W], Section 2.6). Therefore, it is sufficient to imbed M as an A-bimodule into an injective A-bimodule I (it is a classical construction, see e.g. [W], Section 2.3). Then we apply the functor R to this this imbedding of A-bimodules. It gives an imbedding $j \colon M \to Q(M)$. $\square$

The main fact about the induced and the coinduced tetramodules is the following:

PROPOSITION 3.6. *Let A be an associative bialgebra. Then the functor $X \mapsto \mathrm{Hom}_{\mathcal{T}etra(A)}(X,Q)$ for fixed $Q \in \mathcal{T}etra_{\mathrm{Coind}}(A)$ is an exact functor from $\mathcal{T}etra_{\mathrm{Ind}}(A)^{\mathrm{opp}}$ to $\mathcal{V}ect$. As well, the functor $Y \mapsto \mathrm{Hom}_{\mathcal{T}etra(A)}(P,Y)$ for fixed $P \in \mathcal{T}etra_{\mathrm{Ind}}(A)$ is an exact functor from $\mathcal{T}etra_{\mathrm{Coind}}(A)$ to $\mathcal{V}ect$.*

Proof. We prove the first statement. Let

$$0 \to LN' \to LN \to LN'' \to 0 \tag{3.7}$$

be an exact sequence of tetramodules, $N, N', N'' \in \mathcal{B}icomod(A)$. We need to prove that the sequence

$$0 \to \mathrm{Hom}_{\mathcal{T}etra(A)}(LN'', RM) \to \mathrm{Hom}_{\mathcal{T}etra(A)}(LN, RM) \to \mathrm{Hom}_{\mathcal{T}etra(A)}(LN', RM) \to 0 \tag{3.8}$$

is exact for any $M \in \mathcal{B}imod(A)$.

By the adjunction, the exactness of (3.8) is equivalent to the exactness of the sequence

$$0 \to \mathrm{Hom}_{\mathcal{B}imod(A)}(F_2LN'', M) \to \mathrm{Hom}_{\mathcal{B}imod(A)}(F_2LN, M) \to \mathrm{Hom}_{\mathcal{B}imod(A)}(F_2LN', M) \to 0 \tag{3.9}$$

The latter sequence is indeed exact as for any $N \in \mathcal{B}icomod(A)$ the A-bimodule F_2LN is free and, therefore, projective.

The second statement is proven analogously. $\square$

3.3 Some homological algebra

We recall here some construction of homological algebra, based on the Grothendieck's interpretation of the derived functors as "universal δ-functors" [Tohoku, 2.1-2.2]. This construction provides a useful way of computation of Ext's functors in abelian categories, using resolutions from "semi-projective" and "semi-injective" objects.

3.3.1 A $(\mathcal{P}, \mathcal{Q})$-pair

DEFINITION 3.7. Let $\mathcal{A}$ be an abelian category, and let $\mathcal{P}$, $\mathcal{Q}$ be two additive subcategories of $\mathcal{A}$. We say that $\mathcal{P}$ and $\mathcal{Q}$ form a (P,Q)-pair, if the following conditions are fulfilled:

1. the functor $\mathrm{Hom}(?,Q)$ is exact on $\mathcal{P}^{\mathrm{opp}}$ for any $Q \in \mathcal{Q}$;

2. the functor $\mathrm{Hom}(P,?)$ is exact on $\mathcal{Q}$ for any $P \in \mathcal{P}$;

3. for any object $M \in \mathcal{A}$, there is an epimorphism $P \to M$ for $P \in \mathcal{P}$, and there is a monomorphism $M \to Q$ for $Q \in \mathcal{Q}$;

4. a stronger version of 3: for any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in $\mathcal{A}$ the epimorphisms $P_i \to M_i$, $p_i \colon P_i \in \mathcal{P}$, can be chosen such that there is a map of complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \longrightarrow & P_2 & \longrightarrow & P_3 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_3} & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0
\end{array}
\qquad (3.10)
$$

where the upper line is an exact sequence. As well, also the dual condition for $\mathcal{Q}$ for monomorphisms $q_i \colon M_i \to Q_i$, $Q_i \in \mathcal{Q}$ is required.

Note that the third condition guarantees that each object $M \in \mathcal{A}$ has a $\mathbb{Z}_{\leq 0}$-graded resolution in $\mathcal{P}$ and a $\mathbb{Z}_{\geq 0}$-graded resolution in $\mathcal{Q}$.

EXAMPLE 3.8. If $\mathcal{A}$ has enough projectives and $\mathcal{P}$ is the additive subcategory of projective objects, $\mathcal{Q} = \mathcal{A}$ gives a (P, Q)-pair. Analogously, if $\mathcal{A}$ has enough injectives and $\mathcal{Q}$ is the additive subcategory of injective objects, $\mathcal{P} = \mathcal{A}$ gives a (P, Q)-pair.

PROPOSITION 3.9. *Let A be an associative bialgebra, and let $\mathcal{A} = \mathcal{T}etra(A)$. Then the pair $(\mathcal{T}etra_{\mathrm{Ind}}(A), \mathcal{T}etra_{\mathrm{Coind}}(A))$ is a (P, Q)-pair.*

Proof. The first two properties were proven in Proposition 3.6, and the third property follows from Lemma 3.4. Moreover, the construction in the Lemma gives immediately the fourth assertion in Definition 3.7. $\qquad\square$

3.3.2 THE KEY-LEMMA

The main fact about (P, Q)-pairs is the following lemma:

KEY-LEMMA 3.10. *Let $\mathcal{A}$ be an abelian category, having enough projective or injective objects. Suppose we are given a (P, Q)-pair $(\mathcal{P}, \mathcal{Q})$ in $\mathcal{A}$, and let $M, N \in \mathcal{A}$ be two objects. Suppose $P^\bullet \to M$ is a resolution of M of objects in $\mathcal{P}$, and $N \to Q^\bullet$ is a resolution of N by objects in $\mathcal{Q}$. Then*

$$
\mathrm{Ext}^\bullet_{\mathcal{A}}(M, N) = H^\bullet(\mathrm{Hom}_{\mathcal{A}}(P^\bullet, Q^\bullet))
\qquad (3.11)
$$

Proof. The proof consists from several steps. We give the proof for the case when $\mathcal{A}$ has enough injectives, the case of enough projectives is analogous.

The rough idea is: any derived functor in an abelian category (if it exists) enjoys the property of being of universal δ-functor, and as such, is uniquely defined up to an isomorphism by its zero degree component. We prove that the functor $(M, N) \mapsto H^\bullet(\mathrm{Hom}_{\mathcal{A}}(P^\bullet, Q^\bullet))$ is a δ-functor, and then prove that it is a universal δ-functor.

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, and let $\{T_n \colon \mathcal{A} \to \mathcal{B}\}$, $n \geq 0$ be a collection of additive functors. One says that this collection is *a cohomological δ-functor* if for any exact sequence

$$0 \to M \to N \to L \to 0 \tag{3.12}$$

in $\mathcal{A}$, one has a morphism $\delta \colon T_n(L) \to T_{n+1}(M)$, $n \geq 0$ in $\mathcal{B}$, forming a long exact sequence:

$$\cdots \to T_{n-1}(L) \xrightarrow{\delta} T_n(M) \to T_n(N) \to T_n(L) \xrightarrow{\delta} T_{n+1}(M) \to \ldots \tag{3.13}$$

$n \geq 1$, depending functorially on the short exact sequence (3.12).

Consider a δ-functor $\{T_n\}$. We say that this δ-functor is *universal* if for any other δ-functor $\{S_n\}$ with the natural transformation $f_0 \colon T_0 \to S_0$ there is a unique morphism of δ-functors $\{f_n \colon T_n \to S_n\}$ extending f_0. It follows immediately that the universal δ-functor with $T_0 = F$, if it exists, is unique. This point of view, independent on existence of enough projective objects, was emphasized by Grothendieck in [Tohoku] (see also [W], Chapter 2).

Now consider the functor $\text{Hom}_{\mathcal{A}}(M, ?)$ as a functor of the second argument. If $\mathcal{A}$ has enough injectives, the functors $T_n(M, N) = \text{Ext}_{\mathcal{A}}^n(M, N)$ is a universal cohomological δ-functor (see [W], Theorem 2.4.7).

Now the proof of Key-Lemma goes in several steps:

Step 1. $T_k \colon (M, N) \mapsto H^k(\text{Hom}_{\mathcal{A}}(P^\bullet, Q^\bullet))$ with $P^\bullet \in \mathcal{P}$, $Q^\bullet \in \mathcal{Q}$ is well-defined, that is does not depend on the choice of $P^\bullet$ and $Q^\bullet$;

Step 2. it is a cohomological δ-functor with $T_0 = \text{Hom}_{\mathcal{A}}(M, N)$;

Step 3. it is a universal cohomological δ-functor.

Clearly the Key-Lemma follows from these three claims.

Step 1: it easily follows from the conditions 1. and 2. in the definition of a (P, Q)-pair.

Step 2: it follows easily from condition 4. in the definition of a $(\mathcal{P}, \mathcal{Q})$-pair.

Step 3: this is a bit more tricky. An additive functor $F \colon \mathcal{A} \to \mathcal{B}$ is called *effaceable* if for any object $N \in \mathcal{A}$ there is a monomorphism $j \colon M \to I$ such that $F(j) = 0$. It is proven in [Tohoku] that a cohomological δ-functor $\{T_n\}$ for which all T_n for $n \geq 1$ are effaceable, is universal. It remains to prove that our functors $T_n(N) = H^n(\text{Hom}_{\mathcal{A}}(P^\bullet(M), Q^\bullet(N)))$, $n \geq 1$, are effaceable. We can choose a monomorphism $j \colon N \to I$ with $I \in \mathcal{Q}$ by condition 3. in Definition 3.7. Now the effaceability follows as $T_n(I) = 0$, $n \geq 1$, by 1. and 2. in Definition 3.7.

Thus, it is proven that the functors $H^n(\text{Hom}_{\mathcal{A}}(P^\bullet, Q^\bullet))$ form a universal cohomological δ-functor which has the same the same 0-component as $\text{Ext}_{\mathcal{A}}^n(M, N)$. Therefore, the two δ-functors are isomorphic.

$\square$

Theorem 3.1 is proven.

3.4 Example: a computation of the Gerstenhaber-Schack cohomology for $A = S(V)$

Here we compute, as an application of previous results, the Gerstenhaber-Schack cohomology for the (co)free commutative and cocommutative bialgebra $A = S(V)$. For simplicity, we assume V is a finite-dimensional, although the result (being slightly modified) remains true for general V.

PROPOSITION 3.11. *Let $A = S(V)$ be (co)free commutative cocommutative bialgebra, V finite-dimensional. Then the Gerstenhaber-Schack cohomology of A is*

$$H^k_{\mathrm{GS}}(A) = \bigoplus_{\substack{i+j=k \\ i,j \geq 0}} \Lambda^i V \otimes \Lambda^j(V^*)$$

Proof. We compute $\mathrm{Ext}^{\bullet}_{\mathcal{T}etra(A)}(A, A)$ for $A = S(V)$, as $H^{\bullet}(\mathrm{Hom}(P^{\bullet}, Q^{\bullet}))$, where $P^{\bullet}$ is a resolution of A by tetramodules which are free as bimodules, and $Q^{\bullet}$ is a resolution of A by tetramodules which are cofree as bicomodules. We are allowed to use such resolutions by Proposition 3.9 and Key-lemma 3.10.

We now use more "economic" resolutions $P^{\bullet}$ and $Q^{\bullet}$, than the bar- and cobar-resolutions from Theorem 3.1. Namely, we use suitable Koszul resolutions.

Recall these resolutions $P^{\bullet}$ and $Q^{\bullet}$. We have:

$$
\begin{aligned}
&P^{-k} = S(V) \boxtimes_1 \Lambda^k(V) \boxtimes_1 S(V), \\
&\partial_k(f \otimes (e_0 \wedge \cdots \wedge e_k) \otimes g) = \\
&\sum_{i=0}^{k} (-1)^i \left((v_i \cdot f) \otimes (e_0 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_k) \otimes g - f \otimes (e_0 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_k) \otimes (v_i \cdot g) \right)
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
&Q^k = S(V) \boxtimes_2 \Lambda^k(V) \boxtimes_2 S(V), \\
&d_k(f \otimes (e_0 \wedge \cdots \wedge e_k) \otimes g) = \\
&\Delta^1_r(f) \otimes (\Delta^2_r(f) \wedge e_0 \wedge \cdots \wedge e_k) \otimes g - f \otimes (e_0 \wedge \cdots \wedge e_k \wedge \Delta^1_\ell(g)) \otimes \Delta^2_\ell(g)
\end{aligned}
\tag{3.15}
$$

where in the both equations the tetramodule structure on $\Lambda^k(V)$ is trivial for any k. In the second equation $\Delta_\ell(f)$ denotes the projection of $\Delta(f) \in S(V) \otimes S(V)$ to $V \otimes S(V)$, and $\Delta_r(f)$ denotes the projection of $\Delta(f)$ to $S(V) \otimes V$.

One easily sees, that when $\mathrm{Hom}_{\mathcal{T}etra(S(V))}(P^{\bullet}, Q^{\bullet})$ is computed, the differential in this complex vanishes. This gives the result.

$\square$

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