BPS INVARIANTS OF SEMI-STABLE SHEAVES ON RATIONAL SURFACES

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ABSTRACT. BPS invariants are computed, capturing topological invariants of moduli spaces of semi-stable sheaves on rational surfaces. For a suitable stability condition, it is proposed that the generating function of BPS invariants of a Hirzebruch surface Σ_ℓ takes the form of a product formula. BPS invariants for other stability conditions and other rational surfaces are obtained using Harder-Narasimhan filtrations and the blow-up formula. Explicit expressions are given for rank ≤ 3 sheaves on Σ_ℓ or the projective plane $\mathbb{P}^2$. The applied techniques can be applied iteratively to compute invariants for higher rank.

1. INTRODUCTION

Topological invariants of moduli spaces of semi-stable sheaves on complex surfaces are a rich subject with links to many topics in physics and mathematics. Closely related topics in physics are gauge theory, instantons, electric-magnetic duality [31] and also (multi-center) black holes [6, 24, 25]. Instantons saturate the bound on their minimal action, the so-called Bogomolnyi-Prasad-Sommerfeld (BPS) bound. The prime interest of this article are topological invariants of moduli spaces of instantons, in particular their Poincaré polynomials, which are commonly referred to as "BPS invariants". These invariants correspond also to (refined) supersymmetric indices enumerating supersymmetric or BPS states.

Instantons on complex surfaces are described algebraically as semi-stable vector bundles and coherent sheaves [14, 8]. Generating functions of BPS invariants of sheaves on surfaces are computed for rank 1 by Göttsche [10] and rank 2 by Yoshioka [33, 34]. These generating functions lead to intriguing connections with (mock) modular forms [31, 11, 12, 3], which are a manifestation of electric-magnetic duality of the gauge theory [31]. Refs. [26, 28] compute BPS invariants for rank 3 sheaves with Chern classes such that stability coincides with semi-stability.

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The present article computes the BPS invariants of *semi-stable* sheaves with rank 3 on Hirzebruch surfaces Σ_ℓ and on the projective plane $\mathbb{P}^2$, and explains how to generalize the computations to higher rank. The developed techniques can be applied straightforwardly to compute BPS invariants of the other rational and ruled surfaces. Although the extension from stable to semi-stable might seem a minor one, it requires to deal with various subtle but fundamental aspects of the moduli spaces of semi-stable sheaves, which could be neglected in Ref. [26]. Having resolved how to deal with these aspects for $r = 3$, the computations can in principle be extended to any rank.

This introduction continues with summarizing the contents of the paper, after recalling the computations in Ref. [26] which were inspired by [33, 34, 11, 12]. A crucial fact for the computations is that the blow-up $\phi : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2$ is isomorphic to the Hirzebruch surface $\Sigma_\ell \to C$ with $\ell = 1$. The fibre f and base C of Σ_ℓ are both isomorphic to $\mathbb{P}^1$. As explained in more detail in Section 5.1, the BPS invariants of Σ_ℓ with polarization J chosen sufficiently close to f (a so-called "suitable" polarization, see Definition 5.1) vanish for sheaves with first Chern class c_1 and rank r such that $c_1 \cdot f \neq 0 \mod r$.

Wall-crossing then allowed to compute the BPS invariants for other choices of J. The BPS invariants of $\mathbb{P}^2$ were obtained from those of $\tilde{\mathbb{P}}^2$ by application of the blow-up formula [35, 12, 23], which is a simple relation between the generating functions of the invariants for $\mathbb{P}^2$ and $\tilde{\mathbb{P}}^2$. However, its original form is only valid for $\gcd(c_1 \cdot \phi^* H, r) = 1$ and $J = \phi^* H$, with H the hyperplane class of $\mathbb{P}^2$.

The present paper describes how to deal with the cases when c_1 and r do not satisfy the constraints for vanishing of the BPS invariant or the blow-up formula. The formal theory of invariants of moduli spaces (or stacks) of semi-stable sheaves is developed by Kontsevich and Soibelman [21] and Joyce [15, 16, 17]. We will in particular use the notion of virtual Poincaré functions for moduli stacks, which are a generalization of Poincaré polynomials of manifolds. The virtual Poincaré function of a moduli stack is (conjecturally) related to the BPS invariant by (3.5). The BPS invariant is most natural from physics and leads to generating functions with modular properties.

Two novel ingredients of this paper are:

(1) Eq. (4.2) which provides for any rank $r \geq 1$ the generating function of virtual Poincaré functions of the moduli stack of sheaves on a Hirzebruch surface Σ_ℓ whose

restriction to the fibre f is semi-stable. Eq. (4.9) gives the generalization to virtual Hodge functions for more general ruled surfaces $\Sigma_{g,\ell}$.

(2) *Extended* Harder-Narasimhan filtrations $0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell = F$, whose definition (Def. 5.3) differs from the usual definition (5.2) of HN filtrations by allowing quotients $E_i = F_i/F_{i-1}$ with equal (Gieseker) stability $p_J(E_i, n) \succeq p_J(E_{i+1}, n)$. These filtrations in combination with the associated invariants (5.8) are particularly useful to compute generating functions of BPS invariants starting from Conjecture 4.2 and their changes across walls of marginal stability.

To obtain the BPS invariants for a suitable polarization, one subtracts from Eq. (4.2) generating functions corresponding to extended HN-filtrations given by (5.8), analogous to the seminal papers about vector bundles on curves [13, 1]. Naturally, these techniques are also applicable to compute invariants of semi-stable invariants for other mathematical objects like vector bundles on curves and quivers. Also a solution to this recursive procedure is given analogous to Ref. [37]. Then repeated application of the formula for filtrations (which is equivalent with the wall-crossing formulas [21, 17]) gives the BPS invariants for other choices of the polarization.

Finally, the blow-up formula provides the invariants on $\mathbb{P}^2$. The earlier mentioned condition $\gcd(c_1 \cdot \phi^* H, r) = 1$ is a consequence of the fact that the blow-up formula is applicable for the Poincaré functions $\mathcal{I}^\mu(\Gamma, w; J)$ with respect to μ-stability instead of the more refined Gieseker stability. However with the invariant for filtrations (5.8), it is straightforward to transform the BPS invariants $\Omega(\Gamma, w; J)$ to $\mathcal{I}^\mu(\Gamma, w; J)$ for μ-stability. The rational factors in Eq. (5.8) appear naturally in the relation between the generating functions of these invariants.

The paper illustrates in detail the above steps for sheaves with rank 2 and 3, and shows their agreement with various consistency conditions, e.g. the blow-up formula, integrality and $w \leftrightarrow w^{-1}$ symmetry of the Poincaré polynomial.

The outline of the paper is as follows. Section 2 reviews some necessary properties of sheaves on surfaces including stability conditions. Section 3 discusses the invariants and generating functions. Section 4 presents the generating function (4.2) of the virtual Poincaré functions of the stack of sheaves whose restriction to the fibre is semi-stable. Then we continue with the computation of the invariants of Σ_ℓ for any choice of polarization in Section 5. Finally Section 6 presents the blow-up formula (6.1) and computes the generating function for sheaves on $\mathbb{P}^2$ with $(r, c_1) = (3, 0)$.

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2. Sheaves on surfaces

We consider sheaves on a smooth projective surface S. The Chern character of the sheaf F is given by $\mathrm{ch}(F) = r(F) + c_1(F) + \frac{1}{2}c_1(F)^2 - c_2(F)$ in terms of the rank $r(F)$ and its Chern classes $c_1(F)$ and $c_2(F)$. The vector $\Gamma(F)$ parametrizes in the following the topological classes of the sheaf $\Gamma(F) := (\, r(F), \mathrm{ch}_1(F), \mathrm{ch}_2(F)\,)$. Other frequently occuring quantities are the determinant $\Delta(F) = \frac{1}{r(F)}(c_2(F) - \frac{r(F)-1}{2r(F)}c_1(F)^2)$, and $\mu(F) = c_1(F)/r(F) \in H^2(S, \mathbb{Q})$.

Given a filtration $0 \subset F_1 \subset \cdots \subset F_\ell = F$, let $E_i = F_i/F_{i-1}$ and $\Gamma_i = \Gamma(E_i)$. The discriminant of F is given in terms of the subobjects and quotients by:

$$\Delta(\,\Gamma(F)\,) = \sum_{i=1}^{\ell} \frac{r(E_i)}{r(F)}\Delta(E_i) - \frac{1}{2r(F)} \sum_{i=2}^{\ell} \frac{r(F_i)\, r(F_{i-1})}{r(E_i)} \left(\mu(F_i) - \mu(F_{i-1})\right)^2. \tag{2.1}$$

We are interested in the moduli space (or moduli stack) of semi-stable sheaves with respect to Gieseker stability, but also the coarser μ-stability appears in order to apply the blow-up formula. To define these two stability conditions, let $C(S) \subset H^2(S, \mathbb{R})$ be the ample cone of S, and the (reduced) Hilbert polynomial $p_J(F, n) = \chi(F \otimes J^n)/r(F)$. For a surface S, we have [8]:

$$p_J(F,n) = J^2n^2/2 + \left(\frac{c_1(F)\cdot J}{r(F)} - \frac{K_S \cdot J}{2}\right) n + \frac{1}{r(F)}\left(\frac{c_1(F)^2 - K_S\cdot c_1(F)}{2} - c_2(F)\right) + \chi(\mathcal{O}_S). \tag{2.2}$$

Note that this function can be obtained from the physical central charge as in [6, 24]. In the large volume limit, the stability condition asymptotes to the lexicographic ordering of polynomials based on their coefficients. This ordering is denoted by $\prec$. Then,

Definition 2.1. *A torsion free sheaf F is Gieseker stable (respectively semi-stable) if for every subsheaf $F' \subsetneq F$, $p_J(F', n) \prec p_J(F, n)$ (respectively $p_J(F', n) \preceq p_J(F, n)$).*

and

Definition 2.2. *Given a choice $J \in C(S)$, a torsion free sheaf F is called μ-stable if for every subsheaf $F' \subset F$, $\mu(F') \cdot J < \mu(F) \cdot J$, and μ-semi-stable if for every subsheaf F', $\mu(F') \cdot J \leq \mu(F) \cdot J$.*

Thus μ-stability is a coarser stability condition then Gieseker stability, although the walls of marginal stability for both stability conditions are the same. A wall of marginal stability $W(F', F) \subset H^2(S, \mathbb{R})$ is the codimension 1 subspace of $C(S)$, such that $(\mu(F') - \mu(F)) \cdot J = 0$, but $(\mu(F') - \mu(F)) \cdot J \neq 0$ away from $W(F', F)$. The invariants based on Gieseker stability exhibit better integrality and polynomial properties then the ones based on μ-stability. On the other hand, operations like restriction to a curve and blowing-up a point of S are most natural for μ-semi-stable sheaves.

The moduli space $\mathcal{M}_J(\Gamma)$ of Gieseker stable sheaves on S (with respect to the ample class J) whose rank and Chern classes are determined by Γ has expected dimension:

$$d_{\mathrm{exp}}(\Gamma) = \dim_{\mathbb{C}}(\mathrm{Ext}^1(F, F)) - \dim_{\mathbb{C}}(\mathrm{Ext}^2(F, F)) = 2r^2\Delta - r^2\chi(\mathcal{O}_S) + 1. \tag{2.3}$$

When $\mathrm{Ext}^2(F, F) = 0$ the moduli space is smooth and of the expected dimension. Vanishing of $\mathrm{Ext}^2(F, F)$ for semi-stable sheaves on surfaces can be proven if the polarization satisfies $J \cdot K_S < 0$. More generally, we have

Proposition 2.3. *Let $J \in C(S)$ such that $J \cdot K_S < 0$ and let F and G be Gieseker semi-stable sheaves with respect to polarization J such that $p_J(F, n) \preceq p_J(G, n)$. Then:*

$$\mathrm{Ext}^2(F, G) = 0.$$

Proof. Due to Serre duality $\mathrm{Ext}^2(F, G) = \mathrm{Hom}(G, F \otimes K_S)^\vee$. Assume contrary to the proposition that $\mathrm{Ext}^2(F, G) \neq 0$, such that a non-vanishing morphism $\psi : G \to F \otimes K_S$ exists. Then $F \otimes K_S$ is a quotient of G, and semi-stability of G implies $p_J(F \otimes K_S, n) \succeq p_J(G, n)$. Now we find a contradiction, since the assumption $J \cdot K_S < 0$ implies $p_J(F \otimes K_S, n) \prec p_J(F, n) \preceq p_J(G, n)$. Therefore a non-vanishing ψ cannot exist and the proposition follows. □

Dimension estimates for (coarse) moduli spaces of semi-stable sheaves are more subtle due to endomorphisms. We will find that BPS invariants computed in Sections 5 and 6 are in agreement with the expected dimension (if non-vanishing).

Twisting a sheaf E by a line bundle $\mathcal{L}$ is an isomorphism of moduli spaces. The Chern classes of the twisted sheaf $E' = E \otimes \mathcal{L}$ are:

$$r(E') = r(E), \quad c_1(E') = c_1(E) + r(E)c_1(\mathcal{L}),$$
$$c_2(E') = c_2(E) + (r(E)-1)c_1(\mathcal{L})c_1(E) + c_1(\mathcal{L})^2\frac{r(E)(r(E)-1)}{2}.$$

The discriminant remains invariant: $\Delta(E') = \Delta(E)$. This shows that it suffices to compute the generating functions for $c_1(E) \in H^2(S, \mathbb{Z}/r\mathbb{Z})$.

Determination of generating functions of BPS invariants for $r \geq 2$ is an open problem in general. To make progress, we specialize in the following to the set of smooth ruled surfaces. A ruled surface is a surface $\Sigma_{g,\ell}$ together with a surjective morphism $\pi : \Sigma_{g,\ell} \to C_g$ to a curve C_g with genus g, such that the fibre over each point of C_g is a smooth irreducible rational curve and such that π has a section. Let f be the fibre of π, then $H_2(\Sigma_{g,\ell}, \mathbb{Z}) = \mathbb{Z}C_g \oplus \mathbb{Z}f$, with intersection numbers $C_g^2 = -\ell$, $f^2 = 0$ and $C_g \cdot f = 1$. The canonical class is $K_{\Sigma_{g,\ell}} = -2C_g + (2g-2-\ell)f$. The holomorphic Euler characteristic $\chi(\mathcal{O}_{\Sigma_{g,\ell}})$ is $1-g$. An ample divisor $J \in C(\Sigma_{g,\ell})$ is parametrized by $J_{m,n} = m(C_g + \ell f) + nf$ with $m, n > 0$. The condition $J \cdot K_S < 0$ translates to $m(2g-2-\ell) < 2n$.

Most of this article will further specialize to the Hirzebruch surfaces $\Sigma_{0,\ell} = \Sigma_\ell$. For these surfaces $J \cdot K_S < 0$ is satisfied for all $J \in C(\Sigma_\ell)$. The surface Σ_1 playes a special role since besides being a ruled surface, Σ_1 is also the blow-up $\phi : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2$ of the projective plane $\mathbb{P}^2$. The exceptional divisor of ϕ is $C_0 = C$, and the pullback of the hyperplane class H of $\mathbb{P}^2$ is given by $\phi^* H = C + f$. Due to the simplicity of $\mathbb{P}^2$, it is of intrinsic interest to determine the generating functions of its BPS invariants.

3. BPS invariants and generating functions

This section defines the generating functions of the BPS invariants and discusses some of its properties. Physically, the BPS invariant arises by considering topologically twisted $\mathcal{N} = 4$ Yang-Mills on the surface S [31]. The path integral of this theory localizes on the BPS solutions, including the instantons, due to the topologically twisted supersymmetry [31]. The BPS invariant is given by a weighted sum over the BPS Hilbert space $\mathcal{H}(\Gamma, J)$, and based on the path integral one can show that the (numerical) BPS invariant corresponds to the Euler number of the BPS moduli space.

Alternatively one can consider the $\mathcal{N} = 2$ supersymmetric theory in $\mathbb{R}^{3,1}$ obtained from the compactification of IIA theory on a non-compact Calabi-Yau $\mathcal{O}(-K_S) \to S$. The $\mathcal{N} = 2$

theory with gauge group $SU(2)$ and without hypermultiplets can be engineered by any of the Hirzebruch surfaces Σ_ℓ [18]. Sheaves supported on Σ_ℓ correspond to magnetic monopoles and dyons in $\mathcal{N}=2$ gauge theory. In this theory, the BPS invariant can be refined with an additional parameter w [9]:

$$\Omega(\Gamma, w; J) = \frac{\mathrm{Tr}_{\mathcal{H}(\Gamma,J)}\, 2\hat{J}_3(-1)^{2\hat{J}_3}(-w)^{2\hat{I}_3+2\hat{J}_3}}{(w-w^{-1})^2}, \tag{3.1}$$

with $\hat{J}_3$ a generator of the $SU(2) \cong \mathrm{Spin}(3)$ group arising from rotations in $\mathbb{R}^{3,1}$, and $\hat{I}_3$ is a generator of the $SU(2)_R$ R-symmetry group. BPS representations have the form $\left[(\frac{1}{2},0)\oplus(0,\frac{1}{2})\right]\otimes\omega$ with $\omega=(j,j')$ a vacuum representation of $\mathrm{Spin}(3)\oplus SU(2)_R$ with spins j and j'. One factor of $w-w^{-1}$ in the denominator will vanish due to the factor $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ (the half-hypermultiplet) present for every BPS state [9]. Since $\Omega(\Gamma,w;J)$ is thus essentially an $SU(2)$ character, this shows that $\Omega(\Gamma,w;J)$ is a polynomial divided by $w-w^{-1}$; the polynomial has integer coefficients and is invariant under $w\leftrightarrow w^{-1}$. The positivity conjectures of Ref. [9] assert furthermore that the coefficients are positive. We choose to divide by the factor $w-w^{-1}$ in order to have nice modular properties of the generating functions. See for example Eq. (3.8).

The $\mathcal{N}=2$ picture shows that the refined BPS invariant provides more information than the Euler number of the moduli space. The w-expansion is expected to give the χ_y-genus of the BPS moduli space [4]. To make this more precise, we let $\mathcal{M}_J(\Gamma)$ be the suitably compactified moduli space of semi-stable sheaves on S with topological classes Γ and for polarization $J\in C(S)$, i.e. the Gieseker-Maruyama compactification. If we assume that $J\cdot K_S<0$ and that semi-stable is equivalent to stable, the moduli space is smooth and the BPS invariant corresponds mathematically to [4]:

$$\Omega(\Gamma, w; J) := \frac{w^{-\dim_{\mathbb{C}}\mathcal{M}_J(\Gamma)}}{w-w^{-1}}\,\chi_{w^2}(\mathcal{M}_J(\Gamma)), \qquad w^2\neq 1, \tag{3.2}$$

with on the right hand side the χ_y-genus, which is defined in terms of the virtual Hodge numbers $h^{p,q}(X)=\dim H^{p,q}(X,\mathbb{Z})$ of the quasi-projective variety X by $\chi_y(X)=\sum_{p,q=0}^{\dim_{\mathbb{C}}(X)}(-1)^{p-q}\,y^p\,h^{p,q}(X)$. Eq. (2.3) provides us with the degree of $\chi_{w^2}(\mathcal{M}_J(\Gamma))$, and since $\mathcal{M}_J(\Gamma)$ is compact, orientable and without boundary $h^{p,q}(X)=h^{\dim_{\mathbb{C}}(X)-p,\dim_{\mathbb{C}}(X)-q}(X)$. For rational surfaces, which include the ruled surfaces with $g=0$, the non-vanishing cohomology of smooth moduli spaces of semi-stable sheaves has Hodge type (p,p) [2, 12]. Therefore, $\chi_{w^2}(X)=P(X,w)=\sum_{i=0}^{2\dim_{\mathbb{C}}(X)} b_i(X)\,w^i$ with $P(X,w)$ the Poincaré polynomial and $b_i(X)=\sum_{p+q=i}h^{p,q}(X)$ the Betti numbers of X.

If semi-stable is not equivalent to stable, $\mathcal{M}_J(\Gamma)$ contains singularities due to non-trivial automorphisms of the sheaves. The formal mathematical framework for the integer BPS invariants or motivic Donaldson-Thomas invariants is developed by Kontsevich and Soibelman [21]. For our purposes it is useful to introduce also two other invariants, $\bar\Omega(\Gamma, w; J)$ and $\mathcal{I}(\Gamma, w; J)$. These invariants are defined using the notion of moduli stack $\mathfrak{M}_J(\Gamma)$ which properly deals with the mentioned singularities in the moduli space of semi-stable sheaves by keeping track of the automorphism groups of the semi-stable sheaves.

The invariant $\mathcal{I}(\Gamma, w; J)$ is an example of a motivic invariant. In general an invariant of a quasi-projective variety X is called 'motivic' if $\Upsilon(X)$ satisfies:

- If $Y \subseteq X$ is a closed subset then $\Upsilon(X) = \Upsilon(X \setminus Y) + \Upsilon(Y)$,
- If X and Y are quasi-projective varieties $\Upsilon(X \times Y) = \Upsilon(X)\,\Upsilon(Y)$.

Ref. [16] defines a motivic invariant, the virtual Poincaré function Υ', for Artin stacks, which are stacks whose stabilizer groups are algebraic groups. The virtual Poincaré function $\mathcal{I}(\Gamma, w; J)$ is a rational function in w and a natural generalization of the Poincaré polynomial of smooth projective varieties to stacks. The definition of these invariants for stacks is such that for a quotient stack $[X/G]$ with G an algebraic group, one has $\Upsilon'([X/G]) = \Upsilon(X)/\Upsilon(G)$.

Using the virtual Poincaré function Υ', Definition 6.20 of Ref. [15] defines the virtual Poincaré function $\mathcal{I}(\Gamma, w; J)$ (in Ref. [15] denoted by $I^{\alpha}_{\rm ss}(\tau)^{\Lambda}$) for the moduli stacks of semi-stable sheaves on surfaces with $\mathrm{Ext}^2(X, Y) = 0$ for $p_J(X, n) \prec p_J(Y, n)$. Definition 6.22 of Ref. [15] also defines a second invariant $\bar\Omega(\Gamma, w; J)$ (denoted by $\bar J^{\alpha}(\gamma)^{\Lambda}$ in Ref. [15]). These appear in fact rather natural from the physical perspective [27, 19]. See also [29] for related discussions of invariants.

The invariants $\bar\Omega(\Gamma, w; J)$ are the rational multi-cover invariants of $\Omega(\Gamma, w; J)$:

$$\bar\Omega(\Gamma, w; J) \;:=\; \sum_{m|\Gamma} \frac{\Omega(\Gamma/m, -(-w)^m; J)}{m}. \tag{3.3}$$

They can be expressed in terms of $\mathcal{I}(\Gamma, w; J)$ and vice versa (Theorem 6.8 in [15]):

$$\bar\Omega(\Gamma_i, w; J) := \sum_{\substack{\Gamma_1+\dots+\Gamma_\ell=\Gamma \\ p_J(\Gamma_i,n)=p_J(\Gamma,n) \text{ for } i=1,\dots,\ell}} \frac{(-1)^{\ell+1}}{\ell} \prod_{i=1}^{\ell} \mathcal{I}(\Gamma_i, w; J). \tag{3.4}$$

with inverse relation:

$$\mathcal{I}(\Gamma, w; J) = \sum_{\substack{\Gamma_1+\dots+\Gamma_\ell=\Gamma \\ p_J(\Gamma_i,n)=p_J(\Gamma,n) \text{ for } i=1,\dots,\ell}} \frac{1}{\ell!} \prod_{i=1}^{\ell} \bar\Omega(\Gamma_i, w; J), \tag{3.5}$$

Note that $\mathcal{I}(\Gamma, w^{-1}; J) \neq -\mathcal{I}(\Gamma, w; J)$ and that $\mathcal{I}(\Gamma, w; J)$ in general has higher order poles in w compared to $\Omega(\Gamma, w; J)$.

It is an interesting question what geometric information the integer invariants $\Omega(\Gamma, w; J)$ carry if $m|\Gamma$ with $m > 1$. For $r = m = 2$, Remark 4.6 of Ref. [34] argues that $\Omega(\Gamma, w; J)$ computes the Betti numbers of rational intersection cohomology of the singular moduli space $\mathcal{M}_J(\Gamma)$. The generating function in Remark 4.6 of Ref. [34] is very closely related to the one obtained for moduli spaces of semi-stable vector bundles over Riemann surfaces in (the Corrigendum to) Ref. [20]. Intersection cohomology is a cohomology theory for manifolds with singularities which satisfies Poincaré duality if the manifolds are complex and compact. It is therefore natural to expect that the BPS invariant (3.2) for $r \geq 3$ also provides Betti numbers of intersection cohomology groups. This issue is left for further research.

The seminal papers [10, 33, 34] compute moduli space and stack invariants by explicitly counting sheaves on the surface S defined over a finite field $\mathbb{F}_s$ with s elements. The Poincaré function $\mathcal{I}(\Gamma, s^{\frac{1}{2}}; J)$ is upto an overall monomial computed by:

$$\sum_{E \in M_J(\Gamma, \mathbb{F}_s)} \frac{1}{\#\mathrm{Aut}(E)}, \tag{3.6}$$

where $M_J(\Gamma, \mathbb{F}_s)$ is the set of semi-stable sheaves with characteristic classes Γ. The Weil conjectures imply that the expansion coefficients in s are the Betti numbers of the moduli spaces. The parameter s is related to the w in this article by $s = w^2$. Eq. (3.6) shows that poles of $\mathcal{I}(\Gamma, w; J)$ in w appear when the sheaves have non-trivial automorphism groups. If semi-stable is equivalent to stable $\mathcal{I}(\Gamma, w; J) = \Omega(\Gamma, w; J)$; the factor $(w - w^{-1})^{-1}$ in Eq. (3.2) is due to the automorphisms which are multiplication by $\mathbb{C}^*$. The automorphism group of semi-stable and unstable bundles or sheaves is in general $GL(n)$, whose number of elements over $\mathbb{F}_s$ is $(1-s)(1-s^2)\dots(1-s^n)$ and thus lead to higher order poles.

We continue now by defining the generating function $h_{r,c_1}(z, \tau; S, J)$ of $\bar{\Omega}(\Gamma, w; J)$:

$$h_{r,c_1}(z, \tau; S, J) = \sum_{c_2} \bar{\Omega}(\Gamma, w; J)\, q^{r\Delta(\Gamma) - \frac{r\chi(S)}{24}}. \tag{3.7}$$

where $q := e^{2\pi i \tau}$, with $\tau \in \mathcal{H}$ and $w := e^{2\pi i z}$ with $z \in \mathbb{C}$. Since twisting by a line bundle (2.4) is an isomorphism of moduli spaces, it suffices to compute $h_{r,c_1}(z, \tau; S, J)$ for $c_1 \in H_2(S, \mathbb{Z}/r\mathbb{Z})$. The expansion parameter t for c_2 in Refs. [10, 33, 34] is related to q by $q = s^r t$.

The generating function $h_{1,c_1}(z,\tau;S)$ depends only on $b_2(S)$ for S a smooth projective surface with $b_1(S)=b_3(S)=0$ [10]:

$$h_{1,c_1}(z,\tau;S)=\frac{i}{\theta_1(2z,\tau)\,\eta(\tau)^{b_2(S)-1}}, \tag{3.8}$$

where the Dedekind eta function $\eta(\tau)$ and Jacobi theta function $\theta_1(z,\tau)$ are defined by:

$$\begin{aligned}\eta(\tau) &:= q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n),\\ \theta_1(z,\tau) &:= iq^{\frac{1}{8}}(w^{\frac{1}{2}}-w^{-\frac{1}{2}})\prod_{n\geq 1}(1-q^n)(1-wq^n)(1-w^{-1}q^n).\end{aligned}$$

The dependence on J is omitted in Eq. (3.8), since all rank 1 torsion free sheaves are stable throughout $C(S)$. Similarly, J is omitted in the following from $h_{r,c_1}(z,\tau;\mathbb{P}^2,J)$, since $b_2(\mathbb{P}^2)=1$ and therefore the BPS invariants do not vary as function of J. For clarity of exposition, Σ_ℓ is omitted from the arguments of $h_{r,c_1}(z,\tau;\Sigma_\ell,J)$.

We will be mainly concerned with the invariants $\bar{\Omega}(\Gamma,w;J)$ since the generating functions are defined in terms of these invariants. However, some formulas are most naturally phrased in terms of $\mathcal{I}(\Gamma,w;J)$. For example, the product formula of Conjecture 4.1 is a generating function for $\mathcal{I}(\Gamma,w;f)$ and the blow-up formula in Section 6 is phrased in terms of $\mathcal{I}^\mu(\Gamma,w;J)$, which are invariants with respect to μ-stability instead of Gieseker stability.

4. Restriction to the fibre of Hirzebruch surfaces

This subsection deals with the set $M_f(\Gamma)$ of sheaves whose restriction to the (generic) fibre f of $\pi:\Sigma_\ell\to C$ is semi-stable. Inspired by the existing results for $r=1$ and 2 [10, 34] and moduli stack invariants for vector bundles over Riemann surfaces [13, 1], a generating function for $r\geq 1$ is proposed enumerating virtual Poincaré functions $\mathcal{I}(\Gamma,w;f)$ of moduli stacks $\mathfrak{M}_f(\Gamma)$ of sheaves whose restriction to the fibre is semi-stable. We do not present a derivation of this generating function based on $\mathfrak{M}_f(\Gamma)$ for $r\geq 3$, nor an analysis of the properties of $\mathfrak{M}_f(\Gamma)$. Section 5 computes the BPS invariants starting from these generating functions, and shows that they pass various non-trivial consistency checks implied by the blow-up and wall-crossing formulas.

We define the generating function $H_{r,c_1}(z,\tau;f)$ of $\mathcal{I}(\Gamma,w;f)$ by:

$$H_{r,c_1}(z,\tau;f):=\sum_{c_2}\mathcal{I}(\Gamma,w;f)\,q^{r\Delta(\Gamma)-\frac{\chi(S)}{24}}. \tag{4.1}$$

The following conjecture gives $H_{r,c_1}(z,\tau;f)$ for any $r\geq 1$ and $c_1\in H_2(\Sigma_\ell,\mathbb{Z})$:

Conjecture 4.1. *The function $H_{r,c_1}(z,\tau;f)$ is given by:*

(4.2)

$$H_{r,c_1}(z,\tau;f)=\begin{cases}\frac{i\,(-1)^{r-1}\,\eta(\tau)^{2r-3}}{\theta_1(2z,\tau)^2\,\theta_1(4z,\tau)^2\ldots\theta_1((2r-2)z,\tau)^2\,\theta_1(2rz,\tau)}, & \text{if } c_1\cdot f=0 \mod r, \quad r\geq 1,\\ 0, & \text{if } c_1\cdot f\neq 0 \mod r, \quad r>1.\end{cases}$$

The above expressions for $H_{r,c_1}(z,\tau;f)$ are not conjectural for all (r,c_1). Vanishing of $H_{r,c_1}(z,\tau;f)$ for $c_1\cdot f\neq 0 \mod r$ is well known. See for example Section 5.3 of [14]. The vanishing is a consequence of the fact that all bundles F on $\mathbb{P}^1$ are isomorphic to a sum of line bundles $F\cong\mathcal{O}(d_1)\oplus\mathcal{O}(d_2)\ldots\mathcal{O}(d_r)$. Therefore, a bundle F on $\mathbb{P}^1$ can only be semi-stable[1] if its degree d is equal to $0 \mod r$ such that the degrees of the line bundles are $d_i=d/r$. The degree $d(E_{|f})$ of the restriction of a sheaf E on Σ_ℓ to f is equal to $c_1(E)\cdot f$. Therefore, the only cases for which $H_{r,c_1}(z,\tau;f)$ does not vanish is for $c_1\cdot f=0 \mod r$.

For $r=1$, Eq. (4.2) reduces to Eq. (3.8). Ref. [34] proved the conjecture for $(r,c_1)=(2,f)$, which is now briefly recalled. Ref. [34] considers the ruled surface $\tilde{\mathbb{P}}^2$ over a finite field $\mathbb{F}_s$, and utilizes the fact that any vector bundle in F can be obtained from $\pi^*\pi_*F$, which is a vector bundle on $\tilde{\mathbb{P}}^2$ supported on C, by successive elementary transformations.

An elementary transformation is defined by [14]:

Definition 4.2. *Let D be an effective divisor on the surface S. If F and G are vector bundles on S and D respectively, then a vector bundle F' on S is obtained by an elementary transformation of F along G if there exists an exact sequence:*

$$0\to F'\to F\to i_*G\to 0, \tag{4.3}$$

where i denotes the embedding $D\subset S$.

This shows that the contribution to $h_{2,c_1}(z,\tau;J)$ from $M_f(\Gamma)$ is the product of the total set of vector bundles on C, multiplied by the number of elementary transformations. The total set of vector bundles with $r=2$ on C is enumerated by [13]:

$$\frac{s^{-3}}{1-s}\,\zeta_C(2) \tag{4.4}$$

where $\zeta_C(n)$ is the zeta function of the Riemann surface C_0. One has for general genus g:

$$\zeta_{C_g}(n)=\frac{\prod_{j=1}^{2g}(1-\omega_j s^{-n})}{(1-s^{-n})(1-s^{1-n})}. \tag{4.5}$$

[1]Recall that a vector bundle F of rank r and degree d on a curve C is stable (respectively semi-stable) if for every subbundle $F'\subsetneq F$ (with rank r' and degree d') $d'/r'<d/r$ (respectively $d'/r'\leq d/r$).

Multiplication of (4.4) by the factor due to elementary transformations gives [34]:

$$\sum_{c_2} \sum_{E\in M_f(2,mf,c_2)} \frac{t^{c_2}}{\#\mathrm{Aut}(E)} = \frac{s^{-3}}{1-s}\zeta_C(2) \prod_{a\geq 1} Z_s(S, s^{2a-2}t^a) Z_s(S, s^{2a}t^a), \tag{4.6}$$

with $Z_s(S,t)$ the zeta function of the surface S:

$$Z_s(S,t) = \frac{1}{(1-t)(1-st)^{b_2(S)}(1-s^2t)}. \tag{4.7}$$

The parameter substitutions $q = s^r t$ and $w^2 = s$ give then Eq. (4.2) (upto an overal monomial in w and q).

This derivation for $r = 2$ indicates that $H_{r,c_1}(z,\tau;f)$ is closely related to that of the virtual Poincaré function of the stack of vector bundles on a Riemann surface C_g with genus g [13, 1]:

$$H_r(z;C_g) := -w^{r^2(1-g)} \frac{(1+w^{2r-1})^{2g}}{1-w^{2r}} \prod_{j=1}^{r-1} \frac{(1+w^{2j-1})^{2g}}{(1-w^{2j})^2}. \tag{4.8}$$

The first term in the q-expansion of Eq. (4.2) starts with Eq. (4.8) for $g = 0$. One could thus understand $H_{r,c_1}(z,\tau;f)$ as an extension of $H_r(z;C_0)$ to a modular infinite product. It is conceivable that Conjecture 4.1 for $r > 2$ can be proven in a similar manner as for $r = 2$. The following sections show that at least for $r = 3, 4$, it is consistent with various other results. Moreover, it continues to hold for the other Hirzebruch surfaces with $\ell \geq 0$.

As an aside we mention the generalization of the conjecture to ruled surfaces $\Sigma_{g,\ell}$ over a Riemann surface C_g with $g > 0$. These surfaces are not rational and the moduli spaces of semi-stable sheaves for these surfaces also have cohomology $H^{p,q}(\mathcal{M}_J(\Gamma),\mathbb{Z})$ for $p \neq q$. In order to capture this more refined information we recall the refinement of Eq. (4.8) to the virtual Hodge function [7]:

$$H_r(u,v;C_g) := -\frac{(xy)^{r^2(1-g)/2}}{1-x^r y^r} \frac{\prod_{j=1}^r (1+x^j y^{j-1})^g (1+x^{j-1}y^j)^g}{\prod_{k=1}^{r-1}(1-x^k y^k)^2}. \tag{4.9}$$

with $x := e^{2\pi i u}$ and $y := e^{2\pi i v}$. The structure of this function directly suggests the following generalization of Conjecture 4.1 for the generating function $H_{r,c_1}(u,v,\tau;f,\Sigma_{g,\ell})$ of virtual Hodge functions $\mathcal{I}(\Gamma_i,x,y;f)$ of the moduli stack $\mathfrak{M}_f(\Gamma;\Sigma_{g,\ell})$:

Conjecture 4.3. *The function $H_{r,c_1}(u,v,\tau;f,\Sigma_{g,\ell})$ is given by:*

$$\begin{cases} \frac{i\,(-1)^{r-1}\,\eta(\tau)^{2r(1-g)-3}}{\theta_1(r(u+v),\tau)} \frac{\prod_{j=1}^r \theta_1(ju+(j-1)v+\frac12,\tau)^g\,\theta_1((j-1)u+jv+\frac12,\tau)^g}{\prod_{k=1}^{r-1}\theta_1(k(u+v),\tau)^2}, & \text{if } c_1\cdot f = 0 \mod r, \quad r\geq 1, \\ 0, & \text{if } c_1\cdot f \neq 0 \mod r, \quad r > 1. \end{cases} \tag{4.10}$$

5. BPS invariants of Hirzebruch surfaces

5.1. **BPS invariants for a suitable polarization.** This subsection computes for $c_1 \cdot f = 0 \mod r$ the BPS invariants of Σ_ℓ for a polarization $J \in C(\Sigma_\ell)$ sufficiently close to $J_{0,1} = f$. The BPS invariants are for this choice of J independent of ℓ. "Sufficiently close" depends on the topological classes of the sheaf. Generalizing Def. 5.3.1 of [14] to general $r \geq 1$, we define a Γ-suitable polarization by:

Definition 5.1. *A polarization J is called Γ-suitable if and only if:*

- *J does not lie on a wall for $\Gamma = (r, \mathrm{ch}_1, \mathrm{ch}_2)$ and,*
- *for any J-semi-stable subsheaf $F' \subset F$ with $\Gamma(F) = \Gamma$, $(\mu(F') - \mu(F)) \cdot f = 0$ or $(\mu(F') - \mu(F)) \cdot f$ and $(\mu(F') - \mu(F)) \cdot J$ have the same sign.*

We will keep the dependence on the Chern classes implicit in the following and denote a suitable polarization by $J_{\varepsilon,1}$ with ε positive but sufficienty small. From the definition follows that if $J_{\varepsilon,1}$ is a $\Gamma(F)$-suitable polarization, and $F_{|f}$ is unstable, then F is μ-unstable. Thus we need to subtract from $M_f(\Gamma)$, i.e. the set of sheaves with topological classes Γ whose restriction to the fibre f is semi-stable, the subset of $M_f(\Gamma)$ which is Gieseker unstable for $J_{\varepsilon,1}$. We continue by explaining this for $r = 2$. Then the general formula is proposed for the invariant enumerating extended HN-filtrations, which is consequently applied to $r = 3$.

A crucial tool to obtain the invariants enumerating semi-stable sheaves are Harder-Narasimhan filtrations [13], which can be defined for either Gieseker or μ-stability. To define these filtrations, let φ denote either Gieseker, $\varphi(F) = p_J(F, n)$, or μ-stability, $\varphi(F) = \mu(F) \cdot J$. Then:

Definition 5.2. *A Harder-Narasimhan filtration (HN-filtration) with respect to the stability condition φ is a filtration $0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell = F$ of the sheaf F such that the quotients $E_i = F_i/F_{i-1}$ are semi-stable with respect to φ and satisfy $\varphi(E_i) > \varphi(E_{i+1})$ for all i.*

Since μ-stability is coarser then Gieseker stability, the length $\ell_{\mathrm{G}}(F)$ of the HN-filtration with respect to Gieseker stability is in general larger than the length $\ell_\mu(F)$ of its HN-filtration with respect to μ-stability.

Using the additive and multiplicative properties of motivic invariants discussed below (3.2), one can determine the BPS invariants for a suitable polarization. The Poincaré function of the stack of HN-filtrations with respect to Gieseker stability and prescribed $\Gamma_i = \Gamma(E_i)$ is

[35]:

$$w^{-\sum_{i<j} r_ir_j(\mu_j-\mu_i)\cdot K_S} \prod_{i=1}^{\ell} \mathcal{I}(\Gamma_i, w; J), \tag{5.1}$$

where $r_ir_j(\mu_j - \mu_i) \cdot K_S$ is the Euler form for semi-stable sheaves on the projective surface S. One could define a similar function for the stack of filtrations with respect to μ-stability. For the generalization to Hodge numbers, one replaces w^2 by xy in $w^{-\sum_{i<j} r_ir_j(\mu_j-\mu_i)\cdot K_S}$ and $\mathcal{I}(\Gamma_i, w; J)$ by $\mathcal{I}(\Gamma_i, x, y; J)$.

For $(r, c_1) = (2, f)$, the only HN-filtrations with respect to $J_{\varepsilon,1}$ have length $\ell_{\rm G} = 2$. Denoting $c_1(E_2) = bC - af$, and thus $c_1(E_1) = -bC + (a + 1)f$, one easily verifies that the HN-filtrations correspond to $a \geq 0$ and $b = 0$. Since $b = 0$ the dependence of K_S in Eq. (5.1) does not lead to a dependence on ℓ. Using that Eq. (3.8) is also the generating function of $\mathcal{I}(\Gamma, w; J)$ for $r = 1$, Eq. (5.1) becomes:

$$\sum_{a\geq 0} w^{-2(2a+1)}\, h_{1,0}(z,\tau)^2 = -\frac{w^2}{1-w^4}\, h_{1,0}(z,\tau)^2, \tag{5.2}$$

where we assumed $|w| > 1$. Subtracting this from Eq. (4.2) for $r = 2$ gives:

$$h_{2,f}(z,\tau; J_{\varepsilon,1}) = \frac{-1}{\theta_1(2z,\tau)^2\,\eta(\tau)^2}\left(\frac{i\,\eta(\tau)^3}{\theta_1(4z,\tau)} + \frac{w^2}{1-w^4}\right), \tag{5.3}$$

which is easily verified to enumerate invariants $\bar\Omega(\Gamma; J_{\varepsilon,1})$ satisfying the expected properties mentioned below Eq. (3.2).

For $(r, c_1) = (2, 0)$, the HN-filtrations with respect to $J_{\varepsilon,1}$ and $\ell_{\rm G} = 2$ split naturally in two subsets: the first set has length $\ell_\mu = 2$ with respect to μ-stability, and the second set has $\ell_\mu = 1$. Similarly to (5.2), the first set gives rise to:

$$-\frac{1}{1-w^4}\, h_{1,0}(z,\tau)^2, \tag{5.4}$$

and the second set to:

$$\frac{1}{2}h_{1,0}(z,\tau)^2 - \frac{1}{2}\sum_{n\geq 0} \Omega((1,0,n),w)^2\, q^{2n}, \tag{5.5}$$

where the second term subtracts from the first the Gieseker semi-stable sheaves which should not be subtracted from $H_2(z,\tau;f)$. Subtraction of Eqs. (5.4) and (5.5) from $H_2(z,\tau;f)$ gives the generating function of $\mathcal{I}(\,(2,0,c_2), w; J)$, which corresponds by Eq. (3.5) to:

$$h_{2,0}(z,\tau; J_{\varepsilon,1}) = \frac{-1}{\theta_1(2z,\tau)^2\,\eta(\tau)^2}\left(\frac{i\,\eta(\tau)^3}{\theta_1(4z,\tau)} + \frac{1}{1-w^4} - \frac{1}{2}\right), \tag{5.6}$$

Again one can verify that the invariants satisfy the expected integrality properties. Remark 4.6 of Ref. [34] determines the Betti numbers of the intersection cohomology of the singular moduli spaces and arrives at the same generating function (5.6).

The Betti numbers for the intersection cohomology of the moduli space of semi-stable vector bundles on Riemann surfaces were earlier computed in Ref. [20]. The above procedure gives these Betti numbers with much less effort. For example, one can easily verify that

$$H_2(z,C_g)+\left(\frac{1}{1-w^4}-\frac{1}{2}\right)H_1(z,C_g)^2, \tag{5.7}$$

with $H_r(z,C_g)$ as in Eq. (4.8), is equivalent with Proposition 5.9 in the Corrigendum to [20].

Since the invariants $\mathcal{I}(\Gamma,w;J)$ are not so compatible with modular generating functions for $r\geq 2$, it is useful to work as much as possible with the invariants $\bar{\Omega}(\Gamma,w;J)$. To this end an extension of the HN-filtration is necessary:

Definition 5.3. *An* extended *Harder-Narasimhan filtration (with respect to Gieseker stability) is a filtration* $0\subset F_1\subset F_2\subset\cdots\subset F_\ell=F$ *whose quotients* $E_i=F_i/F_{i-1}$ *are semi-stable and satisfy* $p_J(E_i,n)\succeq p_J(E_{i+1},n)$.

An example of an extended Harder-Narashimhan filtration can be obtained by considering a Jordan-Hölder filtration of the semi-stable quotients of a standard HN-filtration. Recall that a Jordan-Hölder filtrations is a filtration $0\subset F_1\subset F_2\subset\cdots\subset F_\ell=F$ of a semi-stable bundle F such that the quotients $E_i=F_i/F_{i-1}$ are stable and satisfy $p_J(E_i,n)=p_J(F,n)$. However, not all extended HN-filtrations are obtained this way since Definition 5.3 allows for semi-stable quotients.

From Eq. (3.5) follows that the natural invariant $\bar{\Omega}(\{\Gamma_i\},w;J)$ associated to the stack $\mathfrak{M}_J(\{\Gamma_i\})$ of extended HN-filtrations with prescribed Chern classes $\Gamma_i=\Gamma(E_i)$ is:

$$\bar{\Omega}(\{\Gamma_i\};w,J):=\frac{1}{|\mathrm{Aut}(\{\Gamma_i\};J)|}w^{-\sum_{i<j}r_ir_j(\mu_j-\mu_i)\cdot K_S}\prod_{i=1}^{\ell}\bar{\Omega}(\Gamma_i,w;J). \tag{5.8}$$

The number $|\mathrm{Aut}(\{\Gamma_i\};J)|$ is equal to $\prod_a m_a!$, where m_a is the total number of quotients E_i with equal reduced Hilbert polynomial $p_J(E_a,n)$. Thus only for HN-filtrations $|\mathrm{Aut}(\{\Gamma_i\};J)|=1$.

If the sum over all extended HN-filtrations contains a group $\{E_i\}$ with equal $p_J(E_i,n)$ but unequal Γ_i, the factor $\frac{1}{|\mathrm{Aut}(\{\Gamma_i\};J)|}$ divides out a number of permutations. To avoid this overcounting, one could introduce a further ordering on the vectors Γ_i, which should be obeyed by the set of filtrations to be summed over. Then one would divide by $|\mathrm{Aut}(\{\Gamma_i\})|=$

$\prod_p n_p!$, where n_p is the number of equal vectors Γ_p appearing among the Γ_i, $i = 1, \dots, \ell$. This is the origin of the "Boltzmann statistics" in wall-crossing formulas [27] in the work of Joyce [15].

The functions $h_{r,c_1}(z,\tau;J_{\varepsilon,1})$ with $c_1 \cdot f = 0 \mod r$ are given by the recursive formula

$$h_{r,c_1}(z,\tau;J_{\varepsilon,1}) = H_{r,c_1}(z,\tau;f) - \sum_{\mathrm{ch}_2} \sum_{\substack{\Gamma_1+\dots+\Gamma_\ell=(r,c_1,\mathrm{ch}_2)\\ p_J(\Gamma_i,n)\succeq p_J(\Gamma_{i+1},n),\, \ell>1}} \bar\Omega(\{\Gamma_i\};w,J_{\varepsilon,1})\, q^{r\Delta(\Gamma)-\frac{r\chi(S)}{24}}, \tag{5.9}$$

with $\Delta(\Gamma)$ given in terms of Γ_i by Eq. (2.1) and $H_{r,c_1}(z,\tau;f)$ defined by Eq. (4.2).

We continue by applying Eq. (5.8) to compute $h_{3,c_1}(z,\tau;J_{1,\varepsilon})$, with $c_1 = f$ and 0. One obtains:

Proposition 5.4.

$$\begin{aligned} h_{3,f}(z,\tau;J_{\varepsilon,1}) &= \frac{i\,\eta(\tau)^3}{\theta_1(2z,\tau)^2\,\theta_1(4z,\tau)^2\,\theta_1(6z,\tau)} + \frac{w^2+w^4}{1-w^6}\frac{1}{\theta_1(2z,\tau)^3\,\theta_1(4z,\tau)} \\ &\quad -\frac{w^4}{(1-w^4)^2}\frac{i}{\theta_1(2z,\tau)^3\eta(\tau)^3}, \end{aligned} \tag{5.10}$$

$$\begin{aligned} h_{3,0}(z,\tau;J_{\varepsilon,1}) &= \frac{i\,\eta(\tau)^3}{\theta_1(2z,\tau)^2\,\theta_1(4z,\tau)^2\,\theta_1(6z,\tau)} + \frac{1+w^6}{1-w^6}\,\frac{1}{\theta_1(2z,\tau)^3\,\theta(4z,\tau)} \\ &\quad -\left(\frac{w^4}{(1-w^4)^2}+\frac{1}{3}\right)\,\frac{i}{\theta_1(2z,\tau)^3\,\eta(\tau)^3}. \end{aligned} \tag{5.11}$$

Proof. We start by proving Eq. (5.10). Denote the length of an extended HN-filtration by ℓ, its length with respect to μ-stability by ℓ_μ and Gieseker stability ℓ_{G}. We first consider the unstable filtrations with $\ell = \ell_\mu = 2$, and parametrize $c_1(E_2)$ by $bC - af$. These are parametrized by $a \geq 0$ and $b = 0$. There are four possibilities to be distinguished: whether $r(E_1) = 1$ or 2, and whether the quotient with rank 2 has $c_1 = 0$ or $f \mod 2$. Adding up these contributions, one obtains:

$$-\frac{w^4+w^8}{1-w^{12}}\,h_{1,0}(z,\tau)\,h_{2,0}(z,\tau;J_{\varepsilon,1}) - \frac{w^2+w^{10}}{1-w^{12}}\,h_{1,0}(z,\tau)\,h_{2,f}(z,\tau;J_{\varepsilon,1}), \tag{5.12}$$

The filtrations with $\ell = 3$ consist of 3 subsets: one set with $\ell_\mu = 3$, one with $\ell_\mu = 2$ but $\ell_{\mathrm{G}} = 3$, and one with $\ell_{\mathrm{G}} = 2$. Parametrizing $c_1(E_i) = b_iC - a_if$, the first set is parametrized by $a_i - a_{i+1} > 0$, $\sum_{i=1}^3 a_i = 1$ and $b_i = 0$. These are counted by:

$$\sum_{\substack{k_1,k_2>0\\ k_2=k_1-1 \mod 3}} w^{-4(k_1+k_2)}\,h_{1,0}(z,\tau)^3 = \frac{w^4}{(1-w^4)(1-w^{12})}\,h_{1,0}(z,\tau)^3. \tag{5.13}$$

For the second and third sets, one needs to distinguish between equality of the stability condition of E_2 with E_1 or E_3. These two sets are enumerated by:

$$-\frac{1}{2}\frac{w^4+w^8}{1-w^{12}}\,h_{1,0}(z,\tau)^3. \tag{5.14}$$

Note that the factor $\frac{1}{|\mathrm{Aut}(\{\Gamma_i\},J_{1,\varepsilon})|}$ naturally combines the contributions of filtrations with $\ell_\mu < \ell$. Another observation is that the term $-\frac{1}{2}$ in the second factor of $h_{2,0}(z,\tau;J_{\varepsilon,1})$ (5.6) cancels against (5.14) in the total sum. After subtraction of the terms (5.12)-(5.14) from Eq. (4.2) for $r=3$, and writing the whole series in terms of modular functions, one obtains (5.10).

For $(r,c_1)=(3,0)$, one needs to subtract the following terms:

- due to unstable filtrations with $\ell=\ell_\mu=2$:

$$-\frac{2}{1-w^{12}}\,h_{1,0}(z,\tau)\,h_{2,0}(z,\tau;J_{\varepsilon,1})-\frac{2w^6}{1-w^{12}}\,h_{1,0}(z,\tau)\,h_{2,f}(z,\tau;J_{\varepsilon,1}),$$

- due to unstable filtrations with $\ell=2$, $\ell_\mu=1$ and $\ell_{\mathrm{G}}=1$ or 2:

$$\frac{2}{2}\,h_{1,0}(z,\tau)\,h_{2,0}(z,\tau;J_{\varepsilon,1}),$$

- due to unstable filtrations with $\ell=\ell_\mu=3$:

$$\frac{1+w^{12}}{(1-w^8)(1-w^{12})}\,h_{1,0}(z,\tau)^3,$$

- due to unstable filtrations with $\ell=3$, $\ell_\mu=2$ and $\ell_{\mathrm{G}}=2$ or 3:

$$-\frac{2}{2}\frac{1}{1-w^{12}}\,h_{1,0}(z,\tau)^3,$$

- due to unstable filtrations with $\ell=3$, $\ell_\mu=1$ and $1\leq\ell_{\mathrm{G}}\leq 3$:

$$\frac{1}{6}\,h_{1,0}(z,\tau)^3.$$

Subtracting the terms above from (4.2) gives (5.11). Subtracting further $\frac{1}{3}h_{1,0}(3z,3\tau)=\frac{i}{3\,\theta_1(6z,3\tau)\,\eta(3\tau)}$ from (5.11) provides integer invariants in agreement with the definition (3.3). □

The recursive procedure explained above can be solved, such that $h_{r,c_1}(z,\tau;J_{\varepsilon,1})$ can be directly expressed in terms of the $H_{r'}(z,\tau;f)$ with $r'\leq r$, without computing first the $h_{r',c_1}(z,\tau;J_{\varepsilon,1})$, and moreover giving more compact expressions. The solution follows from Ref. [37] (the solution to the recursion for vector bundles over Riemann surfaces) and Eq.

(3.4) one obtains:

$$\begin{aligned} h_{r,c_1}(z,\tau;J_{\varepsilon,1}) &= \sum_{\substack{(r_1,c_{1,1})+\dots+(r_\ell,c_{1,\ell})=(r,c_1),\\ \mu_i\cdot J_{\varepsilon,1}\geq \mu_{i+1}\cdot J_{\varepsilon,1}}} \frac{(-1)^{m-1}}{m} w^{-\sum_{i<j} r_ir_j(\mu_j-\mu_i)\cdot K_S} \prod_{i=1}^m H_{r_i,0}(z,\tau;f) \\ &= \sum_{\substack{(r_1,a_1)+\dots+(r_\ell,a_\ell)=(r,c_1\cdot C),\\ a_i\geq a_{i+1}}} \frac{(-1)^{m-1}}{m} w^{-2\sum_{i<j} r_ir_j(a_j-a_i)} \prod_{i=1}^m H_{r_i,0}(z,\tau;f) \end{aligned} \tag{5.15}$$

This becomes after carrying out the sums over a_i [37]:

$$\begin{aligned} h_{r,-af}(z,\tau;J_{\varepsilon,1}) &= \sum_{\substack{(r_1,a_1)+\dots+(r_m,a_m)=(r,a)\\ a_i/r_i=a/r}} \frac{(-1)^{m-1}}{m} \\ &\prod_{i=1}^m \left(\sum_{r_1+\dots+r_\ell=r_i} \frac{w^{2M(r_1,\dots,r_\ell;a_i/r_i)}}{(1-w^{2(r_1+r_2)})\dots(1-w^{2(r_{\ell-1}+r_\ell)})} H_{r_1,0}(z,\tau;f)\dots H_{r_\ell,0}(z,\tau;f)\right), \end{aligned} \tag{5.16}$$

where

$$M(r_1,\dots,r_\ell;\lambda) = \sum_{j=1}^{\ell-1} (r_j+r_{j+1})\,\{(r_1+\dots+r_j)\lambda\}, \tag{5.17}$$

with $\{\lambda\} := \lambda - \lfloor \lambda \rfloor$.

One can verify that Eq. (5.16) for $r=3$ is in agreement with Eqs. (5.10) and (5.11). As an example we give here $h_{4,0}(z,\tau;J_{\varepsilon,1})$:

$$\begin{aligned} h_{4,0}(z,\tau) &= H_{4,0}(\tau,z;f) + \frac{1}{2}\frac{1+w^8}{1-w^8} H_{2,0}(\tau,z;f)^2 + \frac{1+w^8}{1-w^8} H_{1,0}(\tau,z;f)\, H_{3,0}(\tau,z;f) \\ &\quad + \frac{1-w^{16}}{(1-w^4)(1-w^6)^2} H_{1,0}(\tau,z;f)^2\, H_{2,0}(\tau,z;f) + \frac{1}{4}\frac{1-w^{16}}{(1-w^4)^4} H_{1,0}(\tau,z;f)^4, \end{aligned} \tag{5.18}$$

which is to be compared with:

$$\begin{aligned} h_{4,0}(z,\tau) &= H_{4,0}(\tau,z;f)-\Bigg(-\frac{w^{12}}{(1-w^8)\,(1-w^{12})^2}+\frac{1}{2}\frac{1+w^{24}}{(1-w^{12})\,(1-w^{24})} \\ &\quad -\frac{1}{3}\frac{1}{1-w^{24}}-\frac{1}{4}\frac{1}{1-w^{16}}+\frac{1}{24}\Bigg)\,h_{1,0}(z,\tau)^4 \\ &\quad -\Bigg(\frac{2(1+w^{20})}{(1-w^{16})\,(1-w^{24})}+\frac{1+w^{24}}{(1-w^{12})\,(1-w^{24})}-\frac{2}{1-w^{24}} \\ &\quad -\frac{1}{1-w^{16}}+\frac{1}{2}\Bigg)\,h_{1,0}(z,\tau)^2\,h_{2,0}(z,\tau;J_{\varepsilon,1}) \\ &\quad -\left(\frac{2\,(w^{10}+w^{30})}{(1-w^{16})\,(1-w^{24})}+\frac{2\,w^{18}}{(1-w^{12})\,(1-w^{24})}\right)\,h_{1,0}(z,\tau)^2\,h_{2,f}(z,\tau;J_{\varepsilon,1}) \\ &\quad -\left(-\frac{1}{1-w^{16}}+\frac{1}{2}\right)\,h_{2,0}(z,\tau;J_{\varepsilon,1})^2-\left(-\frac{w^8}{1-w^{16}}\right)\,h_{2,f}(z,\tau;J_{\varepsilon,1})^2 \\ &\quad -\left(-\frac{2}{1-w^{24}}+1\right)\,h_{1,0}(z,\tau)\,h_{3,0}(z,\tau;J_{\varepsilon,1}) \\ &\quad -\left(-\frac{2\,(w^8+w^{16})}{1-w^{24}}\right)\,h_{1,0}(z,\tau)\,h_{3,0}(z,\tau;J_{\varepsilon,1}). \end{aligned}$$

5.2. **Wall-crossing.** This subsection explains how to compute $h_{r,c_1}(z,\tau;J)$ for a generic choice of polarization J from the generating functions for $J=J_{\varepsilon,1}$. The BPS invariants $\Omega(\Gamma,w;J)$ for J differ in general from those for $J=J_{\varepsilon,1}$, since sheaves might become semi-stable or unstable by changing the polarization. The change of the BPS invariants depends on the Hirzebruch surface Σ_ℓ through the canonical class K_{Σ_ℓ}. Knowing how $h_{r,c_1}(z,\tau;J)$ varies in the ample cone $C(\Sigma_1)$ is particularly important for the computation of $h_{r,c_1}(z,\tau;\mathbb{P}^2)$ since the blow-up formula is to be applied for the polarization $J_{1,0}=\phi^*H$, where H is the hyperplane class of $\mathbb{P}^2$ (see the next section). The change of the invariants can be obtained recursively from Eq. (5.8) after determining which filtrations change from semi-stable to unstable or vice versa.

More quantitatively one has for J and J' sufficiently close:

$$\begin{aligned} \Delta\bar\Omega(\Gamma,w;J\to J') &= \sum_{\substack{\Gamma=\Gamma_1+\dots+\Gamma_\ell,\\ p_{J'}(\Gamma_i)\preceq p_{J'}(\Gamma_{i+1}),\\ p_J(\Gamma_i)\succeq p_J(\Gamma_{i+1})}} \frac{1}{|\mathrm{Aut}(\{\Gamma_i\};J)|}\,w^{-\sum_{i<j} r_ir_j(\mu_j-\mu_i)\cdot K_S}\prod_{i=1}^{\ell}\bar\Omega(\Gamma_i;w,J) \\ &\quad -\sum_{\substack{\Gamma=\Gamma_1+\dots+\Gamma_\ell,\\ p_{J'}(\Gamma_i)\succeq p_{J'}(\Gamma_{i+1}),\\ p_J(\Gamma_i)\preceq p_J(\Gamma_{i+1})}} \frac{1}{|\mathrm{Aut}(\{\Gamma_i\};J')|}\,w^{-\sum_{i<j} r_ir_j(\mu_j-\mu_i)\cdot K_S}\prod_{i=1}^{\ell}\bar\Omega(\Gamma_i;w,J'), \end{aligned} \tag{5.19}$$

with $|\mathrm{Aut}(\{\Gamma_i\};J)|$ defined below Eq. (5.8). Note that the invariants are evaluated on both sides of the wall. This makes this formula a recursive formula as it requires knowledge of $\Omega(\Gamma_i,w;J')$, but since we are only interested in small rank this is not a serious obstacle. A solution to the recursion is given by Theorem 6.24 of [15]. Other ways to determine $\Omega(\Gamma_i,w;J')$ in terms of $\Omega(\Gamma_i,w;J)$ is using a graded Lie algebra [21] or the Higgs branch analysis of Ref. [27] based on Ref. [30].

Since generating functions capturing wall-crossing are already described in the literature, the explicit expressions of $h_{r,c_1}(z,\tau;J_{m,n})$ for $r=2$ and 3, are presented here without further details. We have for $r=2$ [33, 11]:

$$\begin{aligned} h_{2,\beta C-\alpha f}(z,\tau;J_{m,n}) \;=\;& h_{2,\beta C-\alpha f}(z,\tau;J_{\varepsilon,1})+ \\ & \tfrac{1}{2}\textstyle\sum_{a,b\in\mathbb{Z}}\tfrac{1}{2}\left(\,\mathrm{sgn}((2b-\beta)n-(2a-\alpha)m)-\mathrm{sgn}((2b-\beta)-(2a-\alpha)\varepsilon)\,\right) \\ & \times\left(w^{-(\ell-2)(2b-\beta)-2(2a-\alpha)}-w^{(\ell-2)(2b-\beta)+2(2a-\alpha)}\right)\,q^{\frac{\ell}{4}(2b-\beta)^2+\frac{1}{2}(2b-\beta)(2a-\alpha)}\,h_{1,0}(z,\tau)^2, \end{aligned}$$

and for $r=3$ [26, 25]:

$$\begin{aligned} h_{3,\beta C-\alpha f}(z,\tau;J_{m,n}) \;=\;& h_{3,\beta C-\alpha f}(z,\tau;J_{\varepsilon,1})+ \\ & \sum_{a,b\in\mathbb{Z}}\tfrac{1}{2}\left(\,\mathrm{sgn}((3b-2\beta)n-(3a-2\alpha)m)-\mathrm{sgn}((3b-2\beta)-(3a-2\alpha)\varepsilon)\,\right) \\ & \times\left(w^{-(\ell-2)(3b-2\beta)-2(3a-2\alpha)}-w^{(\ell-2)(3b-2\beta)+2(3a-2\alpha)}\right)\,q^{\frac{\ell}{12}(3b-2\beta)^2+\frac{1}{6}(3b-2\beta)(3a-2\alpha)} \\ & \times h_{2,bC-af}(z,\tau;\Sigma_\ell,J_{|3b-2\beta|,|3a-2\alpha|})\,h_{1,0}(z,\tau). \end{aligned}$$

6. BPS invariants of $\mathbb{P}^2$

The Hirzebruch surface Σ_1 can be obtained as a blow-up $\phi:\Sigma_1\to\mathbb{P}^2$ of the projective plane $\mathbb{P}^2$. Interestingly, we can compute the BPS invariants of $\mathbb{P}^2$ from those of Σ_1 from the blow-up formula. This formula is a remarkable result which states that the ratio of generating functions of BPS invariants of a surface S and its blow-up $\tilde{S}$ is a (theta) function independent of S or J [35, 12, 23]. The underlying reason for this relation is that every semi-stable sheaf on $\tilde{S}$ can be obtained from one on S by an elementary transformation along the exceptional divisor of the blow-up.

Two subtle issues of the blow-up formula are (Proposition 3.4 of [35]):

- the stability condition is μ-stability rather than Gieseker stability,
- it involves the virtual Poincaré functions $\mathcal{I}(\Gamma,w;J)$ of the moduli stack.

To take these two issues into account let $\bar{\Omega}^\mu(\Gamma, w; J)$ be the invariant enumerating μ-semi-stable sheaves which is obtained from $\bar{\Omega}^\mu(\Gamma, w; J)$ by addition of the Gieseker unstable sheaves which are μ-semi-stable using Eq. (5.8). Moreover, let $\mathcal{I}^\mu(\Gamma, w; J)$ be the corresponding virtual Poincaré function with corresponding generating function $H^\mu_{r,c_1}(z, \tau; \tilde{S}, J)$.

The blow-up formula now reads [35, 12, 23]:

Proposition 6.1. *Let S be a smooth projective surface and $\phi : \tilde{S} \to S$ the blow-up at a non-singular point, with $C_{\rm e}$ the exceptional divisor of ϕ. The generating functions $H^\mu_{r,c_1}(z, \tau; S, J)$ and $H^\mu_{r,c_1}(z, \tau; \tilde{S}, J)$ are related by the "blow-up formula":*

$$H^\mu_{r,\phi^*c_1 - kC_{\rm e}}(z, \tau; \tilde{S}, \phi^* J) = B_{r,k}(z, \tau)\, H^\mu_{r,c_1}(z, \tau; S, J), \tag{6.1}$$

with

$$B_{r,k}(z, \tau) = \frac{1}{\eta(\tau)^r} \sum_{\substack{\sum_{i=1}^r a_i = 0 \\ a_i \in \mathbb{Z} + \frac{k}{r}}} q^{-\sum_{i<j} a_i a_j} w^{\sum_{i<j} a_i - a_j}.$$

The blow-up formula for generating functions of Hodge numbers is identical except with the replacement of z by $\frac{1}{2}(u + v)$ in $B_{r,k}(z, \tau)$.

The two relevant cases for this article are $r = 2, 3$:

$$B_{2,k}(z, \tau) = \frac{\sum_{n \in \mathbb{Z}+k/2} q^{n^2} w^n}{\eta(\tau)^2}, \qquad B_{3,k}(z, \tau) = \frac{\sum_{m,n \in \mathbb{Z}+k/3} q^{m^2+n^2+mn} w^{4m+2n}}{\eta(\tau)^3}. \tag{6.2}$$

Note that $B_{r,k}(z, \tau)$ does not depend on S or J.

The computation of $h_{r,c_1}(z, \tau; \mathbb{P}^2)$ from $h_{r,\phi^*c_1 - kC}(z, \tau; \Sigma_1)$ in general involves the following three steps:

1. (1) Compute $h^\mu_{r,\phi^*c_1-kC}(z, \tau; J_{1,0})$ by adding to $h_{r,\phi^*c_1-kC}(z, \tau; J_{1,\varepsilon})$ terms due to sheaves on Σ_1 which are not Gieseker stable for $J_{1,\varepsilon}$, but μ-semistable for $\phi^* H = J_{1,0}$, and consequently compute $H^\mu_{r,\phi^*c_1-kC}(z, \tau; J_{1,0})$ by adding the terms prescribed by Eq. (3.5). The generating functions and the factorial factors in Eq. (3.5) combine these two steps very naturally into one.
2. (2) Divide by $B_{r,k}(z, \tau)$ to obtain $H^\mu_{r,c_1}(z, \tau; \mathbb{P}^2)$.
3. (3) Determine $h_{r,c_1}(z, \tau; \mathbb{P}^2)$ from $H^\mu_{r,c_1}(z, \tau; \mathbb{P}^2)$ by reversing step (1).

For $c_1 = \beta C + f$, $\beta = 0$ or 1, and $J = J_{1,0}$, μ-stability is equivalent to Gieseker stability, and therefore steps 1) and 3) become trivial. For example, one can compute $h_{3,H}(z, \tau; \mathbb{P}^2)$ starting from $h_{3,C+f}(z, \tau; J_{1,0})$ as was done in Ref. [26], or from from $h_{3,f}(z, \tau; J_{1,0})$ which requires Conjecture 4.1 and Eq. (5.8). One can verify that the first terms of both q-expansions of

$h_{3,H}(z,\tau;\mathbb{P}^2)$ are equal, which is in agreement with Proposition 6.1. A proof of the equality of these expressions for $h_{3,H}(z,\tau;\mathbb{P}^2)$ would imply a proof of Conjecture 4.1 for $(r,c_1)=(3,f)$ since $h_{3,f}(z,\tau;J_{\varepsilon,1})$ is related to $h_{3,C+f}(z,\tau;J_{1,\varepsilon})$ by the blow-up formula and wall-crossing.

When μ- and Gieseker stability are not equivalent, steps 1) and 3) are not trivial. We will first explain them for $r=2$ following [34]. One obtains:

Proposition 6.2.

$$h_{2,0}(z,\tau;\mathbb{P}^2)=\frac{1}{B_{2,1}(z,\tau)}\left[h_{2,C}(z,\tau;J_{1,\varepsilon})+\sum_{\substack{b<0\\ b=-1 \mod 2}} w^b q^{\frac14 b^2} h_{1,0}(z,\tau)^2\right]-\frac{1}{2}h_{1,0}(z,\tau;\mathbb{P}^2)^2.$$

Proof. The only extended HN-filtrations which are Gieseker unstable for $J=J_{1,\varepsilon}$ and μ-semi-stable for $J=J_{1,0}$ have $\ell=\ell_\mu=2$. For the parametrization $c_1(E_2)=bC-af$, the set of sheaves which is unstable for $J_{1,\varepsilon}$ but μ-semistable for $J_{1,0}$ corresponds to $b<0$ and $a=0$. This gives the second term inside the brackets. Consequently, step (2) divides by $B_{2,1}(z,\tau)$, and step (3) subtracts the μ-semi-stable sheaves which are not Gieseker semi-stable with $\ell=2$ and $\ell_\mu=1$. □

Alternatively, one can compute $h_{2,0}(z,\tau;\mathbb{P}^2)$ starting from $h_{2,0}(z,\tau;J_{1,\varepsilon})$. In that case the term due to step (1) in the brackets is $\left(\sum_{\substack{b<0\\ b=0 \mod 2}} w^b q^{\frac14 b^2}+\frac12\right) h_{1,0}(z,\tau)^2$, and one divides by $B_{2,0}(z,\tau)$. Addition of $\frac12 h_{1,0}(z,\tau;\mathbb{P}^2)$ provides the expected integer invariants, in agreement with [34]. Accidentily, the terms due to step (1) and step (3) can simply be incorporated by replacing $J_{1,\varepsilon}$ by $J_{1,0}$ in $h_{2,\beta C}(z,\tau;J_{1,\varepsilon})$, and can be written in terms of the Lerch sum [3].

The remainder of this section discusses $r=3$. In terms of $h_{3,C}(z,\tau;J_{1,\varepsilon})$, $h_{3,0}(z,\tau;\mathbb{P}^2)$ is given by:

Proposition 6.3.

$$\begin{aligned} h_{3,0}(z,\tau;\mathbb{P}^2) &= \frac{1}{B_{3,1}(z,\tau)}\Bigg[h_{3,C}(z,\tau;J_{1,\varepsilon}) + \Bigg(\sum_{\substack{b<0\\ b=-2,-4 \mod 6}} w^b q^{\frac{1}{12}b^2}\Bigg) h_{1,0}(z,\tau)\, h_{2,0}(z,\tau;J_{1,\varepsilon}) \\ &+ \Bigg(\sum_{\substack{b<0\\ b=-1,-5 \mod 6}} w^b q^{\frac{1}{12}b^2}\Bigg) h_{1,0}(z,\tau)\, h_{2,C}(z,\tau;J_{1,\varepsilon}) \\ &+ \Bigg(\sum_{\substack{k_1,k_2<0\\ k_2=k_1+1 \mod 3}} w^{2(k_1+k_2)} q^{\frac{1}{3}(k_1^2+k_2^2+k_1k_2)} + \frac{1}{2}\sum_{\substack{k<0,\\ k=-1,-2 \mod 3}} w^{2k} q^{\frac{1}{3}k^2}\Bigg) h_{1,0}(z,\tau)^3\Bigg] \\ &- \frac{1}{6}h_{1,0}(z,\tau;\mathbb{P}^2)^3 - \frac{2}{2}\, h_{1,0}(z,\tau;\mathbb{P}^2)\, h_{2,0}(z,\tau;\mathbb{P}^2). \end{aligned} \tag{6.3}$$

The desired integer invariants are obtained from $h_{3,0}(z,\tau;\mathbb{P}^2)$ after subtraction of $\frac{1}{3}h_{1,0}(3z,3\tau;\mathbb{P}^2) = \frac{1}{3}\frac{i}{\theta(6z,3\tau)}$. The first non-vanishing coefficients are presented in Table 1. They are in agreement with the expected dimension of $\mathcal{M}(\Gamma)$ (2.3).

c_2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}	b_{16}	b_{18}	b_{20}	b_{22}	b_{24}	b_{26}	b_{28}	χ
3	1	1	2	2	2	2										18
4	1	2	5	9	15	19	22	23	24							216
5	1	2	6	12	25	43	70	98	125	142	154	156				1512
6	1	2	6	13	28	53	99	165	264	383	515	631	723	774	795	8109

TABLE 1. The Betti numbers b_n (with $n \leq \dim_{\mathbb{C}} \mathcal{M}$) and the Euler number χ of the moduli spaces of semi-stable sheaves on $\mathbb{P}^2$ with $r = 3$, $c_1 = 0$, and $3 \leq c_2 \leq 6$.

Proof. The terms added to $h_{3,C}(z,\tau;J_{1,\varepsilon})$ in the brackets are due to step (1). The last term on the first line and the term on the second line are due to filtrations with $\ell = \ell_\mu = 2$. If one chooses $c_1(E_2) = bC - af$ as for $r = 2$, the set of sheaves which are unstable for $J_{1,\varepsilon}$ but μ-semistable for $J_{1,0}$ corresponds to $b < 0$ and $a = 0$. Similarly, the first term in parentheses on the third line is due to $\ell = \ell_\mu = 3$, and the second term due to $\ell = 3$ and $\ell_\mu = 2$. The sum of the terms in the bracket is $H^\mu_{3,C}(z,\tau;J_{1,\varepsilon})$, and is divided by $B_{3,1}(z,\tau)$ following step (2). Finally, step (3) corresponds to the last line. □

As a consistency check, $h_{3,0}(z,\tau;\mathbb{P}^2)$ can also be computed from $h_{3,0}(z,\tau;J_{1,\varepsilon})$. Then the terms due to step (1) are for $\ell=2$:

$$\left(\frac{2}{2}+2\sum_{\substack{b<0\\ b=0 \mod 6}} w^b q^{\frac{1}{12}b^2}\right) h_{1,0}(z,\tau)\, h_{2,0}(z,\tau;J_{1,\varepsilon}) + \left(2\sum_{\substack{b<0\\ b=-3 \mod 6}} w^b q^{\frac{1}{12}b^2}\right) h_{1,0}(z,\tau)\, h_{2,C}(z,\tau;J_{1,\varepsilon}), \tag{6.4}$$

and for $\ell=3$:

$$\left(\sum_{\substack{k_1,k_2<0\\ k_1=k_2 \mod 3}} w^{2(k_1+k_2)} q^{\frac{1}{3}(k_1^2+k_2^2+k_1k_2)} + \frac{2}{2}\sum_{\substack{k<0\\ k=0 \mod 3}} w^{2k} q^{\frac{1}{3}k^2} + \frac{1}{6}\right) h_{1,0}(z,\tau)^3. \tag{6.5}$$

We conclude by briefly comparing the BPS invariants computed above to the results obtained by Refs. [22, 32] for Euler numbers of moduli spaces using toric localization of the moduli spaces. Ref. [32] computed such Euler numbers for μ-stable vector bundles [32] with rank $r\leq 3$ on $\mathbb{P}^2$, whereas Ref. [22] computed such Euler numbers for μ-stable torsion free sheaves with rank $r\leq 3$ on various smooth toric surfaces. If $\gcd(r,c_1)=1$ and for a generic choice of polarization, the moduli space of μ-stable sheaves is isomorphic to the moduli space of Gieseker semi-stable sheaves. Otherwise, the moduli space of μ-stable sheaves is a smooth open subset of the moduli space of Gieseker semi-stable sheaves. The difference between generating functions of Euler numbers for vector bundles and torsion free sheaves is an overall factor $\eta(\tau)^{r\chi(S)}$.

For Chern classes such that μ-stability is equivalent to Gieseker semi-stability, agreement of Refs. [22, 32] with the techniques described in this paper is expected. This is indeed established in Refs. [26, 28]. In particular, Eq. (4.5) and Table 1 of Ref. [26] agree with Corollary 4.10 in Ref. [32] and Corollary 4.9 in Ref. [22]. If $\gcd(r,c_1)>1$ strictly Gieseker semi-stable sheaves can occur and therefore agreement of $h_{r,0}(z,\tau;\mathbb{P}^2)$ with Refs. [22, 32] is not expected. Indeed the numbers in Table 1 above appear to be different from the Euler numbers computed by Theorem 4.14 of Ref. [32] and Corollary 4.9 of Ref. [22]. It would be interesting to precisely understand the difference between the Euler numbers of the μ-stable loci and the BPS invariants computed above.

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