ON CONCENTRATORS AND RELATED APPROXIMATION CONSTANTS

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ABSTRACT. Pippenger ([Pip77]) showed the existence of $(6m, 4m, 3m, 6)$-concentrator for each positive integer m using a probabilistic method. We generalize his approach and prove existence of $(6m, 4m, 3m, 5.05)$-concentrator (which is no longer regular, but has fewer edges). We apply this result to improve the constant of approximation of almost additive set functions by additive set functions from 44.5 (established by Kalton and Roberts in [KR83]) to 39. We show a more direct connection of the latter problem to the Whitney type estimate for approximation of continuous functions on a cube in $\mathbb{R}^d$ by linear functions, and improve the estimate of this Whitney constant from 802 (proved by Brudnyi and Kalton in [BK00]) to 73.

1. INTRODUCTION

Our original motivation was the following Whitney-type inequality, valid for each $f \in C([0,1]^d)$:

$$\min_L \max_{x \in [0,1]^d} |f(x) - L(x)| \le w_2(d) \max_{x,y \in [0,1]^d} |f(x) + f(y) - 2f((x+y)/2)|,$$

where the minimum is taken over all polynomials L in d variables of total degree ≤ 1 (linear polynomials), and $C([0,1]^d)$ is the space of all continuous real-valued functions on the unit cube $[0,1]^d$. Brudnyi and Kalton (see [BK00]) showed that $w_2(d) \le 802$ and conjectured that $w_2(d) \le 2$. We will show here that $w_2(d) \le 73$, and improve some other constants along the way.

The above estimates, however, stem from seemingly irrelevant combinatorial problem of existence of certain concentrators. An (m, p, q, r)-concentrator is a bipartite graph with m inputs and p outputs, not more than mr edges, such that for any set of $k \le q$ inputs, there

exist k disjoint edges to some k outputs. Using a probabilistic argument, Pippenger [Pip77] showed that $(6m, 4m, 3m, 6)$-concentrators exist for any integer $m \geq 1$. Reducing the average degree of inputs for large m is of primary interest in our context. Our main result is the following theorem.

Theorem 1.1. *For any large enough integer m there exists a $(6m, 4m, 3m, 5.05)$-concentrator.*

For the proof, we use a modification of Pippenger's approach, but this requires much more technical estimates. Unfortunately, our method does not allow to prove that $(6m, 4m, 3m, 5)$-concentrators exist for large m, but we conjecture that this is so, see Remark 2.2.

Pippenger's concentrators were used by Kalton and Roberts in [KR83] to prove the following. There exists an absolute constant $K \leq 44.5$ such that for any algebra $\mathfrak{A}$ of finite sets and any map $\nu : \mathfrak{A} \to \mathbb{R}$ satisfying $|\nu(A \cup B) - \nu(A) - \nu(B)| \leq 1$ whenever $A \cap B = \emptyset$, there exists an additive set-function $\mu : \mathfrak{A} \to \mathbb{R}$ (i.e., $\mu(A \cup B) = \mu(A) + \mu(B)$ for $A \cap B = \emptyset$), satisfying $|\nu(A) - \mu(A)| \leq K$ for any $A \in \mathfrak{A}$. We remark that the same is true if one does not restrict the elements of $\mathfrak{A}$ to be *finite* sets, see [KR83, Proof of Theorem 4.1, p. 809]. From Theorem 1.1, we immediately obtain the following improvement.

Corollary 1.2. $K < 39$.

Since Brudnyi and Kalton [BK00] reduced the problem of estimating $w_2(d)$ to the problem of estimating K, Corollary 1.2 would provide an immediate (but insignificant) improvement of the estimate on $w_2(d)$. We establish a more direct connection between these two questions and prove the following.

Theorem 1.3. $w_2(d) < 73$.

Using Corollary 1.2 and Theorem 1.3, one can follow [BK00] to obtain an improvement of other approximation constants, including Whitney constant for unit balls of finite dimensional l_p-spaces, homogeneous Whitney constants, etc.

The paper is organized as follows. In Section 2, we state the main technical lemma and use it to prove Theorem 1.1. The lemma itself is proved in Section 4 using reduction to a non-linear optimization problem, which was resolved with the aid of a computer. The proof of Corollary 1.2 and Theorem 1.3 can be found in Section 3.

2. Concentrators

Let $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ be the binomial coefficient, and we set $\binom{n}{m} = 0$ if $m < 0$ or $m > n$. The most technical part of our result is the following lemma, which will be proved later in Section 4.

Lemma 2.1. *For any large integer m, with $s = \lceil 5.7m \rceil$, we have*

$$(2.1) \qquad \sum_{k=1}^{3m} \sum_{l=0}^{k} \sum_{r=0}^{k} \binom{s}{l} \binom{6m-s}{k-l} \binom{s-4m}{r} \binom{8m-s}{k-r} \frac{\binom{8k-r}{6k-l}}{\binom{36m-s}{6k-l}} < 1.$$

Now we show how Lemma 2.1 implies our main result closely following the idea of [Pip77] with some extra necessary calculations appearing from non-regularity of the graph.

Proof of Theorem 1.1. Let $s = \lceil 5.7m \rceil$, $N := 36m - s$, and $\mathcal{M} := \{0, 1, \ldots, N - 1\}$. Any permutation π on $\mathcal{M}$ defines a bipartite graph $G(\pi)$ with inputs $\{0, 1, \ldots, 6m - 1\}$ and outputs $\{0, 1, \ldots, 4m - 1\}$, where for every $x \in \mathcal{M}$ there is an edge from $(x \mod 6m)$ to $(\pi(x) \mod 4m)$. There are $6m - s$ inputs of degree 6 and s inputs of degree 5; $s - 4m$ outputs of degree 7 and $8m - s$ outputs of degree 8. Total average degree of the inputs is at most $\frac{36m - 5.7m}{6m} = 5.05$.

Following Pippenger, we want to compute the probability that a random (with respect to the uniform distribution) permutation π is "bad", that is for some k, $1 \leq k \leq 3m$, there exists a set A of k inputs and a set B of k outputs in $G(\pi)$ such that every edge out of A goes into B. Let l, $0 \leq l \leq k$, be the number of vertices from A that have degree 5, and let r, $0 \leq r \leq k$, be the number of vertices from B that have degree 7. Then A corresponds to a set $\mathcal{A}$ of $6(k - l) + 5l = 6k - l$ elements from $\mathcal{M}$, and B corresponds to a set $\mathcal{B}$ of $8(k - r) + 7r = 8k - r$ elements from $\mathcal{M}$. Note that $\mathcal{A}$ can be chosen in $\binom{s}{l}\binom{6m-s}{k-l}$ ways, while $\mathcal{B}$ can be chosen in $\binom{s-4m}{r}\binom{8m-s}{k-r}$ ways, which is reflected in the first four factors of (2.1) (for some values of k and r one or more of these binomial coefficients may be zero). The probability that a permutation π sends each element of $\mathcal{A}$ into $\mathcal{B}$ is equal to

$$(8k - r)(8k - r - 1) \ldots ((8k - r) - (6k - l) + 1) \frac{(N - (6k - l))!}{N!} = \frac{\binom{8k-r}{6k-l}}{\binom{N}{6k-l}} = \frac{\binom{8k-r}{6k-l}}{\binom{36m-s}{6k-l}}.$$

This shows that the probability that a permutation is "bad" is bounded by the left-hand side of (2.1), and by Lemma 2.1, it is bounded by one. Hence, a "good" permutation exists, and the existence of the required concentrator is proved. $\qquad\square$

Remark 2.2. Essentially, [Pip77] considers the case of $s = 0$, and here we find the largest possible s permitting generalization. It is easy to see from the proof of Theorem 1.1, that

if (2.1) is satisfied with $s = 6m$, then a $(6m, 4m, 3m, 5)$-concentrator exists. Let $s(m)$ be the largest value of s so that (2.1) is satisfied. For small values of m, the quotient $s(m)/m$ appears to be larger, and in fact, computer computations show that $s(m)/m \geq 6$ for all $m \leq 150$ (but not for $m = 151$). However, as $m \to \infty$, we have $s(m)/m \to c^* \approx 5.72489$, see Remark 4.4. Hence, our refinement of Pippenger's probabilistic approach allows to prove asymptotic existence of $(6m, 4m, 3m, 5.05)$-concentrators, but does not imply the existence of $(6m, 4m, 3m, 5)$-concentrators for large m. We conjecture that $(6m, 4m, 3m, 5)$-concentrators *do* exist for large m, since our method shows that a random graph from certain configuration space will provide "almost" the required concentrator. If an "average" object is "almost good", it is reasonable to expect that some "best" object will be "good", but the proof may require a completely different, and, perhaps, non-probabilistic approach.

3. Constants

Proof of Corollary 1.2. Following the proof of [KR83, Theorem 4.1, p. 811], we see that if $(6m, 4m, 3m, \gamma)$-concentrators exists for large enough m, then

$$K \leq \frac{7 + 4\gamma - 4/3}{2/3}.$$

For $\gamma = 5.05$, we obtain $K \leq 38.8 < 39$. □

The following lemma is a slight modification of [KR83, Theorem 4.1] combined with new concentrators, which uses a stronger condition on the function being approximated and achieves a better constant.

Lemma 3.1. *For any algebra $\mathfrak{A}$ of sets and any map $\nu : \mathfrak{A} \to \mathbb{R}$ satisfying*

(3.1) $$|\nu(A) + \nu(B) - \nu(A \cap B) - \nu(A \cup B)| \leq 1 \quad \textit{for any } A, B \in \mathfrak{A},$$

and $\nu(\emptyset) = 0$, there exists an additive set-function $\mu : \mathfrak{A} \to \mathbb{R}$, satisfying $|\nu(A) - \mu(A)| \leq \widetilde{K}$ for any $A \in \mathfrak{A}$, where $\widetilde{K} < 36$.

Proof. Note that when $\nu(\emptyset) = 0$, the condition (3.1) implies $|\nu(A) + \nu(B) - \nu(A \cup B)| \leq 1$ for any $A \cap B = \emptyset$. Therefore, we can follow the proof of [KR83, Theorem 4.1] verbatim with a small change that will be described now. Below g, a, A and S are the same as in the proof of [KR83, Theorem 4.1]. We can replace the inequality $g(A \cap S) \geq a - \frac{5}{2}$ on [KR83, Theorem 4.1,

p. 810] by a stronger $g(A \cap S) \geq a - \frac{3}{2}$ using (3.1) for g as follows:

$$g(A \cap S) \geq g(A) + g(S) - g(A \cup S) - 1 \geq \left(a - \frac{1}{2}\right) + a - a - 1 = a - \frac{3}{2}.$$

We used $g(A) \geq a - \frac{1}{2}$, $g(S) = a$, and $g(A \cup S) \leq a$. Consequently, we can replace $\frac{9}{2}$ by $\frac{7}{2}$ everywhere in the proof of [KR83, Theorem 4.1]. Accordingly, if $(6m, 4m, 3m, \gamma)$-concentrators exist for large enough m, then

$$\widetilde{K} \leq \frac{5 + 4\gamma - 4/3}{2/3}.$$

Hence, with $\gamma = 5.05$, we obtain $\widetilde{K} \leq 35.8 < 36$. $\qquad \square$

Proof of Theorem 1.3. We can assume that

$$(3.2) \qquad \max_{x,y \in [0,1]^d} |f(x) + f(y) - 2f((x+y)/2)| = \frac{1}{2},$$

and prove that for some linear polynomial L we have $|f(x) - L(x)| \leq \frac{73}{2}$, $x \in [0,1]^d$.

Let $\mathfrak{A}$ be the algebra of all subsets of $\{1, 2, \ldots, d\}$. Each element of $\mathfrak{A}$ can be naturally assigned to exactly one element of $\{0,1\}^d$ (the set of all vertices of the cube $[0,1]^d$) as follows. For any $A \in \mathfrak{A}$, let $\tau(A) = (x_1, \ldots, x_d)$, where $x_j = 1$ if $j \in A$, and $x_j = 0$ otherwise. For any $f \in C([0,1]^d)$, we define a mapping $\nu : \mathfrak{A} \to \mathbb{R}$ as $\nu(A) = f(\tau(A)) - f(0)$, $A \in \mathfrak{A}$. Under the assumption (3.2), we first claim that (3.1) holds. Indeed, it is easy to see that

$$\hat{x} := \frac{\tau(A) + \tau(B)}{2} = \frac{\tau(A \cap B) + \tau(A \cup B)}{2} \in [0,1]^d,$$

so by (3.2),

$$|\nu(A) + \nu(B) - \nu(A \cap B) - \nu(A \cup B)| = |f(\tau(A)) + f(\tau(B)) - f(\tau(A \cap B)) - f(\tau(A \cup B))|$$
$$\leq |f(\tau(A)) + f(\tau(B)) - 2f(\hat{x})|$$
$$+ |f(\tau(A \cap B)) + f(\tau(A \cup B)) - 2f(\hat{x})|$$
$$\leq \frac{1}{2} + \frac{1}{2} = 1.$$

Applying Lemma 3.1, we obtain an additive set-function μ satisfying $|\nu(A) - \mu(A)| \leq 36$ for all $A \in \mathfrak{A}$. Note that by additivity of μ, the linear function

$$\widetilde{L}(x_1, \ldots, x_d) := \mu(\{1\})x_1 + \cdots + \mu(\{d\})x_d$$

satisfies $\widetilde{L}(\tau(A)) = \mu(A)$, for any $A \in \mathfrak{A}$. Therefore, for the linear polynomial L defined as $L(x) := \widetilde{L}(x) + f(0)$, we have the following estimate at the vertices of the cube:

$$|f(x) - L(x)| \le 36, \quad x \in \{0, 1\}^d.$$

Now we show that this implies the required estimate for all $x \in [0, 1]^d$. Let

$$|f(\tilde{x}) - L(\tilde{x})| = \max_{x \in [0,1]^d} |f(x) - L(x)|.$$

Without loss of generality, assume that $\tilde{x} \in [0, \frac{1}{2}]^d$ (otherwise we replace 0 in the arguments below by an appropriate vertex of the cube). Since $2\tilde{x} \in [0, 1]^d$, we use (3.2) and $L(0) + L(2\tilde{x}) - 2L(\tilde{x}) = 0$ to conclude that

$$2|f(\tilde{x}) - L(\tilde{x})| \le |f(2\tilde{x}) - L(2\tilde{x})| + |f(0) - L(0)| + |f(0) + f(2\tilde{x}) - 2f(\tilde{x})|$$

$$\le |f(\tilde{x}) - L(\tilde{x})| + 36 + \frac{1}{2}.$$

Hence, $|f(\tilde{x}) - L(\tilde{x})| \le \frac{73}{2}$, as required. $\qquad\square$

4. Proof of Lemma 2.1

We need to prove (2.1), which is

$$\sum_{k=1}^{3m} \sum_{l=0}^{k} \sum_{r=0}^{k} \binom{s}{l} \binom{6m-s}{k-l} \binom{s-4m}{r} \binom{8m-s}{k-r} \frac{\binom{8k-r}{6k-l}}{\binom{36m-s}{6k-l}} =: \sum_{k=1}^{3m} \sum_{l=0}^{k} \sum_{r=0}^{k} a(m, s, k, l, r) < 1.$$

Let us give an outline of the proof. The main idea is to show that $a(m, s, k, l, r) \le e^{-cm}$ for some $c > 0$. This will imply the required bound for large m, because there are at most Cm^3 terms of summation. We begin with relating binomial coefficients to a more convenient function $h(n, m)$ in Lemma 4.1. Then we treat "smaller" values of k, i.e., $k \le \lceil 2.6m \rceil$, in Lemma 4.2. This case is easier, since there is a simple estimate for $\sum_{l=0}^{k} \sum_{r=0}^{k} a(m, s, k, l, r)$ such that the bounding function (of k) attains maximum at the boundary of the domain. For the remaining more difficult case $\lceil 2.6m \rceil < k \le 3m$, we reduce the problem to optimization of a certain function φ, as described in Lemma 4.3. First, we show analytically that φ attains its maximum when k is largest. Then we show that the largest value of φ over the remaining two variables l and r will be attained at the only critical point of the domain, which is a solution of an algebraic system of equations of degree 5. Numerical computations are used to verify the required conclusion on the maximum value of φ.

Denote $g(x) := x \ln x$, if $x > 0$, and $g(0) := g(0+) = 0$. Let $h(x, y) := g(x) - g(y) - g(x - y)$. Note that h is defined and continuous on $\{(x, y) : 0 \le y \le x\}$, and also

$$(4.1) \qquad h(\lambda x, \lambda y) = \lambda h(x, y), \quad \lambda > 0.$$

The following lemma relates the binomial coefficient $\binom{n}{m}$ with $h(n, m)$.

Lemma 4.1. *For any integer $n \ge 1$ and $0 \le m \le n$,*

$$\frac{1}{5\sqrt{n}} \exp(h(n, m)) \le \binom{n}{m} \le \exp(h(n, m)).$$

Proof. Stirling's formula gives that for $n \ge 1$

$$\ln(n!) = \ln(\sqrt{2\pi}) + n \ln n + \frac{1}{2} \ln n - n + r(n),$$

where $0 < r(n) < \frac{1}{12n}$. This immediately implies the required estimates. $\qquad\square$

Now we estimate the required sum when k is not large.

Lemma 4.2. *There is an integer m_0 such that for any integers $m \ge m_0$ and $s \le 6m$, we have*

$$(4.2) \qquad \sum_{k=1}^{\lceil 2.6m \rceil} \sum_{l=0}^{k} \sum_{r=0}^{k} \binom{s}{l}\binom{6m-s}{k-l}\binom{s-4m}{r}\binom{8m-s}{k-r} \frac{\binom{8k-r}{6k-l}}{\binom{36m-s}{6k-l}} < \frac{1}{2}.$$

Proof. For simplicity, let $q = q(m) := \lceil 2.6m \rceil$. Since

$$\sum_{l=0}^{k} \binom{s}{l}\binom{6m-s}{k-l} = \binom{6m}{k},$$

and

$$\sum_{r=0}^{k} \binom{s-4m}{r}\binom{8m-s}{k-r} = \binom{4m}{k},$$

it is enough to prove that

$$\sum_{k=1}^{q} \binom{6m}{k}\binom{4m}{k} \frac{\binom{8k}{5k}}{\binom{30m}{5k}} < \frac{1}{2}.$$

Using Lemma 4.1, for $k \le q < 3m$, we obtain

$$(4.3) \qquad \binom{6m}{k}\binom{4m}{k} \frac{\binom{8k}{5k}}{\binom{30m}{5k}} \le 5\sqrt{30m} \exp(f(k, m)),$$

where

$$f(k, m) := h(6m, k) + h(4m, k) + h(8k, 5k) - h(30m, 5k).$$

We have

$$\frac{\partial^2}{\partial k^2} f(k, m) = \frac{3}{k} + \frac{4}{6m - k} - \frac{1}{4m - k} > 0, \quad k \in (0, 3m].$$

Therefore the maximum of the right hand side in (4.3) is attained for $k = 1$ or $k = q$. Hence,

$$(4.4) \qquad \sum_{k=1}^{q} \binom{6m}{k}\binom{4m}{k} \frac{\binom{8k}{5k}}{\binom{30m}{5k}} < 15\sqrt{30}m^{3/2}\big(\exp(f(1,m)) + \exp(f(q,m))\big).$$

It is easy to see that $\lim_{m\to\infty} m^3 \exp(f(1,m)) = C$, for some $C > 0$, hence $\lim_{m\to\infty} m^{3/2}\exp(f(1,m)) = 0$. Also, by (4.1) and continuity of h,

$$\lim_{m\to\infty} \frac{f(a,m)}{m} = h(6,2.6) + h(4,2.6) + h(8 \cdot 2.6, 5 \cdot 2.6) - h(30, 5 \cdot 2.6) < -0.07,$$

and so $\lim_{m\to\infty} m^{3/2}\exp(f(q,m)) = 0$. Therefore, the limit of the right hand side of (4.4) is zero as $m \to \infty$, hence, it is smaller than $\frac{1}{2}$ for large enough m, as required. $\qquad\square$

The estimate of the remaining terms of (2.1) will be deduced from an optimization problem, which we will describe now. The idea is to use Lemma 4.1 and (4.1) to establish asymptotics of each term of the required sum.

Let

$$(4.5)$$
$$\varphi(c,k,l,r) := h(c,l) + h(6-c,k-l) + h(c-4,r) + h(8-c,k-r) + h(8k-r,6k-l) - h(36-c,6k-l).$$

Clearly, for $c = 5.7$ and $k \in [2.6, 3]$ the above function φ is defined when

$$(4.6) \qquad\qquad k + c - 6 \leq l \leq k \quad \text{and} \quad k + c - 8 \leq r \leq c - 4.$$

Our optimization problem is described in the next lemma.

Lemma 4.3. *The absolute maximum value of φ for $c = 5.7$ and any $k \in [2.6, 3]$ over all l and r given by (4.6) is a negative number.*

Proof of Lemma 4.3. We claim that the absolute maximum of φ for $c = 5.7$ and $k \in [2.6, 3]$ over l and r given by (4.6) is achieved when $k = 3$. To simplify exposition, we will often present computations for a general fixed c first, and then substitute $c = 5.7$ in the end.

Observe that under the change of variables

$$x = k - l, \quad y = \frac{c - 4 - r}{4 - k},$$

the inequalities (4.6) can be rewritten as

$$0 \leq x \leq 6 - c \quad \text{and} \quad 0 \leq y \leq 1.$$

Therefore, we only need to show that for any fixed x, y specified above, we have

$$\frac{\partial \varphi(c, k, x, y)}{\partial k} \geq 0, \quad k \in [2.6, 3].$$

It is straightforward to compute that

$$\begin{aligned}
\frac{\partial \varphi(c, k, x, y)}{\partial k} &= (\ln(c - k + x) - \ln(k - x)) \\
&\quad + y(\ln((4 - k)y) - \ln(c - 4 - (4 - k)y)) \\
&\quad + (1 - y)(\ln(4 - 4y - k(1 - y)) - \ln(k(1 - y) + 4 + 4y - c)) \\
&\quad + \Big[(8 - y)\ln((8 - y)k + 4 + 4y - c) \\
&\quad - (3 - y)\ln((3 - y)k + 4 + 4y - c - x) - 5\ln(36 - c - 5k - x)\Big] \\
&=: D_1(c, k, x) + D_2(c, k, y) + D_3(c, k, y) + D_4(c, k, x, y).
\end{aligned}$$

Many intermediary estimates below directly follow from monotonicity of the logarithm and bounds on the involved variables. We have

$$\begin{aligned}
D_2(c, k, y) &= \frac{(c - 4)}{(4 - k)} \frac{(4 - k)y}{(c - 4)} \left[\ln\left(\frac{(4 - k)y}{(c - 4)}\right) - \ln\left(1 - \frac{(4 - k)y}{(c - 4)}\right)\right] \\
&\geq \frac{(c - 4)}{(4 - k)} \frac{(4 - k)y}{(c - 4)} \left[\ln\left(\frac{(4 - k)y}{(c - 4)}\right)\right] \geq \frac{-(c - 4)}{e(4 - k)},
\end{aligned}$$

where we used the fact that $\min_{t \in (0, 1]} t \ln t = -1/e$. Similarly, we get

$$\begin{aligned}
D_3(c, k, y) &= \frac{(8 - c)}{(4 - k)} \frac{(4 - k)(1 - y)}{(8 - c)} \left[\ln\left(\frac{(4 - k)(1 - y)}{(8 - c)}\right) - \ln\left(1 - \frac{(4 - k)(1 - y)}{(8 - c)}\right)\right] \\
&\geq \frac{(8 - c)}{(4 - k)} \frac{(4 - k)(1 - y)}{(8 - c)} \left[\ln\left(\frac{(4 - k)(1 - y)}{(8 - c)}\right)\right] \geq \frac{-(8 - c)}{e(4 - k)}.
\end{aligned}$$

Clearly $D_1(c, k, x) \geq D_1(c, 3, 0)$, and similarly $D_4(c, k, x, y) \geq D_4(c, k, 0, y)$. With fixed c and k, we claim that $D_4(c, k, 0, y)$ attains minimum at $y = 1$. Indeed,

$$\begin{aligned}
\frac{\partial D_4(c, k, 0, y)}{\partial y} &= \ln\left(1 - \frac{5k}{8k + 4 - c + (4 - k)y}\right) \\
&\quad + \frac{5(4 - k)(4 - c + 4y)}{(3k + 4 - c + (4 - k)y)(8k + 4 - c + (4 - k)y)} \\
&=: S_1(c, k, y) + \frac{S_2(c, k, y)}{S_3(c, k, y)} \leq S_1(5.7, 2.6, 1) + \frac{S_2(5.7, 2.6, 1)}{S_3(5.7, 2.6, 0)} \\
&= \ln\frac{15}{41} + \frac{16.1}{116.51} < 0.
\end{aligned}$$

Hence,

$$D_4(c,k,0,y) \geq D_4(c,k,0,1) = 5\ln\left(\frac{7k+8-c}{36-c-5k}\right) + 2\ln\left(1+\frac{5k}{2k+8-c}\right)$$

$$=: T_1(c,k) + T_2(c,k) \geq T_1(5.7,2.6) + T_2(5.7,2.6)$$

$$= 5\ln\frac{20.5}{17.3} + 2\ln\frac{20.5}{7.5} > 2.$$

In summary,

$$\frac{\partial\varphi(c,k,x,y)}{\partial k} \geq D_1(c,3,0) - \frac{(c-4)}{e(4-k)} - \frac{(8-c)}{e(4-k)} + 2 = \ln\frac{2.7}{3} - \frac{4}{e} + 2 > 0,$$

so $\varphi(5.7,k,x,y) \leq \varphi(5.7,3,x,y)$, and we can now focus on the case $k=3$.

With $c=5.7$ and $k=3$ the restrictions (4.6) become $l \in [2.7,3]$ and $r \in [0.7,1.7]$. To find the critical points of φ inside the domain we compute the partial derivatives of φ:

$$(4.7) \qquad \frac{\partial\varphi(c,3,l,r)}{\partial l} = \ln\left(\frac{(c-l)(3-l)(18-c-l)}{l(3-c+l)(6-r+l)}\right),$$

$$(4.8) \qquad \frac{\partial\varphi(c,3,l,r)}{\partial r} = \ln\left(\frac{(c-4-r)(3-r)(6-r+l)}{r(5-c+r)(24-r)}\right).$$

The system of equations $\{\frac{\partial\varphi}{\partial l} = 0,\ \frac{\partial\varphi}{\partial r} = 0\}$ can be reduced to the following algebraic equation of degree 5 on l:

$$(2c-18)l^5 + (-2c^2 - 69c + 846)l^4 + (-2c^3 + 123c^2 + 189c - 11448)l^3$$

$$(4.9) \quad +(2c^4 + 12c^3 - 2349c^2 + 14256c + 95256)l^2 + (-48c^4 + 1089c^3 + 2916c^2 - 125388c)l$$

$$+126c^4 - 4536c^3 + 40824c^2 = 0.$$

This reduction and some further computations were performed using Maple software[1]. When l is found, r can be obtained from $\frac{\partial\varphi}{\partial l} = 0$, which is a linear equation on r. This allows us to compute all critical points numerically with any given precision. In particular, for $c=5.7$, we get that there is only one critical point $(l^*, r^*) \in (2.7,3) \times (0.7,1.7)$, and it satisfies

$$|l^* - \bar{l}| < 10^{-7}, \quad |r^* - \bar{r}| < 10^{-7},$$

where $(\bar{l}, \bar{r}) = (2.8959102, 1.078108)$ is an approximate numerical solution.

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[1]A copy of the corresponding Maple worksheet is available at `http://prymak.net/concentrators.pdf`

We want to prove that the value of φ at the critical point is negative, that is $\varphi(5.7, 3, l^*, r^*) <$
0. At the approximation of the critical point we have $\varphi(5.7, 3, \bar{l}, \bar{r}) < -0.004$, so it suffices to
show that φ cannot change much around our point, more precisely, we need

$$|\varphi(5.7, 3, \bar{l}, \bar{r}) - \varphi(5.7, 3, l^*, r^*)| < 0.004.$$

This can be done by estimating the partial derivatives of φ in a rectangle that contains both
(l^*, r^*) and $(\bar{l}, \bar{r})$, say in $[2.89, 2.9] \times [1.07, 1.08]$. Rewriting (4.7) and (4.8) as sums of logarithms,
using monotonicity of the logarithm and the restrictions $l \in [2.89, 2.9]$ and $r \in [1.07, 1.08]$, it is
straightforward to show that

$$\left|\frac{\partial \varphi}{\partial l}\right| < 10 \quad \text{and} \quad \left|\frac{\partial \varphi}{\partial r}\right| < 10.$$

Therefore, as required,

$$|\varphi(5.7, 3, \bar{l}, \bar{r}) - \varphi(5.7, 3, l^*, r^*)| < 20 \cdot 10^{-7} < 0.004.$$

We proved that φ is negative at the only critical point inside the domain $[2.7, 3] \times [0.7, 1.7]$.

It remains to show that φ cannot achieve its maximum on the boundary of $[2.7, 3] \times [0.7, 1.7]$.
Indeed, from (4.7), it is easy to see that for any fixed $r \in (0.7, 1.7)$, we have

$$\lim_{l \to 2.7^+} \frac{\partial \varphi(5.7, 3, l, r)}{\partial l} = +\infty, \quad \text{and} \quad \lim_{l \to 3^-} \frac{\partial \varphi(5.7, 3, l, r)}{\partial l} = -\infty.$$

Similar arguments apply to $\frac{\partial \varphi}{\partial r}$, for a fixed $l \in (2.7, 3)$. This completes the proof of the lemma.

$\square$

Finally, we are ready for a formal proof of the required estimate.

Proof of Lemma 2.1. In view of Lemma 4.2, we only need to show that

$$\sum_{k=\lceil 2.6m \rceil + 1}^{3m} \sum_{l=0}^{k} \sum_{r=0}^{k} \binom{s}{l} \binom{6m-s}{k-l} \binom{s-4m}{r} \binom{8m-s}{k-r} \frac{\binom{8k-r}{6k-l}}{\binom{36m-s}{6k-l}} < \frac{1}{2}.$$

Each term of the sum can be estimated by Lemma 4.1 as follows:

$$\binom{s}{l} \binom{6m-s}{k-l} \binom{s-4m}{r} \binom{8m-s}{k-r} \frac{\binom{8k-r}{6k-l}}{\binom{36m-s}{6k-l}} < 30\sqrt{m} \exp(\psi(m, s, k, l, r)),$$

where

$$\psi(m, s, k, l, r) := h(s, l) + h(6m - s, k - l) + h(s - 4m, r) + h(8m - s, k - r)$$

$$+ h(8k - r, 6k - l) - h(36m - s, 6k - l).$$

Recalling that $s = s(m) = \lceil 5.7m \rceil$, h is continuous, and using (4.1), we see that

$$\lim_{m \to \infty} \frac{\psi(m, s, k, l, r)}{m} = \varphi(5.7, k, l, r).$$

According to Lemma 4.3, $\varphi(5.7, k, l, r) \leq -\delta$, for some $\delta > 0$. Therefore, for large enough m and some $\delta_1 > 0$, we have

$$\sum_{k=\lceil 2.6m \rceil+1}^{3m} \sum_{l=0}^{k} \sum_{r=0}^{k} \binom{s}{l} \binom{6m-s}{k-l} \binom{s-4m}{r} \binom{8m-s}{k-r} \frac{\binom{8k-r}{6k-l}}{\binom{36m-s}{6k-l}} < 0.4 \cdot 3^2 \cdot 30 m^{7/2} e^{-\delta_1 m},$$

which tends to zero as $m \to \infty$, and so the required sum is smaller than $\frac{1}{2}$ for large enough m. $\qquad\square$

Remark 4.4. Denote by c^* the supremum of all c such that the statement of Lemma 4.3 remains true. One can prove that c^* is the unique solution of the equation

$$\varphi(c, 3, l(c), r(c)) = 0, \quad c \in [5.7, 6],$$

where φ is given by (4.5), and $l = l(c) \in [2.7, 3]$ and $r = r(c) \in [0.7, 1.7]$ is the solution of the system $\{\frac{\partial \varphi}{\partial l} = 0, \frac{\partial \varphi}{\partial r} = 0\}$, see (4.7), (4.8). More detailed numerical computations show that $c^* \in (5.724889, 5.72489)$. Hence, the maximum value of $s = s(m)$ for which (2.1) holds satisfies $\lim_{m \to \infty} s(m)/m = c^*$. For simplicity, we stated and proved the lemma for $c = 5.7$, as the optimal value c^* provides only slight improvement to the constants in Section 3.

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