

On an equivariant analogue of the monodromy zeta function ^{*}

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Abstract

We offer an equivariant analogue of the monodromy zeta function of a germ invariant with respect to an action of finite group G as an element of the Grothendieck ring of finite $(\mathbb{Z} \times G)$ -sets. We formulate equivariant analogues of the Sebastiani–Thom theorem and of the A’Campo formula.

This paper (in a slightly shorter form) was essentially written in February 2008, when it was lost with an external hard disk in Madrid. For a rather long time there was no mood to restore it. However recently some ideas from it were used, in particular, in [3], [5]. This urged to return to the work. The most part of the paper was written when the author used the hospitality of the Max Plank Institute for Mathematics in Bonn.

A number of invariants of singularities of analytic spaces and of germ of analytic functions on them have equivariant versions for singularities with an action of a finite group G and for G -invariant (or G -equivariant) function germs on them. For example, in [10], as an equivariant analogue of the Milnor number of a G -invariant function germ on the space $(\mathbb{C}^n, 0)$ (with an action of the group G) one considers a certain element of the ring $R(G)$ of (virtual) representations of the group G . An important invariant of a germ of an analytic function on a germ of an analytic space (in general, singular) is the classical monodromy zeta function of the germ.

Let $f : (V, 0) \rightarrow (\mathbb{C}, 0)$ be a germ of a complex analytic function on a germ $(V, 0) \subset (\mathbb{C}^N, 0)$ of an analytic space (singular, in general). For a small enough

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$\delta > 0$ the restriction of the function f to the intersection $V \cap B_\delta(0)$ of the space V with the (closed) ball $B_\delta(0)$ of radius δ with the centre at the origin in \mathbb{C}^N is a locally trivial fibration over a punctured neighbourhood of zero in \mathbb{C} (*the Milnor fibration* of the germ f ; see [7]). The fibre $M_f = f^{-1}(\varepsilon) \cap B_\delta(0)$ of this fibration ($0 < |\varepsilon| \ll \delta$ are small enough) is called *the Milnor fibre* of the germ f . A monodromy transformation $\Gamma_f : M_f \rightarrow M_f$ of the Milnor fibration of the germ f corresponding to the loop $\varepsilon \cdot \exp(2\pi i\tau)$ ($\tau \in [0, 1]$) is called a (*classical*) *monodromy transformation* of the germ f . (It is well defined up to isotopy.) The action of a monodromy transformation in the (co)homology groups of the Milnor fibre is called *the (classical) monodromy operator* of the germ f . The zeta function of a transformation $\varphi : X \rightarrow X$ (of a good enough space X , say, of a union of cells in a finite CW-complex or of a real semi-analytic space) is the rational function

$$\zeta_\varphi(t) := \prod_{q \geq 0} \left\{ \det \left(id - t\varphi^*_{|H_c^q(X; \mathbb{R})} \right) \right\}^{(-1)^q},$$

where φ^* is the action of the transformation φ in the cohomology groups $H_c^q(X; \mathbb{R})$ with compact support. The degree of the zeta function $\zeta_\varphi(t)$ (the degree of the numerator minus the degree of the denominator) is equal to the Euler characteristic $\chi(X)$ of the space X (also defined through the cohomologies with compact support). The classical monodromy zeta function $\zeta_f(t) := \zeta_{\Gamma_f}(t)$ of a germ f can be represented in the form $\prod_{m \geq 1} (1 - t^m)^{s_m}$,

where s_m are integers, finitely many of which are different from zero. This follows, for example, from an appropriate version of the A'Campo formula [2].

Remark. The use of the cohomology groups with compact support in the definitions is connected with the fact that in this case the Euler characteristic possesses the property of additivity with respect to a partition into non-intersecting subspaces, and the zeta function of a transformation possesses the corresponding property of multiplicativity, an equivariant version of which is formulated below in Proposition 3.

The zeta function $\zeta_\varphi(t)$ of a transformation $\varphi : X \rightarrow X$ can be expressed through the Lefschetz numbers $L(\varphi^k)$ of the powers φ^k of the transformation φ , $k \geq 1$. Let us define integers r_i , $i = 1, 2, \dots$, by the recurrent equations

$$L(\varphi^m) = \sum_{i|m} r_i. \quad (1)$$

The number r_m “counts” the points $x \in X$, whose φ -orbits have order m (i.e., $\varphi^m(x) = x$, $\varphi^i(x) \neq x$ for $0 < i < m$). Therefore r_m is divisible by m :

$r_m = ms_m$. Then

$$\zeta_\varphi(t) = \prod_{m \geq 1} (1 - t^m)^{s_m}.$$

(In this equation for the zeta function of an arbitrary transformation, in contrast to the formula for the zeta function of a classical monodromy transformation, one may have infinitely many exponents s_m different from zero.

Assume that, on the germ $(V, 0)$, one has an (analytic) action of a finite group G and that the germ f is invariant with respect to this action: $f(gx) = f(x)$ for any $g \in G$. The Milnor fibre M_f of the germ f is a G -space, i.e. a space with an action of the group G , the cohomology groups $H_c^q(M_f)$ are spaces of G -representations (G -modules). Let us denote by $[H^q(M_f)]$ the class of the G -module $H^q(M_f)$ in the ring $R(G)$ of (virtual) representations of the group G . The equivariant Euler characteristic of the Milnor fibre M_f of the germ f is usually defined as $\sum_q (-1)^q [H^q(M_f)] \in R(G)$ (see, e.g., [10]). In this

setting a natural generalization of the notion of the monodromy zeta function of a germ to the equivariant case is absent. We offer a possible generalization and describe some of its properties.

First, let us somewhat modify the definition of the (equivariant) Euler characteristic. For convenience we will consider it on the category of real semi-analytic spaces. (The Milnor fibre, of course, is such a space.) The Grothendieck group $K_0(\Omega)$ of an algebra Ω of spaces or of sets (possibly with additional structures) is the Abelian group generated by the isomorphism classes $[X]$ of the spaces or of the sets from Ω modulo the relations $[X] = [Y] + [X \setminus Y]$ for $Y \subset X$. Usually the Grothendieck group of an algebra of spaces or of sets is a ring with the multiplication defined by the Cartesian product. The Euler characteristic may be regarded as a homomorphism from the Grothendieck ring of real semi-analytic spaces (or of sets which are unions of cells in finite CW-complexes) into the ring \mathbb{Z} of integers. In its turn, the ring \mathbb{Z} in a natural way can be identified with the Grothendieck ring $K_0(\text{f.s.})$ of finite sets.

In a similar way the equivariant Euler characteristic can be defined as a homomorphism from the Grothendieck ring of real semi-analytic spaces with actions of a (finite) group G to the Grothendieck ring $K_0(\text{f.}G - \text{s.})$ of finite G -sets, called also the Burnside ring of the group G : see, e.g., [9], [8]. (A finite G set is a finite set with an action of the group G . The Grothendieck ring $K_0(\text{f.}G - \text{s.})$, as a free Abelian group, is generated by irreducible finite G sets. Such G -sets are in one-to-one correspondence with the conjugacy classes $\mathfrak{H} \in \text{consub}(G)$ of subgroups of the group G : to a subgroup $H \subset G$ ($H \in \mathfrak{H}$) there corresponds the class of the G -set G/H .) If X is a real semi-analytic

space with an action of a finite group G , then there exists a representation of X as a union of cells in a finite CW-complex (and even of simplices in a finite simplicial complex) such that the action of the elements of the group G is coordinated with the partition of X into cells and, moreover, if $g\sigma = \sigma$ for a cell σ , then $g|_\sigma = id$. The set C_n of cells of dimension n is a finite G -set. The equivariant Euler characteristic $\chi_G(X)$ of the G -space X can be defined as the alternating sum $\sum_{n \geq 0} (-1)^n [C_n] \in K_0(f.G - s.)$ of the classes of the G -sets C_n . An equivalent definition of the equivariant Euler characteristic $\chi_G(X)$ (which, in particular, shows that it is well defined) is the following one. For a conjugacy class $\mathfrak{H} \in \text{consub}(G)$ of subgroups of the group G , let $X^{(\mathfrak{H})} := \{x \in X : G_x \in \mathfrak{H}\}$, where $G_x := \{g \in G : gx = x\}$ is the isotropy subgroup of the point x . Then

$$\chi_G(X) = \sum_{\mathfrak{H}} \chi(X^{(\mathfrak{H})}/G)[G/H] = \sum_{\mathfrak{H}} (\chi(X^{(\mathfrak{H})})|H|/|G|)[G/H], \quad (2)$$

where the summation is over all the classes $\mathfrak{H} \in \text{consub}(G)$, H is a representative of the conjugacy class \mathfrak{H} .

Let $K_0(\text{perm.})$ be the Grothendieck ring of permutations on finite sets. It can be regarded as the Grothendieck ring of finite \mathbb{Z} -sets. As an Abelian group, $K_0(\text{perm.})$ is the free Abelian group generated by the cyclic permutations. For a permutation, as a map from a finite set (zero-dimensional topological space) into itself, there is defined its zeta-function. The zeta function of the cyclic permutation on m elements is equal to $(1 - t^m)$. To a virtual permutation $(A_1, \sigma_1) - (A_2, \sigma_2)$, let us associate its zeta function defined as the ratio $\zeta_{\sigma_1}(t)/\zeta_{\sigma_2}(t)$ of the zeta functions of the permutations σ_i . It is not difficult to see that this way one defines a one-to-one map from the Grothendieck ring $K_0(\text{perm.})$ onto the set of rational functions of the form $\prod (1 - t^m)^{s_m}$ with a finite number of the exponents s_m different from zero. (This defines a structure of a ring on the set of the latter ones. Moreover the ‘‘sum’’ of two functions of the indicated form is their product. This should be kept in mind for understanding why Proposition 3 is mentioned above as an equivariant analogue of the multiplicativity property of the zeta function of a map.) Thus the zeta function of a function germ can be regarded as an element of the Grothendieck ring of permutations $K_0(\text{perm.})$. (The zeta function of a germ as an element of the ring $K_0(\text{perm.})$ can be defined as the virtual permutation whose ‘‘usual’’ zeta function coincides with the ‘‘usual’’ zeta function of the germ. Another definition: this is the virtual permutation, Lefschetz numbers of the powers of which coincide with the Lefschetz numbers of the powers of the classical monodromy transformation.) In a similar way the zeta function of a transformation $\varphi : X \rightarrow X$ can be regarded as an element of the Grothendieck ring

$K_0(\text{l.f.perm.})$ of “locally finite” permutations. (The details can be found below in the equivariant setting.)

An equivariant Lefschetz number $L^G(\varphi)$ of a (G -equivariant) map of a G -space X into itself was defined in [8] as an element of the Grothendieck ring $K_0(\text{f.}G - \text{s.})$. It is defined for maps of finite G -CW-complexes and also of the complements to G -subcomplexes in finite G -CW-complexes. If X is a closed G -manifold then $L^G(\varphi)$ can be defined as a (finite) G -set of fixed points of a generic G -equivariant perturbation of the map φ whose points are taken with appropriate signs. If X is a compact G -manifold with boundary, one has the same statement if one requires that the perturbation under consideration has no fixed points on the boundary ∂X . then $L^G(\varphi)$ can be defined as a (finite) G -set of fixed points of a generic G -equivariant perturbation of the map φ whose points are taken with appropriate signs. (In [6], one considers another equivariant analogue of the Lefschetz number adapted to the problem solved there.)

A G -permutation on a G -set A (finite or infinite one) is a one-to-one G -equivariant map σ of the set A into itself or, what is the same, a structure of a $(\mathbb{Z} \times G)$ -set on A . A G -permutation σ is *finite*, if the corresponding G -set A is finite. We shall say that a G -permutation σ is *locally finite*, if the \mathbb{Z} -orbit of each point is finite and, for any natural M , the set of points whose \mathbb{Z} -orbits have order not exceeding M is finite. (A power of a locally finite permutation is locally finite as well.) The sets of isomorphism classes of finite and of locally finite G -permutations are semirings with respect to the usual operations: the disjoint union and the Cartesian product. The set of irreducible (i.e. not representable as the disjoint union of nontrivial elements) G -permutations is in one-to-one correspondence with the set $\text{consubf}(\mathbb{Z} \times G)$ of the conjugacy classes of subgroups of finite index in the group $\mathbb{Z} \times G$. A subgroup $\hat{H} \subset \mathbb{Z} \times G$ of finite index can be described by a triple (H, m, α) , where H is a subgroup of the group G , m is a natural number, α is an element of the group C_H/H , where C_H is the centralizer of the subgroup H : $C_H = \{a \in G : a^{-1}Ha = H\}$. A triple (H, m, α) corresponds to the subgroup $\hat{H}(H, m, \alpha)$ of the group $\mathbb{Z} \times G$ generated by $\{0\} \times H$ and (m, α) . For $g \in C_H$, we permit ourselves to write $\hat{H}(H, m, g)$ instead of $\hat{H}(H, m, [g])$, where $[g]$ is the class of the element g in C_H/H . Let us denote the conjugacy class of the subgroup $\hat{H}(H, m, \alpha)$ in $\mathbb{Z} \times G$ by $\hat{\mathfrak{H}}(H, m, \alpha)$. The triples (H, m, α) and (H', m', α') are equivalent (i.e. the corresponding subgroups in $\mathbb{Z} \times G$ are conjugate to each other) if $m' = m$ and there exists $g \in G$ such that $H' = g^{-1}Hg$, $\alpha' = g^{-1}\alpha g$. Let $K_0(\text{f.}G - \text{perm.})$ and $K_0(\text{l.f.}G - \text{perm.})$ be the Grothendieck rings of finite and of locally finite G -permutations respectively. As a group, $K_0(\text{f.}G - \text{perm.})$ is the free Abelian

group generated by the isomorphism classes of the irreducible G -permutations;

$$K_0(\text{l.f.}G - \text{perm.}) = \left\{ \sum_{\hat{\mathfrak{H}} \in \text{consubf}(\mathbb{Z} \times G)} k_{\hat{\mathfrak{H}}}[\mathbb{Z} \times G/\hat{H}] \right\},$$

where, in general, one may have infinitely many non-zero coefficients $k_{\hat{\mathfrak{H}}}$ (\hat{H} is a representative of the class $\hat{\mathfrak{H}}$).

For a locally finite G -permutation σ regarded as a map of a discrete topological space into itself, there is defined the equivariant Lefschetz number $L^G(\sigma) \in K_0(\text{l.f.}G - \text{s.})$ in the sense of [8]. This is simply the set of fixed points of the permutation σ (i.e. of the action of the element $(1, e) \in \mathbb{Z} \times G$), regarded as a G -set. For a virtual G -permutation $[\sigma_1] - [\sigma_2] \in K_0(\text{l.f.}G - \text{perm.})$, it should be defined as $L^G(\sigma_1) - L^G(\sigma_2)$. (For $g \in G$, the map $g\sigma$ (i.e. the action of the element $(1, g) \in \mathbb{Z} \times G$) is also a locally finite G -permutation and therefore the equivariant Lefschetz number $L^G(g\sigma)$ is defined.)

Let $\varphi : X \rightarrow X$ be a G -equivariant map of a G -space X (“good” in the sense described above) into itself.

Definition: The equivariant zeta function ζ_φ^G of the map φ is a virtual G -permutation $[\sigma] \in K_0(\text{l.f.}G - \text{perm.})$ such that $L^G(g\sigma^m) = L^G(g\varphi^m)$ for all $m \geq 1$ and $g \in G$.

Proposition 1 *The equivariant zeta function of a G -equivariant map φ is well defined (i.e. the corresponding (virtual) G -permutation $[\sigma]$ exists and is unique).*

Proof. Assume that

$$\zeta_\varphi^G = \sum_{\hat{\mathfrak{H}} \in \text{consubf}(\mathbb{Z} \times G)} k_{\hat{\mathfrak{H}}}[(\mathbb{Z} \times G)/\hat{H}].$$

Then, for $g \in C_H$, the coefficient $\ell_h(m, g)$ at the generator $[G/H]$ in $L^G(g\varphi^m)$ is equal to

$$\sum_{\hat{\mathfrak{H}} \subset \hat{\mathfrak{H}}(H, m, g)} k_{\hat{\mathfrak{H}}}.$$

The system

$$\ell_{\hat{\mathfrak{H}}}(m, g) = \sum_{\hat{\mathfrak{H}} \subset \hat{\mathfrak{H}}(H, m, g)} k_{\hat{\mathfrak{H}}}$$

is triangular (with respect to the natural ordering of the triples $(\hat{\mathfrak{H}}, m, \alpha)$) system of equations, which permits to find the coefficients $k_{\hat{\mathfrak{H}}}$ recurrently. \square

The following properties of the equivariant zeta function are easy corollaries from the definition.

Proposition 2 *If G -maps $\varphi_i : X \rightarrow X$, $i = 1, 2$, are homotopic in the class of G -maps, then $\zeta_{\varphi_1}^G = \zeta_{\varphi_2}^G$.*

Proposition 3 *Suppose that a G -map $\varphi : X \rightarrow X$ sends to itself a G -invariant subspace $Y \subset X$ and its complement $X \setminus Y$. Then $\zeta_{\varphi}^G = \zeta_{\varphi|_Y}^G + \zeta_{\varphi|_{X \setminus Y}}^G$.*

Proposition 4 *For G -maps $\varphi_i : X_i \rightarrow X_i$, $i = 1, 2$, one has the equation $\zeta_{\varphi_1 \times \varphi_2}^G = \zeta_{\varphi_1}^G \cdot \zeta_{\varphi_2}^G$. (Here $\varphi_1 \times \varphi_2$ is the map of the space $X_1 \times X_2$ into itself sending $(x_1, x_2) \in X_1 \times X_2$ to $(\varphi(x_1), \varphi(x_2))$.)*

Remark. Propositions 3 and 4 permit to define the equivariant monodromy zeta function as a homomorphism from the Grothendieck ring of germs of constructible G -invariant functions on germs of constructible G -sets into the Grothendieck ring of finite $(\mathbb{Z} \times G)$ -sets.

Proposition 5 *Let $X = X^{(5)}$, i.e. the isotropy groups of all points of X are conjugate to a subgroup $H \in \mathfrak{H}$, and let $\varphi : X \rightarrow X$ be a map such that $g\varphi^m(x) \neq x$ for all $x \in X$, $g \in G$, $m = 1, \dots, m_0 - 1$ (i.e., for $1 \leq m < m_0$, the map φ^m does not send any G -orbit into itself); $g_0\varphi^{m_0} = id$. (Notice that in this case g_0 lies in the centralizer C_H of the subgroup H .) Then*

$$\zeta_{\varphi}^G = \frac{\chi(X/G)}{m_0} [(\mathbb{Z} \times G)/\hat{H}(H, m_0, g_0)] = \chi(X/(\mathbb{Z} \times G)) [(\mathbb{Z} \times G)/\hat{H}(H, m_0, g_0)]$$

(in this case the factor $X/(\mathbb{Z} \times G)$ is defined.).

Let us formulate equivariant analogues of some well-known statements about the monodromy and its zeta function.

Let $f_1 : (V_1, 0) \rightarrow (\mathbb{C}, 0)$ and $f_2 : (V_2, 0) \rightarrow (\mathbb{C}, 0)$ be two germs of G -invariant complex analytic functions. The *Sebastiani–Thom sum* of the germs f_1 and f_2 is the germ $f_1 \oplus f_2 : (V_1 \times V_2, 0) \rightarrow (\mathbb{C}, 0)$ given by the equation $(f_1 \oplus f_2)(x, y) = f_1(x) + f_2(y)$. The germ $f_1 \oplus f_2$ is invariant with respect to the natural (diagonal) G -action on $(V_1 \times V_2, 0)$. The following statement is an equivariant analogue of the Sebastiani–Thom theorem which says that the monodromy operator of the Sebastiani–Thom sum of function germs on the affine spaces $(\mathbb{C}^n, 0)$ with isolated critical points at the origin is equal to the tensor product of the monodromy operators of the germs (or rather of its corollary for the monodromy zeta function).

Theorem 1 *(the equivariant Sebastiani–Thom theorem)*

$$\zeta_{f_1 \oplus f_2}^G = \zeta_{f_1}^G + \zeta_{f_2}^G - \zeta_{f_1}^G \cdot \zeta_{f_2}^G.$$

Proof. The statement follows from the fact that the Milnor fibre $V_{f_1 \oplus f_2}$ of the germ $f_1 \oplus f_2$ can be G -equivariantly retracted (by a deformation) onto the join $V_{f_1} \star V_{f_2}$ of the Milnor fibres of the germs f_1 and f_2 , see, e.g. [1, section 2.7]. Moreover, the monodromy transformation on $V_{f_1} \star V_{f_2}$ is the join of the corresponding transformations on V_{f_1} and on V_{f_2} . Finally, the join $V_{f_1} \star V_{f_2}$ can be represented as the (disjoint) union of the G -invariant subspaces V_{f_1}, V_{f_2} and $(V_{f_1} \star V_{f_2}) \setminus (V_{f_1} \cup V_{f_2}) \simeq V_{f_1} \times V_{f_2} \times (0, 1)$ (with the trivial action of G on $(0, 1)$), to which one can apply Proposition 3. \square

Let $(V, 0)$ be a germ of a complex analytic G -space of pure dimension n and let $f : (V, 0) \rightarrow (\mathbb{C}, 0)$ be a G -invariant germ of a complex analytic function such that the set V_{sing} of the singularities of the space V is contained in the zero-level set $\{f = 0\}$ of the germ f (for example, V has an isolated singularity at the origin). Let $\pi : (X, D) \rightarrow (V, 0)$ ($D = \pi^{-1}(0)$) be a G -resolution of the singularity f which is an isomorphism outside of $\{f = 0\}$. This means that:

- 1) X is a (non-singular) complex analytic manifold of dimension n with an action of the group G ;
- 2) π is a proper analytic map commuting with the action of the group G ;
- 3) the total transform $\pi^{-1}(\{f = 0\})$ of the zero-level set of the germ f is a normal crossing divisor on X .

The last condition is equivalent to the fact that, in a neighbourhood of any point $x \in \pi^{-1}(\{f = 0\})$, there exists a local coordinate system u_1, \dots, u_n on X centred at the point x , such that the lifting $\tilde{f} := f \circ \pi$ of the germ f to the space of resolution X has the form $\tilde{f}(u_1, \dots, u_n) = u_1^{m_1} \cdot \dots \cdot u_n^{m_n}$, where m_i are non-negative integers not all equal to zero. We also will assume that *the exceptional divisor* $D = \pi^{-1}(0)$ is a normal crossing divisor on X as well and that for any point $x \in D$ all the components E_i of the exceptional divisor D passing through the point x are invariant with respect to the isotropy subgroup G_x of the point x .

Remark. The last condition is not too essential and is required here for convenience of reasonings. One can always achieve this by additional blow-ups along pairwise intersections of the components E_i . There appear new components whose non-singular parts (in the exceptional divisor) have zero Euler characteristics. It is not difficult to see that they don't contribute to the right hand side of the equation (3) below.

Let $D = \bigcup_{i=1}^s E_i$ be the decomposition of the exceptional divisor D into the irreducible components and let $\overset{\circ}{E}_i$ be the nonsingular part of the component E_i in the total transform $\pi^{-1}(\{f = 0\})$ of the zero-level set, i.e. the component

E_i itself minus its intersections with all other components of $\pi^{-1}(\{f = 0\})$. Let $\mathring{D} = \bigcup_{i=1}^s \mathring{E}_i$ be the nonsingular part of the exceptional divisor D in $\pi^{-1}(\{f = 0\})$, let $\hat{D} = \mathring{D}/G$ be the corresponding factor space, i.e. the space of orbits of the action of the group G on \mathring{D} , and let $j : \mathring{D} \rightarrow \hat{D}$ be the factorization map.

Let $\{\Xi\}$ be a stratification of the space \hat{D} such that:

- 1) each stratum Ξ is a connected complex analytic manifold and therefore it is contained in the image of one of the components \mathring{E}_i under the action of the map j ;
- 2) over each stratum Ξ the map j is a covering.

As above, for a point $x \in X$, let G_x be its isotropy subgroup. The property 2) implies that the isotropy subgroups of all the points of a connected component of the preimage $j^{-1}(\Xi)$ of a stratum Ξ coincide and the isotropy subgroups of all the points of the preimage $j^{-1}(\Xi)$ are conjugate to each other in G . Let us denote the corresponding conjugacy class by \mathfrak{H}_Ξ . For a stratum Ξ , let us take a point $x \in j^{-1}(\Xi)$. Let $m_x = m_\Xi$ be the multiplicity of zero of the lifting $\tilde{f} = f \circ \pi$ of the function f to the space X of the resolution along the component of \mathring{D} containing the point x . (It is also the multiplicity of the function \tilde{f} at the point x .) The isotropy subgroup G_x of the point x acts on the fibre over the point x of the (one-dimensional) normal fibration to the non-singular part \mathring{D} of the exceptional divisor. Let H_x be the subgroup of the group G_x consisting of the elements whose action on the fibre is trivial. Notice that $G_x \subset C_{H_x}$, the factor G_x/H_x is a cyclic group whose order $n_x = |G_x/H_x| =: n_\Xi$ divides the multiplicity m_x . The action (a representation) of the group G_x/H_x on the fibre defines a homomorphism (even a monomorphism) of the group G_x/H_x to $\mathbb{C}^* = \text{GL}(1, \mathbb{C})$. Let α_x^{-1} be the element of this group which maps to $\exp(2\pi i/n_x)$. Let $\hat{H}_x = \hat{H}(H_x, \frac{m_x}{n_x}, \alpha_x)$. It is not difficult to see that the conjugacy class $\hat{\mathfrak{H}}(H_x, \frac{m_x}{n_x}, \alpha_x)$ of the subgroup $\hat{H}(H_x, \frac{m_x}{n_x}, \alpha_x)$ in $\mathbb{Z} \times G$ does not depend on the choice of a point $x \in j^{-1}(\Xi)$. A representative of this class we shall denote by \hat{H}_Ξ .

Theorem 2 (*the equivariant A'Campo formula*)

$$\zeta_f^G = \sum_{\Xi} \chi(\Xi) \cdot [(\mathbb{Z} \times G)/\hat{H}_\Xi]. \quad (3)$$

Proof. Since the resolution π is an isomorphism outside of $f^{-1}(0)$, the map π defines an identification of the Milnor fibre M_f with its preimage $\pi^{-1}(M_f) \subset X$. In the same way as in the usual (non-equivariant) situation (see [4, 2]) one can construct a G -equivariant deformation retraction of a neighbourhood of the total transform $\pi^{-1}(f^{-1}(0)) = \tilde{f}^{-1}(0)$ of the zero-level set $f^{-1}(0)$ onto the total transform $\tilde{f}^{-1}(0)$ itself such that, outside of a neighbourhood of the union of all the intersections of the components of $\tilde{f}^{-1}(0)$, i.e. where in local coordinates $\tilde{f}(u_1, \dots, u_n) = u_1^m$, the retraction sends the point (u_1, u_2, \dots, u_n) to the point $(0, u_2, \dots, u_n)$ (in some coordinates). One can also construct a G -equivariant monodromy transformation $\tilde{\Gamma}$ on $\pi^{-1}(M_f)$ commuting with the retraction and such that, in a neighbourhood of a point, where $\tilde{f}(u_1, \dots, u_m) = u_1^m$, it sends $(u_1, u_2, \dots, u_n) \in \pi^{-1}(M_f)$ to $(\exp(2\pi i/m)u_1, u_2, \dots, u_n)$. From Proposition 3, it follows that the equivariant zeta function $\zeta_f^G = \zeta_{\tilde{\Gamma}}^G$ is equal to the sum of the equivariant zeta functions of the monodromy transformation on the preimages of the strata Ξ and on neighbourhoods of the intersections of the components of the total transform $\tilde{f}^{-1}(0)$.

It is not difficult to show that the equivariant zeta function of the monodromy transformation on a neighbourhood of an intersection of the components of $\tilde{f}^{-1}(0)$ is equal to zero. This can be deduced from the fact that, in such a neighbourhood, the Milnor fibre can be represented as a fibration by circles coordinated with the group action and the monodromy transformation can be constructed such that it preserves this fibration.

The equivariant zeta function of the monodromy transformation $\tilde{\Gamma}$ on the preimage of a stratum Ξ is given by Proposition 5. Indeed, the intersection of the Milnor fibre $\pi^{-1}(M_f)$ with the normal fibre to \mathring{D} at a point from $\pi^{-1}(\Xi)$ consists of m_{Ξ} points permuted by the monodromy transformation $\tilde{\Gamma}$ cyclically. The intersections with the orbits of the group G divide these points into n_{Ξ} groups with m_{Ξ}/n_{Ξ} points in each. (Here, of course, the term ‘‘group’’ does not have the same meaning as at other places, but, to my surprise, I did not find an appropriate synonym.) The monodromy transformation $\tilde{\Gamma}$ permutes these groups cyclically. Therefore, for $m = 1, \dots, (\frac{m_{\Xi}}{n_{\Xi}} - 1)$, the m -th power of the monodromy transformation does not preserve any G -orbit. In its turn the $\frac{m_{\Xi}}{n_{\Xi}}$ -th power of the monodromy transformation sends each point y from the normal fibre at the point x to the point $a^{-1}y$, where $a \in \alpha_x$. Therefore the transformation $a \circ \tilde{\Gamma}^{\frac{m_{\Xi}}{n_{\Xi}}}$ acts on the intersections of the Milnor fibre $\pi^{-1}(M_f)$ with the normal fibres to \mathring{D} at all the point of $\pi^{-1}(\Xi)$ identically, what is required in Proposition 5. Therefore the equivariant zeta function under consideration is equal to $\chi(\Xi) \cdot [(\mathbb{Z} \times G)/\hat{H}_{\Xi}]$, what proves the statement. \square

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