

EXISTENCE OF COMPATIBLE CONTACT STRUCTURES ON G_2 -MANIFOLDS

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ABSTRACT. In this paper, we show the existence of (co-oriented) contact structures on certain classes of G_2 -manifolds, and that these two structures are compatible in certain ways. Moreover, we prove that any seven-manifold with a spin structure (and so any manifold with G_2 -structure) admits an almost contact structure. We also construct explicit almost contact metric structures on manifolds with G_2 -structures.

1. INTRODUCTION

Let (M, g) be a Riemannian 7-manifold whose holonomy group $Hol(g)$ is the exceptional Lie group G_2 (or, more generally, a subgroup of G_2). Then M is naturally equipped with a covariantly constant 3-form φ and 4-form $*\varphi$. We can define (M, φ, g) as the G_2 -manifold with G_2 structure φ .

We can also define a (co-oriented) contact manifold as a pair (N, ξ) where N is an odd dimensional manifold and ξ , called a (co-oriented) contact structure, is a totally non-integrable (co-oriented) hyperplane distribution on N .

In dimension 7, so far contact geometry and G_2 geometry have been studied independently and each geometry has very distinguished characteristics which are rather different than those in the other. A basic example of such differences is the following: In contact geometry there are no local invariants, in other words, every contact 7-manifold is locally contactomorphic to \mathbb{R}^7 equipped with the standard contact structure. On the other hand, in G_2 geometry it is the G_2 structure itself that determines how local neighborhoods of points look like, and as a result, manifolds with G_2 structures can look the same only at a point, [7], [9].

The aim of this paper is to initiate a new interdisciplinary research area between contact and G_2 geometries. More precisely, we study the existence of (almost) contact structures on 7-dimensional manifolds with (torsion free) G_2 -structures.

2000 *Mathematics Subject Classification.* 53C38,53D10,53D15,57R17.

Key words and phrases. (Almost) contact structures, G_2 structures.

The first named author is partially supported by NSF FRG grant DMS-1065910.

The third named author is partially supported by NSF grant DMS-1105663.

The paper is organized as follows: After the preliminaries (Section 2), we show the existence of almost contact structures on 7-manifolds with spin structures in Section 3. In particular, we prove the following theorem:

Theorem: *Every manifold with G_2 -structure admits an almost contact structure.*

In Section 4, we define A - and B -compatibility between contact and G_2 structures, and also present the motivating example for \mathbb{R}^7 . We also prove the nonexistence result:

Theorem: *Let (M, φ) be a manifold with G_2 -structure such that $d\varphi = 0$. If M is closed (i.e., compact and $\partial M = \emptyset$), then there is no contact structure on M which is A -compatible with φ .*

In Section 5, for any non-vanishing vector field R on a manifold M with G_2 -structure φ , we construct explicit almost contact structure, denoted by $(J_R, R, \alpha_R, g_\varphi)$, and indeed prove the following theorems:

Theorem: *Let (M, φ) be a manifold with G_2 -structure. Then the quadruple $(J_R, R, \alpha_R, g_\varphi)$ defines an almost contact metric structure on M for any non-vanishing vector field R on M . Moreover, such a structure exists on any manifold with G_2 -structure.*

Theorem: *Let (M, φ) be a manifold with G_2 -structure. Suppose that ξ is a contact structure on M such that $(J_R, R, \alpha_R, g_\varphi)$ is an associated almost contact metric structure for ξ . Then ξ is A -compatible.*

In Section 6, we define contact- G_2 -structures on 7-manifolds and analyze their relations with A -compatible contact structures, the main results of that section are:

Theorem: *Let (M, φ) be a manifold with G_2 -structure. Assume that there are nowhere-zero vector fields X, Y and Z on M satisfying $\iota_Z\varphi = Y^\flat \wedge X^\flat$ where X^\flat (resp. Y^\flat) is the covariant 1-form of X (resp. Y) with respect to the G_2 -metric g_φ . Also suppose that $d(i_X i_Y \varphi) = i_X i_Y * \varphi$. Then the 1-form $\alpha := Z^\flat = g_\varphi(Z, \cdot)$ is a contact form on M and it defines an A -compatible contact structure $\text{Ker}(\alpha)$ on (M, φ) .*

Theorem: *Let $(\varphi, R, \alpha, f, g)$ be a contact- G_2 -structure on a smooth manifold M^7 . Then α is a contact form on M . Moreover, $\xi = \text{Ker}(\alpha)$ is an A -compatible contact structure on (M, φ) . In particular, if M is closed, then it does not admit a contact- G_2 -structure with $d\varphi = 0$.*

Theorem: *Let (M, φ) be any manifold with G_2 -structure. Then every A -compatible contact structure on (M, φ) determines a contact- G_2 -structure on M .*

Finally, in Section 7, we present some examples of A -compatible structures and contact- G_2 -structures.

2. PRELIMINARIES

2.1. G_2 -structures and G_2 -manifolds. A smooth 7-dimensional manifold M has a G_2 -structure, if the structure group of TM can be reduced to G_2 . The group G_2 is one of the five exceptional Lie groups which is the group of all linear automorphisms of the imaginary octonions $im\mathbb{O} \cong \mathbb{R}^7$ preserving a certain cross product. Equivalently, it can be defined as the subgroup of $GL(7, \mathbb{R})$ which preserves the 3-form

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

where (x_1, \dots, x_7) are the coordinates on \mathbb{R}^7 , and $e^{ijk} = dx^i \wedge dx^j \wedge dx^k$. As an equivalent definition, a manifold with a G_2 -structure φ is a pair (M, φ) , where φ is a 3-form on M , such that $(T_p M, \varphi)$ is isomorphic to $(\mathbb{R}^7, \varphi_0)$ at every point p in M . Such a φ defines a Riemannian metric g_φ on M . We say φ is *torsion-free* if $\nabla\varphi = 0$ where ∇ is the Levi-Civita connection of g_φ . A Riemannian manifold with a torsion free G_2 -structure is called a G_2 -manifold. Equivalently, the pair (M, φ) is called a G_2 -manifold if its holonomy group (with respect to g_φ) is a subgroup of G_2 . As another characterization, one can show that φ is torsion-free if and only if $d\varphi = d(*\varphi) = 0$ where “ $*$ ” is the Hodge star operator defined by the metric g_φ .

The 3-form φ also determines the cross product and the orientation top (volume) form Vol on M . In fact, for any vector fields u, v, w on M , we have

$$(1) \quad \varphi(u, v, w) = g_\varphi(u \times v, w),$$

$$(2) \quad (\iota_u \varphi) \wedge (\iota_v \varphi) \wedge \varphi = 6g_\varphi(u, v) \text{Vol}.$$

Also we will make use of the following formula as well:

$$(3) \quad u \times (u \times v) = -\|u\|^2 v + g_\varphi(u, v)u.$$

See [2], [3], [9] and [10] for more details on G_2 geometry.

2.2. Contact and almost contact structures. A *contact structure* on a smooth $(2n+1)$ -dimensional manifold M is a global $2n$ -plane field distribution ξ which is totally non-integrable. Non-integrability condition is equivalent to the fact that locally ξ can be given as the kernel of a 1-form α such that $\alpha \wedge (d\alpha)^n \neq 0$. If α is globally defined (in such a case, it is called a *contact form*), then one can define the *Reeb vector field* of α to be the unique global nowhere-zero vector field R on M satisfying the equations

$$(4) \quad \iota_R d\alpha = 0, \quad \alpha(R) = 1$$

where “ ι ” denotes the interior product.

Using R , we can co-orient ξ and, as a result, the structure group of the tangent frame bundle can be reduced to $U(n) \times 1$. Such a reduction of the structure group is called an *almost contact structure* on M . Therefore, for the existence of a co-oriented contact structure on M , one should first ask the existence of an almost contact structure. We refer the reader to [1] and [7] for more on contact geometry.

Definition 2.1 ([8]). Let M^{2n+1} be a smooth manifold. If the structure group of its tangent bundles TM^{2n+1} reduces to $U(n) \times 1$, then M^{2n+1} is said to have an *almost contact structure*.

3. ALMOST CONTACT STRUCTURES ON 7-MANIFOLDS WITH A SPIN STRUCTURE

Although no explicit description is given, nevertheless the following result shows the existence of almost contact structures not only on manifolds with G_2 -structures but also on a much wider family of 7-manifolds. Recall that if a manifold admits a spin structure, then its second Stiefel-Whitney class is zero.

Theorem 3.1. *Every 7-manifold with a spin structure admits an almost contact structure.*

Proof. Assume that M is a 7-manifold with spin structure. By definition, M admits an almost contact structure if and only if the structure group of TM can be reduced to $U(3) \times 1$. Equivalently, the associated fiber bundle $TM[SO(7)/U(3)]$ with fiber $SO(7)/U(3)$ admits a cross-section [13]. If s is a cross section of fiber bundle over the the $(i - 1)$ -skeleton of M , then the cohomology class

$$o^i(s) \in H^i(M, \pi_{i-1}(SO(7)/U(3))).$$

is the obstruction to extending s over the i -skeleton. Since we have

$$\pi_i(SO(7)/U(3)) = 0$$

unless $i = 2, 6$, the only obstructions to the existence of such a cross section arise in $H^i(M, \mathbb{Z})$ for $i = 3, 7$. In [11], Massey shows that these obstructions are the integral Stiefel-Whitney classes of the associated dimensions. Recall that the integral Stiefel-Whitney classes are defined as the images $\beta(w_i)$ of the Stiefel-Whitney classes under the Bockstein homomorphism. Here the Bockstein homomorphism is the connecting homomorphism $\beta : H^i(M, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{i+1}(M, \mathbb{Z})$ which arises from the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Therefore, the obstructions o^3, o^7 to the existence of an almost contact structures on 7-manifolds are 2-torsion classes.

Now we know that $w_2(M) = 0$ (since M is spin), and hence the third integral Stiefel-Whitney class vanishes, i.e., $o^3 = W_3(M) = \beta(w_2) = 0$. Therefore, the only obstruction lies in the cohomology group $H^7(M)$.

We consider the following cases: First, if M is a closed manifold, then by Poincaré duality $H^7(M) \cong H_0(M) \cong \mathbb{Z}$ and hence the top dimensional obstruction o^7 vanishes. Secondly, if M has a boundary, then (again by the duality) we have $o^7 \in H^7(M) \cong H_0(M, \partial M) \cong 0$. Now, if M is non-compact without a boundary, then the cohomology group $H^7(M) \cong (H_{cs}^0(M))^*$ where H_{cs} denotes the compactly supported cohomology. Hence, it is torsion-free. \square

Since every manifold with G_2 -structure is spin, we get

Corollary 3.2. *Every manifold with G_2 -structure admits an almost contact structure.* \square

4. COMPATIBILITY AND THE MOTIVATING EXAMPLE

Assuming the existence of a contact structure on a manifold with a G_2 -structure, we can also ask if and how these two structures are related. We define *two* different notions of *compatibility* between them as follows:

Definition 4.1. A (co-oriented) contact structure ξ on (M, φ) is said to be *A-compatible* with the G_2 -structure φ if there exist a vector field R on M and a nonzero function $f : M \rightarrow \mathbb{R}$ such that $d\alpha = \iota_R\varphi$ for some contact form α for ξ and fR is the Reeb vector field of a contact form for ξ .

Definition 4.2. A (co-oriented) contact structure ξ on (M, φ) is said to be *B-compatible* with the G_2 -structure φ if there are (global) vector fields X, Y on M such that $\alpha = \iota_Y\iota_X\varphi$ is a contact form for ξ .

In this paper, we will mainly consider *A-compatible* contact structures. We remark that if φ is torsion-free or at least $d\varphi = 0$, then Definition 4.1 makes sense only if M is noncompact or compact with boundary. Indeed, we can easily prove the following:

Theorem 4.3. *Let (M, φ) be a manifold with G_2 -structure such that $d\varphi = 0$. If M is closed (i.e., compact and $\partial M = \emptyset$), then there is no contact structure on M which is *A-compatible* with φ .*

Proof. Suppose ξ is an *A-compatible* contact structure on (M, φ) . Therefore, $d\alpha = \iota_R\varphi$ for some contact form α for ξ and some nonvanishing vector field R . Using the equation (2), we have

$$d\alpha \wedge d\alpha \wedge \varphi = (\iota_R\varphi) \wedge (\iota_R\varphi) \wedge \varphi = 6\|R\|^2 \text{Vol.}$$

Since $d\varphi = 0$, we have $d\alpha \wedge d\alpha \wedge \varphi = d(\alpha \wedge d\alpha \wedge \varphi)$. Now by Stoke's Theorem,

$$0 \not\leq \int_M 6\|R\|^2 \text{Vol} = \int_M d(\alpha \wedge d\alpha \wedge \varphi) = \int_{\partial M} \alpha \wedge d\alpha \wedge \varphi = 0$$

(as $\partial M = \emptyset$). This gives a contradiction. \square

For another application of this argument on specific vector fields on manifolds with G_2 structures, see [5].

We now explore the relation between the standard contact structure ξ_0 and the standard G_2 -structure φ_0 on \mathbb{R}^7 . Indeed, the notion of A- and B-compatibility relies on this motivating example.

Fix the coordinates $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ on \mathbb{R}^7 . In these coordinates,

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

where e^{ijk} denotes the 3-form $dx_i \wedge dx_j \wedge dx_k$. Consider the standard contact structure ξ_0 on \mathbb{R}^7 as the kernel of the 1-form

$$\alpha_0 = dx_1 - x_3 dx_2 - x_5 dx_4 - x_7 dx_6.$$

For simplicity, through out the paper we will denote $\partial/\partial x_i$ by ∂x_i (so we have $dx_i(\partial x_j) = \delta_{ij}$). Consider the vector fields

$$R = \partial x_1, X = \partial x_7 \text{ and } Y = -x_7 \partial x_1 + x_5 \partial x_3 - x_3 \partial x_5 - \partial x_6 + f \partial x_7$$

where $f : \mathbb{R}^7 \rightarrow \mathbb{R}$ is any smooth function (in fact, it is enough to take $f \equiv 0$ for our purpose). By a straightforward computation, we see that

$$d\alpha_0 = \iota_R(\varphi_0), \quad \alpha_0 = \iota_Y \iota_X(\varphi_0).$$

Also observe that R is the Reeb vector field of α_0 . Note that this contact structure is not unique A-compatible with φ_0 . In fact we have various ways of choosing the contact structures by rotating indexes and signes. For example, the contact structure $\alpha = dx_2 + x_3 dx_1 - x_6 dx_4 + x_7 dx_5$ with $R = \partial x_2$ is another A-compatible contact structure with φ_0 and by choosing two vectors $X = \partial x_7, Y = \partial x_5 - x_3 \partial x_6 + x_6 \partial x_3 - x_7 \partial x_2 + f \partial x_7$ it is easily seen as being B-compatible with φ_0 . Therefore, we have proved:

Theorem 4.4. *There are contact structures ξ on \mathbb{R}^7 which are both A- and B-compatible with the standard G_2 -structure φ_0 . \square*

5. AN EXPLICIT ALMOST CONTACT METRIC STRUCTURE

We first give an alternative definition of an almost contact structure, and then construct an explicit almost contact structure on a manifold with G_2 -structure. The reader is referred to [1] for the equivalence between the previous definition (Definition 2.1) and this new one.

Definition 5.1 ([12]). An *almost contact structure* on a differentiable manifold M^{2n+1} is a triple (J, R, α) consists of a field J of endomorphisms of the tangent spaces, a vector field R , and a 1-form α satisfying

- (i) $\alpha(R) = 1$,
- (ii) $J^2 = -I + \alpha \otimes R$

where I denotes the identity transformation.

For completeness, we provide the proof of the following lemma.

Lemma 5.2 ([12]). *Suppose that (J, R, α) is an almost contact structure on M^{2n+1} . Then $J(R) = 0$ and $\alpha \circ J = 0$*

Proof. Since $J^2(R) = -R + \alpha(R)R = -R + 1 \cdot R = 0$, we have either $J(R) = 0$ or $J(R)$ is nonzero vector field whose image is 0. Suppose $J(R)$ is nonzero vector field which is mapped to 0 by J . Then from

$$0 = J^2(J(R)) = -J(R) + \alpha(J(R)) \cdot R$$

we get $J(R) = \alpha(J(R)) \cdot R$, and so $\alpha(J(R)) \neq 0$ (as $J(R) \neq 0$). But then

$$J^2(R) = J(J(R)) = J(\alpha(J(R))R) = \alpha(J(R)) \cdot J(R) = [\alpha(J(R))]^2 \cdot R \neq 0$$

which contradicts to assumption that $J^2(R) = J(J(R)) = 0$. Hence, we conclude that $J(R) = 0$ must be the case.

Now for any vector X , we see that

$$J^3(X) = J(J^2(X)) = J((-X) + \alpha(X)R) = -J(X) + J(\alpha(X)R)$$

and also we have

$$J^3(X) = J^2(J(X)) = -J(X) + \alpha(J(X))R.$$

So combining these we compute

$$\begin{aligned} \alpha(J(X))R &= J^3(X) + J(X) \\ &= -J(X) + J(\alpha(X)R) + J(X) = J(\alpha(X)R). \end{aligned}$$

But using the fact $J(R) = 0$ we have

$$J(\alpha(X)R) = \alpha(X)J(R) = 0.$$

Therefore, $\alpha(J(X)) = 0$ as $R \neq 0$. Hence, $\alpha \circ J = 0$ for any vector X . □

We can also introduce a Riemannian metric into the picture as suggested in the following definition.

Definition 5.3 ([12]). An *almost contact metric structure* on a differentiable manifold M^{2n+1} is a quadruple (J, R, α, g) where (J, R, α) is an almost contact structure on M and g is a Riemannian metric on M satisfying

$$(5) \quad g(Ju, Jv) = g(u, v) - \alpha(u)\alpha(v)$$

for all vector fields u, v in TM . Such a g is called a *compatible metric*.

Remark 5.4. Every manifold with an almost contact structure admits a compatible metric (see [1], for a proof). Also setting $u = R$ in Equation (5) gives $g(JR, Jv) = g(R, v) - \alpha(R)\alpha(v)$. Since $J(R) = 0$, an immediate consequence is that α is the covariant form of R , that is, $\alpha(v) = g(R, v)$.

Definition 5.5 ([12]). Let M be an odd dimensional manifold, and α be a contact form on M with the Reeb vector field R . Therefore, $d\alpha$ is a symplectic form on the contact structure (or distribution) $\xi = \text{Ker}(\alpha)$. We say that the triple (J, R, α) is an *associated almost contact structure* for ξ if J is $d\alpha$ -compatible almost complex structure on the complex bundle ξ , that is

$$d\alpha(JX, JY) = d\alpha(X, Y) \text{ and } d\alpha(X, JX) > 0 \text{ for all } X, Y \in \xi.$$

Furthermore, if g is a metric on M , we consider two equations :

$$(6) \quad g(JX, JY) = g(X, Y) - \alpha(X)\alpha(Y)$$

$$(7) \quad d\alpha(X, Y) = g(JX, Y)$$

for all $X, Y \in TM$. We say that (J, R, α, g) is an *associated almost contact metric structure* if two equations (6) and (7) hold. In this case, g is called an *associated metric*.

Suppose that (M, φ) is a manifold with G_2 -structure. There might be many ways to construct almost contact metric structures on (M, φ) . Here we give a particular way of constructing almost contact metric structures on (M, φ) . Denote the Riemannian metric and the cross product (determined by φ) by $g_\varphi = \langle \cdot, \cdot \rangle_\varphi$ and \times_φ , respectively. Suppose that R is a nowhere vanishing vector field on M . By normalizing R using g_φ , we may assume that $\|R\| = 1$. Then using the metric, we define the 1-form α_R as the metric dual of R , that is,

$$\alpha_R(u) = g_\varphi(R, u) = \langle R, u \rangle_\varphi.$$

Moreover, using the cross product and R , we can define an endomorphism $J_R : TM \rightarrow TM$ of the tangent spaces by

$$J_R(u) = R \times_\varphi u.$$

Note that $J_R(R) = 0$, and so J_R , indeed, defines a complex structure on the orthogonal complement R^\perp of R with respect to g_φ . With these, we have

Theorem 5.6. *Let (M, φ) be a manifold with G_2 -structure. Then the quadruple $(J_R, R, \alpha_R, g_\varphi)$ defines an almost contact metric structure on M for any non-vanishing vector field R on M . Moreover, such a structure exists on any manifold with G_2 -structure.*

Proof. As before, we will assume that R is already normalized using g_φ . First, note that $\alpha_R(R) = g_\varphi(R, R) = \|R\|^2 = 1$. Also we have

$$J_R^2(u) = J_R(R \times_\varphi u) = R \times_\varphi (R \times_\varphi u) = -\|R\|^2 u + g_\varphi(R, u)R = -u + \alpha(u)R$$

where we made use of the identity (3). This shows that the endomorphism $J_R : TM \rightarrow TM$ satisfies the condition

$$J_R^2 = -I + \alpha \otimes R.$$

Therefore, the triple (J_R, R, α_R) is an almost contact structure on M . Next, we check g_φ is a compatible metric with this structure. Using (1) and (3), we compute

$$\begin{aligned} g_\varphi(J_R u, J_R v) &= g_\varphi(R \times_\varphi u, R \times_\varphi v) = \varphi(R, u, R \times_\varphi v) = -\varphi(R, R \times_\varphi v, u) \\ &= -g_\varphi(R \times_\varphi (R \times_\varphi v), u) = -g_\varphi(-\|R\|^2 v + g_\varphi(R, v)R, u) \\ &= -g_\varphi(-v + g_\varphi(R, v)R, u) = g_\varphi(v, u) - g_\varphi(\alpha_R(v)R, u) \\ &= g_\varphi(u, v) - \alpha_R(v) \underbrace{g_\varphi(R, u)}_{\alpha_R(u)} = g_\varphi(u, v) - \alpha_R(u)\alpha_R(v) \end{aligned}$$

which holds for all vector fields u, v in TM . This proves that g_φ satisfies (5). Hence, $(J_R, R, \alpha_R, g_\varphi)$ is an almost contact metric structure on M .

For the last statement, we know by [14] that there exists a nowhere vanishing vector field R on any 7-dimensional manifold. In particular, $(J_R, R, \alpha_R, g_\varphi)$ can be constructed on any manifold M with G_2 -structure φ . □

Theorem 5.7. *Let (M, φ) be a manifold with G_2 -structure, and $(J_R, R, \alpha_R, g_\varphi)$ be an almost contact metric structure on M constructed as above. Suppose that ξ is a contact structure on M such that $(J_R, R, \alpha_R, g_\varphi)$ is an associated almost contact metric structure for ξ . Then ξ is A-compatible.*

Proof. By assumption $(J_R, R, \alpha_R, g_\varphi)$ is an associated almost contact metric structure for ξ . Therefore, g_φ is an associated metric and satisfies

$$d\alpha_R(u, v) = g_\varphi(J_R(u), v) \text{ for all } u, v \in TM.$$

But then using the equation defining J_R and (1), we obtain

$$d\alpha_R(u, v) = g_\varphi(R \times_\varphi u, v) = \varphi(R, u, v) = i_R \varphi(u, v), \quad \forall u, v \in TM.$$

Therefore, we have $d\alpha_R = i_R \varphi$. Also R is the Reeb vector field of α_R by assumption. Hence, ξ is A-compatible by definition. □

Corollary 5.8. *Let (M, φ) be a manifold with G_2 -structure such that $d\varphi = 0$, and $(J_R, R, \alpha_R, g_\varphi)$ be an almost contact metric structure on M constructed as above. If M is closed, then there is no contact structure on M whose associated almost contact metric structure is $(J_R, R, \alpha_R, g_\varphi)$.*

Proof. On the contrary, suppose that $\xi = \text{Ker}(\alpha_R)$ is a contact structure on a closed manifold M equipped with a G_2 -structure φ and $d\varphi = 0$, and also that $(J_R, R, \alpha_R, g_\varphi)$ is an associated almost contact metric structure. Then, by Theorem 5.7, ξ is A-compatible, but this contradicts to Theorem 4.3. □

6. CONTACT- G_2 -STRUCTURES ON 7-MANIFOLDS

Suppose that (M, φ) is a manifold with G_2 -structure. Let us recall the decomposition of the space Λ^2 of 2-forms on M obtained from G_2 -representation and some other useful formulas which we will use. A good source for these is [2] and also [9]. According to irreducible G_2 -representation, $\Lambda^2 = \Lambda_7^2 \oplus \Lambda_{14}^2$ where

$$\begin{aligned} \Lambda_7^2 &= \{i_v \varphi; v \in \Gamma(TM)\} \\ &= \{\beta \in \Lambda^2; *(\varphi \wedge \beta) = -2\beta\} \\ (8) \quad &= \{\beta \in \Lambda^2; *(*\varphi \wedge (*(\varphi \wedge \beta))) = 3\beta\} \end{aligned}$$

$$\begin{aligned} \Lambda_{14}^2 &= \{\beta \in \Lambda^2; *\varphi \wedge \beta = 0\} \\ &= \{\beta \in \Lambda^2; *(\varphi \wedge \beta) = \beta\} \end{aligned}$$

Also on any Riemannian n -manifold, for any k -form α and a vector field v , the following equalities hold:

$$(9) \quad i_v * \alpha = (-1)^k * (v^\flat \wedge \alpha) \quad \text{and}$$

$$(10) \quad i_v \alpha = (-1)^{nk+n} * (v^\flat \wedge * \alpha).$$

As a last one we recall a very useful equality: For any k -form λ , and any $(n+1-k)$ -form μ and any vector field v on a smooth manifold of dimension n , we have

$$(11) \quad (\iota_v \lambda) \wedge \mu = (-1)^{k+1} \lambda \wedge (\iota_v \mu).$$

Now we are ready to prove:

Theorem 6.1. *Let (M, φ) be a manifold with G_2 -structure. Assume that there are nowhere-zero vector fields X, Y and Z on M satisfying*

$$(12) \quad \iota_Z \varphi = Y^\flat \wedge X^\flat$$

where X^\flat (resp. Y^\flat) is the covariant 1-form of X (resp. Y) with respect to the G_2 -metric g_φ . Also suppose that

$$(13) \quad d(i_X i_Y \varphi) = i_X i_Y * \varphi.$$

Then the 1-form $\alpha := Z^\flat = g_\varphi(Z, \cdot)$ is a contact form on M and it defines an A -compatible contact structure $\text{Ker}(\alpha)$ on (M, φ) .

Proof. From (8) we know that $\iota_Z \varphi$ is an element of Λ_7^2 . Set $\iota_Z \varphi = \beta \in \Lambda_7^2$, and so we have $\iota_Z \varphi = \beta = Y^\flat \wedge X^\flat$ by (12). Also applying (9) twice gives

$$i_X i_Y * \varphi = -i_X (* (Y^\flat \wedge \varphi)) = - * (X^\flat \wedge Y^\flat \wedge \varphi) = *(Y^\flat \wedge X^\flat \varphi) = *(\beta \wedge \varphi)$$

from which we get

$$(14) \quad i_X i_Y * \varphi = -2\beta$$

where we use the second line in (8). Moreover, by (10) followed by (9),

$$(15) \quad i_X i_Y \varphi = i_X (* (Y^\flat \wedge * \varphi)) = - * (X^\flat \wedge Y^\flat \wedge * \varphi) = * (\beta \wedge * \varphi).$$

Now putting (14) and (15) into (13) gives us

$$(16) \quad d * (\beta \wedge * \varphi) = -2\beta = -2\iota_Z \varphi.$$

Recall the formula $(i_v \varphi) \wedge * \varphi = 3 * v^\flat$ which is true for any vector field v . By taking $v = Z$, we compute the left-hand side in (16) as

$$d * (\beta \wedge * \varphi) = d * (3 * Z^\flat) = 3 dZ^\flat = 3 d\alpha.$$

Combining these together we obtain

$$(17) \quad d\alpha = -\frac{2}{3} \iota_Z \varphi.$$

Next, consider the identity (11) by taking $\lambda = \varphi$, $v = Z$ and $\mu = \alpha \wedge (d\alpha)^2$: Using (17), we compute the left-hand side as

$$(\iota_Z \varphi) \wedge \alpha \wedge (d\alpha)^2 = -\frac{3}{2} \alpha \wedge (d\alpha)^3,$$

and the right-hand side as

$$\varphi \wedge \iota_Z (\alpha \wedge (d\alpha)^2) = \alpha(Z) \varphi \wedge d\alpha \wedge d\alpha = \frac{4}{9} \|Z\|^2 \varphi \wedge (\iota_Z \varphi) \wedge (\iota_Z \varphi).$$

Therefore, by using the identity (2) in the right-hand side, we obtain

$$\alpha \wedge (d\alpha)^3 = -\frac{16}{9} \|Z\|^4 \text{Vol}.$$

Hence, we conclude that $\alpha \wedge (d\alpha)^3$ is a volume form on M (as being a nonzero function multiple of the volume form Vol on M determined by the metric g_φ). Equivalently, α is a contact form on M . Moreover, it follows from (17) that $(1/\|Z\|^2)Z$ is the Reeb vector field of α , i.e., it satisfies (4). Hence, $\text{Ker}(\alpha)$ is an A -compatible contact structure on (M, φ) . □

With the inspiration we get from the proof of Theorem 6.1, we define a new structure on 7-manifolds as follows:

Definition 6.2. Let M^7 be a smooth manifold. A *contact- G_2 -structure* on M is a quintuple $(\varphi, R, \alpha, f, g)$ where φ is a G_2 -structure, R is a nowhere-zero vector field, α is a 1-form on M , and $f, g : M \rightarrow \mathbb{R}$ are nowhere-zero smooth functions such that

- (i) $\alpha(R) = f$
- (ii) $d(g\alpha) = \iota_R \varphi$.

Observe that we have already seen an example of a contact- G_2 -structure in the above proof (of course under the assumptions of Theorem 6.1) with $R = Z, \alpha = Z^\flat, f = \|Z\|^2, g \equiv -3/2$. The reason why we call the quintuple $(\varphi, R, \alpha, f, g)$ “contact- G_2 -structure” is given by the following theorem.

Theorem 6.3. *Let $(\varphi, R, \alpha, f, g)$ be a contact- G_2 -structure on a smooth manifold M^7 . Then α is a contact form on M . Moreover, $\xi = \text{Ker}(\alpha)$ is an A-compatible contact structure on (M, φ) . In particular, if M is closed, then it does not admit a contact- G_2 -structure with $d\varphi = 0$.*

Proof. We first show that α is a contact form on M . Consider the 1-form

$$\alpha' := g\alpha.$$

Note that $\text{Ker}(\alpha) = \text{Ker}(\alpha')$ as g is a nowhere-zero function. Therefore, if we show that α' is a contact form on M , then it will imply that so is α . The conditions in Definition 6.2 translate into

$$\alpha'(R) = fg \quad \text{and} \quad d\alpha' = \iota_R\varphi.$$

Also from the equation (2) we get

$$(d\alpha')^2 \wedge \varphi = (\iota_R\varphi) \wedge (\iota_R\varphi) \wedge \varphi = 6\|R\|^2 \text{Vol}.$$

Now if we write the equation (11) by taking $\lambda = \varphi, \mu = \alpha' \wedge (d\alpha')^2$ and $v = R$, then the left-hand side gives

$$(\iota_R\varphi) \wedge \alpha' \wedge (d\alpha')^2 = (d\alpha') \wedge \alpha' \wedge (d\alpha')^2 = \alpha' \wedge (d\alpha')^3,$$

and from the right-hand side we have

$$\varphi \wedge \iota_R(\alpha' \wedge (d\alpha')^2) = \alpha'(R) \varphi \wedge (d\alpha')^2 = fg \varphi \wedge (d\alpha')^2 = 6fg\|R\|^2 \text{Vol}.$$

Therefore, we conclude

$$\alpha' \wedge (d\alpha')^3 = 6fg\|R\|^2 \text{Vol}$$

which implies that α' (and so α) is a contact form on M as $6fg\|R\|^2$ is a nowhere-zero function on M .

Next, we consider the vector field $R' = (1/fg)R$. Clearly, $\alpha'(R') = 1$. Also we compute

$$\iota_{R'}d\alpha' = (1/fg)\iota_Rd\alpha' = (1/fg)\iota_R(\iota_R\varphi) = 0$$

as φ is skew-symmetric. Therefore, R' is the Reeb vector field of α' , and so $\xi = \text{Ker}(\alpha')$ is an A-compatible contact structure on (M, φ) by definition. Finally, the last statement now follows from Theorem 4.3. \square

The next result shows that we can go also in the reverse direction.

Theorem 6.4. *Let (M, φ) be any manifold with G_2 -structure. Then every A-compatible contact structure on (M, φ) determines a contact- G_2 -structure on M .*

Proof. Let ξ be a given A-compatible contact structure on (M, φ) . By definition, there exist a non-vanishing vector field R on M , a contact form α for ξ and a nowhere-zero function $h : M \rightarrow \mathbb{R}$ such that $d\alpha = \iota_R\varphi$ and hR is the Reeb vector field of some contact form (possibly different than α) for ξ . Being a Reeb vector field, hR is transverse to the contact distribution ξ . Therefore,

R is also transverse to ξ because h is nowhere-zero on M . As a result, there must be a nowhere-zero function $f : M \rightarrow \mathbb{R}$ such that

$$\alpha(R) = f.$$

To check this, assume, on the contrary, that the function $M \rightarrow \mathbb{R}$ given by $x \mapsto \alpha_x(R_x)$ has a zero, say at p . So, we have $\alpha_p(R_p) = 0$ which means that $R_p \in \text{Ker}(\alpha_p) = \xi_p$. But this contradicts to the fact that R is everywhere transverse to ξ . Hence, we obtain a contact- G_2 -structure $(\varphi, R, \alpha, f, 1)$. This finishes the proof. □

7. SOME EXAMPLES

In this final section, we give some examples of G_2 -manifolds admitting A -compatible contact structures. In fact, by Theorem 6.4, in each example we will also have a corresponding contact- G_2 -structure.

7.1. $CY \times S^1$ (or $CY \times \mathbb{R}$). Consider a well-known example of G_2 -manifold $(CY \times S^1, \varphi)$ where we assume $CY(\Omega, \omega)$ is a 3-fold Calabi-Yau manifold which is either noncompact or compact with boundary. Assume Kähler form ω on CY is exact, i.e. $\omega = d\lambda$ for some $\lambda \in \Omega^1(CY)$ and set $\alpha = dt + \lambda$ where t is the coordinate on S^1 . Then $\alpha \wedge (d\alpha)^3 = \omega^3 \wedge dt$ is a volume form, and so α is a contact 1-form on $CY \times S^1$. Moreover, ∂t is the Reeb vector field of α as $\iota_{\partial t}\alpha = 1$ and $\iota_{\partial t}d\alpha = \iota_{\partial t}\omega = 0$. Also observe that since $\varphi = \text{Re}(\Omega) + \omega \wedge dt$ (see [10], for instance), we compute

$$\iota_{\partial t}\varphi = \iota_{\partial t}(\text{Re}(\Omega) + \omega \wedge dt) = \iota_{\partial t}\text{Re}(\Omega) + \iota_{\partial t}(\omega \wedge dt) = \omega \iota_{\partial t}dt = \omega = d\lambda = d\alpha.$$

Thus, $\xi = \text{Ker}(\alpha)$ is an A -compatible contact structure on $(CY \times S^1, \varphi)$, or in other words, $(\varphi, \partial t, \alpha, 1, 1)$ is a contact- G_2 -structure on $CY \times S^1$. We note that, by considering t as a coordinate on \mathbb{R} , the above argument also gives a contact- G_2 -structure on $CY \times \mathbb{R}$.

7.2. $W \times S^1$ (or $W \times \mathbb{R}$). We now give a special case of the above example. First, we need some definitions: A *Stein manifold* of complex dimension n is a triple (W^{2n}, J, ψ) where J is a complex structure on W and $\psi : W \rightarrow \mathbb{R}$ is a smooth map such that the 2-form $\omega_\psi = -d(d\psi \circ J)$ is non-degenerate (and so an exact symplectic form) on W . Indeed, (W, J, ω_ψ) is an exact Kähler manifold. We say that (M^{2n-1}, ξ) is *Stein fillable* if there is a Stein manifold (W^{2n}, J, ψ) such that ψ is bounded from below, M is a non-critical level of ψ , and $-(d\psi \circ J)$ is a contact form for ξ .

Next, consider a parallelizable Stein manifold (W, J, ψ) of complex dimension three. By a result of [6], we know that $c_1(W, J) = 0$, i.e., the first Chern class of (W, J) vanishes. Therefore, W admits a Calabi-Yau structure with associated

Kähler form $\omega_\psi = -d(d\psi \circ J)$. Let Ω be the non-vanishing holomorphic 3-form on W corresponding to this Calabi-Yau structure. Then by the previous example, $(W \times S^1, \varphi)$ is a G_2 -manifold with $\varphi = \operatorname{Re}(\Omega) + \omega_\psi \wedge d\theta$ (where θ is the coordinate on S^1), $\alpha = d\theta - (d\psi \circ J)$ is a contact 1-form on $W \times S^1$ with the Reeb vector field $\partial\theta$, and $\xi = \operatorname{Ker}(\alpha)$ is an A -compatible contact structure on $(W \times S^1, \varphi)$. Again by considering θ as a coordinate on \mathbb{R} , we obtain an A -compatible contact structure on $(W \times \mathbb{R}, \varphi)$. Note that the corresponding contact- G_2 -structure in both cases is $(\varphi, \partial\theta, \alpha, 1, 1)$.

Now consider the unit disk $\mathbb{D}^2 \subset \mathbb{C}$. Then $(W \times \mathbb{D}^2, J \times i, \psi + |z|^2)$ is a Stein manifold where i is the usual complex structure and $z = re^{i\theta}$ is the coordinate on \mathbb{C} . Let η be the induced contact structure on the boundary

$$\partial(W \times \mathbb{D}^2) = (\partial W \times \mathbb{D}^2) \cup (W \times S^1).$$

Then we remark that the restriction of the Stein fillable structure η on $W \times S^1$ is the contact structure ξ constructed above.

7.3. $\mathbb{R}^3 \times K^4$. Let K be a Kähler manifold with an exact Kähler form ω , i.e. $\omega = d\lambda$ for some $\lambda \in \Omega^1(K)$. Note that K is either noncompact or compact with boundary. Consider the G_2 -manifold $\mathbb{R}^3 \times K^4$ with the G_2 -structure

$$\varphi = dx_1 dx_2 dx_3 + \omega \wedge dx_1 + \operatorname{Re}(\Omega) \wedge dx_2 - \operatorname{Im}(\Omega) \wedge dx_3$$

where (x_1, x_2, x_3) are the coordinates on \mathbb{R}^3 (see [10]). Then $\alpha = dx_1 + x_2 dx_3 + \lambda$ is a contact 1-form as $\alpha \wedge (d\alpha)^3 = dx_1 dx_2 dx_3 \wedge \omega^2$ is a volume form on $\mathbb{R}^3 \times K^4$. One can easily check that ∂x_1 is the Reeb vector field of α . Furthermore,

$$\begin{aligned} i_{\partial x_1} \varphi &= i_{\partial x_1} (dx_1 dx_2 dx_3 + \omega \wedge dx_1 + \omega \wedge dx_2 + \omega \wedge dx_3) \\ &= dx_2 dx_3 + i_{\partial x_1} (\omega dx_1) = dx_2 dx_3 + \omega = d(x_2 dx_3 + \lambda) = d\alpha \end{aligned}$$

Hence, $\xi = \operatorname{Ker}(\alpha)$ is an A -compatible contact structure on $(\mathbb{R}^3 \times K^4, \varphi)$ and the corresponding contact- G_2 -structure on $\mathbb{R}^3 \times K^4$ is $(\varphi, \partial x_1, \alpha, 1, 1)$.

7.4. $T^*M^3 \times \mathbb{R}$. Let M be any oriented Riemannian 3-manifold and T^*M denote the cotangent bundle of M . It is shown in [4] that $T^*M \times \mathbb{R}$ has a G_2 -structure φ with $d\varphi = 0$. To describe φ , let (x_1, x_2, x_3) be local coordinates on M around a given point, and consider the corresponding standard local coordinates $(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)$ on the cotangent bundle T^*M . These define the standard symplectic structure $\omega = -d\lambda$ on T^*M where $\lambda = \sum_{i=1}^3 \xi_i dx_i$ is the tautological 1-form on T^*M . Let t denote the coordinate on \mathbb{R} . Then $\varphi = \operatorname{Re}(\Omega) - \omega \wedge dt$ where $\Omega = (dx_1 + id\xi_1) \wedge (dx_2 + id\xi_2) \wedge (dx_3 + id\xi_3)$ is the complex-valued $(3, 0)$ -form on M . On the other hand, the 1-form $\alpha = dt + \lambda$ is a contact form on $T^*M \times \mathbb{R}$ with the Reeb vector field ∂t . Now it is straightforward to check that $\xi = \operatorname{Ker}(\alpha)$ is an A -compatible contact structure on $(T^*M \times \mathbb{R}, \varphi)$ and also that $(\varphi, \partial t, \alpha, 1, 1)$ is the corresponding contact- G_2 -structure on $T^*M \times \mathbb{R}$.

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