Yang-Baxter equation, parameter permutations, and the elliptic beta integral

S. E. Derkachov[a1], V. P. Spiridonov[b2]

[a] St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences, Fontanka 27, 191023 St. Petersburg, Russia.

[b] Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Moscow reg. 141980, Russia and Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111, Bonn, Germany.

Abstract

We construct an infinite-dimensional solution of the Yang-Baxter equation (YBE) of rank 1 which is represented as an integral operator with an elliptic hypergeometric kernel acting in the space of functions of two complex variables. This R-operator intertwines the product of two standard L-operators associated with the Sklyanin algebra, an elliptic deformation of $s\ell(2)$-algebra. It is built from three basic operators $\mathrm{S}_1, \mathrm{S}_2$, and S_3 generating the permutation group of four parameters $\mathfrak{S}_4$. Validity of the key Coxeter relations (including the star-triangle relation) is based on the elliptic beta integral evaluation formula and the Bailey lemma associated with an elliptic Fourier transformation. The operators S_j are determined uniquely with the help of the elliptic modular double.

Contents

[1]e-mail: derkach@pdmi.ras.ru
[2]e-mail: spiridon@theor.jinr.ru

1 Introduction

The Yang-Baxter equation (YBE)

$$\mathbb{R}_{12}(u-v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{23}(v) = \mathbb{R}_{23}(v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{12}(u-v) \tag{1.1}$$

plays a key role in the theory of completely integrable quantum systems [1–7]. Its general solution is described by the operators $\mathbb{R}_{ik}(u)$ acting in the tensor product of three (in general different) spaces $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3$. The indices i and k show that $\mathbb{R}_{ik}(u)$ acts nontrivially in the tensor product $\mathbb{V}_i \otimes \mathbb{V}_k$ and it is the unity operator in the remaining part of $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3$. The operator $\mathbb{R}_{ik}(u)$ depend on the complex spectral parameter u and is called the R-matrix (or R-operator).

For spaces $\mathbb{V}_i$ of finite dimension there are three increasing levels of complexity of known YBE solutions described by rational, trigonometric, and elliptic functions. Investigation of the most complicated elliptic level was initiated by Baxter [8] for the case when all $\mathbb{V}_i$-spaces are two-dimensional. In a more general setting, when one of the spaces becomes infinite-dimensional, a major role is played by the Sklyanin algebra [9, 10]. Our main goal consists in the construction of a solution of the Yang-Baxter equation when all spaces $\mathbb{V}_i$ are infinite-dimensional and $\mathbb{R}_{ik}(u)$ are described by integral operators. In this case the hierarchy of solutions of YBE is attached to the plain hypergeometric, q-hypergeometric, and elliptic hypergeometric functions [11], in the increasing order of complexity. An explicit realization of the R-matrix as an integral operator in the simplest situation of rank 1 symmetry algebra $s\ell(2)$ has been considered in detail in [12]. In the present work we discuss only rank 1 R-operators related to the most complicated elliptic level.

Elliptic hypergeometric integrals were introduced in [13, 14]. They define the general form of elliptic hypergeometric functions which cannot be approached by infinite series [15] because of the convergence problems. The discovery of such functions and various relations for them formed a breakthrough in the theory of special functions. These functions generalize all previously constructed functions of hypergeometric type and they still obey the properties characteristic to classical special functions [16]. In particular, elliptic beta integrals [11,13] form a new class of exactly computable integrals generalizing the Euler, Selberg and other known beta integrals and their q-analogues. A kind of elliptic Fourier transform was introduced in [17] as an integral generalization of the well known Bailey chain transformation [16]. An elliptic extension of Faddeev's modular double [18] was introduced in [19]. All these constructions will play a major role in our consideration below.

We use the general strategy of building integral operator solutions of YBE whose initial steps were discovered in [20]. Its formulation was completed at the rational level in [21, 22], where an $SL(N,\mathbb{C})$-invariant solution of YBE related to A_n-root system has been constructed. In [23] this method was used also at the elliptic level employing some formal infinite series. Here we apply it for constructing solutions of YBE related to elliptic hypergeometric integrals. First we define some useful formal operators $\mathrm{S}_1\,, \mathrm{S}_2$, and S_3 performing elementary permutations of parameters in the defining RLL-relation. One of these operators is an intertwining operator of the Sklyanin algebra. Then we build these operators explicitly as integral operators with elliptic hypergeometric kernels. Finally we prove the Coxeter relations for these operators, for which the elliptic beta integral and the related integral Bailey lemma play a crucial role, and confirm that they indeed generate the group $\mathfrak{S}_4$. The cubic Coxeter relation represents the star-triangle relation. The operators S_1 , S_2, and S_3 are determined essentially uniquely, if one implements the elliptic modular doubling principle. As discussed in the concluding section, our results have applications to an interplay between integrable $2d$ spin systems and $4d$ supersymmetric gauge field theories.

2 Sklyanin algebra

In the simplest case of equation (1.1) all $\mathbb{V}_i$-spaces are two-dimensional, $\mathbb{V}_i = \mathbb{C}^2$. For this case, in solving the eight-vertex model Baxter has found the following R-matrix [1,2,8]

$$\mathbb{R}_{12}(u) = \sum_{a=0}^{3} w_a(u)\, \sigma_a \otimes \sigma_a \quad ; \quad w_a(u) = \frac{\theta_{a+1}(u+\eta)}{\theta_{a+1}(\eta)}\,, \tag{2.1}$$

where $\sigma_0 = \mathbb{1}$ and $\sigma_\alpha, \alpha = 1, 2, 3$, are the standard Pauli matrices. We use the shorthand notation $\theta_j(x) \equiv \theta_j(x|\tau)$ for the Jacobi theta-functions with modular parameter τ. The definitions and useful formulae for θ_j-functions are listed in the Appendix. This R-matrix depends on the spectral parameter $u \in \mathbb{C}$ and two additional free variables $\eta \in \mathbb{C}$, $\theta_j(\eta) \neq 0, j = 1, \ldots, 4$, and $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$. Another 4×4 matrix solution of YBE has been found by Felderhof [24]. As shown by Krichever [25], the Baxter and Felderhof R-matrices represent all solutions of YBE for $\mathbb{V}_i = \mathbb{C}^2$.

At the next level of complexity of relation (1.1) one of the spaces, say $\mathbb{V}_3$, is arbitrary, and $\mathbb{V}_1, \mathbb{V}_2 = \mathbb{C}^2$. In this case the R-matrix $\mathbb{R}_{13}(u) \equiv \mathrm{L}_{13}(u)$ (and $\mathbb{R}_{23}(u) \equiv \mathrm{L}_{23}(u)$) is known as the quantum L-operator or the Lax matrix. It is a matrix with two rows and two columns acting in $\mathbb{V}_1$

$$\mathrm{L}_{13}(u) = \mathrm{L}(u) := \sum_{a=0}^{3} w_a(u)\, \sigma_a \otimes \mathbf{S}^a = \begin{pmatrix} w_0(u)\, \mathbf{S}^0 + w_3(u)\, \mathbf{S}^3 & w_1(u)\, \mathbf{S}^1 - \mathrm{i} w_2(u)\, \mathbf{S}^2 \\ w_1(u)\, \mathbf{S}^1 + \mathrm{i} w_2(u)\, \mathbf{S}^2 & w_0(u)\, \mathbf{S}^0 - w_3(u)\, \mathbf{S}^3 \end{pmatrix}, \tag{2.2}$$

where the matrix element entries $\mathbf{S}^a$ are some operators acting in $\mathbb{V}_3$. The same $\mathbf{S}^a$-operators enter $\mathrm{L}_{23}(u)$ which acts as a 2×2 matrix in $\mathbb{V}_2$. In this case the equation for L-operator is the Yang-Baxter equation of the form

$$\mathbb{R}_{12}(u-v)\, \mathrm{L}_{13}(u)\, \mathrm{L}_{23}(v) = \mathrm{L}_{23}(v)\, \mathrm{L}_{13}(u)\, \mathbb{R}_{12}(u-v)\,, \tag{2.3}$$

where $\mathbb{R}_{12}(u)$ is Baxter's R-matrix (2.1). This equation is equivalent to the following set of relations for four operators $\mathbf{S}^0, \mathbf{S}^1, \mathbf{S}^2, \mathbf{S}^3$ forming the Sklyanin algebra [9,10]:

$$\mathbf{S}^\alpha\, \mathbf{S}^\beta - \mathbf{S}^\beta\, \mathbf{S}^\alpha = \mathrm{i} \cdot \left(\mathbf{S}^0\, \mathbf{S}^\gamma + \mathbf{S}^\gamma\, \mathbf{S}^0 \right) \,,$$

$$\mathbf{S}^0\, \mathbf{S}^\alpha - \mathbf{S}^\alpha\, \mathbf{S}^0 = \mathrm{i}\, \mathbf{J}_{\beta\gamma} \cdot \left(\mathbf{S}^\beta\, \mathbf{S}^\gamma + \mathbf{S}^\gamma\, \mathbf{S}^\beta \right) \,,$$

where the triplet (α, β, γ) is an arbitrary cyclic permutation of $(1, 2, 3)$ and the structure constants $\mathbf{J}_{\beta\gamma}$ are parameterized in terms of theta functions as

$$\mathbf{J}_{12} = \frac{\theta_1^2(\eta)\theta_4^2(\eta)}{\theta_2^2(\eta)\theta_3^2(\eta)} \ ; \quad \mathbf{J}_{23} = \frac{\theta_1^2(\eta)\theta_2^2(\eta)}{\theta_3^2(\eta)\theta_4^2(\eta)} \ ; \quad \mathbf{J}_{31} = -\frac{\theta_1^2(\eta)\theta_3^2(\eta)}{\theta_2^2(\eta)\theta_4^2(\eta)} \, . \tag{2.4}$$

One can write $\mathbf{J}_{\alpha\beta} = \frac{\mathbf{J}_\beta - \mathbf{J}_\alpha}{\mathbf{J}_\gamma}$, $\gamma \neq \alpha, \beta$, where

$$\mathbf{J}_1 = \frac{\theta_2(2\eta)\theta_2(0)}{\theta_2^2(\eta)} \ ; \quad \mathbf{J}_2 = \frac{\theta_3(2\eta)\theta_3(0)}{\theta_3^2(\eta)} \ ; \quad \mathbf{J}_3 = \frac{\theta_4(2\eta)\theta_4(0)}{\theta_4^2(\eta)} \, .$$

There are two Casimir operators commuting with all generators: $[\mathbf{K}_0, \mathbf{S}^a] = [\mathbf{K}_2, \mathbf{S}^a] = 0$,

$$\mathbf{K}_0 = \sum_{a=0}^{3} \mathbf{S}^a \, \mathbf{S}^a \ ; \qquad \mathbf{K}_2 = \sum_{\alpha=1}^{3} \mathbf{J}_\alpha \, \mathbf{S}^\alpha \, \mathbf{S}^\alpha \, .$$

We shall use the explicit realization of generators as difference operators found in [10]

$$[\mathbf{S}^a \, \Phi](z) = \frac{(\mathrm{i})^{\delta_{a,2}}\theta_{a+1}(\eta)}{\theta_1(2z)} \Big[\theta_{a+1}(2z - 2\eta\ell) \cdot \Phi(z+\eta) - \theta_{a+1}(-2z - 2\eta\ell) \cdot \Phi(z-\eta) \Big] , \tag{2.5}$$

where $\Phi(z)$ are some (supposedly meromorphic) functions of $z \in \mathbb{C}$. In this realization the Casimir operators reduce to the following scalar expressions

$$\mathbf{K}_0 = 4\,\theta_1^2\big((2\ell+1)\,\eta\big) \ ; \quad \mathbf{K}_2 = 4\,\theta_1\big(2(\ell+1)\,\eta\big)\,\theta_1(2\ell\,\eta) \, .$$

The variable $\ell \in \mathbb{C}$ is called the *spin*. It labels the Sklyanin algebra representations since it fixes (together with η and τ) the Casimir operator values. Note that R-matrix (2.1) is invariant under the change of variables $u \to -u, \eta \to -\eta$. However, for operators (2.5) the reflection $\eta \to -\eta$ changes their sign, $\mathbf{S}^a \to -\mathbf{S}^a$. Therefore, the L-operator changes the sign if one negates simultaneously the spectral parameter u and η.

For $\mathbf{S}^a$-operators (2.5) there exists a useful factorized representation for the L-operator

$$\mathrm{L}(u_1, u_2) = \frac{1}{\theta_1(2z)} \begin{pmatrix} \bar\theta_3(z - u_1) & -\bar\theta_3(z + u_1) \\ -\bar\theta_4(z - u_1) & \bar\theta_4(z + u_1) \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\eta\partial_z} & 0 \\ 0 & \mathrm{e}^{-\eta\partial_z} \end{pmatrix} \begin{pmatrix} \bar\theta_4(z + u_2) & \bar\theta_3(z + u_2) \\ \bar\theta_4(z - u_2) & \bar\theta_3(z - u_2) \end{pmatrix},$$

where $\mathrm{e}^{\eta\partial_z}$ is a shift operator, $\mathrm{e}^{\eta\partial_z} f(z) = f(z+\eta)$. New parameters $u_1 = \frac{u}{2} + \eta(\ell + \frac{1}{2})$ and $u_2 = \frac{u}{2} - \eta(\ell + \frac{1}{2})$ are simple linear combinations of the spectral parameter u and the spin ℓ. Here we use notation $\bar\theta_a(x) \equiv \theta_a(x|\frac{\tau}{2})$ for theta-functions with the modular parameter $\frac{\tau}{2}$.

When $2\ell + 1$ is a positive integer there exists $(2\ell+1)$-dimensional space $\Theta_{4\ell}^+$ of even theta-functions of order 4ℓ (having 4ℓ zeros in the fundamental parallelogram of periods) which is invariant under the action of generators $\mathbf{S}^a$. For $\ell = 1/2$ one can choose the basis of Θ_2^+-functions as $e_1 = \bar\theta_4(x)$ and $e_2 = \bar\theta_3(x)$. Then the Sklyanin algebra generators reduce in this basis to sigma-matrices $\mathbf{S}^a = \theta_1(2\eta|\tau)\sigma_a$ and the L-operator becomes proportional to the Baxter R-matrix (2.1).

The L-operator (2.2) is not unique. For instance, the operator $\sigma_\beta\,\mathrm{L}$, where σ_β is any Pauli matrix, is also a solution of equation (2.3). This follows from the fact that the matrix $X_\beta :=$

$\sigma_\beta \otimes \sigma_\beta$ obeys the properties $X_\beta^2 = 1$ and $X_\beta \mathbb{R}_{12}(u) X_\beta = \mathbb{R}_{12}(u)$. This freedom leads also to the existence of nontrivial automorphisms of the Sklyanin algebra [10].

The top level of complexity of the R-matrix corresponds to the situation when all spaces $\mathbb{V}_i$ are infinite-dimensional. In this case one deals with the most general solutions of the Yang-Baxter equation.

In the next section we explain step-by-step our strategy for building this solution, which is essentially the same as in [23] where the important role is played by an intertwining operator. For $2\ell \in \mathbb{Z}_{\geq 0}$ such an intertwining operator was constructed first by Zabrodin [26] as a finite sum of the powers of the finite-difference operator $e^{-\eta\partial_z}$. Its straightforward extension to arbitrary values of ℓ as an infinite series proposed in [27] has only a formal meaning due to the convergency problem. Nevertheless, the needed Coxeter relations were verified in [23] by checking the equality of coefficients in two such formal infinite series with the help of the Frenkel-Turaev summation formula [15].

Here we put the construction on a firm mathematical ground by using a different general ansatz for the intertwining operator which is more useful for practical applications. The key ingredients needed for the completion of this program is the elliptic beta integral [11, 13], the most general known exact integration formula generalizing the Euler beta integral, and the elliptic Fourier transformation of [17] which is defined precisely with the help of needed intertwining operator.

3 General construction

We solve YBE (1.1) when all spaces $\mathbb{V}_i$ are infinite-dimensional in two steps:

- on the first stage, we solve a defining RLL-relation using as elementary building blocks some simple operators S_1, S_2, and S_3. The key structural entries at this step are Coxeter relations for S_i validity of which is guaranteed by the elliptic beta integral evaluation formula [13];
- on the second stage, we prove that the operator $\mathbb{R}_{12}(u)$ found from the defining RLL-relation obeys YBE (1.1).

Consider a realization of YBE different from the previous ones, namely, when the spaces $\mathbb{V}_1$ and $\mathbb{V}_2$ are arbitrary and the space $\mathbb{V}_3$ is two-dimensional. Then equation (1.1) is reduced to the defining equation for an infinite-dimensional (unknown) R-matrix called RLL-relation [9, 28]:

$$\mathbb{R}_{12}(u-v)\,\mathrm{L}_1(u)\,\mathrm{L}_2(v) = \mathrm{L}_2(v)\,\mathrm{L}_1(u)\,\mathbb{R}_{12}(u-v)\,. \tag{3.6}$$

Here we use compact notation: the index k in L_k indicates that the Sklyanin algebra generators $\mathbf{S}_k^a$ entering this matrix are the operators acting in the space $\mathbb{V}_k$, i.e. $\mathbf{S}_k^a : \mathbb{V}_k \to \mathbb{V}_k$. The operators $\mathbf{S}_1^a$ and $\mathbf{S}_2^b$ act in different spaces and, evidently, commute, $\left[\mathbf{S}_1^a, \mathbf{S}_2^b\right] = 0$. Matrices L_k in equation (3.6) are multiplied as usual 2×2 matrices acting in the space $\mathbb{V}_3 = \mathbb{C}^2$.

Due to the non-uniqueness of representation of the L-operator, there are several possible forms of equation (3.6) with different $\mathbb{R}$-operators labeled by two indices a and b enumerating possible L-operators

$$\mathbb{R}_{12}^{ab}(u-v)\,\sigma_a\,\mathrm{L}_1(u)\,\sigma_b\,\mathrm{L}_2(v) = \sigma_b\,\mathrm{L}_2(v)\,\sigma_a\,\mathrm{L}_1(u)\,\mathbb{R}_{12}^{ab}(u-v)\,.$$

For a technical reason, which will be clear *a posteriori*, we fix $a = b = 3$ from the very beginning and denote $\mathbb{R}_{jk}(u) := \mathbb{R}^{33}_{jk}(u)$. In this case it is possible to cancel one σ_3 and our main defining RLL-relation takes the form

$$\mathbb{R}_{12}(u-v)\,\mathrm{L}_1(u)\,\sigma_3\,\mathrm{L}_2(v) = \mathrm{L}_2(v)\,\sigma_3\,\mathrm{L}_1(u)\,\mathbb{R}_{12}(u-v)\,. \tag{3.7}$$

We assume that $\mathbb{V}_1$ is the space of functions of a complex variable z_1 and $\mathbb{V}_2$ is the space of functions of a complex variable z_2. Respectively, the space $\mathbb{V}_1 \otimes \mathbb{V}_2$, where $\mathbb{R}_{12}$ is acting, is the space of functions $\Phi(z_1, z_2)$ of two independent variables z_1 and z_2.

It is convenient to extract from the R-matrix the permutation operator $\mathbb{R}_{12}(u) := \mathbb{P}_{12}\,\mathrm{R}_{12}(u)$, where the permutation operator interchanges arguments, $\mathbb{P}_{12}\,\Phi(z_1, z_2) = \Phi(z_2, z_1)$. Then the defining equation for the operator R_{12} has the following explicit form

$$\mathrm{R}_{12}(u-v)\,\mathrm{L}_1(u_1,u_2)\,\sigma_3\,\mathrm{L}_2(v_1,v_2) = \mathrm{L}_1(v_1,v_2)\,\sigma_3\,\mathrm{L}_2(u_1,u_2)\,\mathrm{R}_{12}(u-v)\,, \tag{3.8}$$

where the operators z, ∂_z and ℓ in the Sklyanin algebra generators (2.5) entering L_1 are replaced by z_1, ∂_{z_1} and ℓ_1, whereas in L_2 they are replaced by z_2, ∂_{z_2} and ℓ_2. We use also the following notation for combinations of the spectral parameters and spin variables

$$\begin{aligned} u_1 &= \frac{u}{2} + \eta\left(\ell_1 + \frac{1}{2}\right), \qquad & u_2 &= \frac{u}{2} - \eta\left(\ell_1 + \frac{1}{2}\right), \\ v_1 &= \frac{v}{2} + \eta\left(\ell_2 + \frac{1}{2}\right), \qquad & v_2 &= \frac{v}{2} - \eta\left(\ell_2 + \frac{1}{2}\right). \end{aligned} \tag{3.9}$$

For a subsequent use it is convenient to assume that these parameters do not depend on η and τ (i.e., to assume that the spectral parameters u, v and the variables $g_{1,2} := \eta(2\ell_{1,2} + 1)$ are independent on η and τ).

Equation (3.8) admits a natural interpretation: the operator R_{12} interchanges the set of parameters (u_1, u_2) from the first L-operator with the set of parameters (v_1, v_2) in the second L-operator. It is useful to combine these four parameters in one set in the natural order $\mathbf{u} \equiv (u_1, u_2, v_1, v_2)$. Then the operator

$$\mathrm{R}_{12}(u-v) \equiv \mathrm{R}_{12}(\mathbf{u}) \equiv \mathrm{R}_{12}(u_1, u_2|v_1, v_2)$$

corresponds to a particular permutation s in the group of permutations of four parameters $\mathfrak{S}_4$:

$$s \to \mathrm{R}_{12}(\mathbf{u})\ ;\ s\mathbf{u} \equiv s(u_1, u_2, v_1, v_2) = (v_1, v_2, u_1, u_2).$$

Any permutation from the group $\mathfrak{S}_4$ can be composed from the elementary transpositions s_1, s_2, and s_3:

$$s_1\mathbf{u} = (u_2, u_1, v_1, v_2)\ ,\ s_2\mathbf{u} = (u_1, v_1, u_2, v_2)\ ,\ s_3\mathbf{u} = (u_1, u_2, v_2, v_1),$$

which interchange only two nearest neighboring elements in the set (u_1, u_2, v_1, v_2). For example, the permutation s has the following decomposition $s = s_2 s_1 s_3 s_2$. It is natural to search for the operators $\mathrm{S}_i(u_1, u_2, v_1, v_2) \equiv \mathrm{S}_i(\mathbf{u})$ representing these elementary transpositions in L-operators

$$(\overbrace{u_1\ ,\ u_2}^{\mathrm{S}_1}, \overbrace{v_1\ ,\ v_2}^{\mathrm{S}_3})\ ;\ (u_1\ , \overbrace{u_2, v_1}^{\mathrm{S}_2},\ v_2).$$

Namely, we demand that S_i obey the following defining relations

$$\mathrm{S}_1(\mathbf{u})\,\mathrm{L}_1(u_1,u_2) = \mathrm{L}_1(u_2,u_1)\,\mathrm{S}_1(\mathbf{u})\ ; \quad \mathrm{S}_3(\mathbf{u})\,\mathrm{L}_2(v_1,v_2) = \mathrm{L}_2(v_2,v_1)\,\mathrm{S}_3(\mathbf{u}), \tag{3.10}$$

$$\mathrm{S}_2(\mathbf{u})\,\mathrm{L}_1(u_1,u_2)\,\sigma_3\,\mathrm{L}_2(v_1,v_2) = \mathrm{L}_1(u_1,v_1)\,\sigma_3\,\mathrm{L}_2(u_2,v_2)\,\mathrm{S}_2(\mathbf{u})\,. \tag{3.11}$$

Since R_{12}-matrix acts in the space $\mathbb{V}_1 \otimes \mathbb{V}_2$, operators S_i should be scalars with respect to $\mathbb{V}_3 = \mathbb{C}^2$. Moreover, it is natural to demand that S_1 commutes with L_2 and S_3 commutes with L_1:

$$\mathrm{S}_1(\mathbf{u})\mathrm{L}_2(v_1,v_2) = \mathrm{L}_2(v_1,v_2)\mathrm{S}_1(\mathbf{u}), \qquad \mathrm{S}_3(\mathbf{u})\mathrm{L}_1(u_1,u_2) = \mathrm{L}_1(u_1,u_2)\mathrm{S}_3(\mathbf{u}). \tag{3.12}$$

Our first step consists in the direct construction of these operators (see the next section). Having these operators we can build the R-matrix.

Theorem 1. *Suppose that formal scalar operators* S_i *satisfy relations (3.10)–(3.12). Then the composite operator* $\mathrm{R}_{12}(\mathbf{u})$*,*

$$\mathrm{R}_{12}(\mathbf{u}) = \mathrm{S}_2(s_1 s_3 s_2\mathbf{u})\,\mathrm{S}_1(s_3 s_2\mathbf{u})\,\mathrm{S}_3(s_2\mathbf{u})\,\mathrm{S}_2(\mathbf{u}), \tag{3.13}$$

satisfies equation (3.8).

Proof. The proof reduces to a direct check, which is quite simple. Namely, using equation (3.11) we have

$$\mathrm{R}_{12}(\mathbf{u})\,\mathrm{L}_1(u_1,u_2)\,\sigma_3\,\mathrm{L}_2(v_1,v_2) = \mathrm{S}_2(s_1 s_3 s_2\mathbf{u})\,\mathrm{S}_1(s_3 s_2\mathbf{u})\,\mathrm{S}_3(s_2\mathbf{u})\,\mathrm{L}_1(u_1,v_1)\,\sigma_3\,\mathrm{L}_2(u_2,v_2)\mathrm{S}_2(\mathbf{u}).$$

Using the commutativity of $\mathrm{S}_3(s_2\mathbf{u})$ with σ_3 and $\mathrm{L}_1(u_1,v_1)$ (3.12) and the second equation in (3.10), we can rewrite this expression as

$$\mathrm{S}_2(s_1 s_3 s_2\mathbf{u})\,\mathrm{S}_1(s_3 s_2\mathbf{u})\,\mathrm{L}_1(u_1,v_1)\,\sigma_3\,\mathrm{L}_2(v_2,u_2)\mathrm{S}_3(s_2\mathbf{u})\,\mathrm{S}_2(\mathbf{u}).$$

Now we apply the first equation in (3.10) and commutativity of $\mathrm{S}_1(s_3 s_2\mathbf{u})$ with σ_3 and $\mathrm{L}_2(v_2,u_2)$ (3.12) and obtain the expression

$$\mathrm{S}_2(s_1 s_3 s_2\mathbf{u})\,\mathrm{L}_1(v_1,u_1)\,\sigma_3\,\mathrm{L}_2(v_2,u_2)\mathrm{S}_1(s_3 s_2\mathbf{u})\,\mathrm{S}_3(s_2\mathbf{u})\,\mathrm{S}_2(\mathbf{u}).$$

Finally, applying equation (3.11) with (u_1,u_2,v_1,v_2) replaced by (v_1,u_1,v_2,u_2) we obtain the right-hand side of equation (3.8). □

Expression (3.13) for the R-matrix corresponds to a special decomposition of the permutation s: $s = s_2 s_1 s_3 s_2$. We will see that operators S_i depend on their parameters in a special way

$$\mathrm{S}_1(\mathbf{u}) = \mathrm{S}_1(u_1 - u_2)\ ;\ \mathrm{S}_2(\mathbf{u}) = \mathrm{S}_2(u_2 - v_1)\ ;\ \mathrm{S}_3(\mathbf{u}) = \mathrm{S}_3(v_1 - v_2), \tag{3.14}$$

so that the operator $\mathrm{R}_{12}(\mathbf{u})$ depends on the difference of spectral parameters $u - v$ as it should,

$$\mathrm{R}_{12}(u_1,u_2|v_1,v_2) = \mathrm{S}_2(u_1 - v_2)\,\mathrm{S}_1(u_1 - v_1)\,\mathrm{S}_3(u_2 - v_2)\,\mathrm{S}_2(u_2 - v_1)\,. \tag{3.15}$$

We have thus the following correspondence between permutations s_i and our operators S_i:

$$s_i \longrightarrow \mathrm{S}_i(\mathbf{u})\ ; \quad s_i s_j \longrightarrow \mathrm{S}_i(s_j\mathbf{u})\,\mathrm{S}_j(\mathbf{u}). \tag{3.16}$$

In order to prove that we have a representation of the permutation group $\mathfrak{S}_4$ it remains to prove the defining Coxeter relations for the generators

$$s_i^2 = \mathbb{1} \longrightarrow \mathrm{S}_i(s_i\mathbf{u})\,\mathrm{S}_i(\mathbf{u}) = \mathbb{1}\,; \quad s_1s_3 = s_3s_1 \longrightarrow \mathrm{S}_1(s_3\mathbf{u})\,\mathrm{S}_3(\mathbf{u}) = \mathrm{S}_3(s_1\mathbf{u})\,\mathrm{S}_1(\mathbf{u})\,, \tag{3.17}$$

$$s_1s_2s_1 = s_2s_1s_2 \longrightarrow \mathrm{S}_1(s_2s_1\mathbf{u})\,\mathrm{S}_2(s_1\mathbf{u})\,\mathrm{S}_1(\mathbf{u}) = \mathrm{S}_2(s_1s_2\mathbf{u})\,\mathrm{S}_1(s_2\mathbf{u})\,\mathrm{S}_2(\mathbf{u})\,, \tag{3.18}$$

$$s_2s_3s_2 = s_3s_2s_3 \longrightarrow \mathrm{S}_2(s_3s_2\mathbf{u})\,\mathrm{S}_3(s_2\mathbf{u})\,\mathrm{S}_2(\mathbf{u}) = \mathrm{S}_3(s_2s_3\mathbf{u})\,\mathrm{S}_2(s_3\mathbf{u})\,\mathrm{S}_3(\mathbf{u})\,. \tag{3.19}$$

One can try to work with the equivalent power form of these relations connected to reflection groups

$$(s_is_j)^{m_{ij}} = 1, \quad m_{ii} = 1, \quad m_{ij} = 2,\ |i-j| > 1, \quad m_{i,i\pm1} = 3,$$

but it is much less efficient. The explicit form of the operators S_i will be determined in the next section. The proof of relations (3.17)-(3.19) will be given in Sect. 5.

Consider now the space $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3 \otimes \mathbb{C}^2$, where $\mathbb{V}_3$ is a new infinite-dimensional space of functions depending on $z_3 \in \mathbb{C}$. Introduce the third L-matrix $\mathrm{L}_3(w_1, w_2)$, $w_{1,2} = \frac{w}{2} \pm \eta(\ell_3 + \frac{1}{2})$, where w is a new spectral parameter and ℓ_3 is a new spin variable in the Sklyanin algebra generators (2.5) with z, ∂_z replaced by z_3, ∂_{z_3}.

It is natural to form the set $\mathbf{u} = (u_1, u_2, v_1, v_2, w_1, w_2)$ and consider the group of permutations of six parameters $\mathfrak{S}_6$. In addition to the previous case we have two more elementary permutation generators $s_4\mathbf{u} = (u_1, u_2, v_1, w_1, v_2, w_2)$ and $s_5\mathbf{u} = (u_1, u_2, v_1, v_2, w_2, w_1)$. We define operators S_4 and S_5 such that the triple $\{\mathrm{S}_3, \mathrm{S}_4, \mathrm{S}_5\}$ has the same properties as the triple $\{\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3\}$ after the replacement of parameters (u_1, u_2, v_1, v_2) by (v_1, v_2, w_1, w_2). More precisely, we demand that

$$\begin{aligned}
&\mathrm{S}_5(\mathbf{u})\,\mathrm{L}_3(w_1, w_2) = \mathrm{L}_3(w_2, w_1)\,\mathrm{S}_5(\mathbf{u})\,,\\
&\mathrm{S}_4(\mathbf{u})\,\mathrm{L}_2(v_1, v_2)\,\sigma_3\,\mathrm{L}_3(w_1, w_2) = \mathrm{L}_2(v_1, w_1)\,\sigma_3\,\mathrm{L}_3(v_2, w_2)\,\mathrm{S}_4(\mathbf{u})\,,
\end{aligned} \tag{3.20}$$

and that S_5 commutes with $\mathrm{S}_{1,2,3}$ and S_4 commutes with $\mathrm{S}_{1,2}$.

Introduce the composite operator similar to $\mathrm{R}_{12}(\mathbf{u})$,

$$\begin{aligned}
\mathrm{R}_{23}(\mathbf{u}) \equiv \mathrm{R}_{23}(v_1, v_2|w_1, w_2) &= \mathrm{S}_4(s_3s_5s_4\mathbf{u})\,\mathrm{S}_3(s_5s_4\mathbf{u})\,\mathrm{S}_5(s_4\mathbf{u})\,\mathrm{S}_4(\mathbf{u})\\
&= \mathrm{S}_4(v_1 - w_2)\,\mathrm{S}_3(v_1 - w_1)\,\mathrm{S}_5(v_2 - w_2)\,\mathrm{S}_4(v_2 - w_1).
\end{aligned} \tag{3.21}$$

To define the R-matrix R_{13} we consider the action of permutation operators $\mathbb{P}_{jk}$ on $\mathrm{S}_i(\mathbf{u})$. Conjugating relations (3.10) by $\mathbb{P}_{12}$, one can see that $\mathbb{P}_{12}\mathrm{S}_3\mathbb{P}_{12}$ should be identified with S_1 having the same argument. Namely,

$$\mathbb{P}_{12}\mathrm{S}_1(u_1 - u_2) = \mathrm{S}_3(u_1 - u_2)\mathbb{P}_{12}, \qquad \mathbb{P}_{12}\mathrm{S}_3(v_1 - v_2) = \mathrm{S}_1(v_1 - v_2)\mathbb{P}_{12}.$$

Conjugating similarly (3.11), one cannot directly deduce properties of S_2. As we will see from the explicit form of this operator derived later, $\mathbb{P}_{12}\mathrm{S}_2(\mathbf{u}) = \mathrm{S}_2(\mathbf{u})\mathbb{P}_{12}$. Relations $\mathbb{P}_{12}\mathrm{S}_{4,5}(\mathbf{u}) = \mathrm{S}_{4,5}(\mathbf{u})\mathbb{P}_{12}$ are evident. Analogous considerations yield nontrivial commutation relations

$$\mathbb{P}_{13}\mathrm{S}_2(u_2 - v_1) = \mathrm{S}_4(u_2 - v_1)\mathbb{P}_{13}, \qquad \mathbb{P}_{23}\mathrm{S}_5(w_1 - w_2) = \mathrm{S}_3(w_1 - w_2)\mathbb{P}_{23},$$

etc. However, the operator $\mathbb{P}_{12}\mathrm{S}_4\mathbb{P}_{12} = \mathbb{P}_{23}\mathrm{S}_2\mathbb{P}_{23}$ cannot be expressed in terms of $\mathrm{S}_i(\mathbf{u})$-operators. Now we define

$$\begin{aligned}
&\mathrm{R}_{13}(\mathbf{u}) \equiv \mathrm{R}_{13}(u_1, u_2|w_1, w_2) = \mathbb{P}_{12}\mathrm{R}_{23}(u_1, u_2|w_1, w_2)\mathbb{P}_{12}\\
&= \mathbb{P}_{12}\mathrm{S}_4(u_1 - w_2)\,\mathrm{S}_3(u_1 - w_1)\,\mathrm{S}_5(u_2 - w_2)\,\mathrm{S}_4(u_2 - w_1)\mathbb{P}_{12}\\
&= \mathbb{P}_{12}\mathrm{S}_4(u_1 - w_2)\mathbb{P}_{12}\,\mathrm{S}_1(u_1 - w_1)\,\mathrm{S}_5(u_2 - w_2)\,\mathbb{P}_{12}\mathrm{S}_4(u_2 - w_1)\mathbb{P}_{12}.
\end{aligned} \tag{3.22}$$

Analogously,

$$\mathrm{R}_{13}(\mathbf{u}) = \mathbb{P}_{23}\mathrm{R}_{12}(u_1,u_2|w_1,w_2)\mathbb{P}_{23} = \mathbb{P}_{23}\mathrm{S}_2(u_1-w_2)\mathbb{P}_{23}\,\mathrm{S}_1(u_1-w_1)\,\mathrm{S}_5(u_2-w_2)\,\mathbb{P}_{23}\mathrm{S}_2(u_2-w_1)\mathbb{P}_{23}.$$

We thus see that the operator $\mathrm{R}_{13}(\mathbf{u})$ cannot be factorized purely in terms of the operators S_i.

Theorem 2. *Suppose we have a set of well-defined operators* $\mathrm{S}_i(\mathbf{u}), i = 1,\ldots,5,$ *satisfying intertwining relations* (3.10)-(3.12), (3.20) *and the* $\mathfrak{B}_6$*-braid group generating relations*

$$\mathrm{S}_j\mathrm{S}_k = \mathrm{S}_k\mathrm{S}_j,\ |j-k|>1, \qquad \mathrm{S}_j\mathrm{S}_{j+1}\mathrm{S}_j = \mathrm{S}_{j+1}\mathrm{S}_j\mathrm{S}_{j+1}. \tag{3.23}$$

Then the R*-matrices*

$$\mathbb{R}_{12}(u-v) = \mathbb{P}_{12}\mathrm{R}_{12}(\mathbf{u}), \quad \mathbb{R}_{23}(v-w) = \mathbb{P}_{23}\mathrm{R}_{23}(\mathbf{u}), \quad \mathbb{R}_{13}(u-w) = \mathbb{P}_{13}\mathrm{R}_{13}(\mathbf{u}),$$

where operators $\mathrm{R}_{ij}(\mathbf{u})$ *are fixed in* (3.13), (3.21), *and* (3.22), *satisfy the Yang-Baxter equation*

$$\mathbb{R}_{12}(u-v)\,\mathbb{R}_{13}(u-w)\,\mathbb{R}_{23}(v-w) = \mathbb{R}_{23}(v-w)\,\mathbb{R}_{13}(u-w)\,\mathbb{R}_{12}(u-v). \tag{3.24}$$

Proof. Consider the following permutation of parameters in the product of three L-operators:

$$\mathrm{L}_1(u_1,u_2)\,\sigma_3\,\mathrm{L}_2(v_1,v_2)\,\sigma_3\,\mathrm{L}_3(w_1,w_2) \to \mathrm{L}_1(w_1,w_2)\,\sigma_3\,\mathrm{L}_2(v_1,v_2)\,\sigma_3\,\mathrm{L}_3(u_1,u_2).$$

It can be realized in two different ways as shown on the figure below

$$\begin{tikzcd}
\mathrm{L}_1(v_1,v_2)\sigma_3\mathrm{L}_2(u_1,u_2)\sigma_3\mathrm{L}_3(w_1,w_2) \arrow[r, "\mathrm{R}_{23}(u_1,u_2|w_1,w_2)"] & \mathrm{L}_1(v_1,v_2)\sigma_3\mathrm{L}_2(w_1,w_2)\sigma_3\mathrm{L}_3(u_1,u_2) \arrow[d, "\mathrm{R}_{12}(v_1,v_2|w_1,w_2)"'] \\
\mathrm{L}_1(u_1,u_2)\sigma_3\mathrm{L}_2(v_1,v_2)\sigma_3\mathrm{L}_3(w_1,w_2) \arrow[u, "\mathrm{R}_{12}(u_1,u_2|v_1,v_2)"'] \arrow[d, "\mathrm{R}_{23}(v_1,v_2|w_1,w_2)"'] & \mathrm{L}_1(w_1,w_2)\sigma_3\mathrm{L}_2(v_1,v_2)\sigma_3\mathrm{L}_3(u_1,u_2) \\
\mathrm{L}_1(u_1,u_2)\sigma_3\mathrm{L}_2(w_1,w_2)\sigma_3\mathrm{L}_3(v_1,v_2) \arrow[r, "\mathrm{R}_{12}(u_1,u_2|w_1,w_2)"'] & \mathrm{L}_1(w_1,w_2)\sigma_3\mathrm{L}_2(u_1,u_2)\sigma_3\mathrm{L}_3(v_1,v_2) \arrow[u, "\mathrm{R}_{23}(u_1,u_2|v_1,v_2)"]
\end{tikzcd}$$

The condition of commutativity of this diagram indicates that

$$\begin{aligned}
&\mathrm{R}_{23}(u_1,u_2|v_1,v_2)\,\mathrm{R}_{12}(u_1,u_2|w_1,w_2)\,\mathrm{R}_{23}(v_1,v_2|w_1,w_2) \\
&= \mathrm{R}_{12}(v_1,v_2|w_1,w_2)\,\mathrm{R}_{23}(u_1,u_2|w_1,w_2)\,\mathrm{R}_{12}(u_1,u_2|v_1,v_2).
\end{aligned} \tag{3.25}$$

Let us prove this equality using the braid group generating relations (3.23) for operators S_j. We start from the identity

$$\mathrm{S}_2\mathrm{S}_3\mathrm{S}_1\mathrm{S}_2 \cdot \mathrm{S}_4\mathrm{S}_3\mathrm{S}_5\mathrm{S}_4 \cdot \mathrm{S}_2\mathrm{S}_1\mathrm{S}_3\mathrm{S}_2 = \mathrm{S}_2\mathrm{S}_3\mathrm{S}_4\mathrm{S}_1 \cdot \mathrm{S}_3\mathrm{S}_2\mathrm{S}_3 \cdot \mathrm{S}_1\mathrm{S}_5\mathrm{S}_4\mathrm{S}_3\mathrm{S}_2.$$

The left-hand side is equal to the product of R-matrices in the left-hand side of (3.25) under the taken convention $\mathrm{S}_j\mathrm{S}_k := \mathrm{S}_j(s_k\mathbf{u})\mathrm{S}_k(\mathbf{u})$. The right-hand side is obtained by permuting $\mathrm{S}_1\mathrm{S}_2$ with neighboring S_4, $\mathrm{S}_5\mathrm{S}_4$ with neighboring $\mathrm{S}_2\mathrm{S}_1$, and application of the cubic relation from (3.23) to the emerging product $\mathrm{S}_2\mathrm{S}_3\mathrm{S}_2$. Now we replace $\mathrm{S}_1\cdot\mathrm{S}_3$ by $\mathrm{S}_3\mathrm{S}_1$, permute $\mathrm{S}_1\mathrm{S}_5$ with neighboring S_3, apply the cubic relation to the emerging product $\mathrm{S}_1\mathrm{S}_2\mathrm{S}_1$ and obtain the left hand side of the relation

$$\mathrm{S}_2\mathrm{S}_3\mathrm{S}_4\mathrm{S}_3\cdot\mathrm{S}_2\mathrm{S}_1\mathrm{S}_2\cdot\mathrm{S}_5\mathrm{S}_3\mathrm{S}_4\mathrm{S}_3\mathrm{S}_2 = \mathrm{S}_4\mathrm{S}_2\mathrm{S}_3\mathrm{S}_2\cdot\mathrm{S}_4\mathrm{S}_1\mathrm{S}_5\mathrm{S}_4\cdot\mathrm{S}_2\mathrm{S}_3\mathrm{S}_2\mathrm{S}_4.$$

The right-hand side is obtained after applying the cubic relation to two products $\mathrm{S}_3\mathrm{S}_4\mathrm{S}_3$ and permuting three operators S_2 with neighboring S_4's and one S_2 with neighboring $\mathrm{S}_5\mathrm{S}_4$. Now we permute neighboring S_4 and S_1 and apply cubic relations to the products $\mathrm{S}_2\mathrm{S}_3\mathrm{S}_2$ (twice) and $\mathrm{S}_4\mathrm{S}_5\mathrm{S}_4$. This yields the left-hand side of the relation

$$\mathrm{S}_4\mathrm{S}_3\mathrm{S}_2\mathrm{S}_3\cdot\mathrm{S}_1\mathrm{S}_5\mathrm{S}_4\mathrm{S}_5\cdot\mathrm{S}_3\mathrm{S}_2\mathrm{S}_3\mathrm{S}_4 = \mathrm{S}_4\mathrm{S}_3\mathrm{S}_5\mathrm{S}_4\cdot\mathrm{S}_2\mathrm{S}_1\mathrm{S}_3\mathrm{S}_2\cdot\mathrm{S}_4\mathrm{S}_3\mathrm{S}_5\mathrm{S}_4.$$

The right-hand side expression is obtained after pulling S_5-operators to the left and right from S_4, permuting neighboring S_3 and S_1, applying the cubic relation to the emerging product $\mathrm{S}_3\mathrm{S}_4\mathrm{S}_3$, and, finally, pulling S_4-operators to the left and right from S_3. And, evidently, it coincides with the right-hand side expression of equality (3.25).

Let us multiply the left-hand side expression in (3.25) by the operator $\mathbb{P}_{12}\mathbb{P}_{13}\mathbb{P}_{23}$ and the right-hand side expression by the equal operator $\mathbb{P}_{23}\mathbb{P}_{13}\mathbb{P}_{12}$. Pulling permutation operators $\mathbb{P}_{jk}$ to appropriate R-matrices using relations

$$\begin{aligned}\mathbb{P}_{13}\mathbb{P}_{23}\mathrm{R}_{23}(u_1,u_2|v_1,v_2)\mathbb{P}_{23}\mathbb{P}_{13} &= \mathbb{P}_{13}\mathrm{S}_4(u_1-v_2)\,\mathrm{S}_5(u_1-v_1)\,\mathrm{S}_3(u_2-v_2)\,\mathrm{S}_4(u_2-v_1)\mathbb{P}_{13}\\ &= \mathrm{S}_2(u_1-v_2)\,\mathrm{S}_1(u_1-v_1)\,\mathrm{S}_3(u_2-v_2)\,\mathrm{S}_2(u_2-v_1) = \mathrm{R}_{12}(\mathbf{u})\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}_{13}\mathbb{P}_{12}\mathrm{R}_{12}(v_1,v_2|w_1,w_2)\mathbb{P}_{12}\mathbb{P}_{13} &= \mathbb{P}_{13}\mathrm{S}_2(v_1-w_2)\,\mathrm{S}_3(v_1-w_1)\,\mathrm{S}_1(v_2-w_2)\,\mathrm{S}_2(v_2-w_1)\,\mathbb{P}_{13}\\ &= \mathrm{S}_4(v_1-w_2)\,\mathrm{S}_3(v_1-w_1)\,\mathrm{S}_5(v_2-w_2)\,\mathrm{S}_4(v_2-w_1) = \mathrm{R}_{23}(\mathbf{u}),\end{aligned}$$

one comes to the desired equation (3.24). □

From this consideration we conclude that the Yang-Baxter relation (3.25) is nothing else than a word identity in the group algebra of the braid group $\mathfrak{B}_6$. Equation (3.24) is more complicated since it involves the external operators $\mathbb{P}_{jk}$. Note that the described proof does not require the condition $\mathrm{S}_j^2 = \mathbb{1}$ reducing $\mathfrak{B}_6$ to the permutation group $\mathfrak{S}_6$. The operators S_j which we construct below do satisfy relations $\mathrm{S}_j^2 = \mathbb{1}$ after the analytical continuation in parameters and, so, they generate the $\mathfrak{S}_6$-group.

4 Elementary transpositions and intertwining operators

We shall use the factorized form of the L-operator which allows one to simplify considerably all calculations

$$\mathrm{L}(u_1,u_2) = \frac{1}{\theta_1(2z)}\cdot\mathrm{M}(z-u_1;z+u_1)\cdot\begin{pmatrix}\mathrm{e}^{\eta\partial} & 0\\ 0 & \mathrm{e}^{-\eta\partial}\end{pmatrix}\cdot\mathrm{N}(z-u_2;z+u_2), \tag{4.1}$$

where

$$\mathrm{M}(a;b) = \begin{pmatrix} \bar\theta_3\,(a) & -\bar\theta_3\,(b) \\ -\bar\theta_4\,(a) & \bar\theta_4\,(b) \end{pmatrix} \quad ; \quad \mathrm{N}(a;b) = \begin{pmatrix} \bar\theta_4\,(b) & \bar\theta_3\,(b) \\ \bar\theta_4\,(a) & \bar\theta_3\,(a) \end{pmatrix} . \tag{4.2}$$

To prove this factorization of the L-operator one has to multiply explicitly all three matrices involved in it and use the addition formula

$$\theta_1(x+y)\theta_1(x-y) + \theta_4(x+y)\theta_4(x-y) = \bar\theta_4(x)\bar\theta_3(y)$$

and its variations which are listed in the Appendix. The product of matrices N and M has the form

$$\mathrm{N}(a_1;b_1)\cdot \mathrm{M}(a_2;b_2) = 2\cdot \begin{pmatrix} \theta_1\,(b_1-a_2)\,\theta_1\,(b_1+a_2) & -\theta_1\,(b_1-b_2)\,\theta_1\,(b_1+b_2) \\ \theta_1\,(a_1-a_2)\,\theta_1\,(a_1+a_2) & -\theta_1\,(a_1-b_2)\,\theta_1\,(a_1+b_2) \end{pmatrix}, \tag{4.3}$$

in particular,

$$\mathrm{N}(a;b)\cdot \mathrm{M}(a;b) = -2\cdot \theta_1\,(a-b)\,\theta_1\,(b+a) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{4.4}$$

To avoid lengthy formulae we use compact notation $\mathrm{N}(a\mp b) \equiv \mathrm{N}(a-b;a+b)$, $\mathrm{M}(a\mp b) \equiv \mathrm{M}(a-b;a+b)$ and $\theta_j(a,b) = \theta_j(a)\theta_j(b)$, $\theta_j(a\mp b) = \theta_j\,(a-b)\,\theta_j\,(a+b)$, so that

$$\mathrm{L}(u_1,u_2) = \frac{1}{\theta_1(2z)}\cdot \mathrm{M}(z\mp u_1)\cdot \begin{pmatrix} \mathrm{e}^{\eta\partial} & 0 \\ 0 & \mathrm{e}^{-\eta\partial} \end{pmatrix} \cdot \mathrm{N}(z\mp u_2).$$

Consider first the defining relation for operator S_2 (3.11),

$$\mathrm{S}_2\, \underline{\mathrm{M}(z_1\mp u_1)} \begin{pmatrix} \mathrm{e}^{\eta\partial_1} & 0 \\ 0 & \mathrm{e}^{-\eta\partial_1} \end{pmatrix} \mathrm{N}(z_1\mp u_2)\,\sigma_3\, \mathrm{M}(z_2\mp v_1) \begin{pmatrix} \mathrm{e}^{\eta\partial_2} & 0 \\ 0 & \mathrm{e}^{-\eta\partial_2} \end{pmatrix} \underline{\mathrm{N}(z_2\mp v_2)} =$$

$$= \underline{\mathrm{M}(z_1\mp u_1)} \begin{pmatrix} \mathrm{e}^{\eta\partial_1} & 0 \\ 0 & \mathrm{e}^{-\eta\partial_1} \end{pmatrix} \mathrm{N}(z_1\mp v_1)\,\sigma_3\, \mathrm{M}(z_2\mp u_2) \begin{pmatrix} \mathrm{e}^{\eta\partial_2} & 0 \\ 0 & \mathrm{e}^{-\eta\partial_2} \end{pmatrix} \underline{\mathrm{N}(z_2\mp v_2)}\, \mathrm{S}_2.$$

We underlined the matrices which can be canceled under the commutativity condition $[\mathrm{S}_2, z_1] = [\mathrm{S}_2, z_2] = 0$. This observation suggests that S_2 is just a multiplication operator:

$$[\mathrm{S}_2\Phi]\,(z_1,z_2) = \mathrm{S}(z_1,z_2)\cdot \Phi(z_1,z_2)\,.$$

Consequently, the operator S_2 commutes with the matrices $\mathrm{M}(z_1\mp u_1)$ and $\mathrm{N}(z_2\mp v_2)$, so that they both cancel from the equation and we obtain a much simpler defining relation for the function $\mathrm{S}(z_1,z_2)$:

$$\mathrm{S}(z_1,z_2) \begin{pmatrix} \mathrm{e}^{\eta\partial_1} & 0 \\ 0 & \mathrm{e}^{-\eta\partial_1} \end{pmatrix} \cdot \mathrm{N}(z_1\mp u_2)\,\sigma_3\, \mathrm{M}(z_2\mp v_1) \begin{pmatrix} \mathrm{e}^{\eta\partial_2} & 0 \\ 0 & \mathrm{e}^{-\eta\partial_2} \end{pmatrix} =$$

$$= \begin{pmatrix} \mathrm{e}^{\eta\partial_1} & 0 \\ 0 & \mathrm{e}^{-\eta\partial_1} \end{pmatrix} \cdot \mathrm{N}(z_1\mp v_1)\,\sigma_3\, \mathrm{M}(z_2\mp u_2) \begin{pmatrix} \mathrm{e}^{\eta\partial_2} & 0 \\ 0 & \mathrm{e}^{-\eta\partial_2} \end{pmatrix} \mathrm{S}(z_1,z_2),$$

or in the equivalent form

$$\begin{pmatrix} \mathrm{S}(z_1-\eta,z_2) & 0 \\ 0 & \mathrm{S}(z_1+\eta,z_2) \end{pmatrix} \mathrm{N}(z_1\mp u_2)\,\sigma_3\, \mathrm{M}(z_2\mp v_1) =$$

$$= \mathrm{N}(z_1 \mp v_1)\, \sigma_3\, \mathrm{M}(z_2 \mp u_2) \begin{pmatrix} \mathrm{S}(z_1, z_2+\eta) & 0 \\ 0 & \mathrm{S}(z_1, z_2-\eta) \end{pmatrix}.$$

Using a theta functions identity given in the Appendix one can see that

$$\mathrm{N}(a_1; b_1) \cdot \sigma_3 \cdot \mathrm{M}(a_2; b_2) = 2 \cdot \begin{pmatrix} \theta_4\,(b_1 \mp a_2) & -\theta_4\,(b_1 \mp b_2) \\ \theta_4\,(a_1 \mp a_2) & -\theta_4\,(a_1 \mp b_2) \end{pmatrix}. \tag{4.5}$$

The derived matrix equation can be simplified further on, since a number of theta functions depending on the combination of parameters $u_2 + v_1$ drops out from it. As a result, we come to a system of four linear finite difference equations of the first order

$$\theta_4(z_1 + z_2 + u_2 - v_1)\, \mathrm{S}(z_1 - \eta, z_2) = \theta_4(z_1 + z_2 + v_1 - u_2)\, \mathrm{S}(z_1, z_2 + \eta)\ ,$$

$$\theta_4(z_1 + z_2 - u_2 + v_1)\, \mathrm{S}(z_1 + \eta, z_2) = \theta_4(z_1 + z_2 - v_1 + u_2)\, \mathrm{S}(z_1, z_2 - \eta)\ ,$$

$$\theta_4(z_1 - z_2 + u_2 - v_1)\, \mathrm{S}(z_1 - \eta, z_2) = \theta_4(z_1 - z_2 + v_1 - u_2)\, \mathrm{S}(z_1, z_2 - \eta)\ ,$$

$$\theta_4(z_1 - z_2 - u_2 + v_1)\, \mathrm{S}(z_1 + \eta, z_2) = \theta_4(z_1 - z_2 - v_1 + u_2)\, \mathrm{S}(z_1, z_2 + \eta)\ .$$

Their structure suggests to look for the solution in the factorized form

$$\mathrm{S}(z_1, z_2) = \Phi_+(z_1 + z_2) \cdot \Phi_-(z_1 - z_2).$$

Then, in each equality one of the $\Phi_\pm$-factors drops out and we obtain equations

$$\theta_4(z + u_2 - v_1)\, \Phi_\pm(z - \eta) = \theta_4(z - u_2 + v_1)\, \Phi_\pm(z + \eta), \tag{4.6}$$

or, in the equivalent form,

$$\Phi_\pm(z + 2\eta) = \mathrm{e}^{2\pi \mathrm{i}(u_2 - v_1)}\, \frac{\theta_1(z + u_2 - v_1 + \eta + \frac{\tau}{2})}{\theta_1(z - u_2 + v_1 + \eta + \frac{\tau}{2})} \cdot \Phi_\pm(z).$$

In this section we suppose that $\mathrm{Im}(\eta) > 0$. Then a particular solution of this equation is described by a ratio of elliptic gamma functions $\Gamma(z) = \Gamma(z|\tau, 2\eta)$ (see formula (11.14) in the Appendix),

$$\Phi(z + 2\eta) = \mathrm{e}^{\pi \mathrm{i}(a-b)}\, \frac{\theta_1(z + a)}{\theta_1(z + b)} \cdot \Phi(z)\ ; \qquad \Phi(z) = \frac{\Gamma(z + a|\tau, 2\eta)}{\Gamma(z + b|\tau, 2\eta)},\ \mathrm{Im}(\eta), \mathrm{Im}(\tau) > 0. \tag{4.7}$$

The equation for $\Phi(z)$ does not assume restrictions on η. Its solution and corresponding intertwining operators valid for $\mathrm{Im}(\eta) < 0$ use the function $\Gamma(z|\tau, -2\eta)$ and for $\mathrm{Im}(\eta) = 0$ one needs the modified elliptic gamma function [11, 14]. These solutions will be described in a special section below.

Using (4.7) we can write the general solution $\mathrm{S}(z_1, z_2)$ in the form

$$\mathrm{S}(z_1, z_2) = \frac{\Gamma(z_1 + z_2 + u_2 - v_1 + \eta + \frac{\tau}{2})}{\Gamma(z_1 + z_2 - u_2 + v_1 + \eta + \frac{\tau}{2})} \frac{\Gamma(z_1 - z_2 + u_2 - v_1 + \eta + \frac{\tau}{2})}{\Gamma(z_1 - z_2 - u_2 + v_1 + \eta + \frac{\tau}{2})} \varphi_2(z_1, z_2), \tag{4.8}$$

where φ_2 is an arbitrary function satisfying the periodicity conditions

$$\varphi_2(z_1 + 2\eta, z_2) = \varphi_2(z_1, z_2 + 2\eta) = \varphi_2(z_1 + \eta, z_2 + \eta) = \varphi_2(z_1, z_2).$$

Using the reflection formula for the elliptic gamma function (11.18) and the notation

$$\Gamma\left(\pm x \pm y + a\right) \equiv \Gamma\left(-x - y + a\right)\, \Gamma\left(-x + y + a\right)\, \Gamma\left(x - y + a\right)\, \Gamma\left(x + y + a\right),$$

it is possible to rewrite formula (4.8) in a much more compact form

$$\mathrm{S}(z_1, z_2) = \Gamma\left(\pm z_1 \pm z_2 + u_2 - v_1 + \eta + \tfrac{\tau}{2} | \tau, 2\eta\right) \varphi_2(z_1, z_2).$$

The functional freedom $\varphi_2(z_1, z_2)$ strongly influencing the final results will be fixed in the section dedicated to the elliptic modular double. We remark also that our choice of the L-operator in the form $\sigma_3\mathrm{L}$ is done for technical reasons in order to have permutational symmetry $z_1 \leftrightarrow z_2$ in (4.8), i.e. $\mathbb{P}_{12}\mathrm{S}_2(\mathbf{u}) = \mathrm{S}_2(\mathbf{u})\mathbb{P}_{12}$, absent for other choices.

Let us consider now defining equations (3.10) for operators S_1 and S_3. Permutation of the parameters $u_1 = \frac{u}{2} + \eta\left(\ell_1 + \frac{1}{2}\right)$ and $u_2 = \frac{u}{2} - \eta\left(\ell_1 + \frac{1}{2}\right)$ is equivalent to the change of the spin $\ell_1 \to -1 - \ell_1$ and similarly the permutation of parameters $v_1 = \frac{v}{2} + \eta\left(\ell_2 + \frac{1}{2}\right)$ and $v_2 = \frac{u}{2} - \eta\left(\ell_2 + \frac{1}{2}\right)$ is equivalent to the change $\ell_2 \to -1 - \ell_2$. In the L-matrix (2.3) only generators $\mathbf{S}^a$ depend on the spin, therefore defining equations (3.10) can be rewritten in terms of the $\mathbf{S}^a$-generators alone:

$$\mathrm{S}_1 \cdot \mathbf{S}^a(\ell_1) = \mathbf{S}^a(-1 - \ell_1) \cdot \mathrm{S}_1\,, \qquad \mathrm{S}_3 \cdot \mathbf{S}^a(\ell_2) = \mathbf{S}^a(-1 - \ell_2) \cdot \mathrm{S}_3\,, \tag{4.9}$$

where we explicitly indicate the spin ℓ dependence of $\mathbf{S}^a$-operators. The meaning of these relations is the following: the operator S_1 intertwines representations with the spins ℓ_1 and $-1 - \ell_1$ realized in the space of functions of the variable z_1 and the operator S_3 intertwines representations with the spins ℓ_2 and $-1 - \ell_2$ realized in the space of functions of variable z_2. Note that in terms of the variable $g = \eta(2\ell + 1)$ this corresponds to the simple sign change $g \to -g$.

Evidently the operators S_1 and S_3 are equivalent to each other differing only by the spaces where they are acting. Let us construct the general intertwining operator W defined in two equivalent ways: either as a solution of the matrix equation for the L-operator

$$\mathrm{W} \cdot \mathrm{L}(u_1, u_2) = \mathrm{L}(u_2, u_1) \cdot \mathrm{W}\,,$$

or, alternatively, as a solution of the system of equations involving generators of the Sklyanin algebra

$$\mathrm{W} \cdot \mathbf{S}^a(\ell) = \mathbf{S}^a(-1 - \ell) \cdot \mathrm{W}\,. \tag{4.10}$$

For $2\ell \in \mathbb{Z}_{\geq 0}$ such an intertwining operator W was constructed in [26] as a finite sum of the powers of the finite-difference operator $e^{-\eta\partial_z}$ (see below). A formal extension of this W to infinite series and arbitrary values of ℓ proposed in [27] does not represent a well defined operator.

In this paper we use a different approach inspired by the elliptic hypergeometric integrals [13, 14]. Namely, we construct the intertwining operator W using a quite general ansatz for it as an integral operator

$$\left[\mathrm{W}\, \Phi\right](z) = \int_\alpha^\beta \Delta(z, x)\, \Phi(x)\; dx \tag{4.11}$$

for some integration interval $[\alpha,\beta]\in\mathbb{R}$.

We are going to solve thus the equation

$$\int_\alpha^\beta \Delta(z,x)[\mathbf{S}^a(\ell)\Phi(x)]dx = \mathbf{S}^a(-1-\ell)\int_\alpha^\beta \Delta(z,x)\Phi(x)dx,$$

where $\mathbf{S}^a$-operators on the left-hand side act on the functions of variable x, whereas on the right-hand side — on the functions of variable z. More explicitly, we have

$$\begin{aligned}&\int_\alpha^\beta \frac{\Delta(z,x)}{\theta_1(2x)}\Big(\theta_a(2x-2\eta\ell)\Phi(x+\eta)-\theta_a(-2x-2\eta\ell)\Phi(x-\eta)\Big)dx\\ &=\int_{\alpha+\eta}^{\beta+\eta}\frac{\theta_a(2x-2\eta(\ell+1))}{\theta_1(2x-2\eta)}\Delta(z,x-\eta)\Phi(x)dx-\int_{\alpha-\eta}^{\beta-\eta}\frac{\theta_a(-2x-2\eta(\ell+1))}{\theta_1(2x+2\eta)}\Delta(z,x+\eta)\Phi(x)dx\\ &=\int_\alpha^\beta\Big(\frac{\theta_a(2z+2\eta(\ell+1))}{\theta_1(2z)}\Delta(z+\eta,x)-\frac{\theta_a(-2z+2\eta(\ell+1))}{\theta_1(2z)}\Delta(z-\eta,x)\Big)\Phi(x)dx,\end{aligned}$$

where $a=1,2,3,4$. To get consistent equations for the kernel $\Delta(z,x)$ we impose the constraint that the integrations $\int_{\alpha\pm\eta}^{\beta\pm\eta}dx$ give the same result as the integration $\int_\alpha^\beta dx$. For $\mathrm{Im}(\eta)>0$ or $\mathrm{Im}(\eta)<0$ this is so if the integrals over intervals $[\alpha,\alpha+\eta]$ and $[\beta,\beta+\eta]$ as well as over $[\alpha,\alpha-\eta]$ and $[\beta,\beta-\eta]$ coincide and if the contour integrals over the parallelograms $[\alpha,\beta,\beta+\eta,\alpha+\eta]$ and $[\alpha,\beta,\beta-\eta,\alpha-\eta]$ are equal to zero. The former constraint is satisfied if the integrands are periodic with the period $\beta-\alpha$ and the latter condition is fulfilled if the integrands are analytical in the respective parallelograms and have no simple poles there (or the sum of their residues is equal to zero). The case $\mathrm{Im}(\eta)=0$ will be considered separately.

Supposing that these demands are satisfied, which will be analyzed a posteriori, we obtain the equation

$$\begin{aligned}\int_\alpha^\beta\Big(&\frac{\theta_a(2x-s)}{\theta_1(2x-2\eta)}\Delta(z,x-\eta)-\frac{\theta_a(-2x-s)}{\theta_1(2x+2\eta)}\Delta(z,x+\eta)\\ &-\frac{\theta_a(2z+s)}{\theta_1(2z)}\Delta(z+\eta,x)+\frac{\theta_a(-2z+s)}{\theta_1(2z)}\Delta(z-\eta,x)\Big)\Phi(x)dx=0,\end{aligned}\tag{4.12}$$

where $s=2\eta(\ell+1)$. Since this integral should vanish for arbitrary admissible function $\Phi(x)$, its integrand should vanish on its own. Therefore the following system of four finite-difference equations should be true

$$\begin{aligned}&\frac{\theta_a(2x-s)}{\theta_1(2x-2\eta)}\Delta(z,x-\eta)-\frac{\theta_a(-2x-s)}{\theta_1(2x+2\eta)}\Delta(z,x+\eta)\\ &\qquad=\frac{\theta_a(2z+s)}{\theta_1(2z)}\Delta(z+\eta,x)-\frac{\theta_a(-2z+s)}{\theta_1(2z)}\Delta(z-\eta,x).\end{aligned}\tag{4.13}$$

Let us multiply the equation with $a=3$ by $\theta_4(2z+s)$ and the equation with $a=4$ by $\theta_3(2z+s)$ and subtract them from each other. Using theta-functions identity (11.7) from the Appendix we obtain the equality

$$\begin{aligned}&\frac{\theta_1(2z+2x,2z-2x+2s|2\tau)}{\theta_1(2x-2\eta|\tau)}\Delta(z,x-\eta)-\frac{\theta_1(2z-2x,2z+2x+2s|2\tau)}{\theta_1(2x+2\eta|\tau)}\Delta(z,x+\eta)\\ &\qquad=-\frac{\theta_1(2s,4z|2\tau)}{\theta_1(2z|\tau)}\Delta(z-\eta,x).\end{aligned}\tag{4.14}$$

Similarly, we multiply now the equation with $a = 1$ by $\theta_2(2z + s)$ and the equation with $a = 2$ by $\theta_1(2z + s)$ and subtract them from each other. Applying theta-functions identity (11.11) from the Appendix we obtain

$$\begin{aligned} &\frac{\theta_4(2z+2x|2\tau)\theta_1(2z-2x+2s|2\tau)}{\theta_1(2x-2\eta|\tau)}\Delta(z,x-\eta) - \frac{\theta_4(2z-2x|2\tau)\theta_1(2z+2x+2s|2\tau)}{\theta_1(2x+2\eta|\tau)}\Delta(z,x+\eta) \\ &\qquad = -\frac{\theta_4(2s|2\tau)\theta_1(4z|2\tau)}{\theta_1(2z|\tau)}\Delta(z-\eta,x). \end{aligned} \tag{4.15}$$

Exclude the term $\Delta(z - \eta, x)$ from the obtained equations. Namely, divide equation (4.14) by $\theta_1(2s|2\tau)$, equation (4.15) by $\theta_4(2s|2\tau)$ and subtract them. This yields the equation

$$\Big(\theta_1(2z+2x|2\tau)\theta_4(2s|2\tau) - \theta_4(2z+2x|2\tau)\theta_1(2s|2\tau)\Big)\frac{\theta_1(2z-2x+2s|2\tau)}{\theta_1(2x-2\eta|\tau)}\Delta(z,x-\eta)$$

$$= \Big(\theta_1(2z-2x|2\tau)\theta_4(2s|2\tau) - \theta_4(2z-2x|2\tau)\theta_1(2s|2\tau)\Big)\frac{\theta_1(2z+2x+2s|2\tau)}{\theta_1(2x+2\eta|\tau)}\Delta(z,x+\eta).$$

Applying theta-function identities (11.12) and (11.13) from the Appendix, we come to the following compact linear first order finite difference equation

$$\frac{\Delta(z,x+\eta)}{\Delta(z,x-\eta)} = \frac{\theta_1(2x+2\eta, z+x-s, z-x+s)}{\theta_1(2x-2\eta, z-x-s, z+x+s)}. \tag{4.16}$$

Exclude now from system (4.13) the terms $\Delta(z, x \pm \eta)$. Repeating similar steps as before, we exclude first $\Delta(z, x-\eta)$, then $\Delta(z, x+\eta)$, and come to analogous equations with $\theta_1(2x\pm 2\eta)$ replaced by $\theta_1(2z)$ and slightly changed arguments in other theta functions. The final result is

$$\frac{\Delta(z+\eta,x)}{\Delta(z-\eta,x)} = \frac{\theta_1(x-z+s, x+z-s)}{\theta_1(x+z+s, x-z-s)}. \tag{4.17}$$

Excluding the $\Delta(z, x-\eta)$-term from equations (4.14) and (4.15) we obtain the third needed equation

$$\frac{\Delta(z,x+\eta)}{\Delta(z-\eta,x)} = \frac{\theta_1(2x+2\eta, z+x-s)}{\theta_1(2x, z+x+s)}. \tag{4.18}$$

The same equation emerges (in a different way) if one excludes $\Delta(z + \eta, x)$-term from the pair of equations leading to relation (4.17).

After these considerations it is not difficult to find the general solution of equations (4.16), (4.17), and (4.18) valid for $\mathrm{Im}(\eta) > 0$:

$$\Delta(z,x) = \mathrm{e}^{\frac{\pi \mathrm{i}}{\eta}(x^2-z^2)}\,\frac{\Gamma(\pm z \pm x + \eta - s|\tau, 2\eta)}{\Gamma(\pm 2x|\tau, 2\eta)}\varphi(z,x), \tag{4.19}$$

where $\varphi(z+2\eta, x) = \varphi(z, x+2\eta) = \varphi(z+\eta, x+\eta) = \varphi(z, x)$ is an arbitrary periodic function. A way to fix this functional freedom by imposing an additional symmetry will be considered in the section on elliptic modular double.

The key ingredient in expression (4.19) described by the ratio of elliptic gamma functions is periodic in x with the period 1. Similarly, $\theta_a(2x)$-functions entering the intertwining relation

(4.12) have this period. Therefore we have to demand that the rest of the integrands in (4.12) be periodic with the same period 1. This condition forces the length of the integration interval to be equal to 1, $\beta - \alpha = 1$. This periodicity allows us to fix the point α arbitrarily, and we take $\alpha = 0$ and $\beta = 1$. After the shifts $x \to x \pm \eta$ the factor $e^{\pi \mathrm{i} x^2/\eta}$ gets multiplied by the function $e^{\pm 2\pi \mathrm{i} x + \pi \mathrm{i}\eta}$ which has period 1. Therefore we have to demand that the products $e^{\pi \mathrm{i} x^2/\eta}\varphi(z, x \pm \eta)\Phi(x)$ are periodic functions of x with period 1.

Now we note that the ratio of elliptic gamma functions in $\Delta(z, x)$ is invariant under the transformations $z \to z+1$ and $z \to -z$. If we require that similar properties are obeyed by the function $\varphi(z, x)$, then the functions $\Psi(z) = e^{\pi \mathrm{i} z^2/\eta}[W\Phi](z)$ become periodic $\Psi(z+1) = \Psi(z)$ and even $\Psi(-z) = \Psi(z)$. Therefore it is natural to demand that the original functions $\Phi(x)$ belong to the same class of functions, i.e. that $e^{\pi \mathrm{i} x^2/\eta}\Phi(x)$ are invariant under the transformations $x \to x + 1$ and $x \to -x$. This assumes that $\varphi(z, x + 1) = \varphi(z, -x) = \varphi(z, x)$, which resolves all the periodicity restrictions and forces $\varphi(z, x)$ to be an even elliptic function of z and x with periods 2η and 1. This fixes the space of functions $\Phi(x)$ where our operator W can work as an intertwining operator of the Sklyanin algebra generators. Note that its structure does not contradict with the property that for $2\ell \in \mathbb{Z}_{\geq 0}$ the Sklyanin algebra has finite-dimensional representations in the space of even theta functions of order 4ℓ, since $e^{\pi \mathrm{i} x^2/\eta}$ is an even theta function of order zero.

It remains to consider singularities of the integrand in (4.12). The reflection equation for elliptic gamma function (11.18) shows that the product $\theta_1(2x|\tau)\Gamma(2x|\tau, 2\eta)\Gamma(-2x|\tau, 2\eta)$ has not zeros, i.e. the poles at $x = \eta$ or $x = -\eta$ in the integrands of relation (4.12) are spurious. The divisor structure of the elliptic gamma function shows that if

$$e^{2\pi \mathrm{i}(\pm x \pm z + \eta - s + \tau j + 2\eta k)} \neq 1, \quad j, k = 0, 1, 2, \ldots,$$

for any choice of signs when x varies in the rectangle $[-\mathrm{Im}(\eta), 1-\mathrm{Im}(\eta), 1+\mathrm{Im}(\eta), \mathrm{Im}(\eta)]$, then no poles enter the needed domain. In the multiplicative notation $X = e^{2\pi \mathrm{i} x}$, $Z = e^{2\pi \mathrm{i} z}$, $p = e^{2\pi \mathrm{i}\tau}$, $q = e^{4\pi \mathrm{i}\eta}$, one has the constraint $|XZ^{\pm 1}tp^jq^k| \neq 1$, $t = e^{2\pi \mathrm{i}(\eta - s)} = e^{-2\pi \mathrm{i}\eta(2\ell+1)} = e^{2\pi \mathrm{i}(u_2 - u_1)}$, when $|q|^{1/2} \leq |X| \leq |q|^{-1/2}$. Evidently, for such values of X the annuli $|Xq^j|, j = 0, 1, \ldots$, cover the whole disk of radius $|q|^{-1/2}$. Therefore we escape poles, if

$$|Z^{\pm 1}t| < |q|^{1/2} \quad \text{or} \quad \mathrm{Im}(u_1 - u_2 \pm z) < \mathrm{Im}(\eta). \tag{4.20}$$

If z is a real number, i.e. $|Z| = 1$, then we come to the constraint $|t| < |q|^{1/2}$ or $\mathrm{Im}(\eta(\ell+1)) < 0$. For real ℓ this means that $\ell < -1$, since $\mathrm{Im}(\eta) > 0$. The finite-dimensional realizations of the R-matrix emerging for half-integer spins, $2\ell \in \mathbb{Z}_{\geq 0}$, require thus a special treatment.

Demanding that our basic space functions $\Phi(x)$ and the periodic factors $\varphi(z, x)$ have no simple poles in the domain of interest, we satisfy completely the conditions guaranteeing validity of equations (4.13). We stress that the function obtained after action of our operator $\int_0^1 \Delta(z, x)\Phi(x)dx$ satisfies the demands imposed on $\Phi(x)$. Indeed, since $\mathrm{Im}(x) = 0$ the function of interest is analytical when z varies in the rectangle $[-\mathrm{Im}(\eta), 1 - \mathrm{Im}(\eta), 1 + \mathrm{Im}(\eta), \mathrm{Im}(\eta)]$.

Presence of the exponential $e^{\pi i x^2/\eta}$ in the definition of base space functions is annoying and we wish to pass to a more natural setting. In order to do that we conjugate all our operators by this exponential and define new realization of the Sklyanin algebra generators

$$\mathbf{S}^a_{mod} = e^{\pi \mathrm{i} z^2/\eta} \mathbf{S}^a e^{-\pi \mathrm{i} z^2/\eta} = e^{-\pi \mathrm{i}\eta} \frac{(\mathrm{i})^{\delta_{a,2}} \theta_{a+1}(\eta)}{\theta_1(2z)} \Big[\theta_{a+1}\left(2z - g + \eta\right) \cdot e^{-2\pi \mathrm{i} z} \cdot \mathrm{e}^{\eta \partial_z} - \theta_{a+1}\left(-2z - g + \eta\right) \cdot e^{2\pi \mathrm{i} z} \cdot \mathrm{e}^{-\eta \partial_z} \Big], \tag{4.21}$$

where we denoted $g = \eta(2\ell + 1)$. Analogously, we define

$$\mathrm{W}_{mod} = e^{\pi \mathrm{i} z^2/\eta} \mathrm{W} e^{-\pi \mathrm{i} x^2/\eta},$$

acting as

$$[\mathrm{W}_{mod} \Psi](z) = \int_0^1 \frac{\Gamma(\pm z \pm x + \eta - s|\tau, 2\eta)}{\Gamma(\pm 2x|\tau, 2\eta)} \varphi(z, x) \cdot \Psi(x) dx.$$

Then the intertwining relation $\mathrm{W}_{mod} \cdot \mathbf{S}^a_{mod}(\ell) = \mathbf{S}^a_{mod}(-1 - \ell) \cdot \mathrm{W}_{mod}$ is true provided our operators act in the space of even periodic functions $\Psi(x) = \Psi(-x) = \Psi(x + 1)$ which do not have simple poles in the domain $-\mathrm{Im}(\eta) \leq \mathrm{Im}(x) \leq \mathrm{Im}(\eta)$. Let us summarize obtained results.

Theorem 3. *Let* $\mathrm{Im}(\eta) > 0$ *(or* $|q| < 1$*). Denote as* V *the space of functions of two complex variables* $\Psi(z_1, z_2)$ *which are even and periodic in each variable with the period 1 and which do not have simple poles in the domains* $-Im(\eta) \leq Im(z_1), Im(z_2) \leq Im(\eta)$*. Define three operators*

$$[\mathrm{S}_2(u_2 - v_1)\Psi](z_1, z_2) = \Gamma\left(\pm z_1 \pm z_2 + u_2 - v_1 + \eta + \tfrac{\tau}{2}\right) \varphi_2(z_1, z_2) \cdot \Psi(z_1, z_2), \tag{4.22}$$

where $Im(u_2 - v_1 + \tau/2 - \eta) > 0$ *(or* $|\sqrt{pq} e^{2\pi \mathrm{i}(u_2 - v_1)}| < |q|$*),*

$$[\mathrm{S}_1(u_1 - u_2)\Psi](z_1, z_2) = \frac{\kappa}{\Gamma(2u_2 - 2u_1)} \int_0^1 \frac{\Gamma(\pm z_1 \pm x + u_2 - u_1)}{\Gamma(\pm 2x)} \varphi_1(z_1, x) \cdot \Psi(x, z_2) dx, \tag{4.23}$$

where $Im(u_2 - u_1 \pm z_1 - \eta) > 0$ *(or* $|e^{2\pi \mathrm{i}(u_2 - u_1 \pm z_1)}| < |q|^{1/2}$*),*

$$\kappa = \frac{(q;q)_\infty \, (p;p)_\infty}{2}, \qquad (t;q)_\infty := \prod_{j=0}^{\infty} (1 - t q^j),$$

and

$$[\mathrm{S}_3(v_1 - v_2)\Psi](z_1, z_2) = \frac{\kappa}{\Gamma(2v_2 - 2v_1)} \int_0^1 \frac{\Gamma(\pm z_2 \pm x + v_2 - v_1)}{\Gamma(\pm 2x)} \varphi_3(z_2, x) \cdot \Psi(z_1, x) dx, \tag{4.24}$$

where $Im(v_2 - v_1 \pm z_2 - \eta) > 0$ *(or* $|e^{2\pi \mathrm{i}(v_2 - v_1 \pm z_2)}| < |q|^{1/2}$*). Here* $\varphi_k(z, x)$ *are arbitrary even elliptic functions of* z *and* x *with periods* 1 *and* 2η *satisfying additional constraints*

$$\varphi_k(z + \eta, x + \eta) = \varphi_k(z, x), \quad k = 1, 2, 3,$$

and not having simple poles in the domains $-Im(\eta) \leq Im(z), Im(x) \leq Im(\eta)$*.*

Then the operators $\mathrm{S}_k(\mathbf{u})$ *map the space* V *onto itself and satisfy the defining intertwining relations* (3.10)*,* (3.11)*, and* (3.12)*, provided in the corresponding* L*-operator* (2.2) *one uses the Sklyanin algebra generators in the form* (4.21)*.*

Here we do not use the notation $\mathrm{S}_{k,mod}$ for brevity assuming that there will be no confusion in the following which particular form of the intertwining operators is used. The functions φ_k may depend on the parameters u_1, u_2, v_1, v_2 in arbitrary way. If they would not depend on the coordinates z and x, then we can drop φ_k completely, since they become irrelevant for solutions of YBE (without consideration of the unitarity condition). Then the operators $\mathrm{S}_{1,3}$ do not depend on the spectral parameter since they involve only the differences $u_1 - u_2$ or $v_1 - v_2$. Similarly, S_2 will depend only on the difference $u_2 - v_1$. The above choice of the normalization constant κ, as well as of the elliptic gamma function prefactors in $\mathrm{S}_{1,3}$ are dictated by the elliptic beta integral [13], as described explicitly in the next section for a special choice $\varphi_k(z, x) = 1$.

Denote $y_{1,2} = e^{2\pi \mathrm{i} z_{1,2}}$. Fourier series expansions for our basic functions $\Psi(z_1, z_2)$ show that our space is equivalent to the space of meromorphic functions of $y_1, y_2 \in \mathbb{C}^*$ satisfying the constraints $f(y_1^{-1}, y_2) = f(y_1, y_2^{-1}) = f(y_1, y_2)$. Therefore we can pass in the definition of $\mathrm{S}_{1,3}$-operators from real integrals over $[0, 1]$ to contour integrals over the unit circle $\mathbb{T}$ of positive orientation

$$[\mathrm{S}_1(u_1 - u_2)f](y_1, y_2) = \kappa \int_{\mathbb{T}} \frac{\Gamma(t y_1^{\pm 1} y^{\pm 1}; p, q)}{\Gamma(t^2, y^{\pm 2}; p, q)} \varphi_1(y_1, y) \cdot f(y, y_2) \frac{dy}{2\pi \mathrm{i} y}, \tag{4.25}$$

where $t = e^{2\pi \mathrm{i}(u_2 - u_1)}$, $|t y_1^{\pm 1}| < |q|$, $\varphi_1(q y_1, y) = \varphi_1(y_1, q y) = \varphi_1(q^{1/2} y_1, q^{1/2} y) = \varphi_1(y_1, y)$, and $\Gamma(t; p, q)$ is the elliptic gamma function in multiplicative notation (11.19). Evidently, sequential actions of the S_k-operators create multiple contour integral operators. Deforming the integration contours one can relax the constraints on parameters and define resulting functions by analytical continuation in parameters.

5 Coxeter relations and the elliptic beta integral

In this section we prove that the derived operators S_i obey relations (3.17), (3.18), and (3.19) generating the permutation group $\mathfrak{S}_4$ for a special choice of the periodic factors

$$\varphi_k(z, x) = 1, \qquad k = 1, 2, 3.$$

Under these conditions operators (4.22)-(4.24) depend on the differences of parameters. Therefore Coxeter relations for them can be represented in a simpler form:

$$\mathrm{S}_k(a)\, \mathrm{S}_k(-a) = \mathbb{1}\,; \qquad \mathrm{S}_1(a)\, \mathrm{S}_2(a+b)\, \mathrm{S}_1(b) = \mathrm{S}_2(b)\, \mathrm{S}_1(a+b)\, \mathrm{S}_2(a)\,, \tag{5.1}$$

$$\mathrm{S}_1(a)\, \mathrm{S}_3(b) = \mathrm{S}_3(b)\, \mathrm{S}_1(a)\,; \qquad \mathrm{S}_2(a)\, \mathrm{S}_3(a+b)\, \mathrm{S}_2(b) = \mathrm{S}_3(b)\, \mathrm{S}_2(a+b)\, \mathrm{S}_3(a). \tag{5.2}$$

In the theory of quantum integrable systems the cubic relations are known as the star-triangle relations [22]. There are two evident equalities. The operator $\mathrm{S}_3(a)$ differs from the operator $\mathrm{S}_1(a)$ only by the change of variable $z_1 \to z_2$ and therefore these operators commute, $\mathrm{S}_1(a)\, \mathrm{S}_3(b) = \mathrm{S}_3(b)\, \mathrm{S}_1(a)$. If one would stick to the generating relations in the form $(s_i s_{i\pm 1})^3 = 1$, then the cubic relation in (5.1) is replaced by

$$\mathrm{S}_1(-b)\, \mathrm{S}_2(-a-b)\, \mathrm{S}_1(-a)\, \mathrm{S}_2(b)\, \mathrm{S}_1(a+b)\, \mathrm{S}_2(a) \; = \mathbb{1}$$

and a similar replacement holds for (5.2).

The equality $\mathrm{S}_2(a)\,\mathrm{S}_2(-a) = \mathbb{1}$ is easily verified with the help of reflection formula for elliptic gamma function (we set $\varphi_2 = 1$), since the operator S_2 reduces to the multiplication by a given function. If we would have an arbitrary periodic function $\varphi_2(z_1, z_2; a)$ in the definition of $\mathrm{S}_2(a)$ (in fact, φ_2 may depend on u_2 and v_1 separately, not only on their difference), then this condition does not fix φ_2, there are many nontrivial elliptic functions of z_1, z_2 obeying the constraint $\varphi_2(z_1, z_2; a)\varphi_2(z_1, z_2; -a) = 1$. Only if $\varphi_2(z_1, z_2; a)$ does not depend on z_1, z_2, this condition fixes S_2 up to a constant multiplier $\varphi(a)$ satisfying the constraint $\varphi_2(a)\varphi_2(-a) = 1$.

Let us show that the remaining nontrivial Coxeter relations follow from the elliptic beta integral evaluation formula [13]. Denote

$$[\mathrm{S}_2(a)\,\Psi]\,(z_1, z_2) = \mathrm{D}_a(z_1, z_2)\cdot\Psi(z_1, z_2) \quad ; \quad \mathrm{D}_a(z_1, z_2) = \Gamma(\pm z_1 \pm z_2 + a + \eta + \tfrac{\tau}{2})\,,$$

and

$$[\mathrm{S}_1(b)\,\Psi]\,(z_1, z_2) = \int_0^1 \mathrm{d}z\,\mathrm{W}_b(z_1, z)\,\Psi(z, z_2) \quad ; \quad \mathrm{W}_b(z_1, z) = \kappa\frac{\Gamma(\pm z \pm z_1 - b)}{\Gamma(-2b, \pm 2z)}\,,$$

where $\Gamma(a, b) = \Gamma(a|2\eta, \tau)\Gamma(b|2\eta, \tau)$. Similarly, $\mathrm{S}_3(b)$ has the same form as $\mathrm{S}_1(b)$, but it acts in the space of functions of z_2.

The first Coxeter relation

$$\mathrm{S}_1(a)\,\mathrm{S}_2(a+b)\,\mathrm{S}_1(b) = \mathrm{S}_2(b)\,\mathrm{S}_1(a+b)\,\mathrm{S}_2(a) \tag{5.3}$$

is equivalent to the following equation for the kernels

$$\int_0^1 \mathrm{d}z\,\mathrm{W}_a(z_1, z)\,\mathrm{D}_{a+b}(z, z_2)\,\mathrm{W}_b(z, x) = \mathrm{D}_b(z_1, z_2)\,\mathrm{W}_{a+b}(z_1, x)\,\mathrm{D}_a(x, z_2)\,. \tag{5.4}$$

The second Coxeter relation

$$\mathrm{S}_3(a)\,\mathrm{S}_2(a+b)\,\mathrm{S}_3(b) = \mathrm{S}_2(b)\,\mathrm{S}_3(a+b)\,\mathrm{S}_2(a) \tag{5.5}$$

is equivalent to a similar equation for the kernels

$$\int_0^1 \mathrm{d}z\,\mathrm{W}_a(z_2, z)\,\mathrm{D}_{a+b}(z_1, z)\,\mathrm{W}_b(z, x) = \mathrm{D}_b(z_1, z_2)\,\mathrm{W}_{a+b}(z_2, x)\,\mathrm{D}_a(z_1, x)\,. \tag{5.6}$$

This equation can be obtained from the first one after permutation $z_1 \leftrightarrow z_2$ provided the function $\mathrm{D}_a(z_1, z_2)$ is symmetric, $\mathrm{D}_a(z_1, z_2) = \mathrm{D}_a(z_2, z_1)$, which is evident in our case. Therefore we have to prove the first Coxeter relation only.

Let us compare the elliptic beta integral evaluation formula [13]

$$\kappa\int_0^1 \frac{\prod_{k=1}^6 \Gamma(g_k \pm z)}{\Gamma(\pm 2z)}\,\mathrm{d}z = \prod_{1\le j<k\le 6} \Gamma(g_j + g_k)\,, \tag{5.7}$$

where

$$g_1 + \cdots + g_6 = 2\eta + \tau\,(\mathrm{mod}\ \mathbb{Z}), \qquad \mathrm{Im}(g_k), \mathrm{Im}(\eta), \mathrm{Im}(\tau) > 0, \tag{5.8}$$

with relation (5.6) which has the following explicit form

$$\kappa \int_0^1 \mathrm{d}z \, \frac{\Gamma(\pm z \pm z_1 - a)}{\Gamma(-2a, \pm 2z)} \cdot \Gamma(\pm z \pm z_2 + a + b + \eta + \tfrac{\tau}{2}) \cdot \frac{\Gamma(\pm x \pm z - b)}{\Gamma(-2b, \pm 2x)} =$$

$$= \Gamma(\pm z_1 \pm z_2 + b + \eta + \tfrac{\tau}{2}) \cdot \frac{\Gamma(\pm x \pm z_1 - a - b)}{\Gamma(-2a - 2b, \pm 2x)} \cdot \Gamma(\pm x \pm z_2 + a + \eta + \tfrac{\tau}{2}) \,.$$

After evident simplifications we arrive at the elliptic beta integral (5.7) for the following choice of parameters

$$\begin{aligned} &g_1 = z_1 - a \ , \ g_2 = -z_1 - a \ , \ g_3 = z_2 + a + b + \eta + \tfrac{\tau}{2} \ , \\ &g_4 = -z_2 + a + b + \eta + \tfrac{\tau}{2} \ , \ g_5 = x - b \ , \ g_6 = -x - b \,. \end{aligned}$$

The constraints on values of z_1, z_2, x, a, and b used in the construction of intertwining operators guarantee that $|e^{2\pi \mathrm{i} g_{1,2,5,6}}| < |q|^{1/2}$ and $|e^{2\pi \mathrm{i} g_{3,4}}| < |q|$. Since $\prod_{k=1}^6 e^{2\pi \mathrm{i} g_k} = pq$, we conclude that we should have $|p| < |q|^3$. However, by analytical continuation one can see that identity (5.4) is valid for a wider region of parameters $\mathrm{Im}(g_k) > 0$ which is symmetric in 2η and τ (or p and q). Thus, the cubic Coxeter relations hold true as a consequence of the exact integration formula (5.7).

The same result is obtained for the original Sklyanin algebra generators realization (2.5) leading to the exponential factors $e^{\pi \mathrm{i}(z^2 - x^2)/\eta}$ in the definition of $\mathrm{S}_{1,3}$-operators. Namely, all such exponentials cancel from the integral identity.

It is not clear whether our normalization $\varphi_k = 1$ (or, more generally, the demand that φ_k do not depend on z_1, z_2) is crucial for the obtained result. Cubic Coxeter relation is valid for all φ_k-functions satisfying the constraint

$$\varphi_1(z_1, z; a)\varphi_2(z, z_2; a + b)\varphi_1(z, x; b) = \varphi_2(z_1, z_2; b)\varphi_1(z_1, x; a + b)\varphi_2(x, z_2; a).$$

It is necessary to understand how strongly this relation restricts functions $\varphi_k(z, x)$. Perhaps there are nontrivial solutions, so that it is desirable to fix the functional freedom in somewhat different way.

Let us take the limit $b \to -a$ in relation (5.3). It is evident that $\mathrm{S}_2(0) = \mathbb{1}$, i.e. the left-hand side expression reduces to the product $\mathrm{S}_1(a)\,\mathrm{S}_1(-a)$ and the right-hand side – to $\mathrm{S}_2(-a)\mathrm{S}_1(0)\mathrm{S}_2(a)$. As it will be shown below in the section on finite-dimensional reductions, one has $\mathrm{S}_1(0) = \mathbb{1}$ (this follows after careful resolution of the ambiguity arising from vanishing multiplier in front of the integral and diverging value of the integral itself). Since $\mathrm{S}_2(-a)\mathrm{S}_2(a) = \mathbb{1}$, we formally come to the remaining quadratic Coxeter relations

$$\mathrm{S}_1(a)\,\mathrm{S}_1(-a) = \mathbb{1}\,; \quad \mathrm{S}_3(a)\,\mathrm{S}_3(-a) = \mathbb{1}\,.$$

The problem is that for the moment we have defined operators $\mathrm{S}_{1,3}(a)$ only in the restricted domain of values of a and z_1, namely, $\mathrm{Im}(a \pm z_1) < -\mathrm{Im}(\eta)$. Therefore we should give proper definition to $\mathrm{S}_{1,3}(-a)$-operators for the inversion relations to be true.

Suppose that the integral operator $\mathrm{S}_1(a)$ in the form (4.25) acts on a holomorphic function of $y \in \mathbb{C}^*$ so that all the singularities of the resulting function are determined by the kernel of

$\mathrm{S}_1(a)$. From the divisor structure of elliptic gamma functions one can see that the latter kernel has poles at

$$ in: \ y = t y_1^{\pm 1} p^j q^k, \qquad out: \ y = t^{-1} y_1^{\pm 1} p^{-j} q^{-k}, \tag{5.9} $$

where $j, k \in \mathbb{Z}_{\geq 0}$, with the first sequence converging to zero $y = 0$ and the second one going to infinity. For $|tx^{\pm 1}| < |q|^{1/2}$ the contour $\mathbb{T}$ separates these two sets of poles. Let us replace now the integration contour $\mathbb{T}$ in (4.25) by an arbitrary contour C separating poles *in* from *out*. By Cauchy theorem no singularities emerge after changing values of variables t and x as soon as the kernel poles do not cross the integration contour. Evidently, this procedure extends the definition of the operator S_1 to arbitrary values of t and x guaranteeing existence of the contour C. The latter condition is violated if the poles from *in* and *out* sets pinch the integration contour, which can happen if

$$ t y_1^{\pm 1} p^{j_1} q^{k_1} = (t y_1^{\pm 1})^{-1} p^{-j_2} q^{-k_2} \quad \text{or} \quad t y_1^{\pm 1} p^{j_1} q^{k_1} = (t y_1^{\mp 1})^{-1} p^{-j_2} q^{-k_2}. \tag{5.10} $$

The S_1-operator kernel contains also the multiplier $1/\Gamma(t^2; p, q)$ vanishing for $t^2 \to p^{-j} q^{-k}$ and diverging for $t^2 \to p^{j+1} q^{k+1}$ with $j, k \in \mathbb{Z}_{\geq 0}$. As a result, we come to the constraints

$$ t^2 \neq p^{-j} q^{-k}, \ p^{j+1} q^{k+1}, \quad j, k \in \mathbb{Z}_{\geq 0}, \tag{5.11} $$

and $y_1^2 \neq t^2 p^j q^k, \ t^{-2} p^{-j} q^{-k}$ for arbitrary fixed t and $j, k \in \mathbb{Z}_{\geq 0}$. Thus we have defined the action of operator $[\mathrm{S}_1(a)\Phi](z)$ for arbitrary generic values of z and $a \neq \eta j + \tau k/2$ or $a \neq 1/2 + \eta j + \tau k/2$, $j, k \in \mathbb{Z}$, $(j, k) \neq (1, 0), (0, 1)$.

For $t^2 = q^{-n} p^{-m}$ with integer $n, m \geq 0$ and generic values of q and p, which we always assume, the second set of equalities in (5.10) is satisfied if $j_1 + j_2 = m$ and $k_1 + k_2 = n$, i.e. there are $(n+1) \times (m+1)$ pairs of poles pinching the contour C. As will be shown in Sect. 8 for these exceptional values of t, with the exclusion of the points $t = \pm 1$ (i.e., $a = 0, 1/2$), the operator $\mathrm{S}_1(a)$ has nontrivial zero modes. Analogously, the operator $\mathrm{S}_1(-a)$ has zero modes for $t^2 = q^n p^m$, with integer $n, m \geq 0$, $(n, m) \neq (0, 0)$. As a result, the relation $\mathrm{S}_1(a)\mathrm{S}_1(-a) = \mathbb{1}$ cannot be true for $t^2 = q^n p^m$, $n, m \in \mathbb{Z}$, $(n, m) \neq 0$.

The following rigorous inversion statement was established in [29].

Theorem 4. *Let $p, q, t \in \mathbb{C}$ such that $\max\{|p|, |q|\} < |t|^2 < 1$. For fixed $w \in \mathbb{T}$ let C_w denote a contour inside the annulus $\mathbb{A} = \{z \in \mathbb{C}; \ |t| - \epsilon < |z| < |t|^{-1} + \epsilon\}$ for infinitesimally small but positive ϵ, such that C_w has the points $wt^{-1}, (wt)^{-1}$ in its interior and excludes their reciprocals. Let $f(z) = f(z^{-1})$ be a function holomorphic on $\mathbb{A}$. Then for $|t| < |x| < |t|^{-1}$ there holds*

$$ \frac{(p;p)_\infty^2 (q;q)_\infty^2}{(4\pi \mathrm{i})^2} \int_{\mathbb{T}} \bigg(\int_{C_w} \frac{\Gamma(t w^{\pm 1} x^{\pm 1}, t^{-1} w^{\pm 1} z^{\pm 1}; p, q)}{\Gamma(t^{\pm 2}, z^{\pm 2}, w^{\pm 2}; p, q)} f(z) \frac{dz}{z} \bigg) \frac{dw}{w} = f(x). \tag{5.12} $$

Rewriting relation (5.12) as

$$ \kappa \int_{\mathbb{T}} \frac{\Gamma(t w^{\pm 1} x^{\pm 1}; p, q)}{\Gamma(t^2, w^{\pm 2}; p, q)} \frac{dw}{2\pi \mathrm{i} w} \bigg(\kappa \int_{C_w} \frac{\Gamma(t^{-1} w^{\pm 1} z^{\pm 1}; p, q)}{\Gamma(t^{-2}, z^{\pm 2}; p, q)} f(z) \frac{dz}{2\pi \mathrm{i} z} \bigg) = f(x), $$

one evidently comes to the equality $[\mathrm{S}_1(a)\mathrm{S}_1(-a)f](x) = f(x)$ for analytically continued $\mathrm{S}_1(a)$-operators. Here $\mathrm{S}_1(a)$ is extended to the domain of parameters $|tx^{\pm 1}| < 1$ and $\mathrm{S}_1(-a)$ is a continuation of $\mathrm{S}_1(a)$-operator to the domain $|tw^{\pm 1}| < 1/\max\{|p|^{1/2}, |q|^{1/2}\} + \epsilon$.

In (5.12) it is assumed, of course, that the integrand poles at $z = t^{-1}w^{\pm1}p^jq^k$ for $j,k \in \mathbb{Z}_{\geq 0}$ sit inside C_w and their reciprocals – outside of C_w. The lower bound $\max\{|p|,|q|\} < |t^2|$ was imposed in order to guarantee that only poles $t^{-1}w^{\pm1}$ and $tw^{\mp1}$ are crossed over when one deforms the contour C_w to $\mathbb{T}$. Under the weaker condition $|q|,|p| < |t|$ a number of poles may enter the annulus $\mathbb{A}$, from both sides, but they still do not cross the unit circle. Therefore the same arguments as in [29] apply and we obtain the needed inversion relation $\mathrm{S}_1(a)\mathrm{S}_1(-a) = \mathbb{1}$ in a wider region of parameter a. In particular, for $|x| = 1$ and $\max\{|p|,|q|\} < |t| < |q|^{1/2}$ we can satisfy even the original restrictions for $\mathrm{S}_1(a)$-operator parameters, $|tx^{\pm1}| < |q|^{1/2}$. Thus, we have proved that the analytically continued operators $\mathrm{S}_{1,2,3}$ do satisfy Coxeter relations of the permutation group $\mathfrak{S}_4$.

If one substitutes into relation $\mathrm{S}_1(a)\mathrm{S}_1(-a) = \mathbb{1}$ the expression (4.23) without taking care of existing restrictions and assumes that z_1 and z_2 are real, then the straightforward consideration yields

$$\int_0^1 \mathrm{W}_a(z_1,z)\,\mathrm{W}_{-a}(z,z_2)\,\mathrm{d}z = \frac{1}{2}\,\left[\,\delta(z_1-z_2)+\delta(z_1+z_2)\,\right]\,, \tag{5.13}$$

where $\delta(z)$ is the Dirac delta-function. Using explicit expressions for W_a-functions, one obtains the following formal integral identity

$$\kappa\int_0^1 \frac{\Gamma(\pm z_1 \pm z - a, \pm z_2 \pm z + a)}{\Gamma(\pm 2z)}\mathrm{d}z = \frac{\Gamma(\pm 2a, \pm 2z_2)}{2\kappa}\,\left[\,\delta(z_1-z_2)+\delta(z_1+z_2)\,\right]\,, \tag{5.14}$$

which was partially considered in [19,30,31].

On the right-hand side of equality (5.14) one has the product

$$\Gamma(\pm 2a) = \Gamma(t^{\pm 2};p,q) = \frac{1}{(t^2;p)_\infty(pt^{-2};p)_\infty(t^{-2};q)_\infty(qt^2;q)_\infty},$$

which diverges for $t^2 = p^j$ or $t^2 = q^j$, $j \in \mathbb{Z}$. Therefore, for generic values of z_1, z_2 one would expect problems with the inversion relation at least for these values of the parameter t. However, as mentioned above, the inversion relation remains true for $t = \pm 1$ and it breaks down for $t^2 = p^jq^k,\ j,k \in \mathbb{Z},\ (j,k) \neq (0,0)$. Therefore relation (5.14) cannot be true for these values of the t-parameter. It is necessary to perform a careful investigation of the inversion relation for wider regions of parameters than it was discussed above and understand properly when formula (5.14) is valid.

A more detailed discussion of the intertwining operator properties is given in the following sections.

6 Connection with the elliptic Fourier transform

The Bailey chains technique is a well known tool for generating infinite sequences of identities for plain and q-hypergeometric series [16]. It is very useful for proving the Rogers-Ramanujan identities needed for solving $2d$ statistical mechanics models [1]. An integral analogue of the Bailey chains was discovered in [17] directly at the elliptic level using a new universal integral transform for functions depending on one parameter t. As shown in [29] the inverse of this transform is substantially equivalent to the reflection of the parameter $t \to t^{-1}$ which resembles the Fourier transform. Let us describe the key ingredients of this elliptic Fourier transformation technique.

For a given function $\alpha(z,t)$ analytic near the unit circle $z\in\mathbb{T}$ define the integral transformation

$$\beta(w,t)=M(t)_{wz}\alpha(z,t):=\frac{(p;p)_\infty(q;q)_\infty}{4\pi\mathrm{i}}\int_{\mathbb{T}}\frac{\Gamma(tw^{\pm1}z^{\pm1};p,q)}{\Gamma(t^2,z^{\pm2};p,q)}\alpha(z,t)\frac{dz}{z},\tag{6.1}$$

where $|tw|,|t/w|<1$. In [17] the functions related in this way were said to form an integral elliptic Bailey pair with respect to the parameter t (this particular renormalized form of the definition was presented in [29]).

An integral analogue of the Bailey lemma provides a method to generate an infinite sequence of Bailey pairs from a given germ pair. Namely, suppose that $\alpha(z,t)$ and $\beta(w,t)$ form an integral elliptic Bailey pair with respect to the parameter t. Then for $|s|,|t|<1, |\sqrt{pq}y^{\pm1}|<|st|$ the new functions

$$\alpha'(w,st)=D(s;y,w)\alpha(w,t),\quad D(s;y,w)=\Gamma(\sqrt{pq}s^{-1}y^{\pm1}w^{\pm1};p,q),\tag{6.2}$$
$$\beta'(w,st)=D(t^{-1};y,w)M(s)_{wx}D(st;y,x)\beta(x,t),\tag{6.3}$$

where $w\in\mathbb{T}$, form an integral elliptic Bailey pair with respect to the parameter st.

The proof of this statement is easy. Indeed, the demand

$$\beta'(w,st)=M(st)_{wz}\alpha'(z,st)$$

is equivalent to the equality

$$D(t^{-1};y,w)M(s)_{wx}D(st;y,x)M(t)_{xz}\alpha(z,t)=M(st)_{wz}D(s;y,z)\alpha(z,t).$$

Since $D(t^{-1};y,w)D(t;y,w)=1$ due to the reflection equation for the elliptic gamma function, we can rewrite it as an operator identity

$$M(s)_{wx}D(st;y,x)M(t)_{xz}=D(t;y,w)M(st)_{wz}D(s;y,z).\tag{6.4}$$

Substitute explicit expressions for M and D-operators and change the order of integrations in the left-hand side expression. Then one can check that relation (6.4) is equivalent to the elliptic beta integral evaluation formula with correct restrictions on the parameters, which proves the statement.

Relation (6.4) is nothing else that the operator form of the star-triangle relation [22]. Equality (6.4) was explicitly presented in [11] in the matrix form as relation (6.5) (which is connected with some finite dimensional reduction of the intertwining operators).

Comparing the operator $M(t)$ with the intertwining operator of the previous section in multiplicative form (4.25), we see that

$$M(t)_{wy}f(y)=[\mathrm{S}_1(u_1-u_2)f](w)\tag{6.5}$$

for $t=e^{2\pi\mathrm{i}(u_2-u_1)}$ and $\varphi_1=1$. Also evidently

$$\mathrm{S}_2(a)=D(e^{-2\pi\mathrm{i}a};e^{2\pi\mathrm{i}z_1},e^{2\pi\mathrm{i}z_2}),$$

provided in (6.2) one has $+\sqrt{pq}$ in the elliptic gamma function arguments, since $e^{2\pi \mathrm{i}(\eta+\tau/2)} = +\sqrt{pq}$. Another square root sign choice in (6.2) yields the S_2-operator corresponding to the Sklyanin algebra generators differing from (2.5) by the addition of 1/2 to the arguments of theta functions depending on the spin ℓ (see below).

The operators entering the integral Bailey lemma D and M define thus elementary transposition operators satisfying inversion relations $\mathrm{S}_2(a)\mathrm{S}_2(-a) = \mathbb{1}$, $\mathrm{S}_1(a)\mathrm{S}_1(-a) = \mathbb{1}$ and the basic relation (6.4) is equivalent to the Coxeter relation

$$\mathrm{S}_1(a)\mathrm{S}_2(a+b)\mathrm{S}_1(b) = \mathrm{S}_2(b)\mathrm{S}_1(a+b)\mathrm{S}_2(a). \tag{6.6}$$

Take this operator identity (or (6.4)), use the additive notation for S_k-operators (4.22)–(4.24) (with $\varphi_k = 1$) and act by it onto the Dirac delta-function $(\delta(x-z)+\delta(x+z))/2$. Then one obtains the equality (5.4) which can be written as

$$\begin{aligned}\int_0^1 &\rho(u)\mathrm{D}_{\xi-a}(x,u)\mathrm{D}_{a+b}(y,u)\mathrm{D}_{\xi-b}(w,u)du \\ &= \chi(a,b)\mathrm{D}_b(x,y)\mathrm{D}_{\xi-a-b}(x,w)\mathrm{D}_a(y,w),\end{aligned} \tag{6.7}$$

where

$$\mathrm{D}_a(x,u) = \mathrm{S}_2(a) = D(e^{-2\pi \mathrm{i}a}; e^{2\pi \mathrm{i}x}, e^{2\pi \mathrm{i}u}) = \Gamma(e^{2\pi \mathrm{i}(a-\xi\pm x\pm u)}; p,q) \tag{6.8}$$

and

$$e^{-4\pi \mathrm{i}\xi} := pq, \quad \rho(u) = \frac{(p;p)_\infty(q;q)_\infty}{2\Gamma(e^{\pm 4\pi \mathrm{i}u}; p,q)}, \quad \chi(a,b) = \Gamma(e^{-4\pi \mathrm{i}a}, e^{-4\pi \mathrm{i}b}, e^{4\pi \mathrm{i}(a+b-\xi)}; p,q).$$

Equality (6.7) represents a functional form of the star-triangle relation considered in [31], with $\mathrm{D}_a(x,u)$ being the Boltzmann weight for edges connecting spins x and u sitting in the neighboring vertices of a lattice, $\rho(u)$ is related to the self-energy for spins, ξ is the crossing parameter. In this picture the integration means a computation of the partition function for an elementary star-shaped cell with contributions coming from all possible values of the continuous "spin" sitting in the central vertex. So, we may conclude that the Bailey lemma established in [17] is equivalent to the star-triangle relation for particular elliptic hypergeometric Boltzmann weights.

Defining $\mathrm{S}_3(a)$ as an elliptic Fourier transformation operator for the variable z_2 we obtain another generator of the group $\mathfrak{S}_4$. In this way, the algebraic relations for the Bailey lemma ingredients appear to be equivalent to the Coxeter relations for permutation group generators. This fact was not understood in the original paper [17], however the connection of the integral Bailey transformation to the star-triangle relation was briefly remarked in [30].

7 Uniqueness of solutions and the elliptic modular double

The concept of elliptic modular double introduced in [19] allows us to fix the functional freedom in the definition of the $\mathfrak{S}_4$-permutation group generators.

The general infinite-dimensional space solution of the Yang-Baxter e quation we obtain

$$\mathbb{R}_{12}(\mathbf{u}) = \mathbb{P}_{12}\mathrm{S}_2(u_1-v_2)\,\mathrm{S}_1(u_1-v_1)\,\mathrm{S}_3(u_2-v_2)\,\mathrm{S}_2(u_2-v_1)$$

is symmetric in the parameters p and q, since we assume that $u_{1,2}$ and $v_{1,2}$ are independent variables (or, equivalently, u and $g = \eta(2\ell+1)$ should be considered as independent of η and τ). Using the multiplicative notation and the modified forms of the Sklyanin algebra generators and intertwining operators, this R-operator has the following explicit action on the functions of two variables

$$[\mathbb{R}_{12}(\mathbf{u})f](z_1,z_2) = \frac{(p;p)_\infty^2(q;q)_\infty^2}{(4\pi \mathrm{i})^2}\Gamma(\sqrt{pq}z_1^{\pm1}z_2^{\pm1}e^{2\pi\mathrm{i}(u_1-v_2)};p,q) \tag{7.1}$$
$$\times\int_{\mathbb{T}^2}\frac{\Gamma(e^{2\pi\mathrm{i}(v_1-u_1)}z_2^{\pm1}x^{\pm1}, e^{2\pi\mathrm{i}(v_2-u_2)}z_1^{\pm1}y^{\pm1}, \sqrt{pq}e^{2\pi\mathrm{i}(u_2-v_1)}x^{\pm1}y^{\pm1};p,q)}{\Gamma(e^{4\pi\mathrm{i}(v_1-u_1)}, e^{4\pi\mathrm{i}(v_2-u_2)}, x^{\pm2}, y^{\pm2};p,q)}f(x,y)\frac{dx}{x}\frac{dy}{y}.$$

Because of the symmetry in p and q, we have not one but two RLL-intertwining relations. The second one is obtained from (3.7) simply by permuting 2η and τ (or p and q):

$$\mathbb{R}_{12}(u-v)\,\mathrm{L}_1^{doub}(u)\,\sigma_3\,\mathrm{L}_2^{doub}(v) = \mathrm{L}_2^{doub}(v)\,\sigma_3\,\mathrm{L}_1^{doub}(u)\,\mathbb{R}_{12}(u-v)\,. \tag{7.2}$$

Remind that after our similarity transformation the initial L-operator is given by the expression (2.2) where $w_a(u) = \theta_{a+1}(u+\eta|\tau)/\theta_{a+1}(\eta|\tau)$ and Sklyanin generators $\mathbf{S}^a$ have the form (4.21). The operator $\mathrm{L}^{doub}(u)$ is obtained from $\mathrm{L}(u)$ simply by permuting 2η and τ. This means that $\mathrm{L}^{doub}(u)$ also has the form (2.2), but now $w_a(u) = \theta_{a+1}(u+\tau/2|2\eta)/\theta_{a+1}(\tau/2|2\eta)$ and the operators $\mathbf{S}^a_{mod}$ (4.21) should be replaced by

$$\widetilde{\mathbf{S}}^a_{mod} = e^{-\pi\mathrm{i}\frac{\tau}{2}}\frac{(\mathrm{i})^{\delta_{a,2}}\theta_{a+1}(\tau/2|2\eta)}{\theta_1(2z|2\eta)}\Big[\,\theta_{a+1}\left(2z-g+\frac{\tau}{2}|2\eta\right)\cdot e^{-2\pi\mathrm{i}z}\cdot \mathrm{e}^{\frac{1}{2}\tau\partial_z}$$
$$-\,\theta_{a+1}\left(-2z-g+\frac{\tau}{2}|2\eta\right)\cdot e^{2\pi\mathrm{i}z}\cdot \mathrm{e}^{-\frac{1}{2}\tau\partial_z}\,\Big]\,, \tag{7.3}$$

where g-parameter is the same arbitrary parameter as in (4.21).

Such a direct product of two Sklyanin algebras was introduced in [19] under the name *an elliptic modular double*. It represents an elliptic analogue of Faddeev's modular double [18] described as a direct product of two q-analogues of $sl(2)$-algebra, $U_q(sl(2))\otimes U_{\widetilde{q}}(sl(2))$, with $q = e^{4\pi\mathrm{i}\eta}$ and $\widetilde{q} = e^{-\pi\mathrm{i}/\eta}$.

The modified operators $\mathrm{S}_{1,3}(a)$ are are invariant under permutation of p and q and, therefore, they satisfy in addition to (4.9) the $\widetilde{\mathbf{S}}^a$-operator intertwining relations as well:

$$\mathrm{S}_1\cdot\widetilde{\mathbf{S}}^a(\ell_1) = \widetilde{\mathbf{S}}^a(-1-\ell_1)\cdot\mathrm{S}_1\,,\qquad \mathrm{S}_3\cdot\widetilde{\mathbf{S}}^a(\ell_2) = \widetilde{\mathbf{S}}^a(-1-\ell_2)\cdot\mathrm{S}_3\,, \tag{7.4}$$

This bonus symmetry in p and q in $\mathrm{S}_k(a)$ and R-operator originates from a particular choice of the arbitrary elliptic functions $\varphi_k(z_1,z_2)$ emerging in solutions of corresponding finite-difference equations. We can invert the logic and demand from the very beginning existence of the elliptic modular double (7.2). Then we can repeat the same considerations as before and get the same solutions of the finite-difference equations, but now the phase factors are restricted by additional periodicity requirements

$$\varphi_k(z_1+\tau,z_2) = \varphi_k(z_1,z_2+\tau) = \varphi_k(z_1+\tau/2,z_2+\tau/2) = \varphi(z_1,z_2). \tag{7.5}$$

Then we use the well-known Jacobi theorem stating that any function with three incommensurate periods

$$\varphi(z+\omega_1) = \varphi(z+\omega_2) = \varphi(z+\omega_3) = \varphi(z),\quad \sum_{k=1}^{3} n_k\omega_k \neq 0,\quad n_k\in\mathbb{Z},$$

must be a constant, $\varphi = const$. Since our φ_k-functions were fixed already to be elliptic functions with periods 1 and 2η we conclude that the constraints (7.5) with generic values of τ and η enforce $\varphi_k(z_1, z_2) = const$. Thus φ_k may depend only on the parameters $u_{1,2}, v_{1,2}$, and we have chosen the normalization $\varphi_k = 1$ consistent with the unitarity constraint.

Thus the elliptic modular double (i.e., two RLL-relations for a given R-operator) fixes the intertwining operators S_k uniquely (up to the multiplication by a constant). Note that this double algebra generators $\mathbf{S}^a$ and $\widetilde{\mathbf{S}}^a$ do not commute with each other, however they satisfy some simple anticommutation relations [19]. As described also in [19], there exists a second modular double obtained by employing a modular transformation of theta functions and the modified elliptic gamma function [14], which will be considered below.

8 Reduction to a finite-dimensional case

An intertwining operator for the Sklyanin algebra generators with positive integer values of $2\ell+1$ was constructed in [26] as a finite-difference operator of a finite order. Its formal extension to infinite order was used in [23] for building solutions of YBE along the same lines as described above using the realization of $\mathrm{S}_{1,3}(a)$. In particular, the operator $\mathrm{S}_1(a)$ had the following form (see the beginning of Sect. 5.1 in [23]):

$$\mathrm{S}_1(a) = e^{2\pi \mathrm{i} a\eta x + \pi \mathrm{i}\eta a^2} \frac{\Gamma(2\eta x)}{\Gamma(2\eta(x+a))} \sum_{k=0}^{\infty} \frac{[2k-x-a]}{[-x-a]} \prod_{j=0}^{k-1} \frac{[j-x-a][j-a]}{[j+1-x][j+1]} e^{(a-2k)\partial_x}, \quad (8.1)$$

where $[x] = \theta_1(2\eta x|\tau)$. However, this operator is not well defined unless the infinite series terminates. In particular, the formal action of this operator on a meromorphic function of x yields in general a diverging series. Therefore we assume, as in [26], that $a = 2\ell+1$ is a positive integer.

Let us denote

$$q = e^{4\pi \mathrm{i}\eta}, \quad p = e^{2\pi \mathrm{i}\tau}, \quad w = e^{-2\pi \mathrm{i}\eta x}, \quad t = e^{-2\pi \mathrm{i}\eta a},$$

and assume that $|q| < 1$. Then, using the theta function $\theta(x;p) = (x;p)_\infty (px^{-1};p)_\infty$, we can write

$$\begin{aligned} \mathrm{S}_1(a) f(w) = {} & q^{\frac{ax}{2}+\frac{a^2}{4}} \frac{\Gamma(w^{-2};p,q)}{\Gamma(t^{-2}w^{-2};p,q)} \\ & \times \sum_{k=0}^{\infty} t^{-2k} \frac{\theta(t^2 w^2 q^{2k};p)}{\theta(t^2 w^2;p)} \prod_{j=0}^{k-1} \frac{\theta(t^2 w^2 q^j, t^2 q^j;p)}{\theta(w^2 q^{j+1}, q^{j+1};p)} f(tq^k w), \end{aligned} \quad (8.2)$$

where $\theta(t_1, \ldots, t_k;p) = \theta(t_1;p) \ldots \theta(t_k;p)$ and $\Gamma(x;p,q)$ is the elliptic gamma function (11.19). Because of the series termination, operator (8.2) is well defined for $|q| < 1$ and $|q| > 1$, and even $|q| = 1$, provided q is not a root of unity.

Introduce now an integral transformation operator acting in the space of $z \to z^{-1}$ invariant functions, $f(z^{-1}) = f(z)$,

$$[B(t)f](w) = \frac{\kappa}{\Gamma(t^2;p,q)} \int_C \frac{dz}{2\pi \mathrm{i} z} g(z,w,t) \frac{\Gamma(tw^{\pm 1} z^{\pm 1};p,q)}{\Gamma(z^{\pm 2};p,q)} f(z),$$

where $g(z,w,t)$ is some fixed function to be determined from the comparison with operator $\mathrm{S}_1(a)$ (8.1) and κ was fixed earlier. The contour of integration C is chosen in such a way that it separates geometric progressions of the integrand poles converging to zero $z=0$ from their $z\to 1/z$ reciprocals. In particular, if $|tw|,|t/w|<1$ and the product $g(z,w,t)f(z)$ is an analytical function near the unit circle with positive orientation $\mathbb{T}$, then one can choose $C=\mathbb{T}$.

Now we take $|twq^k|>1$, $k=0,\dots,N$, $|twq^{N+1}|<1$, $|twp|<1$, $|t/w|<1$, and the contour of integration C as a deformation of $\mathbb{T}$ containing in its interior the poles at $z=twq^k$, $k=0,\dots,N$, lying outside $\mathbb{T}$ and excluding the poles at $z=t^{-1}w^{-1}q^{-k}$, $k=0,\dots,N$, which enter $\mathbb{T}$. Now we pull C to $\mathbb{T}$ and pick up the residues at $z^{\pm1}=twq^k$, $k=0,\dots,N$. This yields the formula

$$
\begin{aligned}
[B(t)f](w) &= \frac{\kappa}{\Gamma(t^2;p,q)}\sum_{k=0}^{N}\lim_{z\to twq^k}(z-twq^k)\frac{\Gamma(tw^{\pm1}z^{\pm1};p,q)}{z\Gamma(z^{\pm2};p,q)}\\
&\quad\times\left(g(twq^k,w,t)+g(t^{-1}w^{-1}q^{-k},w,t)\right)f(twq^k)\\
&+\frac{\kappa}{\Gamma(t^2;p,q)}\int_{\mathbb{T}}\frac{dz}{2\pi \mathrm{i} z}g(z,w,t)\frac{\Gamma(tw^{\pm1}z^{\pm1};p,q)}{\Gamma(z^{\pm2};p,q)}f(z).
\end{aligned}
$$

Using the relation

$$
\lim_{z\to1}(1-z)\Gamma(z;p,q)=\frac{1}{(p;p)_\infty(q;q)_\infty},
$$

we find

$$
\lim_{z\to twq^k}\left(1-\frac{twq^k}{z}\right)\Gamma(twz^{-1};p,q)=\frac{1}{(p;p)_\infty(q;q)_\infty}\frac{1}{\theta(q^{-k},\dots,q^{-1};p)}.
$$

As a result, we obtain

$$
\begin{aligned}
[B(t)f](w) &= \frac{\Gamma(w^{-2};p,q)}{\Gamma(t^{-2}w^{-2};p,q)}\sum_{k=0}^{N}\frac{g(twq^k,w,t)+g(t^{-1}w^{-1}q^{-k},w,t)}{2}\\
&\quad\times t^{-4k}w^{-2k}q^{-k^2}\frac{\theta(t^2w^2q^{2k};p)}{\theta(t^2w^2;p)}\prod_{j=0}^{k-1}\frac{\theta(t^2q^j,t^2w^2q^j;p)}{\theta(q^{j+1},w^2q^{j+1};p)}f(twq^k)\\
&+\frac{\kappa}{\Gamma(t^2;p,q)}\int_{\mathbb{T}}\frac{dz}{2\pi \mathrm{i} z}g(z,w,t)\frac{\Gamma(tw^{\pm1}z^{\pm1};p,q)}{\Gamma(z^{\pm2};p,q)}f(z).
\end{aligned}
$$

Now we take the limit $t^2\to q^{-2\ell-1}$, which means $a\to 2\ell+1\in\mathbb{Z}_{>0}$. Since $1/\Gamma(t^2;p,q)=1/\Gamma(q^{-2\ell-1};p,q)=0$, and the integral $\int_{\mathbb{T}}$ is finite, we see that the second term disappears with the final result

$$
\begin{aligned}
[B(t)f](w)\Big|_{t^2=q^{-2\ell-1}} &= \frac{\Gamma(w^{-2};p,q)}{\Gamma(q^{2\ell+1}w^{-2};p,q)}\sum_{k=0}^{2\ell+1}\frac{g(wq^{k-\frac{2\ell+1}{2}},w,q^{-\frac{2\ell+1}{2}})+g(w^{-1}q^{-k+\frac{2\ell+1}{2}},w,q^{-\frac{2\ell+1}{2}})}{2}\\
&\times q^{2k(2\ell+1)}w^{-2k}q^{-k^2}\frac{\theta(w^2q^{2k-2\ell-1};p)}{\theta(q^{-2\ell-1}w^2;p)}\prod_{j=0}^{k-1}\frac{\theta(q^{j-2\ell-1},w^2q^{j-2\ell-1};p)}{\theta(q^{j+1},w^2q^{j+1};p)}f(wq^{k-\frac{2\ell+1}{2}}). \qquad (8.3)
\end{aligned}
$$

This series terminates automatically at $k=2\ell+1=N$.

We have derived this result under the following constraints for w-variable $|twq^{N+1}| < 1$ and $|t/w| < 1$, or $|q^{-\ell-1/2}| < |w| < |q^{\ell-N-1/2}|$. However, the derived result is independent on these restrictions since in the limit $t^2 \to q^{-2\ell-1}$ there are $2\ell+2$ pairs of poles which pinch the integration contour and one inevitably has to pass to the residues sum of the above form independently on w-values.

Equating this expression with the terminating series operator $\mathrm{S}_1(a)$, we come to the relation

$$\frac{1}{2}\left(g(z,w,t)+g(z^{-1},w,t)\right)t^{-4k}w^{-2k}q^{-k^2} = q^{\frac{ax}{2}+\frac{a^2}{2}}t^{-2k}, \quad z = twq^k.$$

It has unique solution invariant under the transformation $z \to 1/z$

$$g(z,w,t) = q^{\frac{ax}{2}+\frac{a^2}{2}} \exp\left[\frac{(\log z)^2-(\log tw)^2}{\log q}\right],$$

as wanted. However, it is not an analytical function of z. But for validity of our consideration we needed only that the product $g(z,w,t)f(z)$ be analytical. Let us denote $z = e^{2\pi i\eta u}$. Then we can write

$$g(z = e^{2\pi i\eta u}, w = e^{-2\pi i\eta x}, t = e^{-2\pi i\eta a}) = e^{\pi i\eta(u^2-x^2)}.$$

Thus we have to demand that $\phi(z) := e^{\pi i\eta u^2}f(z)$ is a meromorphic function of z. Let us demand additionally that $\phi(z^{-1}) = \phi(z)$. Then the operator $B_{mod}(t)$ defined after the similarity transformation

$$B_{mod}(t) = e^{\pi i\eta x^2}B(t)e^{-\pi i\eta u^2}$$

maps the space of meromorphic A_1-invariant functions $\phi(z)$ onto itself. Let us replace in $B_{mod}(t)$ the variables $u \to u/\eta, x \to x/\eta$, i.e. we pass to the parameterization $z = e^{2\pi i u}, w = e^{-2\pi i x}$. Then explicitly we have

$$[B_{mod}(t)\psi](w) = \frac{(p;p)_\infty (q;q)_\infty}{2\Gamma(t^2;p,q)} \int_0^1 \frac{\Gamma(te^{2\pi i(\pm x \pm u)};p,q)}{\Gamma(e^{\pm 4\pi i u};p,q)}\psi(e^{2\pi i u})du.$$

Evidently, for $t = e^{-2\pi i(u_1-u_2)}$ this operator coincides with the intertwining operator $\mathrm{S}_1(u_1-u_2)$ (4.23) (for $\varphi_1 = 1$) in such a way that

$$[\mathrm{S}_1(b)\Psi](x) = [B_{mod}(e^{-2\pi i b})\psi](e^{2\pi i x}),$$

where $\Psi(x) = \psi(e^{2\pi i x})$ (i.e., the difference is only in the additive or multiplicative notation). We conclude that the discrete intertwining operator of [26] is a special limiting case of our intertwining operator $\mathrm{S}_1(b)$.

Consider the identity

$$\mathrm{S}_3(u_1-u_2)\mathrm{R}_{12}(u_1,u_2|v_1,v_2) = \mathrm{R}_{12}(u_2,u_1|v_1,v_2)\mathrm{S}_1(u_1-u_2),$$

which is easy to check using the R-matrix factorization. Multiplying it by the permutation operator $\mathbb{P}_{12}$ and using relation $\mathbb{P}_{12}\mathrm{S}_3 = \mathrm{S}_1\mathbb{P}_{12}$, we obtain the identity

$$\mathrm{S}_1(u_1-u_2)\mathbb{R}_{12} = \mathbb{R}'_{12}\mathrm{S}_1(u_1-u_2),$$

where $\mathbb{R}_{12} = \mathbb{P}_{12}\mathrm{R}_{12}(u_1, u_2|v_1, v_2)$ and $\mathbb{R}'_{12} = \mathbb{P}_{12}\mathrm{R}_{12}(u_2, u_1|v_1, v_2)$. It shows that the kernel space of S_1-operator is mapped onto itself by our R-matrix $\mathbb{R}_{12}$, i.e. zero modes of S_1 form an invariant space for the action of operator (7.1).

In the same way, intertwining relations (4.9) and (7.4) show that the kernel space of S_1-operator forms an invariant space for the elliptic modular double, i.e. it is mapped onto itself by the Sklyanin algebra generators $\mathbf{S}^a_{mod}$ (4.21) and $\widetilde{\mathbf{S}}^a_{mod}$ (7.3). The standard Sklyanin algebra admits finite-dimensional representations in the space of theta functions of modulus τ and, naturally, its modular partner has similar representations in the space of theta functions of modulus 2η. Therefore, there should exist finite-dimensional representations of the Sklyanin algebra in the space of products of theta functions of moduli τ and 2η. As shown in [19], the elliptic modular double has a non-trivial automorphism permuting these theta-function submodules.

In general, the latter factorization of Sklyanin-algebra modules is related to the fact that sums of residues of poles of elliptic hypergeometric integrals factorizes to the product of two elliptic hypergeometric series with permuted modular parameters p and q, which leads to the concept of two-index biorthogonality relation [14]. Let us show that the corresponding residue calculus demonstrates existence of nontrivial finite-dimensional kernel space for $\mathrm{S}_1(a)$-operator for $a = \eta(2\ell_q+1)+\tau(\ell_p+1/2)$ and $a = 1/2+\eta(2\ell_q+1)+\tau(\ell_p+1/2)$, where $2\ell_q, 2\ell_p \in \mathbb{Z}_{\geq 0}$ (i.e., the reduction considered above is only a special case of the much more general finite-dimensional reduction).

For this, let us repeat our consideration with a different set of poles taken into account. Namely, let us take the limit $t^2 \to q^{-N}p^{-M}$ with $N, M \in \mathbb{Z}_{\geq 0}$. Now a number of poles leave the unit circle and a number of them enter it. As indicated during the discussion of analytical continuation of the S_1-operator, there will be precisely $(N+1)(M+1)$ pairs of poles pinching the integration contour C in B_{mod} (we choose $g(z, w, t) = 1$). Let us pull the integration contour through one half of the poles approaching it, say, through $z = tp^jq^k$, $j = 0, \dots, M$, $k = 0, \dots, N$, and sum the corresponding residues. Denote also $N = 2\ell_q + 1$ and $M = 2\ell_p + 1$ with half integer $\ell_p, \ell_q \geq -1/2$ in order to match notation with the spins of Sklyanin algebras entering the elliptic modular double. Now, using relations

$$\lim_{z\to twq^kp^j}\left(1-\frac{twq^kp^j}{z}\right)\Gamma\Big(\frac{tw}{z};p,q\Big) = \frac{(-1)^{jk+j+k}q^{(j+1)k(k+1)/2}p^{(k+1)j(j+1)/2}}{(p;p)_\infty(q;q)_\infty\theta(q,\dots,q^k;p)\theta(p,\dots,p^j;q)} \tag{8.4}$$

and

$$\theta(p^kz;p) = (-z)^{-k}p^{-\frac{k(k-1)}{2}}\theta(z;p), \quad k \in \mathbb{Z},$$

we find

$$[B_{mod}(t)f](w) = \frac{\Gamma(w^{-2};p,q)}{\Gamma(t^{-2}w^{-2};p,q)}\sum_{k=0}^{2\ell_q+1}\frac{\theta((tw)^2q^{2k};p)}{\theta((tw)^2;p)}\prod_{b=0}^{k-1}\frac{\theta(t^2q^b,(tw)^2q^b;p)}{\theta(q^{b+1},w^2q^{b+1};p)}$$
$$\times\sum_{j=0}^{2\ell_p+1}\frac{\theta((tw)^2p^{2j};q)}{\theta((tw)^2;q)}\prod_{a=0}^{j-1}\frac{\theta(t^2p^a,(tw)^2p^a;q)}{\theta(p^{a+1},w^2p^{a+1};q)}\frac{f(tq^kp^jw)}{t^{4(jk+j+k)}w^{2(j+k)}p^{2jk+j^2}q^{2jk+k^2}}, \tag{8.5}$$

where we should substitute the actual value of $t = \pm q^{-\ell_q-1/2}p^{-\ell_p-1/2}$.

Note that for $\ell_p = \ell_q = -1/2$, when $t = \pm 1$ (or $a = 0, 1/2$), this series contains only one term. Thus, for $t = 1$ (or $a = 0$) the intertwining operator becomes the unity operator,

$B_{mod}(1) = \mathbb{1}$ (or $\mathrm{S}_1(0) = \mathbb{1}$). For $t = -1$ (or $a = 1/2$) the intertwining operator becomes the parity operator,

$$B_{mod}(-1) = P, \qquad Pf(w) = f(-w).$$

In the additive notation, we can write $\mathrm{S}_1(1/2) = e^{\frac{1}{2}\partial_x}$, which is the half-period shift for the variable $w = e^{-2\pi \mathrm{i} x}$. All our functions are analytical in w (after removal of the exponential factors from $\mathrm{S}_{1,3}$-operators) and $t = e^{-2\pi \mathrm{i} a}$. Therefore $\mathrm{S}_1(a+1) = \mathrm{S}_1(a)$ and $\mathrm{S}_1(-1/2) = \mathrm{S}_1(1/2)$, so that

$$\mathrm{S}_1^2(1/2) := \mathrm{S}_1(-1/2)\mathrm{S}_1(1/2) = P^2 = e^{\partial_x},$$

which is equivalent to the unity operator, since it is the operator of shifting by the period 1.

If we remove the constraint $t = \pm q^{-\ell_q - 1/2} p^{-\ell_p - 1/2}$ in (8.5) (which is not legitimate in our procedure since we would need to restore an integral part in B_{mod}) and set formally $\ell_q, \ell_p \to \infty$, we would obtain the double infinite series operator which sharply differs from the univariate infinite series operator used in [23]. This fact shows the principle difference of the rigorously defined integral operator $B_{mod}(t)$ of [17] from the formal infinite elliptic hypergeometric series realization of the S_1-operator of [23].

From (8.5) we see that for special quantized values of t the $B_{mod}(t)$-operator has an almost factorized form, the only non-factorizable pieces being the multiplier $(pqt^2)^{-2jk}$ and the action on the function itself $f(tq^k p^j w)$. Suppose now that we work in the space of functions of the form

$$f(z) = \theta^+_{4\ell_q}(z;p)\theta^+_{4\ell_p}(z;q), \tag{8.6}$$

where $\theta^+_{4\ell}(z;q)$ is an arbitrary A_1-symmetric theta-function of order $4\ell \geq 0$ with the modular parameter q, i.e. a holomorphic function of $z \in \mathbb{C}^*$ satisfying the properties

$$\theta^+_{4\ell}(z^{-1};q) = \theta^+_{4\ell}(z;q), \qquad \theta^+_{4\ell}(qz;q) = \frac{1}{(qz^2)^{2\ell}}\theta^+_{4\ell}(z;q). \tag{8.7}$$

As mentioned already, such a consideration is inspired by our starting intertwining relation (4.9) and its partner (7.4) with $\mathrm{S}_1 \sim B(t)$, which show that the space of zero modes of the B-operator forms an invariant space for two Sklyanin algebra generators. It is known that the standard Sklyanin algebra generators leave invariant the space formed by theta functions $\theta^+_{4\ell}(z;q)$ [10]. But our operator $B_{mod}(t)$ is symmetric in p and q. Therefore it is natural to consider the above ansatz for $f(z)$ to have an explicit realization of the automorphism for the elliptic modular double permuting two Sklyanin algebras [19].

Substitute now expression (8.6) into formula (8.5), use second relation in (8.7), explicitly substitute the value of t, and remove where possible extra powers of p or q from theta-function arguments. Then for integer ℓ_p and ℓ_q we obtain

$$\begin{aligned} [B_{mod}f](w) = {}& \frac{(-1)^{(2\ell_q+1)(2\ell_p+1)} p^{\ell_p(1-2\ell_p\ell_q+2\ell_p-2\ell_q)} q^{\ell_q(1-2\ell_p\ell_q+2\ell_q-2\ell_p)}}{w^{2(2\ell_p+2\ell_q+1)} \prod_{b=0}^{2\ell_q} \theta(w^{-2}q^b;p) \prod_{a=0}^{2\ell_p} \theta(w^{-2}p^a;q)} \\ & \times \left(\sum_{k=0}^{2\ell_q+1} q^k \frac{\theta(w^2 q^{2k-2\ell_q-1};p)}{\theta(w^2 q^{-2\ell_q-1};p)} \prod_{b=0}^{k-1} \frac{\theta(q^{b-2\ell_q-1}, w^2 q^{b-2\ell_q-1};p)}{\theta(q^{b+1}, w^2 q^{b+1};p)} \theta^+_{4\ell_q}(p^{-\frac{1}{2}} q^{k-\ell_q-\frac{1}{2}} w;p) \right) \\ & \times \left(\sum_{j=0}^{2\ell_p+1} p^j \frac{\theta(w^2 p^{2j-2\ell_p-1};q)}{\theta(w^2 p^{-2\ell_p-1};q)} \prod_{a=0}^{j-1} \frac{\theta(p^{a-2\ell_p-1}, w^2 p^{a-2\ell_p-1};q)}{\theta(p^{a+1}, w^2 p^{a+1};q)} \theta^+_{4\ell_p}(q^{-\frac{1}{2}} p^{j-\ell_p-\frac{1}{2}} w;q) \right) \end{aligned} \tag{8.8}$$

Similar expressions are found when one of the parameters ℓ_p or ℓ_q (or both) is a half-integer. We see a complete factorization of the action of our operator to the proper subspaces $\theta^+_{4\ell_q}(z;p)$ and $\theta^+_{4\ell_p}(z;q)$, so that each of its factors is independent on the other one. This means that after finding zero modes for one of the factors other zero modes are obtained simply by the interchange $p \leftrightarrow q$ and $\ell_p \leftrightarrow \ell_q$.

Note that it is not legitimate to choose $\ell_p = -1/2$ (or $\ell_q = -1/2$) in formula (8.8), since the second relation in (8.7) is valid only for $\ell \geq 0$. For $2\ell_p + 1 = 0$ (or $2\ell_q + 1 = 0$), when the second sum is absent, we obtain the previously considered operator (8.3). We have found thus a space of zero modes of the elliptic Fourier transformation operator $B_{mod}(t)$ of dimension $d_{zm} = (2\ell_p + 1)(2\ell_q + 1)$ for $\ell_p, \ell_q \geq 0$. For $2\ell_p + 1 = 0$ one has $d_{zm} = 2\ell_q + 1$ and, vice versa, for $2\ell_q + 1 = 0$ one has $d_{zm} = 2\ell_p + 1$. For $2\ell_p + 1 = 2\ell_q + 1 = 0$ (i.e., $t = \pm 1$) there are no zero modes since $B_{mod}(1) = 1$ and $B_{mod}(-1) = P$, the parity operator. This space forms a nontrivial finite-dimensional invariant subspace of the R-operator $\mathbb{R}_{12}$ which we plan to investigate in detail in the future.

It would be interesting to characterize the full space of zero modes of the integral operator $B_{mod}(t)$. We conjecture that for holomorphic functions $f(z)$, $z \in \mathbb{C}^*$, satisfying the property $f(z^{-1}) = f(z)$, and generic values of bases p and q such zero modes exist only for $t^2 = q^N p^M$ with $N, M \in \mathbb{Z}$, $(N, M) \neq (0, 0)$. The A_1-symmetric products of theta functions of moduli p and q described above are conjectured to form the full finite-dimensional subspace of these zero modes. Respectively, we conjecture that the latter space describes all finite-dimensional modules for our R-operator $\mathbb{R}_{12}$ bases on holomorphic functions. In this respect it would be interesting to understand how expression (7.1) is reduced to Baxter's R-matrix (2.1), Sklyanin's L-operator [9], and how it is related to Felderhof's solution of YBE [24].

Examples of the meromorphic zero modes for $B_{mod}(t)$ with generic continuous values of the parameter t are found from the elliptic beta integral evaluation (5.7). Indeed, let us rewrite this formula in the form

$$[B_{mod}(t)f](w) = g(w), \qquad f(z) = \prod_{j=1}^{4} \Gamma(t_j z^{\pm 1}; p, q),$$

$$g(w) = \prod_{j=1}^{4} \Gamma(t w^{\pm 1} t_j; p, q) \prod_{1 \leq j < k \leq 4} \Gamma(t_j t_k; p, q),$$

where $t^2 \prod_{j=1}^{4} t_j = pq$ and the integration contour C in $B_{mod}(t)$ is chosen in an appropriate way. Take now, for instance, $t_3 t_4 = pq$ and assume that parameters t_1 and t_2 do not depend on w and $t_1 t_2 p^j q^k \neq 1$, $j, k \in \mathbb{Z}_{\geq 0}$ (or $t^2 \neq p^{-j} q^{-k}$ since we have the constraint $t^2 t_1 t_2 = 1$). Then $\Gamma(t_3 t_4; p, q) = 0$ and no singularities emerge from other elliptic gamma functions in $g(w)$. Therefore one obtains $g(w) = 0$, i.e. $f(z) = \Gamma(t_1 z^{\pm 1}, t_2 z^{\pm 1}; p, q)$ is a meromorphic zero mode of the integral operator $B_{mod}(t)$. If the values of t_1 and t_2 depend on w then instead of vanishing the function $g(w)$ may diverge, which is illustrated by the inversion relation $B_{mod}(t) B_{mod}(t^{-1}) = \mathbb{1}$.

9 Solutions for $\mathrm{Im}(\eta) < 0$ and $\mathrm{Im}(\eta) = 0$

Suppose now that $\mathrm{Im}(\eta) < 0$. Then a particular solution of the starting equation for $\Phi(z)$ in (4.7) has the form

$$\Phi(z) = \frac{\Gamma(z + b - 2\eta|\tau, -2\eta)}{\Gamma(z + a - 2\eta|\tau, -2\eta)}.$$

Note the flip of gamma functions and a shift of z by 2η with respect to the $\mathrm{Im}(\eta) > 0$ case. Using this fact it is not difficult to find solutions of equations (4.6), (4.16), (4.17), and (4.18) for $\mathrm{Im}(\eta) < 0$. For instance, instead of (4.19) one finds

$$\Delta(z, x) = \mathrm{e}^{-\frac{\pi \mathrm{i}}{\eta}(x^2 - z^2)} \frac{\Gamma(\pm z \pm x - \eta + s|\tau, -2\eta)}{\Gamma(\pm 2x|\tau, -2\eta)} \varphi(z, x),$$

where $\varphi(z, x)$ has the same properties as before. Now S_k-operators should act on functions of the form $\mathrm{e}^{\frac{\pi \mathrm{i}}{\eta} x^2} \Psi(x)$ where $\Psi(x + 1) = \Psi(-x) = \Psi(x)$. Passing to the space of functions $\Psi(x)$ we come to the following result.

Theorem 5. *Let* $\mathrm{Im}(\eta) < 0$ *(or* $|q| > 1$*) and* V *be the space of even and periodic functions of two complex variables* $\Psi(z_1, z_2)$ *with the period 1 which do not have simple poles in the domains* $Im(\eta) \leq Im(z_1), Im(z_2) \leq -Im(\eta)$*. Define three operators*

$$[\mathrm{S}_2(u_2 - v_1)\Psi](z_1, z_2) = \Gamma\left(\pm z_1 \pm z_2 - u_2 + v_1 - \eta + \tfrac{\tau}{2}|\tau, -2\eta\right) \varphi_2(z_1, z_2) \cdot \Psi(z_1, z_2), \qquad (9.1)$$

where $|\sqrt{p/q} e^{2\pi \mathrm{i}(v_1 - u_2)}| < |q|^{-1}$,

$$[\mathrm{S}_1(u_1 - u_2)\Psi](z_1, z_2) = \frac{\kappa'}{\Gamma(2u_1 - 2u_2|\tau, -2\eta)} \int_0^1 \frac{\Gamma(\pm z_1 \pm x + u_1 - u_2|\tau, -2\eta)}{\Gamma(\pm 2x|\tau, -2\eta)} \varphi_1(z_1, x) \cdot \Psi(x, z_2) dx, \qquad (9.2)$$

where $|e^{2\pi \mathrm{i}(u_1 - u_2 \pm z_1)}| < |q|^{-1/2}$ *and* $\kappa' = (p; p)_\infty (q^{-1}; q^{-1})_\infty / 2$,

$$[\mathrm{S}_3(v_1 - v_2)\Psi](z_1, z_2) = \frac{\kappa'}{\Gamma(2v_1 - 2v_2|\tau, -2\eta)} \int_0^1 \frac{\Gamma(\pm z_2 \pm x + v_1 - v_2|\tau, -2\eta)}{\Gamma(\pm 2x|\tau, -2\eta)} \varphi_3(z_2, x) \cdot \Psi(z_1, x) dx, \qquad (9.3)$$

where $|e^{2\pi \mathrm{i}(v_1 - v_2 \pm z_2)}| < |q|^{-1/2}$*. Here* $\varphi_k(z, x)$*-functions have the same properties as in the case* $Im(\eta) > 0$.

Then the operators $\mathrm{S}_k(\mathbf{u})$ *map the space* V *onto itself and satisfy the defining intertwining relations* (3.10), (3.11), *and* (3.12), *provided in the corresponding* L*-operator* (2.2) *one uses the Sklyanin algebra generators realization*

$$\mathbf{S}^a_{mod} = e^{\pi \mathrm{i} \eta} \frac{(\mathrm{i})^{\delta_{a,2}} \theta_{a+1}(\eta)}{\theta_1(2z)} \Big[\theta_{a+1}(2z - 2\eta\ell) \cdot e^{2\pi \mathrm{i} z} \cdot \mathrm{e}^{\eta \partial} - \theta_{j+1}(-2z - 2\eta\ell) \cdot e^{-2\pi \mathrm{i} z} \cdot \mathrm{e}^{-\eta \partial} \Big].$$

They satisfy also the Coxeter relations (3.17), (3.18), and (3.19) as a consequence of the elliptic beta integral evaluation formula with q *replaced by* q^{-1}.

For the choice $\varphi_k = 1$ the R-matrix has the following explicit form

$$[\mathbb{R}_{12}(\mathbf{u})f](z_1,z_2) = \frac{(p;p)_\infty^2 (q^{-1};q^{-1})_\infty^2}{(4\pi\mathrm{i})^2} \Gamma(\sqrt{p/q}\, z_1^{\pm 1} z_2^{\pm 1} e^{2\pi\mathrm{i}(v_2-u_1)}; p, q^{-1}) \tag{9.4}$$
$$\times \int_{\mathbb{T}^2} \frac{\Gamma(e^{2\pi\mathrm{i}(u_1-v_1)} z_2^{\pm 1} x^{\pm 1}, e^{2\pi\mathrm{i}(u_2-v_2)} z_1^{\pm 1} y^{\pm 1}, \sqrt{p/q}\, e^{2\pi\mathrm{i}(v_1-u_2)} x^{\pm 1} y^{\pm 1}; p, q^{-1})}{\Gamma(e^{4\pi\mathrm{i}(u_1-v_1)}, e^{4\pi\mathrm{i}(u_2-v_2)}, x^{\pm 2}, y^{\pm 2}; p, q^{-1})} f(x,y) \frac{dx}{x}\frac{dy}{y}.$$

Evidently, this expression is symmetric in p and q^{-1}, i.e. there exists the second RLL-intertwining relation obtained from (3.7) simply by permuting -2η and τ (or p and q^{-1}). The demand of existence of this modular double forces the functions φ_k to be constants independent on z_1 and z_2. Note that the operator (9.4) is formally obtained from (7.1) simply by the changes $u-v \to v-u$ and $\eta \to -\eta$. However, in difference from the Baxter R-matrix (2.1), this is not a symmetry transformation since both expressions are defined only for a particularly fixed sign of $\mathrm{Im}(\eta)$.

Consideration of the regime $\mathrm{Im}(\eta) = 0$, or $|q| = 1$, is substantially more complicated. One has to use the modified elliptic gamma function [14]. Before passing to corresponding considerations, we would like to consider the situation when $|q| < 1$ and the Sklyanin algebra generators have the form

$$\mathbf{S}^a = \frac{(\mathrm{i})^{\delta_{a,2}} \theta_{a+1}(\eta)}{\theta_1(2z)} \Big[\, \theta_{a+1}\left(2z - 2\eta\ell + \frac{1}{2}\right) \cdot e^{\eta\partial_z} - \theta_{a+1}\left(-2z - 2\eta\ell + \frac{1}{2}\right) \cdot e^{-\eta\partial_z} \,\Big], \tag{9.5}$$

which differs from (2.5) by the addition of 1/2 to arguments of theta functions depending on the spin ℓ. These operators represent a particular automorphism of the algebra [10] with the Casimir operators changed to

$$\mathbf{K}_0 = 4\,\theta_2^2\big((2\ell+1)\,\eta\big)\;; \qquad \mathbf{K}_2 = 4\,\theta_2\big(2(\ell+1)\,\eta\big)\,\theta_2(2\ell\,\eta)\,.$$

One can check that in this case the factorization (4.1) has the same form with the replacements $u_1 \to u_1 - 1/4$ and $u_2 \to u_2 + 1/4$. Similar shifts $v_1 \to v_1 - 1/4$ and $v_2 \to v_2 + 1/4$ take place in the second L-operator entering intertwining relation (3.11). Substituting these shifts in the appropriate places of derivation of the S_2-operator, this time we come to the following equations

$$\begin{aligned}
\theta_3(z_1+z_2+u_2-v_1)\,\mathrm{S}(z_1-\eta,z_2) &= \theta_3(z_1+z_2+v_1-u_2)\,\mathrm{S}(z_1,z_2+\eta)\;,\\
\theta_3(z_1+z_2-u_2+v_1)\,\mathrm{S}(z_1+\eta,z_2) &= \theta_3(z_1+z_2-v_1+u_2)\,\mathrm{S}(z_1,z_2-\eta)\;,\\
\theta_3(z_1-z_2+u_2-v_1)\,\mathrm{S}(z_1-\eta,z_2) &= \theta_3(z_1-z_2+v_1-u_2)\,\mathrm{S}(z_1,z_2-\eta)\;,\\
\theta_3(z_1-z_2-u_2+v_1)\,\mathrm{S}(z_1+\eta,z_2) &= \theta_3(z_1-z_2-v_1+u_2)\,\mathrm{S}(z_1,z_2+\eta)
\end{aligned} \tag{9.6}$$

with the general solution for S_2-operator

$$\mathrm{S}_2(a) = \Gamma\left(\pm z_1 \pm z_2 + a + \tfrac{1}{2} + \eta + \tfrac{\tau}{2} | \tau, 2\eta\right) \varphi_2(z_1,z_2), \quad a = u_2 - v_1. \tag{9.7}$$

For $\varphi_2 = 1$, one still has $\mathrm{S}_2(-a)\mathrm{S}_2(a) = \mathbb{1}$, as needed. Similar picture holds for $|q| > 1$ regime as well.

As to the operators S_1 and S_3, they do not change their form at all. Indeed, the intertwining relations (4.10) lead to equations (4.13) with the replacements $s \to s - 1/2$ in the first row theta functions and $s \to s + 1/2$ in the second row. As a result of the latter inhomogeneity, it happens that equation (4.14) does not change apart of the overall sign for all terms, equation (4.15) does not change at all. As a result, the final equations (4.16), (4.17), and (4.18) do not change at all. Therefore, the shape of the S_1-operator does not change. Validity of the cubic Coxeter relation is guaranteed again by the elliptic beta integral with the replacement of corresponding parameters $g_3 \to g_3 + 1/2$ and $g_4 \to g_4 + 1/2$, which does not spoil the balancing condition (5.8) defined modulo $\mathbb{Z}$. As a result the R-operator (7.1) is slightly changed — it is necessary to replace in the kernel of this operator $+\sqrt{pq}$ by $-\sqrt{pq}$. The modular double exists as well with the partner Sklyanin algebra generators being obtained from (7.3) after the replacement $g \to g - 1/2$.

Now it is straightforward to build a solution of equations (9.6) which is well defined for $\mathrm{Im}(\eta) = 0$. First we set $2\eta = \omega_1/\omega_2, \tau = \omega_3/\omega_2$ and renormalize all other variables in Sklyanin's L-operator

$$z_1 \to \frac{z_1}{\omega_2}, \quad z_2 \to \frac{z_2}{\omega_2}, \quad u \to \frac{u}{\omega_2}, \quad v \to \frac{v}{\omega_2}.$$

Then equation (4.7) takes the form

$$\Phi(z+\omega_1) = \mathrm{e}^{\pi \mathrm{i} \frac{a-b}{\omega_2}} \frac{\theta_1(\frac{z+a}{\omega_2})}{\theta_1(\frac{z+b}{\omega_2})} \cdot \Phi(z) \ ,$$

which has a particular solution of the form

$$\Phi(z) = \frac{G(z+a;\omega)}{G(z+b;\omega)},$$

where $G(z;\omega)$ is the modified elliptic gamma function (11.20) well defined for $|q| \leq 1$ and satisfying the same key equation as $\Gamma(e^{2\pi \mathrm{i} u/\omega_2}; p, q)$ (11.23). Using these facts, we can immediately write out the final general expression for S_2-operator following from equations (9.6) and valid for $\mathrm{Im}(\tau) > 0$, $\mathrm{Im}(\eta/\tau) < 0$ (which admits $\mathrm{Im}(\eta) = 0$):

$$\mathrm{S}_2(a) = G(\pm z_1 \pm z_2 + a + \tfrac{1}{2}\textstyle\sum_{k=1}^{3} \omega_k\,; \omega)\varphi_2(z_1, z_2), \quad a = u_2 - v_1,$$

where $\varphi_2(z_1 + \omega_1, z_2) = \varphi_2(z_1, z_2 + \omega_1) = \varphi_2(z_1 + \omega_1/2, z_2 + \omega_1/2)$. Because of the inversion formula for $G(z;\omega)$-function, for $\varphi_2 = 1$ one has $\mathrm{S}_2(-a)\mathrm{S}_2(a) = \mathbb{1}$.

A solution of equations (4.16), (4.17), and (4.18) for the Δ-kernel valid for $\mathrm{Im}(\eta) = 0$ has the form

$$\Delta(z,x) = \mathrm{e}^{\frac{2\pi \mathrm{i}}{\omega_1\omega_2}(x^2 - z^2)} \frac{G(\pm x \pm z - u_1 + u_2;\omega)}{G(\pm 2x;\omega)} \varphi_1(z,x), \tag{9.8}$$

where φ_1-function has the same periodicity properties as φ_2.

Substitute now the second form of $G(x;\omega)$-function (11.21) into these expressions. Then we can write

$$\mathrm{S}_2(a) = e^{-\frac{4\pi \mathrm{i} a}{\omega_1\omega_2\omega_3}(z_1^2+z_2^2)} \cdot \Gamma\big(-\tfrac{1}{\omega_3}(\pm z_1 \pm z_2 + a + \tfrac{1}{2}\textstyle\sum_{k=1}^{3}\omega_k) \,\big|\, -\tfrac{\omega_2}{\omega_3}, -\tfrac{\omega_1}{\omega_3}\big) \cdot \varphi_2'(z_1,z_2),$$

where

$$\varphi_2'(z_1,z_2) = \mathrm{e}^{-\frac{4\pi \mathrm{i}}{3} B_{3,3}(a+\frac{1}{2}\sum_{k=1}^{3}\omega_k;\omega)} \cdot \varphi_2(z_1,z_2).$$

Analogously,

$$\Delta(z,x) = \mathrm{e}^{\frac{2\pi \mathrm{i}}{\omega_1\omega_2}(x^2-z^2)}\mathrm{e}^{\frac{4\pi \mathrm{i}}{\omega_1\omega_2\omega_3}[x^2(b-\frac{1}{2}\sum_{k=1}^3\omega_k)+z^2(b+\frac{1}{2}\sum_{k=1}^3\omega_k)]}$$
$$\times \frac{\Gamma\big(-\frac{1}{\omega_3}(\pm x\pm z-b)\big|-\frac{\omega_2}{\omega_3},-\frac{\omega_1}{\omega_3}\big)}{\Gamma\big(\pm\frac{2x}{\omega_3}\big|-\frac{\omega_2}{\omega_3},-\frac{\omega_1}{\omega_3}\big)}\cdot\varphi_1'(z,x)$$

where $b=u_1-u_2$ and

$$\varphi_1'(z,x) = \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}(B_{3,3}(0;\omega)-2B_{3,3}(-b;\omega))}\cdot\varphi_1(z,x).$$

After derivation of the Δ-kernel we have to fix the integration interval $[\alpha,\beta]$ and the space of functions for which the S_1-operator really satisfies the intertwining relations. First, as it was done earlier, we pass to the modified Sklyanin algebra generators (4.21) with additional shift by 1/2 in the arguments of ℓ-dependent theta functions and conjugate similarly S_k-operators. This does not change the operator S_2, but removes the exponential $\mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega_1\omega_2}(x^2-z^2)}$ from $\Delta(z,x)$. Then we note that the ratio of elliptic gamma functions in Δ is a periodic function of z and x with the period ω_3. Therefore we set $\alpha=0$ and $\beta=\omega_3$ and demand that the modified operator S_1 acts on functions $\Phi(x)$ such that $\Psi(x) := \mathrm{e}^{\frac{4\pi\mathrm{i}}{\omega_1\omega_2\omega_3}x^2(b-\frac{1}{2}\sum_{k=1}^3\omega_k)}\Phi(x)$ is an even ω_3-periodic function of x, $\Psi(-x)=\Psi(x+\omega_3)=\Psi(x)$. Finally, equations for Δ-kernel are true provided $\Delta(z,x)$ has no poles in the parallelogram $x\in[-\omega_1,\omega_3-\omega_1,\omega_3+\omega_1,\omega_1]$ which, by complete analogy with the previous cases, is guaranteed for $|e^{2\pi\mathrm{i}(b\pm z)/\omega_3}|<|e^{\pi\mathrm{i}\omega_1/\omega_3}|$.

So, we have found operators S_1, S_2, and S_3 (it differs from S_1 only by the space where it acts). Returning back to the original notation, i.e. renormalizing back $x\to x\omega_2, u\to u\omega_2$, etc, we come to the following theorem.

Theorem 6. *Let $Im(\tau)>0$ (i.e., $|p|<1$) and $\mathrm{Im}(\eta/\tau)<0$ (for $\mathrm{Im}(\eta)=0$, i.e. $|q|=1$, this assumes $\mathrm{Re}(\eta)>0$). Denote $\varphi_k'(z,x)$, $k=1,2,3$, arbitrary even elliptic functions of z and x with periods τ and 2η satisfying additional constraints*

$$\varphi_k'(z+\eta,x+\eta)=\varphi_k'(z,x),\quad k=1,2,3,$$

and not having simple poles in the domains $Im(\eta/\tau)\le Im(z/\tau), Im(x/\tau)\le -Im(\eta/\tau)$.

Define the operators

$$\mathrm{S}_2(u_2-v_1)=e^{\frac{2\pi\mathrm{i}(v_1-u_2)}{\eta\tau}(z_1^2+z_2^2)}\cdot\Gamma\big(\frac{\pm z_1\pm z_2+v_1-u_2}{\tau}-\tfrac{1}{2}-\tfrac{\eta}{\tau}-\tfrac{1}{2\tau}\,|-\tfrac{1}{\tau},-\tfrac{2\eta}{\tau}\big)\cdot\varphi_2'(z_1,z_2),\quad(9.9)$$

where $|\sqrt{\widetilde{p}\widetilde{r}}e^{2\pi\mathrm{i}(v_1-u_2)/\tau}|<|\widetilde{r}|$ with $\widetilde{p}=e^{-2\pi\mathrm{i}/\tau}$ and $\widetilde{r}=e^{-4\pi\mathrm{i}\eta/\tau}$, and

$$[\mathrm{S}_1(a)\Phi](z_1,z_2)=\widetilde{\kappa}\int_0^\tau \mathrm{e}^{\frac{2\pi\mathrm{i}}{\eta\tau}[x^2(a-\frac{1}{2}-\eta-\frac{\tau}{2})+z_1^2(a+\frac{1}{2}+\eta+\frac{\tau}{2})]}\frac{\Gamma\big(\frac{1}{\tau}(\pm x\pm z_1+a)\big|-\frac{1}{\tau},-\frac{2\eta}{\tau}\big)}{\Gamma\big(\frac{2a}{\tau},\pm\frac{2x}{\tau}\big|-\frac{1}{\tau},-\frac{2\eta}{\tau}\big)}\cdot\varphi_1'(z_1,x)\Phi(x,z_2)\frac{dx}{\tau},$$

where $a=u_1-u_2$, $|e^{2\pi\mathrm{i}(a\pm z_1)/\tau}|<|\widetilde{r}|^{1/2}$ and $\widetilde{\kappa}=(\widetilde{p};\widetilde{q})_\infty(\widetilde{r};\widetilde{r})_\infty/2$,

$$[\mathrm{S}_3(b)\Phi](z_1,z_2)=\widetilde{\kappa}\int_0^\tau \mathrm{e}^{\frac{2\pi\mathrm{i}}{\eta\tau}[x^2(b-\frac{1}{2}-\eta-\frac{\tau}{2})+z_2^2(b+\frac{1}{2}+\eta+\frac{\tau}{2})]}\frac{\Gamma\big(\frac{1}{\tau}(\pm x\pm z_2+b)\big|-\frac{1}{\tau},-\frac{2\eta}{\tau}\big)}{\Gamma\big(\frac{2b}{\tau},\pm\frac{2x}{\tau}\big|-\frac{1}{\tau},-\frac{2\eta}{\tau}\big)}\cdot\varphi_3'(z_2,x)\Phi(z_1,x)\frac{dx}{\tau},\quad(9.10)$$

where $b = v_1 - v_2$, $|e^{2\pi\mathrm{i}(b\pm z_2)/\tau}| < |\widetilde{r}|^{1/2}$.

Denote as $V_{a,b}$ *the space of functions of two complex variables* $\Phi(z_1, z_2)$ *such that the products*

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{\eta\tau}z_1^2(a-\frac{1}{2}-\eta-\frac{\tau}{2})}\mathrm{e}^{\frac{2\pi\mathrm{i}}{\eta\tau}z_2^2(b-\frac{1}{2}-\eta-\frac{\tau}{2})}\Phi(z_1, z_2)$$

are even and periodic in z_1 *and* z_2 *with the period* τ *and which do not have simple poles in the domains* $Im(\eta/\tau) \leq Im(z_1/\tau), Im(z_2/\tau) \leq -Im(\eta/\tau)$. *Then the operators* S_1, S_2 *and* S_3 *map the space* $V_{a,b}$ *for* $a = u_1 - u_2$ *and* $b = v_1 - v_2$ *onto itself and they satisfy the defining intertwining relations* (3.10), (3.11) *and* (3.12) *provided in the corresponding* L*-operator* (2.2) *one uses the Sklyanin algebra generators of the form*

$$\mathbf{S}^a_{mod} = e^{-\pi\mathrm{i}\eta}\frac{(\mathrm{i})^{\delta_{a,2}}\theta_{a+1}(\eta)}{\theta_1(2z)}\Big[\,\theta_{a+1}\left(2z - 2\eta\ell + \tfrac{1}{2}\right)\cdot e^{-2\pi\mathrm{i}z}\cdot \mathrm{e}^{\eta\partial_z} - \theta_{a+1}\left(-2z - 2\eta\ell + \tfrac{1}{2}\right)\cdot e^{2\pi\mathrm{i}z}\cdot \mathrm{e}^{-\eta\partial_z}\,\Big]. \tag{9.11}$$

It remains to confirm the Coxeter relations for the choice $\varphi'_k = 1$. The relation $\mathrm{S}_2(a)\mathrm{S}_2(-a) = \mathbb{1}$ is evident. It is not difficult to check the cubic Coxeter relation (5.3) as it leads to a solution of the star-triangle relation for $\mathrm{Im}(\eta) = 0$ considered in [30]. We shall not present corresponding details – although it is neater than before, all the exponential factors cancel and the identity is reduced again to the computation of the elliptic beta integral in a particular parameterization [13]. Also, it follows from the integral analogue of the Bailey lemma formulated in terms of the $G(z;\omega)$-function. In a similar way, equalities $\mathrm{S}_1(a)\mathrm{S}_1(-a) = \mathrm{S}_3(a)\mathrm{S}_3(-a) = \mathbb{1}$ are reduced to the previously considered inversion relations in a different parameterization because of the cancellation of exponential factors. In general, in the arbitrary product $\ldots \mathrm{S}_i\mathrm{S}_j\mathrm{S}_k \ldots$ one never violates the restrictions on space of functions $\Phi(z_1, z_2)$ needed for operators S_1 and S_3, i.e. all the exponential factors can be pulled out to the far left and far right.

We could write out the explicit form of the R-operator using its factorized form, but it is skipped since this is a straightforward procedure leading to a somewhat cumbersome expression. After dropping the exponential factors from this expression one would come to the R-operator which is obtained from (7.1) (with $\sqrt{pq}$ replaced by $-\sqrt{pq}$) by a simple modular transformation $(\omega_2, \omega_3) \to (-\omega_3, \omega_2)$.

As to the elliptic modular double, the R-operator written in terms of the $G(u;\omega)$-functions (i.e., the form obtained after scalings $z \to z/\omega_2$, etc) is symmetric with respect to the permutation $\omega_1 \leftrightarrow \omega_2$. In the original notation $z, g := \eta(2\ell + 1), \eta, \tau$, the permutation of these quasiperiods is equivalent to the changes $\eta \to 1/4\eta$, $\tau \to \tau/2\eta, z \to z/2\eta, g \to g/2\eta$ (here g is considered as an independent variable). Therefore the derived R-operator also has second RLL-relation, where L^{doub} is composed of a new Sklyanin algebra generators of the form

$$\widetilde{\mathbf{S}}^a_{mod} = e^{-\frac{\pi\mathrm{i}}{4\eta}}\frac{(\mathrm{i})^{\delta_{a,2}}\theta_{a+1}(\frac{1}{4\eta}|\frac{\tau}{2\eta})}{\theta_1(\frac{z}{\eta}|\frac{\tau}{2\eta}))}\Big[\,\theta_{a+1}\left(\frac{2z-g+1}{2\eta}\Big|\frac{\tau}{2\eta}\right)\cdot e^{-\pi\mathrm{i}z/\eta}\cdot \mathrm{e}^{\frac{1}{4\eta}\partial_z} - \theta_{a+1}\left(\frac{-2z-g+1}{2\eta}\Big|\frac{\tau}{2\eta}\right)\cdot e^{\pi\mathrm{i}z/\eta}\cdot \mathrm{e}^{-\frac{1}{4\eta}\partial_z}\,\Big]. \tag{9.12}$$

This elliptic modular double has been introduced in [19] as well. Thus we have found solutions of the Yang-Baxter equation (1.1) for all possible regions of the key complex parameter η.

10 Conclusion

In this paper we have merged two constructions from the theory of quantum integrable systems and the theory of special functions. One construction is a specific approach to building YBE solutions developed in [20–23]. It is based on the realization of the permutation group generators by various operators acting in the functional spaces and it directly leads to the factorized form of the R-matrices as products of elementary transposition operators. Another construction is the elliptic beta integral evaluation [13] and its various consequences formulated as an elliptic Fourier transformation and integral Bailey lemma [17]. This result formed a basis for developing the theory of a principally new and very powerful class of special functions — elliptic hypergeometric integrals [11, 14]. As a result of our considerations, both fields have benefited and mutually enriched each other. The most complicated known R-matrix at the elliptic level appeared to be defined by an integral operator with an elliptic hypergeometric kernel and algebraic properties of the integral Bailey lemma ingredients got a natural interpretation as Coxeter relations for the permutation group generators. Moreover, the key integral operator defining the elliptic Fourier transformation appeared to be an intertwining operator for the Sklyanin algebra. The general construction shows that YBE is a simple consequence of a particular word identity in the group algebra for the braid group $\mathfrak{B}_6$ or, in our case, of the symmetric group $\mathfrak{S}_6$, whose generators are realized as integral operators.

Our results can be applied to all known forms of YBE [7]. In particular, a generalization of our construction to root systems is relatively straightforward due to the abundance of elliptic beta integrals on root systems [11] and corresponding elliptic Fourier transformations [29]. In the rational case the most general known R-operator for A_n-root system was constructed in [21, 22]. Star-triangle and star-star type relations for the root systems following from the elliptic hypergeometric integral identities were considered in [30, 32].

In [33], Faddeev and Volkov constructed a solution of YBE at the q-hypergeometric level with the help of the pentagon relation for noncompact quantum dilogarithms. A generalization of this model has been found in [30] and a question was posed — is it still related to the pentagon relation and does there exist an elliptic analogue of the latter ? As shown in [23], the method used in the present paper works at the compact q-hypergeometric level using q-exponential functions. The noncompact situation can be treated as well after appropriate replacement of q-exponentials by the noncompact quantum dilogarithms. However, it is easy to degenerate our elliptic results to both compact and non-compact q-levels. From our analysis of the elliptic hypergeometric constructions it is not clear which relation can be taken as a direct elliptic analogue of the pentagon relation. Instead of a potential five-term relation, the key role is played by the hexagon relation (6.4) emerging in the theory of elliptic Fourier transformation [17] and defining the Coxeter relation for permutation operators (or the star-triangle relation in integrable models of statistical mechanics).

We would like to stress that the spin variable ℓ in our analysis takes continuous values. Therefore, strictly speaking, we deal not with the discrete Ising-type models, but with two-dimensional quantum field theories. In this context the Yang-Baxter equation can be interpreted as a condition of factorizing the N-body S-matrix to the product of two-body scattering matrices [7].

Let us discuss briefly an application of our results to four-dimensional ($4d$) supersymmetric gauge field theories. The key discovery of [34] consists in the fact that superconformal indices of these theories are described by the elliptic hypergeometric integrals. For example, the elliptic beta integral evaluation formula gets a remarkable interpretation as a direct indication on the confinement phenomenon in the simplest $4d$ supersymmetric quantum chromodynamics.

A relation between $4d$ Nekrasov instanton partition function and conformal blocks in $2d$ Liouville field theory was empirically discovered in [35]. In [36], superconformal indices of $4d$ $\mathcal{N} = 2$ supersymmetric field theories were tied to correlation functions of $2d$ topological field theories. Our results are relevant to a different type of $4d/2d$ correspondence discovered in [30], where $4d$ superconformal indices coincide with partition functions of integrable models of $2d$ spin systems.

Seiberg duality is a special electric-magnetic duality of $4d$ supersymmetric non-abelian gauge field theories. In the language of elliptic hypergeometric integrals there are two qualitatively different situations. When the dual theory confines corresponding superconformal index is identical to some elliptic beta integral on a root system or, from the statistical mechanics point of view, to star-triangle relation for multicomponent spin systems. When the dual theory is a nontrivial interacting field theory, one deals with symmetry transformations for integrals equivalent to the star-star relations in statistical mechanics [30].

Since superconformal indices for simple gauge groups coincide with the statistical sums of elementary cells for spin systems on the plane, the Seiberg duality transformations represent the Kramers-Wannier type duality transformations for corresponding new $2d$ integrable models. As indicated in [37] sequential integral transformations following from the Bailey lemma define superconformal indices of particular quiver gauge theories. This procedure corresponds to building full two-dimensional lattice partition functions as prescribed in the theory of quantum integrable systems.

Acknowledgments

The authors are deeply indebted to L. D. Faddeev and A. M. Vershik for useful discussions and general support. The work of V. S. is supported by RFBR grant no. 11-01-00980 and NRU HSE scientific fund grant no. 12-09-0064. The work of S. D. is supported by RFBR grants 11-01-00570, 11-01-12037, 12-02-91052 and Deutsche Forschungsgemeinschaft (KI 623/8-1).

11 Appendix

In this Appendix we collect some useful formulae. The standard infinite q-product is defined as

$$(x;q)_\infty = \prod_{k=0}^{+\infty}(1 - q^k \cdot x) \; ; \quad q \in \mathbb{C} \, , \quad |q| < 1. \tag{11.1}$$

The general theta-function with characteristics has the form

$$\theta_{a,b}(z|\tau) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi \mathrm{i}(n+\frac{a}{2})^2\tau} \cdot \mathrm{e}^{2\pi \mathrm{i}(n+\frac{a}{2})(z+\frac{b}{2})}.$$

We use four standard theta-functions

$$\theta_1(z|\tau) = -\theta_{1,1}(z|\tau) = -\sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi \mathrm{i}\left(n+\frac{1}{2}\right)^2\tau} \cdot \mathrm{e}^{2\pi \mathrm{i}\left(n+\frac{1}{2}\right)\left(z+\frac{1}{2}\right)}$$

$$= \mathrm{i}p^{1/8}e^{-\pi \mathrm{i}z}\,(p;p)_\infty\,\theta(e^{2\pi \mathrm{i}z};p), \tag{11.2}$$

where $p = e^{2\pi\mathrm{i}\tau}$ and

$$\theta(t;p) = (t;p)_\infty (pt^{-1};p)_\infty, \tag{11.3}$$

$$\theta_2(z|\tau) = \theta_{1,0}(z|\tau) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi\mathrm{i}\left(n+\frac{1}{2}\right)^2\tau} \cdot \mathrm{e}^{2\pi\mathrm{i}\left(n+\frac{1}{2}\right)z} \tag{11.4}$$

$$\theta_3(z|\tau) = \theta_{0,0}(z|\tau) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi\mathrm{i}n^2\tau} \cdot \mathrm{e}^{2\pi\mathrm{i}nz} \tag{11.5}$$

$$\theta_4(z|\tau) = \theta_{0,1}(z|\tau) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi\mathrm{i}n^2\tau} \cdot \mathrm{e}^{2\pi\mathrm{i}n\left(z+\frac{1}{2}\right)}. \tag{11.6}$$

The following identities are used to factorize the L-operator and to derive defining equations for the operators S_1, S_2, and S_3:

$$2\,\theta_1(x+y)\,\theta_1(x-y) = \bar\theta_4(x)\,\bar\theta_3(y) - \bar\theta_4(y)\,\bar\theta_3(x), \tag{11.7}$$
$$2\,\theta_2(x+y)\,\theta_2(x-y) = \bar\theta_3(x)\,\bar\theta_3(y) - \bar\theta_4(y)\,\bar\theta_4(x), \tag{11.8}$$
$$2\,\theta_3(x+y)\,\theta_3(x-y) = \bar\theta_3(x)\,\bar\theta_3(y) + \bar\theta_4(y)\,\bar\theta_4(x), \tag{11.9}$$
$$2\,\theta_4(x+y)\,\theta_4(x-y) = \bar\theta_4(x)\,\bar\theta_3(y) + \bar\theta_4(y)\,\bar\theta_3(x), \tag{11.10}$$
$$2\,\theta_4(x+y)\,\theta_1(x-y) = \bar\theta_1(x)\,\bar\theta_2(y) - \bar\theta_1(y)\,\bar\theta_2(x), \tag{11.11}$$
$$\bar\theta_1(x-y)\,\bar\theta_2(x+y) = \theta_1(2x)\,\theta_4(2y) - \theta_1(2y)\,\theta_4(2x), \tag{11.12}$$

where $\bar\theta_a(z) \equiv \theta_a\left(z|\frac{\tau}{2}\right)$. We need also the duplication formula

$$\theta_1(2x|2\tau) = \frac{(-p;p)_\infty}{(p;p)_\infty}\theta_1(x|\tau)\theta_2(x|\tau), \qquad p = e^{2\pi\mathrm{i}\tau}. \tag{11.13}$$

For $\mathrm{Im}(\tau) > 0$, $\mathrm{Im}(\eta) > 0$ the elliptic gamma function is defined by the double infinite product

$$\Gamma(z|\tau,2\eta) \equiv \prod_{n,m=0}^{\infty} \frac{1-\mathrm{e}^{2\pi\mathrm{i}(\tau(n+1)+2\eta(m+1)-z)}}{1-\mathrm{e}^{2\pi\mathrm{i}(\tau n+2\eta m+z)}}. \tag{11.14}$$

It is symmetric in its modular parameters $\Gamma(z|\tau,2\eta) = \Gamma(z|2\eta,\tau)$ and satisfies equations

$$\Gamma(z+1|\tau,2\eta) = \Gamma(z|\tau,2\eta), \tag{11.15}$$

$$\Gamma(z+\tau|\tau,2\eta) = \theta(e^{2\pi\mathrm{i}z};e^{4\pi\mathrm{i}\eta})\cdot\Gamma(z|\tau,2\eta), \tag{11.16}$$

$$\Gamma(z+2\eta|\tau,2\eta) = \theta(e^{2\pi\mathrm{i}z};e^{2\pi\mathrm{i}\tau})\cdot\Gamma(z|\tau,2\eta), \tag{11.17}$$

and the normalization condition $\Gamma(\eta+\tau/2|\tau,2\eta) = 1$. One can evidently replace in these equations

$$\theta(e^{2\pi\mathrm{i}z};e^{2\pi\mathrm{i}\tau}) = \mathrm{R}(\tau)\cdot\mathrm{e}^{\pi iz}\theta_1(z|\tau), \quad \theta(e^{2\pi\mathrm{i}z};e^{4\pi\mathrm{i}\eta}) = \mathrm{R}(2\eta)\cdot\mathrm{e}^{\pi iz}\theta_1(z|2\eta),$$

where the constant $\mathrm{R}(\tau)$ does not depend on z: $\mathrm{R}(\tau) = -\mathrm{i}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{4}}\cdot\left(\mathrm{e}^{2\pi\mathrm{i}\tau};\mathrm{e}^{2\pi\mathrm{i}\tau}\right)_\infty^{-1}$.

Zeros of $\Gamma(z|\tau, 2\eta)$ are located at $z = \mathbb{Z} + \tau\mathbb{Z}_{>0} + 2\eta\mathbb{Z}_{>0}$ and poles at $z = \mathbb{Z} + \tau\mathbb{Z}_{\leq 0} + 2\eta\mathbb{Z}_{\leq 0}$. The reflection equation for this function has the form

$$\Gamma(z|\tau, 2\eta)\Gamma(-z + 2\eta + \tau|\tau, 2\eta) = 1. \tag{11.18}$$

In the multiplicative notation one has

$$\Gamma(t; p, q) = \prod_{j,k=0}^{\infty} \frac{1 - t^{-1}p^{j+1}q^{k+1}}{1 - tp^j q^k}, \quad |p|, |q| < 1, \tag{11.19}$$

so that $\Gamma(t; p, q)\Gamma(pq/t; p, q) = 1$ and

$$\Gamma(qt; p, q) = \theta(t; p)\Gamma(t; p, q), \quad \Gamma(pt; p, q) = \theta(t; q)\Gamma(t; p, q).$$

For incommensurate $\omega_1, \omega_2, \omega_3 \in \mathbb{C}$ define three base variables,

$$\begin{aligned} q &= e^{2\pi \mathrm{i} \frac{\omega_1}{\omega_2}}, \quad p = e^{2\pi \mathrm{i} \frac{\omega_3}{\omega_2}}, \quad r = e^{2\pi \mathrm{i} \frac{\omega_3}{\omega_1}}, \\ \widetilde{q} &= e^{-2\pi \mathrm{i} \frac{\omega_2}{\omega_1}}, \quad \widetilde{p} = e^{-2\pi \mathrm{i} \frac{\omega_2}{\omega_3}}, \quad \widetilde{r} = e^{-2\pi \mathrm{i} \frac{\omega_1}{\omega_3}}, \end{aligned}$$

where $\widetilde{q}, \widetilde{p}, \widetilde{r}$ denote particular modular transformed bases. The condition that $\sum_{k=1}^{3} n_k \omega_k \neq 0$, $n_k \in \mathbb{Z}$, implies that none of p, q, and r is a root unity.

For $|q|, |p| < 1$ (which assumes $|r| < 1$) the modified elliptic gamma function is defined as

$$G(u; \omega) = \Gamma(e^{2\pi \mathrm{i} u/\omega_2}; p, q)\Gamma(re^{-2\pi \mathrm{i} u/\omega_1}; \widetilde{q}, r) = \frac{\Gamma(e^{2\pi \mathrm{i} u/\omega_2}; p, q)}{\Gamma(\widetilde{q}e^{2\pi \mathrm{i} u/\omega_1}; \widetilde{q}, r)}. \tag{11.20}$$

This is a meromorphic function of u even for $\omega_1/\omega_2 > 0$, when $|q| = 1$, which is easily seen from its another representation

$$G(u; \omega) \;=\; e^{-\frac{\pi \mathrm{i}}{3} B_{3,3}(u;\omega)} \Gamma(e^{-2\pi \mathrm{i} u/\omega_3}; \widetilde{r}, \widetilde{p}), \tag{11.21}$$

where $B_{3,3}$ is a Bernoulli polynomial of the third order

$$B_{3,3}(u; \omega) = \frac{1}{\omega_1 \omega_2 \omega_3} \Big(u - \frac{1}{2}\sum_{k=1}^{3} \omega_k\Big) \Big((u - \frac{1}{2}\sum_{k=1}^{3} \omega_k)^2 - \frac{1}{4}\sum_{k=1}^{3} \omega_k^2\Big). \tag{11.22}$$

Multiple Bernoulli polynomials are defined in the theory of Barnes multiple zeta-function from the following expansion

$$\frac{x^m e^{xu}}{\prod_{k=1}^{m}(e^{\omega_k x} - 1)} = \sum_{n=0}^{\infty} B_{m,n}(u; \omega_1, \ldots, \omega_m) \frac{x^n}{n!}.$$

This function satisfies the equations

$$G(u + \omega_1) = \theta(e^{2\pi \mathrm{i} u/\omega_2}; p)G(u), \tag{11.23}$$

$$G(u + \omega_2) = \theta(e^{2\pi \mathrm{i} u/\omega_1}; r)G(u), \qquad G(u + \omega_3) = e^{-\pi \mathrm{i} B_{2,2}(u;\omega)} G(u).$$

and the normalization condition $G(\sum_{m=1}^{3}\omega_m/2)=1$. Here

$$B_{2,2}(u;\omega)=\frac{u^2}{\omega_1\omega_2}-\frac{u}{\omega_1}-\frac{u}{\omega_2}+\frac{\omega_1}{6\omega_2}+\frac{\omega_2}{6\omega_1}+\frac{1}{2}$$

is the second order Bernoulli polynomial appearing in the modular transformation law for the theta function

$$\theta\left(e^{-2\pi\mathrm{i}\frac{u}{\omega_1}};e^{-2\pi\mathrm{i}\frac{\omega_2}{\omega_1}}\right)=e^{\pi\mathrm{i}B_{2,2}(u;\omega)}\theta\left(e^{2\pi\mathrm{i}\frac{u}{\omega_2}};e^{2\pi\mathrm{i}\frac{\omega_1}{\omega_2}}\right). \tag{11.24}$$

The reflection equation for $G(u)$ has the form

$$G(a,b;\omega):=G(a;\omega)G(b;\omega)=1, \qquad a+b=\sum_{k=1}^{3}\omega_k.$$

References.

[1] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London, 1982.

[2] L. D. Faddeev and L. A. Takhtadzhan, *The quantum method of inverse problem and the Heisenberg XYZ model*, Uspekhi Mat. Nauk **34** (5), 13–63 (Russian Math. Surveys **34** (5) (1979), 11–68).

[3] P. P. Kulish and E. K. Sklyanin, *On the solutions of the Yang-Baxter equation*, Zap. Nauchn. Sem. LOMI **95** (1980), 129–160.

[4] P. P. Kulish and E. K. Sklyanin, *Quantum spectral transform method. Recent developments*, Lect. Notes in Physics, vol. **151** (1981), 61–119.

[5] M. Jimbo (ed), *Yang-Baxter equation in integrable systems*, Adv. Ser. Math. Phys., 10, World Scientific (Singapore), 1990.

[6] L. D. Faddeev, *How algebraic Bethe ansatz works for integrable model*, Quantum Symmetries, Proc. Les-Houches summer school, LXIV, North-Holland, 1998, 149–211.

[7] J. H. H. Perk and H. Au-Yang, *Yang-Baxter equations*, `math-ph/0606053`.

[8] R. J. Baxter, *Partition function of the eight-vertex lattice model*, Ann. Phys. **70** (1972), 193–228 (reprinted in Ann. Phys. **281** (2000), 187–222).

[9] E. K. Sklyanin, *On some algebraic structures related to Yang-Baxter equation*, Funkz. Analiz i ego Pril. **16** (1982), 27–34.

[10] E. K. Sklyanin, *On some algebraic structures related to Yang-Baxter equation: representations of the quantum algebra*, Funkz. Analiz i ego Pril. **17** (1983), 34–48.

[11] V. P. Spiridonov, *Essays on the theory of elliptic hypergeometric functions*, Uspekhi Mat. Nauk **63** (3) (2008), 3–72 (Russian Math. Surveys **63** (3) (2008), 405–472), `arXiv:0805.3135 [math.CA]`.

[12] S. E. Derkachov, D. Karakhanyan, and R. Kirschner, *Universal R-matrix as integral operator*, Nucl. Phys. **B618** (2001), 589–616.

[13] V. P. Spiridonov, *On the elliptic beta function*, Uspekhi Mat. Nauk **56** (1) (2001), 181–182 (Russian Math. Surveys **56** (1) (2001), 185–186).

[14] V. P. Spiridonov, *Theta hypergeometric integrals*, Algebra i Analiz **15** (6) (2003), 161–215 (St. Petersburg Math. J. **15** (6) (2004), 929–967), `math.CA/0303205`.

[15] I. Frenkel and V. Turaev, *Elliptic solutions of the Yang-Baxter equation and modular hypergeometric functions*, The Arnold-Gelfand Mathematical Seminars (Cambridge, MA:Birkhauser Boston) (1997), pp. 171–204.

[16] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia of Math. Appl. **71**, Cambridge Univ. Press, Cambridge, 1999.

[17] V. P. Spiridonov, *A Bailey tree for integrals*, Teor. Mat. Fiz. **139** (2004), 104–111 (Theor. Math. Phys. **139** (2004), 536–541), `math.CA/0312502`.

[18] L. D. Faddeev, *Modular double of a quantum group*, Conf. Moshé Flato 1999, vol. I, Math. Phys. Stud. **21**, Kluwer, Dordrecht, 2000, pp. 149–156.

[19] V. P. Spiridonov, *Continuous biorthogonality of the elliptic hypergeometric function*, Algebra i Analiz **20** (5) (2008), 155–185 (St. Petersburg Math. J. **20** (5) (2009) 791–812), `arXiv:0801.4137 [math.CA]`.

[20] S. E. Derkachov, *Factorization of the R-matrix. I.* Zap. Nauchn. Sem. POMI **335** (2006), 134–163 (J. Math. Sciences **143** (1) (2007), 2773-2790), `math.QA/0503396`.

[21] S. E. Derkachov and A. N. Manashov, *R-matrix and Baxter Q-operators for the noncompact* $\mathrm{SL}(n, \mathbb{C})$ *invariant spin chain*, SIGMA **2** (2006), 084.

[22] S. Derkachov and A. Manashov, *General solution of the Yang-Baxter equation with the symmetry group* $\mathrm{SL}(n, \mathbb{C})$, Algebra i Analiz **21** (4) (2009), 1–94 (St. Petersburg Math. J. **21** (2010), 513–577).

[23] S. Derkachov, D. Karakhanyan, and R. Kirschner, *Yang-Baxter $\mathcal{R}$-operators and parameter permutations*, Nucl. Phys. **B785** (2007), 263–285.

[24] B. Felderhof, *Diagonalization of the transfer matrix of the free-fermion model*, Physica **66** (1973), no. 2, 279–198.

[25] I. M. Krichever, *Baxter's equations and algebraic geometry*, Funct. An. and its Appl. **15** (1981), no. 2, 22–35.

[26] A. Zabrodin, *On the spectral curve of the difference Lame operator*, Int. Math. Research Notices, no. 11 (1999), 589–614.

[27] A. Zabrodin, *Commuting difference operators with elliptic coefficients from Baxter's vacuum vectors*, J. Phys. A: Math. Gen. **33** (2000), 3825–3850.

[28] P. P. Kulish, N. Yu. Reshetikhin, and E. K. Sklyanin, *Yang-Baxter equation and representation theory*, Lett. Math. Phys. **5** (1981), 393–403.

[29] V. P. Spiridonov and S. O. Warnaar, *Inversions of integral operators and elliptic beta integrals on root systems*, Adv. Math. **207** (2006), 91–132.

[30] V. P. Spiridonov, *Elliptic beta integrals and solvable models of statistical mechanics*, Contemp. Math. **563** (2012), 181–211, `arXiv:1011.3798 [hep-th]`.

[31] V. V. Bazhanov and S. M. Sergeev, *A master solution of the quantum Yang-Baxter equation and classical discrete integrable equations*, `arXiv:1006.0651 [math-ph]`.

[32] V. V. Bazhanov and S. M. Sergeev, *Elliptic gamma-function and multi-spin solutions of the Yang-Baxter equation*, `arXiv:1106.5874 [math-ph]`.

[33] A. Yu. Volkov and L. D. Faddeev, *Yang-Baxterization of the quantum dilogarithm*, Zapiski POMI **224** (1995), 146–154 (J. Math. Sciences **88** (2) (1998), 202–207).

[34] F. A. Dolan and H. Osborn, *Applications of the superconformal index for protected operators and q-hypergeometric identities to $\mathcal{N} = 1$ dual theories*, Nucl. Phys. **B818** (2009), 137–178.

[35] L. F. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, Lett. Math. Phys. **91** (2010), 167–197.

[36] A. Gadde, E. Pomoni, L. Rastelli, and S. S. Razamat, *S-duality and 2d topological QFT*, J. High Energy Phys. **03** (2010), 032.

[37] V. P. Spiridonov and G. S. Vartanov, *Elliptic hypergeometry of supersymmetric dualities*, Commun. Math. Phys. **304** (2011), 797–874.