

Differential algebras with Banach-algebra coefficients I: From C^* -algebras to the K-theory of the spectral curve

Maurice J. Dupré
 Department of Mathematics, Tulane University
 New Orleans, LA 70118 USA
 mdupre@tulane.edu

James F. Glazebrook
 (Primary Inst.)
 Department of Mathematics and Computer Science
 Eastern Illinois University
 600 Lincoln Ave., Charleston, IL 61920–3099 USA
 jfglazebrook@eiu.edu
 (Adjunct Faculty)
 Department of Mathematics
 University of Illinois at Urbana–Champaign
 Urbana, IL 61801, USA

Emma Previato*
 Department of Mathematics and Statistics, Boston University
 Boston, MA 02215–2411, USA
 ep@math.bu.edu

Abstract

We present an operator-coefficient version of Sato’s infinite-dimensional Grassmann manifold, and τ -function. In this context, the Burchnell-Chaundy ring of commuting differential operators becomes a C^* -algebra, to which we apply the Brown-Douglas-Fillmore theory, and topological invariants of the spectral ring become readily available. We construct KK classes of the spectral curve of the ring and, motivated by the fact that all isospectral Burchnell-Chaundy rings make up the Jacobian of the curve, we compare the (degree-1) K-homology of the curve with that of its Jacobian. We show how the Burchnell-Chaundy C^* -algebra extension of the compact operators provides a family of operator-valued τ -functions.

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1 Introduction

This work arises from [20, 21] where we started an operator-theoretic, Banach algebra approach to the Sato-Segal-Wilson theory in the setting of Hilbert modules with the extension of the classical Baker and τ -functions to operator-valued functions. Briefly recalled, the Sato-Segal-Wilson theory [37, 38] in the context of integrable Partial Differential Equations* (PDEs) produces a linearization of the time flows, as one-parameter groups acting on Sato’s ‘Universal Grassmann Manifold’, via conjugation of the trivial solution ($\tau \equiv 0$) by a formal pseudodifferential operator. An important issue, which underlies our work, is that the Grassmannian considered is analytic as opposed to formal. The algebro-geometric solutions, which we focus on, come from a Riemann surface, or from an algebraic curve (the former case is our default assumption, that is to say, the curve is smooth) which is a ‘spectral curve’ in the sense of J. L. Burchnell and T. W. Chaundy (see e.g. [28] for an updated account of their work). These authors in the 1920’s gave a classification of the rank-one[†] commutative rings of Ordinary Differential Operators (ODOs) with fixed spectral curve, as the Jacobi variety of the curve (line bundles of fixed degree, up to isomorphism), using differential algebra. The Krichever map [28, 38] sends a quintuple of holomorphic data associated to the ring to Sato’s infinite-dimensional Grassmann manifold.

Using the Sato correspondence we used a conjugating action by an integral operator and showed how the conjugated Burchnell-Chaundy ring \mathbb{A} of pseudodifferential operators can be represented as a commutative subring of a certain Banach $*$ -algebra A , a ‘restricted’ subalgebra of the bounded linear operators $\mathcal{L}(H_{\mathcal{A}})$ where $H_{\mathcal{A}}$ is a *Hilbert module* over a C^* -algebra \mathcal{A} . A particular feature in this work is the Banach Grassmannian $\text{Gr}(p, A)$ which resembles in a more general sense the restricted Grassmannians formerly introduced in [33, 38] and these latter spaces can be recovered as subspaces of $\text{Gr}(p, A)$. Some of our results require the algebra \mathcal{A} to be commutative and therefore $\mathcal{A} \cong C(Y)$, for some compact Hausdorff space Y . The commutative ring \mathbb{A} has as its joint spectrum $X' = \text{Spec}(\mathbb{A})$ an irreducible complex curve whose one-point compactification is a non-singular (by assumption) algebraic curve X of genus $g_X \geq 1$. With \mathcal{A} being commutative, we obtain a Y -parametrized version of the Krichever correspondence (originally given for the case $\mathcal{A} = \mathbb{C}$).

To transfer information between the geometry of integrable PDEs and that of operator algebras, we show that naturally associated to the ring \mathbb{A} is a C^* -algebra $\underline{\mathbb{A}}$ (which we refer to as the *Burchnell-Chaundy C^* -algebra*) which, by the Gelfand-Neumark-Segal theorem, can be realized as a C^* -algebra of operators on an associated Hilbert space $H(\mathcal{A})$. For any generating s -tuple of commuting operators in $\underline{\mathbb{A}}$ we consider its joint spectrum and show that it is homeomorphic to the joint spectrum X' .

This paper also considers the K-homology of $C(X)$ and related theory. In particular, by the Brown-Douglas-Fillmore (BDF) theorem, the group $K^1(C(X))$ is the extension group $\text{Ext}(X)$, of the compact operators $\mathcal{K}(H)$ by $C(X)$, in the sense that there is a natural transformation of covariant functors

$$\text{Ext}(X) \longrightarrow \text{Hom}(K^1(X), \mathbb{Z}),$$

and this extension group, in turn, can be identified with the C^* -isomorphisms of $C(X)$ into the Calkin algebra $\mathcal{Q}(H)$. Using only the commutative subalgebra we constructed from the Burchnell-

*The KP hierarchy is the model we use; variations would utilize slightly different objects of abstract algebra, e.g. matrices instead of scalars.

[†]Namely, such that the orders of the operators in the ring are not all divisible by some number $r > 1$; if they are, the ring corresponds to a rank- r vector bundle.

Chaundy ring, for the case $\mathcal{A} = \mathbb{C}$, we obtain some of these C*-isomorphisms, parametrizing in fact the Jacobian $J(X)$ of X (which is the group $\text{Ext}^1(\mathcal{O}_X, \mathcal{O}_X^*) \cong H^1(X, \mathcal{O}^*)$). Finally, using the Abel map which embeds $X \rightarrow J(X)$, we fit the three C*-algebras $\underline{\mathbb{A}}, C(X)$ and $C(J(X))$ into the framework of BDF theory [10]. We show, using the basic instrumentation of K-homology, that there exists a natural injection of Ext groups $\text{Ext}(X) \rightarrow \text{Ext}(J(X))$ (an isomorphism when the genus g_X is 1). Finally, in this Part I we consider the operator (\mathcal{A} -) valued τ -function as defined on a group of multiplication operators and show that extensions of the compact operators by the Burchnell-Chaundy C*-algebra $\underline{\mathbb{A}}$, yield a family of τ -functions attached to ‘transverse’ subspaces $W \in \text{Gr}(p, A)$, one for each such extension.

Part II [22] of this work involves a number of extensions of Part I, in particular a geometric setting for predeterminants of the τ -function, called \mathcal{T} -functions. In this setting it is possible pull back the tautological bundle over the Grassmannian, and obtain, much as in the algebraic case of [1], a Poincaré bundle over a product of homogeneous varieties for operator group flows, whose fibres will be related by the BDF theorem to the Jacobians of the spectral curves. An operator cross-ratio on the fibres provides the τ -function, and we investigate the Schwarzian derivative associated to the cross-ratio. As a result, operator-valued objects defined with a minimum of topological assumptions obey the same hierarchies as the corresponding objects arising from the theory of special functions. Then we bring together the main results of this Part I to show in Part II [22] that solutions to the KP hierarchy can in fact be parametrized by elements of $\text{Ext}(\underline{\mathbb{A}})$. Thus we view our results as an enhancement for operator theory and a potential enhancement for the theory under development of geometric quantization and integrable PDEs, where functional Hamiltonians are replaced by operators. Other approaches in the operator-theoretic setting as applied to control theory are studied in [26] and to Lax-Phillips scattering in [4]. The present work could in part be viewed as extensions of the Cowen-Douglas theory [14, 31] linking complex analytic/algebraic techniques to operator theory, thus suggesting future developments.

This Part I is organized as follows: for the reader’s sake we recall the necessary setting and details from [20] and other relevant works in an Appendix §A including our parametrized version of the Krichever correspondence, and proceed from §2 to the construction of the above-described C*-algebras in §3. In §4 we pursue the BDF theory and attach topological objects to the spectral curve and Burchnell-Chaundy ring.

The new results we have obtained in this Part I are Theorem 3.1, Propositions 4.1, A.1, Theorems 4.2, 4.3, 4.4, 4.5.

2 The basic setting

2.1 The restricted Banach *-algebra A and the space $\text{Gr}(p, A)$

Let A be a unital complex Banach(able) algebra with group of units $G(A)$ and space of idempotents $P(A)$. An important carrier for what follows is an infinite dimensional Grassmannian, denoted $\text{Gr}(p, A)$, whose structure as a Banach homogeneous space is briefly described in Appendix §A.2 as well as its universal bundle. Similar spaces and related objects have been used by a number of authors in dealing with a variety of different operator-geometric problems (see §A.2 Remark A.2).

Remark 2.1. We point out that we use the term projection to simply mean an idempotent even though functional analysts often use the term ‘projection’ to mean a self-adjoint idempotent when

dealing with a $*$ -algebra. Consequently we will explicitly state when an idempotent is or should be required to be self-adjoint. In the latter case we denote the set of these as $P^{\text{sa}}(A)$.

Let H be a separable (infinite dimensional) Hilbert space. Given a unital separable C^* -algebra \mathcal{A} , we take the standard (free countable dimensional) Hilbert module $H_{\mathcal{A}}$ over \mathcal{A} (see Appendix §A.1) and consider a *polarization* of $H_{\mathcal{A}}$ given by a pair of submodules (H_+, H_-) , such that

$$H_{\mathcal{A}} = H_+ \oplus H_- \text{ , and } H_+ \cap H_- = \{0\}. \quad (2.1)$$

If we have a unitary \mathcal{A} -module map J satisfying $J^2 = 1$, there is an induced eigenspace decomposition $H_{\mathcal{A}} = H_+ \oplus H_-$, for which $H_{\pm} \cong H_{\mathcal{A}}$. This leads to the (restricted) Banach algebra $A = \mathcal{L}_J(H_{\mathcal{A}})$ as described in [20] (generalizing that of $\mathcal{A} = \mathbb{C}$ in [33]). Specifically, we have

$$A = \mathcal{L}_J(H_{\mathcal{A}}) := \{\mathbb{T} \in \mathcal{L}_{\mathcal{A}}(H_{\mathcal{A}}) : \mathbb{T} \text{ is adjointable and } [J, \mathbb{T}] \in \mathcal{L}_2(H_{\mathcal{A}})\}, \quad (2.2)$$

(by definition $\mathcal{L}_2(H_{\mathcal{A}})$ denotes the Hilbert-Schmidt operators) where for $\mathbb{T} \in A$, we assign the norm (cf. [33, §6.2]):

$$\|\mathbb{T}\|_J = \|\mathbb{T}\| + \|[J, \mathbb{T}]\|_2. \quad (2.3)$$

The algebra A can thus be seen as a Banach $*$ -algebra with isometric involution (when $\mathcal{A} \cong \mathbb{C}$ we simply write $\mathcal{L}_J(H)$). This is our ‘restricted’ algebra which we will use henceforth. Together with the topology induced by $\|\cdot\|_J$, the group of units $G(A)$ is a complex Banach Lie group for which we have the unitary Lie subgroup $U(A) \subset G(A)$. Further, $\text{Gr}(p, A)$ as a Banach manifold, can also be equipped with a Hermitian structure (see §A.2 and references therein).

We denote by \bar{A} , the C^* -algebra norm closure of A in the C^* -algebra $\mathcal{L}(H_{\mathcal{A}})$. By standard principles, A is thus a Banach $*$ -algebra with isometric involution dense in the C^* -algebra \bar{A} . In particular, as A is a Banach algebra, it is closed under the holomorphic functional calculus, and as it is a $*$ -subalgebra of the C^* -algebra \bar{A} , it is therefore a (unital) *pre- C^* -algebra* in the sense of [9, 11].

Remark 2.2. For the most part of this work we shall be taking the Hilbert space to be $H = L^2(S^1, \mathbb{C})$ without loss of generality.

2.2 The case where \mathcal{A} is commutative

We shall henceforth assume that \mathcal{A} is a commutative separable C^* -algebra. The Gelfand transform implies there exists a compact metric space Y such that $Y = \text{Spec}(\mathcal{A})$ and $\mathcal{A} \cong C(Y)$. Setting $B = \mathcal{L}_J(H)$, we can now express the Banach $*$ -algebra A in the form

$$A \cong \{\text{continuous functions } Y \longrightarrow B\} = C(Y, B), \quad (2.4)$$

for which the $\|\cdot\|_2$ -trace in the norm of A is regarded as continuous as a function on Y . Note the Banach algebra $B = \mathcal{L}_J(H)$ corresponds to taking $\mathcal{A} = \mathbb{C}$, and with respect to the polarization $H = H_+ \oplus H_-$, we recover the usual restricted Grassmannians, denoted $\text{Gr}(H_+, H)$ of [33, 38].

3 The Burchnell-Chaundy C*-algebra

Appendices §A.3 and §A.4 reviews the Burchnell-Chaundy algebra and related geometry. We henceforth use the notation introduced there, referencing to it as we go along.

Motivated by Theorem A.1 we obtain a natural C*-algebra associated to the ring \mathbb{A} (§A.3). Firstly, in view of Theorem A.1, let

$$i_K : \mathbb{A} \longrightarrow A, \quad (3.1)$$

be the inclusion map induced via conjugation by the integral operator K (A.6) in Appendix (A.16). We also recall the C*-algebra \bar{A} of §2.1.

Definition 3.1. The *Burchnell-Chaundy C* algebra*, denoted $\underline{\mathbb{A}}$, is the C*-subalgebra of \bar{A} generated by $i_K(\mathbb{A})$.

Thus $\underline{\mathbb{A}}$ is a separable C*-subalgebra of \bar{A} (this uses the fact that \bar{A} is separable). We also endow $\underline{\mathbb{A}}$ with an identity as induced from that of \bar{A} . Further, let

$$\underline{i}_K : \mathbb{A} \longrightarrow \underline{\mathbb{A}}, \quad (3.2)$$

be the induced map of (3.1) into the C*-algebra $\underline{\mathbb{A}}$ where both maps in (3.1), (3.2) factor through the tensor product $\mathbb{A} \otimes \mathcal{A}$.

Remark 3.1. Note that the relevant algebras considered above, are already inside the C*-algebra $\mathcal{L}\mathcal{A}(H\mathcal{A})$. In view of Remark A.1, applying a completely positive state to pass to $H(\mathcal{A})$ serves to make $\mathcal{L}\mathcal{A}(H\mathcal{A})$ a C*-subalgebra of $\mathcal{L}(H(\mathcal{A}))$. Thus we may, if we wish, identify $\underline{\mathbb{A}}$ as a particular C*-subalgebra of $\mathcal{L}(H(\mathcal{A}))$.

In the following we will also consider a commutative Banach subalgebra $\underline{\mathbb{B}}$ of $\underline{\mathbb{A}}$ as generated by elements of $i_K(\mathbb{A})$ that closes the image of i_K in the norm topology. As maximal ideals are proper closed ideals, the image of i_K and $\underline{\mathbb{B}}$ have the same spectrum in the weak *-topology (recall that as the hull kernel topology is completely regular and Hausdorff, it follows that the weak *-topology and hull kernel topology are the same [35]). We shall use this fact in establishing Theorem 3.1 below.

Suppose we have some finite number s of commuting operators $L_i \in \mathbb{A}$, for $1 \leq i \leq s$. Recalling (3.2) and setting $T_i = \underline{i}_K(L_i) \in \underline{\mathbb{B}}$, leads to

$$0 = \underline{i}_K([L_i, L_j]_{\mathbb{A}}) = [\underline{i}_K(L_i), \underline{i}_K(L_j)]_{\underline{\mathbb{A}}} = [T_i, T_j]_{\underline{\mathbb{A}}}, \quad (3.3)$$

and therefore to a commuting s -tuple $\mathcal{T}(s) \equiv (T_1, \dots, T_s) \in \underline{\mathbb{B}}^s$.

We next specify the connection between the joint spectrum of the Burchnell-Chaundy ring \mathbb{A} and the joint spectrum $\sigma(\mathcal{T}(s), \underline{\mathbb{B}})$ of a generating commuting s -tuple $\mathcal{T}(s)$ of operators in the Banach subalgebra $\underline{\mathbb{B}}$.

Theorem 3.1. *For any s -tuple $\mathcal{T}(s) \in \underline{\mathbb{B}}^s$ of commuting operators that generate $\underline{\mathbb{B}}$, the map of spectra*

$$\sigma(\mathcal{T}(s), \underline{\mathbb{B}}) \longrightarrow (X' = \text{Spec}(\mathbb{A})) \times Y, \quad (3.4)$$

is a homeomorphism.

Proof. Initially, it is enough to deal with the case $Y = \{\text{pt}\}$. Recall that each $T_i = \underline{i}_K(L_i) \in \underline{\mathbb{B}}$, for $1 \leq i \leq s$. By definition of the spectrum, any point of the latter can be regarded as a non-trivial algebra homomorphism $\underline{\mathbb{B}} \xrightarrow{f} \mathbb{C}$, which by definition restricts to the ring of generators \mathbb{A} , that is, we have a restricted homomorphism $f|_{\mathbb{A}} : \mathbb{A} \xrightarrow{f} \mathbb{C}$. Note that if f_1, f_2 are two such homomorphisms, then if $f_1 = f_2$ on \mathbb{A} , we have $f_1 = f_2$ on $\underline{i}_K(\mathbb{A})$.

Now given a complex homomorphism on the image of \underline{i}_K , which is a normed continuous ring dense in $\underline{\mathbb{B}}$, then viewing this as a linear functional, it admits a closed kernel which splits the normed ring as a topological vector space, and consequently the closure of this kernel in $\underline{\mathbb{B}}$ will be a closed maximal ideal of $\underline{\mathbb{B}}$ which therefore defines a complex homomorphism extending that from the image of \underline{i}_K . We have then $f_1 = f_2$ on $\underline{\mathbb{B}}$. Thus the restriction map is a continuous bijection of the spectrum $\sigma(\mathcal{T}(s), \underline{\mathbb{B}})$ onto the spectrum of \underline{i}_K , and hence that of \mathbb{A} . As both spaces are compact Hausdorff spaces, this map is a homeomorphism. The more general conclusion follows on parametrizing by Y . \square

Remark 3.2. The s -tuple $\mathcal{T}(s) = (T_1, \dots, T_s)$ of commuting operators in $\underline{\mathbb{B}}$, being images $T_i = \underline{i}_K(L_i)$ of operators that commute with L of order n , provide a solution to the n -th generalized KdV-hierarchy, simply reproducing the construction in [38, Proposition 4.12, Corollary 5.18, and §6].

4 Extension of compact operators and K-homology

4.1 Ext and KK-classes

Let us now return to the (non-singular) algebraic curve $X = X' \cup \{x_\infty\}$, of genus g_X , associated to $\text{Spec}(\mathbb{A})$. Initially, we shall consider a fixed $y \in Y$ as in the case $\mathcal{A} \cong \mathbb{C}$, and in that case, replace $H_{\mathcal{A}}$ by $H = L^2(S^1, \mathbb{C})$. Application of the BDF extension theory [10] shows that an extension of the compact operators $\mathcal{K}(H)$ by $C(X)$ yields a unital *-monomorphism to the Calkin algebra $\mathcal{Q}(H)$:

$$\varrho : C(X) \longrightarrow \mathcal{Q}(H) := \mathcal{L}(H)/\mathcal{K}(H). \quad (4.1)$$

The group $\text{Ext}(X)$ of these extensions is the same as the degree-1 K-homology group $K^1(C(X))$ [9, 10]. More generally, $K^*(C(X))$ can be identified with Kasparov's KK-group $KK_*(C(X), \mathbb{C})$; we refer the reader to e.g. [9, 11, 27, 39] for details.

The relevance of this last observation is as follows. We recall that in §A.3 there are subspaces $W \in \text{Gr}(q, B)$ which are images of the Krichever correspondence (these are characterized by the size of the ring B_W in (A.10); see [38, Remark 6.3]). One datum of the Krichever correspondence was a holomorphic line bundle $\mathcal{L} \longrightarrow X$. Restricting to rank-1 projections we replace $\text{Gr}(q, B)$ by the (infinite dimensional) projective space $\mathbb{P}(H_{\mathcal{A}})$, which can be given a Hermitian structure together with its universal (Hermitian) line bundle $\mathcal{L}(p, A) \longrightarrow \mathbb{P}(H_{\mathcal{A}})$ (see Appendix §A.2). The projective space $\mathbb{P}(H_{\mathcal{A}})$ is a classifying space, so that given a suitable holomorphic map $f : X \longrightarrow \mathbb{P}(H_{\mathcal{A}})$, the holomorphic line bundle $\mathcal{L} \cong f^*\mathcal{L}(p, A)$ admits a Hermitian structure induced by this pullback.

On such a Hermitian bundle we endow a Hermitian connection ∇ (see e.g. [42, III Theorem 1.2]), so we now have a Hermitian holomorphic line bundle with connection $(\mathcal{L}, \nabla) \longrightarrow X$. In particular, the $(0, 1)$ -component ∇'' of ∇ , is taken to be the $\bar{\partial}$ -operator on sections (see e.g. [42, III Theorem 2.1]). Since X , regarded as a compact Riemann surface, admits the Kähler property, there is a Dirac operator $D = \sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ on sections (see e.g. [36, (2.20)]). Thus we are in

the setting of elliptic operators on sections of vector bundles over a compact manifold from which KK-classes can be constructed by standard procedures [9, 12, 39].

In §A.7, we discussed families of Y -parametrized line bundles. Now instead we fix the bundle and parametrize possible Hermitian connections ∇ on $\mathcal{L} \rightarrow X$, by Y . That is, we have a family of connections $\{\nabla_y\}_{y \in Y}$, and hence a Y -parametrized family of Dirac operators $\{D_y\}_{y \in Y}$. Thus by [12, 39] such data leads to constructing elements u of the the group $KK_*(C(X), C(Y)) = KK_*(C(X), \mathcal{A})$. We summarize matters as follows.

Proposition 4.1. *The data $\{(\mathcal{L}, \nabla_y)\}_{y \in Y}$ of a Hermitian holomorphic line bundle on X with Hermitian connections parametrized by Y , determines a class $u \in KK_*(C(X), C(Y))$ parametrized by $\mathbb{P}(H_{\mathcal{A}}) \times Y$.*

In particular, the degree-1 component $u^{[1]}$ provides an element of the group $\text{Ext}(C(X), \mathcal{A}) \cong KK_1(C(X), \mathcal{A})$, and in the special case $\mathcal{A} = \mathbb{C}$, an element of $\text{Ext}(X) = K^1(C(X))$.

Remark 4.1. The Krichever quintuple yields the data of Proposition 4.1 with Y reduced to one point. In particular, we can associate to any point of the Grassmannian in the image of the Krichever map and to a given Hermitian structure over the universal bundle a class u .

4.2 Extensions by the Burchnall-Chaundy C^* algebra

We recall the C^* -Burchnall-Chaundy algebra $\underline{\mathbb{A}}$ and consider a more general situation than §4.1 using the Hilbert module $H_{\mathcal{A}}$ and the ring \mathbb{A} . Concerning the spectrum $\text{Spec}(\mathcal{Q}(H))$ of the Calkin algebra, we refer to [15] for the study of spectra of (quotient) C^* -algebras.

Theorem 4.1.

- (1) *With respect to the subring inclusion (3.2) induced by the integral operator K (see (A.16)), there exists an extension of the compact operators $\mathcal{K}(H_{\mathcal{A}})$ by the Burchnall-Chaundy C^* -algebra $\underline{\mathbb{A}}$. Moreover, there also exists a well-defined map $\underline{q}_K : \mathbb{A} \rightarrow \mathcal{Q}(H_{\mathcal{A}})$.*
- (2) *Recalling $X' = \text{Spec}(\mathbb{A})$, there exists a well-defined map of spectra*

$$\text{Spec}(\mathcal{Q}(H_{\mathcal{A}})) \cong \text{Spec}(\mathcal{Q}(H)) \times Y \rightarrow X' \times Y. \quad (4.2)$$

Proof. Firstly, in relationship to $H_{\mathcal{A}}$, the Calkin C^* -algebra $\mathcal{Q}(H_{\mathcal{A}})$ can be expressed as follows:

$$\begin{aligned} \mathcal{Q}(H_{\mathcal{A}}) &= \mathcal{L}(H_{\mathcal{A}})/\mathcal{K}(H_{\mathcal{A}}) \\ &\cong \mathcal{L}(H \otimes \mathcal{A})/\mathcal{K}(H \otimes \mathcal{A}) \\ &\cong (\mathcal{L}(H) \otimes \mathcal{A})/(\mathcal{K}(H) \otimes \mathcal{A}) \\ &\cong (\mathcal{L}(H)/\mathcal{K}(H)) \otimes \mathcal{A} \\ &= \mathcal{Q}(H) \otimes \mathcal{A}. \end{aligned} \quad (4.3)$$

Again, an application of the BDF extension theory [10] shows that an extension of the compact operators $\mathcal{K}(H_{\mathcal{A}})$ by $\underline{\mathbb{A}}$, yields a unital $*$ -monomorphism

$$\underline{q} : \underline{\mathbb{A}} \rightarrow \mathcal{Q}(H_{\mathcal{A}}). \quad (4.4)$$

We also recall that a Fredholm module $(H_{\mathcal{A}}, S)$ over $\underline{\mathbb{A}}$ (where S an essentially unitary operator) determines an element in K-homology $K^*(\underline{\mathbb{A}})$ [9, 10]. Combining the morphism $\underline{\varrho}$ in (4.4) with the inclusion (3.2) induced by the integral operator K in §A.6, we produce a well defined map from the ring \mathbb{A} to $\mathcal{Q}(H) \otimes \mathcal{A}$ viewed as the composition

$$\underline{\varrho}_K : \mathbb{A} \longrightarrow \mathbb{A} \otimes \mathcal{A} \xrightarrow{i_K} \underline{\mathbb{A}} \xrightarrow{\underline{\varrho}} \mathcal{Q}(H_{\mathcal{A}}). \quad (4.5)$$

Furthermore, using (4.3), (4.5) and the contravariance of the ‘Spec’ functor, we have a well-defined map

$$\begin{aligned} \varrho_K^* : \text{Spec}(\mathcal{Q}(H_{\mathcal{A}})) &\cong \text{Spec}(\mathcal{Q}(H) \otimes \mathcal{A}) \\ &\cong \text{Spec}(\mathcal{Q}(H)) \times Y \longrightarrow (X' = \text{Spec}(\mathbb{A})) \times Y. \end{aligned} \quad (4.6)$$

□

4.3 The C*-algebra of the Jacobian and extensions

We return now to the Jacobian torus $J(X)$ of X and recall that there exists a holomorphic embedding (the Abel map) $\mu : X \longrightarrow J(X)$ (see e.g.[24]). The following theorem establishes a relationship between the respective commutative C*-algebras of continuous functions $C(J(X))$ and $C(X)$ and the dual K_* -functor:

Theorem 4.2. *There exists a short exact sequence of C*-algebras*

$$0 \rightarrow \mathfrak{J} \xrightarrow{i} C(J(X)) \xrightarrow{p} C(X) \rightarrow 0, \quad (4.7)$$

where \mathfrak{J} is a two-sided ideal in $C(J(X))$, i is injective and p is surjective. Furthermore, (4.7) induces a periodic sequence of K_* -groups:

$$\begin{array}{ccccc} K_0(\mathfrak{J}) & \xrightarrow{i_*} & K_0(C(J(X))) & \xrightarrow{p_*} & K_0(C(X)) \\ \uparrow & & & & \downarrow \\ K_1(C(X)) & \xleftarrow{p_*} & K_1(C(J(X))) & \xleftarrow{i_*} & K_1(\mathfrak{J}) \end{array} \quad (4.8)$$

Proof. Firstly, from the embedding $\mu : X \longrightarrow J(X)$ we identify X as a closed subset of $J(X)$ and by standard principles, the induced map on continuous functions $p : C(J(X)) \longrightarrow C(X)$ is surjective. Alternatively, we note that the transcendence degrees of the rings of meromorphic functions $\text{Mero}(J(X))$ and $\text{Mero}(X)$ are $g_X \geq 1$ and 1, respectively. Thus with respect to a two-sided ideal \mathfrak{J}_0 in $\text{Mero}(J(X))$ we have a short exact sequence:

$$0 \rightarrow \mathfrak{J}_0 \xrightarrow{i} \text{Mero}(J(X)) \xrightarrow{p} \text{Mero}(X) \rightarrow 0, \quad (4.9)$$

where in each case the elements of the ring can be approximated by Laurent polynomials extendable to the continuous functions. Hence (4.7) follows. The last part follows from the periodicity of the K_* -functor [11, 23]. □

The periodicity sequence in (4.8) corresponds to the analogous sequence in the K-theory of spaces:

$$\begin{array}{ccccc}
K^0(J(X)/\text{Im } \mu) & \longrightarrow & K^0(J(X)) & \longrightarrow & K^0(X) \\
\uparrow & & & & \downarrow \\
K^1(X) & \longleftarrow & K^1(J(X)) & \longleftarrow & K^1(J(X)/\text{Im } \mu)
\end{array} \tag{4.10}$$

Now we return to K-homology and in particular, in degree 1 the following establishes a relationship between the ‘Ext’ classes of X and $J(X)$ on introducing the homology Chern character [5, 11]. Note that this is the case in which the BDF theory guarantees that elements of ‘Ext’ correspond to *-monomorphisms of the C*-algebra to the Calkin algebra (see (4.1), (4.4) and (4.5)). Further, on recalling the algebra B_W in (A.10) with respect to $W \in \text{Gr}(p, A)$, the set of monomorphisms of the Burchall-Chaundy algebra into the algebras $B_W \subset \mathcal{L}(H_{\mathcal{A}})$ is indeed the Jacobian $J(X)$, as proved by Burchall and Chaundy and formalized in [28, 38]. It is from this point of view that $J(X)$ can be mapped to elements of ‘Ext’. We make matters more precise by the following:

Theorem 4.3. *There exists a well-defined map $\Phi : J(X) \rightarrow \text{Ext}(\underline{A})$ which assigns to each morphism $\varphi_W : \mathbb{A} \rightarrow B_W$, an extension of the compact operators $\mathcal{K}(H_{\mathcal{A}})$ by \underline{A} , with respect to subspaces $W \in \text{Gr}(p, A)$, and the subring inclusion (3.2) induced via the integral operator K .*

Proof. In order to construct Φ we first assign to a point of $J(X)$, a morphism $\varphi_W : \mathbb{A} \rightarrow B_W$. Using the compact extensions map $\underline{\varrho}_K$ in (4.5) and the fact that $B_W \subset \mathcal{L}(H_{\mathcal{A}})$, the desired map is that for which the diagram

$$\begin{array}{ccc}
\mathbb{A} & \xrightarrow{\varphi_W} & B_W \subset \mathcal{L}(H_{\mathcal{A}}) \\
& \searrow \underline{\varrho}_K & \downarrow \Pi \\
& & \mathcal{L}(H_{\mathcal{A}})/\mathcal{K}(H_{\mathcal{A}}) = \mathcal{Q}(H_{\mathcal{A}})
\end{array} \tag{4.11}$$

commutes on restricting the projection Π to B_W . Note that the subring inclusion (3.2) is implicit in the definition of $\underline{\varrho}_K$ in (4.5). More specifically, $\Phi : \mathbb{A} \rightarrow \text{Ext}(\underline{A})$ is determined via the assignment $\varphi_W \mapsto \underline{\varrho}_K$ for which $\underline{\varrho}_K = (\Pi|_{B_W}) \circ \varphi_W$ defines an extension of $\mathcal{K}(H_{\mathcal{A}})$ by \underline{A} . \square

In diagram (4.12) below, the vertical maps ‘ ch_1 ’ denote the (degree 1) homology Chern character morphisms (see e.g. [5, 11]).

Theorem 4.4. *The following diagram is commutative*

$$\begin{array}{ccc}
K^1(C(X)) & \xrightarrow{\hat{\mu}_*} & K^1(C(J(X))) \\
\text{ch}_1 \downarrow & & \downarrow \text{ch}_1 \\
H_1(X, \mathbb{Q}) & \xrightarrow{\mu_* \cong} & H_1(J(X), \mathbb{Q})
\end{array} \tag{4.12}$$

Here the map μ_* is an isomorphism and the map $\hat{\mu}_*$ is injective. Equivalently, the map $\hat{\mu}_* : \text{Ext}(X) \rightarrow \text{Ext}(J(X))$, is injective. In particular, when the genus $g_X = 1$, the map $\hat{\mu}_*$ is an isomorphism.

Proof. Applying the functor K^* to (4.7), the resulting long exact sequence yields an injective map $\hat{\mu}_* : K^1(C(X)) \rightarrow K^1(C(J(X)))$. The commutativity of the diagram arises from applying the homology Chern character homomorphism to each side. Regarding the lower horizontal arrow μ_* , we recall some elementary facts concerning X and its Jacobian $J(X)$ (see e.g. [24]). Setting the genus $g_X = g$, if $\delta_1, \dots, \delta_{2g}$ are 1-cycles in X forming a (canonical) basis for $H_1(X, \mathbb{Z})$, then $H_1(X, \mathbb{Z}) \cong \mathbb{Z}\{\delta_1, \dots, \delta_{2g}\}$. Next, we identify $J(X) = \mathbb{C}^g/\Lambda$ where $\Lambda \subset \mathbb{C}^g$ is a discrete lattice of maximal rank $2g$. Accordingly, $H_1(J(X), \mathbb{Z}) \cong \Lambda \cong \mathbb{Z}\{\lambda_1, \dots, \lambda_{2g}\}$, for a basis $\lambda_1, \dots, \lambda_{2g}$ for Λ . Thus, from the (one-to-one) assignment $\delta_i \mapsto \lambda_i$, $1 \leq i \leq 2g$, it follows that we have an isomorphism $H_1(X, \mathbb{Z}) \cong H_1(J(X), \mathbb{Z})$. Since both X and $J(X)$ are quotients of their respective covering spaces by torsion-free discrete groups, then on tensoring the respective integral homology groups by \mathbb{Q} , it also follows that $H_1(X, \mathbb{Q}) \cong H_1(J(X), \mathbb{Q})$. In the case of genus $g_X = 1$, X is, up to the choice of a base point, an elliptic curve (a complex 1-dimensional torus), the map μ is an isomorphism and consequently induces an isomorphism $\hat{\mu}_* : K^1(C(X)) \rightarrow K^1(C(J(X)))$, in other words, $\text{Ext}(X) \cong \text{Ext}(J(X))$. \square

4.4 The flow of multiplication operators and the τ -function

The action on $\text{Gr}(p, A)$ of the group of multiplication operators $\Gamma_+(\mathcal{A})$ (see Appendix A.3) is induced via its restriction to the subspace H_+ in any polarization. Here we start by considering certain elements of $\Gamma_+(\mathcal{A})$ on which an operator-valued τ -function can be defined.

Definition 4.1. We say that $W \in \text{Gr}(p, A)$ is *transverse* if it is the graph of a linear operator $C : \mathsf{H}_+ \rightarrow \mathsf{H}_-$.

Consider the element $q_\zeta \in \Gamma_+(\mathcal{A})$ given by a map $q_\zeta(z) = (1 - z\zeta^{-1})$, for $|\zeta| > 1$, whose inverse is given by

$$q_\zeta^{-1} = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a \in \text{Fred}(\mathsf{H}_+), d \in \text{Fred}(\mathsf{H}_-), b \in \mathcal{K}(H_{\mathcal{A}}) \text{ (in fact, } b \in \mathcal{L}_2(\mathsf{H}_-, \mathsf{H}_+)) \right\}. \quad (4.13)$$

We define the *operator-valued τ -function*

$$\tau_W : \Gamma_+(\mathcal{A}) \rightarrow \mathbb{C} \otimes 1_{\mathcal{A}}, \quad (4.14)$$

relative to a transverse subspace $W \in \text{Gr}(p, A)$ by

$$\tau_W(q_\zeta) = \det(1 + a^{-1}bC) \otimes 1_{\mathcal{A}}, \quad (4.15)$$

where $C : \mathsf{H}_+ \rightarrow \mathsf{H}_-$ is the map whose graph is W . The following lemma (and its proof) is essentially that of [38, Lemma 5.15]:

Lemma 4.1. *Let $W \in \text{Gr}(p, A)$ be transverse and let f_0 be the unique element of H_- such that $1 + f_0 \in W$. Then for $|\zeta| > 1$, we have $\tau_W(q_\zeta) = 1 + f_0(\zeta)$.*

Proof. We start with the expression of (4.13) in which $b : \mathsf{H}_- \rightarrow \mathsf{H}_+$ takes z^{-k} to $\zeta^{-k}q_\zeta^{-1}$. Thus $a^{-1}b$ is a rank-1 map that takes $f \in \mathsf{H}_-$ to the constant function $f(\zeta)$. The map $a^{-1}bC$ is also of rank-1 and the infinite determinant

$$\tau_W(q_\zeta) = \det(1 + a^{-1}bC) \otimes 1_{\mathcal{A}} = (1 + \text{Tr}(a^{-1}bC)) \otimes 1_{\mathcal{A}}. \quad (4.16)$$

Since C maps 1 to $f_0(z)$, the lemma follows. \square

Now for $g \in \Gamma_+(\mathcal{A})$, such that $g^{-1}W$ is transverse to H_- , the *operator-valued Baker function* $\psi_W(g, \zeta)$ (see Appendix A.5) is characterized as the unique function of the form $1 + \sum_{i=1} a_i \zeta^{-i}$, whose boundary value $|\zeta| \rightarrow 1$, lies in the transverse space $g^{-1}W$. On the other hand, we have as in [38, §3] the relationship

$$\tau_W(g) = \sigma(g^{-1}W)(g^{-1}\sigma(W))^{-1}, \quad (4.17)$$

where σ is a global section of the determinant line bundle $\text{Det}_{\text{Gr}} \rightarrow \text{Gr}(p, A)$. But then it is straightforward to see from (4.17) that

$$\tau_W(g \cdot q_\zeta)(\tau_W(g))^{-1} = \tau_{g^{-1}W}(q_\zeta) = \psi_W(g, \zeta), \quad (4.18)$$

by the definition of ψ_W and Lemma 4.1. Thus (4.18) relates the τ and Baker functions in this operator-valued setting.

Remark 4.2. For $t = (t_1, \dots, t_n)$, (4.18) can be expanded to give (cf. [1, 37, 38]):

$$\begin{aligned} \psi_W(\zeta, t) &= \tau(t - [\zeta^{-1}])(\tau(t))^{-1} \\ &= \tau(t_1 - 1/\zeta, t_2 - 1/2\zeta^2, t_3 - 1/3\zeta^3, \dots)(\tau(t_1, t_2, t_3, \dots))^{-1}. \end{aligned} \quad (4.19)$$

Since applying the BDF theorem provides for each element of the Burchnell-Chaundy C^* -algebra $\underline{\mathbb{A}}$ an extension of the compact operators $\mathcal{K}(H_{\mathcal{A}})$, we summarize matters by the following:

Theorem 4.5. *Each element $e \in \text{Ext}(\underline{\mathbb{A}})$ determines a corresponding τ -function $(\tau_W)_e$ relative to a transverse subspace $W \in \text{Gr}(p, A)$. In other words, $\text{Ext}(\underline{\mathbb{A}})$ parametrizes a family of τ -functions $\{(\tau_W)_e\}_{e \in \text{Ext}(\underline{\mathbb{A}})}$ for such subspaces.*

Proof. Observe that by considering the space $J_{\mathcal{A}}(X)$ of monomorphisms $\mathbb{A} \otimes \mathcal{A} \rightarrow B_W$ (see Appendix §A.4), we obtain from the same principles of Theorem 4.3, a map $\Phi_{\mathcal{A}} : J_{\mathcal{A}}(X) \rightarrow \text{Ext}(\underline{\mathbb{A}})$. Then on recalling the map $\Xi : \Gamma_+(\mathcal{A}) \rightarrow J_{\mathcal{A}}(X)$ in (A.12), we have a well-defined map

$$\Upsilon_{\mathcal{A}} : \Gamma_+(\mathcal{A}) \rightarrow \text{Ext}(\underline{\mathbb{A}}), \quad (4.20)$$

given via composition $\Upsilon_{\mathcal{A}} = \Phi_{\mathcal{A}} \circ \Xi$. The result follows by essentially finding an inverse for this map. Effectively, for each $e \in \text{Ext}(\underline{\mathbb{A}})$ providing an extension of $\mathcal{K}(H_{\mathcal{A}})$, we have from (4.13) a corresponding element of $\Gamma_+(\mathcal{A})$ given by

$$(q_\zeta^{-1})_e = \begin{bmatrix} a & b_e \\ 0 & d \end{bmatrix}. \quad (4.21)$$

Hence, from (4.15), we obtain an element $(q_\zeta)_e \in \Gamma_+(\mathcal{A})$ to which there corresponds a τ -function

$$\tau_W((q_\zeta)_e) = \det(1 + a^{-1}b_e C) \otimes 1_{\mathcal{A}} = (1 + \text{Tr}(a^{-1}b_e C)) \otimes 1_{\mathcal{A}}, \quad (4.22)$$

associated to a transverse subspace $W \in \text{Gr}(p, A)$. \square

A APPENDIX

A.1 Hilbert modules

Take H to be a separable Hilbert space. Given a unital separable C^* -algebra \mathcal{A} one may consider the standard (free countable dimensional) Hilbert module $H_{\mathcal{A}}$ over \mathcal{A} ,

$$H_{\mathcal{A}} = \{ \{ \zeta_i \}, \zeta_i \in \mathcal{A}, i \geq 1 : \sum_{i=1}^{\infty} \zeta_i \zeta_i^* \in \mathcal{A} \} \cong \oplus \mathcal{A}_i, \quad (\text{A.1})$$

where each \mathcal{A}_i represents a copy of \mathcal{A} . We can form the algebraic tensor product $H \otimes_{\text{alg}} \mathcal{A}$ on which there is an \mathcal{A} -valued inner product

$$\langle x \otimes \zeta, y \otimes \eta \rangle = \langle x, y \rangle \zeta^* \eta, \quad x, y \in H, \zeta, \eta \in \mathcal{A}. \quad (\text{A.2})$$

Thus $H \otimes_{\text{alg}} \mathcal{A}$ becomes an inner-product \mathcal{A} -module whose completion is denoted by $H \otimes \mathcal{A}$. Given an orthonormal basis for H , we have the following identification (a unitary equivalence) given by $H \otimes \mathcal{A} \approx H_{\mathcal{A}}$ (see e.g. [29, 17]). For properties of compact and Fredholm operators over Hilbert modules, see e.g. [9]. We take $\mathcal{L}(H_{\mathcal{A}})$ to denote the C^* -algebra of adjointable linear operators on $H_{\mathcal{A}}$ and $\text{Fred}(H_{\mathcal{A}})$ to denote the space of Fredholm operators (see e.g. [9, 27]). We also make use of the Schatten ideals of operators: the Banach spaces $\mathcal{L}_{\ell}(H_{\mathcal{A}})$ are defined as the subspaces of $\mathcal{L}(H_{\mathcal{A}})$ consisting of operators S with norm satisfying $\|S\|_{\ell}^{\ell} = \text{Tr}(S^*S)^{\ell/2} < \infty$, (for $0 \leq \ell \leq \infty$, where $\mathcal{L}_{\infty}(H_{\mathcal{A}}) = \mathcal{K}(H_{\mathcal{A}})$ is the compact operators) [40]. Since this will turn out to be an essential ingredient (for example, in the Gelfand transform §2.2) it is assumed that $H_{\mathcal{A}}$ is a separable Hilbert $*$ -module.

Remark A.1. If ϕ denotes a state of \mathcal{A} , then we can produce a positive semi-definite pre-Hilbert space structure on $H_{\mathcal{A}}$ via an induced inner product $\langle v|w \rangle_{\phi} = \phi(\langle v|w \rangle_{\mathcal{A}})$. From the Gelfand-Neumark-Segal (GNS) Theorem there is an associated Hilbert space H_{ϕ} for which $\ell^2 \otimes H_{\phi}$ is the completion of $H_{\mathcal{A}}$ under this induced inner product (see e.g. [9, 23]). When ϕ is understood we shall simply denote the latter by $H(\mathcal{A})$. Observe that $H(\mathcal{A})$ contains $H_{\mathcal{A}}$ as a dense vector subspace which is a right \mathcal{A} -module, if ϕ is completely positive. The assignment of Hilbert spaces $H \mapsto H(\mathcal{A})$ and use of the \mathcal{A} -valued inner product in (A.2), thus allows us to modify various results that were originally established for the various objects $(H, \langle \cdot, \cdot \rangle)$ (such as the bounded linear operators, the compact operators, etc. in the case $\mathcal{A} = \mathbb{C}$).

A.2 On the geometry of the Grassmannian $\text{Gr}(p, A)$ and related properties

Let A be a unital complex Banach(able) algebra with group of units $G(A)$ and space of idempotents $P(A)$; for basic facts about Banach algebras we refer to [16, 35]. For $p \in P(A)$, we say that the orbit of p under the inner automorphic action of $G(A)$ is *the similarity class of p* and denote the latter by $\Lambda = \text{Sim}(p, A) = G(A) * p$. Following [18] there exists a space $V(p, A)$ modeled on the space of proper partial isomorphisms of A which is the total space of a locally trivial principal $G(pAp)$ -bundle

$$G(pAp) \hookrightarrow V(p, A) \longrightarrow \text{Gr}(p, A), \quad (\text{A.3})$$

where the base is the Grassmannian $\text{Gr}(p, A)$ viewed as the image of $\Lambda = \text{Sim}(p, A)$ with respect to the projection map. The space $\text{Gr}(p, A)$ is a (complex) Banach manifold and the construction

of the bundle in (A.3) generalizes the usual Stiefel bundle construction in finite dimensions. If $A[p]$ denotes the commutant of $p \in A$, we set $G[p] = G(A[p])$. If we further assume that A has the unital associative $*$ -algebra property, there is the corresponding unitary group $U(A) \subset G(A)$. Thus on setting $U[p] = U(A) \cap G[p]$, we can specialize (A.3) as an analytic, principal $U[p]$ -bundle $U[p] \hookrightarrow V(p, A) \longrightarrow \text{Gr}(p, A) \cong U(A)/U[p]$.

Remark A.2. The references [18, 21] describe the possible analytic structures of the spaces $P(A)$, $\text{Gr}(p, A)$, $V(p, A)$ and Λ , and related objects in some depth. For such spaces we wish to point out that in the Banach manifold and C^* -algebra setting there is available related work by other authors, using different techniques, that is applicable to a broad scope of independent problems to those considered here; see for instance [2, 6, 7, 8, 13, 25, 30, 31, 32, 34, 41, 43] (and the references to related work cited therein). The present authors thus acknowledge an inevitable overlap with certain technical details in this regard.

We proceed with A as defined in (2.2) and take E to be its underlying complex Banach space that admits a splitting subspace decomposition compatible with the polarization (2.1) (for the theory of compact and Fredholm operators extended to Banach spaces we refer to [44]). In the following we restrict to the $\ell = 2$ (Hilbert-Schmidt) case and consider as in [20, 33, 38] the explicit form of $G(A)$ as the group of automorphisms of $\text{Gr}(p, A)$ as given by

$$G(A) = \left\{ \begin{bmatrix} T_1 & S_1 \\ S_2 & T_2 \end{bmatrix} : T_1 \in \text{Fred}(\mathbf{H}_+), T_2 \in \text{Fred}(\mathbf{H}_-), S_1 \in \mathcal{L}_2(\mathbf{H}_-, \mathbf{H}_+), \right. \\ \left. S_2 \in \mathcal{L}_2(\mathbf{H}_+, \mathbf{H}_-) \right\}, \quad (\text{A.4})$$

whereby $\text{Gr}(p, A) = G(A)/G[p]$ (see [18, 20]). Note that for $p \in P^{\text{sa}}$, we have $U[p] \cong U(A) \cap G[p] \cong U(A) \cap (U(\mathbf{H}_+) \times U(\mathbf{H}_-))$. If we take ρ to denote the left action of $U[p]$ on $H(p)$ where the latter denotes the underlying Hilbert space of $pA \cong \mathbf{H}_+$, then following [19, §5] we have the associated universal vector bundle to (A.3)

$$\gamma(p, A) = U(A) \times_{U[p]} H(p) \longrightarrow U(A)/U[p] = \text{Gr}(p, A), \quad (\text{A.5})$$

as was established in [19, §5]. We remark that homogeneous vector bundles (such as (A.5)) arising from certain representations have been more recently studied in [8]. From works such as [3, 7, 8, 25, 33] it is straightforward to deduce that $\text{Gr}(p, A)$ admits the structure of a Hermitian manifold and that the universal vector bundle $\mathcal{E}(p, A) \longrightarrow \text{Gr}(p, A)$ admits the structure of a Hermitian vector bundle.

The space $\text{Gr}(p, A)$ may be realized more specifically in the following way. Suppose that a fixed $p \in P(A)$ acts as the projection of $H_{\mathcal{A}}$ on \mathbf{H}_+ along \mathbf{H}_- . Then in a similar way to [33, 38], the space $\text{Gr}(p, A)$ is the Grassmannian consisting of subspaces $W = r(H_{\mathcal{A}})$, for $r \in P(A)$, such that:

- (1) the projection $p_+ = pr : W \longrightarrow \mathbf{H}_+$ is in $\text{Fred}(H_{\mathcal{A}})$, and
- (2) the projection $p_- = (1 - p)r : W \longrightarrow \mathbf{H}_-$ is in $\mathcal{L}_2(\mathbf{H}_+, \mathbf{H}_-)$.

Alternatively, for (2) we may take projections $q \in P(A)$ such that for the fixed $p \in P(A)$, the difference $q - p \in \mathcal{L}_2(\mathbf{H}_+, \mathbf{H}_-)$. Further, there is the *big cell* $C_b = C_b(p, A) \subset \text{Gr}(p, A)$ defined as the collection of all subspaces $W \in \text{Gr}(p, A)$ such that the projection $p_+ \in \text{Fred}(H_{\mathcal{A}})$ is an isomorphism.

In regard of $*$ -subalgebras of A , the following lemma is easily deduced from [20, §2.4]

Lemma A.1. *Let $B \subset A$ be a Banach $*$ -subalgebra of A , with inclusion $h : B \rightarrow A$. Then there is an induced inclusion of Grassmannians $\text{Gr}(q, B) \subset \text{Gr}(p, A)$ where for $q \in P(B)$ we have set $p = h(q) \in P(A)$.*

Example A.1. Let $\mathcal{B} \subset \mathcal{A}$ be a C^* -subalgebra. Then it is straightforward to see that we have an inclusion $\mathcal{L}_J(H_{\mathcal{B}}) \rightarrow A = \mathcal{L}_J(H_{\mathcal{A}})$. In particular, when $\mathcal{B} \cong \mathbb{C}$ and $H = L^2(S^1, \mathbb{C})$ for which there is a polarization $H = H_+ \oplus H_-$ ($H_+ \cap H_- = \{0\}$), the inclusion $\mathcal{L}_J(H) \rightarrow \mathcal{L}_J(H) \otimes \mathcal{A}$ induces an inclusion $\text{Gr}(H_+, H) \subset \text{Gr}(p, A)$ where $\text{Gr}(H_+, H)$ is the ‘restricted’ Grassmannian as used in [33, 38].

A.3 The Burchall-Chaundy ring and holomorphic geometry

Let us briefly recall how for $\mathcal{A} = \mathbb{C}$ the relevant algebra of differential operators of the KP flows, is related to the Grassmannian $\text{Gr}(q, B)$ [38]. Let \mathbb{B} denote the algebra of analytic functions $U \rightarrow \mathbb{C}$ where U is a connected open neighborhood of the origin in \mathbb{C} . The (noncommutative) algebra $\mathbb{B}[\partial]$ of linear differential operators with coefficients in \mathbb{B} , consists of expressions

$$\sum_{i=0}^N a_i \partial^i, \quad (a_i \in \mathbb{B}, \quad \text{for some } N \in \mathbb{Z}). \quad (\text{A.6})$$

Here $\partial := \partial/\partial x$ and the a_i can be regarded as operators on functions, with multiplication

$$[\partial, a] = \partial a - a\partial = \partial a/\partial x. \quad (\text{A.7})$$

More generally, we pass to the algebra $\mathbb{B}[[\partial^{-1}]]$ of formal pseudodifferential operators with coefficients in \mathbb{B} . This algebra is obtained by formally inverting the operator ∂ (see e.g. [37]) and taking Laurent series as in (A.6), with $-\infty < i \leq N$.

The ring \mathbb{A} is assumed to be a commutative subring of $\mathbb{B}[\partial]$, whose joint spectrum is a complex irreducible curve $X' = \text{Spec}(\mathbb{A})$ with completion a non-singular algebraic curve $X = X' \cup \{x_\infty\}$ of genus $g_X \geq 1$. We recall from [38] the following associated quintuple of data $(X, x_\infty, z, \mathcal{L}, \varphi)$: $\mathcal{L} \rightarrow X$ is a holomorphic line bundle, x_∞ is a smooth point of X , z the inverse of a local parameter on X at x_∞ , where z is used to identify a closed neighborhood X_∞ of x_∞ in X with a neighborhood of the disk $D_\infty = \{z : |z| \geq 1\}$ in the Riemann sphere, and φ is a holomorphic trivialization of \mathcal{L} over $X_\infty \subset X$. Subsequently, we consider L^2 -boundary values of \mathcal{L} over $X \setminus D_\infty$ and φ identifies sections of \mathcal{L} over S^1 with \mathbb{C} -valued functions. Thus we arrive at the (separable) Hilbert space $H = L^2(S^1, \mathbb{C})$ together with a polarization $H = H_+ \oplus H_-$ (with $H_+ \cap H_- = \{0\}$) in the case $B = \mathcal{L}_J(H)$. Further, the quintuple $(X, x_\infty, z, \mathcal{L}, \varphi)$ can be mapped, by the *Krichever correspondence*, to a point $W \in \text{Gr}(q, B)$ (cf [28, 38]). Following [38], the properties of the projections p_\pm in §A.2 apply and the kernel (respectively, cokernel) of the projection $p_+ : W \rightarrow H_+$ is isomorphic to the sheaf-cohomology group $H^0(X, \mathcal{L}_\infty)$ (respectively, $H^1(X, \mathcal{L}_\infty)$) where $\mathcal{L}_\infty = \mathcal{L} \otimes \mathcal{O}(-x_\infty)$.

Let Γ_+ denote the group of real analytic mappings $f : S^1 \rightarrow \mathbb{C}^*$, extending to holomorphic functions $f : D_0 \rightarrow \mathbb{C}^*$ in the closed unit disk $D_0 = \{z \in \mathbb{C} : |z| \leq 1\}$, such that $f(0) = 1$. The natural action of Γ_+ on $\text{Gr}(q, H)$ corresponds to its action on $(X, x_\infty, z, \mathcal{L}, \varphi)$, and further from [38](Prop. 6.9) there exists a surjective homomorphism

$$\Gamma_+ \rightarrow J(X), \quad (\text{A.8})$$

where $J(X)$ is the Jacobian torus of X (recall that $J(X)$ is the commutative group of holomorphic line bundles of zero degree over X). The Krichever correspondence [28] (see also [38, Proposition 6.2]) links the data of the quintuple to a flow of multiplication operators Γ_+ on H , thus inducing a linear flow on $J(X)$ of X . Furthermore, this flow corresponds to the evolution of solutions of the generalized KdV flow.

We now take the maps in the definition of Γ_+ to be \mathcal{A} -valued and denote the resulting group by $\Gamma_+(\mathcal{A})$. An action of $\Gamma_+(\mathcal{A}) \subset G(A)$ on $\text{Gr}(p, A)$ is induced via its restriction to the subspace H_+ in any polarization. Recalling that $\mathcal{A} = C(Y)$, then clearly, in the special case $Y = \{pt\}$ we recover the above group Γ_+ . In view of (2.4) and $\text{Gr}(q, B) = \text{Gr}(H_+, H)$, the restriction $\Gamma_+(\mathcal{A})|_{\text{Gr}(q, B)}$ as a Y -parametrized family $\{(\Gamma_+)_y\}_{y \in Y}$. Effectively, we then have a flow of multiplication operators as given by

$$\begin{aligned} \Gamma_+(\mathcal{A}) &= \{(\exp(\sum_{\alpha} a_{\alpha} z^{\alpha}) : a_{\alpha} \in \mathcal{A})\} \\ &\cong \{(\exp(\sum_a t_a z^a))_y\}_{y \in Y}. \end{aligned} \tag{A.9}$$

We also have the group $\Gamma_-(\mathcal{A})$ of multiplication operators on $H_{\mathcal{A}}$ given by the group of continuous maps $g : \mathbb{C} \setminus \text{Int}(D) \rightarrow \mathcal{A}$ such that g is real analytic in $x \in \mathbb{C} \setminus \text{Int}(D)$ extending to $g(z)$ holomorphic in $z \in \mathbb{C} \setminus \text{Int}(D)$, and $g(\infty) = 1$.

A.4 The Burchall-Chaundy ring with coefficients in \mathcal{A}

Continuing with $\mathcal{A} \cong C(Y)$ a commutative (separable) C^* -algebra, we now consider the algebra $\mathcal{O}(U, \mathcal{A})$ of \mathcal{A} -valued analytic functions $U \rightarrow \mathcal{A}$ where U is a connected open neighborhood of the origin in \mathbb{C} . In turn, we consider the algebra $\mathcal{O}(Y, \mathbb{B}[\partial])$ of linear differential operators with coefficients in $\mathcal{O}(Y, \mathbb{B})$ consisting of expressions (A.6) where essentially the same discussion in §A.3 applies verbatim (note that this algebra contains the algebraic tensor product $\mathbb{B}[\partial] \otimes \mathcal{A}$). In particular, the coefficients a_i are now thought of as \mathcal{A} -valued functions.

Relative to $W \in \text{Gr}(p, A)$, the algebra

$$B_W = \{f(z) = \sum_{s=-\infty}^N c_s z^s : s \in \mathbb{N}, c_s \in \mathcal{A}, f(z)W \subset W\}, \tag{A.10}$$

(denoted by A_W in [38]) contains the coordinate ring of the curve $X \setminus \{x_{\infty}\}$. In fact, the set of monomorphisms $\mathbb{A} \rightarrow B_W$ is identified with $J(X)$ [28, 38]. Let then $J_{\mathcal{A}}(X)$ denote the set of monomorphisms $\mathbb{A} \otimes \mathcal{A} \rightarrow B_W$. In view of (A.8) and the appropriate inclusions, we obtain a commutative diagram

$$\begin{array}{ccc} \Gamma_+ & \longrightarrow & \Gamma_+(\mathcal{A}) \\ \downarrow & & \downarrow \\ J(X) & \longrightarrow & J_{\mathcal{A}}(X) \end{array} \tag{A.11}$$

and thus from (A.8) a well-defined surjective homomorphism

$$\Xi : \Gamma_+(\mathcal{A}) \longrightarrow J_{\mathcal{A}}(X). \tag{A.12}$$

Observe that since $\mathcal{A} = C(Y)$, (A.12) is equivalent to a family of maps $Y \rightarrow \{\Gamma_+ \rightarrow J(X)\}$.

A.5 The abstract operator-valued Baker function

Here we consider subspaces $W \in \text{Gr}(p, A)$ of the form $W = gh_g H_+$ with $g \in \Gamma_+(\mathcal{A})$ and $h_g \in \Gamma_-(\mathcal{A})$. Let $\Gamma_+^W(\mathcal{A})$ denote the subgroup of elements $g \in \Gamma_+(\mathcal{A})$ such that $g^{-1}W$ is transverse to H_- . Also, for $g \in \Gamma_+^W(\mathcal{A})$, we consider orthogonal projections

$$p_1^g = p|g^{-1}(W) : g^{-1}(W) \longrightarrow H_+, \quad p_1^g \in \text{Fred}(H_{\mathcal{A}}), \quad (\text{A.13})$$

(i.e. $p_1^g \in \text{Fred}(H_{\mathcal{A}} \cap P^{\text{sa}}(A))$) and define in relationship to the big cell C_b , the subgroup of $\Gamma_+^W(\mathcal{A})$ as given by

$$\tilde{\Gamma}_+^W(\mathcal{A}) = \{g \in \Gamma_+(\mathcal{A}) : p_1^g \text{ is an isomorphism}\}. \quad (\text{A.14})$$

Definition A.1. The operator-valued Baker function ψ_W associated to the subspace $W \in C_b \subset \text{Gr}(p, A)$ is defined formally as:

$$\psi_W = (p_1^g)^{-1}(\mathbf{1}) = \left(\sum_{s=0}^{\infty} a_s(g) \zeta^{-s} \right) g(\zeta) \in Wg(\zeta), \quad (\text{A.15})$$

where $g \in \tilde{\Gamma}_+^W(\mathcal{A})$ and the a_s are analytic \mathcal{A} -valued (operator-matrix) functions on $\Gamma_+^W(\mathcal{A})$ extending to all \mathcal{A} -valued functions $g \in \Gamma_+(\mathcal{A})$ meromorphic in z (cf. [20]).

A.6 A formal integral operator

As in [20, §6], following [38], there exists a formal integral operator $K \in \mathbb{B}[[\partial^{-1}]]$ given by

$$K = 1 + \sum_{s=1}^{\infty} a_s(x) \partial^{-s}, \quad (\text{A.16})$$

(where the a_s are \mathcal{A} -valued functions) unique up to a constant coefficient operator such that $L = K(\partial^n)K^{-1}$ belongs to \mathbb{A} . Under the above correspondence the (formal) Baker function ψ_W is defined as $\psi_W = Ke^{xz}$, the main point being that the function ψ_W will be an eigenfunction for the operator $L^{1/n} = \partial + [\text{lower-order terms}]$, that is, $L^{1/n}\psi_W = z\psi_W$, and accordingly

$$\psi_W(x, z) = \left(1 + \sum_{s=1}^{\infty} a_s(x) z^{-s} \right) e^{xz}. \quad (\text{A.17})$$

Using a form of the Sato correspondence [37], we established (for \mathcal{A} not necessarily commutative):

Theorem A.1. [20, Theorem 6.1] *Given the Baker function ψ_W associated to a subspace $W \in \text{Gr}(p, A)$, the Burchall-Chaundy ring $\mathbb{A} \subset \mathbb{B}[[\partial^{-1}]] \otimes \mathcal{A}$ is conjugated into $A \subset \mathcal{L}_J(H_{\mathcal{A}})$ as a commutative subring, the conjugating integral operator K being unique up to constant coefficient operators.*

A.7 The Y -parametrized holomorphic data

Since we have effectively tensored the coefficients of \mathbb{A} by $\mathcal{A} = C(Y)$, we can modify the discussion in §A.3 using the expression for A in (2.4). Specifically, the same construction involving the data

in §A.3 yields a Y -parametrized map to $\text{Gr}(q, B) = \text{Gr}(H_+, H)$ where we recall $B = \mathcal{L}_J(H)$ from §2.2. Consequently, for a Y -parametrized quintuple in §A.3, $y \in Y$, we obtain the assignment

$$\{(X, x_\infty, z_y, \mathcal{L}_y, \varphi_y)\}_{y \in Y} \mapsto \text{certain points } W_y \in Y \times \text{Gr}(q, B). \quad (\text{A.18})$$

As was noted in §4.4, the restriction $\Gamma_+(\mathcal{A})|\text{Gr}(q, B)$ acts as a family of multiplication operators $\{(\Gamma_+)_y\}_{y \in Y}$ on subspaces $W \in \text{Gr}(q, B)$. Following [38, §6], an element $g \in \Gamma_+$ serves as a transition function for a line bundle over X (that is, $g \in \Gamma_+$ determines a point in the Picard group $\text{Pic}^{g_X}(X)$ [24] which is then twisted into a point of the Jacobian $J(X)$); we recall that g_X denotes the genus of X). The restricted action $\Gamma_+(\mathcal{A})|\text{Gr}(q, B)$ gives a parametrized version:

Proposition A.1. *Let $\mathcal{J}_0(X \times Y)$ denote the space of topologically trivial line bundles on $X \times Y$. Then there exists a well-defined map*

$$\Gamma_+(\mathcal{A})|\text{Gr}(q, B) \longrightarrow \mathcal{J}_0(X \times Y), \quad (\text{A.19})$$

given by $g_y \mapsto \mathcal{L}_{g_y}$, and an induced action of $\Gamma_+(\mathcal{A})|\text{Gr}(q, B)$ on $\mathcal{J}_0(X \times Y)$.

Proof. We follow [38, Proposition 6.9] closely. Let $X_0 = X \setminus X_\infty$, and let $\mathcal{L}_{g_y} \longrightarrow X \times Y$ be the line bundle obtained by taking topologically trivial line bundles over $X_0 \times Y$ and $X_\infty \times Y$ and glueing them by $g_y = (g, y)$ over an open neighborhood of $S^1 \times Y$, where $g \in \Gamma_+$. This line bundle has degree g_X , so it is not topologically trivial, but by changing g_y by an element of Γ_- we achieve degree zero. Thus we obtain a map

$$\begin{aligned} \Gamma_+(\mathcal{A})|\text{Gr}(q, B) &\longrightarrow \mathcal{J}_0(X \times Y), \\ g_y &\mapsto \mathcal{L}_{g_y} \end{aligned} \quad (\text{A.20})$$

where \mathcal{L}_{g_y} has a φ_g -induced trivialization

$$\varphi_{g_y} = (\varphi_g, y) : \mathcal{L}_{g_y}|X_\infty \times Y \longrightarrow \mathbb{C} \times X_\infty \times Y. \quad (\text{A.21})$$

Consequently, $(\Gamma_+)_y$ acts on (X, x_∞, z) and on (\mathcal{L}, φ) via the tensor product with $(\mathcal{L}_{g_y}, \varphi_{g_y})$. \square

In this way the action of $\Gamma_+(\mathcal{A})|\text{Gr}(q, B)$ on Y -parametrized solutions of the generalized n -th KdV equations corresponds to $\mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{L}_{g_y}$. For a fixed $y \in Y$, the assignment $g \mapsto \mathcal{L}_g$ defines a surjective group homomorphism $\Gamma_+ \longrightarrow J(X)$, as in the case $\mathcal{A} \cong \mathbb{C}$ [38, Remark 6.8].

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