

Non-Levi closed conjugacy classes of $SP_q(2n)$

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Abstract

We construct an explicit quantization of semi-simple conjugacy classes of the complex symplectic group $SP(2n)$ with non-Levi isotropy subgroups through an operator realization on highest weight modules over the quantum group $U_q(\mathfrak{sp}(2n))$.

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1 Introduction

We construct a quantization of closed conjugacy classes of the complex algebraic group $SP(2n)$ whose isotropy subgroup is not of Levi type. Such classes are not isomorphic to adjoint orbits in the Lie algebra $\mathfrak{sp}(2n)$, and their Poisson structure is not exactly $SP(2n)$ -invariant. The quantization features a quantum group symmetry, which is a deformation of the conjugation action of $SP(2n)$ on itself. The present study is based on [1] and develops the ideas of [2], where the simplest non-Levi conjugacy class $SP(4)/SP(2) \times SP(2)$ has been worked out in details.

The conjugacy classes of interest form a family that is as large as of Levi type: they involve diagonalizable symplectic matrices whose eigenvalues simultaneously include $+1$ and -1 (a Levi class may have at most one of them). Note that among the classical matrix groups only

symplectic and orthogonal groups admit classes of this type: for the special linear group they are all isomorphic to adjoint orbits in the Lie algebra and have Levi isotropy subgroups. In the present article we address only symplectic groups. Orthogonal groups are given a special treatment in [3] based on a similar approach. Besides the basic similarities, there are certain technical distinctions, and we have chosen to separate the orthogonal case from symplectic in order to simplify the presentation.

The Poisson structure on the conjugacy classes comes from a Poisson structure on the group, which is analogous to the canonical invariant Poisson structure on the Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n)$ (we assume the natural isomorphism between the adjoint and coadjoint representations of \mathfrak{g}). Quantization of this structure is analogous to quantization of the Kostant-Kirillov-Souriau bracket on the coadjoint orbits, with the difference that the former allows for quantum group symmetry rather than classical.

Quantization of conjugacy classes with Levi isotropy subgroups has been constructed in [1] using the representation theory of quantum groups. We should stress that the methods of [1] are inapplicable, as they are, for the non-Levi classes, whose quantization is still an open problem. In our recent paper [2] we have shown how to approach it on the simplest example of $SP(4)/SP(2) \times SP(2)$. In this work, we develop those ideas further and cover all non-Levi conjugacy classes of $SP(2n)$. Along with the Levi type worked out in [1], this is solving the problem for all diagonalizable classes of $SP(2n)$.

Let us we explain our methods. It is natural to seek a quantization of an affine variety in terms of generators and relations, in other words, as a quotient of a free algebra. Supposedly this projection factors through a projection from a quantized coordinate ring $\mathbb{C}_\hbar[G]$ of the group $G = SP(2n)$. It is an equivariant quantization of the Poisson structure on G whose restriction to conjugacy classes is the Poisson structure of our interest. The algebra $\mathbb{C}_\hbar[G]$ is well studied and its explicit description in generators and relations is available. It is related to the "reflection equation" or "boundary Yang-Baxter equation", which is well established in the mathematical physics literature, [4, 5]. To ensure that the quotient of $\mathbb{C}_\hbar[G]$ is a flat deformation, we seek to realize it in an algebra that is flat over the ring of formal power series in the deformation parameter \hbar . Due to certain structural properties of $\mathbb{C}_\hbar[G]$, this would also yield the defining relations, provided we have managed to find an ideal in the kernel turning into the defining ideal in the classical limit (such an ideal shall automatically coincide with the kernel).

The algebra $\mathbb{C}_\hbar[G]$ can be also realized as a subalgebra in the quantized universal enveloping algebra $U_\hbar(\mathfrak{g})$. This fact enables one to construct the quantization of classes through

a realization of $\mathbb{C}_\hbar[G]$ in the algebra of endomorphisms of an appropriate $U_\hbar(\mathfrak{g})$ -module. This approach was successfully applied to conjugacy classes with Levi isotropy subgroups, which were quantized via parabolic Verma modules. However, there is no immediate analog of parabolic Verma modules for non-Levi subalgebras in \mathfrak{g} . The obstructions are two-fold. Firstly, there is no natural candidate for the quantized stabilizer as a subalgebra in $U_\hbar(\mathfrak{g})$. Secondly, even in the classical case there is no parabolic extension of the non-Levi stabilizer. These are the principal properties that facilitate the parabolic induction in the Levi case. Therefore the key step is to find a suitable replacement of the parabolic Verma modules for non-Levi conjugacy classes. We take a quotient of a special auxiliary parabolic Verma module for it, which is chosen as follows.

Let $K \subset G$ denote the stabilizer of the initial point of the class. It contains a maximal Levi subgroup $L \subset K$. There are actually two such subgroups, which correspond to the two symplectic blocks $SP(2m)$ and $SP(2p)$ in K rotating the -1 -eigenspace and, respectively, the $+1$ -eigenspace of the initial point. We obtain L by reducing $SP(2m)$ to $GL(m)$. In the classical situation, the isotropy subalgebra $U(\mathfrak{k}) \subset U(\mathfrak{g})$ is generated over the Levi subalgebra $U(\mathfrak{l})$ by a certain pair of root vectors e_δ, f_δ . We construct the parabolic Verma $U_\hbar(\mathfrak{g})$ -module \hat{M}_λ relative to $U_\hbar(\mathfrak{l})$, where the highest weight λ is conditioned by the presence of a singular vector of weight $\lambda - \delta$. The quotient M_λ of \hat{M}_λ over the submodule generated by that singular vector is the module where we realize the quantization of $\mathbb{C}[G/K]$.

Intuitively the passage from \hat{M}_λ to M_λ can be interpreted as follows. The functional dimension of \hat{M}_λ is $\frac{1}{2} \dim G/L$. Its algebra of endomorphisms (locally finite part of) is isomorphic to the tensor product $\hat{M}_{-\lambda}^* \otimes \hat{M}_\lambda$, where $\hat{M}_{-\lambda}^*$ is the restricted dual $U_\hbar(\mathfrak{g})$ -module. It can be realized through a parabolic induction, and the pairing between $\hat{M}_{-\lambda}^*$ and \hat{M}_λ is a kind of Shapovalov form. The module $\hat{M}_{-\lambda}^* \otimes \hat{M}_\lambda$ is isomorphic to $\text{Ind}_\mathfrak{q}^{\mathfrak{g}} \mathbb{C}$, where the induction is understood in the quantum group setting. It is a deformation of the classical induced module $\text{Ind}_\mathfrak{q}^{\mathfrak{g}} \mathbb{C}$, which is in duality with the function algebra on the coset space G/L ; hence the idea to realize the polynomial algebra $\mathbb{C}[G/L]$ in the self-dual module $\hat{M}_{-\lambda}^* \otimes \hat{M}_\lambda$. This qualitative consideration explains why we should eliminate the part of \hat{M}_λ generated by $f_\delta v$. Doing so we kill the extra degrees of freedom along $\text{ad}(\mathfrak{l})f_\delta$ and obtain a module, M_λ , with the proper functional dimension $\frac{1}{2} \dim G/K$.

The subalgebra $\mathbb{C}_\hbar[G] \subset U_\hbar(\mathfrak{g})$ is generated by the entries of an invariant matrix $\mathcal{Q} \in \text{End}(\mathbb{C}^{2n}) \otimes \mathbb{C}_\hbar[G]$ canonically constructed from the universal R-matrix of $U_\hbar(\mathfrak{g})$. The problem of the quantized ideal of the class boils down to determining the minimal polynomial of \mathcal{Q} regarded as an operator on $\mathbb{C}^{2n} \otimes M_\lambda$. The matrix \mathcal{Q} is semi-simple on $\mathbb{C}^{2n} \otimes \hat{M}_\lambda$, and

its eigenvalues are known. Clearly \mathcal{Q} satisfies the same polynomial equation on $\mathbb{C}^{2n} \otimes M_\lambda$, which is however not necessarily minimal. We prove that the extra eigenvalue drops from the spectrum of \mathcal{Q} in the transition from \hat{M}_λ to M_λ and obtain the minimal polynomial on $\mathbb{C}^{2n} \otimes M_\lambda$ from that on $\mathbb{C}^{2n} \otimes \hat{M}_\lambda$, through this reduction.

The passage from \hat{M}_λ to M_λ is analogous to the passage from G/L to G/K , where the class G/L is obtained from G/K by splitting the $2m$ eigenvalues -1 into m pairs of reciprocals $\mu, \mu^{-1} \neq \pm 1$. In the limit as $\mu \rightarrow -1$ they glue up, and the isotropy subgroup jumps from L to K . The minimal polynomial of G/L acquires a non-simple factor $(x+1)^2$, which should be reduced in the minimal polynomial of G/K . Similarly, we check that the extra divisor of the minimal polynomial of \mathcal{Q} is canceled in the projection $\mathbb{C}^{2n} \otimes \hat{M}_\lambda \rightarrow \mathbb{C}^{2n} \otimes M_\lambda$, and the classical limit yields the minimal polynomial of G/K . This implies the second important step of our strategy: the analysis of the $U_\hbar(\mathfrak{g})$ -module $\mathbb{C}^{2n} \otimes M_\lambda$ and the invariant operator \mathcal{Q} on it.

Putting the non-Levi conjugacy classes into a common quantization scheme with the classes of Levi type implies several far reaching consequences. First of all, recall that the latter (along with quantum semi-simple coadjoint orbits) gave rise to the theory of dynamical Yang-Baxter equation over a general non-Abelian base, [6]. To a large extent, that theory is based on the properties of the parabolic \mathcal{O}^l -category. Recall that it is a module category over that of finite dimensional representations of $U_\hbar(\mathfrak{g})$ under the tensor product multiplication (for a definition of module categories over monoidal categories, see e.g. [7]). We observe an analogous category \mathcal{O}^t associated with a non-Levi quantum conjugacy class, which is generated (as a module category) by M_λ of a feasible weight λ . It is natural to expect that the study of \mathcal{O}^t will make a significant contribution to the present theory of dynamical Yang-Baxter equation. Further, the parabolic category \mathcal{O}^l consists of $U_\hbar(\mathfrak{g})$ -modules that are parabolically induced from $U_\hbar(\mathfrak{l})$ -modules. At the same time, the algebra $U(\mathfrak{k})$ is not quantized as a Hopf subalgebra in $U_\hbar(\mathfrak{g})$, and there is no *a priori* natural quantum counterpart for it. It is therefore interesting to understand its quantization, which would be a \mathfrak{k} -analog of the Levi subalgebra $U_\hbar(\mathfrak{l})$. This might help to understand the category \mathcal{O}^t .

An important special case of non-Levi conjugacy classes comprises the symmetric spaces $SP(2n)/SP(2m) \times SP(2p)$, $m+p=n$. There is an extended literature on their quantization in connection with integrable models, [8], and representation theory, [9, 10, 11]. The quantized function algebras were basically viewed as subalgebras in the Hopf dual to $U_\hbar(\mathfrak{g})$ annihilated by certain coideal subalgebras. Such subalgebras play the role of quantum stabilizers of the "initial point". An advanced theory of quantum symmetric pairs (equivalently,

quantum stabilizers) was developed in [12, 13, 14, 15]. In the present paper, we adopt a different approach to quantization realizing it by endomorphisms in a $U_\hbar(\mathfrak{g})$ -module. The two approaches are complementary, as in the classical geometry a closed conjugacy class can be alternatively presented as a subalgebra and a quotient algebra of $\mathbb{C}[G]$. That is also possible in the special case of symmetric classes, because they possess a "classical point", i.e. a one-dimensional representation of $\mathbb{C}_\hbar[G]$. Equivalently, it is a numerical matrix solving the reflection equation. Such a matrix is annihilated by the coideal subalgebra and serves as the initial point of the class in the theory of symmetric pairs. Contrary to our setting, this initial point is not diagonal. As we already mentioned, our approach is lacking the quantum version of stabilizer, and that complicates the further study of the quantized non-Levi classes. It is an interesting problem to match our approach with the theory of quantum symmetric pairs. That could help to identify the quantum stabilizer within the present approach and facilitate further advances in the theory of quantized non-Levi conjugacy classes.

2 Classical conjugacy classes

Throughout the paper, G designates the algebraic group $SP(2n)$ of symplectic matrices preserving a non-degenerate skew symmetric form $\|C_{ij}\|_{i,j=1}^{2n}$ in the complex vector space \mathbb{C}^{2n} ; the Lie algebra of G is denoted by \mathfrak{g} . We choose the realization $C_{ij} = \epsilon_i \delta_{ij'}$, where δ_{ij} is the Kronecker symbol, $i' = 2n + 1 - i$, and $\epsilon_i = -\epsilon_{i'} = 1$ for $i = 1, \dots, n$.

The polynomial ring $\mathbb{C}[G]$ is generated by the matrix coordinate functions $\|A_{ij}\|_{i,j=1}^{2n}$, modulo the set of $2n \times 2n$ relations written in the matrix form as

$$ACA^t = C. \tag{2.1}$$

The right conjugacy action of G on itself induces a left action on $\mathbb{C}[G]$ by duality; the matrix A is invariant as an element of $\text{End}(\mathbb{C}^{2n}) \otimes \mathbb{C}[G]$.

The group G is equipped with the Drinfeld-Sklyanin bivector field

$$\{A_1, A_2\} = \frac{1}{2}(A_2 A_1 r - r A_1 A_2), \tag{2.2}$$

where $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a solution of the classical Yang-Baxter equation, [16]. Equation (2.2) is understood in $\text{End}(\mathbb{C}^{2n}) \otimes \text{End}(\mathbb{C}^{2n}) \otimes \mathbb{C}[G]$. The subscripts label the natural embeddings of $\text{End}(\mathbb{C}^{2n})$ in $\text{End}(\mathbb{C}^{2n}) \otimes \text{End}(\mathbb{C}^{2n})$, as usual in the quantum groups literature.

The bivector field (2.2) defines a Poisson bracket on $\mathbb{C}[G]$ making G a Poisson group. There is a variety of solutions of the classical Yang-Baxter equation, which are parameterized

by combinatorial objects (Manin triples) and certain Cartan bivectors, [17]. We choose the so called standard solution

$$r = \sum_{i=1}^{2n} (e_{ii} \otimes e_{ii} - e_{ii} \otimes e_{i'i'}) + 2 \sum_{\substack{i,j=1 \\ i>j}}^{2n} (e_{ij} \otimes e_{ji} - \epsilon_i \epsilon_j e_{ij} \otimes e_{i'j'}), \quad (2.3)$$

which is the simplest of all. The corresponding quantum group is pretty similar to the classical universal enveloping algebra. In particular, one can define quantum Levi subalgebras, parabolic subalgebras *etc* facilitating the parabolic induction. The restriction to the standard solution (2.3) is methodological. If one is concerned with an abstract quantization, in terms of generators and relations rather than an operator realization, the other cases can be readily obtained from the standard. At the end of the article, we indicate what modifications to the resulting formulas should be made in order to include an arbitrary r-matrix.

We regard the group G as a G -space under the conjugation action. The object of our study is another Poisson structure on G ,

$$\{A_1, A_2\} = \frac{1}{2}(A_2 r_{21} A_1 - A_1 r A_2 + A_2 A_1 r - r_{21} A_1 A_2), \quad (2.4)$$

in the shortcut matrix form. It is compatible with the conjugation action and makes G a Poisson space over the Poisson group G equipped with the Drinfeld-Sklyanin bracket (2.2). The bivector (2.4) restricts to every conjugacy class making it a Poisson homogeneous space over G .

A closed conjugacy class $O \subset G$ consists of diagonalizable matrices and is determined by the set of their eigenvalues $S_O = \{\mu_i, \mu_i^{-1}\}_{i=1}^n$. Every eigenvalue μ enters S_O along with its reciprocal μ^{-1} . In particular, there may be $\mu = \mu^{-1} = \pm 1$. One should distinguish two situations: a) S_O contains either $+1$ or -1 or none, and b) both $+1$ and -1 belong to S_O . In the first case, O is isomorphic to an orbit in \mathfrak{g} via the Cayley transformation, and its isotropy subgroup is of Levi type. A conjugacy class of second type is not isomorphic to an adjoint orbit. In terms of Dynkin diagram, every Levi subgroup is obtained by scraping out a subset of nodes, while for non-Levi isotropy subgroups one should scrape out a set of nodes from the affine Dynkin diagram of \mathfrak{g} .



Informally, a non-Levi centralizer necessarily contains two symplectic blocks rotating the eigenspaces of eigenvalues ± 1 .

We associate with a class O an integer valued vector $\mathbf{n} = (n_1, \dots, n_\ell, m, p)$ and a complex valued vector $\boldsymbol{\mu} = (\mu_1, \dots, \mu_\ell, -1, 1)$ assuming $\mu_i, i = 1, \dots, \ell$, all invertible, not a square root of 1, and $\mu_i \neq \mu_j^{\pm 1}, i \neq j$. The initial point $o \subset O$ will be fixed to the diagonal matrix with the entries

$$\underbrace{\mu_1, \dots, \mu_1}_{n_1}, \dots, \underbrace{\mu_\ell, \dots, \mu_\ell}_{n_\ell}, \underbrace{-1, \dots, -1}_m, \underbrace{1, \dots, 1}_{2p}, \underbrace{-1, \dots, -1}_m, \underbrace{\mu_\ell^{-1}, \dots, \mu_\ell^{-1}}_{n_\ell}, \dots, \underbrace{\mu_1^{-1}, \dots, \mu_1^{-1}}_{n_1},$$

so that $\sum_{i=1}^{\ell} n_i + m + p = n$. We reserve the integers $m = n_{\ell+1}, p = n_{\ell+2}$ to denote respectively, the ranks of the blocks corresponding to $-1 = \mu_{\ell+1}$ and $+1 = \mu_{\ell+2}$ (we view ± 1 as degenerations of the parameters $\mu_{\ell+1}$ and $\mu_{\ell+2}$). The specialization $n_1 = \dots = n_\ell = 0$ is formally encoded by $\ell = 0$ and referred to as the *symmetric case*, because the corresponding conjugacy class is a symmetric space.

The stabilizer subgroup of the initial point o is the direct product

$$K = GL(n_1) \times \dots \times GL(n_\ell) \times SP(2m) \times SP(2p) \quad (2.5)$$

and it is determined by the vector \mathbf{n} . The positive integer ℓ counts the number of the GL -blocks in K . In the symmetric case, (2.5) reduces to $SP(2m) \times SP(2p)$.

Let \mathcal{M}_K denote the moduli space of conjugacy classes with the fixed isotropy subgroup (2.5), regarded as Poisson spaces. The set of all $\ell + 2$ -tuples $\boldsymbol{\mu}$ with invertible components such that $\mu_{\ell+1}^2 = \mu_{\ell+2}^2 = 1$ and $\mu_i \neq \mu_j^{\pm 1}$ for distinct i, j parameterize \mathcal{M}_K albeit not uniquely. Multiplication by the scalar matrix $-1 \in G$ preserves this set and swaps $\mu_{\ell+1}$ with $\mu_{\ell+2}$. This transformation is an automorphism of G as a G -space and preserves the Poisson structure (2.4). Therefore, the subset $\hat{\mathcal{M}}_K$ of $\boldsymbol{\mu}$ with fixed $\mu_{\ell+1} = -1$ and $\mu_{\ell+2} = 1$ can also be used for parametrization of \mathcal{M}_K . The residual ambiguity is due to permutations of the components $\mu_i \neq \pm 1$ with equal multiplicities.

The class O associated with $\boldsymbol{\mu}$ and \mathbf{n} is determined by the set of polynomial equations

$$(A - \mu_1) \dots (A - \mu_\ell)(A + 1)(A - 1)(A - \mu_\ell^{-1}) \dots (A - \mu_1^{-1}) = 0, \quad (2.6)$$

$$\text{Tr}(A^k) = \sum_{i=1}^{\ell} n_i (\mu_i^k + \mu_i^{-k}) + 2m(-1)^k + 2p, \quad k = 1, \dots, 2n, \quad (2.7)$$

on the entries of the matrix A . In fact, the ideal in $\mathbb{C}[G]$ generated by this set of relations is radical and therefore coincides with the defining ideal of $\mathbb{C}[O]$ in $\mathbb{C}[G]$. This is a consequence of the following general fact.

Consider a smooth variety X in an affine space Y of dimension $\dim(Y)$. Suppose that X is defined by a system of polynomial equations $F_i(x) = 0, i \in I$, where I is a finite set

of indices. The ideal $J' = (F_i)_{i \in I}$ is contained in the defining ideal J of X , i.e. the ideal of all polynomial functions vanishing on X . In general, J' might be less than J , and then the quotient $\mathbb{C}[Y]/J'$ cannot be regarded as a ring of functions on X : there will be nilpotent elements in $\mathbb{C}[Y]/J'$. It is essential for our approach to quantization to ensure that the ideal J' is exactly J . The latter obeys a certain maximality requirement, which plays a role in the construction. We will use the following criterion of radicality of J' .

Proposition 2.1. *Suppose that the rank of the differential $\{dF_i\}_{i \in I}$ is equal to $\dim(Y) - \dim(X)$ at every point $x \in X$. Then the ideal $J' \subset \mathbb{C}[Y]$ generated by $\{F_i\}_{i \in I}$ coincides with the defining ideal J of X .*

Proof. Denote by $A' = \mathbb{C}[Y]/J'$ and $A = \mathbb{C}[Y]/J$ the quotient algebras and consider their affine schemes with the structure sheafs \mathcal{O}' and \mathcal{O} , respectively. Since J is the radical of J' , the natural embedding $\text{Spec}(A') \rightarrow \text{Spec}(A)$ is an isomorphism making \mathcal{O} a subsheaf in \mathcal{O}' . The condition on the rank of $\{dF_i\}_{i \in I}$ implies, by the Jacobian criterion of smoothness [18], that \mathcal{O}_a and \mathcal{O}'_a are regular local rings at every point $a \in \text{Spec}(A')$, and $\mathcal{O}_a = \mathcal{O}'_a$. As the two sheafs coincide locally, they coincide globally. Hence $A' \simeq A$, and $J' = J$. \square

Proposition 2.1 provides a convenient test for verification if a particular system of equations gives rise to the defining ideal. That is especially so for homogeneous varieties, as it suffices to look at the initial point only. Remark that the condition on the rank can be replaced with a more practical condition on the kernel: $\dim(\cap_i \ker dF_i) = \dim(X)$.

First we consider the general linear group, which result was obtained in [19] within a different approach.

Theorem 2.2. *Let \tilde{G} be the general linear group of the vector space \mathbb{C}^N and let $\tilde{O} \subset \tilde{G}$ be the conjugacy class of diagonalizable matrices with pairwise distinct eigenvalues (μ_1, \dots, μ_l) of multiplicities (n_1, \dots, n_l) . The system of polynomial equations*

$$(A - \mu_1) \dots (A - \mu_l) = 0, \quad \text{Tr}(A^k) = \sum_{i=1}^l n_i \mu_i^k, \quad k = 1, \dots, N \quad (2.8)$$

has rank $N^2 - \dim(\tilde{O})$ everywhere on \tilde{O} , hence it generates the defining ideal of \tilde{O} .

Proof. Take $o = \text{diag}(\underbrace{\mu_1, \dots, \mu_1}_{n_1}, \dots, \underbrace{\mu_l, \dots, \mu_l}_{n_l}) \in \tilde{O}$ for the initial point in \tilde{O} . The matrix o can be written as $o = \sum_{i=1}^l \mu_i P_i$, where $P_i: \mathbb{C}^N \rightarrow \mathbb{C}^{n_i}$ is the diagonal projector to the μ_i -eigenspace of o . Denote by E_{ij} , $i, j = 1, \dots, l$, the subspace of matrices $P_i \text{End}(\mathbb{C}^N) P_j$. We

have $\text{End}(\mathbb{C}^N) = \bigoplus_{i,j=1}^l E_{ij}$ for the entire matrix algebra, and $\tilde{\mathfrak{k}} = \bigoplus_{i=1}^l E_{ii}$ for the Lie algebra $\tilde{\mathfrak{k}}$ of the stabilizer of o . The tangent space $T_o(\tilde{G})$ is naturally identified with $\tilde{\mathfrak{m}} = \bigoplus_{\substack{i,j=1 \\ i \neq j}}^l E_{ij}$. The class \tilde{O} is the zero locus of the system of equations (2.8). To prove the statement, it is sufficient to check the rank of the system (2.8) at the point o .

Denote by F the matrix polynomial $\prod_{i=1}^l (A - \mu_i)$ and by ϑ_k the trace $\text{Tr}(A^k)$, $k = 1, \dots, N$. The system of relations involves $N \times N$ matrix entries F_{ij} and N differences $\vartheta_k - \sum_{i=1}^l n_i \mu_i^k$. It is easy to check that

$$dF(\xi) = 0, \quad d\vartheta_k(\xi) = 0, \quad k = 1, \dots, N,$$

for all $\xi \in E_{ij}$ with $i \neq j$ and

$$dF(\xi) = \prod_{\substack{i=1 \\ i \neq j}}^l (\mu_j - \mu_i) \xi, \quad d\vartheta_k(\xi) = k \mu_j^k \text{Tr}(\xi), \quad k = 1, \dots, N,$$

for all $\xi \in E_{jj}$. Note that the equations with $d\vartheta_k$ are redundant as $\ker(dF) \subset \ker d\vartheta_k$. To see this, one should differentiate the trace of $F(\xi)$. The left equation tells us that $\text{im } dF = \tilde{\mathfrak{k}}$, as the numerical coefficient before ξ does not vanish. This proves the assertion. \square

Based on Theorem 2.2, we apply Proposition 2.1 to describe the defining ideals of closed conjugacy classes of the symplectic groups.

Theorem 2.3. *The system of polynomial relations (2.6) and (2.7) along with the defining relations of the group (2.1) generate the defining ideal of the class $O \subset SP(2n)$.*

Proof. As shown in the proof of Theorem 2.2, the differential on the trace functions is linear dependent of the differential of the minimal polynomials. Therefore, the essential part of the Jacobian comes from the minimal polynomial and the equation of the group. The tangent space $T_o(G)$ is the set of fixed points of the linear automorphism $\sigma_o: \xi \mapsto -o\sigma(\xi)o$, where σ is the involutive algebra anti-automorphism $\xi \mapsto -C\xi^t C$. Clearly the tangent space can be presented as $o\mathfrak{g}$ using the matrix multiplication and the embedding $\mathfrak{g} \subset \text{End}(2n)$.

The automorphism σ_o is an involution, so the tangent space $T_o(G)$ is the image of the idempotent $\frac{\text{id} + \sigma_o}{2}$. Using the same notation as in the proof of Theorem 2.2, we split the tangent space into the direct sum $o\mathfrak{g} = \mathfrak{k}_o \oplus \mathfrak{m}_o$, where $\mathfrak{k}_o = o\mathfrak{g} \cap \tilde{\mathfrak{k}}$ and $\mathfrak{m}_o = o\mathfrak{g} \cap \tilde{\mathfrak{m}}$. This is possible because the spaces $\tilde{\mathfrak{k}}$ and $\tilde{\mathfrak{m}}$ are stable under multiplication by o .

Let $H(A)$ denote the endomorphism $\text{End}(\mathbb{C}^{2n}) \rightarrow \text{End}(\mathbb{C}^{2n})$, $A \mapsto ACA^t C + 1$. We need to find the rank of the differential of the mapping $\text{End}(\mathbb{C}^{2n}) \rightarrow \text{End}(\mathbb{C}^{2n}) \oplus \text{End}(\mathbb{C}^{2n})$,

$A \mapsto H(A) \oplus F(A)$, at the point $A = o$. Equivalently we can find its kernel, which is the intersection $\ker dH_o \cap \ker dF_o$. The tangent space $T_o(\text{End}(\mathbb{C}^{2n}))$ splits into the direct sum $\mathfrak{og}^\perp \oplus \mathfrak{og}$. The kernel of dH_o is exactly \mathfrak{og} , hence $\ker(dH_o \oplus dF_o)$ is just $\ker dF_o|_{\mathfrak{og}}$. In the course of the proof of Theorem 2.2 we saw that $\mathfrak{m}_o \subset \ker dF_o|_{\mathfrak{og}}$. This inclusion is, in fact, an equality. Indeed, $\mathfrak{k}_o \subset \tilde{\mathfrak{k}}$, and dF_o is injective on $\tilde{\mathfrak{k}}$. Hence it is injective on \mathfrak{k}_o . Thus, the kernel of the differential $dH_o \oplus dF_o$ is exactly \mathfrak{m}_o . But $\mathfrak{m}_o \simeq T_o(O)$, and the rank of the map $dH_o \oplus dF_o: \text{End}(\mathbb{C}^{2n}) \rightarrow \text{End}(\mathbb{C}^{2n}) \oplus \text{End}(\mathbb{C}^{2n})$ is equal to the codimension of G . This completes the proof. \square

Although non-Levi conjugacy classes are of our main concern, Theorem 2.3 holds true for any semi-simple conjugacy class. It generalizes for the orthogonal groups in the obvious way, with the only stipulation for the D -series: the traces of matrix powers are not enough to fix a class, and one has to add one more condition on the invariants of G , see e.g. [1].

3 Quantum group $U_{\hbar}(\mathfrak{sp}(2n))$

Recall the definition of the quantum group $U_{\hbar}(\mathfrak{sp}(2n))$, which is a deformation of the universal enveloping algebra $U(\mathfrak{sp}(2n))$ along the formal parameter \hbar in the class of Hopf algebras, [16]. Let R and R^+ denote respectively the root system and the subset of positive roots of the Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n)$. Let $\Pi_+ = (\alpha_1, \alpha_1, \dots, \alpha_n)$ be the set of simple positive roots. By (\cdot, \cdot) we designate the canonical inner form on the linear span of Π^+ . The set Π^+ can be conveniently expressed through an orthogonal basis $(\varepsilon_i)_{i=1}^n$ by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, \dots, n-1$, $\alpha_n = 2\varepsilon_n$. We reserve the special notation β for the long root α_n .

The inner product establishes a linear isomorphism between the linear span $\mathbb{C}\Pi^+$ and its dual, \mathfrak{h} . We define $h_\lambda \in \mathfrak{h}$ for every $\lambda \in \mathfrak{h}^* = \mathbb{C}\Pi^+$ as the image of λ under this isomorphism: $\mu(h_\lambda) = (\lambda, \mu)$ for all $h \in \mathfrak{h}$. In particular, we set h_ρ for the half-sum of all positive roots $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$.

The quantum group $U_{\hbar}(\mathfrak{g})$ is a $\mathbb{C}[[\hbar]]$ -algebra generated by the simple root vectors (Chevalley generators) e_μ, f_μ , and the Cartan generators $h_\mu \in \mathfrak{h}$, $\mu \in \Pi^+$. The vector space \mathfrak{h} generates the commutative Cartan subalgebra $U_{\hbar}(\mathfrak{h})$ in $U_{\hbar}(\mathfrak{g})$. The elements h_μ obey the following commutation relations with e_ν, f_ν :

$$[h_\mu, e_\nu] = (\mu, \nu)e_\nu, \quad [h_\mu, f_\nu] = -(\mu, \nu)f_\nu, \quad \mu, \nu \in \Pi^+.$$

In these formulas, the only non-zero inner products are

$$(\alpha_i, \alpha_i) = 2, \quad (\alpha_{i-1}, \alpha_i) = -1, \quad (\beta, \beta) = 4, \quad (\beta, \alpha_{n-1}) = -2,$$

where i takes all admissible values in the range $1, \dots, n-1$. Note that the Cartan generators h_μ are different from those of [22], which are obtained from h_μ via division by $\frac{(\mu, \mu)}{2}$.

The positive and negative Chevalley generators commute to $U_{\hbar}(\mathfrak{h})$:

$$[e_\mu, f_\nu] = \delta_{\mu, \nu} \frac{q^{h_\mu} - q^{-h_\mu}}{q_\mu - q_\mu^{-1}}, \quad \mu \in \Pi^+,$$

where $q_\mu = q = e^{\hbar}$ for $\mu \neq \beta$ and $q_\beta = q^2$.

The non-adjacent positive Chevalley generators commute. The adjacent generators satisfy the Serre relations

$$\begin{aligned} e_\mu^2 e_\nu - (q + q^{-1}) e_\mu e_\nu e_\mu + e_\nu e_\mu^2 &= 0, \quad \text{for } \mu, \nu \neq \beta, \quad \text{and} \\ e_\beta^2 e_\mu - (q^2 + q^{-2}) e_\beta e_\mu e_\beta + e_\mu e_\beta^2 &= 0, \\ e_\mu^3 e_\beta - (q^2 + 1 + q^{-2}) e_\mu^2 e_\beta e_\mu + (q^2 + 1 + q^{-2}) e_\mu e_\beta e_\mu^2 - e_\beta e_\mu^3 &= 0 \end{aligned}$$

for $\mu = \alpha_{n-1}$. Similar relations holds for the negative Chevalley generators f_μ .

The involution $\omega: e_\mu \leftrightarrow f_\mu$ and $\omega(h_\mu) = -h_\mu$, $\mu \in \Pi^+$, extends to an algebra automorphism of $U_{\hbar}(\mathfrak{g})$

The comultiplication Δ and antipode γ are defined on the generators by

$$\begin{aligned} \Delta(h_\mu) &= h_\mu \otimes 1 + 1 \otimes h_\mu, \quad \gamma(h_\mu) = -h_\mu, \\ \Delta(e_\mu) &= e_\mu \otimes 1 + q^{h_\mu} \otimes e_\mu, \quad \gamma(e_\mu) = -q^{-h_\mu} e_\mu, \\ \Delta(f_\mu) &= f_\mu \otimes q^{-h_\mu} + 1 \otimes f_\mu, \quad \gamma(f_\mu) = -f_\mu q^{h_\mu}, \end{aligned}$$

for all $\mu \in \Pi_+$. The counit homomorphism $\varepsilon: U_{\hbar}(\mathfrak{g}) \rightarrow \mathbb{C}[[\hbar]]$ annihilates e_μ, f_μ, h_μ . Note that we use the opposite coalgebra structure on $U_{\hbar}(\mathfrak{g})$ as compared to [22].

Besides the Cartan subalgebra $U_{\hbar}(\mathfrak{h})$, the quantum group $U_{\hbar}(\mathfrak{g})$ contains the following Hopf subalgebras. The positive and negative Borel subalgebras $U_{\hbar}(\mathfrak{b}^\pm)$ are generated over $U_{\hbar}(\mathfrak{h})$ by $\{e_\mu\}_{\mu \in \Pi^+}$ and $\{f_\mu\}_{\mu \in \Pi^+}$, respectively. For any Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$ corresponding to a subset $\Pi_{\mathfrak{l}}^+ \subset \Pi^+$, the universal enveloping algebra $U(\mathfrak{l})$ is quantized to a Hopf subalgebra $U_{\hbar}(\mathfrak{l}) \subset U_{\hbar}(\mathfrak{g})$, along with the parabolic subalgebras $U_{\hbar}(\mathfrak{p}^\pm)$ generated by $U_{\hbar}(\mathfrak{b}^\pm)$ over $U_{\hbar}(\mathfrak{l})$.

The triangular decomposition $\mathfrak{g} = \mathfrak{n}_1^- \oplus \mathfrak{l} \oplus \mathfrak{n}_1^+$ gives rise to the factorization

$$U_{\hbar}(\mathfrak{g}) = U_{\hbar}(\mathfrak{n}_1^-) U_{\hbar}(\mathfrak{l}) U_{\hbar}(\mathfrak{n}_1^+), \quad (3.9)$$

where $U_{\hbar}(\mathfrak{n}_1^\pm)$ are certain subalgebras in $U_{\hbar}(\mathfrak{b}^\pm)$, [20]. This factorization makes $U_{\hbar}(\mathfrak{g})$ a free $U_{\hbar}(\mathfrak{n}_1^-) - U_{\hbar}(\mathfrak{n}_1^+)$ -bimodule generated by $U_{\hbar}(\mathfrak{l})$. In the special case of this decomposition

relative to $\mathfrak{l} = \mathfrak{h}$, we use the notation $U_{\hbar}(\mathfrak{g}_{\pm}) = U_{\hbar}(\mathfrak{n}_{\mathfrak{g}}^{\pm})$. Note that, contrary to the classical situation, $U_{\hbar}(\mathfrak{n}_{\mathfrak{g}}^{\pm})$ are not Hopf subalgebras in $U_{\hbar}(\mathfrak{g})$.

We shall also deal with the Hopf subalgebra $U_q(\mathfrak{g}) \subset U_{\hbar}(\mathfrak{g})$ generated by the simple root vectors and the exponentials $t_{\alpha_i}^{\pm} = q^{\pm h\alpha_i}$, $\alpha_i \in \Pi_+$. Let us stress that we regard $U_q(\mathfrak{g})$ as a $\mathbb{C}[[\hbar]]$ -algebra. The other mentioned subalgebras of $U_{\hbar}(\mathfrak{g})$ have their counterparts in $U_q(\mathfrak{g})$, and we use the subscript q for their notation. The roles of quantum groups $U_{\hbar}(\mathfrak{g})$ and $U_q(\mathfrak{g})$ are different in what follows. While $U_q(\mathfrak{g})$ is a source of non-commutative functions on quantum geometric spaces, $U_{\hbar}(\mathfrak{g})$ is a measure of their symmetry. This difference is somewhat camouflaged in the classical geometry but becomes more distinctive in quantum.

4 Quantum subgroup $U_{\hbar}(\mathfrak{gl}(n))$

The quantum group $U_{\hbar}(\mathfrak{sp}(2n))$ contains the quantum subgroup $U_{\hbar}(\mathfrak{gl}(n))$ corresponding to the positive simple roots $(\alpha_1, \dots, \alpha_{n-1}) \subset \Pi^+$. We need a few technical facts about this subalgebra, which are used in the sequel.

Fix a pair of integers i, j such that $i < j < n$ and put $\mu = \alpha_i + \dots + \alpha_j \in R^+$. Along with $(\alpha_1, \dots, \alpha_{n-1})$, such μ exhaust all of positive roots of $\mathfrak{gl}(n)$. The integer $j - i + 1$ is called height of the root μ and denoted by $\text{ht}(\mu)$. Define elements $f_{\mu}, \tilde{f}_{\mu} \in U_{\hbar}(\mathfrak{gl}(n))$ by

$$f_{\mu} = [f_{\alpha_i}, \dots [f_{\alpha_{j-1}}, f_{\alpha_j}]_q \dots]_q, \quad \tilde{f}_{\mu} = [f_{\alpha_i}, \dots [f_{\alpha_{j-1}}, f_{\alpha_j}]_{q^{-1}} \dots]_{q^{-1}},$$

were $[x, y]_a$ designates the combination $xy - ayx$ with a scalar a . It is also convenient to put $\tilde{f}_{\alpha_i} = f_{\alpha_i}$ for the simple roots α_i , $i = 1, \dots, n-1$. Here are some commutation relations involving these root vectors.

Lemma 4.1. *Let $\mu = \alpha_i + \dots + \alpha_j$ and f_{μ} and \tilde{f}_{μ} be as above. Suppose the integer k is such that $i < k < j$. Then $[e_{\alpha_k}, f_{\mu}] = 0$, $[e_{\alpha_k}, \tilde{f}_{\mu}] = 0$. Further,*

$$[e_{\alpha_i}, f_{\mu}] = f_{\mu'} q^{-h\alpha_i}, \quad [e_{\alpha_j}, f_{\mu}] = -q f_{\mu''} q^{h\alpha_j}, \quad [e_{\alpha_i}, \tilde{f}_{\mu}] = \tilde{f}_{\mu'} q^{h\alpha_i}, \quad [e_{\alpha_j}, \tilde{f}_{\mu}] = -q^{-1} \tilde{f}_{\mu''} q^{-h\alpha_j},$$

where $\mu' = \alpha_{i+1} + \dots + \alpha_j$ and $\mu'' = \alpha_i + \dots + \alpha_{j-1}$.

Proof. It is sufficient to check only the group of equalities involving f_{μ} , as the equalities with \tilde{f}_{μ} can be obtained by the formal replacement $q \rightarrow q^{-1}$. Let us start with the special case of $k = i + 1$, $j = i + 2$:

$$[e_{\alpha_{i+1}}, f_{\mu}] \sim [f_{\alpha_i}, [q^{h\alpha_{i+1}} - q^{-h\alpha_{i+1}}, f_{\alpha_{i+2}}]_q]_q \sim [f_{\alpha_i}, f_{\alpha_{i+2}} q^{-h\alpha_{i+1}}]_q = [f_{\alpha_i}, f_{\alpha_{i+2}}]_q q^{-h\alpha_{i+1}} = 0.$$

The general case is verified in a similar way based on the formula $f_\mu = [f_{\mu_1}, [f_{\alpha_k}, f_{\mu_2}]_q]_q$, where $\mu_1 = \alpha_i + \dots + \alpha_{k-1}$ and $\mu_2 = \alpha_{k+1} + \dots + \alpha_j$. This formula is an elementary corollary of the definition of f_μ . Further,

$$\begin{aligned} [e_{\alpha_i}, f_\mu] &= \left[\frac{q^{h_{\alpha_i}} - q^{-h_{\alpha_i}}}{q - q^{-1}}, f_{\mu'} \right]_q = -\frac{1}{q - q^{-1}} [q^{-h_{\alpha_i}}, f_{\mu'}]_q = f_{\mu'} q^{-h_{\alpha_i}}, \\ [e_{\alpha_j}, f_\mu] &= \left[f_{\mu''}, \frac{q^{h_{\alpha_j}} - q^{-h_{\alpha_j}}}{q - q^{-1}} \right]_q = \frac{1}{q - q^{-1}} [f_{\mu''}, q^{h_{\alpha_j}}]_q = \frac{(1 - q^2)}{q - q^{-1}} f_{\mu''} q^{h_{\alpha_j}} = -q f_{\mu''} q^{h_{\alpha_j}}, \end{aligned}$$

as required. \square

In particular, a positive Chevalley generator e_α commutes with f_μ , $\text{ht}(\mu) > 1$, unless $\mu - \alpha$ is a root. If $\mu - \alpha$ is a root, then α is either the smallest or the greatest simple root in μ (under the natural ordering of $\{\alpha_i\}_{i=1}^{n-1}$). The commutator $[e_\alpha, f_\mu]$ includes the factor $f_{\mu-\alpha}$ in both cases, but we find it convenient to keep reference to the position of α within μ using notation μ' and μ'' .

Lemma 4.2. *Suppose $\mu = \alpha_i + \dots + \alpha_j$ and g is a monomial (word) in the simple root vectors $\{f_{\alpha_k}\}_{k=i}^j$ that contains f_{α_i} and f_{α_j} at most once. Then*

1. $[g, f_\mu] = 0$ if both f_{α_i} and f_{α_j} enter g or none,
2. $[g, f_\mu]_{q^{-1}} = 0$ if g contains only f_{α_i} ,
3. $[g, f_\mu]_q = 0$ if g contains only f_{α_j} .

In particular, $[\tilde{f}_\mu, f_\mu] = 0$, $[\tilde{f}_{\mu'}, f_\mu]_q = 0$, and $[\tilde{f}_{\mu''}, f_\mu]_{q^{-1}} = 0$.

Proof. It is known that f_{α_k} commutes with \tilde{f}_γ if $i < k < j$, see e.g. [21]. Further, the "higher order Serre relations"

$$\begin{aligned} f_{\alpha_i} f_\mu &= f_{\alpha_i} [f_{\alpha_i}, f_{\mu'}]_q = q^{-1} [f_{\alpha_i}, f_{\mu'}]_q f_{\alpha_i} = q^{-1} f_\mu f_{\alpha_i}, \\ f_{\alpha_j} f_\mu &= f_{\alpha_j} [f_{\mu''}, f_{\alpha_j}]_q = q [f_{\mu''}, f_{\alpha_j}]_q f_{\alpha_j} = q f_\mu f_{\alpha_j}. \end{aligned}$$

in $U_h(\mathfrak{gl}(n))$ readily imply the statement. \square

5 Generalized Verma module M_λ

We need to set up a few conventions about representations of quantum groups. We assume that they are free modules over the ring of scalars, and their rank will be referred to as

dimension. We call a $U_{\hbar}(\mathfrak{g})$ -module irreducible if it is irreducible over $U(\mathfrak{g})$ in the classical limit. As $U_q(\mathfrak{g})$ and $U_{\hbar}(\mathfrak{g})$ have different Cartan subalgebras, their sets of weights are different. Still we prefer to use additive language for $U_q(\mathfrak{g})$ -weights, which are then parameterized by the assignment $\lambda \mapsto q^\lambda$.

We shall be dealing with weight modules over $U_{\hbar}(\mathfrak{g})$ generated by a weight vector v annihilated by the positive Chevalley generators. Under our convention, all weights in such modules belong to $-\mathbb{Z}\Pi^+ + \lambda$, where $\lambda \in \mathfrak{h}^*[[\hbar]]$ is the highest weight supported by v . However, such representations are not sufficient for us, because they yield $\lim_{q \rightarrow 1} q^{2\lambda} = 1$. In our further constructions, $\lim_{q \rightarrow 1} \text{diag}(q^{2(\lambda, \varepsilon_1)}, \dots, q^{2(\lambda, \varepsilon_n)}, q^{-2(\lambda, \varepsilon_n)}, \dots, q^{-2(\lambda, \varepsilon_1)}) \in G$ is the initial point of the conjugacy class under study. Therefore, we should include highest weights from $\hbar^{-1}\mathfrak{h}^*[[\hbar]]$. These are actually weights of $U_q(\mathfrak{g})$ as they are well defined on $q^{\pm h\alpha_i} \in U_q(\mathfrak{g})$. It is sufficient for our purposes to restrict λ to $\hbar^{-1}\mathfrak{h}^* \oplus \mathfrak{h}^* \subset \hbar^{-1}\mathfrak{h}^*[[\hbar]]$.

The presence of the \hbar -irregular term in $\hbar^{-1}\mathfrak{h}^* \oplus \mathfrak{h}^*$ implies that the representation of $U_q(\mathfrak{g})$ on such a module, V , cannot be extended to a representation of $U_{\hbar}(\mathfrak{g})$. We have to extend V over the field of Laurent series $\mathbb{C}((\hbar))$ (bounded from below), where the action of $U_{\hbar}(\mathfrak{g})$ can be defined. The corresponding extension $\text{End}(V)((\hbar))$ contains $\text{End}(V)$ as a $\mathbb{C}[[\hbar]]$ -submodule. It turns out that the locally finite part of $\text{End}(V)$, which is still a $\mathbb{C}[[\hbar]]$ -submodule, does admit a $U_{\hbar}(\mathfrak{g})$ -action, cf. Section 9.

Let $L \subset K$ denote the Levi subgroup

$$L = GL(n_1) \times \dots \times GL(n_\ell) \times GL(m) \times SP(2p)$$

for K as in (2.5). The difference between L and K is only one Cartesian factor $GL(m) \subset SP(2m)$. By \mathfrak{l} we denote the Lie algebra of L . It is a reductive subalgebra in \mathfrak{g} of maximal rank n .

By $\mathfrak{c}_{\mathfrak{l}} \subset \mathfrak{h}$ we denote the centre of \mathfrak{l} . In the presence of the canonical inner product on \mathfrak{h}^* , we identify its dual $\mathfrak{c}_{\mathfrak{l}}^*$ with the subspace in \mathfrak{h}^* which is orthogonal to the annihilator of $\mathfrak{c}_{\mathfrak{l}}$. Any element $\lambda \in \hbar^{-1}\mathfrak{c}_{\mathfrak{l}}^* \oplus \mathfrak{c}_{\mathfrak{l}}^*$ defines a one-dimensional representation of $U_q(\mathfrak{l})$ denoted by \mathbb{C}_λ . It assigns nil to the Chevalley generators of $U_q(\mathfrak{l})$ and acts by $q^{h\alpha} \mapsto q^{(\lambda, \alpha)}$ on the Cartan subalgebra. This representation extends to the parabolic subalgebra $U_q(\mathfrak{p}^+)$ as trivial on $U_q(\mathfrak{b}^+) \subset U_q(\mathfrak{p}^+)$. Denote by $\hat{M}_\lambda = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p}^+)} \mathbb{C}_\lambda$ the parabolic Verma $U_q(\mathfrak{g})$ -module induced from \mathbb{C}_λ . It is generated by the highest weight vector, which we denote by v . We impose the condition on λ that \hat{M}_λ admits a singular vector of weight $-\delta + \lambda$, where $\delta = 2\alpha_{n-p} + \dots + 2\alpha_{n-1} + \beta \in R^+$. Such a vector generates a submodule $\hat{M}_{\lambda-\delta} \subset \hat{M}_\lambda$. The quotient module $\hat{M}_\lambda / \hat{M}_{\lambda-\delta}$ is the subject of our interest.

We consider $\hat{M}_\lambda/\hat{M}_{\lambda-\delta}$ as a candidate in order to realize the quantization of G/K in $\text{End}(\hat{M}_\lambda/\hat{M}_{\lambda-\delta})$, by the analogy with the Levi class G/L . The tangent space to G/L at the initial point is naturally identified with $\mathfrak{n}_\mathfrak{l}^+ \oplus \mathfrak{n}_\mathfrak{l}^-$. The locally finite part of $\text{End}(\hat{M}_\lambda)$ is isomorphic to $\hat{M}_\lambda^* \otimes \hat{M}_\lambda$, where \hat{M}_λ^* is the dual parabolic Verma module $U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p}^-)} \mathbb{C}_{-\lambda}$. For generic λ , a quantization of G/L can be realized in $\text{End}(\hat{M}_\lambda)$, [1]. As a vector space, $\text{End}(\hat{M}_\lambda)$ is isomorphic to $U_h(\mathfrak{n}_\mathfrak{l}^+) \otimes U_h(\mathfrak{n}_\mathfrak{l}^-)$, and the functional dimension of \hat{M}_λ is equal to $\dim \mathfrak{n}_\mathfrak{l}^- = \frac{1}{2} \dim G/L$. The tangent space at the initial point of G/K is transversal to $\mathfrak{m}^- \oplus \mathfrak{m}^+$, where $\mathfrak{m}^- = \text{ad}(\mathfrak{l})(f_\delta)$ and $\mathfrak{m}^+ = \text{ad}(\mathfrak{l})(e_\delta)$. It is presentable as $\mathfrak{n}_\mathfrak{k}^+ \oplus \mathfrak{n}_\mathfrak{k}^-$ on setting $\mathfrak{n}_\mathfrak{k}^\pm = \mathfrak{n}_\mathfrak{l}^\pm \ominus \mathfrak{m}^\pm$. A module that supports quantization of G/K should have the functional dimension $\dim \mathfrak{n}_\mathfrak{k}^- = \frac{1}{2} \dim G/K$. Such is the quotient $\hat{M}_\lambda/\hat{M}_{\lambda-\delta}$, where the vector f_δ vanishes along with a q -version of \mathfrak{m}^- , cf. Section 6.

For the sake of technical convenience, we assume that $\ell = 0$, $m = 1$, $n = 1 + p$. This restriction will be relaxed later on. In this setting, the root α_1 is distinguished, as f_{α_1} is the only negative Chevalley generator which does not belong to $U_q(\mathfrak{l})$ and does not kill the highest weight vector $v \in \hat{M}_\lambda$.

Lemma 5.1. *For all $i = 1, \dots, p$ and $k = 2, \dots, p+1$, $k \neq i+1$, the vector $f_{\alpha_k} f_{\alpha_i} \cdot \triangleright \cdot f_{\alpha_1} v \in \hat{M}_\lambda$ is nil.*

Proof. We prove it by induction on i . The case $i = 1$ is obvious, as f_{α_k} commutes with f_{α_1} for all $k = 3, \dots, p+1$. Suppose we have done it for all $i = 1, \dots, l-1$ and let us prove it for $i = l$. Consider the following three cases first.

- a) $k \geq l+2$. The generator f_{α_k} can be freely pushed to the right until it kills v .
- b) $k = l$. Plugging $f_{\alpha_l}^2 f_{\alpha_{l-1}} = (q + q^{-1}) f_{\alpha_l} f_{\alpha_{l-1}} f_{\alpha_l} - f_{\alpha_{l-1}} f_{\alpha_l}^2$ into $f_{\alpha_l} f_{\alpha_l} \cdot \triangleright \cdot f_{\alpha_1} v$ moves at least one copy of f_{α_l} to the right of $f_{\alpha_{l-1}}$. This proves b) by reducing it to a) for $i = l-2$ and applying the induction assumption.
- c) $k = l-1$. Plug $f_{\alpha_{l-1}} f_{\alpha_l} f_{\alpha_{l-1}} = \frac{1}{q+q^{-1}} (f_{\alpha_{l-1}}^2 f_{\alpha_l} + f_{\alpha_l} f_{\alpha_{l-1}}^2)$ into $f_{\alpha_{l-1}} f_{\alpha_l} f_{\alpha_{l-1}} \cdot \triangleright \cdot f_{\alpha_1} v$. The first summand falls into a) for $i = l-2$, and the second into the b) for $i = l-1$. They both vanish by the induction assumption.

Finally, if $k < l-1$, then f_{α_k} is pushed to the right till it meets $f_{\alpha_{k+1}}$. This falls into the case c) for $i = k+1$ and vanishes by the induction assumption. \square

We apply Lemma 5.1 to analyze the structure of certain weight subspaces in \hat{M}_λ .

Lemma 5.2. Put $\delta' = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_p + \beta$. The subspace of weight $-\delta' + \lambda$ in \hat{M}_λ is spanned by the vector $f_{\alpha_2} \cdot \prec \cdot f_{\alpha_p} f_\beta f_{\alpha_p} \cdot \succ \cdot f_{\alpha_2} f_{\alpha_1} v$.

Proof. The subspace of weight $-\delta' + \lambda$ is spanned by Chevalley monomials applied to the highest weight vector v . They are products of f_{α_1}, f_β , and two copies of $f_{\alpha_i}, i = 2, \dots, p$, each. Every monomial must have the rightmost factor f_{α_1} as it is the only generator that does not kill v . By Lemma 5.1 all the monomials should have the factor $f_\beta f_{\alpha_p} \cdot \succ \cdot f_{\alpha_1}$ on the right. We conclude that vectors of weight $-\delta' + \lambda$ should be combinations of $\phi_\sigma = g_\sigma f_\beta f_{\alpha_p} \cdot \succ \cdot f_{\alpha_2} f_{\alpha_1} v$ with $g_\sigma = \sigma(f_{\alpha_2} \cdot \prec \cdot f_{\alpha_p})$, where σ is a permutation of the factors. Suppose $g_\sigma = \dots f_{\alpha_i} f_{\alpha_k} \cdot \prec \cdot f_{\alpha_p}$ for some $k = 2, \dots, p+1$ and $i < k-1$ (we assume formally that f_{α_i} is in the rightmost position in g_σ if $k = p+1$). Then f_{α_i} can be pushed through to the right of f_β , and this falls under Lemma 5.1. Thus, the factors in g_σ are all ordered as stated, and the permutation σ is identical.

Finally, the vector of concern is not zero. Indeed, the subspace of weight $-\delta' + \lambda$ in \hat{M}_λ has the same dimension as the subspace of weight $-\delta'$ in $U(\mathfrak{n}_1^-)$, which is exactly 1, due to the Poincaré-Birkhoff-Witt (PBW) basis in $U(\mathfrak{n}_1^-)$. \square

Put $\gamma = \alpha_1 + \dots + \alpha_p, \delta = 2\gamma + \beta$, and introduce the vector

$$f_\delta = [f_\gamma, [\tilde{f}_\gamma, f_\beta]_{q^{-2}}]_{q^2} = [\tilde{f}_\gamma, [f_\gamma, f_\beta]_{q^2}]_{q^{-2}}, \quad (5.10)$$

Remark that the right equality holds by virtue of Lemma 4.2. Note that f_δ is not a member of the standard PBW basis associated with a reduced decomposition of the longest Weyl group element, see Section 6 for more details.

Lemma 5.3. The vector f_δ is presentable in the form

$$[f_{\alpha_1}, [f_{\alpha_2}, \dots [f_{\alpha_p}, [f_{\alpha_1}, \dots [f_{\alpha_{p-1}}, [f_{\alpha_p}, f_\beta]_{q^2}]_q \dots]_q]_{q^{-1}} \dots]_{q^{-1}}]_{q^{-2}}.$$

Proof. First of all, remark that p internal commutators amount to $[f_\gamma, f_\beta]_{q^2}$. Further, fix $i = 2, \dots, p+1$ and define the root ν from the equality $\gamma = \nu + \alpha_i + \dots + \alpha_p$ (ν is simply γ if $i = p+1$). Suppose we have proved that f_δ is presentable in the form $[[f_\nu, f_{\alpha_i}]_{q^{-1}}, z]_{q^{-2}}$, where $z = [f_{\alpha_{i+1}}, \dots [f_{\alpha_p}, [f_\gamma, f_\beta]_{q^2}]_{q^{-1}} \dots]_{q^{-1}}$. In particular, this is true for $i = p+1$. The vector f_ν commutes with everything in z but f_γ . All the Chevalley generators in f_ν except from f_{α_1} commute with f_γ , and f_{α_1} enters f_ν exactly once. Applying Lemma 4.2, we conclude that $[f_\nu, z]_{q^{-1}} = 0$. Using the "Jacobi identity"

$$[x, [y, z]_a]_b = [[x, y]_c, z]_{\frac{ab}{c}} + c[y, [x, z]_{\frac{b}{c}}]_{\frac{a}{c}}, \quad (5.11)$$

which holds true in any associative algebra with scalar a, b , and invertible c , we write

$$[f_\nu, [f_{\alpha_i}, z]_{q^{-1}}]_{q^{-2}} = [[f_\nu, f_{\alpha_i}]_{q^{-1}}, z]_{q^{-2}} + q^{-1}[[f_\nu, f_{\alpha_i}]_{q^{-1}}, z],$$

for $a = c = q^{-1}$, $b = q^{-2}$. The second term vanishes, and we come to the equality $[[f_\nu, f_{\alpha_i}]_{q^{-1}}, z]_{q^{-2}} = [f_\nu, [f_{\alpha_i}, z]_{q^{-1}}]_{q^{-2}}$. Descending induction on $i = p + 1, \dots, 2$ completes the proof. \square

Now we lift the assumption $\ell = 0$, $m = 1$ and work out the case of general \mathfrak{k} and \mathfrak{l} :

$$\mathfrak{k} = \mathfrak{gl}(n_1) \oplus \dots \oplus \mathfrak{gl}(n_\ell) \oplus \mathfrak{sp}(2m) \oplus \mathfrak{sp}(2p), \quad (5.12)$$

$$\mathfrak{l} = \mathfrak{gl}(n_1) \oplus \dots \oplus \mathfrak{gl}(n_\ell) \oplus \mathfrak{gl}(m) \oplus \mathfrak{sp}(2p). \quad (5.13)$$

Let $\mathfrak{g}' = \mathfrak{sp}(2 + 2p)$ be the Lie subalgebra in \mathfrak{g} corresponding to the positive simple roots $(\alpha_{n-p}, \dots, \alpha_n)$. The vectors $f_\gamma, \tilde{f}_\gamma, f_\delta \in U_q(\mathfrak{g}')$ are carried over to $U_q(\mathfrak{g})$, where we use the same notation for them. This relates the case $\ell = 0$, $m = 1$ to the general setting. The root α_{n-p} plays the same role as α_1 in the symmetric case with $m = 1$. We will use the notation α for it in order to emphasize the general meaning of formulas with it.

Proposition 5.4. *Suppose that $q^{2(\lambda, \alpha)} = -q^{-2p}$. Then $f_\delta v$ is a singular vector in \hat{M}_λ .*

Proof. At first, we return to the symmetric case $\ell = 0$ with $m = 1$. Furthermore, as the case $p = 1$ has been studied in [2], we assume $p > 1$.

Applying e_β to $f_\delta v$ we obtain, up to a non-zero scalar factor,

$$[f_\gamma, [\tilde{f}_\gamma, q^{h_\beta} - q^{-h_\beta}]_{q^{-2}}]_{q^2} \sim [f_\gamma, [\tilde{f}_\gamma, q^{-h_\beta}]_{q^{-2}}]_{q^2} \sim [f_\gamma, \tilde{f}_\gamma q^{-h_\beta}]_{q^2} = [f_\gamma, \tilde{f}_\gamma q^{-h_\beta}]_{q^2} = [f_\gamma, \tilde{f}_\gamma]_{q^2} q^{-h_\beta}.$$

The last commutator is zero, by Lemma 4.2.

If $1 < i < n - 1$, then e_{α_i} commutes with $f_\gamma, \tilde{f}_\gamma$, and hence with $[f_\gamma, [\tilde{f}_\gamma, f_\beta]_{q^{-2}}]_{q^2}$, by Lemma 4.1. Therefore $e_{\alpha_i} f_\delta v = 0$ for such i . Thus, we only need to check that $f_\delta v$ is annihilated by $e_\alpha = e_{\alpha_1}$ and $e_{\alpha_p} = e_{\alpha_{n-1}}$.

Using the formulas $[e_{\alpha_p}, f_\gamma] = -q f_{\gamma''} q^{h_{\alpha_p}}$, $[e_{\alpha_p}, \tilde{f}_\gamma] = -q^{-1} \tilde{f}_{\gamma''} q^{-h_{\alpha_p}}$ from Lemma 4.1 we get for $[e_{\alpha_p}, f_\delta]$ an expression, which is proportional to

$$q[f_{\gamma''} q^{h_{\alpha_p}}, [\tilde{f}_\gamma, f_\beta]_{q^{-2}}]_{q^2} + q^{-1}[f_\gamma, [\tilde{f}_{\gamma''} q^{-h_{\alpha_p}}, f_\beta]_{q^{-2}}]_{q^2}.$$

The second term vanishes, because $[\tilde{f}_{\gamma''} q^{-h_{\alpha_p}}, f_\beta]_{q^{-2}} \sim [\tilde{f}_{\gamma''}, f_\beta]_{q^{-2}} q^{-h_{\alpha_p}} = 0$. Let us check that the first term is nil too.

For $p = 2$, the first term is proportional to

$$[f_{\alpha_{p-1}}q^{h_{\alpha_p}}, [\tilde{f}_\gamma, f_\beta]_{q^{-2}}]_{q^2} = q[f_{\alpha_{p-1}}, [\tilde{f}_\gamma, f_\beta]_{q^{-2}}]_q q^{h_{\alpha_p}} = q[[f_{\alpha_{p-1}}, \tilde{f}_\gamma]_q, f_\beta]_{q^{-2}} q^{h_{\alpha_p}} = 0,$$

as $[f_{\alpha_{p-1}}, \tilde{f}_\gamma]_q = [f_{\alpha_1}, \tilde{f}_\gamma]_q = 0$ by Lemma 4.2.

If $p > 2$, we present f_δ as $f_\delta = [\tilde{f}_\nu, [f_{\alpha_{p-1}}, [f_{\alpha_p}, [f_\gamma, f_\beta]_{q^2}]_{q^{-1}}]_{q^{-2}}$, where $\nu = \gamma - \alpha_{p-1} - \alpha_p$ (we did it in the proof of Lemma 5.3). This way we unveil $f_{\alpha_{p-1}}$ hidden in \tilde{f}_γ . Although it commutes with e_{α_p} , it does not commute with $q^{\pm h_{\alpha_p}}$ arising from $[e_{\alpha_p}, f_{\alpha_p}]$. Observe that $[e_{\alpha_p}, [f_\gamma, f_\beta]_{q^2}] \sim [f_{\gamma''}q^{h_{\alpha_p}}, f_\beta]_{q^2} \sim [f_{\gamma''}, f_\beta]_{q^2} = 0$. Then $[e_{\alpha_p}, f_\delta]$ is proportional to

$$[\tilde{f}_\nu, [f_{\alpha_{p-1}}, [q^{h_{\alpha_p}} - q^{-h_{\alpha_p}}, [f_\gamma, f_\beta]_{q^2}]_{q^{-1}}]_{q^{-2}} = [\tilde{f}_\nu, [f_{\alpha_{p-1}}, [q^{h_{\alpha_p}}, [f_\gamma, f_\beta]_{q^2}]_{q^{-1}}]_{q^{-2}} = 0,$$

because $[f_{\alpha_{p-1}}, [q^{h_{\alpha_p}}, [f_\gamma, f_\beta]_{q^2}]_{q^{-1}} \sim [[f_{\alpha_{p-1}}, f_\gamma], f_\beta]_{q^2} q^{h_{\alpha_p}} = 0$, by Lemma 4.2.

We have shown that $f_\delta v$ is annihilated by $e_{\alpha_i} \in U_q(\mathfrak{sp}(2p)) \subset U_q(\mathfrak{g})$. Next we check that it is killed by $e_{\alpha_1} = e_\alpha$. Based on Lemma 4.1, we find that $e_\alpha f_\delta v = [e_\alpha, f_\delta]v$ is equal to

$$\begin{aligned} & [f_{\gamma'}q^{-h_\alpha}, [\tilde{f}_\gamma, f_\beta]_{q^{-2}}]_{q^2} v + [f_{\gamma'}, [\tilde{f}_{\gamma'}q^{h_\alpha}, f_\beta]_{q^{-2}}]_{q^2} v = \\ & = q^{1-(\alpha, \lambda)} [f_{\gamma'}, [\tilde{f}_\gamma, f_\beta]_{q^{-2}}]_q v + q^{(\alpha, \lambda)} [f_{\gamma'}, [\tilde{f}_{\gamma'}, f_\beta]_{q^{-2}}]_q v. \end{aligned}$$

Using Lemma 5.2, we develop the commutators in $[e_\alpha, f_\delta]v$ and find it proportional to

$$(q^{-(\alpha, \lambda)-p} + q^{(\alpha, \lambda)+p}) f_{\alpha_2} \cdot \prec \cdot f_{\alpha_p} f_\beta f_{\alpha_p} \cdot \succ \cdot f_{\alpha_2} f_\alpha v.$$

It turns zero if and only if $q^{2(\alpha, \lambda)} = -q^{-2p}$. This completes the proof for $\ell = 0$, $m = 1$.

The vector $f_\delta v \in \hat{M}_\lambda$ has been shown to be singular with respect to the subalgebra $U_q(\mathfrak{g}')$, where $\mathfrak{g}' = \mathfrak{sp}(2+2p)$ is defined above. Therefore $f_\delta v$ is singular with respect to entire $U_q(\mathfrak{g})$, as $f_\delta \in U_q(\mathfrak{g}')$ commutes with simple root vectors $e_\mu \neq U_q(\mathfrak{g}')$. \square

The following statement presents $f_\delta v$ as a linear combination of Chevalley monomials.

Proposition 5.5. *The vector $f_\delta v$ is a linear combination of the monomials*

$$f_{\alpha_i} \cdot \prec \cdot f_{\alpha_{n-1}} f_\beta f_{\alpha_{i-1}} \cdot \succ \cdot f_{\alpha_{n-p}} f_{\alpha_{n-1}} \cdot \succ \cdot f_{\alpha_{n-p}} v, \quad i = n - p + 1, \dots, n.$$

Proof. Follows from Lemma 5.1 applied to $f_\delta = f_\gamma \tilde{f}_\gamma f_\beta - q^{-2} f_\gamma f_\beta \tilde{f}_\gamma - q^2 \tilde{f}_\gamma f_\beta f_\gamma + f_\beta f_\gamma \tilde{f}_\gamma$. \square

From now on, saying the vector $f_\delta v \in \hat{M}_\lambda$ is singular, we assume that the weight λ satisfies the hypothesis of Proposition 5.4.

We define certain weight subspaces in order to formalize further exposition. For all $k = 1, \dots, \ell + 2$ let ν_k be an element of the standard orthogonal basis $\{\varepsilon_i\}_{i=1}^n \subset \mathfrak{h}^*$ which is

not vanishing on the Cartan subalgebra of the i -th block in (5.13), counting from the left. Put $\mu_k^0 = e^{2(\eta, \nu_k)}$ assuming $\eta \in \mathfrak{c}_\ell^*$. This definition is independent of the choice of ν_k : for any other ν_k , call it ν'_k , the difference $\nu_k - \nu'_k$ is a root of \mathfrak{l} and orthogonal to $\eta \in \mathfrak{c}_\ell^*$. In particular, one can take $\nu_{\ell+2} = \varepsilon_n = \frac{1}{2}\beta$. Since \mathfrak{c}_ℓ^* is orthogonal to β , one has $\mu_{\ell+2}^0 = 1$.

Let $\mathfrak{c}_{\ell,reg}^*$ denote the set of all weights $\eta \in \mathfrak{c}_\ell^*$ such that $\mu_k^0 \neq (\mu_j^0)^{\pm 1}$ for $k \neq j$. Denote by \mathfrak{c}_ℓ^* the subset in \mathfrak{c}_ℓ^* such that $\mu_{\ell+1}^0 = -1$ and by $\mathfrak{c}_{\ell,reg}^*$ the subspace of \mathfrak{c}_ℓ^* such that $\mu_k^0 \neq (\mu_j^0)^{\pm 1}$ for $k, j = 1, \dots, \ell + 2$ and $k \neq j$. Clearly $\mathfrak{c}_{\ell,reg}^*$ is dense in \mathfrak{c}_ℓ^* , being complementary to a discrete family of hyperplanes. Remark that the vector $\boldsymbol{\mu}^0 = (\mu_i^0)$ belongs to $\hat{\mathcal{M}}_K$, which parameterizes the moduli space \mathcal{M}_K of classes with fixed K . All elements of $\hat{\mathcal{M}}_K$ can be obtained this way.

The sets \mathfrak{c}_ℓ^* and $\mathfrak{c}_{\ell,reg}^*$ are explicitly described as follows. Introduce $\ell + 2$ weights $\mathcal{E}_i \in \mathfrak{h}^*$:

$$\mathcal{E}_1 = \varepsilon_1 + \dots + \varepsilon_{n_1}, \quad \mathcal{E}_2 = \varepsilon_{n_1+1} + \dots + \varepsilon_{n_1+n_2}, \quad \dots, \quad \mathcal{E}_{\ell+2} = \varepsilon_{n-p+1} + \dots + \varepsilon_n,$$

so that $(\mathcal{E}_i, \nu_k) = \delta_{ik}$. Then \mathfrak{c}_ℓ^* is formed by the combinations $\sum_{i=1}^{\ell+1} \Lambda_i \mathcal{E}_i$ with arbitrary complex coefficients Λ_i . The subset $\mathfrak{c}_\ell^* \subset \mathfrak{c}_\ell^*$ is characterized by the condition $\Lambda_{\ell+1} = \frac{\sqrt{-1}\pi}{2} + \sqrt{-1}\mathbb{Z}\pi$. The subsets $\mathfrak{c}_{\ell,reg}^* \subset \mathfrak{c}_\ell^*$ and $\mathfrak{c}_{\ell,reg}^* \subset \mathfrak{c}_\ell^*$ are specified by $\Lambda_k \notin \sqrt{-1}\pi\mathbb{Z}$ and $\Lambda_k \pm \Lambda_j \notin \sqrt{-1}\pi\mathbb{Z}$, for $k, j = 1, \dots, \ell + 1$, $k \neq j$.

Finally, we introduce two subsets $\mathfrak{C}_{\ell,reg}^* \subset \mathfrak{C}_\ell^*$ in $\frac{1}{\hbar}\mathfrak{c}_\ell^* \oplus \mathfrak{c}_\ell^*$ by $\mathfrak{C}_\ell^* = \hbar^{-1}\mathfrak{c}_\ell^* - p\mathcal{E}_{\ell+1}$ and $\mathfrak{C}_{\ell,reg}^* = \hbar^{-1}\mathfrak{c}_{\ell,reg}^* - p\mathcal{E}_{\ell+1}$; obviously $\mathfrak{C}_{\ell,reg}^*$ is dense in \mathfrak{C}_ℓ^* . By construction, all weights from \mathfrak{C}_ℓ^* satisfy $q^{2(\alpha, \lambda)} = -q^{-2p}$.

Definition 5.6. Assuming $\lambda \in \mathfrak{C}_\ell^*$, we denote by $\hat{M}_{\lambda-\delta} \subset \hat{M}_\lambda$ the submodule generated by $f_\delta v$ and we denote by M_λ the quotient module $\hat{M}_\lambda / \hat{M}_{\lambda-\delta}$.

The module M_λ is the subject of our further study.

6 On a basis in M_λ

In this section we prove that the quotient module M_λ is free over $\mathbb{C}[[\hbar]]$ and construct a PBW basis for it. To that end, we need a version of the PBW theorem for the nilpotent subalgebra $U_\hbar(\mathfrak{g}_-) \subset U_\hbar(\mathfrak{g})$. Our consideration is based on the standard PBW theorem for $U_\hbar(\mathfrak{g})$ with the advantage of working over $\mathbb{C}[[\hbar]]$, which makes us more flexible in the choice of basis.

Recall that the quantum version of the higher root vectors in \mathfrak{g} can be associated with a reduced decomposition of the longest element w_0 of the Weyl group of \mathfrak{g} , [22]. Every such decomposition gives rise to an ordered set of root vectors f_i , $i = 1, \dots, r = |R^+|$, generating a

PBW basis in $U_q(\mathfrak{g}_-)$. By construction, higher root vectors are expressed through deformed commutators of the Chevalley generators. By a deformed commutator, we mean $[x, y]_a$, with $a \in \mathbb{C}[[\hbar]]$ satisfying $a = 1 \pmod{\hbar}$.

We argue that, over $\mathbb{C}[[\hbar]]$, we can a) arbitrarily change the order of the f_i 's and b) redefine f_i in an appropriate although rather general way. Namely, for $\mu \in R^+$ the root vector f_μ is a combination $\sum_l a_l \phi_l$, where ϕ_l is a monomial in $\{f_\alpha\}_{\alpha \in \Pi^+}$ and $a_l \in \mathbb{C}[[\hbar]]$ is a scalar. Let \hat{f}_μ be an element of $U_\hbar(\mathfrak{g}_-)$ obtained from f_μ by replacing a_l with their deformations $\hat{a}_l \in \mathbb{C}[[\hbar]]$, i.e. with any $\hat{a}_l = a_l \pmod{\hbar}$. Note that a_l are themselves deformations of the classical coefficients participating in the classical counterpart of f_μ . Moreover, if the classical prototype of f_μ is a combination of commutators in the Chevalley generators, we can replace them with any deformed commutators. This way we get rid of the sequence of deformed commutators prescribed by the reduced decomposition of w_0 . Then \hat{f}_μ is still a deformation of f_μ , i.e. $\hat{f}_\mu = f_\mu \pmod{\hbar}$, and the omitted terms have the same weight μ .

Consider, for instance, the case of $\mathfrak{sp}(4)$ treated in [2]. Let α, β be the short and long simple roots, respectively. There are two more positive roots $\gamma = \alpha + \beta$ and $\delta = 2\alpha + \beta$. The classical root vectors are $f_\gamma = [f_\alpha, f_\beta]$, $f_\delta = [f_\alpha, [f_\alpha, f_\beta]]$, and their standard q -counterparts associated with the decomposition $w_0 = s_\beta s_\alpha s_\beta s_\alpha$ (simple reflections) are $f_\gamma = [f_\alpha, f_\beta]_{q^2}$, $f_\delta = [f_\alpha, [f_\alpha, f_\beta]_{q^2}]$. Our version is $\hat{f}_\gamma = [f_\alpha, f_\beta]_{q^2}$, $\hat{f}_\delta = [f_\alpha, [f_\alpha, f_\beta]_{q^2}]_{q^{-2}}$, i. e. the external commutator in \hat{f}_δ is deformed as compared to f_δ . In terms of Chevalley monomials, the δ -root vectors read

$$f_\delta = f_\alpha^2 f_\beta - (q^2 + 1) f_\alpha f_\beta f_\alpha + q^2 f_\beta f_\alpha^2, \quad \hat{f}_\delta = f_\alpha^2 f_\beta - (q^2 + q^{-2}) f_\alpha f_\beta f_\alpha + f_\beta f_\alpha^2.$$

The coefficients in \hat{f}_δ are deformation of the coefficients in f_δ . In the standard PBW basis corresponding to the ordering $f_\beta, f_\gamma, f_\delta, f_\alpha$, one has $\hat{f}_\delta = f_\delta + (1 - q^{-2}) f_\gamma f_\alpha = f_\delta \pmod{\hbar}$.

We fix an arbitrary order on the set $\{\hat{f}_i\}$. Let $\mathbf{k} = (k_1, \dots, k_r)$ denote a multivector with non-negative integer components.

Proposition 6.1. *The monomials $\hat{f}_{\mathbf{k}} = \hat{f}_1^{k_1} \dots \hat{f}_r^{k_r} \in U_\hbar(\mathfrak{g}_-)$ form a basis.*

Proof. First suppose that $(\hat{f}_i) = (f_i)$ is the standard ordered system of root vectors associated with a reduced decomposition of w_0 . This PBW basis establishes a $\mathbb{C}[[\hbar]]$ -linear isomorphism between $U_\hbar(\mathfrak{g}_-)$ and $U(\mathfrak{g}_-) \otimes \mathbb{C}[[\hbar]]$. It is also an isomorphism of \mathfrak{h}^* -graded spaces, as it preserves weights. This isomorphism makes the multiplication of $U_\hbar(\mathfrak{g}_-)$ a deformation of the multiplication of $U(\mathfrak{g}_-) \otimes \mathbb{C}[[\hbar]]$ (the trivial extension of the ring of scalars), and makes $U_\hbar(\mathfrak{g}_-)$ a deformation of the \mathfrak{h}^* -graded algebra $U(\mathfrak{g}_-)$. Fix a permutation σ of $1, \dots, r$ and

define $g_{\mathbf{k}} = f_{\sigma(1)}^{k_{\sigma(1)}} \dots f_{\sigma(r)}^{k_{\sigma(r)}}$ to be a PBW monomials in the reordered system of standard root vectors ($f_{\sigma(i)}$). Let us show that $\{g_{\mathbf{k}}\}$ form a basis over $\mathbb{C}[[\hbar]]$.

The linear operator $\Phi: f_{\mathbf{k}} \mapsto g_{\mathbf{k}} = \sum_{\mathbf{m}} \Phi_{\mathbf{k},\mathbf{m}} f_{\mathbf{m}}$ preserves the weight subspace, and we must check that it is invertible in every weight subspace. Modulo \hbar , Φ is relating two PBW bases in the classical universal enveloping algebra $U(\mathfrak{g}_-)$. It is invertible modulo \hbar , hence it is invertible in every weight subspace in $U_{\hbar}(\mathfrak{g}_-)$, which is a free finite $\mathbb{C}[[\hbar]]$ -module.

We have proved that we can arbitrarily change the order of the generators of the standard PBW basis, when working over $\mathbb{C}[[\hbar]]$. Further, we have $\hat{f}_i = f_i \pmod{\hbar}$ by construction, and hence $\hat{f}_{\mathbf{k}} = f_{\mathbf{k}} + \hbar \sum_{\mathbf{m}} \Psi_{\mathbf{k},\mathbf{m}} f_{\mathbf{m}}$ for some $\mathbb{C}[[\hbar]]$ -linear operator Ψ . Again, we restrict the consideration to the weight subspace containing $f_{\mathbf{k}}$. It is free and finite over $\mathbb{C}[[\hbar]]$, hence the operator $\text{id} + \hbar\Psi$ is invertible. This completes the proof. \square

Further on we drop the symbol $\hat{\cdot}$ from \hat{f}_i . Specifically, we define the higher root vectors as follows. At the first step, let all $\{f_{\mu}\}_{\mu \in R^+}$ be the standard root vectors. Next we redefine the root vectors of every semisimple block of $U_{\hbar}(\mathfrak{l})$ according to its Weyl group element and its decomposition. Finally, we construct a q -analog of classical $\mathfrak{m}^- = \text{ad}(\mathfrak{l})(f_{\delta})$ from f_{δ} defined in (5.10) through the operators $\text{ad}_{op}(f_{\mu}): U_{\hbar}(\mathfrak{g}) \rightarrow U_{\hbar}(\mathfrak{g})$, $\text{ad}_{op}(f_{\mu}): y \mapsto f_{\mu}^{(2)} y \gamma^{-1}(f_{\mu}^{(1)})$, where μ is a simple positive root of $\mathfrak{gl}(m) \subset \mathfrak{sp}(2m)$. To simplify the enumeration, we restrict to the symmetric case $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{sp}(2p)$.

All roots from $R_{\mathfrak{k}}^+ - R_{\mathfrak{l}}^+$ have the form $\mu = \sum_{j=i}^{k-1} \alpha_j + 2 \sum_{j=k}^{m-1} \alpha_k + \delta$, $1 \leq i \leq k \leq m$, where the left (resp. the right) sum is present only if $i < k$ (resp. $k < m$). We construct the root vector f_{μ} as

$$f_{\mu} = (\text{ad}_{op}(f_{\alpha_i}) \circ \dots \circ \text{ad}_{op}(f_{\alpha_{k-1}})) \circ (\text{ad}_{op}^2(f_{\alpha_k}) \circ \dots \circ \text{ad}_{op}^2(f_{\alpha_{m-1}}))(f_{\delta}).$$

It is a linear combination of Chevalley monomials with scalar coefficients, since $\text{ad}_{op}(f_{\alpha_i})x = f_{\alpha_i}x - q^{-(\alpha_i, \nu)} x f_{\alpha_i}$ for any element $x \in U_{\hbar}(\mathfrak{g})$ of weight ν and all $\alpha_i \in \Pi_+$. The coefficients coincide with classical modulo \hbar because the commutators are deformations of the classical ones. We denote by \mathfrak{m}^- the $\mathbb{C}[[\hbar]]$ -linear span of f_{μ} , $\mu \in R_{\mathfrak{k}}^+ - R_{\mathfrak{l}}^+$.

Let \mathfrak{l}_- denote the linear span of the negative root vectors of $U_{\hbar}(\mathfrak{l})$. We regard the module \hat{M}_{λ} as that over $U_{\hbar}(\mathfrak{g}_-)$. It is induced from the trivial representation of the subalgebra $U_{\hbar}(\mathfrak{l}_-)$ and isomorphic to the quotient by the left ideal $U_{\hbar}(\mathfrak{g}_-)\mathfrak{l}_-$. The $U_{\hbar}(\mathfrak{g}_-)$ -module M_{λ} is isomorphic to the quotient by left ideal $U_{\hbar}(\mathfrak{g}_-)(f_{\delta} \oplus \mathfrak{l}_-)$. This ideal is equal to $U_{\hbar}(\mathfrak{g}_-)(\mathfrak{m}^- \oplus \mathfrak{l}_-)$ since $f_{\delta} \in \mathfrak{m}^-$ and $\mathfrak{m}^- \subset \text{ad}_{op}(U_{\hbar}(\mathfrak{l}_-))f_{\delta} \subset U_{\hbar}(\mathfrak{g}_-)(f_{\delta} \oplus \mathfrak{l}_-)$. Now we can establish the main result of this section.

Proposition 6.2. *The module M_λ is free over $\mathbb{C}[[\hbar]]$.*

Proof. The module M_λ is isomorphic to the quotient of the left regular $U_\hbar(\mathfrak{g}_-)$ -module by the left ideal $U_\hbar(\mathfrak{g}_-)(\mathfrak{m}^- \oplus \mathfrak{L}_-)$, hence it is spanned by the BPW monomials with no root vectors from $\mathfrak{m}^- \oplus \mathfrak{L}_-$. Such monomials form a basis in M_λ . \square

7 Module $\mathbb{C}^{2n} \otimes M_\lambda$: the symmetric case

In this section we put $\ell = 0$ and work with the Levi subalgebra $U_q(\mathfrak{gl}(m)) \otimes U_q(\mathfrak{sp}(2p))$, $m + p = n$. In this setting, the distinguished root α is $\alpha_{n-p} = \alpha_m$. It is complementary to the Dynkin sub-diagram of \mathfrak{l} in the diagram of \mathfrak{g} .

Consider the natural vector representation of $U_q(\mathfrak{sp}(2n))$ in \mathbb{C}^{2n} and denote by π the homomorphism $U_q(\mathfrak{sp}(2n)) \rightarrow \text{End}(\mathbb{C}^{2n})$. Let $(w_i)_{i=1}^{2n} \subset \mathbb{C}^{2n}$ be the standard basis, whose elements carry the weights $(\varepsilon_1, \dots, \varepsilon_n, -\varepsilon_n, \dots, -\varepsilon_1)$. In this basis, the matrices $\pi(h_\mu)$, $\pi(e_\mu)$ and $\pi(f_\mu)$, $\mu \in \Pi_+$, are independent of q and are the same as in the classical representation of $U(\mathfrak{sp}(2n))$.

For generic weight $\lambda \in \mathfrak{C}_{\mathfrak{l}, reg}^*$, the tensor product $\mathbb{C}^{2n} \otimes \hat{M}_\lambda$ is the direct sum of three submodules of highest weights $\nu_1 = \varepsilon_1 + \lambda$, $\nu_2 = \varepsilon_{m+1} + \lambda$, $\nu_3 = \varepsilon_{n+p+1} + \lambda$, see. e.g. [1]. Let u_{ν_i} , $i = 1, 2, 3$, denote their generators, which are singular vectors in $\mathbb{C}^{2n} \otimes \hat{M}_\lambda$. We are going to prove the direct sum decomposition $\mathbb{C}^{2n} \otimes M_\lambda = M_1 \oplus M_2$, where M_i are the images of \hat{M}_i under the projection $\mathbb{C}^{2n} \otimes \hat{M}_\lambda \rightarrow \mathbb{C}^{2n} \otimes M_\lambda$. The submodule M_1 is generated by $u_{\nu_1} = w_1 \otimes v$, which is the only singular vector of weight ν_1 . We shall see that the submodule \hat{M}_3 is annihilated under the projection $\mathbb{C}^{2n} \otimes \hat{M}_\lambda \rightarrow \mathbb{C}^{2n} \otimes M_\lambda$ (in fact, u_{ν_3} degenerates to $w_1 \otimes f_\delta v$ once $\lambda \in \mathfrak{C}_{\mathfrak{f}}^*$ and vanishes in $\mathbb{C}^{2n} \otimes M_\lambda$, see [2] for the special case of $\mathfrak{g} = \mathfrak{sp}(4)$).

Lemma 7.1. *The vector*

$$u_{\nu_2} = \frac{q^{(\alpha, \lambda)} - q^{-(\alpha, \lambda)}}{q - q^{-1}} w_{m+1} \otimes v - q^{-1} w_m \otimes f_{\alpha_m} v + \dots + (-q)^{-m} w_1 \otimes f_{\alpha_1} \cdot \prec \cdot f_{\alpha_m} v \quad (7.14)$$

of weight $\nu_2 = \varepsilon_{m+1} + \lambda$ is singular.

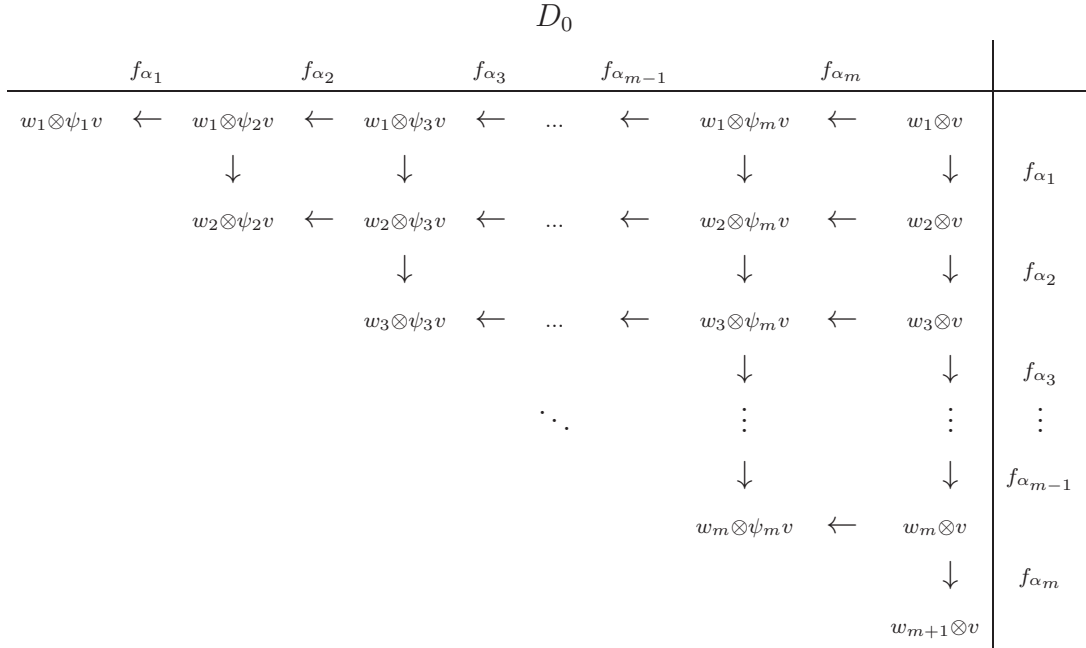
Proof. A straightforward calculation that (7.14) is annihilated by all e_μ , $\mu \in \Pi^+$. \square

Further we develop a diagram technique which will help us study the module $\mathbb{C}^{2n} \otimes M_\lambda$. Introduce the monomials $\psi_i = f_{\alpha_i} \cdot \prec \cdot f_{\alpha_m} \in U_q(\mathfrak{g}_-)$, $i = 1, \dots, m$, and write

$$u_{\nu_2} = \frac{q^{(\alpha, \lambda)} - q^{-(\alpha, \lambda)}}{q - q^{-1}} w_{m+1} \otimes v + \sum_{i=1}^m (-q)^{i-m-1} w_i \otimes \psi_i v.$$

When restricted to the Levi subalgebra $\mathfrak{l} = \mathfrak{gl}(m) \oplus \mathfrak{sp}(2p)$, the natural representation of \mathfrak{g} on \mathbb{C}^{2n} splits into three irreducible sub-representations, $\mathbb{C}^{2n} = \mathbb{C}^m \oplus \mathbb{C}^{2p} \oplus \mathbb{C}^m$. The block $\mathfrak{sp}(2p)$ acts on \mathbb{C}^{2p} by the natural representation and trivially on the other subspaces. The first copy of \mathbb{C}^m supports the natural representation of $\mathfrak{gl}(m)$, while the second copy of \mathbb{C}^m is the dual representation; the $\mathfrak{gl}(m)$ -action on \mathbb{C}^{2p} is trivial.

The action of $f_{\alpha_1}, \dots, f_{\alpha_m}$ on the highest block $\mathbb{C}^m \otimes M_\lambda$ can be conveniently illustrated by the directed diagram



The origin of the diagram is the vertex $w_1 \otimes v$ in the north-east corner. We call the north-west to south-east lines diagonals and count them from the origin down to south-west. The nodes on the diagram designate the one-dimensional subspaces in $\mathbb{C}^m \otimes M_\lambda$ spanned by the corresponding tensors. The horizontal arrows symbolize the action of the Chevalley generators on the tensor factor M_λ while the vertical arrows indicate the action on the tensor factor \mathbb{C}^m . Each node has two arrows directed from it. The horizontal arrow yields the action on the whole $\mathbb{C}^m \otimes M_\lambda$ (up to an invertible scalar factor) when the associated generator is distinct from the generator assigned to the vertical arrow. When both arrows are labeled with the same generator, the latter sends the node to the two-dimensional space spanned by the nodes down and to the left. Such nodes lie on the m -th diagonal, which is straight above the principal. The rightmost vertical arrows excepting f_{α_m} amount to the action on $\mathbb{C}^m \otimes M_\lambda$, as the associated generators kill v .

The sub-triangle above the principal diagonal belongs to M_1 , the submodule in $\mathbb{C}^m \otimes M_\lambda$ generated by u_{ν_1} . That is clear for its rightmost column $\{w_i \otimes v\}_{i=1}^m$, which nodes are obtained

from $w_1 \otimes v$ by $\{f_{\alpha_i}\}_{i=1}^{m-1} \subset \mathcal{L}_-$. The horizontal and vertical arrows directed from $w_i \otimes \psi_j v$, $j > i + 1$, are marked differently. Therefore, every node $w_i \otimes \psi_j v$, $j > i$, is proportional to $f_{\alpha_j}(w_i \otimes \psi_{j+1} v) \sim \psi_j(w_i \otimes v) \in M_1$. Application of f_{α_i} to $w_i \otimes \psi_{i+1} v \in M_1$ (which is on the diagonal of the sub-triangle) gives

$$w_i \otimes \psi_i v + w_{i+1} \otimes q^{-1} \psi_{i+1} v \in M_1, \quad i < m, \quad w_m \otimes \psi_m v + w_{m+1} \otimes q^{-(\alpha, \lambda)} v \in M_1. \quad (7.15)$$

Lemma 7.2. *The singular vector u_{ν_2} is equal to $q^{-m} \frac{q^{(\alpha, \lambda) + m} - q^{-(\alpha, \lambda) - m}}{q - q^{-1}} w_{m+1} \otimes v$ modulo M_1 .*

Proof. All nodes above the main diagonal in D_0 lie in M_1 . Formulas (7.15) imply that $w_i \otimes \psi_i v = -q^{-1} w_{i+1} \otimes \psi_{i+1} v$ modulo M_1 , for $i \leq m$, if we set $\psi_{m+1} = 1$. Therefore $w_i \otimes \psi_i v = -(-q)^{i-m} q^{-(\alpha, \lambda)} w_{m+1} \otimes v$ modulo M_1 , for $i = 1, \dots, m$. Then (7.14) gives

$$u_{\nu_2} = \left(\frac{q^{(\alpha, \lambda)} - q^{-(\alpha, \lambda)}}{q - q^{-1}} + q^{-(\alpha, \lambda) - 1} \sum_{i=0}^{m-1} q^{2(i-m)} \right) w_{m+1} \otimes v \quad \text{mod } M_1.$$

Now the proof is immediate. □

In order to prove the direct decomposition $\mathbb{C}^{2n} \otimes M_\lambda = M_1 \oplus M_2$, we develop our diagram technique further. Introduce monomials $\phi_i \in U_q(\mathfrak{g}_-)$, $i = 1, \dots, p + 1$, of degree $2p + 1$ by the formulas (recall that $m = n - p$ in this section)

$$\phi_i := (f_{\alpha_{m+i-1}} \cdot \leftarrow \cdot f_{\alpha_{n-1}} f_\beta f_{\alpha_{m+i-2}} \cdot \rightarrow \cdot f_{\alpha_m}) (f_{\alpha_{n-1}} \cdot \rightarrow \cdot f_{\alpha_m}), \quad i = 1, \dots, p + 1.$$

According to Proposition 5.5, the root vector f_δ is a linear combination of ϕ_i .

Denote by f_i^l , $l = 1, \dots, 2p + 1$, the l -th factor in ϕ_i counting from the right. By construction, $f_i^l = f_{\alpha_{l+m-1}}$ for $1 \leq l \leq p$ and all i . The other $p = n - m$ elements f_i^l for $l = p + 1, \dots, 2p + 1$ are obtained by a permutation of the leftmost p terms, including f_β , of the sequence

$$f_{\alpha_m}, \dots, f_{\alpha_{n-1}}, f_\beta, f_{\alpha_{n-1}}, \dots, f_{\alpha_m}. \quad (7.16)$$

Denote by ϕ_i^l the product $f_i^l \cdot \rightarrow \cdot f_i^1$ for all $l = 1, \dots, 2p + 1$. In particular, $\phi_i^l = f_{\alpha_{l-1+m}} \cdot \rightarrow \cdot f_{\alpha_m}$ for all $1 \leq l \leq p$, and $\phi_i^{2p+1} = \phi_i$. It is also convenient to put $\phi_i^0 := 1$ for all i .

With every $i = 1, \dots, p + 1$, we associate a diagram D_i of $p + 1$ rows if $i > 1$ and of $2p + 2$ rows if $i = 1$. The lengths of the rows vary from $2p + 2$ to 1 in D_1 and to $p + 2$ in D_i , $i > 1$, from top to bottom. The rows are leveled on the right, so D_1 is a triangle and D_i is a trapezoid for $i > 1$. All D_i can be extended further down as D_1 , but we need only their first $p + 1$ rows.

The rightmost column in D_i is formed by the tensors $w_{m+l-1} \otimes v$, where l runs from 1 to $p+1$ if $i > 1$ and to $2p+2$ in D_1 . The intersection of l -th row and j -th column is the tensor $w_{m+l-1} \otimes \phi_i^{j-1} v$. As before, the nodes span one-dimensional subspaces in $\mathbb{C}^{2n} \otimes M_\lambda$ and the arrows designate the action of f_{α_i} : horizontal on M_λ and vertical on \mathbb{C}^{2n} . In all diagrams the vertical arrow applied to the j -th row is labeled with f_1^j , i.e. the j -th term in (7.16) from the right. The horizontal arrows are the factors f_i^l constituting ϕ_i .

If the generators assigned to the two arrows directed from a node are distinct, the horizontal arrow gives the action on $\mathbb{C}^{2n} \otimes M_\lambda$, up to an invertible scalar factor. If they coincide, the node is sent to the span of the two nodes: next down and next to the left. Modulo the down node, the horizontal arrow still gives the action on $\mathbb{C}^{2n} \otimes M_\lambda$, up to an invertible scalar. This follows from the coproduct $\Delta(f_\mu) = f_\mu \otimes q^{-h_\mu} + 1 \otimes f_\mu$, $\mu \in \Pi^+$.

$$\begin{array}{c}
D_1 \\
\hline
\begin{array}{ccccccc|c}
& f_1^{2p+1} & & f_1^{2p} & \dots & f_1^2 & & f_1^1 & \\
w_m \otimes \phi_1^{2p+1} v & \leftarrow & w_m \otimes \phi_1^{2p} v & \leftarrow & \dots & \leftarrow & w_m \otimes \phi_1^1 v & \leftarrow & w_m \otimes v \\
& & \downarrow & & & & \downarrow & & \downarrow & f_1^1 \\
& & w_{m+1} \otimes \phi_1^{2p} v & \leftarrow & \dots & \leftarrow & w_{m+1} \otimes \phi_1^1 v & \leftarrow & w_{m+1} \otimes v \\
& & & & & & \downarrow & & \downarrow & f_1^2 \\
& & & & \ddots & & \vdots & & \vdots & \vdots \\
& & & & & & \downarrow & & \downarrow & f_1^{2p} \\
& & & & & & w_{n+p} \otimes \phi_1^1 v & \leftarrow & w_{n+p} \otimes v \\
& & & & & & & & \downarrow & f_1^{2p+1} \\
& & & & & & & & w_{n+p+1} \otimes v & \\
\end{array} \\
\\
D_i, \quad i > 1 \\
\hline
\begin{array}{ccccccc|c}
& f_i^{2p+1} & & f_i^{2p} & \dots & & & f_i^{p+1} & \dots & f_i^1 & \\
w_m \otimes \phi_i^{2p+1} v & \leftarrow & w_m \otimes \phi_i^{2p} v & \leftarrow & \dots & & & w_m \otimes \phi_i^{p+1} v & \leftarrow & \dots & \leftarrow & w_m \otimes v \\
& & \downarrow & & & & & \downarrow & & & & \downarrow & f_1^1 \\
& & w_{m+1} \otimes \phi_i^{2p} v & \leftarrow & & & & \vdots & & & & \downarrow & \vdots \\
& & & & \ddots & & & \downarrow & & & & \downarrow & f_1^p \\
& & & & & & & w_{m+p} \otimes \phi_i^{p+1} v & \leftarrow & \dots & \leftarrow & w_{m+p} \otimes v \\
\end{array}
\end{array}$$

We present the diagrams D_1, D_2, D_3 in Appendix, in order to illustrate the formalism in the case of $m = 1, p = 2, n = 3$.

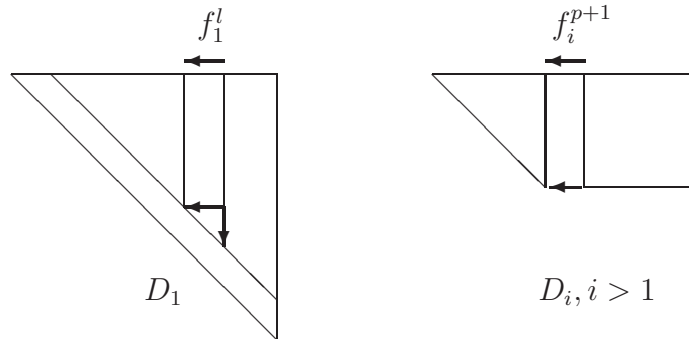
In the following theorem we regard $U_q(\mathfrak{g})$ as a \mathbb{C} -algebra generated by $e_\mu, f_\mu, t_\mu, \mu \in \Pi^+$, assuming $q^4 \neq 1$.

Theorem 7.3. *Suppose that $q^{-2p+2m} \neq -1$. Then the $U_q(\mathfrak{g})$ -module $\mathbb{C}^{2n} \otimes M_\lambda$ is isomorphic to the direct sum $M_1 \oplus M_2$.*

Proof. There exists a $U_q(\mathfrak{g})$ -invariant operator $\mathcal{Q} = (\pi \otimes \text{id})(\mathcal{R}_{21}\mathcal{R})$ on $\mathbb{C}^{2n} \otimes U_{\hbar}(\mathfrak{g})$, where \mathcal{R} is the universal R-matrix of $U_{\hbar}(\mathfrak{g})$. This operator plays an important role in this exposition and is discussed at length in Section 9. Here we need the information on its eigenvalues, which are found in [1]. Specifically on \hat{M}_1 and \hat{M}_2 , the operator \mathcal{Q} is a scalar multiplier by, respectively, $q^{2(\lambda, \varepsilon_1)}$ and $q^{2(\lambda, \varepsilon_{m+1})-2m}$. Hence its eigenvalues on M_1 and M_2 are $-q^{-2p}$ and q^{-2m} , assuming $\lambda \in \mathfrak{C}_\dagger^*$. Under the hypothesis of the theorem, they are distinct, therefore the submodules M_1 and M_2 have zero intersection.

We must show that the sum $M = M_1 \oplus M_2$ exhausts all of $\mathbb{C}^{2n} \otimes M_\lambda$. It is sufficient to prove that $\mathbb{C}^{2n} \otimes v$ lies in M . Since v is annihilated by $f_{\alpha_i}, i = 1, \dots, m-1$, we have $f_{\alpha_i}(w_i \otimes v) = w_{i+1} \otimes v$. Therefore, $w_i \otimes v \in M_1$ for $i = 1, \dots, m$. By Lemma 7.2, the vector $w_{m+1} \otimes v$ belongs to M if $q^{2(\alpha, \lambda)+2m} = -q^{-2p+2m} \neq 1$. The Chevalley generators $f_{\alpha_i}, i = m+1, \dots, n$, belong to the Levi subalgebra and kill v . Applying them repeatedly to $w_{m+1} \otimes v \in M$ we get $w_l \otimes v$ for all $l = m+2, \dots, n+p$ and prove that they are in M .

The crucial step is to show that $w_{n+p+1} \otimes v \in M$. First of all, the triangle above the principal diagonal of D_1 lies in M . This is checked by induction on the column number. Let C_l be the linear span of the nodes of column l above the principal diagonal and C'_l the linear span of nodes from C_l without the bottom one. We have proved that C_1 lies in M . Suppose it is true for some column $l \geq 1$. Let f_1^l be the Chevalley generator assigned to the horizontal arrow from column l to column $l+1$. It sends C'_l isomorphically to C_{l+1} modulo C_l , which lies in M by the induction assumption. Therefore C_{l+1} lies in M . The left diagram below displays schematically the induction transition.



Now we are going to prove that the nodes $w_{m+l} \otimes \phi_1^{2p+1-l}v$, $l = 0, \dots, 2p+1$, on the main diagonal belong to M . To compare vectors modulo M we will use the symbol \equiv , i.e. $x \equiv y$ if and only if $x - y \in M$. Consider the diagonal next to the main. We already know that its nodes are in M . The Chevalley generators assigned to the horizontal and vertical arrows coincide at every node on this diagonal, hence each of them is mapped to a linear combination of two nodes on the principal diagonal. Their images are

$$a_1 w_m \otimes \phi_1^{2p+1}v + w_{m+1} \otimes \phi_1^{2p}v \equiv 0, \quad \dots, \quad a_{2p+1} w_{n+p} \otimes \phi_1^1v + w_{n+p+1} \otimes \phi_1^0v \equiv 0,$$

where a_i are non-zero scalars. Thus, all nodes on the principal diagonal of D_1 are proportional to each other, modulo M .

Now we turn to the diagram D_i , $i = 2, \dots, p+1$. Observe that its first p columns on the right coincide with the corresponding square part of D_1 (the first p horizontal arrows are the same in all diagrams). That part is situated above the principal diagonal in D_1 and therefore lies in M . Since $f_i^{p+1}w_{n+p} = 0$ for $i > 1$, the operator f_i^{p+1} is mapping the node $w_{m+p} \otimes \phi_i^p v$ onto $w_{m+p} \otimes \phi_i^{p+1}v$. This implies that the column $p+1$ in D_i lies in M . A simple induction proves that the leftmost triangular part of D_i including column $p+1$ lies in M . The reasoning is similar to what we did for the triangle in D_1 above the main diagonal. In particular, the tensor $w_m \otimes \phi_i^{2p+1} = w_m \otimes \phi_i$, $i > 1$, belongs to M .

Now recall from Proposition 5.5 that f_δ is a linear combination of ϕ_i . Adding the equalities $w_m \otimes \phi_i^{2p+1}v \equiv 0$ for $i = 2, \dots, 2p+1$, with appropriate multipliers, to the equality $a_1 w_m \otimes \phi_1^{2p+1}v + w_{m+1} \otimes \phi_1^{2p}v \equiv 0$ we replace $w_m \otimes \phi_1^{2p+1}v$ with $w_m \otimes f_\delta v$, which is nil in $\mathbb{C}^{2n} \otimes M_\lambda$. This gives $w_{m+1} \otimes \phi_1^{2p}v \equiv 0$. On the other hand, we proved that all the nodes on the main diagonal of D_1 are proportional modulo M . Moving down the diagonal we eventually conclude that $w_{n+p+1} \otimes v \equiv 0$.

To complete the proof, we must check that $w_j \otimes v \in M$ for $j > n+p+1$. These tensors belong to $\mathbb{C}^m \otimes M_\lambda$, where $\mathbb{C}^m \subset \mathbb{C}^{2n}$ is the irreducible module of the Levi subalgebra generated by w_{n+p+1} . We have $M \supset U_{\mathfrak{h}}(\mathfrak{t})(w_{n+p+1} \otimes v) = \mathbb{C}^m \otimes v$. Thus, $\mathbb{C}^{2n} \otimes v$ is contained in M , and $M = \mathbb{C}^{2n} \otimes M_\lambda$. \square

The direct sum decomposition is a strong property, which is hard to prove for general \mathfrak{k} . For our purposes, it is sufficient to replace it with an increasing filtration, which construction is easier. We rephrase Theorem 7.3 for the symmetric case in this milder setting, which will be a part of a construction for general \mathfrak{k} further on.

Set $V_1 = M_1$ to be the $U_q(\mathfrak{g})$ -module generated by $w_1 \otimes v$ and denote by V_2 the $U_q(\mathfrak{g})$ -module generated by $\{w_1 \otimes v, w_{m+1} \otimes v\}$, so that $V_1 \subset V_2$. While $w_{m+1} \otimes v$ is not a singular

vector, it is so modulo V_1 . Identified with its projection to $V_2/V_1 \simeq M_2$, it is a highest weight vector in the quotient V_2/V_1 .

Proposition 7.4. *The module V_2 coincides with $\mathbb{C}^{2n} \otimes M_\lambda$, and $V_2/V_1 \simeq M_2$.*

Proof. Using a similar reasoning as in the proof of Theorem 7.3, we show that $\mathbb{C}^{2n} \otimes v$ and hence $\mathbb{C}^{2n} \otimes M_\lambda$ lie in V_2 . The only difference is that the inclusion $w_{m+1} \otimes v \subset V_2$ holds by the very construction, and this is a simplification. \square

8 Module $\mathbb{C}^{2n} \otimes M_\lambda$: general case

For the general Levi subalgebra \mathfrak{l} , the vector space \mathbb{C}^{2n} decomposes in the direct sum of irreducible \mathfrak{l} -submodules,

$$\mathbb{C}^{2n} = W_1 \oplus \dots \oplus W_{\ell+1} \oplus W_{\ell+2} \oplus W_{\ell+3} \oplus \dots \oplus W_{2\ell+3},$$

of dimensions $n_1, \dots, n_\ell, m, 2p, m, n_\ell, \dots, n_1$. This decomposition corresponds to the block-diagonal structure of \mathfrak{l} . For $i = 1, \dots, \ell+1$, the block $\mathfrak{gl}(n_i)$ acts by the natural representation on $W_i \simeq \mathbb{C}^{n_i}$, by the dual representation on $W_{2\ell+4-i} \simeq \mathbb{C}^{n_i}$, and trivially on the other spaces. The block $\mathfrak{sp}(2p)$ acts by the natural representation on $W_{\ell+2} \simeq \mathbb{C}^{2p}$ and trivially on the other subspaces. The highest weights ν_i of W_i are

$$\varepsilon_1, \varepsilon_{n_1+1}, \dots, \varepsilon_{n_1+\dots+n_\ell+1}, \varepsilon_{n_1+\dots+n_\ell+m+1}, -\varepsilon_{n_1+\dots+n_\ell+m}, -\varepsilon_{n_1+\dots+n_\ell}, \dots, -\varepsilon_{n_1}. \quad (8.17)$$

The highest weight vectors w_{ν_i} , $i = 1, \dots, 2\ell+3$, belong to the standard basis $\{w_i\}_{i=1}^{2n} \subset \mathbb{C}^{2n}$.

For generic weight $\lambda \in \mathfrak{t}_{\text{reg}}^*$ the decomposition of \mathbb{C}^{2n} induces the decomposition

$$\mathbb{C}^{2n} \otimes \hat{M}_\lambda = \bigoplus_{i=1}^{2\ell+3} \hat{M}_i \quad (8.18)$$

of $U_q(\mathfrak{g})$ -submodules. The blocks are generated by singular vectors of weights $\nu_i + \lambda$, where ν_i are given by (8.17). This follows from non-degeneracy of the contravariant form on \hat{M}_λ and \hat{M}_i at generic λ , see e.g. [1].

The \mathfrak{l} -modules $W_{\ell+1}$ and $W_{\ell+3}$ are merged into a single irreducible \mathfrak{k} -module that supports the natural representation of the block $\mathfrak{sp}(2m) \subset \mathfrak{k}$. The other \mathfrak{l} -submodules in \mathbb{C}^{2n} remain irreducible with respect to \mathfrak{k} .

Denote by M_i the images of \hat{M}_i under the projection $\mathbb{C}^{2n} \otimes \hat{M}_\lambda \rightarrow \mathbb{C}^{2n} \otimes M_\lambda$. One should expect that $\hat{M}_{\ell+3}$ is annihilated by the projection, and decomposition (8.18) turns into

$$\mathbb{C}^{2n} \otimes M_\lambda = M_1 \oplus \dots \oplus M_{\ell+1} \oplus M_{\ell+2} \oplus M_{\ell+4} \oplus \dots \oplus M_{2\ell+3}.$$

However, this is not easy to prove in the general case. On the other hand, all we need is the spectrum of the invariant operator $\mathcal{Q} \in \text{End}(\mathbb{C}^{2n} \otimes M_\lambda)$, cf. Section 9. It is sufficient for that to replace the direct sum with a suitable filtration, which is easier.

Denote by V_k the $U_q(\mathfrak{g})$ -submodule in $\mathbb{C}^{2n} \otimes M_\lambda$ generated by $\{w_{\nu_i} \otimes v\}_{i=1, \dots, k}$ assuming $k = 1, \dots, 2\ell + 3$. We have the obvious inclusion $V_{k-1} \subset V_k$. It is convenient to set $V_0 = \{0\}$.

Proposition 8.1. *The $U_q(\mathfrak{g})$ -modules $\{0\} = V_0 \subset V_1 \subset \dots \subset V_{2\ell+3} = \mathbb{C}^{2n} \otimes M_\lambda$ form an ascending filtration. For each $k = 1, \dots, 2\ell+3$ the graded component V_k/V_{k-1} is either $\{0\}$ or a highest weight module generated by (the image of) $w_{\nu_k} \otimes v$. In particular, $V_{\ell+3}/V_{\ell+2} = \{0\}$.*

Proof. Observe that $e_\mu(w_{\nu_k} \otimes v) = 0 \pmod{V_{k-1}}$ for all $\mu \in \Pi^+$, i.e. $w_{\nu_k} \otimes v$ is a singular vector in V_k/V_{k-1} unless it is nil. Since V_k/V_{k-1} is generated by $w_{\nu_k} \otimes v$, it is the highest weight vector unless $V_k/V_{k-1} = \{0\}$.

We will prove the inclusion $\bigoplus_{i=1}^k W_i \otimes v \subset V_k$ for $k = 2\ell + 3$ and $\mathbb{C}^{2n} \otimes v \subset V_{2\ell+3}$ in particular. This will imply $V_{2\ell+3} = \mathbb{C}^{2n} \otimes M_\lambda$.

Suppose we have proved that $W_k \otimes v \subset V_k$ for some $k > 0$. This is also true for $k = 0$ if we set $W_0 = \{0\}$. By construction, $w_{\nu_{k+1}} \otimes v \in V_{k+1}$. Then $U_q(\mathfrak{l}_-)(w_{\nu_{k+1}} \otimes v) = U_q(\mathfrak{l}_-)w_{\nu_{k+1}} \otimes v = W_{k+1} \otimes v \subset V_{k+1}$ because \mathfrak{l}_- annihilates v and $w_{\nu_{k+1}}$ generates the $U_q(\mathfrak{l}_-)$ -submodule W_{k+1} . Induction on k proves $W_k \otimes v \subset V_k$ and therefore $\bigoplus_{i=1}^k W_i \otimes v \subset V_k$ for all k .

Finally, let us prove the equality $V_{\ell+2} = V_{\ell+3}$. It is sufficient to check the inclusion $w_{\nu_{\ell+3}} \otimes v \in V_{\ell+2}$, because $V_{\ell+3}/V_{\ell+2}$ is generated by $w_{\nu_{\ell+3}} \otimes v$. This boils down to the symmetric case studied in Proposition 7.4. The vector w_{n-p} belongs to $W_{\ell+1}$, so $w_{n-p} \otimes v \in V_{\ell+2}$. Also, $w_{n-p+1} \otimes v \in V_{\ell+2}$, because $w_{n-p+1} = w_{\nu_{\ell+2}}$. Consider the Lie subalgebra $\mathfrak{g}' = \mathfrak{sp}(2 + 2p) \subset \mathfrak{sp}(2n)$ defined in Section 5. Let $M'_\lambda \subset M_\lambda$ be the $U_q(\mathfrak{g}')$ -submodule in M_λ generated by v . Consider the natural representation of $U_q(\mathfrak{g}')$ on \mathbb{C}^{2+2p} with the highest weight vector w_{n-p} and the lowest weight vector $w_{n-p+1} = w_{\nu_{\ell+3}}$. Let V'_2 be the $U_q(\mathfrak{g}')$ -submodule generated by $w_{n-p} \otimes v$ and $w_{n-p+1} \otimes v$. By Proposition 7.4, $V'_2 = \mathbb{C}^{2+2p} \otimes M'_\lambda$, hence $w_{n-p+1} \otimes v \in V'_2 \subset V_{\ell+2}$, as required. This completes the proof. \square

9 The matrix of quantum coordinate functions

The classical description of semi-simple conjugacy classes is formulated in terms of operations (multiplication, transposition, trace functional) with the matrix A of coordinate functions on $\text{End}(\mathbb{C}^{2n})$. The matrix A is G -invariant, and its entries generate the polynomial algebra of the class. A similar description of the quantum conjugacy classes involves a matrix A

with non-commutative entries or its image $\mathcal{Q} \in \text{End}(\mathbb{C}^{2n}) \otimes U_q(\mathfrak{g})$, which should be regarded as the "restriction" of A to the "quantum group" G_q . In this section, we study algebraic properties of \mathcal{Q} .

The operator \mathcal{Q} is defined through the universal R-matrix of $U_{\hbar}(\mathfrak{g})$, which is an invertible element of (completed) tensor square of $U_{\hbar}(\mathfrak{g})$, conventionally denoted by \mathcal{R} :

$$\mathcal{Q} = (\pi \otimes \text{id})(\mathcal{R}_{12}\mathcal{R}) \in \text{End}(\mathbb{C}^{2n}) \otimes U_q(\mathfrak{g}).$$

That the entries of \mathcal{Q} lie in $U_q(\mathfrak{g}) \subset U_{\hbar}(\mathfrak{g})$ follows from the explicit expression of the universal R-matrix, see [22]. Regarded as an operator on $\mathbb{C}^{2n} \otimes \hat{M}_{\lambda}$, it satisfies a polynomial equation with the roots [1]

$$q^{2(\lambda+\rho, \nu_i)-2(\rho, \varepsilon_1)} = q^{2(\lambda, \nu_i)+2(\rho, \nu_i-\varepsilon_1)},$$

where $\{\nu_i\}_{i=1}^{2\ell+3}$ are the highest weights of the irreducible \mathfrak{l} -submodules $W_i \subset \mathbb{C}^{2n}$ and ρ the half-sum of the positive roots of \mathfrak{g} .

Assuming $\lambda \in \mathfrak{C}_{\mathfrak{t}, \text{reg}}^*$, put $\Lambda_i = \hbar(\lambda, \varepsilon_{n_1+\dots+n_{i-1}+1}) = (\lambda, \varepsilon_{n_1+\dots+n_i}) \in \mathbb{C}$ for $i = 1, \dots, \ell$. Define $\boldsymbol{\mu} \in \mathbb{C}^{\ell+2}[q, q^{-1}]$ by

$$\mu_i = e^{2\Lambda_i} q^{-2(n_1+\dots+n_{i-1})}, \quad \mu_{\ell+1} = -q^{-2(n_1+\dots+n_{\ell}+p)}, \quad \mu_{\ell+2} = q^{-2(n_1+\dots+n_{\ell}+m)}, \quad (9.19)$$

where $i = 1, \dots, \ell$. The eigenvalues of \mathcal{Q} on $\mathbb{C}^{2n} \otimes \hat{M}_{\lambda}$ are expressed through $\boldsymbol{\mu}$ by

$$\mu_i, \quad \mu_i^{-1} q^{-4n+2(n_i-1)}, \quad i = 1, \dots, \ell+1, \quad \text{and} \quad \mu_{\ell+2}. \quad (9.20)$$

Recall that, for $\lambda \in \mathfrak{C}_{\mathfrak{t}, \text{reg}}^*$, the classical limit $\boldsymbol{\mu}^0 = \lim_{q \rightarrow 1} \boldsymbol{\mu} \subset \hat{M}_K$ parameterizes the moduli space of classes with the stabilizer K . Explicitly, $\mu_i^0 = e^{2\Lambda_i}$ for $i = 1, \dots, \ell$, $\mu_{\ell+1}^0 = -1$, and $\mu_{\ell+2}^0 = 1$.

Proposition 9.1. *For $\lambda \in \mathfrak{C}_{\mathfrak{t}, \text{reg}}^*$, the operator $\mathcal{Q} \in \text{End}(\mathbb{C}^{2n} \otimes M_{\lambda})$ satisfies a polynomial equation of degree $2\ell + 2$ with the roots*

$$\mu_i, \quad \mu_i^{-1} q^{-4n+2(n_i-1)}, \quad i = 1, \dots, \ell, \quad \text{and} \quad \mu_{\ell+1}, \quad \mu_{\ell+2}. \quad (9.21)$$

Proof. The proof is based on the following fact: a linear operator on a complex vector space is semi-simple if and only if it satisfies a polynomial equation with simple roots. This is true for finite dimensional vector spaces, but in our case we can restrict the consideration to every weight subspace in $\mathbb{C}^{2n} \otimes \hat{M}_{\lambda}$, which is finite dimensional and \mathcal{Q} -invariant.

It is known that \mathcal{Q} , as an operator on $\mathbb{C}^{2n} \otimes \hat{M}_{\lambda}$, satisfies a polynomial equation of degree $2\ell + 3$ with the roots (9.20), cf. [1]. Its eigenvalues are pairwise distinct in the classical

limit, apart from $\lim_{q \rightarrow 1} \mu_{\ell+1} = \lim_{q \rightarrow 1} \mu_{\ell+1}^{-1} q^{-4n+2(m-1)} = -1$. However, for $q \neq 1$ this coincidence is no longer the case, and the eigenvalues (9.19) become pairwise distinct for q in a punctured neighborhood of 1: "quantization eliminates degeneration". This implies that \mathcal{Q} is semi-simple on $\mathbb{C}^{2n} \otimes \hat{M}_\lambda$ for all q close to 1 and hence for generic q (for fixed λ , the entries of \mathcal{Q} in every weight subspace are polynomials in $q^{\pm 1}$). Therefore it is semi-simple on the quotient $\mathbb{C}^{2n} \otimes M_\lambda$, where $\mu_{\ell+1}^{-1} q^{-4n+2(m-1)}$ is no longer its eigenvalue, by Proposition 8.1. This proves the statement for generic q and therefore for all q . \square

The matrix \mathcal{Q} produces central elements of $U_\hbar(\mathfrak{g})$ via the q -trace construction. Since \mathcal{Q} commutes with $\Delta U_\hbar(\mathfrak{g})$, the elements

$$\mathrm{Tr}_q(\mathcal{Q}^k) := \mathrm{Tr}((\pi(q^{2h\rho}) \otimes 1)\mathcal{Q}^k) \in U_\hbar(\mathfrak{g}), \quad k = 1, 2, \dots \quad (9.22)$$

are invariant under the adjoint action $\mathrm{ad}(u)x = u^{(1)}x\gamma(u^{(2)})$, $u, x \in U_\hbar(\mathfrak{g})$. It is a standard fact from the Hopf algebra theory that *ad*-invariant elements are central and *vice versa*. We will use the shortcut notation τ_k for $\mathrm{Tr}_q(\mathcal{Q}^k)$, $k = 1, 2, \dots$

A $U_\hbar(\mathfrak{g})$ -module of highest weight λ defines a one-dimensional representation χ^λ of the centre of $U_\hbar(\mathfrak{g})$, which assigns a scalar to each τ_k :

$$\chi^\lambda(\tau_k) = \sum_{\nu} q^{2k(\lambda+\rho, \nu) - 2k(\rho, \nu_1)} \prod_{\alpha \in \mathbb{R}_+} \frac{q^{(\lambda+\nu+\rho, \alpha)} - q^{-(\lambda+\nu+\rho, \alpha)}}{q^{(\lambda+\rho, \alpha)} - q^{-(\lambda+\rho, \alpha)}}. \quad (9.23)$$

The summation is taken over the weights $\nu \in \{\pm \varepsilon_j\}_{j=1}^n$ of the module \mathbb{C}^{2n} . Restriction of λ to $\mathfrak{C}_{\mathfrak{t}, \mathrm{reg}}^*$ makes the right-hand side a function of the vector $\boldsymbol{\mu}$ defined in (9.19). We denote this function by $\vartheta_{\mathbf{n}, q}^k(\boldsymbol{\mu})$, where $\mathbf{n} = (n_1, \dots, n_\ell, m, p)$ is the integer valued vector of multiplicities. In the limit $\hbar \rightarrow 0$ the function $\vartheta_{\mathbf{n}, q}^k(\boldsymbol{\mu})$ goes over into the right-hand side of (2.7), where $\boldsymbol{\mu}$ should be replaced with $\boldsymbol{\mu}^0 = \lim_{q \rightarrow 1} \boldsymbol{\mu}$.

10 Quantum conjugacy classes of non-Levi type

By quantization of a commutative \mathbb{C} -algebra \mathcal{A} we understand a $\mathbb{C}[[\hbar]]$ -algebra \mathcal{A}_\hbar , which is free as a $\mathbb{C}[[\hbar]]$ -module, and $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar \simeq \mathcal{A}$ as a \mathbb{C} -algebra. Note that we do not require \hbar -adic completion because the algebras of our interest are direct sums of $U_\hbar(\mathfrak{g})$ -submodules, which we prefer to preserve under quantization. Below we describe the quantization of $\mathbb{C}[G]$ along the Poisson bracket (2.4).

Recall (see e.g. [23]) that the image of the universal R-matrix in the natural representation is equal, up to a scalar factor, to

$$R = \sum_{i,j=1}^{2n} q^{\delta_{ij}-\delta_{ij'}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{\substack{i,j=1 \\ i>j}}^{2n} (e_{ij} \otimes e_{ji} - q^{\rho_i-\rho_j} \epsilon_i \epsilon_j e_{ij} \otimes e_{i'j'}),$$

where $\rho_i = -\rho_{i'} = (\rho, \varepsilon_i) = n + 1 - i$ for $i = 1, \dots, n$.

Denote by S the operator $PR \in \text{End}(\mathbb{C}^{2n}) \otimes \text{End}(\mathbb{C}^{2n})$, where P is the ordinary flip of $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$. It commutes with the $U_{\hbar}(\mathfrak{g})$ -action on $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$ and generates three invariant idempotents. One of them is a one-dimensional projector κ onto the trivial $U_{\hbar}(\mathfrak{g})$ -submodule; it is proportional to $\sum_{i,j=1}^{2n} q^{\rho_i-\rho_j} \epsilon_i \epsilon_j e_{i'j} \otimes e_{ij'}$.

Denote by $\mathbb{C}_{\hbar}[G]$ the associative algebra generated by the entries of a matrix $A = \|A_{ij}\|_{i,j=1}^{2n} \in \text{End}(\mathbb{C}^{2n}) \otimes \mathbb{C}_{\hbar}[G]$ modulo the relations

$$S_{12}A_2S_{12}A_2 = A_2S_{12}A_2S_{12}, \quad A_2S_{12}A_2\kappa = -q^{-2n-1}\kappa = \kappa A_2S_{12}A_2. \quad (10.24)$$

These relations are understood in $\text{End}(\mathbb{C}^{2n}) \otimes \text{End}(\mathbb{C}^{2n}) \otimes \mathbb{C}_{\hbar}[G]$, and the indices distinguish the two copies of $\text{End}(\mathbb{C}^{2n})$, in the usual way. Note that the factor $-q^{-2n-1}$ before κ is missing in [1].

The algebra $\mathbb{C}_{\hbar}[G]$ is a quantization of $\mathbb{C}[G]$. It is the quotient of the well known "reflection equation algebra" defined through the left identity in (10.24). From the quantization point of view, it was studied in [26] and [24]. Note that it is different from the RTT -quantization of $\mathbb{C}[G]$ and is not a Hopf algebra. The algebra $\mathbb{C}_{\hbar}[G]$ carries a $U_{\hbar}(\mathfrak{g})$ -action, which is a deformation of the coadjoint action of $U(\mathfrak{g})$ on $\mathbb{C}[G]$. This action is set up as

$$(\text{id} \otimes x)(A) = (\pi(\gamma(x^{(1)})) \otimes \text{id})(A)(\pi(x^{(2)}) \otimes \text{id}), \quad x \in U_{\hbar}(\mathfrak{g}),$$

on the entries of the matrix A . When extended further to the free algebra generated by $\{A_{ij}\}$, this action makes it a $U_{\hbar}(\mathfrak{g})$ -module algebra. Relations (10.24) are invariant, therefore $\mathbb{C}_{\hbar}[G]$ becomes a $U_{\hbar}(\mathfrak{g})$ -module algebra, too.

It is important that $\mathbb{C}_{\hbar}[G]$ can be realized as an invariant subalgebra in $U_q(\mathfrak{g})$, where the latter is regarded as the adjoint module. The assignment

$$A \mapsto (\pi \otimes \text{id})(\mathcal{R}_{21}\mathcal{R}) = \mathcal{Q} \in \text{End}(\mathbb{C}^{2n}) \otimes U_q(\mathfrak{g}),$$

preserves the relations (10.24) and determines an embedding $\mathbb{C}_{\hbar}[G] \subset U_q(\mathfrak{g})$, [24]. Relations (10.24) amount to the Yang-Baxter equation on \mathcal{R} and the existence of the one-dimensional invariant in $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$ (quantum "symplectic form").

The following properties of $\mathbb{C}_\hbar[G]$ will be of importance. Denote by $I_\hbar(G) \subset \mathbb{C}_\hbar[G]$ the subalgebra of $U_\hbar(\mathfrak{g})$ -invariants. It coincides with the centre of $\mathbb{C}_\hbar[G]$ and is generated by the q -traces $\text{Tr}_q(A^l)$, $l = 1, \dots, 2n$, which go over to τ_l under the embedding to $U_\hbar(\mathfrak{g})$. Not all traces are independent, but that is immaterial for this exposition. The algebra $\mathbb{C}_\hbar[G]$ is freely generated over $I_\hbar(G)$ by a $U_\hbar(\mathfrak{g})$ -module whose isotypic components are finite dimensional, [24]. This is a quantum version of the Kostant-Richardson theorem, [25].

Our approach to quantization is based on the following strategy that is similar to [1]. Suppose we have constructed two $U_\hbar(\mathfrak{g})$ -algebras S_\hbar and T_\hbar along with an equivariant homomorphism $\varphi: S_\hbar \rightarrow T_\hbar$ obeying the following conditions: 1) all isotypic components in S_\hbar are $\mathbb{C}[[\hbar]]$ -finite, 2) T_\hbar has no \hbar -torsion (multiplication by \hbar is injective), 3) there is an ideal $J_\hbar \subset \ker \varphi$ such that the image J_0^\flat of $J_0 = J_\hbar/\hbar J_\hbar$ in $S_0 = S_\hbar/\hbar S_\hbar$ is a maximal \mathfrak{g} -invariant ideal in S_0 , 4) S_0 is commutative. Then a) the kernel of φ coincides with J_\hbar , b) $\varphi(S_\hbar)$ is a quantization of the algebra S_0/J_0^\flat . Remark that if S_0 is the coordinate ring of a \mathfrak{g} -variety, maximal proper \mathfrak{g} -invariant ideals are exactly the radical ideals of orbits in it.

In our situation, $T_\hbar = \text{End}(M_\lambda)$ is the algebra of linear endomorphisms of M_λ and S_\hbar is the quotient of $\mathbb{C}_\hbar[G]$ by the ideal generated by $\ker \chi^\lambda$. Explicitly, this ideal is determined by the relations (9.23). By Proposition 6.2, M_λ and hence $\text{End}(M_\lambda)$ are free over $\mathbb{C}[[\hbar]]$. Note that we cannot take simply $\mathbb{C}_\hbar[G]$ for the role of S_\hbar , because the isotypic components of $\mathbb{C}_\hbar[G]$ are not finite due to the large centre $I_\hbar(G)$ (the subalgebra of invariants). This centre is reduced to scalars in S_\hbar , which therefore has finite isotypic components by the quantum Richardson theorem.

The composition of the embedding $\mathbb{C}_\hbar[G] \rightarrow U_q(\mathfrak{g})$ and the representation homomorphism $U_q(\mathfrak{g}) \rightarrow \text{End}(M_\lambda)$ yields a representation of $\mathbb{C}_\hbar[G]$ and factors through the homomorphism φ . The defining ideal of a class in G is a maximal G -invariant proper ideal in $\mathbb{C}[G]$, therefore its projection to S_0 is a proper maximal G -invariant ideal too. Thus, to construct the quantization, it is sufficient to check that φ annihilates an ideal that turns into the defining ideal of the class in the classical limit. As $\ker \chi^\lambda$ is already factored out in S_\hbar , we need to check the polynomial equation on \mathcal{Q} . That has been done in Proposition 9.1.

There is an issue about the action of $U_\hbar(\mathfrak{g})$ as mentioned in Section 5. The quantum group $U_\hbar(\mathfrak{g})$ cannot act on the $U_q(\mathfrak{g})$ -module M_λ because the operators from \mathfrak{h} are irregular in \hbar for $\lambda \in \mathfrak{C}_{\mathfrak{t}, reg}^*$. This can be fixed as follows. The image of $\mathbb{C}_\hbar[G]$ is contained in the subalgebra $\text{End}^\circ(M_\lambda)$ of locally finite endomorphisms of M_λ . We extend M_λ by the Laurent series in \hbar , to enable the action of $U_\hbar(\mathfrak{g})$. This action gives rise to the natural adjoint action on $\text{End}^\circ(M_\lambda)((\hbar))$. It is easy to see that the $\mathbb{C}[[\hbar]]$ -submodule $\text{End}^\circ(M_\lambda) \subset \text{End}^\circ(M_\lambda)((\hbar))$ is

$U_{\hbar}(\mathfrak{g})$ -invariant. Indeed, $\text{End}^\circ(M_\lambda)$ is a weight module and all its weights belong to $\mathbb{Z}\Pi^+$. Therefore, the action of \mathfrak{h} on $\text{End}^\circ(M_\lambda)$ is correctly defined. The adjoint action of the Chevalley generators on $\text{End}^\circ(M_\lambda)$ brings about operators $q^{\mathfrak{h}} \subset \text{End}^\circ(M_\lambda)$, through the comultiplication and antipode. This reasoning proves that, albeit $U_{\hbar}(\mathfrak{g})$ is not represented on M_λ , the "adjoint" action on $\text{End}^\circ(M_\lambda)$ is well defined.

Theorem 10.1. *Suppose that $\lambda = \mathfrak{C}_{\mathfrak{t}, \text{reg}}^*$ and let $\boldsymbol{\mu}$ be as defined in (9.19). The quotient of $\mathbb{C}_{\hbar}[G]$ by the ideal of relations*

$$\prod_{i=1}^{\ell} (\mathcal{Q} - \mu_i) \times (\mathcal{Q} - \mu_{\ell+1}) (\mathcal{Q} - \mu_{\ell+2}) \times \prod_{i=1}^{\ell} (\mathcal{Q} - \mu_i^{-1} q^{-4n+2(n_i-1)}) = 0, \quad (10.25)$$

$$\text{Tr}_q(\mathcal{Q}^k) = \vartheta_{\mathbf{n}, q}^k(\boldsymbol{\mu}) \quad (10.26)$$

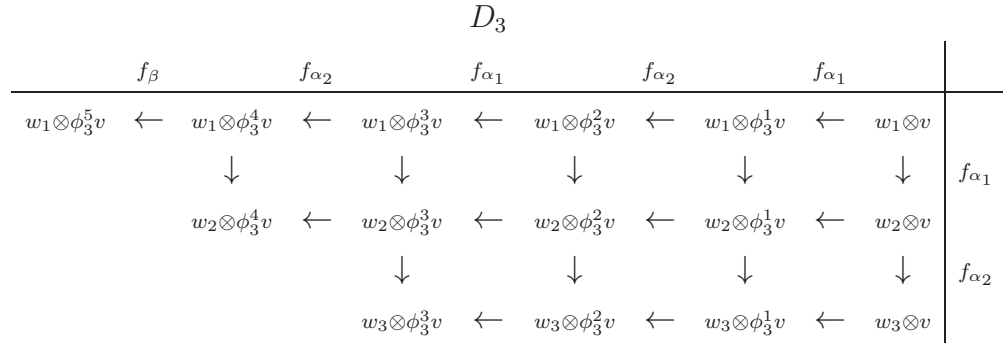
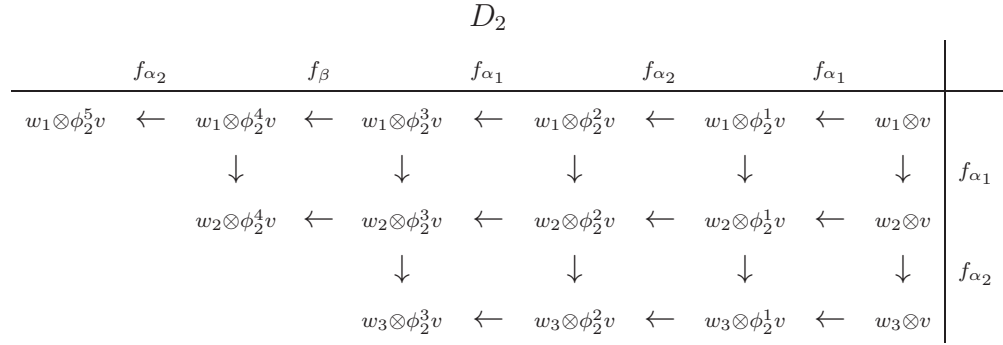
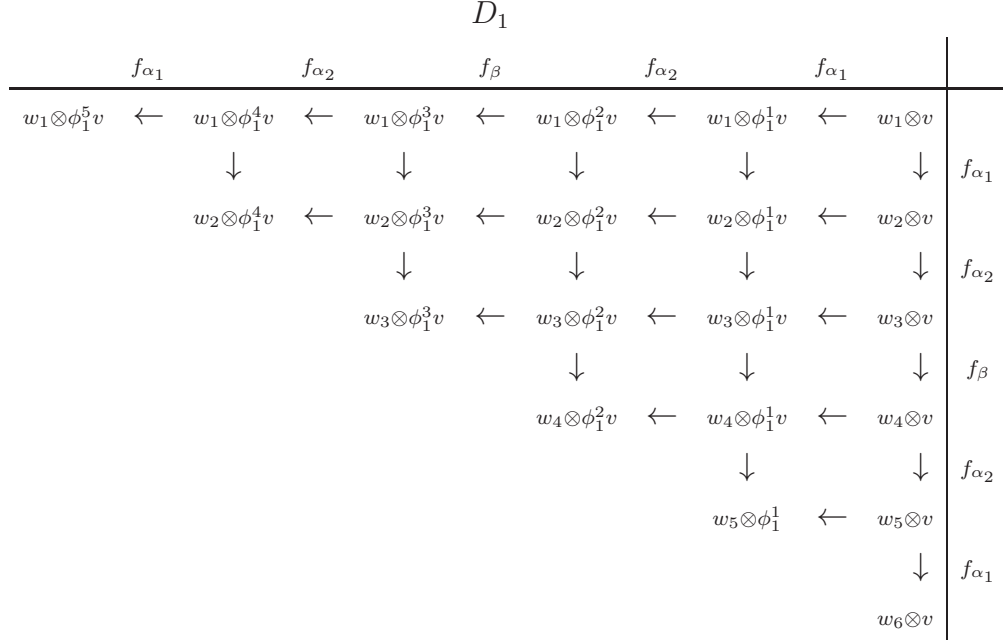
is an equivariant quantization of the class $\boldsymbol{\mu}^0 = \hat{\mathcal{M}}_K$, where $\boldsymbol{\mu}^0 = \lim_{\hbar \rightarrow 0} \boldsymbol{\mu}$. It is the image of $\mathbb{C}_{\hbar}[G]$ in the algebra of endomorphisms of the $U_q(\mathfrak{g})$ -module M_λ .

Theorem 10.1 describes quantization in terms of the matrix \mathcal{Q} , which is the image of the matrix A . To obtain the description in terms of A , one should replace \mathcal{Q} with A and add the relations (10.24).

The constructed quantization is equivariant with respect to the standard or Drinfeld-Jimbo quantum group $U_{\hbar}(\mathfrak{g})$. Other quantum groups are obtained from standard by twist, [27]. Formulas (10.25) and (10.26) are valid for any quantum group $U_{\hbar}(\mathfrak{g})$ upon the following modifications. The matrix \mathcal{Q} is expressed through the universal R-matrix as usual. The q-traces should be redefined as $\tau_k = \text{Tr}_q(\mathcal{Q}^k) = q^{1+2n} \text{Tr}(\pi(\gamma^{-1}(\mathcal{R}_1)\mathcal{R}_2)\mathcal{Q}^k)$, where γ is the antipode and \mathcal{R} is the universal R-matrix of $U_{\hbar}(\mathfrak{g})$. This can be verified along the lines of [28].

11 Appendix

Below we present the diagrams D_1, D_2, D_3 in order to illustrate the formalism of Section 7 for the case of $m = 1, p = 2, n = 3$.



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