

LIE ALGEBRAS IN SYMMETRIC MONOIDAL CATEGORIES

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ABSTRACT. We study algebras defined by identities in symmetric monoidal categories. Our focus is on Lie algebras. Besides usual Lie algebras, there are examples appearing in the study of knot invariants and Rozansky-Witten invariants. Our main result is a proof of Westbury's conjecture for K3-surface: there exists a Lie algebra homomorphism from Vogel's universal simple Lie algebra to the Lie algebra describing the Rozansky-Witten invariants of a K3-surface. Most of the paper involves setting up a proper language to discuss the problem and we formulate nine open questions as we proceed.

In 1996 Deligne made a conjecture that exceptional Lie algebras formed a series [5]. He proposed a possible explanation: he expected an object specializing to all exceptional Lie algebras. Influenced by this conjecture, Vogel proposed such an object in 1999 [26]. Besides exceptional Lie algebras, it specializes to all simple (complex finite dimensional) Lie algebras as well as some Lie superalgebras. The object was named Vogel's universal simple Lie algebra \mathfrak{g}_V . With Vogel's paper still unpublished the object remains a mystery. It is a subject of several recent studies [17, 19, 28].

The aim of this paper is to describe another specialization of \mathfrak{g}_V . In a private conversation Westbury has asked the author whether \mathfrak{g}_V specializes to \mathfrak{g}_X , a certain Lie algebra associated by Kapranov [11] to an irreducible holomorphic symplectic manifold X in the study of Rozansky-Witten invariants [22]. By *Westbury's Conjecture* we understand existence of such a specialization. We give an affirmative answer to Westbury's Conjecture in the case of a K3-surface. It remains open in the general case.

Most of the paper is devoted to setting up a machinery just to ask the question rigorously. There are several levels of generality in which the setup can be laid out. As opposed to a more general operadic language of Hinich and Vaintrob [6], we choose a more elementary language in the spirit of Sawon [24] to benefit a wider audience.

Here is a detailed description of the paper. Section 1 contains all necessary definitions and facts about the categories. In Section 2 we discuss algebras in the categories. We speculate on possible utility of such algebras in the study of identities. In Section 3 we study the Lie algebras. We formulate three different notions of simplicity. We introduce Vogel's universal Lie algebra. In Section 4 we investigate the Lie algebras that appear in the study of the Rozansky-Witten invariants and give an affirmative answer to Westbury's conjecture in the case of a K3-surface.

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1. CATEGORIES

1.1. **Tensor categories.** By a *tensor category over a commutative ring* \mathbb{K} we understand a \mathbb{K} -linear¹ symmetric monoidal category

$$\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{a}, \mathbb{I}, \mathbf{l}, \mathbf{r}, \mathbf{c})$$

where $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is the tensor product bifunctor, $\mathbf{a}_{A,B,C} : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$ is the associativity transformation, $\mathbf{c}_{A,B} : A \otimes B \rightarrow B \otimes A$ is the symmetric braiding, $\mathbb{I} \in \mathcal{C}$ is the monoidal identity, $\mathbf{l}_A : \mathbb{I} \otimes A \rightarrow A$ and $\mathbf{r}_A : A \otimes \mathbb{I} \rightarrow A$ are left and right unity transformations. We require the structures to agree: the tensor

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¹The home-sets are \mathbb{K} -modules and the compositions are \mathbb{K} -bilinear, for instance, if $\mathbb{K} = \mathbb{Z}$ this means that the category is preadditive.

product must be \mathbb{K} -bilinear on morphisms and the natural morphism $\mathbb{K} \rightarrow \text{end}_{\mathcal{C}}(\mathbb{I}) = \text{hom}_{\mathcal{C}}(\mathbb{I}, \mathbb{I})$ must be a ring isomorphism. We attempt to follow notations and terminology of Joyal and Street [9], although their “tensor category” is the same as a monoidal category, while our tensor categories are \mathbb{K} -linear and symmetric.

Observe that \mathcal{C} is not required to be abelian or even additive. Additivity can be easily fixed by completing the category with respect to finite direct sums and the zero object but not being abelian is an important distinction with other “tensor categories” that appear in the literature. Our tensor categories are close to those of Hinich and Vaintrob [6] with an exception of our additional assumption that $\mathbb{K} \rightarrow \text{end}_{\mathcal{C}}(\mathbb{I})$ is an isomorphism. This assumption is not very restrictive.

Proposition 1. *Suppose $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{a}, \mathbb{I}, \mathbf{l}, \mathbf{r}, \mathbf{c})$ is a symmetric monoidal additive category. Then the ring $\mathbb{K}_{\mathcal{C}} = \text{end}_{\mathcal{C}}(\mathbb{I})$ is commutative and the category \mathcal{C} is a tensor category of $\mathbb{K}_{\mathcal{C}}$.*

Proof. The ring $\mathbb{K}_{\mathcal{C}}$ is commutative by the Eckmann-Hilton argument [13, Prop 6.2]. Each $\text{hom}_{\mathcal{C}}(X, Y)$ is a $\mathbb{K}_{\mathcal{C}}$ - $\mathbb{K}_{\mathcal{C}}$ -bimodule via the left action

$$\text{end}_{\mathcal{C}}(\mathbb{I}) \times \text{hom}_{\mathcal{C}}(X, Y) \rightarrow \text{hom}_{\mathcal{C}}(\mathbb{I} \otimes X, \mathbb{I} \otimes Y) \cong \text{hom}_{\mathcal{C}}(X, Y)$$

and the right action

$$\text{hom}_{\mathcal{C}}(X, Y) \times \text{end}_{\mathcal{C}}(\mathbb{I}) \rightarrow \text{hom}_{\mathcal{C}}(X \otimes \mathbb{I}, Y \otimes \mathbb{I}) \cong \text{hom}_{\mathcal{C}}(X, Y).$$

These actions coincide: given $\alpha \in \text{end}_{\mathcal{C}}(\mathbb{I})$, $f \in \text{hom}_{\mathcal{C}}(X, Y)$, both αf and $f\alpha$ can be read off the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\mathbf{l}_X^{-1}} & \mathbb{I} \otimes X & \xrightarrow{\alpha \otimes f} & \mathbb{I} \otimes Y & \xrightarrow{\mathbf{l}_Y} & Y \\ \downarrow = & & \mathbf{c}_{i,X} \downarrow & & \downarrow \mathbf{c}_{i,Y} & & \downarrow = \\ X & \xrightarrow{\mathbf{r}_X^{-1}} & X \otimes \mathbb{I} & \xrightarrow{f \otimes \alpha} & Y \otimes \mathbb{I} & \xrightarrow{\mathbf{r}_Y} & Y \end{array}$$

as compositions from the top left corner to the bottom right corner. The diagram is commutative: the outside squares are commutative by the standard fact [9, Prop 2.1] while the middle square is commutative because the commutativity constraint \mathbf{c} is a natural transformation. Hence, $\alpha f = f\alpha$.

Finally, $\mathbb{K}_{\mathcal{C}}$ -bilinearity of compositions is evident. \square

Tensor categories are ubiquitous in modern mathematics, so we spare the reader of an example now but introduce some as we go along.

1.2. Tensor powers in a tensor category. Given an object A in a tensor category over \mathbb{K} , we define its iterated tensor products by $A^{\otimes 0} := \mathbb{I}$ and $A^{\otimes n} := A^{\otimes(n-1)} \otimes A$. There is an action of the symmetric group S_n on the object $A^{\otimes n}$, i.e. a semigroup homomorphism

$$S_n \rightarrow \text{end}_{\mathcal{C}}(A^{\otimes n}, A^{\otimes n}), \quad \sigma \mapsto \tilde{\sigma}_A,$$

defined as follows. Using a chain of associativity constraints

$$\gamma_i : A^{\otimes n} \rightarrow A^{\otimes(i-1)} \otimes ((A \otimes A) \otimes A^{\otimes(n-i-1)}),$$

we define it on transpositions by

$$(\widetilde{i, i+1})_A := \gamma_i^{-1} \circ (I \otimes (\mathbf{c}_{A,A}) \otimes I) \circ \gamma_i$$

and extend to the whole group: existence of such an extension follows from the axioms of symmetric monoidal category. It gives a $\mathbb{K}S_n$ -module structure on $\text{hom}_{\mathcal{C}}(A^{\otimes n}, B)$ for a pair of objects A, B of \mathcal{C} .

1.3. Extension of scalars. Given a tensor category \mathcal{C} over \mathbb{K} and ring homomorphisms $\mathbb{F} \rightarrow \mathbb{K}$ and $\mathbb{F} \rightarrow \mathbb{K}'$ of commutative rings, we can construct a new tensor category $\mathcal{C}' = \mathcal{C} \otimes_{\mathbb{F}} \mathbb{K}'$ over $\mathbb{K} \otimes_{\mathbb{F}} \mathbb{K}'$. It has the same objects as \mathcal{C} but new morphisms $\text{hom}_{\mathcal{C}'}(X, Y) := \text{hom}_{\mathcal{C}}(X, Y) \otimes_{\mathbb{F}} \mathbb{K}'$. The compositions are defined in the obvious way: $(f \otimes \alpha) \circ (g \otimes \beta) = fg \otimes \alpha\beta$. Similarly, all the natural transformations are extended to \mathcal{C}' . All verifications are routine and left to an interested reader.

1.4. **Tensor functors.** Let \mathcal{C} and \mathcal{D} be tensor categories over the same commutative ring \mathbb{K} . A *tensor functor* $F = (F_1, F_2, F_3) : \mathcal{C} \rightarrow \mathcal{D}$ consists of a \mathbb{K} -linear functor $F_1 : \mathcal{C} \rightarrow \mathcal{D}$, a natural isomorphism $F_{2,A,B} : FA \otimes FB \rightarrow F(A \otimes B)$ and an isomorphism $F_3 : \mathbb{I}_{\mathcal{D}} \rightarrow F\mathbb{I}_{\mathcal{C}}$. They have to satisfy the standard axioms [9].

2. ALGEBRAS

2.1. **Algebras.** We can talk about algebras in \mathcal{C} over a \mathbb{K} -linear operad or even a \mathbb{K} -linear prop² [6]. A *\mathbb{K} -linear prop* is a tensor category \mathcal{P} over \mathbb{K} satisfying the following five conditions:

- (1) its objects form the monoid $(\mathbb{N}, +)$, i.e. $n \otimes m = n + m$,
- (2) it is strict, in this context it means that all $\mathbf{a}_{n,m,k} = I_{n+m+k}$, $\mathbf{l}_n = I_n$, and $\mathbf{r}_n = I_n$,
- (3) for each object $n > 0$ a semigroup embedding $S_n \hookrightarrow \text{hom}_{\mathcal{P}}(n, n)$, $\pi \mapsto \tilde{\pi}$ is chosen,
- (4) for all $\pi \in S_n$, $\sigma \in S_m$ their tensor product is another permutation $\tau \in S_{n+m}$:

$$\tilde{\pi} \otimes \tilde{\sigma} = \tilde{\tau} \text{ where } \tau(k) = \begin{cases} \pi(k) & \text{if } k \leq n \\ \sigma(k-n) + n & \text{if } k > n \end{cases},$$

- (5) the commutativity constraint is the shuffle $\mathbf{sh}_{n,m} \in S_{n+m}$:

$$\mathbf{c}_{n,m} = \widetilde{\mathbf{sh}}_{n,m} \text{ where } \mathbf{sh}_{n,m}(k) = \begin{cases} k+m & \text{if } k \leq n \\ k-n & \text{if } k > n \end{cases}.$$

Now an *algebra in \mathcal{C}* over a prop \mathcal{P} is a tensor functor $\mathcal{P} \rightarrow \mathcal{C}$. This level of generality is not always useful as the prop needs to be constructed. A more elementary approach is more suitable for our purposes. A *signature* $I = (I, h, t)$ is a set of operations I with head and tail functions $h, t : I \rightarrow \mathbb{Z}_{\geq 0}$, so that we think of $i \in I$ as an operation on $t(i)$ variables with $h(i)$ values. An *algebra of signature I* is a triple $(\mathcal{C}, A, (m_i, i \in I))$ where \mathcal{C} is a tensor category over a commutative ring \mathbb{K} , A is an object of \mathcal{C} , and $m_i \in \text{hom}_{\mathcal{C}}(A^{\otimes t(i)}, A^{\otimes h(i)})$ is a family of morphisms. If \mathbb{K} or \mathcal{C} is fixed, we talk about \mathbb{K} -algebras or \mathcal{C} -algebras. We specify a signature by listing the size of I and pairs $(t(i), h(i))$, thinking that $I = \{1, 2, \dots, n\}$ and the k -th pair are the tail and head values of m_k .

Let us contemplate the connection between these two notions of algebra. A prop \mathcal{P} admits a generating set of morphisms $I = \{i \mid i \in \text{hom}_{\mathcal{P}}(h(i), t(i))\}$. An algebra $F : \mathcal{P} \rightarrow \mathcal{C}$ over a prop \mathcal{P} determines the algebra $(\mathcal{C}, F_1(1), (F_1(i), i \in I))$ of the signature $I = (I, h, t)$, determined by the generating set. In the opposite direction, an algebra $(\mathcal{C}, A, (m_i, i \in I))$ of the signature $I = (I, h, t)$ may or may not determine an algebra over \mathcal{P} . In general, it determines only an algebra over a free prop $\mathcal{P}(I)$. The prop \mathcal{P} is a quotient of the free prop $\mathcal{P}(I)$, so to determine an algebra over \mathcal{P} the algebra must satisfy some axioms, coming from \mathcal{P} . Thus, the props provide an implicit method of requesting axioms on algebras, but often it is more convenient to do explicitly.

In the partial case of an algebra A of signature $[1 : (2, 1)]$ the axioms are just polylinear identities. Let $F = \mathbb{K}\{x_1, \dots\}$ be the free nonassociative algebra in countably many variables, F_n the subset of polylinear elements of degree n , linear in fixed variables, say x_1, \dots, x_n . There is a natural ‘‘evaluation’’ map of \mathbb{K} -modules

$$\psi_A : F_n \rightarrow \text{hom}_{\mathcal{C}}(A^{\otimes n}, A),$$

defined recursively on monomials:

$$\psi(x_1) = I_A, \quad \psi(x_1 x_2) = m_1, \quad \psi(v(x_{\sigma(1)}, \dots, x_{\sigma(k)})w(x_{\sigma(k+1)}, \dots, x_{\sigma(n)})) = m_1 \circ (\psi(v) \otimes \psi(w)) \circ \gamma \circ \widetilde{\sigma}^{-1}_A$$

where $v \in F_k$, $w \in F_{n-k}$, $\sigma \in S_n$ and $\gamma : A^{\otimes n} \rightarrow A^{\otimes k} \otimes A^{\otimes (n-k)}$ is the composition of associativity constraints, and then extended to linear combinations by $\psi(\sum_i \alpha_i v_i) = \sum_i \alpha_i \psi(v_i)$. Now we say that A satisfies a polylinear identity f if $\psi_A(f) = 0$.

A *Lie algebra* is an algebra of signature $[1 : (2, 1)]$ satisfying the Jacobi and the anticommutativity identities:

$$(x_1 x_2) x_3 + (x_2 x_3) x_1 + (x_3 x_1) x_2 \quad \text{and} \quad x_1 x_2 + x_2 x_1,$$

which can be written explicitly in the tensor notation as

$$m_1 \circ (m_1 \otimes I) \circ (\widetilde{1} + \widetilde{(2, 3, 1)} + \widetilde{(3, 1, 2)}) = 0, \quad m_1 \circ (\widetilde{1} + \widetilde{(1, 2)}) = 0.$$

²PROP is an abbreviation of ‘‘product and permutation category’’, introduced by MacLane.

Notice that if \mathcal{C} is the category of \mathbb{K} -modules and $\frac{1}{2} \in \mathbb{K}$ then our definition is just the usual definition of a Lie algebra over \mathbb{K} . However, if $\frac{1}{2} \notin \mathbb{K}$, our anticommutativity is weak, i.e., $xy + yx = 0$ but we cannot conclude that $x^2 = 0$.

An *associative algebra* is an algebra of signature $[2 : (2, 1), (0, 1)]$ satisfying the associativity $(x_1 x_2) x_3 = x_1 (x_2 x_3)$ and the left and right unities. The latter cannot be written in terms of ψ_A so we recourse to the tensor notation formulating the axioms:

$$m_1 \circ (m_1 \otimes I) = m_1 \circ (I \otimes m_1) \circ \mathbf{a}_{A,A,A}, \quad m_1 \circ (m_2 \otimes I) = \mathbf{l}_A, \quad m_1 \circ (I \otimes m_2) = \mathbf{r}_A.$$

A *metric object* is an algebra of signature $[2 : (0, 2), (2, 0)]$ such that both structures are symmetric:

$$m_2 = m_2 \circ \mathbf{c}_{A,A}, \quad m_1 = \mathbf{c}_{A,A} \circ m_1,$$

and A is a dual object of A in the sense of monoidal categories, i.e., the identity

$$I_A = \mathbf{r}_A \circ (I_A \otimes m_2) \circ \mathbf{a}_{A,A,A} \circ (m_1 \otimes I_A) \circ \mathbf{l}_A^{-1}$$

holds. Notice that m_2 is “non-degenerate” in this case. If $(A^*, u \in \text{hom}(A \otimes A^*, \mathbb{I}), c \in \text{hom}(\mathbb{I}, A^* \otimes A))$ is another dual object, then

$$A \xrightarrow{\mathbf{l}_A^{-1}} \mathbb{I} \otimes A \xrightarrow{c \otimes I} (A^* \otimes A) \otimes A \xrightarrow{\mathbf{a}_{A^*,A,A}} A^* \otimes (A \otimes A) \xrightarrow{I \otimes m_2} A^* \otimes \mathbb{I} \xrightarrow{\mathbf{r}_{A^*}} A^*$$

is a canonical isomorphism. Following Hinich and Vaintrob [6], we are introducing Casimir and metric Lie algebras in the next section.

2.2. Representations. Let $(\mathcal{C}, \mathfrak{g}, m)$ be a Lie algebra. A *representation* of \mathfrak{g} is a pair (M, ρ) where M is an object in \mathcal{C} , $\rho \in \text{hom}_{\mathcal{C}}(\mathfrak{g} \otimes M, M)$ and the Jacobi identity

$$\rho \circ (m \otimes I_M) = \rho \circ (I_{\mathfrak{g}} \otimes \rho) \circ \mathbf{a}_{\mathfrak{g},\mathfrak{g},M} - \rho \circ (I_{\mathfrak{g}} \otimes \rho) \circ \mathbf{a}_{\mathfrak{g},\mathfrak{g},M} \circ (\mathbf{c}_{\mathfrak{g},\mathfrak{g}} \otimes I_M) \in \text{hom}_{\mathcal{C}}((\mathfrak{g} \otimes \mathfrak{g}) \otimes M, M)$$

holds. Completely parallel to the case of Lie algebras in vector spaces, the representations form a tensor category $\text{mod}_{\mathcal{C}}(\mathfrak{g})$. The morphisms $\text{hom}_{\mathfrak{g}}(M, N)$ are those morphisms in $\text{hom}_{\mathcal{C}}(M, N)$ that commute with the action of \mathfrak{g} . The tensor product is the tensor product in \mathcal{C} with the usual action:

$$\rho_{M \otimes N} = (\rho_M \otimes I_N) \circ \mathbf{a}_{\mathfrak{g},M,N}^{-1} + (I_M \otimes \rho_N) \circ \mathbf{a}_{M,\mathfrak{g},N} \circ (\mathbf{c}_{\mathfrak{g},M} \otimes I_M) \circ \mathbf{a}_{\mathfrak{g},M,N}^{-1} \in \text{hom}_{\mathcal{C}}(\mathfrak{g} \otimes (M \otimes N), M \otimes N).$$

The constraints are inherited from \mathcal{C} . Let us summarise this in a proposition whose proof is left to an interested reader.

Proposition 2. *Let \mathfrak{g} be a Lie \mathcal{C} -algebra. Then the category $\text{mod}_{\mathcal{C}}(\mathfrak{g})$ is a tensor category over $\mathbb{K}_{\mathcal{C}}$. Its identity object is $\mathbb{I}_{\text{mod}_{\mathcal{C}}(\mathfrak{g})} = \mathbb{I}_{\mathcal{C}}$ with zero action. The multiplication in \mathfrak{g} is a morphism in $\text{mod}_{\mathcal{C}}(\mathfrak{g})$, so \mathfrak{g} is a Lie $\text{mod}_{\mathcal{C}}(\mathfrak{g})$ -algebra.*

One could ask what sort of tensor categories one gets by iteration: consider representations of \mathfrak{g} in $\text{mod}_{\mathcal{C}}(\mathfrak{g})$, etc. The answer is rather uninteresting: the reader can easily verify that subject to the existence of finite direct sums in \mathcal{C} , the category of representations of \mathfrak{g} in $\text{mod}_{\mathcal{C}}(\mathfrak{g})$ is canonically equivalent to the category of representations of the semidirect product $\mathfrak{g} \ltimes \mathfrak{g}_a$ in \mathcal{C} (where \mathfrak{g}_a is the adjoint representation of \mathfrak{g} treated as a Lie algebra with the zero Lie product).

Now a *Casimir Lie algebra* is a \mathcal{C} -algebra of signature $[2 : (2, 1), (0, 2)]$ such that m_1 defines a Lie algebra while m_2 is a symmetric morphism in the category of \mathfrak{g} -modules.

Similarly a *metric Lie algebra* is a \mathcal{C} -algebra \mathfrak{g} of signature $[3 : (2, 1), (0, 2), (2, 0)]$ such that (\mathfrak{g}, m_1) is a Lie \mathcal{C} -algebra, and (\mathfrak{g}, m_2, m_3) is a metric $\text{mod}_{\mathcal{C}}(\mathfrak{g})$ -object.

2.3. Homomorphisms. Naturally there are two notions of a homomorphism between algebras of the same signature. A *homomorphism* from (\mathcal{C}, A, m_i) to (\mathcal{C}, B, m'_i) is a morphism $\varphi \in \text{hom}_{\mathcal{C}}(A, B)$ which commutes with all multiplications, i.e. the diagram

$$\begin{array}{ccc} A^{\otimes h(i)} & \xrightarrow{\varphi^{\otimes h(i)}} & B^{\otimes h(i)} \\ m_i \downarrow & & \downarrow m'_i \\ A^{\otimes t(i)} & \xrightarrow{\varphi^{\otimes t(i)}} & B^{\otimes t(i)} \end{array}$$

is commutative for all i .

To avoid confusion we use the term specialization for the second notion. A *specialization from* (\mathcal{C}, A, m_i) to (\mathcal{D}, B, m'_i) is a triple $\psi = (\psi_1, \psi_2, \psi_3)$ where $\psi_1 : \mathbb{K}_{\mathcal{C}} \rightarrow \mathbb{K}_{\mathcal{D}}$ is a ring homomorphism, $\psi_2 : \mathcal{C} \otimes_{\mathbb{K}_{\mathcal{C}}} \mathbb{K}_{\mathcal{D}} \rightarrow \mathcal{D}$ is a tensor functor, and ψ_3 is a homomorphism from $\psi_2(A)$ to B . A specialization such that ψ_3 is an isomorphism is called *isospecialization*.

Notice that we may consider the three types of homomorphisms (or specializations) between metric Lie algebras: a Lie algebra homomorphism, a Casimir Lie algebra homomorphism, and a metric Lie algebra homomorphism. The difference is in the preserved operations.

2.4. Identities of algebras in categories. Varieties of algebras is a topic of an active study in Algebra [3]. Should one study varieties of algebras in categories? In this subsection we make some easy observations and pose some questions.

At the first glance, at least if \mathbb{K} is a \mathbb{Q} -algebra, the varieties of algebras in categories are not much different from the varieties of usual algebras. Indeed, an interested reader could easily verify that the identities of a \mathcal{C} -algebra is equal to the set of polylinear elements in some T -ideal (an ideal closed under substitutions) of the free algebra $\mathbb{K}\langle x_1, \dots \rangle$. For instance, the following Engel-type theorem is an immediate consequence of the same theorem for usual Lie algebras [16, Th 6.4.1].

Theorem 3. *Suppose \mathbb{K} is a commutative \mathbb{Q} -algebra, \mathcal{C} is a tensor category over \mathbb{K} , and \mathfrak{g} is a Lie \mathcal{C} -algebra satisfying the linearised Engel identity. Then \mathfrak{g} satisfies the linearised nilpotency.*

Nevertheless, in our view, it is interesting to understand examples of \mathcal{C} -algebras, in particular, what identities they satisfy. For instance, if X is a rigid object in a tensor category \mathcal{C} then $E_X = X \otimes X^*$ is an associative \mathcal{C} -algebra, so-called *the internal endomorphism algebra*. The identities of E_X in the free associative algebra $\mathbb{K}\langle x_1, x_2, \dots \rangle$ form a T -ideal $I(E_X) \triangleleft \mathbb{K}\langle x_1, x_2, \dots \rangle$. It is interesting to study it [3].

Question 1. *Given a rigid object X in \mathcal{C} , what is the minimal degree of an element of $I(E_X)$?*

If \mathcal{C} is the category of vector spaces, X is an n -dimensional vector space then E_X is the algebra of $n \times n$ matrices, whose minimal identity has degree $2n$ by the Amitsur-Levitzki theorem [1]. If \mathcal{C} is the category of supervector spaces (this is the same as superbundles on a point, see Section 4.1), X is an (n, k) -dimensional supervector space then E_X is the matrix superalgebra whose minimal identity is conjectured to be of the degree $2(nk + n + k) - \min\{n, k\}$ [23].

One reason the matrix superalgebras are of interest is that in characteristic zero they generate all prime varieties of associative algebras [14]. Is it conceivable that algebras E_X generate all varieties?

Question 2. *Give an example of a commutative \mathbb{Q} -algebra \mathbb{K} and a T -ideal in $\mathbb{K}\langle x_1, \dots \rangle$ that is not an ideal of identities of E_X for any rigid object X in any tensor category \mathcal{C} over \mathbb{K} .*

We finish this section explaining the categorical analogue of the Grassmann envelope used to study the varieties of superalgebras. The *global sections functor* $\Gamma : \mathcal{C} \rightarrow \text{mod}(\mathbb{K}_{\mathcal{C}})$ is defined by $\Gamma(A) := \text{hom}_{\mathcal{C}}(\mathbb{I}, A)$. If A is an algebra in \mathcal{C} then $\Gamma(A)$ is a $\mathbb{K}_{\mathcal{C}}$ -algebra in $\mathbb{K}_{\mathcal{C}}$ -modules. The following proposition is evident.

Proposition 4. *Suppose \mathcal{C} is a tensor category, A is a \mathcal{C} -algebra, C is an associative commutative \mathcal{C} -algebra. Then $\Gamma(A \otimes C)$ is a $\text{mod}(\mathbb{K}_{\mathcal{C}})$ -algebra of the same signature as A . Moreover, $\Gamma(A \otimes C)$ satisfies all identities of A .*

The Grassmann algebra $G_{\infty} := \mathbb{K}\langle x_1, x_2, \dots \rangle / (x_i x_j + x_j x_i)$ in countably many variables is an associative commutative algebra in the category \mathcal{C} of supervector spaces. If A is a superalgebra then $\Gamma(A \otimes G_{\infty})$ is the Grassmann envelope of A . The identities of $\Gamma(A \otimes G_{\infty})$ are precisely the identities of A in \mathcal{C} . Can this be carried out in any category?

Question 3. *Let \mathcal{C} be a cocomplete³ tensor category with split idempotents⁴. Does there exist a commutative associative algebra G in \mathcal{C} such that for any \mathcal{C} -algebra A all identities of $\Gamma(A \otimes G)$ hold in A ?*

³closed under arbitrary direct sums.

⁴every idempotent $e \in \text{hom}_{\mathcal{C}}(X, X)$ is the projection operator $X \xrightarrow{\cong} Y \oplus Z \rightarrow Y \hookrightarrow X$.

3. SIMPLE LIE ALGEBRAS

3.1. Quasisimple Lie algebras. Following Vogel [25, 26], a Lie algebra $(\mathcal{C}, \mathfrak{g}, m_1)$ is *quasisimple* if it can be extended to a metric Lie algebra $(\mathcal{C}, \mathfrak{g}, m_1, m_2, m_3)$ and the natural map $\mathbb{K}_{\mathcal{C}} \rightarrow \text{hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g})$ is an isomorphism. We discuss how it is related to the usual simplicity in the next section.

3.2. Simple Lie algebras. If the \mathbb{K} -tensor category \mathcal{C} is abelian, we can talk about subobjects, quotients, kernels, ideals, etc. in the usual way. In particular, there is the usual notion of a simple Lie algebra. However, the category being abelian is not required to introduce simplicity, although the utility of this notion is unclear. Neither is it clear whether this definition is useful for other classes of algebras.

Recall that a morphism $f \in \text{hom}_{\mathcal{C}}(X, Y)$ is a *monomorphism* if it can be cancelled on the left, i.e., $fa = fb$ implies $a = b$ for all $a, b \in \text{hom}_{\mathcal{C}}(W, X)$. A Lie algebra \mathfrak{g} in \mathcal{C} is *simple* if $\mathfrak{g} \not\cong 0$ (notice that if a zero object exists, it is unique up to a canonical isomorphism) and every algebra homomorphism $f : \mathfrak{g} \rightarrow \mathfrak{h}$ is a monomorphism or the zero. Notice the fine difference with the usual simple Lie algebras in vector spaces: the latter definition traditionally excludes the one-dimensional Lie algebra while our definition does not.

In general, there is no relation between simple and quasisimple Lie algebras. Let \mathbb{K} be a commutative ring, not a field, $1/2 \in \mathbb{K}$. It admits a proper ideal I . Then the Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ (in the category of \mathbb{K} -modules) is quasisimple: the trace form $m_2(X \otimes Y) := \text{Tr}(XY)$ gives a metric structure with a coform

$$m_3(\alpha) := \alpha \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \right).$$

It is not simple because the quotient homomorphism $\mathfrak{sl}_2(\mathbb{K}) \rightarrow \mathfrak{sl}_2(\mathbb{K}/I)$ is neither zero, nor a monomorphism.

Here is another example of a quasisimple non-simple Lie algebra. Let X be a set of simple finite dimensional complex Lie algebras, $C = \mathbb{C}X$ the group coalgebra. A C -comodule is just an X -graded vector space $M = \bigoplus_{x \in X} M_x$. We consider the tensor category \mathcal{C} of C -comodules with the standard tensor coproduct as a tensor product: $M \otimes N := \bigoplus_{x \in X} (M_x \otimes_{\mathbb{C}} N_x)$. Now $\mathfrak{g} = \bigoplus_{\mathfrak{h} \in X} \mathfrak{h}$ is a metric Lie \mathcal{C} -algebra under the direct sums of the products, the forms and the coforms

$$\bigoplus_{\mathfrak{h} \in X} m_{\mathfrak{h}} : \bigoplus (\mathfrak{h} \otimes \mathfrak{h}) \rightarrow \bigoplus \mathfrak{h}, \quad \bigoplus_{\mathfrak{h} \in X} m_{\mathfrak{h}} : \bigoplus (\mathfrak{h} \otimes \mathfrak{h}) \rightarrow C, \quad \bigoplus_{\mathfrak{h} \in X} m_{\mathfrak{h}} : C \rightarrow \bigoplus (\mathfrak{h} \otimes \mathfrak{h}).$$

Observe that \mathfrak{g} is quasisimple but, if X contains at least two elements, not simple since every \mathfrak{h} is a proper ideal of \mathfrak{g} resulting in the quotient homomorphism.

In the opposite direction, $\mathfrak{sl}_2(\mathbb{C})$ is a simple Lie algebra in the category of real vector spaces but it is not quasisimple because $\text{hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) = \mathbb{C} \neq \mathbb{R}$. Nevertheless, sometimes there is a relation.

Theorem 5. *Let \mathcal{C} be the category of vector spaces over an algebraically closed field \mathbb{K} of characteristic zero. Then quasisimple Lie \mathcal{C} -algebras are exactly the finite dimensional simple⁵ Lie algebras.*

Proof. A finite dimensional simple Lie algebra \mathfrak{g} is quasisimple because the Killing form is non-degenerate and $\mathbb{K} \rightarrow \text{end}_{\mathfrak{g}}(\mathfrak{g})$ is an isomorphism by the Schur lemma and algebraic closeness of \mathbb{K} .

In the opposite direction, the space of symmetric invariant form $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{K}$ is one-dimensional since it is a subspace of $(\mathfrak{g}^* \otimes \mathfrak{g}^*)^{\mathfrak{g}} \cong \text{hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) = \mathbb{K}$. By the theorem of Bajo and Benayadi [2], \mathfrak{g} is simple. \square

It would be interesting to find some other categories where an analogue of Theorem 5 holds or where the quasisimple Lie algebras admit a meaningful classification. Let us pose two precise questions.

Question 4. *Let G be an affine group scheme over a field \mathbb{K} of characteristic zero. Let \mathcal{C} be the category of rational G -modules. What are simple and quasisimple Lie algebras in \mathcal{C} ?*

To get a feel of this question, let us consider an absolutely irreducible finite dimensional G -module V as a Lie algebra with the zero multiplication in \mathcal{C} . According to our definition, V is simple. On the other hand, V is quasisimple if and only if $V \cong V^*$.

Question 5. *Let \mathcal{C} be the category of supervector spaces⁶ over an algebraically closed field \mathbb{K} of characteristic zero. Is a quasisimple Lie \mathcal{C} -algebra a finite dimensional simple Lie superalgebra?*

⁵Notice that this includes the one-dimensional Lie algebra according to our definition

⁶It is equivalent to the category of rational C_2 -modules as a monoidal category, but not as a tensor category.

Finite dimensional simple Lie superalgebras are classified by Kac [10]. Simple Lie algebras and simple Lie superalgebras of types $A(m, n)$, $n \neq m$, $B(m, n)$, $C(n)$, $D(m, n)$, $m - n \neq 1$, $F(4)$, and $G(3)$ have nondegenerate Killing forms [10, Th. 1]. Hence, they are quasisimple Lie \mathcal{C} -algebras. The remaining simple Lie superalgebras of types $A(n, n)$, $D(n + 1, n)$, $P(n)$, $Q(n)$, and $D(2, 1, \alpha)$ as well as Cartan type superalgebras have zero Killing forms [10, Prop. 2.4.1], although there may be an invariant form distinct from the Killing form. For instance, this happens in types $Q(n)$ and $D(2, 1, \alpha)$. Her is a partial result towards this question.

Proposition 6. *Let \mathfrak{g} be a quasisimple Lie algebra in the category of supervector spaces over a field \mathbb{K} . Then*

- (1) \mathfrak{g} is perfect,
- (2) the centre of \mathfrak{g} is zero,
- (3) every minimal ideal of \mathfrak{g} is abelian.

Proof. Let I be a minimal ideal. Its orthogonal complement I^\perp is also an ideal. Hence, the intersection $I \cap I^\perp$ is zero or I . If $I \cap I^\perp = 0$, then $\mathfrak{g} = I \oplus I^\perp$ and we get nontrivial idempotents in $\mathbb{K} = \text{end}_{\mathfrak{g}}(\mathfrak{g})$. Hence $I \cap I^\perp = I$, so $I \subseteq I^\perp$. This means that $([x, y], a) = (x, [y, a]) = 0$ for all $x, y \in I$, $a \in \mathfrak{g}$. As the form is nondegenerate, $[x, y] = 0$ and I is abelian.

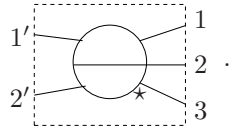
Now $[\mathfrak{g}, \mathfrak{g}]^\perp = \{a \in \mathfrak{g} \mid \forall x, y \in \mathfrak{g} ([x, y], a) = (x, [y, a]) = 0\} = \{a \in \mathfrak{g} \mid \forall y \in \mathfrak{g} [y, a] = 0\} = Z(\mathfrak{g})$. We conclude that \mathfrak{g} is perfect and $Z(\mathfrak{g}) = 0$. Indeed, otherwise there exists a nonzero bilinear form α on $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Extend it to a form $\tilde{\alpha}$ on \mathfrak{g} : $\tilde{\alpha}(x, y) := \alpha(x + [\mathfrak{g}, \mathfrak{g}], y + [\mathfrak{g}, \mathfrak{g}])$. The form $\tilde{\alpha}$ is invariant ($\tilde{\alpha}([x, y], z) = 0 = \tilde{\alpha}(x, [y, z])$), nonzero and degenerate. This contradicts the one-dimensionality of $\text{end}_{\mathfrak{g}}(\mathfrak{g}) \cong (\mathfrak{g} \otimes \mathfrak{g})^{*\mathfrak{g}}$. \square

Using Proposition 6 and Cartan's criteria [7, 4.3 and 5.1], we can furnish an alternative proof of Theorem 5. Consider \mathfrak{g} as a superalgebra with zero odd part, hence Proposition 6 applies. Since \mathfrak{g} is not solvable, the Killing form is nonzero. So the Killing form is a multiple of the non-degenerate form, and itself nondegenerate. Thus, \mathfrak{g} is semisimple. It is simple since $\text{end}_{\mathfrak{g}}(\mathfrak{g}) = \mathbb{K}$. This proof fails for the superalgebras because of the lack of the Cartan criteria.

3.3. Universal metric Lie algebra. The universal metric Lie algebra appears in the study of Vassiliev invariants [20, 26]. The relevant symmetric monoidal category \mathcal{C} is a prop. The hom-set $\text{hom}_{\mathcal{C}}(m, n)$ is the \mathbb{Q} -vector space of Jacobi diagrams⁷ [20]. Recall that an (m, n) -Jacobi diagram is a compact curve X such that

- (1) the boundary of X is the set $\{1, 2, \dots, m, 1', 2' \dots n'\}$,
- (2) X has finitely many trivalent singular points, i.e. points x with a neighbourhood U such that $U \setminus \{x\}$ is a union of three lines,
- (3) for each trivalent singular point x a cyclic ordering on the components of $U \setminus \{x\}$ is fixed.

Here is an example of the Jacobi diagram in $\text{hom}_{\mathcal{C}}(3, 2)$:



The external dashed borders have no significance. This diagram has five trivalent singular points, four of them have the standard counterclockwise ordering, while the remaining vertex marked with \star has the opposite clockwise ordering. The point of transversal intersection has no significance: it is actually two distinct points of the curve which coincided after an immersion into a plane (a.k.a a drawing). Now $\text{hom}_{\mathcal{C}}(m, n)$ is the quotient space of the \mathbb{Q} -span of all (m, n) -Jacobi diagram subject to the *AS*-relation and the *IHX*-relation:

$$\begin{array}{c} \text{Y-shape} \end{array} + \begin{array}{c} \text{Y-shape with } \star \end{array} = 0, \quad \begin{array}{c} \text{IHX relation} \end{array} = 0.$$

⁷also known as Chinese characters

The dashed circles bound neighbourhoods in Jacobi diagrams X_1, X_2 for the first relation and Y_1, Y_2, Y_3 for the second one, which are identical except for these neighbourhoods. The relations mean that

$$X_1 + X_2 = 0 \quad \text{and} \quad Y_1 - Y_2 + Y_3 = 0.$$

The composition and the tensor product of morphisms is done by “stacking” the boxes:

$$\boxed{X} \circ \boxed{Y} = \boxed{X \quad Y} ; \quad \boxed{X} \otimes \boxed{Y} = \begin{array}{|c|} \hline X \\ \hline Y \\ \hline \end{array}$$

Thus, the tensor product of morphisms is just a union while the composition is gluing along the corresponding part of the boundary. The symmetric group S_n embeds in a way that $\tilde{\pi}$ is a union of n intervals connecting k with $\pi(k)'$:

$$\tilde{\pi} = \begin{array}{|c|} \hline 1' \\ \hline \pi(2)' \\ \hline \pi(n)' \\ \hline \pi(1)' \\ \hline \vdots \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array} .$$

The \mathcal{C} -algebra $\mathfrak{g}_M = (1, \mu, \tau, \gamma)$ where

$$\mu = \begin{array}{|c|} \hline 1' \\ \hline \vdots \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline \end{array} , \quad \tau = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline \end{array} , \quad \gamma = \begin{array}{|c|} \hline 1' \\ \hline 2' \\ \hline \vdots \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline \end{array}$$

is a metric Lie algebra [20, Prop. 2.4]. The reader can easily verify this. Jacobi identity and anticommutativity follow from the AS-relation and the IHX-relation. The axioms of a metric object, as well as invariance of the form and co-form are evident on a drawing. The universal property of this metric Lie algebra is more subtle [20, Prop. 2.5]. We state it as a separate proposition.

Proposition 7. *Given a commutative \mathbb{Q} -algebra \mathbb{K} and a metric Lie \mathbb{K} -algebra $(\mathcal{D}, \mathfrak{g}, m, t, c)$, there exists a unique specialization $\psi = (\psi_1, \psi_2, \psi_3) : \mathfrak{g}_M \rightarrow \mathfrak{g}$ such that*

- (1) $\psi_1 : \mathbb{Q} \rightarrow \mathbb{K}$ is the \mathbb{Q} -algebra structure on \mathbb{K} ,
- (2) $\psi_2 = (\psi_{21}, \psi_{22}, \psi_{23}) : \mathcal{C} \otimes \mathbb{K} \rightarrow \mathcal{D}$ is a tensor functor, defined on objects by $\psi_{21}(n) = \mathfrak{g}^{\otimes n}$, and uniquely determined on morphisms by the requirements that
 - (a) $\psi_{21}(\mu) = m$,
 - (b) $\psi_{21}(\tau) = t$,
 - (c) $\psi_{21}(\gamma) = c$,
 - (d) $\psi_{22,n,m} : \mathfrak{g}^{\otimes n} \otimes \mathfrak{g}^{\otimes m} \rightarrow \mathfrak{g}^{\otimes(n+m)}$ is the composition of associativity constraints in \mathcal{D} ,
 - (e) $\psi_{23} = I_{\mathbb{1}}$, the identity morphism of $\mathbb{1}_{\mathcal{D}}$,
- (3) $\psi_3 = I_{\mathfrak{g}}$, the identity morphism of \mathfrak{g} .

The specialization ψ is an example of isospecialization. Its constituent is a functor $\psi_2 : \mathcal{C} \otimes \mathbb{K} \rightarrow \mathcal{D}$ that turns \mathfrak{g} into an algebra over the prop $\mathcal{C} \otimes \mathbb{K}$. In the language of props $\mathcal{C} \otimes \mathbb{K}$ is the universal metric Lie algebra over \mathbb{K} : $\mathcal{C} \otimes \mathbb{K}$ -algebras are the same as metric Lie \mathbb{K} -algebras [20, Cor. 2.2].

In the rest of the section we will show that \mathfrak{g}_M is not quasisimple. Consider $\text{hom}_{\mathcal{C}}^k(n, m)$, the \mathbb{Q} -linear span of all those diagrams in $\text{hom}_{\mathcal{C}}(n, m)$ that contain precisely k connected components. The AS-relation and IHX-relation do not change the number of connected components. Hence the spaces $\text{hom}_{\mathcal{C}}^k(n, m)$ for different k have zero intersections, while $\text{hom}_{\mathcal{C}}(n, m)$ splits into a direct sum $\text{hom}_{\mathcal{C}}(n, m) = \bigoplus_{k=0}^{\infty} \text{hom}_{\mathcal{C}}^k(n, m)$. Notice that $\text{hom}_{\mathcal{C}}^0(n, m) \neq 0$ if and only if $n = m = 0$: it is spanned by the empty diagram. The tensor products preserve this grading, but the compositions, in general, destroy it. Let us summarize properties of the grading.

Proposition 8. *The following statements hold.*

- (1) $\text{hom}_{\mathcal{C}}^k(n, m) \otimes \text{hom}_{\mathcal{C}}^s(n, m) \subseteq \text{hom}_{\mathcal{C}}^{k+s}(n, m)$ for all k, s, n, m .
- (2) $\text{hom}_{\mathcal{C}}^k(0, m) \circ \text{hom}_{\mathcal{C}}^s(n, 0) \subseteq \text{hom}_{\mathcal{C}}^{k+s}(n, m)$ for all k, s, n, m .
- (3) Under this grading the scalar ring $\mathbb{K}_{\mathcal{C}} = \text{hom}_{\mathcal{C}}(0, 0)$ is a graded algebra, isomorphic to the symmetric algebra of $\text{hom}_{\mathcal{C}}^1(0, 0)$.
- (4) Under this grading each $\text{hom}_{\mathcal{C}}(n, m)$ is a graded $\mathbb{K}_{\mathcal{C}}$ -module.

Proof. The tensor product of morphisms is a disjoint union. Hence, the number of components of the tensor product is the sum, and (1) follows. A similar argument proves (2). (4) immediately follows from (1).

It follows from (1) that $\text{hom}_{\mathcal{C}}(0, 0)$ is a graded commutative algebra. Thus, there is a natural homomorphism $\varphi : S(\text{hom}_{\mathcal{C}}^1(0, 0)) \rightarrow \text{hom}_{\mathcal{C}}(0, 0)$ of graded algebras. It is surjective because each diagram is a product of its components.

Let us show that φ is injective. Consider $x_1, x_2 \dots$, all connected Jacobi diagrams without boundary before the *AS*-relation and the *IHX*-relation are imposed. There are natural homomorphisms of graded algebras φ_1, φ_2 making the diagram

$$\begin{array}{ccc}
 S(\text{hom}_{\mathcal{C}}^1(0, 0)) & \xrightarrow{\varphi} & \text{hom}_{\mathcal{C}}(0, 0) \\
 \varphi_1 \uparrow & \nearrow \varphi_2 & \\
 \mathbb{Q}[x_1, x_2 \dots] & &
 \end{array}$$

commutative. The kernel of φ_1 is the ideal I_1 generated by all $y_1 + y_2$ from the *AS*-relation and all $y_1 - y_2 + y_3$ from the *IHX*-relation, where y_1, y_2 and y_3 are connected. Similarly, The kernel of φ_2 is the ideal I_2 generated by all $z_1 + z_2$ from the *AS*-relation and all $z_1 - z_2 + z_3$ from the *IHX*-relation, where z_1, z_2 and z_3 are no longer necessarily connected. Both relations operate on just one component, that is, if $z_1 + z_2$ is an *AS*-relation, $z_1 = x_1 z$ and $z_2 = x_2 z$ where x_i is the component affected by the *AS*-relation and z is the union of the remaining components. Thus, $z_1 + z_2 = (x_1 + x_2)z \in I_1$. Similarly, $z_1 - z_2 + z_3 = (x_1 - x_2 + x_3)z \in I_1$ for all *IHX*-relations. It follows that $I_1 = I_2$ and φ is injective. \square

Since connected Jacobi diagrams span $\text{hom}_{\mathcal{C}}^1(0, 0)$, one can choose a basis $y_1, y_2 \dots$ among them. It follows that $\mathbb{K}_{\mathcal{C}} = \text{hom}_{\mathcal{C}}(0, 0)$ is isomorphic the polynomial algebra $\mathbb{Q}[y_1, y_2 \dots]$ on the chosen connected $(0, 0)$ -Jacobi diagrams such as

$$\delta = \boxed{\text{circle}} , \theta = \boxed{\text{circle with horizontal line}} , \dots \in \text{hom}_{\mathcal{C}}^1(0, 0),$$

while $\text{end}_{\mathfrak{g}_M}(\mathfrak{g}_M) = \text{hom}_{\mathcal{C}}(1, 1)$ is a graded $\mathbb{K}_{\mathcal{C}}$ -module. The homomorphism $\mathbb{K}_{\mathcal{C}} \rightarrow \text{hom}_{\mathcal{C}}(1, 1)$ is the action on the identity element $v \mapsto v \otimes I_1$. Since $I_1 \in \text{hom}_{\mathcal{C}}^1(1, 1)$ and $\text{hom}_{\mathcal{C}}^0(0, 0) = \mathbb{Q}$, any connected non-identity diagram from $\text{hom}_{\mathcal{C}}^1(1, 1)$ is not in the image of the map $\mathbb{K}_{\mathcal{C}} \rightarrow \text{end}_{\mathfrak{g}_M}(\mathfrak{g}_M)$. In particular, the following element ϕ is not in the image:

$$\phi = \boxed{\text{circle with two horizontal lines}} \in \text{hom}_{\mathcal{C}}(1, 1).$$

3.4. Vogel's ring Λ . The symmetric group S_n acts on the vector space $\text{hom}_{\mathcal{C}}^k(n, 0)$ by permuting the inputs. As a vector space, Vogel's ring Λ is $\text{hom}_{\mathcal{C}}^1(3, 0)^{S_3, \varepsilon}$, the skew invariants with respect to the sign character ε of S_3 . The multiplication is via insertion of one diagram into any trivalent point of the second diagram:

$$\boxed{\text{trivalent point}} \cdot \boxed{Y} = \boxed{\text{trivalent point with Y inside}}$$

Observe that connectedness and skew-invariance of elements of $\text{hom}_{\mathcal{C}}^1(3, 0)$ are crucial for this product to be well-defined. Once a trivalent singular point is chosen, there are six ways to insert a diagram from $\text{hom}_{\mathcal{C}}^1(3, 0)$. Skew-invariance removes this dependency. If two singular points are connected by an edge, one

can see that insertions into each of these two points give the same result. Consequently, the connectedness eliminates the dependency of insertion on the choice of a singular point. Vogel proves these facts [25, Prop 3.2]. At the same time he establishes that the insertion defines a structure of a Λ -module on the linear span of connected diagrams with a non-empty set of trivalent singular points $\text{hom}_{\mathcal{C}}^s(m, n) \subseteq \text{hom}_{\mathcal{C}}(m, n)$.

While associativity of Λ is obvious, commutativity requires a subtle argument [26, Prop 4.8]. Notice that Vogel works over any commutative coefficient ring R and proves the identity $12xy = 12yx$ in Λ_R . In our case $1/12 \in R = \mathbb{Q}$, therefore the ring Λ is commutative. Here are some elements of Λ :

$$1 = \begin{array}{|c|} \hline \text{Diagram: A trivalent vertex pointing left.} \\ \hline \end{array}, \quad t = \begin{array}{|c|} \hline \text{Diagram: A circle with three lines extending to the right.} \\ \hline \end{array}, \quad x_n = \begin{array}{|c|} \hline \text{Diagram: A circle with three lines extending to the right, inside a dashed box with a dot and 'n' above it.} \\ \hline \end{array}.$$

Let us note that $x_1 = 2t$ and $x_2 = t^2$ [26].

The ring Λ is naturally graded: Λ_n is spanned by the diagrams with $2n + 1$ trivalent singular points. The Poincare series of Λ is not known.

The ring Λ is closely related to a subalgebra $\tilde{\Lambda} = \mathbb{Q}[\sigma_1] \oplus \omega\mathbb{Q}[z_1, z_2, z_3]^{S_3}$ of $\mathbb{Q}[z_1, z_2, z_3]^{S_3}$ where $\omega = (\sigma_1 + z_1)(\sigma_1 + z_2)(\sigma_1 + z_3) = \sigma_3 + \sigma_2\sigma_1 + 2\sigma_1^3$ and $\sigma_1, \sigma_2, \sigma_3$ are the elementary symmetric polynomials. Kleissler constructs further elements $\chi_n \in \tilde{\Lambda}$ by

$$\chi_0 = 0, \quad \chi_1 = 2\sigma_1, \quad \chi_2 = \sigma_1^2, \quad \chi_{n+3} = \sigma_1\chi_{n+2} - \sigma_2\chi_{n+1} + \sigma_3\chi_n + \frac{\sigma_2\sigma_1^{n+1}}{2} - \frac{\sigma_3\sigma_1^n}{2} - \sigma_3(2\sigma_1)^n$$

and a homomorphism of graded algebras [15]

$$\varphi : \tilde{\Lambda} \rightarrow \Lambda, \quad \varphi(\chi_n) = x_n.$$

Vogel constructs a polynomial $\pi \in \tilde{\Lambda}$ such that $\varphi(\pi) = 0$ and conjectures that $\tilde{\Lambda}/(\pi) \cong \Lambda$ [25].

There is a canonical invariant non-degenerate form (the Killing form) $K : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$ on a simple complex Lie algebra \mathfrak{g} . It determines a canonical isospecialization $\psi : \mathfrak{g}_M \rightarrow \mathfrak{g}$ that, in its turn, defines a canonical ring homomorphism $\Theta_{\mathfrak{g}} : \Lambda \rightarrow \mathbb{C}$. It is sensible to call it *the Vogel character of Lie algebra \mathfrak{g}* . Let us clarify its origin [25, 26]. An element $X \in \Lambda$ gives a \mathfrak{g} -invariant map $\psi_2(X) : \wedge^3 \mathfrak{g} \rightarrow \mathbb{C}$. The space of such maps is one-dimensional, hence, $\psi_2(X) = \alpha\psi_2(1_{\Lambda})$ for some complex number α .

Proposition 9. *In the notation of the last section the map $\Theta_{\mathfrak{g}}(X) = \alpha$ is a ring homomorphism.*

Proof. It is obvious that $\Theta_{\mathfrak{g}}$ is linear. Let $\tilde{Y} \in \text{hom}_{\mathcal{C}}(3, 3)$ be the result of removal of a neighbourhood of a singular point from Y . Now one can represent products in Vogel ring via compositions:

$$XY = X \circ \tilde{Y} \quad \text{and} \quad Y = 1_{\Lambda} \circ \tilde{Y},$$

which implies that $\psi_2(XY) = \psi_2(X) \circ \psi_2(\tilde{Y}) = \Theta_{\mathfrak{g}}(X)\psi_2(1_{\Lambda}) \circ \psi_2(\tilde{Y}) = \Theta_{\mathfrak{g}}(X)\psi_2(Y)$ and, finally, $\Theta_{\mathfrak{g}}(XY) = \Theta_{\mathfrak{g}}(X)\Theta_{\mathfrak{g}}(Y)$. \square

3.5. Vogel's universal Lie algebra. Vogel has realized how to combine the character $\Theta_{\mathfrak{g}} : \Lambda \rightarrow \mathbb{C}$ and the isospecialization $\psi : \mathfrak{g}_M \rightarrow \mathfrak{g}$ into a single structure by forcing the action of Λ on the category \mathcal{C} [26]. The new tensor category \mathcal{C}' is a quotient of $\mathcal{C} \otimes_{\mathbb{Q}} \Lambda$ by the Vogel relations:

$$\begin{array}{|c|} \hline \text{Diagram: Trivalent vertex in a dashed box labeled X.} \\ \hline \end{array} \otimes v - \begin{array}{|c|} \hline \text{Diagram: Trivalent vertex in a dashed box labeled X with a smaller dashed box v inside.} \\ \hline \end{array} \otimes 1; \quad \begin{array}{|c|} \hline \text{Diagram: Circle with horizontal line in a dashed box labeled Y.} \\ \hline \end{array} \otimes 2t - \begin{array}{|c|} \hline \text{Diagram: Circle with horizontal line and a smaller circle inside in a dashed box labeled Y.} \\ \hline \end{array} \otimes 1.$$

Thus, \mathcal{C}' is also a prop with morphisms $\text{hom}_{\mathcal{C}'}(m, n) := \text{hom}_{\mathcal{C}}(m, n) \otimes_{\mathbb{Q}} \Lambda / I_{m,n}$ where $I_{m,n}$ is the Λ -span of all

$$X_1 \otimes v - X_2 \otimes 1, \quad Y_1 \otimes 2t - Y_2 \otimes 1$$

where $v \in \Lambda$, X_1 and X_2 differ as on the diagrams in the first relation, Y_1 and Y_2 differ as on the diagrams in the second relation.

Theorem 10. (1) \mathcal{C}' is a tensor category over $\Lambda[\delta]$.

(2) $\mathfrak{g}_V = (1, \mu \otimes 1, \tau \otimes 1, \gamma \otimes 1)$ is a quasisimple Lie \mathcal{C}' -algebra.

Proof. Observe that the tensor product and the composition of cosets

$$(f + I_{n,m}) \circ (g + I_{k,n}) = fg + I_{k,m}, \quad (f + I_{m,n}) \otimes (g + I_{m',n'}) = f \otimes g + I_{m+m',n+n'}$$

are well defined. To see this, it suffices to establish that the collection of subspaces $I_{n,m}$ is a tensor ideal, that is,

$$f \circ I_{k,n} \subseteq I_{k,m} \supseteq I_{n,m} \circ g, \quad h \otimes I_{m,n} \subseteq I_{m+m',n+n'} \supseteq I_{m,n} \otimes h.$$

By Λ -linearity, we can assume that $f \in \text{hom}_{\mathcal{C}}(n, m)$, $g \in \text{hom}_{\mathcal{C}}(k, n)$ and $h \in \text{hom}_{\mathcal{C}}(m', n')$. The argument used in Proposition 8 is based on the local nature of the relations. It works here as well. Let $X_1 \otimes v - X_2 \otimes 1 \in I_{k,n}$ where X_1 and X_2 are diagrams related by the first Vogel relation. Then $f \circ X_1$ and $f \circ X_2$ are also related by the first Vogel relation, and $(f \otimes 1) \circ (X_1 \otimes v - X_2 \otimes 1) \in I_{k,m}$. The same argument works for the second Vogel relation and the tensor products. Thus, \mathcal{C}' is a tensor category.

To compute the scalars, let us recall that the span of the connected diagrams with at least one singular point $\text{hom}_{\mathcal{C}}^s(n, m)$ is a Λ -module [25, Prop 3.2]. Proposition 8 implies that

$$\mathbb{K}_{\mathcal{C}} = \text{hom}_{\mathcal{C}}(0, 0) \cong S(\text{hom}_{\mathcal{C}}^1(0, 0)) = \mathbb{Q}[\delta] \otimes_{\mathbb{Q}} S(\text{hom}_{\mathcal{C}}^s(0, 0)),$$

because δ is the only connected diagram from $\text{hom}_{\mathcal{C}}^1(0, 0)$ without singular points. The element θ is a free generator of the Λ -module $\text{hom}_{\mathcal{C}}^s(0, 0)$ [25, Cor 4.6], hence,

$$\mathbb{K}_{\mathcal{C}} \cong \mathbb{Q}[\delta] \otimes_{\mathbb{Q}} S(\Lambda \cdot \theta).$$

Let $I'_{0,0}$ be the Λ -span of the first Vogel relations inside $I_{0,0}$. Since the first Vogel relation asserts Λ -linearity on $S(\Lambda \cdot \theta)$,

$$\text{hom}_{\mathcal{C}}(0, 0) \otimes_{\mathbb{Q}} \Lambda/I'_{0,0} \cong \mathbb{Q}[\delta] \otimes_{\mathbb{Q}} \text{Sym}_{\Lambda}(\Lambda \cdot \theta)/(\theta - 2t\delta) \cong \Lambda[\delta].$$

Evidently, the second Vogel relation has no further effect and

$$\mathbb{K}_{\mathcal{C}'} = \text{hom}_{\mathcal{C}}(0, 0) \otimes_{\mathbb{Q}} \Lambda/I_{0,0} = \text{hom}_{\mathcal{C}}(0, 0) \otimes_{\mathbb{Q}} \Lambda/I'_{0,0} \cong \Lambda[\delta],$$

proving the first statement. Since \mathfrak{g}_M is a metric Lie algebra, \mathfrak{g}_V is a metric Lie algebra. Furthermore,

$$\text{end}_{\mathfrak{g}_M}(\mathfrak{g}_M) = \text{hom}_{\mathcal{C}}(1, 1) \cong \mathbb{K}_{\mathcal{C}} \otimes_{\mathbb{Q}} (\text{hom}_{\mathcal{C}}^c(1, 1) \oplus (\text{hom}_{\mathcal{C}}^c(0, 1) \otimes_{\mathbb{Q}} \text{hom}_{\mathcal{C}}^c(1, 0))) \cong \mathbb{K}_{\mathcal{C}} \otimes_{\mathbb{Q}} (\mathbb{Q}I_1 \oplus \Lambda \cdot \phi)$$

because ϕ is a free generator of the Λ -module $\text{hom}_{\mathcal{C}}^s(1, 1)$ [25, Cor 4.6] and $\text{hom}_{\mathcal{C}}^c(0, 1) = 0$, $\text{hom}_{\mathcal{C}}^c(1, 0) = 0$ [25, Prop. 4.3]. Finally,

$$\text{end}_{\mathfrak{g}_V}(\mathfrak{g}_V) = \text{hom}_{\mathcal{C}'}(1, 1) \cong \mathbb{K}_{\mathcal{C}} \otimes_{\mathbb{Q}} (\mathbb{Q}I_1 \oplus \Lambda \cdot \phi) \otimes_{\mathbb{Q}} \Lambda/I_{1,1} \cong \mathbb{K}_{\mathcal{C}'} \otimes_{\mathbb{Q}} (\mathbb{Q}I_1 \oplus \Lambda \cdot \phi)/(\phi - 2tI_1) \cong \mathbb{K}_{\mathcal{C}'}. \quad \square$$

We say that a Lie \mathcal{C} -algebra \mathfrak{g} is *V-simple* if it is quasisimple and a specialization $\mathfrak{g}_M \rightarrow \mathfrak{g}$ extends to a specialization $\mathfrak{g}_V \rightarrow \mathfrak{g}$.

Clearly, all this definition requires is a ring homomorphism $\Theta_{\mathfrak{g}} : \Lambda[\delta] \rightarrow \mathbb{K}_{\mathcal{C}}$ satisfying the Vogel relations. Existence of such a homomorphism is not clear to us, in general. Vogel gives a categorical criterion for a quasisimple Lie algebra to be *V-simple* [26] but it is of limited use. To prove that a quasisimple Lie superalgebra \mathfrak{g} is *V-simple* [26] it is easier to use the homomorphism $\Theta_{\mathfrak{g}}$ constructed at the end of Section 3.4 (notice that $\Theta_{\mathfrak{g}}(\delta) = \dim_{\mathcal{C}} \mathfrak{g}$ in any category). In the final chapter we use the same strategy of constructing a ring homomorphism explicitly while the Vogel categorical criterion fails. So far, all known to us quasisimple Lie algebras are *V-simple*, hence, the following question is interesting.

Question 6. *Let \mathcal{C} be a tensor category. Is a quasisimple Lie \mathcal{C} -algebra *V-simple*?*

Finally, let us discuss the connection between the Vogel plane and the Vogel ring. By *the Vogel plane* one commonly understands the weighted projective plane $\mathbb{P}_{1,2,3}^2 = \mathbb{P}^2/S_3$ [17, 19]. A point⁸ $(\alpha : \beta : \gamma) \in \mathbb{P}_{1,2,3}^2(\mathbb{C})$, where α, β, γ are nonzero eigenvalues of Casimir operator on the symmetric square of the adjoint representation $S^2(\mathfrak{g})$, corresponds to a simple complex Lie algebra \mathfrak{g} . When the invariant form changes, the Casimir operator changes by a scalar, hence, the triple $(\alpha : \beta : \gamma)$ is also defined only up to a scalar. This construction has an issue with exceptional (according to Deligne) Lie algebras by Deligne [5]. A Lie algebra

⁸Landsberg and Manivel [17], following Vogel [26], assign a different point $((2\delta - \alpha) : (2\delta - \beta) : (2\delta - \gamma))$.

of type A_2 , D_4 , E_n , F_4 or G_2 admits only two such eigenvalues: α and β . Nevertheless, it is sensible to define

$$\gamma := 5\delta - \alpha - \beta,$$

where δ is the Casimir eigenvalue on the adjoint representation, because the identity $\alpha + \beta + \gamma = 5\delta$ holds in exceptional Lie algebras. There is a more serious issue with A_1 , it admits only one eigenvalue α . In this case, one assigns a whole line $\beta + \gamma = 5\delta - \alpha$.

Now let us consider a V -simple Lie algebra \mathfrak{g} over a ring \mathbb{K} with Vogel character $\Theta_{\mathfrak{g}} : \Lambda[\delta] \rightarrow \mathbb{K}$. Homomorphisms of graded rings

$$\mathbb{Q}[z_1, z_2, z_3] \leftrightarrow \mathbb{Q}[z_1, z_2, z_3]^{S_3} \leftarrow \tilde{\Lambda} \xrightarrow{\varphi} \Lambda$$

give another homomorphism $\tilde{\Theta}_{\mathfrak{g}} := \Theta_{\mathfrak{g}} \circ \varphi : \tilde{\Lambda} \rightarrow \mathbb{K}$. Rings $\mathbb{Q}[z_1, z_2, z_3]^{S_3}$ and $\tilde{\Lambda}$ have a common ring of fractions $\mathbb{Q}[z_1, z_2, z_3]^{S_3}[\omega^{-1}] = \tilde{\Lambda}[\omega^{-1}]$, hence, if $\tilde{\Theta}_{\mathfrak{g}}(\omega)$ is invertible in \mathbb{K} , then one gets a graded homomorphism

$$\hat{\Theta}_{\mathfrak{g}} : \mathbb{Q}[z_1, z_2, z_3]^{S_3} \rightarrow \mathbb{K}, \quad \hat{\Theta}_{\mathfrak{g}}(f) = \frac{\tilde{\Theta}_{\mathfrak{g}}(f\omega)}{\tilde{\Theta}_{\mathfrak{g}}(\omega)}.$$

Examples of algebras with undefined $\hat{\Theta}_{\mathfrak{g}}$ are \mathfrak{sl}_2 and K3-surfaces in the next chapter. In both cases $\tilde{\Theta}_{\mathfrak{g}}(\omega) = 0$. Another consequence of invertibility of $\tilde{\Theta}_{\mathfrak{g}}(\omega)$ is that the \mathbb{K}^{\times} -orbit $[\hat{\Theta}_{\mathfrak{g}}]$ turns out to be a point of the weighted projective plane $\mathbb{P}_{1,2,3}^2$ over the ring \mathbb{K} . Recall that the multiplicative group \mathbb{K}^{\times} acts on the set $\text{hom}(\mathbb{Q}[z_1, z_2, z_3]^{S_3}, \mathbb{K})$ via the grading of the ring $\mathbb{Q}[z_1, z_2, z_3]^{S_3}$:

$$(\alpha \cdot \Theta)(x_n) = \Theta(\alpha^{-n} x_n) \text{ for } \alpha \in \mathbb{K}^{\times}, x_n \in \mathbb{Q}[z_1, z_2, z_3]_n^{S_3}, \Theta \in \text{hom}(\mathbb{Q}[z_1, z_2, z_3]^{S_3}, \mathbb{K}),$$

and the set of \mathbb{K} -points $\mathbb{P}_{1,2,3}^2(\mathbb{K})$ is a subset of the set of orbits. Note that $\hat{\Theta}_{\mathfrak{g}}$ depends on the metric Lie algebra \mathfrak{g} . A different choice of a metric associated is linked by a scalar $\alpha \in \mathbb{K}^{\times}$, resulting in the homomorphism $\alpha \cdot \hat{\Theta}_{\mathfrak{g}}$. Thus, the orbit $[\hat{\Theta}_{\mathfrak{g}}] \in \mathbb{P}_{1,2,3}^2(\mathbb{K})$ depends only on the Lie algebra itself, but not on its metric structure.

4. ROZANSKY-WITTEN INVARIANTS

4.1. The category of vector superbundles. A holomorphic manifold X admits an associated tensor category, crucial for Kapranov's approach to Rozansky-Witten invariants [11, 21]. We attach a slightly smaller category than Kapranov. We explain the difference after we explain the construction.

The objects in the category $\text{SV}(X)$ of vector superbundles are locally free coherent sheaves $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$. The tensor product is usual:

$$(\mathcal{F} \otimes \mathcal{G})_0 = (\mathcal{F}_0 \otimes \mathcal{G}_0) \oplus (\mathcal{F}_1 \otimes \mathcal{G}_1), \quad (\mathcal{F} \otimes \mathcal{G})_1 = (\mathcal{F}_1 \otimes \mathcal{G}_0) \oplus (\mathcal{F}_0 \otimes \mathcal{G}_1).$$

A locally free coherent sheaf \mathcal{F} gives to two objects: even $\mathcal{F}^+ = \mathcal{F} \oplus 0$ and odd $\mathcal{F}^- = 0 \oplus \mathcal{F}$. The unit object is the even trivial line bundle \mathcal{O}_X^+ . The symmetric braiding on $\text{SV}(X)$ is the usual superbraiding

$$\tau(v_i \otimes w_j) = (-1)^{ij} w_j \otimes v_i$$

where v_i, w_j are homogeneous of degree i or j , i.e. local sections of \mathcal{F}_i and \mathcal{G}_j , correspondingly.

The hom-sets are slightly unusual:

$$\text{hom}_{\text{SV}(X)}(\mathcal{F}_0 \oplus \mathcal{F}_1, \mathcal{G}_0 \oplus \mathcal{G}_1) = \oplus_{i,j,n} \text{Ext}^{(i-j)+2n}(\mathcal{F}_i, \mathcal{G}_j).$$

The composition is the cup-product of extensions. The tensor product of two morphisms $f = (f_n)$ and $g = (g_n)$ is defined by $(f \otimes g)_n = f_n \otimes g_n$.

The category $\text{SV}(X)$ is a full subcategory of the 2-periodic derived category $D^2(X)$ of coherent sheaves on X [12]. A superbundle $\mathcal{F}_0 \oplus \mathcal{F}_1$ corresponds to a complex $\oplus_{n \in \mathbb{Z}} (\mathcal{F}_0[2n] \oplus \mathcal{F}_1[2n+1])$ with zero differentials.

In the literature some "larger" categories are sometimes considered. First, one can get more morphisms in by considering all extensions:

$$\text{hom}_{\text{SV}_1(X)}(\mathcal{F}_0 \oplus \mathcal{F}_1, \mathcal{G}_0 \oplus \mathcal{G}_1) = \oplus_{i,j} \text{Ext}^*(\mathcal{F}_i, \mathcal{G}_j),$$

resulting in a category $\text{SV}_1(X)$: $\text{SV}(X)$ is a subcategory of $\text{SV}_1(X)$ with the same objects but larger sets of morphisms. The category $\text{SV}_1(X)$ is a full subcategory of the unfolded derived category $\widehat{D}(X)$, whose objects and tensor products are from $D^b(X)$, while morphisms account for all extensions:

$$\text{hom}_{\widehat{D}(X)}(\mathcal{F}_*, \mathcal{G}_*) = \bigoplus_{n \in \mathbb{Z}} \text{hom}_{D^b(X)}(\mathcal{F}_*, \mathcal{G}_*[n]).$$

For us all these larger categories are irrelevant since all the necessary objects and morphisms for the Kapranov theorem are in $\text{SV}(X)$.

4.2. Atiyah classes. Let \mathcal{T}_X be the tangent sheaf on X , $\mathcal{D}_X^{<n}$ the sheaf of differential operators with holomorphic coefficients of order less than n . We have an exact sequence of $\mathcal{O}_X - \mathcal{O}_X$ -bimodules

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{D}_X^{<2} \rightarrow \mathcal{T}_X \rightarrow 0.$$

Notice that on both \mathcal{O}_X and \mathcal{T}_X the right and the left actions of \mathcal{O}_X coincide but on $\mathcal{D}_X^{<2}$ they are different. Given a locally free coherent sheaf \mathcal{F} , we get a new exact sequence by tensoring with it

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{D}_X^{<2} \otimes_{\mathcal{O}_X} \mathcal{F} \rightarrow \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{F} \rightarrow 0.$$

This extension is the Atiyah class

$$A_{\mathcal{F}} \in \text{Ext}^1(\mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{F}) \subseteq \text{hom}_{\text{SV}(X)}(\mathcal{T}_X^- \otimes \mathcal{F}, \mathcal{F}).$$

Let us remark that the standard Atiyah class is actually $-A_{\mathcal{F}}$ [11]. The following theorem has essentially been proved by Kapranov [11], but a reader may benefit by looking at later reviews [20, 21].

Theorem 11. (1) $\mathfrak{g}_X = (\text{SV}(X), \mathcal{T}_X^-, A_{\mathcal{T}_X})$ is a Lie algebra.

(2) Every superbundle $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$ is a representation of \mathfrak{g}_X with the action $A_{\mathcal{F}}$.

(3) If ς is a holomorphic symplectic form on X then \mathfrak{g}_X is a metric Lie algebra with a form ς .

Proof. Kapranov's original proof establishes this theorem in the bigger category $\text{SV}_1(X)$ [11]. We just need to point out that everything in Kapranov's proof happens in either odd or even extensions, so the theorem actually holds in $\text{SV}(X)$. In particular, observe that ς is skew-symmetric and \mathcal{T}_X^- is odd, hence ς is symmetric in $\text{SV}(X)$. \square

At the moment, the difference between $\text{SV}(X)$ and $\text{SV}_1(X)$ looks purely cosmetic. It becomes crucial when one addresses quasisimplicity.

4.3. Quasisimplicity. Let $S^n \mathcal{F}$ (or $T^n \mathcal{F}$, or $\Lambda^n \mathcal{F}$) be the n -th symmetric (or tensor, or exterior) power of a locally free coherent sheaf \mathcal{F} . Let us make a couple of general observations before addressing quasisimplicity of \mathfrak{g}_X . First, observe that $H^m(X, \mathcal{F}) = 0$ for all $m > \dim_{\mathbb{C}} X$ [27, Cor 4.39]. Secondly, observe⁹ that for a holomorphic symplectic manifold X

$$\text{end}(\mathcal{T}_X) \cong T^2 \mathcal{T}_X \cong S^2 \mathcal{T}_X \oplus \Lambda^2 \mathcal{T}_X \cong S^2 \mathcal{T}_X \oplus (\mathcal{L}_X \oplus \mathcal{O}_X)$$

with the trace map splitting the morphism $\mathcal{O}_X \rightarrow \Lambda^2 \mathcal{T}_X$, in particular, on an open set $U \subseteq X$

$$\mathcal{L}_X(U) = \{F \in \Lambda^2 \mathcal{T}_X(U) \subseteq \text{end}(\mathcal{T}_X(U)) \mid \text{Tr}(F) = 0\}.$$

Theorem 12. Let X be a holomorphic symplectic manifold. Then \mathfrak{g}_X is quasisimple if and only if $H^{2*}(X, S^2 \mathcal{T}_X \oplus \mathcal{L}_X) = 0$.

Proof. The scalars of the category $\text{SV}(X)$ are

$$\mathbb{K}_{\text{SV}(X)} = \text{end}_{\text{SV}(X)}(\mathcal{O}_X^+) = \text{Ext}^{2*}(\mathcal{O}_X, \mathcal{O}_X) = H^{2*}(X, \mathcal{O}_X).$$

The naturality of the Atiyah class means that every homomorphism in $\text{SV}(X)$ is a homomorphism of representations of \mathfrak{g}_X . Hence,

$$\text{end}_{\mathfrak{g}_X}(\mathfrak{g}_X) = \text{Ext}^{2*}(\mathcal{T}_X, \mathcal{T}_X) = H^{2*}(T^2 \mathcal{T}_X) = H^{2*}(X, S^2 \mathcal{T}_X) \oplus H^{2*}(X, \mathcal{L}_X) \oplus H^{2*}(X, \mathcal{O}_X)$$

and the natural map $\mathbb{K}_{\text{SV}(X)} \rightarrow \text{hom}_{\mathfrak{g}_X}(\mathfrak{g}_X, \mathfrak{g}_X)$ is the identity on the third component. \square

⁹This corresponds to $T^2 L(\omega_1) \cong L(2\omega_1) \oplus L(\omega_2) \oplus L(0)$ for the representations of $\mathfrak{sp}(n)$.

The even cohomology is easy to control in the case of K3-surfaces. This is in sharp contrast to the general behaviour of the symmetric plurigenus $Q_n(X) := H^0(X, S^n \mathcal{T}_X^*)$ that is not a topological invariant of a complex manifold X [4]. The crucial argument for the next theorem has been explained to us by A. Kuznetsov.

Theorem 13. *If X is a K3-surface then \mathfrak{g}_X is quasisimple.*

Proof. By Theorem 12 it suffices to prove that $H^0(X, S^2 \mathcal{T}_X)$ vanishes because $\mathcal{L}_X = 0$ for a surface. Suppose $H^0(X, S^2 \mathcal{T}_X) \neq 0$. Then we have two sections of

$$\text{end}(\mathcal{T}_X) \cong T^2 \mathcal{T}_X = \Lambda^2 \mathcal{T}_X \oplus S^2 \mathcal{T}_X :$$

the skew symmetric identity map $I \in \text{end}(\mathcal{T}_X)$ and some symmetric section $S \in \text{end}(\mathcal{T}_X)$. The determinant $\det(I + \lambda S)$ is a global function on X for each $\lambda \in \mathbb{C}$. Thus, it is constant and one can choose $\lambda_0 \in \mathbb{C}$ such that $\det(I + \lambda_0 S) = 0$. We conclude that the rank of $S' := I + \lambda_0 S$ is 1 at a generic point.

Furthermore, since I is skew-symmetric and S is symmetric, S' does not vanish, so the rank is 1 at each point. Thus, the image of S' is an invertible sheaf \mathcal{L} . Let us show that \mathcal{L} must be trivial.

If X is not algebraic then the only line bundle on X is trivial. If X is algebraic then \mathcal{T}_X is semistable [18] (cf. [8, Ch. 7.4]) and $\mu_H(\mathcal{L}) \leq \mu_H(\mathcal{T}_X) = 0$ for any ample divisor H . The dual of the natural map $\mathcal{T}_X \rightarrow \mathcal{L}$ is an embedding $\mathcal{L}^* \hookrightarrow \mathcal{T}_X^* \cong \mathcal{T}_X$, hence $-\mu_H(\mathcal{L}) = \mu_H(\mathcal{L}^*) \leq \mu_H(\mathcal{T}_X) = 0$. Thus, $\mathcal{L} \cdot H = \mu_H(\mathcal{L}) = 0$ for any ample divisor H , proving that \mathcal{L} is trivial.

Since \mathcal{L} is trivial, \mathcal{L} and consequently \mathcal{T}_X has a nonzero section. This contradicts $H^0(X, \mathcal{T}_X) = 0$. \square

4.4. V -simplicity. The specialization $\text{RW} : \mathfrak{g}_M \rightarrow \mathfrak{g}_X$ is known as the Rozansky-Witten invariants (a.k.a weight system) of a holomorphic symplectic manifold X . It is computed up to various degrees of explicitness for many concrete X [20]. Let $\varsigma \in H^0(X, \Lambda^2 \mathcal{T}^*)$ be the holomorphic symplectic form on X , $\bar{\varsigma} \in H^0(X, \Omega^{0,2})$ its conjugate form, and $[\bar{\varsigma}] \in H^2(X, \mathcal{O}_X)$ the class corresponding to $\bar{\varsigma}$ under the natural isomorphism $H^0(X, \Omega^{0,2}) \cong H^2(X, \mathcal{O}_X)$. Using ς , one gets natural embeddings $\Lambda^{n+m} \mathcal{T}_X^* \hookrightarrow \text{hom}(T^n \mathcal{T}_X, T^m \mathcal{T}_X)$ so that for $\gamma \in \text{hom}_{\mathbb{C}}(\mathfrak{g}_M^{\otimes n}, \mathfrak{g}_M^{\otimes m})$ with k singular trivalent points¹⁰

$$\text{RW}_2(\gamma) \in H^k(X, \Lambda^{n+m} \mathcal{T}_X^*) \hookrightarrow H^k(X, \text{hom}(T^n \mathcal{T}_X, T^m \mathcal{T}_X)) \hookrightarrow \text{hom}_{\text{SV}(X)}(\mathfrak{g}_X^{\otimes n}, \mathfrak{g}_X^{\otimes m}).$$

In particular, if X is a K3-surface, then $\text{RW}_2(\gamma) = 0$ whenever γ has at least three trivalent singular points. For the diagrams with $\text{RW}_2(\gamma) \neq 0$ we have [20]

$$\text{RW}_2(\delta) = -2, \text{RW}_2(\phi) = -24[\bar{\varsigma}], \text{RW}_2(\theta) = 48[\bar{\varsigma}] \in H^2(X, \mathcal{O}_X).$$

Theorem 14. *If X is a K3-surface then \mathfrak{g}_X is a V -simple Lie algebra.*

Proof. By Theorem 13 the Lie algebra \mathfrak{g}_X is quasisimple and

$$\text{end}_{\mathfrak{g}_X}(\mathfrak{g}_X) \cong H^{2*}(X, \mathcal{O}_X) = \mathbb{C}[z]/(z^2) \quad \text{where } z = [\bar{\varsigma}] \in H^2(X, \mathcal{O}_X).$$

A graded homomorphism

$$\Theta_X : \Lambda \rightarrow \mathbb{C}[z]/(z^2), \quad \Theta_X(t) = -24z$$

is uniquely determined since $\Lambda_0 = \mathbb{Q}$ and $\Lambda_1 = \mathbb{Q}t$. We claim that Θ_X defines a required specialization. Indeed, most of the relations defining \mathcal{C}' hold in $\text{SV}(X)$ for the trivial reason: both sides are zero. The relations¹¹ $\theta = 2t\delta$ and $\phi = 2tI$ hold as both sides become $48z$ and $-24z$. \square

Theorem 14 proves Westbury's conjecture for a K3-surface. Westbury has conjectured it for any compact irreducible holomorphic symplectic manifold Y . We finish by stating this conjecture more carefully as a series of questions. The quasisimplicity of \mathfrak{g}_Y can be established by Theorem 12 and boils down to the following question.¹²

Question 7. *Which compact irreducible holomorphic symplectic manifolds Y satisfy $H^{2*}(Y, S^2 \mathcal{T}_Y \oplus \mathcal{L}_Y) = 0$?*

¹⁰Observe that $k + n + m$ is always even.

¹¹It is also clear by applying the metric to $\phi = 2tI$ that $\phi = 2tI$ implies $\theta = 2t\delta$.

¹²Justin Sawon has reported to us an answer to Question 7: surfaces. Since $H^2(Y, \Omega^2) = H^2(Y, \mathcal{O}_Y) \oplus H^2(Y, \mathcal{L}_Y)$, a manifold Y with a positive answer to Question 7 must have $h^{2,2}(Y) = 1$. If $\dim(Y) > 2$, then one can use its three Kahler structures to construct two linearly independent classes in $H^2(Y, \Omega^2)$, in particular, $h^{2,2}(Y) > 1$.

If $\dim_{\mathbb{C}} Y = n$ then the scalars of $\mathrm{SV}(Y)$ are $\mathbb{K}_{\mathrm{SV}(Y)} \cong H^{2*}(Y, \mathcal{O}_Y) = \mathbb{C}[z]/(z^n)$ where $z = [\bar{\zeta}] \in H^2(Y, \mathcal{O}_Y)$. Thus, the V -simplicity of \mathfrak{g}_Y boils down to some explicit identities on the Rozansky-Witten invariants.

Question 8. *Does there exist a graded homomorphism $\Theta_Y : \Lambda \rightarrow \mathbb{C}[z]/(z^n)$ that gives rise to a specialization $\mathfrak{g}_V \rightarrow \mathfrak{g}_Y$?*

It would be interesting if these homomorphisms are compatible for different Y or, at least, for the Hilbert schemes.

Question 9. *Does there exist a V -simple Lie algebra \mathfrak{g}_H over $\mathbb{C}[z]$ that specializes to $\mathfrak{g}_{X^{[n]}}$ for all X and n where $X^{[n]}$ is the Hilbert scheme of n points on a K3-surface X , so that the specialization $\mathfrak{g}_M \rightarrow \mathfrak{g}_{X^{[n]}}$ factorises as $\mathfrak{g}_M \rightarrow \mathfrak{g}_H \rightarrow \mathfrak{g}_{X^{[n]}}$?*

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