

COMBINATORIAL CUBIC SURFACES AND RECONSTRUCTION THEOREMS

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Abstract. This note contains a solution to the following problem: reconstruct the definition field and the equation of a projective cubic surface, using only combinatorial information about the set of its rational points. This information is encoded in two relations: collinearity and coplanarity of certain subsets of points. We solve this problem, assuming mild “general position” properties.

This study is motivated by an attempt to address the Mordell–Weil problem for cubic surfaces using essentially model theoretic methods. However, the language of model theory is not used explicitly.

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0. Introduction and overview

0.1. Cubic hypersurfaces. Let K be a field, finite or infinite. In the finite case we assume cardinality of K to be sufficiently large, the exact lower boundary depending on various particular combinatorial construction.

Let $P = \mathbf{P}_K^N$ be a projective space over K , with a projective coordinate system $(z_1 : z_2 : \cdots : z_{N+1})$. A *cubic hypersurface* $V \subset P$ defined over K is, by definition, the closed subscheme defined by an equation $c = 0$ where $c \in K[z_1 : z_2, \cdots : z_{N+1}]$ is a non-zero cubic form. There is a bijection

between the set of such subschemes and the set $\mathbf{P}^9(K)$ of coefficients of c modulo K^* .

We will say that V is *generically reduced* if after extending K to an algebraic closure \overline{K} , c does not acquire a multiple factor.

In this paper, I will be interested in the following problem:

0.1.1. Problem. *Assuming V generically reduced, reconstruct K and the subscheme $V \subset P = \mathbf{P}_K^N$ starting with the set of its K -points $V(K)$ endowed with some additional combinatorial structures of geometric origin.*

The basic combinatorial data that I will be using are subsets of smooth points of $V(K)$ lying upon various sections of V by projective subspaces of P defined over K . Thus, for the main case treated here, that of cubic surfaces ($N = 3$), I will deal combinatorially with the structure, consisting of

- a) *The subset of smooth (reduced, non-singular) points $S := V_{sm}(K)$.*
- b) *A triple symmetric relation “collinearity”: $\mathcal{L} \subset S^3 := S \times S \times S$.*
- c) *A set \mathcal{P} of subsets of S called “plane sections”.*

In the first approximation, one can imagine \mathcal{L} (resp. \mathcal{P}) as simply subsets of collinear triples (resp. K -points of K -plane sections) of V . However, various limiting and degenerate cases must be treated with care as well.

For example, as a working definition of L we will adopt the following convention: $(p, q, r) \in S^3$ belongs to \mathcal{L} if either $p+q+r$ is the full intersection cycle of V with a K -line $l \subset \mathbf{P}^N$ (with correct multiplicities), or else if there exists a K -line $l \subset V$ such that $p, q, r \in l$.

0.2. Geometric constraints. If an instance of the set-theoretic combinatorial structure such as $(S, \mathcal{L}, \mathcal{P})$ above, comes from a cubic surface V defined over a field K , we will call such a structure *geometric one*.

Geometric structures satisfy additional combinatorial constraints.

The reconstruction problem in this context consists of two parts:

(i) *Find a list of constraints ensuring that each $(S, \mathcal{L}, \mathcal{P})$ satisfying these constraints is geometric.*

(ii) *Devise a combinatorial procedure that reconstructs K and $V \subset \mathbf{P}^3$ realizing $(S, \mathcal{L}, \mathcal{P})$ as a geometric one.*

Besides, ideally we want the reconstruction procedure to be functorial: certain maps of combinatorial structures, in particular, their isomorphisms,

must induce/be induced by morphisms of ground fields and K -linear maps of P .

In the subsection 0.4, I will describe a classical archetype of reconstruction, – combinatorial characterization of projective planes. I will also explain the main motivation for trying to extend this technique to cubic surfaces: *the multidimensional weak Mordell-Weil problem*.

0.3. Reconstruction of K from curves and configurations of curves. One cannot hope to reconstruct the ground field K , if V is zero-dimensional or one-dimensional. Only starting with cubic surfaces ($N=3$), this prospect becomes realistic.

In fact, if $N = 1$, we certainly cannot reconstruct K from any combinatorial information about one K -rational cycle of degree 3 on P_K^1 .

If $N = 2$, then for a smooth cubic curve V , the set $V(K)$ endowed with the collinearity relation is the same as $V(K)$ considered as a principal homogeneous space over the “Mordell–Weil” abelian group, unambiguously obtained from $(V(K), L)$ as soon as we arbitrarily choose the identity (or zero) point: cf. a recollection of classical facts in sec. 1 below. Generally, this group does not carry enough information to get hold of K , if K is finitely generated over \mathbf{Q} .

However, the situation becomes more promising, if we assume V geometrically irreducible and having just *one singular point* which is defined over K . More specifically, assume that this point is either an ordinary double point with two different branches/tangents defined over K each, or a cusp with triple tangent line, which is then automatically defined over K .

In the first case, we will say that V is a curve of *multiplicative type*, in the second, of *additive type*.

Then we can reconstruct, respectively, the multiplicative or the additive group of K , up to an isomorphism. In fact, these two groups are canonically identified with $V_{sm}(K)$ as soon as one smooth K -point is chosen, in the same way as the Mordell–Weil group is geometrically constructed from a smooth cubic curve with collinearity relation.

Finally for $N = 3$, now allowing V to be smooth, and under mild genericity restrictions, we can combine these two procedures and reconstruct both K and a considerable part of the whole geometric picture.

The idea, which is the main new contribution of this note, is this. Choose two points (p_m, p_a) in $V_{sm}(K)$, not lying on a line in V , whose tangent

sections (C_m, C_a) are, respectively, of multiplicative and additive type. (To find such points, one might need to replace K by its finite extension first).

Now, one can intersect the tangent planes to p_m and p_a by elements of a K -rational pencil of planes, consisting of all planes containing p_m and p_a . This produces a birational identification of C_m and C_a .

The combinatorial information, used in this construction, can be extracted from the data \mathcal{L} and \mathcal{P} . The resulting combinatorial object, carrying full information about both K^* and K^+ , can be then processed into K , if a set of additional combinatorial constraints is satisfied.

Using four tangent plane section in place of two, one can then unambiguously reconstruct the whole subscheme V .

For further information, cf. the main text.

0.4. Combinatorial projective planes and weak Mordell–Weil problem. My main motivation for this study was an analog of Mordell–Weil problem for cubic surfaces: cf. [M3], [KaM], [Vi].

Roughly speaking, the classical Mordell–Weil Theorem for elliptic curves can be stated as follows. Consider a smooth plane cubic curve C , i. e. a plane model of an elliptic curve, over a field K finitely generated over its prime subfield. Then the whole set $C(K)$ can be generated in the following way: start with a *finite* subset $U \subset C(K)$ and iteratively enlarge it, adding to already obtained points each point $p \circ q \in C(K)$ that is collinear with two points $p, q \in C(K)$ that were already constructed. If $p = q$, then the third collinear point, by definition, is obtained by drawing the tangent line to C at p .

In the case of a cubic surface V , say, not containing K -lines, there are two versions of this geometric process (“drawing secants and tangents”). We may allow to consecutively add only points collinear to $p, q \in V(K)$ when $p \neq q$. Alternatively, we may also allow to add all K -points of the plane section of V tangent to V at $p = q$.

I will call the respective two versions of finite generation conjecture *strong*, resp. *weak*, *Mordell–Weil problem* for cubic surfaces.

Computer experiments suggest that weak finite generation might hold at least for some cubic surfaces defined over \mathbf{Q} : see [Vi] for the latest data. The same experiments indicate however, that the “descent” procedure, by which Mordell–Weil is proved for cubic curves, will not work in two-dimensional case: a stable percentage of \mathbf{Q} -points of height $\leq H$ remains not expressible in the form $p \circ q$ with p, q of smaller height.

In view of this, I suggested in [M3], [KaM] to use a totally different approach to finite generation, based on the analogy with classical theory of abstract, or combinatorial, projective planes.

The respective finite generation statement can be stated as follows.

For any field K of finite type over its prime subfield, the whole set $\mathcal{P}^2(K)$ can be obtained by starting with a finite subset $U \subset \mathcal{P}^2(K)$ and consecutively adding to it lines through pairs of distinct points, already obtained, and intersection points of pairs of constructed lines.

The strategy of proof can be presented as a sequence of the following steps.

STEP 1. Define a *combinatorial projective plane* (S, \mathcal{L}) as an abstract set S whose elements are called (*combinatorial*) *points*, endowed with a set of subsets of points \mathcal{L} called (*combinatorial*) *lines*, such that each two distinct points are contained in a single line, and each two distinct lines intersect at a single point.

STEP 2. Find combinatorial conditions upon (S, \mathcal{L}) , that are satisfied for K -points of each geometric projective plane $\mathcal{P}^2(K)$, and that exactly characterize geometric planes, so that starting with (S, \mathcal{L}) satisfying these conditions, one can reconstruct from (S, \mathcal{L}) a field K and an isomorphism of (S, \mathcal{L}) with $(\mathcal{P}^2(K), \text{projective } K\text{-lines})$ unambiguously.

In fact, this reconstruction must be also functorial with respect to embeddings of projective planes $S \subset S'$ and the respective combinatorial lines.

These conditions are furnished by the beautiful *Pappus Theorem/Axiom* (at least, if cardinality of S is infinite or finite but large enough).

STEP 3. Given a geometric projective plane $(\mathcal{P}^2(K), \text{projective } K\text{-lines})$, start with four points in general position $U_0 \subset \mathcal{P}^2(K)$ and generate the minimal subset S of $\mathcal{P}^2(K)$ stable with respect to drawing lines through two points and taking intersection point of two lines.

This subset, with induced collinearity structure, is a combinatorial projective plane. It satisfies the Pappus Axiom, because it was satisfied for $\mathcal{P}^2(K)$. It is not difficult to deduce then that S is isomorphic to $\mathcal{P}^2(K_0)$, with $K_0 \subset K$ the prime subfield, and the embeddings $K_0 \rightarrow K$ and $S \rightarrow \mathcal{P}^2(K)$ are compatible with geometry.

STEP 4. Finally, one can iterate this procedure as follows. If K is finitely generated, there exists a finite sequence of subfields $K_0 \subset K_1 \subset \dots \subset K_n = K$ such that each K_i is generated over K_{i-1} by one element, say θ_i . If we already

know a finite generating set of points $U_{i-1} \subset \mathcal{P}^2(K_{i-1})$, define $U_i \subset \mathcal{P}^2(K_i)$ as $U_{i-1} \cup \{(\theta_i : 1 : 0)\}$. One easily sees that U_i generates $\mathcal{P}^2(K_i)$.

0.5. Results of this paper. As was explained in 0.3, results of this paper give partial versions for cubic surfaces of Steps 1 and 2 in the finite generation proof, sketched above. I can now reconstruct the ground field K and the total subscheme $V \subset \mathcal{P}_K^3$, under appropriate genericity assumptions, from the combinatorics of $V(K)$ geometric origin.

However, these results still fall short of a finite generation statement.

The reader must be aware that this approach is essentially model–theoretic, and it was inspired by the successes of [HrZ] and [Z].

My playground is much more restricted, and I do not use explicitly the (meta)language of model theory, working in the framework of Bourbaki structures.

More precisely, constructions, explained in sec. 2 and 3, are oriented to the reconstruction of fields of finite type and cubic surfaces over them. According to [HrZ] and [Z], if one works over an algebraically closed ground field, one can reconstruct combinatorially (that is, in a model theoretic way) much of the classical algebraic geometry.

In sec. 4, I introduce the notion of a *large field*, tailor–made for cubic (hyper)surfaces, and show that large fields can be reconstructed even from (sets of rational points of) smooth plane cubic curves, endowed with collinearity relation and an additional structure consisting of *pencils of collinear points* on such a curve. Any field K having no non–trivial extensions of degree 2 and 3 is large, hence large fields lie between finitely generated and algebraically close ones.

1. Quasigroups and cubic curves

1.1. Definition. Let S be a set and $\mathcal{L} \subset S \times S \times S$ be a subset of triples with the following properties:

- (i) \mathcal{L} is invariant with respect to permutations of factors S .
 - (ii) Each pair $p, q \in S$ uniquely determines $r \in S$ such that $(p, q, r) \in \mathcal{L}$.
- Then (S, \mathcal{L}) is called a symmetric quasigroup.

This structure in fact defines a binary composition law

$$\circ : S \times S \rightarrow S : p \circ q = r \iff (p, q, r) \in \mathcal{L}. \quad (1.1)$$

Properties of \mathcal{L} stated in the Definition 1.1 can be equivalently rewritten in terms of \circ : for all $p, q \in S$

$$p \circ q = q \circ p, \quad p \circ (p \circ q) = q. \quad (1.2)$$

The structure (S, \circ) , satisfying (1.2), will also be called a symmetric quasigroup. The importance of \mathcal{L} for us is that, together with its versions, it naturally comes from geometry.

In terms of (S, \circ) , we can define the following groups. For each $p \in S$, the map $t_p : q \mapsto p \circ q$ is an involutive permutation of S : $t_p^2 = \text{id}_S$.

Denote by $\Gamma = \Gamma(S, \mathcal{L})$ the group generated by all $t_p, p \in S$. Let $\Gamma^0 \subset \Gamma$ be its subgroup, consisting of products of an even number of involutions t_p .

1.2. Theorem–Definition. *A symmetric quasigroup (S, \circ) is called abelian, if it satisfies any (and thus all) of the following equivalent conditions:*

(i) *There exists a structure of abelian group on S , $(p, q) \mapsto pq$, and an element $u \in S$ such that for all $p, q \in S$ we have $p \circ q = up^{-1}q^{-1}$.*

(ii) *The group Γ^0 is abelian.*

(iii) *For all $p, q, r \in S$, $(t_p t_q t_r)^2 = 1$.*

(iv) *For any element $u \in S$, the composition law $pq := u \circ (p \circ q)$ turns S into an abelian group.*

(v) *The same as (iv) for some fixed element $u \in S$.*

Under these conditions, S is a principal homogeneous space over Γ^0 .

For a proof, cf. [M1], Ch. I, sec. 1,2, especially Theorem 2.1.

1.3. Example: plane cubic curves. Let K be a field, $C \subset \mathbf{P}_K^2$ an absolutely irreducible cubic curve defined over K . Denote by $S = C_{sm}(K) \subset C(K)$ the set of non-singular K -points of C . Define the collinearity relation \mathcal{L} by the following condition:

$$(p, q, r) \in \mathcal{L} \text{ iff } p + q + r \text{ is the intersection cycle of } C \text{ with a } K\text{-line.} \quad (1.3)$$

Then (S, \mathcal{L}) is an abelian symmetric quasigroup. This is a classical result.

More precisely, we have the following alternatives. C might be non-singular over an algebraic closure of K . Then C is the plane model of an abstract elliptic curve defined over K , the group Γ^0 can be identified with

K -points of its Picard group. We call the latter also the Mordell–Weil group of C over K .

Singular curves will be more interesting for us, because they carry more information about the ground field K . Each geometrically irreducible singular cubic curve has exactly one singular geometric point, say p , and it is rational over K . More precisely, we will distinguish three cases.

(I) C is of *multiplicative type*. This means that p is a double point two tangents to which at p are rational over K .

(II) C is of *additive type*. This means that p is a cusp: a point with triple tangent.

(III) C is of *twisted type*. This means that p is a double point p two tangents to which at p are rational and conjugate over a quadratic extension of K .

The structure of quasigroups related to singular cubic curves is clarified by the following elementary and well known statement.

1.3.1. Lemma. (i) *If C is of multiplicative type, Γ^0 is isomorphic to K^* .*

(ii) *If C is of additive type, Γ^0 is isomorphic to K^+ .*

(iii) *If C is of twisted type, Γ^0 is isomorphic to the group of K -points of a form of G_m or G_a that splits over the respective quadratic extension of K . The first case occurs when $\text{char}K \neq 2$, the second one when $\text{char}K = 2$.*

Proof. (*Sketch.*) In all cases, the group law $pq := u \circ (p \circ q)$, for an arbitrary fixed $u \in S$ determines the structure of an algebraic group over K upon the curve C_0 which can be defined as the normalization of C with preimage(s) of p deleted. An one-dimensional geometrically connected algebraic group becomes isomorphic to G_m or G_a over any field of definition of its points “at infinity”.

In the next section, we will recall more precise information about the respective isomorphisms in the non-twisted cases.

2. Reconstruction of the ground field and a cubic surface from combinatorics of tangent sections

2.1. The key construction. Let K be a field of cardinality ≥ 4 . Then the set $H := \mathbf{P}^1(K)$ consists of ≥ 5 points.

Consider a family of five pairwise distinct points in H for which we choose the following suggestive notation:

$$0_a, \infty_a, 0_m, 1_m, \infty_m \in \mathbf{P}^1(K). \quad (2.1)$$

In view of its origin, the set $H \setminus \{\infty_a\}$ has a special structure of abelian group A (written additively, with zero 0_a). In fact, the choice of any affine coordinate x_a on \mathbf{P}_K^1 with zero at 0_a and pole at ∞_a defines this structure: it sends $p \in H \setminus \{\infty_a\}$ to the value of x_a at p , and addition is addition in K^+ . The structure does not depend on x_a , but x_a determines the isomorphism of G_a with K^+ , and this isomorphism *does* depend on x_a : the set of all x_a 's is the principal homogeneous space over K^* .

Similarly, the set $H \setminus \{0_m, \infty_m\}$ has a special structure of abelian group M , with identity 1_m . A choice of affine coordinate x_m on \mathbf{P}^1 , with divisor supported by $(0_m, \infty_m)$ and taking value $1 \in K$ at 1_m , defines this structure. Again, it does not depend on x_m , but x_m determines its isomorphism with K^* , and this isomorphism *does* depend on x_m . There are, however, only two choices: x_m and x_m^{-1} . They differ by renaming $0_m \leftrightarrow \infty_m$.

Having said this, consider now *an abstract set* H with a subfamily of five elements denoted as in (2.1). Moreover, assume in addition that we are given composition laws $+$ on $H \setminus \{\infty_a\}$ and \cdot on $H \setminus \{0_m, \infty_m\}$ turning these sets into two abelian groups, A (written additively, with zero 0_a) and M (written multiplicatively, with identity 1_m). Define the inversion map $i : M \rightarrow M$ using this multiplication law: $i(p) = p^{-1}$.

We will encode this extended version of (2.1), with additional data recorded in the notation M, A , as a bijection

$$\mu : M \cup \{0_m, \infty_m\} \rightarrow A \cup \{\infty_a\} \quad (2.2)$$

It is convenient to extend the multiplication and inversion, resp. addition and sign reversal, to commutative partial composition laws on two sets (2.2) by the usual rules: for $p \in M, q \in A$, we set

$$p \cdot 0_m := 0_m, \quad p \cdot \infty_m := \infty_m, \quad i(0_m) := \infty_m, \quad i(\infty_m) := 0_m, \quad (2.3)$$

$$q \pm \infty_a := \infty_a. \quad (2.4)$$

The following two lemmas are our main tool in this section.

2.2. Lemma. *If (2.2) comes from a projective line as above, then the map*

$$\begin{aligned} \nu : M \cup \{0_m, \infty_m\} &\rightarrow A \cup \{\infty_a\}, \\ \nu(p) &:= \mu\{\mu^{-1}[\mu(p) - \mu(0_m)] \cdot i \circ \mu^{-1}[\mu(p) - \mu(\infty_m)]\} \end{aligned} \quad (2.5)$$

is a well defined bijection.

Moreover,

$$\nu(0_m) = 0_a := 0, \quad \nu(\infty_m) = \infty_a := \infty. \quad (2.6)$$

Finally, identifying $M \cup \{0_m, \infty_m\}$ and $A \cup \{\infty_a\}$ with the help of ν and combining addition and multiplication, now (partially) defined on H , we get upon $H \setminus \{\infty\}$ a structure of the commutative field, with zero 0 and identity $1 := \nu(1_m)$. This field is isomorphic to the initial field K .

Proof. In the situation (2.1), if A is identified with K^+ using an affine coordinate x_a , and M is identified with K^* using another affine coordinate x_m as above, these coordinates are connected by the evident fractional linear transformation, bijective on $\mathbf{P}^1(K)$:

$$x_a = c \cdot (x_m - x_m(0_m)) \cdot (x_m - x_m(\infty_m))^{-1}, \quad c \in K^*.$$

The definition (2.5) is just a fancy way to render this relation, taking into account that now we have to add and to multiply in two different locations, passing back and forth via μ and μ^{-1} . Instead of multiplying by c , we normalize multiplication so that $\nu(1_\infty)$ becomes identity.

This observation makes all the statements evident.

The same arguments read in reverse direction establish the following result:

2.3. Lemma on Reconstruction. *Conversely, let M and A be two abstract abelian groups, extended by “improper elements” to the sets with partial composition laws $M \cup \{0_m, \infty_m\}$ and $A \cup \{\infty_a\}$, as in (2.3), (2.4). Assume that we are given a bijection μ as in (2.2), mapping $1, 0_m$, and ∞_m to A . Assume moreover that:*

(i) The respective mapping ν defined by (2.5) is a well defined bijection.

(ii) The set A endowed with its own addition, and multiplication transported by ν from M , is a commutative field K .

Then we get a natural identification $H = \mathbf{P}^1(K)$. This construction is inverse to the one described in sec. 2.1.

2.4. Combinatorial projective lines and functoriality. Let us call an instance of the data (2.2)–(2.4), satisfying the constraints of Lemma 2.2, a *combinatorial projective line* (this name will be better justified in the remainder of this section). Let us call triples $(K, \mathbf{P}^1(K), j)$ where j is a subfamily of five points in $\mathbf{P}^1(K)$ as in (2.1), *geometric projective lines*.

The constructions we sketched above are obviously functorial with respect to various natural maps such as:

a) *On the geometric side:* Morphisms of fields, naturally extended to projective lines with marked points. Fractional linear transformations of $\mathbf{P}^1(K)$, naturally acting upon j and identical on K .

b) *On the combinatorial side:* Embeddings of groups $M \rightarrow M', A \rightarrow A'$, compatible with (μ, μ') and on improper points. Automorphisms of (M, A) , supplied with compatibly changed μ and improper points.

These statements can be made precise and stated as equivalence of categories. We omit details here.

Now we turn to the description of a bare–bones geometric situation, that can be obtained (in many ways) from a cubic surface, directly producing combinatorial projective lines.

2.5. (C_m, C_a) –configurations. Consider a family of subschemes in \mathbf{P}_K^3 , that we will call a *configuration*:

$$\text{Conf} := (p_m, p_a; C_m, C_a; P_m, P_a) \quad (2.7)$$

It consists of the following data:

- (i) Two distinct K –points $p_m, p_a \in \mathbf{P}^3(K)$.
- (ii) Two distinct K –planes $P_m, P_a \subset \mathbf{P}^3$ such that $p_m \in P_m, p_m \notin P_a$ and $p_a \in P_a, p_a \notin P_m$.
- (iii) Two geometrically irreducible cubic K –curves $C_m \subset P_m, C_a \subset P_a$.

We impose on these data the following constraints:

(A) $p_m \in C_m(K)$ is a double point, and C_m is of multiplicative type, in the sense of 1.3.

(B) $p_a \in C_a(K)$ is a cusp, and C_a is of additive type.

(C) Let $l := P_m \cap P_a$. Denote by $0_m, \infty_m \in l$ the intersection points with l of two tangents to C_m at x_m (in the chosen order). Denote by $0_a \in l$ the

intersection point with l of the tangent to C_a at x_a . These three points are pairwise distinct.

Let $M := C_{m,sm}(K)$, $A := C_{a,sm}(K)$ be the respective sets of smooth points, with their group structure, induced by collinearity relation and a choice of 1_m , resp. 0_a , as in sec. 1.

Define the bijection $\alpha : \tilde{C}_m(K) \rightarrow l(K)$, where \tilde{C}_m is the normalization of C_m , by mapping each smooth point $q \in C(K)$ to the intersection point with l of the line, passing through p_m and q . The two tangent lines at p_m define the images of two points of \tilde{C}_m lying over p_m .

Similarly, define the bijection $\beta : C_a(K) \rightarrow l(K)$, by mapping each smooth point $q \in C(K)$ to the intersection point with l of the line, passing through p_a and q . The point on l where the triple tangent at cusp intersects it, is denoted ∞_a .

Finally, put

$$\mu := \beta^{-1} \circ \alpha : M \cup \{0_m, \infty_m\} \rightarrow A \cup \{\infty_a\} \quad (2.8)$$

Thus $l(K)$ acquires both structures: of a combinatorial line and of a geometric line.

2.6. (C_m, C_a) -configurations from cubic surfaces. Let V be a smooth cubic surface defined over K . At each non-singular point $p \in V(K)$, there exists a well defined tangent plane to V defined over K . The intersection of this plane with V , for p outside of a proper Zariski closed subset, is a geometrically irreducible curve C , having p as its single singular point.

Again, generically it is of twisted multiplicative type, if $\text{char}K \neq 2$, and of twisted additive type, when p lies on a curve in V .

Therefore, under these *genericity conditions*, replacing K by its finite extension if need be, and renaming this new field K , we can find two tangent plane sections of V that form a (C_m, C_a) -configuration in the ambient projective space.

2.6.1. Example. Consider the diagonal cubic surface $\sum_{i=1}^4 a_i z_i^3 = 0$ over a field K of characteristic $\neq 3$. Then the discriminant of the quadratic equation defining directions of two tangents of the tangent section at $(z_1 : z_2 : z_3 : z_4)$, up to a factor in K^{*2} , is

$$D := \prod_{i=1}^4 a_i z_i.$$

Hence the set of points of (twisted) additive type consists of four elliptic curves

$$E_i : z_i = \sum_{j \neq i} a_j z_j^3 = 0, \quad i = 1, \dots, 4.$$

The remaining points (outside 27 lines) are of (twisted) multiplicative type. Those for which $D \in K^{*2}$ are of purely multiplicative type.

2.7. Reconstruction of the configuration itself. Returning to the map (2.8), we see that K can be reconstructed from the (C_m, C_a) configuration, using *only* the collinearity relation on the set

$$\tilde{C}_m(K) \cup C_a(K) \cup l(K). \quad (2.9)$$

Moreover, we get the canonical structure of a projective line over K on l , together with the family of five K -points on it.

To reconstruct the whole configuration, as a K -scheme up to an isomorphism, from the same data, it remains to give in addition two 0-cycles on l : its intersection with C_m and C_a respectively. Again, passing to a finite extension of K , if need be, we may and will assume that all intersection points in $C_m \cap l$, $C_a \cap l$ are defined over K . This again means that these cycles belong to the respective collinearity relation on $C_m(K) \cup C_a(K) \cup l(K)$.

To show that knowing these cycles, we can reconstruct C_m and C_a in their respective projective planes, let us look at the equations of these curves.

In P_m , choose projective coordinates $(z_1 : z_2 : z_3)$ over K in such a way that l is given by the equation $z_3 = 0$, p_m is $(0 : 0 : 1)$, equations of two tangents at p_m are $z_1 = 0$, $z_2 = 0$, and the points $0_m, \infty_m$ are respectively $(0 : 1 : 0)$ and $(1 : 0 : 0)$. Then the equation of C_m must be of the form

$$z_1 z_2 z_3 + c(z_1, z_2) = 0$$

where c is a cubic form. To give the intersection $C_m \cap l$ is the same as to give the linear factors of c . Since z_i are defined up to multiplication by constants from K^* , this defines (C_m, P_m) up to isomorphism.

Similar arguments work for C_a ; its equation in coordinates $(z'_1 : z'_2 : z'_3)$ on P_a such that l is defined by $z'_3 = 0$, will now be

$$z_1'^2 z_3 + c'(z'_1, z'_2) = 0.$$

We may normalize z'_2 by the condition that $0_a = (1 : 0 : 0)$, and then reconstruct linear factors of c' from the respective intersection cycle $C_a \cap l$.

2.8. Reconstruction of V from a tangent tetrahedral configuration. Let now V be a cubic surface over K . Assume that $V(K)$ contains four points $p_i, i = 1, \dots, 4$, such that tangent planes P_i at them are pairwise distinct. Moreover, assume that tangent sections C_i are either of multiplicative, or of additive type, and each of these two types is represented by some C_i . One can certainly find such p_i defined over a finite extension of K .

We will call such a family of subschemes (p_i, C_i, P_i) a *tetrahedral configuration*, even when we do not assumed a priori that it comes from a V . If it comes from a V , we will say that it is a *tangent tetrahedral configuration*.

Without restricting generality, we may choose in the ambient \mathbf{P}_K^3 a coordinate system $(z_1 : \dots : z_4)$ in such a way that $z_i = 0$ is an equation of P_i .

If the configuration is tangent to V , let $F(z_1, \dots, z_4) = 0$ be the equation of V . Here F is a cubic form with coefficients in K determined by V up to a scalar factor. For each $i \in \{1, \dots, 4\}$, write F in the form

$$F = \sum_{a=0}^3 z_i^a f_{3-a}^{(i)}(z_j \mid j \neq i), \quad (2.10)$$

where $f_b^{(i)}$ is a form of degree b in remaining variables.

Clearly, $f_3^{(i)} = 0$ is an equation of C_i in the plane P_i . Hence K and this equation can be reconstructed, up to a common factor, from a part of the tetrahedral configuration consisting of P_i , another plane P_j with tangent section of different type, and the induced relation of collinearity on them.

Consider the graph $G = G(V; p_1, \dots, p_4)$ with four vertices labeled $(1, \dots, 4)$, in which i and $j \neq i$ are connected by an edge, if there is a cubic monomial in $(z_k \mid k \neq i, j)$, that enters with nonzero coefficients in both $f_3^{(i)}$ and $f_3^{(j)}$. We want this graph to be connected. This will hold, for example, if in F all four coefficients at z_i^3 do not vanish. It is clear from this remark that connectedness of G is an open condition holding on a Zariski dense subset of all tangent configurations.

2.8.1. Proposition. *If the tetrahedral configuration is tangent to V , with connected graph G , then this V is unique.*

Proof. Let $g^{(i)}$ be a cubic form in $z_k, k \neq i$, such that $z_i = 0, g^{(i)} = 0$ are equations of C_i . We may change $g^{(i)}$ multiplying them by non-vanishing constants $c_i \in K$. If our configuration is tangent to V , given by (2.10), we may find c_i in such a way that $c_i g^{(i)} = f_3^{(i)}$. The obtained family of forms $\{c_i g^{(i)}\}$ is *compatible* in the following sense: if a cubic monomial in only two variables has non-zero coefficients in two $g^{(i)}$'s, then these coefficients coincide. In fact, they are equal to the coefficient of the respective monomial in F .

Conversely, if such a compatible system exists, and moreover, the graph G is connected, then (c_i) is unique up to a common factor. From such $c_i g^{(i)}$ one can reconstruct a cubic form of four variables, which will be necessarily proportional to F : coefficient at any cubic monomial m in (z_1, \dots, z_4) in it will be equal to the coefficient of this monomial in any of $c_i g^{(i)}$, for which z_i does not divide m .

2.9. Summary. This section was dedicated to several key constructions that show how and under what conditions a cubic surface V considered as a scheme, together with a ground field K , can be reconstructed from its set of K -points, endowed with some combinatorial data.

The main part of the data was the collinearity relation on $V_{sm}(K)$, and this relation, when it came from geometry, satisfied some strong conditions stated in Lemma 2.2.

However, this Lemma and the data used in 2.8 made appeal also to information about points on the lines of intersections of tangent planes: cf. specifically constructions of maps α and β before formula (2.8).

We want to get rid of this extra datum and work *only* with points of $V_{sm}(K)$.

This must be compensated by taking in account, besides the collinearity relation, an additional *coplanarity relation* on $V(K)$, essentially given by the sets of K -points of (many) non-tangent plane sections.

The next section is dedicated principally to a description of the relevant abstract combinatorial framework. The geometric situations are used mainly to motivate or illustrate combinatorial definitions and axioms.

3. Combinatorial and geometric cubic surfaces

3.1. Definition. *A combinatorial cubic surface is an abstract set S endowed with two structures:*

(i) A symmetric ternary relation “collinearity”: $\mathcal{L} \subset S^3$. We will say that triples $(p, q, r) \in \mathcal{L}$ are collinear.

(ii) A set \mathcal{P} of subsets $C \subset S$ called plane sections.

These relations must satisfy the axioms made explicit below in the subsections 3.2 and 3.3. Until all the axioms are stated and imposed, we may call a structure $(S, \mathcal{L}, \mathcal{P})$ a *cubic pre-surface*.

3.2. Collinearity Axioms. (i) For any $(p, q) \in S^2$, there exists an $r \in S$ such that $(p, q, r) \in \mathcal{L}$.

Call the triple (p, q, r) strictly collinear, if r is unique with this property, and p, q, r are pairwise distinct.

(ii) The subset $\mathcal{L}_s \subset \mathcal{L}$ of strictly collinear triples is a symmetric ternary relation.

(iii) Assume that $p \neq q$ and that there are two distinct $r_1, r_2 \in S$ with $(p, q, r_1) \in \mathcal{L}$ and $(p, q, r_2) \in \mathcal{L}$. Denote by $l = l(p, q)$ the set of all such r 's. Then $l^3 \subset \mathcal{L}$, that is any triple (r_1, r_2, r_3) of points in l is collinear.

Such sets l are called lines in S .

3.2.1. Example: combinatorial cubic surfaces of geometric origin. Let K be a field, and V a cubic surface in \mathbf{P}^3 over K . Denote by $S = V_{sm}(K)$ the set of nonsingular K -points of V .

We endow S with the following relations:

(a) $(p, q, r) \in \mathcal{L}$ iff either $p + q + r$ is the complete intersection cycle of V with a line in \mathbf{P}^3 defined over K (K -line), or else if p, q, r lie on a K -line \mathbf{P}_K^1 , entirely contained in V .

(b) Let $\mathbf{P} \subset \mathbf{P}^3$ be a K -plane. Assume that it either contains at least two distinct points of S , or is tangent to a K -point p , or else contains the tangent line to one of the branches of the tangent section of multiplicative type. Then $C := \mathbf{P}(K) \cap S$ is an element of \mathcal{P} . All elements of \mathcal{P} are obtained in this way.

3.3. Plane sections. We now return to the general combinatorial situation. Let $(S, \mathcal{L}, \mathcal{P})$ be a cubic pre-surface.

For any $p \in S$, put

$$C_p = C_p(S) := \{q \mid (p, p, q) \in \mathcal{L}\} \cup \{p\}. \quad (3.1)$$

3.3.1. Tangent Plane Sections Axiom. For each $p \in S$, we have $C_p \in \mathcal{P}$. Such plane sections are called tangent ones.

The next geometric property of plane sections of geometric cubics can now be rephrased combinatorially as follows.

3.3.2. Composition Axiom. (i) Let $C \in \mathcal{P}$ be a non-tangent plane section containing no lines in S . Then the collinearity relation \mathcal{L} induces on such C a structure of Abelian symmetric quasigroup (cf. Theorem–Definition 1.2).

(ii) Let $C_p = C_p(S)$ be a tangent plane section containing no lines. Then \mathcal{L} induces on $C_p^0 := C_p \setminus \{p\}$ a structure of Abelian symmetric quasigroup.

Choosing a zero/identity point in C , resp. $C_p \setminus \{p\}$, we get in this way a structure of abelian group on each of these sets.

3.3.3. Pencils of Plane Sections Axiom. Let $\lambda := (p, q, r) \in \mathcal{L}$. Assume that at least two of the points p, q, r are distinct. Denote by $\Pi_\lambda \subset \mathcal{P}$ the set

$$\Pi_\lambda := \{C \in \mathcal{P} \mid p, q, r \in C\}. \quad (3.2)$$

and call such Π_λ 's pencils of plane sections.

Then we have:

(i) If (p, q, r) do not lie on a line in S , then

$$S \setminus \{p, q, r\} = \coprod_{C \in \Pi_\lambda} (C \setminus \{p, q, r\}) \quad (3.3)$$

(disjoint union).

(ii) If (p, q, r) lie on a line l , then

$$S \setminus l = \coprod_{C \in \Pi_\lambda} (C \setminus l) \quad (3.4)$$

(disjoint union).

3.4. Combinatorial plane sections C_p of multiplicative/additive types. First of all we must postulate $(p, p, p) \in \mathcal{L}$, since in the geometric case $(p, p, p) \notin \mathcal{L}$ can happen only in a twisted case.

There are two different approaches to the tentative distinction between multiplicative and additive types. In one, we may try to prefigure the future realization of C_m and C_a as essentially the multiplicative (resp. additive) groups of a field K to be constructed.

Then, restricting ourselves for simplicity by fields of characteristic zero, we see that $C_p \setminus \{p\}$ which is of additive type after a choice of 0_a must become a vector space over \mathbf{Q} (be uniquely divisible), whereas the respective group of multiplicative type is never uniquely divisible.

However, these restrictions are too weak.

Instead, we will define *pairs* of combinatorial tangent plane sections modeled on (C_m, C_a) -configurations of sec. 2. After this is done, we will be able to “objectively”, independently of another member of the pair, distinguish between C_m and C_a using e.g. the divisibility criterion.

3.5. Combinatorial (C_m, C_a) -configurations. We can now give a combinatorial version of those (C_m, C_a) -configurations, that in the geometric case consist of two tangent plane sections of a cubic surface, one of additive, another of multiplicative type.

The main point is to see, how to use combinatorial plane sections in place of “external” lines $l = P_m \cap P_a$. This is possible, because the set of points of this line will now be replaced by bijective to it set of plane sections, belonging to a pencil, defined in terms of (C_m, C_a) , and geometrically consisting just of all sections containing p_m and p_a .

Let $(S, \mathcal{L}, \mathcal{P})$ be a combinatorial pre-surface, satisfying Axioms 3.2, 3.3.1, 3.3.2, 3.3.3.

Start with two distinct points of S , not lying on a line in S , and respective tangent sections of S

$$(p_m, p_a; C_{p_m}, C_{p_a}) \quad (3.5)$$

Let $r \in S$ be the unique third point such that $(p_m, p_a, r) \in \mathcal{L}$, $\lambda := \{p_m, p_a, r\}$. Put $C_{p_m}^0 := C_{p_m} \setminus \{p_m\}$, $C_{p_a}^0 := C_{p_a} \setminus \{p_a\}$.

Denote by Π_λ the respective pencil of plane sections. Consider the following binary relation:

$$R \subset C_{p_a} \times C_{p_m} : (p, q) \in R \iff \exists P \in \Pi_\lambda, p, q \in P. \quad (3.6)$$

3.5.1. Definition. $(p_m, p_a; C_{p_m}, C_{p_a})$ is called a (C_m, C_a) -configuration, if the following conditions are satisfied.

(i) R is a graph of some function

$$\lambda : C_{p_a} \rightarrow C_{p_m} \quad (3.7)$$

This function must be a bijection outside of two distinct points $0_m, \infty_m \in C_{p_a}^0$ which are mapped to p_m . Besides, we must have $\lambda(p_a) \in C_{p_m}^0$.

Assuming (i), put

$$A := C_{p_a}^0, \quad M := C_{p_m}^0$$

Introduce on these sets the structures of abelian groups using the the Composition Axiom 3.3.2 and some choices of zero and identity

$$0_a \in C_{p_a}^0, \quad 1_m \in C_{p_m}^0.$$

Define the map

$$\mu : M \cup \{0_m, \infty_m\} \rightarrow A \cup \{\infty_a\} \quad (3.8)$$

which is λ^{-1} on M and identical on $0_m, \infty_m$. Then

- (ii) Conditions of Lemma 2.2 must be satisfied for this μ .
- (iii) $C_{p_m} \cap C_{p_a}$ consists of three pairwise distinct points.

Thus, if $(p_m, p_a; C_{p_m}, C_{p_a})$ is a (C_m, C_a) -configuration, then we can combinatorially reconstruct the ground field and the isomorphic geometric configuration.

However, passing to tetrahedral configurations, we have to impose additional combinatorial compatibility conditions, that in the geometric case were automatic.

They are of two types:

- (a) If two planes P_i, P_j of the tetrahedron carry plane sections of the same type (both additive or both multiplicative), we must write combinatorially maps, establishing their isomorphism, and postulate this fact in the combinatorial setup.

This can be done similarly to the case of (C_m, C_a) -configurations.

- (b) If a schematic tangent plane section C_i can be reconstructed from two different pairs of tetrahedral plane sections C_i, C_j and (C_i, C_k) , the results must be naturally isomorphic.

It is clear, how to do it in principle, and the respective constraints must be stated explicitly.

I abstain from elaborating all details here for the following reason.

If, as a main application of this technique, one tries to imitate the approach to weak Mordell–Weil problem following the scheme of 0.4, then the necessary combinatorial constraints will probably hold automatically for finitely generated combinatorial subsurfaces of an initial geometric surface.

The real problem is: how to recognize that a given (say, finitely generated) combinatorial subsurface of a geometric surface is actually the whole geometric surface.

I do not know any answer to this problem.

It is well known, however, that such proper combinatorial subsurfaces do exist.

For example, when $K = \mathbf{R}$ and $V(\mathbf{R})$ is not connected, one of the components can be a combinatorial cubic surface in its own right. More generally, some unions of classes of the universal equivalence relation ([M1]) are closed with respect to collinearity and coplanarity relations: this can be extracted from the results of [SwD1].

4. Cubic curves and combinatorial cubic curves over large fields

4.1. Large fields and smooth cubic curves. Consider a smooth cubic curve $C \subset \mathcal{P}_K^2$ defined over K . Put $S := C(K)$ and endow S with the collinearity relation $\mathcal{L} \subset S^3$ defined by (1.3). Let $\mathcal{L}^0 := L/\mathbf{S}_3$, the set of orbits of \mathcal{L} with respect to the permutations. We may and will represent the image in \mathcal{L}^0 of $(p, q, r) \in \mathcal{L}$ as the 0-cycle $p + q + r$.

Now assume that K has no non-trivial extensions of degree 2 and 3. Then all intersection points of any K -line with C lie in $C(K)$. Therefore, we have the canonical bijection

$$\xi: \{K\text{-lines in } \mathbf{P}^2(K)\} \rightarrow \mathcal{L}^0: \quad l \mapsto \text{intersection cycle } l \cap C. \quad (4.1)$$

In this approach, K -points of $\mathbf{P}^2(K)$ have to be characterized in terms of pencils of all lines passing through a given point. Therefore, it is more natural to work with the dual projective plane from the start.

Let $\widehat{\mathbf{P}}^2$ be the projective plane dual to the plane in which C lies. Combinatorially, K -points \widehat{l} (resp. lines \widehat{p}) of $\widehat{\mathbf{P}}^2$ are K -lines l (resp. points p) of \mathcal{P}_K^2 , with inverted incidence relation: $\widehat{l} \in \widehat{p}$ iff $p \in l$. Thus, (4.1) turns into a bijection

$$\widehat{\xi}: \{K\text{-points in } \widehat{\mathbf{P}}^2(K)\} \rightarrow \mathcal{L}^0: \quad \widehat{l} \mapsto \text{intersection cycle } l \cap C. \quad (4.2)$$

Thus, $\widehat{\xi}$ sends lines in $\widehat{\mathbf{P}}^2$ to certain subsets in \mathcal{L}^0 that we also may call *pencils*.

This geometric situation motivates the following definition.

Let (S, \mathcal{L}) be an abelian symmetric quasigroup. Put $\mathcal{L}^0 = \mathcal{L}/\mathbf{S}_3$.

Assume that \mathcal{L}^0 is endowed with a set of its subsets \mathcal{P}^0 , called *pencils*, which turns it into a combinatorial projective plane, with pencils as lines. This means that besides the trivial incidence conditions, Pappus Axiom is also valid. Hence we can reconstruct from $(\mathcal{L}^0, \mathcal{P}^0)$ a field K such that $\mathcal{L}^0 = \mathbf{P}^2(K)$, $\mathcal{P}^0 =$ the set of K -lines in $\mathbf{P}^2(K)$.

The following Definition, inspired by geometry, encodes the interaction between the structures (S, \mathcal{L}) and $(\mathcal{L}^0, \mathcal{P}^0)$.

4.2. Definition. *The structure $(S, \mathcal{L}, \mathcal{P}^0)$ is called a combinatorial cubic curve over a large field, if the following conditions are satisfied.*

(i) *For each fixed $p \in S$, the set of cycles $p + q + r \in \mathcal{L}^0$, $q, r \in S$, is a pencil Π_p .*

(ii) *If a pencil Π is not of the type Π_p , then any two distinct elements in Π do not intersect (as unordered triples of S -points).*

(iii) *For each pencil Π_q (resp. each pencil not of type Π_q) and any $p \in S$, (resp. any $p \neq q$) there exists a unique cycle in \mathcal{L}^0 contained in Π and containing p .*

Obviously, each geometric smooth cubic curve over a large field defines the respective combinatorial object.

4.3. Question. Are there such combinatorial curves not coming from a geometric one? In particular, are fields K coming from such combinatorial objects necessarily “large” in the sense of algebraic definition above (closed under taking square and cubic roots)?

Similar constructions can be done and question asked for cubic surfaces. Notice that over a large field, any point on a smooth surface, not lying on a line, is of either multiplicative, or additive type.

APPENDIX. Mordell–Weil and height: numerical evidence

Let V be a geometrically irreducible cubic curve or a cubic surface over a field K , with the standard geometric collinearity relation (1.3) for curves, (3.2.1a) for surfaces, and the binary composition law (1.1) for curves. For surfaces, we will state the following fancy definition.

A.1. Definition. Let $S \subset V(K)$, and X_1, \dots, X_N, \dots free commuting but nonassociative variables,

$$w = (\dots (X_{i_1} \circ X_{i_2}) \circ (X_{i_3} \circ \dots (\dots \circ X_{i_k}) \dots))$$

a finite word in this variables,

$$ev : \{X_1, \dots, X_N, \dots\} \rightarrow S$$

an evaluation map.

a) A point $p \in V(K)$ is called *the strong value of w at (S, ev)* if during the inductive calculation of

$$p = ev(W) := (\dots (ev(X_{i_1}) \circ ev(X_{i_2})) \circ (ev(X_{i_3})) \circ \dots (\dots \circ ev(X_{i_k}) \dots))$$

we never land in a situation where the result of composition is not uniquely defined, that is $x \circ x$ with singular x for a curve, or $x \circ y$ where $y = x$ or the line through x, y lies in V for a surface.

b) A point $p \in S(K)$ is called *a weak value of w at (S, ev)* if during the inductive calculation of

$$p := ev_{weak}(W) = (\dots (ev(X_{i_1}) \circ ev(X_{i_2})) \circ (ev(X_{i_3})) \circ \dots (\dots \circ ev(X_{i_k}) \dots))$$

whenever we land in a situation where \circ is not defined, we are allowed to choose as a value of $y \circ z$ (resp. $y \circ y$) any point of the line \overline{yz} (resp. any point of intersection of a tangent line to V at y with V .)

Thus, weak evaluation produces a whole set of answers.

A.2. Definition. (i) A subset $S \subset V(K)$ strongly generates $V(K)$, if $V(K)$ coincides with the set of all strong S -values of all words w as above.

(i) A subset $S \subset V(K)$ weakly generates $V(K)$, if $V(K)$ coincides with the set of all weak S -values of all words w .

Now we can state two versions of Mordell–Weil problem for cubic surfaces.

Strong Mordell–Weil problem for V : *Is there a finite S that strongly generates $V(K)$?*

Weak Mordell–Weil problem for V : *Is there a finite S that weakly generates $V(K)$?*

For curves, one often calls the weak Mordell–Weil theorem the statement that $C(K)/2C(K)$ is finite (referring to the group structure $p+q = e \circ (p \circ q)$).

A.3. Proving strong Mordell–Weil for smooth cubic curves over number fields. The classical strategy of proof includes two ingredients.

(a) Introduce an arithmetic *height function* $h : \mathcal{P}^2(K) \rightarrow \mathbf{R}$. E.g. for

$$K = \mathbf{Q}, p = (x_0 : x_1 : x_2) \in \mathcal{P}^2(\mathbf{Q}), x_i \in \mathbf{Z}, \text{g.c.d.}(x_i) = 1$$

put

$$h(p) := \max_i |x_i|.$$

(b) Prove *the descent property*: $\exists H_0$ such that if $h(p) > H_0$, $p \in C(K)$, then $p = q \circ r$ for some $q, r \in C(K)$ with $h(q), h(r) < h(p)$.

The same strategy works for general finitely generated fields. For larger fields, the strong Mordell–Weil generally fails, but the weak one might survive.

Let K be algebraically closed, or \mathbf{R} , or a finite extension of \mathbf{Q}_p . Let C be a smooth cubic curve, V a smooth cubic surface over K .

Then:

- $V(K)$ is weakly finitely generated, but not strongly.
- $C(K)$, if non-empty, is not finitely generated.

A.4. Point count on cubic curves. It is well known that as $H \rightarrow \infty$,

$$\text{card} \{p \in C(K) \mid h(p) \leq H\} = \text{const} \cdot (\log H)^{r/2} (1 + o(1)),$$

$$r := \text{rk } C(K) = \text{rk Pic } C. \tag{A.1}$$

A.5. Point count on cubic surfaces. Here we have only a conjecture and some partial approximations to it:

Conjecture: as $H \rightarrow \infty$,

$$\text{card} \{p \in V_0(K) \mid h(p) \leq H\} = \text{const} \cdot H (\log H)^{r-1} (1 + o(1)),$$

$$V_0 := V \setminus \{\text{all lines}\}, r := \text{rk Pic } V \tag{A.2}$$

A proof of (A.1) can be obtained by a slight strengthening of the technique used in the finite generation proof. Namely, one shows that $\log h(p)$ is “almost a quadratic form” on $C(K)$. In fact, it differs from a positive defined quadratic form by $O(1)$, so that (A.1) follows from the count of lattice point in an ellipsoid.

The descent property used for Mordell–Weil ensures that this quadratic form is positive definite.

How could one attack this conjecture? For the circle method, there are too few variables. Moreover, connections with Mordell–Weil for cubic surfaces are totally missing.

Nevertheless, the inequality

$$\text{card} \{p \in V_0(K) \mid h(p) \leq H\} > \text{const} \cdot H(\log H)^{r-1}$$

is proved in [SlSw–D] for cubic surfaces over \mathbf{Q} with two rational skew lines. There are also results for singular surfaces: cf. [Br], [BrD1], [BrD2].

A.6. Some numerical data. Here I will survey some numerical evidence computed by Bogdan Vioreanu, cf. [Vi].

In the following tables, the following notation is used.

Input/table head: code $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4]$ of the surface

$$V : \sum_{i=1}^4 a_i x_i^3 = 0.$$

Outputs:

(i) **GEN:** Conjectural list of weak generators

$$p := (x_1 : x_2 : x_3 : x_4) \in V(\mathbf{Q})$$

(ii) **Nr:** The length of the list $List_{good}$ of all points x , ordered by the increasing height $h(p) := \sum_i |x_i|$, such that any point of the height \leq maximal height in $List_{good}$, is weakly generated by **GEN**.

(iii) **H_{bad}:** the height of the first point that was NOT shown to be generated by **GEN**.

(iv) **L**: the maximal length of a non-associative word with generators in (\mathbf{GEN}, \circ) one of whose weak values produced an entry in $List_{good}$.

Example: For $V = [\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}]$, we have:

$\mathbf{GEN} = \{p^0 := (1 : -1 : -1 : 1)\}$

Nr = 8521: the first 8521 points in the list of points of increasing height are weakly generated by the single point p^0 .

L = 13: the maximal length of a non-associative word in (p^0, \circ) representing some point of the $List_{good}$ was 13.

H_{bad} = 24677: the first point that was not found to be generated by p^0 was of height 24677.

SELECTED DATA

$[\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}]$, rk Pic = 1

GEN	Nr	L	H_{bad}
(0:1:1:-1)	15222	12	23243
(1:1:-1:0)			
(2:-2:1:1)			

$[\mathbf{1}, \mathbf{1}, \mathbf{5}, \mathbf{25}]$, rk Pic = 2

(1:-1:0:0)	32419	9	30072
(1:4:-2:-1)			

$[\mathbf{1}, \mathbf{1}, \mathbf{7}, \mathbf{7}]$, rk Pic = 2

(0:0:1:-1)	16063	7	2578
(1:-1:0:0)			
(1:-1:-1:1)			
(1:-1:1:-1)			

A.7. Discussion of other numerical data. Bogdan Vioreanu studied all in all 16 diagonal cubic surfaces V ; he compiled lists of all points up to height 10^5 , for some of them up to height $3 \cdot 10^5$.

The conjectural asymptotics (A.2) seems to be confirmed.

There is a *good conjectural expression* for the constant in (A.2) (for appropriately normalized height, not the naive one we used). It goes back to works of E. Peyre. Its theoretical structure very much reminds the Birch–Swinnerton–Dyer constant for elliptic curves. For theory and numerical evidence, see [PeT1], [PeT2], [Sw–D2], [Ch–L].

The (weak) finite generation looks confirmed for most of the considered surfaces, but some stubbornly resist, most notably [17, 18, 19, 20], [4, 5, 6, 7], [9, 10, 11, 12].

If one is willing to believe in weak finite generation (as I am), the reason for failure might be the following observable fact:

When one manages to represent a “bad” point p of large height as a non-associative word in the generators (\mathbf{GEN}, \circ) , the height of intermediate results (represented by subwords) tends to be much higher than $h(p)$, and hence outside of the compiled list of points.

Finally, the relative density of points p for which “one-step descent” works, that is,

$$p = q \circ r, \quad h(q), h(r) < h(p),$$

seems to tend to a certain value $0 < d(V) < 1$.

Question: *Can one guess a theoretical expression for $d(V)$?*

Notice that on each smooth cubic curve C , the “one-step descent” works for all points of sufficiently large height, so that $d(C) = 1$.

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