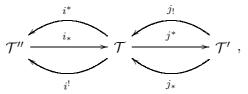
RECOLLEMENTS FROM GENERALIZED TILTING

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ABSTRACT. Let \mathcal{A} be a small dg category over a field k and let \mathcal{U} be a small full subcategory of the derived category $\mathcal{D}\mathcal{A}$ which generate all free dg \mathcal{A} -modules. Let (\mathcal{B}, X) be a standard lift of \mathcal{U} . We show that there is a recollement such that its middle term is $\mathcal{D}\mathcal{B}$, its right term is $\mathcal{D}\mathcal{A}$, and the three functors on its right side are constructed from X. This applies to the pair (\mathcal{A}, T) , where \mathcal{A} is a k-algebra and T is a good n-tilting module, and we obtain a result of Bazzoni–Mantese–Tonolo. This also applies to the pair $(\mathcal{A}, \mathcal{U})$, where \mathcal{A} is an augmented dg category and \mathcal{U} is the category of 'simple' modules, *e.g.* \mathcal{A} is a finite-dimensional algebra or the Kontsevich–Soibelman \mathcal{A}_{∞} -category associated to a quiver with potential. **2010 Mathematics Subject Classifications**: 18E30, 16E45.

A *recollement* of triangulated categories is a diagram of triangulated categories and triangle functors



such that

- $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples;
- $i_*, j_*, j_!$ are fully faithful;
- $j^* \circ i_* = 0;$
- for every object X of \mathcal{T} there are two triangles

 $i_*i^! \longrightarrow X \longrightarrow j_*j^*X \longrightarrow$ and $j_!j^*X \longrightarrow X \longrightarrow i_*i^*X \longrightarrow$,

where the four morphisms are the units and counits.

We also say that this is a recollement of \mathcal{T} in terms of \mathcal{T}' and \mathcal{T}'' . This notion was introduced by Beilinson–Bernstein–Deligne in [4] in geometric contexts, where stratifications of varieties induce recollements of derived categories of sheaves.

In algebraic contexts, recollements are closely related to tilting theory. Let A be a ring. Let $\mathcal{D}(A) = \mathcal{D}(\mathsf{Mod}\,A)$ denote the derived category of (right) A-modules, and per A denote the triangulated subcategory of $\mathcal{D}(A)$ generated by the free module of rank 1. An object Tof per A is called a *partial tilting complex* if $\mathsf{Hom}_{\mathcal{D}(A)}(T, \Sigma^n T) = 0$, and a *tilting complex* if in

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addition tria $(T) = \operatorname{per} A$, where tria(T) is the triangulated subcategory of $\mathcal{D}(A)$ generated by T. Rickard's Morita theorem for derived categories states that the standard functors associated to a tilting complex T over A are triangle equivalences between $\mathcal{D}(A)$ and $\mathcal{D}(\operatorname{End}_{\mathcal{D}(A)}(T))$, see [18]. Later in [12], Koenig proved that under certain conditions a partial tilting complex Tover A yields a recollement of $\mathcal{D}(A)$ in terms of $\mathcal{D}(\operatorname{End}_{\mathcal{D}(A)}(T))$ and a third derived category which measures how far the associated standard functors are from being equivalences (see also [8] [15]). In this sense, a recollement of derived categories can be viewed as a natural generalization of a derived equivalence. The relation between tilting theory and recollements of derived categories has been further studied in [1] [6]. The dg version of Rickard's theorem was developed by Keller in [10], and the result of Koenig was generalized to the dg setting by Jørgensen [9] and Nicolás–Saorín [17], where the role of partial tilting complexes is played by compact objects.

In this paper we deal with a situation which is 'dual' to the one in [12] [9] [17]. Starting from a dg category \mathcal{A} and a set of objets in the derived category $\mathcal{D}\mathcal{A}$ which generates all the compact objects, we construct a dg category \mathcal{B} together with a recollement of $\mathcal{D}\mathcal{B}$ in terms of $\mathcal{D}\mathcal{A}$ and another derived category, see Theorem 1. We identify this third derived category with a certain known category in the special case when \mathcal{A} is the Kontsevich–Soibelman A_{∞} -category associated to a quiver with potential (Corollary 3) or when \mathcal{A} is a finite-dimensional self-injective algebra (Corollary 4). The motivation for our study was to have a better understanding of the 'exterior' case of the Koszul duality (Corollary 2) and a result of Bazzoni–Mantese–Tonolo which says that the right derived Hom-functor associated to an (infinitely generated) good tilting module is fully faithful (Corollary 1).

1. The main result

Let k be a field and let \mathcal{A} be a small dg k-category. Denote by Dif \mathcal{A} the dg category of (right) dg \mathcal{A} -modules. A dg \mathcal{A} -module M is \mathcal{K} -projective if the dg functor Dif $\mathcal{A}(M, ?)$ preserves acyclicity. For example, the free modules $A^{\wedge} = \text{Dif }\mathcal{A}(?, A), A \in \mathcal{A}$, are \mathcal{K} -projective. Let $\mathcal{D}\mathcal{A}$ denote the derived category of \mathcal{A} , which is triangulated with suspension functor Σ the shift functor. For a set of objects or a subcategory \mathcal{S} of $\mathcal{D}\mathcal{A}$ we denote by tria \mathcal{S} the smallest triangulated subcategory of $\mathcal{D}\mathcal{A}$ containing all objects in \mathcal{S} and closed under taking direct summands. Let $\text{per }\mathcal{A} = \text{tria}(\mathcal{A}^{\wedge}, \mathcal{A} \in \mathcal{A})$. An object M of $\mathcal{D}\mathcal{A}$ is compact if the functor $\mathcal{D}\mathcal{A}(M, ?)$ commutes with infinite (set-indexed) direct sums, or equivalently, if M belongs to per \mathcal{A} . See [10].

Let \mathcal{U} be a full small subcategory of \mathcal{DA} such that

Let (\mathcal{B}, X) be a standard lift of \mathcal{U} ([10, Section 7]). Precisely, \mathcal{B} is a dg subcategory of Dif \mathcal{A} consisting of \mathcal{K} -projective resolutions over \mathcal{A} of objects of \mathcal{U} (to avoid confusion, for each object B of \mathcal{B} we will denote by U_B the corresponding dg \mathcal{A} -module) and X is the dg $\mathcal{B}^{op} \otimes \mathcal{A}$ -module defined by $X(B, A) = U_B(A)$. It induces a pair of adjoint dg functors and a pair of adjoint triangle functors

$$\mathsf{Dif}\,\mathcal{B} \xleftarrow{T_X}{\longleftarrow} \mathsf{Dif}\,\mathcal{A}, \qquad \mathcal{D}\mathcal{B} \xleftarrow{\mathbf{L}T_X}{\longleftarrow} \mathcal{D}\mathcal{A}.$$

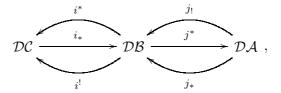
When \mathcal{A} and \mathcal{B} are dg k-algebras (*i.e.* dg k-categories with one object), the functors $\mathbf{L}T_X$ and $\mathbf{R}H_X$ are usually written as ? $\overset{L}{\otimes} X$ and $\mathsf{RHom}(X, ?)$.

Let X^T be the dg $\mathcal{A}^{op} \otimes \mathcal{B}$ -module defined by

$$X^T(A,B) = \mathsf{Dif}\,\mathcal{A}(X^B,A^\wedge),$$

where for $B \in \mathcal{B}$, X^B is by definition the dg \mathcal{A} -module X(B, ?). From the definition of X we see that $X^B = U_B$. The main result of this paper is

Theorem 1. Assume notations as above. There is a dg k-category C and a recollement of triangulated categories



where the adjoint triple $(i^*, i_*, i^!)$ is defined by a dg functor $F : \mathcal{B} \to \mathcal{C}$ (which is bijective on objects) such that $i_* = F^* : \mathcal{DC} \to \mathcal{DB}$ is the pull-back functor, and the adjoint triple (j_1, j^*, j_*) is given by

$$j_! = \mathbf{L}T_{X^T},$$

 $j^* = \mathbf{R}H_{X^T} \simeq \mathbf{L}T_X,$
 $j_* = \mathbf{R}H_X.$

Proof. In view of [17, Theorem 5], it suffices to prove

- (a) $\mathbf{L}T_{X^T}$ is fully faithful,
- (b) $\mathbf{R}H_{X^T} \simeq \mathbf{L}T_X$.

The proof for (a) is the same as the proof of [10, Lemma 10.5 the 'exterior' case c)]. Since (\mathcal{B}, X) is a lift, the restriction of $\mathbf{L}T_X$ on the perfect derived category per \mathcal{B} is fully faithful, and its essential image is tria \mathcal{U} (see [10, Section 7.3]):

$$\mathbf{L}T_X|_{\mathsf{per}\,\mathcal{B}}:\mathsf{per}\,\mathcal{B}\xrightarrow{\sim}\mathsf{tria}\,\mathcal{U}.$$

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It is clear that $\mathbf{R}H_X$ takes an object of tria \mathcal{U} into per \mathcal{B} . Therefore, the restriction $\mathbf{R}H_X|_{\mathsf{tria}\mathcal{U}}$ is a quasi-inverse of $\mathbf{L}T_X|_{\mathsf{per}\mathcal{B}}$, and hence is fully faithful. It follows from [10, Lemma 6.2 a)] that the restriction $\mathbf{L}T_{X^T}|_{\mathsf{per}\mathcal{A}}$ is naturally isomorphic to the restriction of $\mathbf{R}H_X|_{\mathsf{per}\mathcal{A}}$, which is fully faithful by condition (1). Condition (1) also implies that $\mathbf{R}H_X(A^{\wedge}) = (X^T)^A$ $(A \in \mathcal{A})$ belongs to $\mathsf{per}\mathcal{B}$, and hence is compact by [10, Theorem 5.3]. Now applying [10, Lemma 4.2 b)], we obtain that $\mathbf{L}T_{X^T}$ is fully faithful, finishing the proof of (a).

Let $Y \to X^T$ be a \mathcal{K} -projective resolution of dg $\mathcal{A}^{op} \otimes \mathcal{B}$ -modules. Then the specialization $Y^A \to (X^T)^A$ is a \mathcal{K} -projective resolution of dg \mathcal{B} -modules for any object A of \mathcal{A} . Recall that $(X^T)^A$ is compact. It follows from [10, Lemma 6.2 a)] that $\mathbf{L}T_{Y^T} \simeq \mathbf{R}H_Y$. By [10, Lemma 6.1 b)], in order to prove $\mathbf{R}H_{X^T} \simeq \mathbf{L}T_X$, it suffices to prove that as dg $\mathcal{B}^{op} \otimes \mathcal{A}$ -modules Y^T and X are quasi-isomorphic. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We have $H_X(U_B) = B^{\wedge}$, and hence

$$Y^{T}(A, B) = \operatorname{Dif} \mathcal{B}(Y^{A}, B^{\wedge})$$
$$= \operatorname{Dif} \mathcal{B}(Y^{A}, H_{X}(U_{B}))$$
$$\cong \operatorname{Dif} \mathcal{A}(T_{X}(Y^{A}), U_{B}).$$

The composition $T_X(Y^A) \to T_X((X^T)^A) = T_X \circ H_X(A^{\wedge}) \to A^{\wedge}$ is exactly the counit $\mathbf{L}T_X \circ \mathbf{R}H_X(A^{\wedge}) \to A^{\wedge}$, which is an isomorphism in $\mathcal{D}\mathcal{A}$ because the restriction of $\mathbf{R}H_X$ on per \mathcal{A} is fully faithful. Moreover, both $T_X(Y^A)$ and A^{\wedge} are \mathcal{K} -projective dg \mathcal{A} -modules. Therefore we have

$$Y^{T}(A,B) \stackrel{q.rs}{\leftarrow} \operatorname{Dif} \mathcal{A}(A^{\wedge}, U_{B})$$
$$= U_{B}(A)$$
$$= X(A,B).$$

Further, every morphism in the above is functorial in both A and B. This completes the proof of (b). \checkmark

Corollary 1 ([3]). Let A be a k-algebra, and n be a positive integer. Let T be a good n-tilting module, i.e. T is an A-module such that

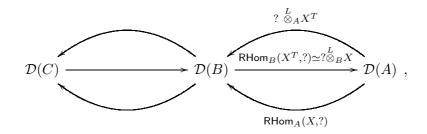
- (T1) the projective dimension of T is less than or equal to n;
- (T2) $\operatorname{Ext}_{A}^{i}(T, T^{(\alpha)}) = 0$ for any integer i > 0 and for any cardinal α ;
- (T3) there is an exact sequence

 $0 \longrightarrow A \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^n \longrightarrow 0 ,$

where T^0, \ldots, T^n are direct summands of direct sums of finite copies of T.

Put $B = \text{End}_A(T)$. Then the right derived functor $\text{RHom}_A(T,?) : \mathcal{D}(A) \to \mathcal{D}(B)$ is fully faithful, and $\mathcal{D}(A)$ is triangle equivalent to the triangle quotient of $\mathcal{D}(B)$ by the kernel of the left derived functor $? \bigotimes_B^L T$.

Proof. Let \mathcal{U} be the full subcategory of $\mathcal{D}(A)$ consisting of one object T. Then the condition (T3) implies the condition (1). Let X be a projective resolution of T over $B^{op} \otimes_k A$, and let \tilde{B} be the dg k-algebra Dif A(X, X). Then X is \mathcal{K} -projective over A, and (\tilde{B}, X) is a standard lift of T. Thanks to (T2), the representation map $B \to \tilde{B}$ of the dg B-A-bimodule X is a quasiisomorphism, inducing mutually quasi-inverse triangle equivalences $? \bigotimes_{\tilde{B}}^{L} \tilde{B} = \mathsf{RHom}_{\tilde{B}}(\tilde{B}, ?) :$ $\mathcal{D}(\tilde{B}) \to \mathcal{D}(B)$ and $? \bigotimes_{B}^{L} \tilde{B} : \mathcal{D}(B) \to \mathcal{D}(\tilde{B})$. Now applying Theorem 1 and composing the resulting recollement with the above triangle equivalences, we obtain a recollement



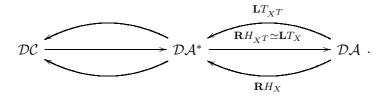
where C is a dg k-algebra. Since X and T are quasi-isomorphic as dg $B^{op} \otimes_k A$ -modules, we have natural isomorphisms ? $\overset{L}{\otimes}_B X \simeq$? $\overset{L}{\otimes}_B T$ and $\mathsf{RHom}_A(X, ?) \simeq \mathsf{RHom}_A(T, ?)$ ([10, Lemma 6.1 b)]). The desired result follows at once.

- **Remark.** a) This result is due to Bazzoni [2] for n = 1 and Bazzoni–Mantese–Tonolo [3] for general n for all rings A.
 - b) By Theorem 1, the left half of the recollement in the proof is induced from a dg homomorphism B → C. For the case n = 1 and for all rings A Chen-Xi obtained in [6] such a recollement with C being an ordinary ring (so that the map B → C becomes a homomorphism of rings). They used some results in [1] and many other results such as the homological properties of the tilting module T.

To state the next corollary, we need to introduce some notions. Let \mathcal{A} be an *augmented* dg k-category ([10, Section 10.2]), *i.e.*

- distinct objects of \mathcal{A} are non-isomorphic,
- for each $A \in \mathcal{A}$, a dg module \overline{A} is given such that $H^0\overline{A}(A) \cong k$ and $H^n\overline{A}(A')$ whenever $n \neq 0$ or $A' \neq A$.

Let (\mathcal{A}^*, X) be a standard lift of $\mathcal{U} = \{\overline{A} | A \in \mathcal{A}\} \subset \mathcal{D}\mathcal{A}$. By abuse of language, we call the dg *k*-category \mathcal{A}^* the *Koszul dual* of \mathcal{A} . Assume that the condition (1) holds, *e.g.* this happens in the 'exterior' case in [10, Section 10.5]. **Corollary 2.** Assume notations as above. There is a recollement of derived categories of dg k-categories

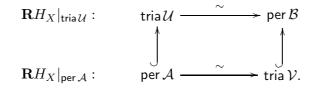


Proof. This is a direct consequence of Theorem 1.

2. The left term

As in the preceding section, we let k be a field, \mathcal{A} be a small dg k-category, \mathcal{U} be a full small subcategory of the derived category $\mathcal{D}\mathcal{A}$ such that tria $\mathcal{U} \supseteq \text{per }\mathcal{A}$, and let (\mathcal{B}, X) be a standard lift of \mathcal{U} . Theorem 1 says that there is a recollement of $\mathcal{D}\mathcal{B}$ in terms of $\mathcal{D}\mathcal{A}$ and a third derived category $\mathcal{D}\mathcal{C}$, where \mathcal{C} is a dg k-category whose objects are in bijection with the objects of \mathcal{U} .

Let $\mathcal{V} = \{(X^T)^A | A \in \mathcal{A}\} \subset \mathcal{DB}$. From the proof of the theorem we obtain a commutative diagram



Therefore $\mathbf{R}H_X$ induces a triangle equivalence between the triangle quotient categories

 $\mathsf{tria}\,\mathcal{U}/\,\mathsf{per}\,\mathcal{A}\xrightarrow{~~}\mathsf{per}\,\mathcal{B}/\,\mathsf{tria}\,\mathcal{V}\ .$

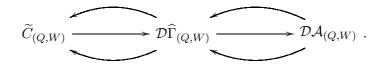
For a triangulated category \mathcal{T} , let \mathcal{T}^c denote the subcategory of compact objects in \mathcal{T} . Let $\operatorname{Tria} \mathcal{V}$ be the localizing subcategory of \mathcal{DB} generated by the objects in \mathcal{V} . We have $(\mathcal{DB})^c = \operatorname{per} \mathcal{B}$, and $(\operatorname{Tria} \mathcal{V})^c = \operatorname{tria} \mathcal{V}$. Thus by [16, Theorem 2.1], the category $(\mathcal{DB}/\operatorname{Tria} \mathcal{V})^c$ is triangle equivalent to the idempotent completion of $\operatorname{per} \mathcal{B}/\operatorname{tria} \mathcal{V}$. Since the essential image of $\operatorname{L} T_{X^T}$ is exactly $\operatorname{Tria} \mathcal{V}$, it follows that \mathcal{DC} is triangle equivalent to the triangle quotient $\mathcal{DB}/\operatorname{Tria} \mathcal{V}$, and hence is an 'unbounded version' of $\operatorname{tria} \mathcal{U}/\operatorname{per} \mathcal{A} \cong \operatorname{per} \mathcal{B}/\operatorname{tria} \mathcal{V}$. Apparently, \mathcal{DC} vanishes if and only if so does $\operatorname{tria} \mathcal{U}/\operatorname{per} \mathcal{A}$, in which case \mathcal{U} consists of a set of compact generators for \mathcal{DA} .

In the following two special cases, we are able to identify \mathcal{DC} with a certain known category (however, the dg category \mathcal{C} is not easy to describe).

Corollary 3. Let (Q, W) be a quiver with potential. Let $\mathcal{A}_{(Q,W)}$ be the Kontsevich–Soibelman A_{∞} -category ([13, Section 3.3]) (or its enveloping dg category), let $\widehat{\Gamma}_{(Q,W)}$ be the complete

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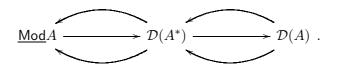
Ginzburg dg category ([7, Section 5]), and let $\widetilde{C}_{(Q,W)}$ be the 'unbounded version' of the generalized cluster category ([11, Remark 4.1]). Then there is a recollement of triangulated categories



Proof. Let $\mathcal{A} = \mathcal{A}_{(Q,W)}$ and let \mathcal{U} be the category of simple \mathcal{A} -modules. Then condition (1) holds since \mathcal{A} is finite-dimensional, and there is a standard lift (\mathcal{B}, X) such that the dg category $\Gamma = \widehat{\Gamma}_{(Q,W)}$ (as the Koszul dual of $\mathcal{A}_{(Q,W)}$) is quasi-isomorphic to \mathcal{B} . By Corollary 2, there is a recollement with the middle term being $\mathcal{D}\Gamma$, the right term being $\mathcal{D}\mathcal{A}$, and the right upper functor being $\mathbf{L}T_{X^T}$. It remains to prove that the left term of this recollement is triangle equivalent to $\widetilde{\mathcal{C}}_{(Q,W)}$. Object sets of \mathcal{A} , of \mathcal{U} , and of Γ can all be identified with the vertice set Q_0 of the quiver Q. For a vertex i of Q, considered as an object of \mathcal{A} , the right dg Γ -module $(X^T)^i$ is isomorphic in $\mathcal{D}\Gamma$ to $\Sigma^{-3}S_i$, where S_i the simple top of the free Γ -module i^{\wedge} . Thus the essential image of $\mathbf{L}T_{X^T}$ is the localizing subcategory $\mathcal{D}_0\Gamma = \operatorname{Tria}(S_i, i \in Q_0)$ of $\mathcal{D}\Gamma$ generated by the S_i , $i \in Q_0$. Thus the left term of the recollement is triangle equivalent to $\mathcal{D}\Gamma/\mathcal{D}_0\Gamma$, which is by definition $\widetilde{\mathcal{C}}_{(Q,W)}$.

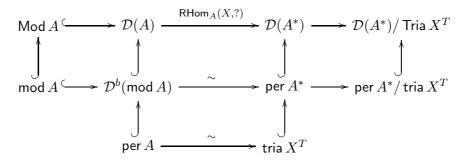
Let A be a finite-dimensional basic k-algebra. Let S be the direct sum of the objects in a set of representatives of isomorphism classes of simple A-modules, and let X be a projective resolution of S. Then $A^* = \text{Dif } A(X, X)$ is the Koszul dual of A.

Corollary 4 ([14]). Let A be a finite-dimensional basic self-injective k-algebra, and A^* its Koszul dual. Let <u>Mod</u>A be the stable category of the category Mod A of A-modules. Then there is a recollement of triangulated categories



Proof. Let mod A be the category of finite-dimensional A-modules, and <u>mod</u>A its stable category. As a triangulated subcategory of $\mathcal{D}(A)$, the bounded derived category $\mathcal{D}^b(\text{mod} A)$ of mod A coincides with tria S. Recall that the essential image of ? $\overset{L}{\otimes}_A X^T$ is Tria X^T . Consider

the following commutative diagram



where the leftmost horizontal functors are the canonical embeddings, and the rightmost horizontal functors are the canonical projections. The restriction of $\mathsf{RHom}_A(X,?)$ on $\mathsf{Mod}\,A$ commutes with infinite direct sums, because X can be chosen such that its component in each degree is a finitely generated projective A-module. Therefore the composition of the three functors in the first row, denoted by F, commutes with infinite direct sums. Since $\mathsf{RHom}_A(X,A) \cong X^T$ belongs to $\mathsf{Tria}\,X^T$, it follows that F factors through the stable category $\mathsf{Mod}A$. In this way, we obtain a triangle functor

$$\overline{F} : \underline{\mathsf{Mod}}A \to \mathcal{D}(A^*)/\operatorname{\mathsf{Tria}} X^T,$$

which commutes with infinite direct sums. It is known that $\underline{\mathsf{Mod}}A$ is compactly generated by $\underline{\mathsf{mod}}A$ and $(\underline{\mathsf{Mod}}A)^c = \underline{\mathsf{mod}}A$. Moreover, the restriction $\bar{F}|_{\underline{\mathsf{mod}}A}$ is the composition of the following three functors

$$\underline{\operatorname{mod}} A \longrightarrow \mathcal{D}^b(\operatorname{mod} A)/\operatorname{per} A \xrightarrow{\sim} \operatorname{per} A^*/\operatorname{tria} X^T { \longrightarrow } \mathcal{D}(A^*)/\operatorname{Tria} X^T$$

The first functor is also an equivalence ([19, Theorem 2.1]). Therefore \overline{F} induces a triangle equivalence between $\underline{\text{mod}}A = (\underline{\text{Mod}}A)^c$ and $\operatorname{per} A^*/\operatorname{tria} X^T = (\mathcal{D}(A^*)/\operatorname{Tria} X^T)^c$. By [10, Lemma 4.2], \overline{F} itself is an equivalence. Now applying Corollary 2 we obtain the desired recollement.

Remark. Let $\mathcal{H}(\ln j A)$ be the homotopy category of injective A-modules and $\mathcal{H}_{ac}(\ln j A)$ be its full subcategory of acyclic complexes. Applying a result of Krause [14, Corollary 4.3] to the algebra A, we obtain a recollement of $\mathcal{H}(\ln j A)$ in terms of $\mathcal{D}(A)$ and $\mathcal{H}_{ac}(\ln j A)$ with the right middle functor being the canonical projection $Q : \mathcal{H}(\ln j A) \to \mathcal{D}(A)$. We claim that this recollement is equivalent to the one in Corollary 4. Indeed, Krause proved in [14] that $\mathcal{H}(\ln j A)$ is compactly generated by (an injective resolution of) the A-module S, and that there is a triangle equivalence $\Theta : \mathcal{H}(\ln j A) \to \mathcal{D}(A^*)$ taking S to A^* . Since both $\Theta(?) \bigotimes_{A^*} X$ and Q commute with infinite direct sums and $\Theta(S) \bigotimes_{A^*} X \cong X \cong S$, it follows that they are isomorphic. Namely, the right middle parts of the two recollements are equivalent via the equivalence Θ . Therefore the two recollements are equivalent. Now let us construct the equivalence Θ by sketching the proof of the assertion that $\mathcal{H}(\ln j A)$ and $\mathcal{D}(A^*)$ are triangle equivalent. Let $\operatorname{Dif}_{\ln j} A$ be the full dg subcategory of $\operatorname{Dif} A$ consisting of complexes of injective A-modules, let iS be an injective resolution of the A-module S, and put $B = \operatorname{Dif} A(\mathbf{iS}, \mathbf{iS})$. Then the dg $B^{op} \otimes A^*$ -module $\operatorname{Dif} A(X, \mathbf{iS})$ yields a triangle equivalence $\Phi : \mathcal{D}(B) \to \mathcal{D}(A^*)$ (see [10, Section 7.3]). Moreover, $\operatorname{Dif}_{\ln j} A$ is a dg enhancement of the triangulated category $\mathcal{H}(\ln j A)$ in the sense of Bondal-Kapranov [5], and there is a dg functor $\operatorname{Dif}_{\ln j} A(\mathbf{iS},?) : \operatorname{Dif}_{\ln j} A \to \operatorname{Dif} B$. Taking zeroth comhomologies gives us a triangle functor $\mathcal{H}(\ln j A) \to \mathcal{H}(B)$, and composing it with the canonical projection $\mathcal{H}(B) \to \mathcal{D}(B)$ we obtain a triangle equivalence $\Psi : \mathcal{H}(\ln j A) \to \mathcal{D}(B)$ (cf. the proof of [10, Theorem 4.3]). Now the composition $\Theta = \Phi \circ \Psi : \mathcal{H}(\ln j A) \to \mathcal{D}(A^*)$ is a triangle equivalence, which takes iS to A^* (up to isomorphism), as desired.

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