# RECOLLEMENTS FROM GENERALIZED TILTING

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Abstract. Let $\mathcal{A}$ be a small dg category over a field $k$ and let $\mathcal{U}$ be a small full subcategory of the derived category $\mathcal{DA}$ which generate all free dg $\mathcal{A}$-modules. Let $(\mathcal{B},X)$ be a standard lift of $\mathcal{U}$. We show that there is a recollement such that its middle term is $\mathcal{DB}$, its right term is $\mathcal{DA}$, and the three functors on its right side are constructed from $X$. This applies to the pair $(A,T)$, where $A$ is a $k$-algebra and $T$ is a good $n$-tilting module, and we obtain a result of Bazzoni–Mantese–Tonolo. This also applies to the pair $(\mathcal{A},\mathcal{U})$, where $\mathcal{A}$ is an augmented dg category and $\mathcal{U}$ is the category of 'simple' modules, *e.g.* $\mathcal{A}$ is a finite-dimensional algebra or the Kontsevich–Soibelman $A_\infty$-category associated to a quiver with potential.

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A *recollement* of triangulated categories is a diagram of triangulated categories and triangle functors

$$\begin{tikzcd}
\mathcal{T}'' \arrow[r, "i_*"] & \mathcal{T} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{T}' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd} ,$$

such that

- $(i^*,i_*,i^!)$ and $(j_!,j^*,j_*)$ are adjoint triples;
- $i_*,j_*,j_!$ are fully faithful;
- $j^*\circ i_*=0$;
- for every object $X$ of $\mathcal{T}$ there are two triangles

$$i_*i^! \longrightarrow X \longrightarrow j_*j^*X \longrightarrow \quad \text{and} \quad j_!j^*X \longrightarrow X \longrightarrow i_*i^*X \longrightarrow \quad ,$$

where the four morphisms are the units and counits.

We also say that this is a recollement of $\mathcal{T}$ in terms of $\mathcal{T}'$ and $\mathcal{T}''$. This notion was introduced by Beilinson–Bernstein–Deligne in [4] in geometric contexts, where stratifications of varieties induce recollements of derived categories of sheaves.

In algebraic contexts, recollements are closely related to tilting theory. Let $A$ be a ring. Let $\mathcal{D}(A)=\mathcal{D}(\mathsf{Mod}\,A)$ denote the derived category of (right) $A$-modules, and $\mathsf{per}\,A$ denote the triangulated subcategory of $\mathcal{D}(A)$ generated by the free module of rank 1. An object $T$ of $\mathsf{per}\,A$ is called a *partial tilting complex* if $\mathsf{Hom}_{\mathcal{D}(A)}(T,\Sigma^nT)=0$, and a *tilting complex* if in

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addition $\mathsf{tria}(T) = \mathsf{per}\,A$, where $\mathsf{tria}(T)$ is the triangulated subcategory of $\mathcal{D}(A)$ generated by $T$. Rickard's Morita theorem for derived categories states that the standard functors associated to a tilting complex $T$ over $A$ are triangle equivalences between $\mathcal{D}(A)$ and $\mathcal{D}(\mathsf{End}_{\mathcal{D}(A)}(T))$, see [18]. Later in [12], Koenig proved that under certain conditions a partial tilting complex $T$ over $A$ yields a recollement of $\mathcal{D}(A)$ in terms of $\mathcal{D}(\mathsf{End}_{\mathcal{D}(A)}(T))$ and a third derived category which measures how far the associated standard functors are from being equivalences (see also [8] [15]). In this sense, a recollement of derived categories can be viewed as a natural generalization of a derived equivalence. The relation between tilting theory and recollements of derived categories has been further studied in [1] [6]. The dg version of Rickard's theorem was developed by Keller in [10], and the result of Koenig was generalized to the dg setting by Jørgensen [9] and Nicolás–Saorín [17], where the role of partial tilting complexes is played by compact objects.

In this paper we deal with a situation which is 'dual' to the one in [12] [9] [17]. Starting from a dg category $\mathcal{A}$ and a set of objets in the derived category $\mathcal{D}\mathcal{A}$ which generates all the compact objects, we construct a dg category $\mathcal{B}$ together with a recollement of $\mathcal{D}\mathcal{B}$ in terms of $\mathcal{D}\mathcal{A}$ and another derived category, see Theorem 1. We identify this third derived category with a certain known category in the special case when $\mathcal{A}$ is the Kontsevich–Soibelman $A_\infty$-category associated to a quiver with potential (Corollary 3) or when $\mathcal{A}$ is a finite-dimensional self-injective algebra (Corollary 4). The motivation for our study was to have a better understanding of the 'exterior' case of the Koszul duality (Corollary 2) and a result of Bazzoni–Mantese–Tonolo which says that the right derived Hom-functor associated to an (infinitely generated) good tilting module is fully faithful (Corollary 1).

## 1. THE MAIN RESULT

Let $k$ be a field and let $\mathcal{A}$ be a small dg $k$-category. Denote by $\mathsf{Dif}\,\mathcal{A}$ the dg category of (right) dg $\mathcal{A}$-modules. A dg $\mathcal{A}$-module $M$ is *$\mathcal{K}$-projective* if the dg functor $\mathsf{Dif}\,\mathcal{A}(M, ?)$ preserves acyclicity. For example, the free modules $A^\wedge = \mathsf{Dif}\,\mathcal{A}(?, A)$, $A \in \mathcal{A}$, are $\mathcal{K}$-projective. Let $\mathcal{D}\mathcal{A}$ denote the derived category of $\mathcal{A}$, which is triangulated with suspension functor $\Sigma$ the shift functor. For a set of objects or a subcategory $\mathcal{S}$ of $\mathcal{D}\mathcal{A}$ we denote by $\mathsf{tria}\,\mathcal{S}$ the smallest triangulated subcategory of $\mathcal{D}\mathcal{A}$ containing all objects in $\mathcal{S}$ and closed under taking direct summands. Let $\mathsf{per}\,\mathcal{A} = \mathsf{tria}(A^\wedge, A \in \mathcal{A})$. An object $M$ of $\mathcal{D}\mathcal{A}$ is *compact* if the functor $\mathcal{D}\mathcal{A}(M, ?)$ commutes with infinite (set-indexed) direct sums, or equivalently, if $M$ belongs to $\mathsf{per}\,\mathcal{A}$. See [10].

Let $\mathcal{U}$ be a full small subcategory of $\mathcal{D}\mathcal{A}$ such that

$$\mathsf{tria}\,\mathcal{U} \supseteq \mathsf{per}\,\mathcal{A}. \tag{1}$$

Let $(\mathcal{B},X)$ be a *standard lift* of $\mathcal{U}$ ([10, Section 7]). Precisely, $\mathcal{B}$ is a dg subcategory of $\mathsf{Dif}\,\mathcal{A}$ consisting of $\mathcal{K}$-projective resolutions over $\mathcal{A}$ of objects of $\mathcal{U}$ (to avoid confusion, for each object $B$ of $\mathcal{B}$ we will denote by $U_B$ the corresponding dg $\mathcal{A}$-module) and $X$ is the dg $\mathcal{B}^{op}\otimes\mathcal{A}$-module defined by $X(B,A)=U_B(A)$. It induces a pair of adjoint dg functors and a pair of adjoint triangle functors

$$\begin{tikzcd} \mathsf{Dif}\,\mathcal{B} \arrow[r, shift left, "T_X"] & \mathsf{Dif}\,\mathcal{A}, \arrow[l, shift left, "H_X"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D}\mathcal{B} \arrow[r, shift left, "\mathbf{L}T_X"] & \mathcal{D}\mathcal{A}. \arrow[l, shift left, "\mathbf{R}H_X"] \end{tikzcd}$$

When $\mathcal{A}$ and $\mathcal{B}$ are dg $k$-algebras (*i.e.* dg $k$-categories with one object), the functors $\mathbf{L}T_X$ and $\mathbf{R}H_X$ are usually written as $?\overset{L}{\otimes}X$ and $\mathsf{RHom}(X,?)$.

Let $X^T$ be the dg $\mathcal{A}^{op}\otimes\mathcal{B}$-module defined by

$$X^T(A,B)=\mathsf{Dif}\,\mathcal{A}(X^B,A^\wedge),$$

where for $B\in\mathcal{B}$, $X^B$ is by definition the dg $\mathcal{A}$-module $X(B,?)$. From the definition of $X$ we see that $X^B=U_B$. The main result of this paper is

**Theorem 1.** *Assume notations as above. There is a dg $k$-category $\mathcal{C}$ and a recollement of triangulated categories*

$$\begin{tikzcd} \mathcal{D}\mathcal{C} \arrow[r, "i_*"] & \mathcal{D}\mathcal{B} \arrow[l, bend right=40, "i^*"'] \arrow[l, bend left=40, "i^!"] \arrow[r, "j^*"] & \mathcal{D}\mathcal{A} \arrow[l, bend right=40, "j_!"'] \arrow[l, bend left=40, "j_*"] \end{tikzcd} ,$$

*where the adjoint triple $(i^*,i_*,i^!)$ is defined by a dg functor $F:\mathcal{B}\to\mathcal{C}$ (which is bijective on objects) such that $i_*=F^*:\mathcal{D}\mathcal{C}\to\mathcal{D}\mathcal{B}$ is the pull-back functor, and the adjoint triple $(j_!,j^*,j_*)$ is given by*

$$\begin{aligned} j_! &= \mathbf{L}T_{X^T}, \\ j^* &= \mathbf{R}H_{X^T} \simeq \mathbf{L}T_X, \\ j_* &= \mathbf{R}H_X. \end{aligned}$$

*Proof.* In view of [17, Theorem 5], it suffices to prove

(a) $\mathbf{L}T_{X^T}$ is fully faithful,

(b) $\mathbf{R}H_{X^T}\simeq\mathbf{L}T_X$.

The proof for (a) is the same as the proof of [10, Lemma 10.5 the 'exterior' case c)]. Since $(\mathcal{B},X)$ is a lift, the restriction of $\mathbf{L}T_X$ on the perfect derived category $\mathsf{per}\,\mathcal{B}$ is fully faithful, and its essential image is $\mathsf{tria}\,\mathcal{U}$ (see [10, Section 7.3]):

$$\mathbf{L}T_X|_{\mathsf{per}\,\mathcal{B}}:\mathsf{per}\,\mathcal{B}\xrightarrow{\sim}\mathsf{tria}\,\mathcal{U}.$$

It is clear that $\mathbf{R}H_X$ takes an object of $\mathsf{tria}\,\mathcal{U}$ into $\mathsf{per}\,\mathcal{B}$. Therefore, the restriction $\mathbf{R}H_X|_{\mathsf{tria}\,\mathcal{U}}$ is a quasi-inverse of $\mathbf{L}T_X|_{\mathsf{per}\,\mathcal{B}}$, and hence is fully faithful. It follows from [10, Lemma 6.2 a)] that the restriction $\mathbf{L}T_{X^T}|_{\mathsf{per}\,\mathcal{A}}$ is naturally isomorphic to the restriction of $\mathbf{R}H_X|_{\mathsf{per}\,\mathcal{A}}$, which is fully faithful by condition (1). Condition (1) also implies that $\mathbf{R}H_X(A^\wedge) = (X^T)^A$ $(A \in \mathcal{A})$ belongs to $\mathsf{per}\,\mathcal{B}$, and hence is compact by [10, Theorem 5.3]. Now applying [10, Lemma 4.2 b)], we obtain that $\mathbf{L}T_{X^T}$ is fully faithful, finishing the proof of (a).

Let $Y \to X^T$ be a $\mathcal{K}$-projective resolution of dg $\mathcal{A}^{op} \otimes \mathcal{B}$-modules. Then the specialization $Y^A \to (X^T)^A$ is a $\mathcal{K}$-projective resolution of dg $\mathcal{B}$-modules for any object $A$ of $\mathcal{A}$. Recall that $(X^T)^A$ is compact. It follows from [10, Lemma 6.2 a)] that $\mathbf{L}T_{Y^T} \simeq \mathbf{R}H_Y$. By [10, Lemma 6.1 b)], in order to prove $\mathbf{R}H_{X^T} \simeq \mathbf{L}T_X$, it suffices to prove that as dg $\mathcal{B}^{op} \otimes \mathcal{A}$-modules $Y^T$ and $X$ are quasi-isomorphic. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We have $H_X(U_B) = B^\wedge$, and hence

$$\begin{aligned} Y^T(A,B) &= \mathsf{Dif}\,\mathcal{B}(Y^A, B^\wedge) \\ &= \mathsf{Dif}\,\mathcal{B}(Y^A, H_X(U_B)) \\ &\cong \mathsf{Dif}\,\mathcal{A}(T_X(Y^A), U_B). \end{aligned}$$

The composition $T_X(Y^A) \to T_X((X^T)^A) = T_X \circ H_X(A^\wedge) \to A^\wedge$ is exactly the counit $\mathbf{L}T_X \circ \mathbf{R}H_X(A^\wedge) \to A^\wedge$, which is an isomorphism in $\mathcal{D}\mathcal{A}$ because the restriction of $\mathbf{R}H_X$ on $\mathsf{per}\,\mathcal{A}$ is fully faithful. Moreover, both $T_X(Y^A)$ and $A^\wedge$ are $\mathcal{K}$-projective dg $\mathcal{A}$-modules. Therefore we have

$$\begin{aligned} Y^T(A,B) &\overset{q.is}{\leftarrow} \mathsf{Dif}\,\mathcal{A}(A^\wedge, U_B) \\ &= U_B(A) \\ &= X(A,B). \end{aligned}$$

Further, every morphism in the above is functorial in both $A$ and $B$. This completes the proof of (b). $\surd$

**Corollary 1** ([3]). *Let $A$ be a $k$-algebra, and $n$ be a positive integer. Let $T$ be a* good $n$-tilting module, i.e. *$T$ is an $A$-module such that*

(T1) *the projective dimension of $T$ is less than or equal to $n$;*
(T2) $\mathsf{Ext}^i_A(T, T^{(\alpha)}) = 0$ *for any integer $i > 0$ and for any cardinal $\alpha$;*
(T3) *there is an exact sequence*

$$0 \longrightarrow A \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^n \longrightarrow 0\ ,$$

*where $T^0, \ldots, T^n$ are direct summands of direct sums of finite copies of $T$.*

*Put* $B = \mathsf{End}_A(T)$. *Then the right derived functor* $\mathsf{RHom}_A(T, ?) : \mathcal{D}(A) \to \mathcal{D}(B)$ *is fully faithful, and* $\mathcal{D}(A)$ *is triangle equivalent to the triangle quotient of* $\mathcal{D}(B)$ *by the kernel of the left derived functor* $? \overset{L}{\otimes}_B T$.

*Proof.* Let $\mathcal{U}$ be the full subcategory of $\mathcal{D}(A)$ consisting of one object $T$. Then the condition (T3) implies the condition (1). Let $X$ be a projective resolution of $T$ over $B^{op} \otimes_k A$, and let $\tilde{B}$ be the dg $k$-algebra $\mathsf{Dif}\, A(X, X)$. Then $X$ is $\mathcal{K}$-projective over $A$, and $(\tilde{B}, X)$ is a standard lift of $T$. Thanks to (T2), the representation map $B \to \tilde{B}$ of the dg $B$-$A$-bimodule $X$ is a quasi-isomorphism, inducing mutually quasi-inverse triangle equivalences $? \overset{L}{\otimes}_{\tilde{B}} \tilde{B} = \mathsf{RHom}_{\tilde{B}}(\tilde{B}, ?) : \mathcal{D}(\tilde{B}) \to \mathcal{D}(B)$ and $? \overset{L}{\otimes}_B \tilde{B} : \mathcal{D}(B) \to \mathcal{D}(\tilde{B})$. Now applying Theorem 1 and composing the resulting recollement with the above triangle equivalences, we obtain a recollement

$$\begin{tikzcd}
\mathcal{D}(C) \arrow[r] & \mathcal{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathsf{RHom}_B(X^T,?) \simeq ? \overset{L}{\otimes}_B X"] & \mathcal{D}(A) \arrow[l, bend right, "? \overset{L}{\otimes}_A X^T"'] \arrow[l, bend left, "\mathsf{RHom}_A(X,?)"] ,
\end{tikzcd}$$

where $C$ is a dg $k$-algebra. Since $X$ and $T$ are quasi-isomorphic as dg $B^{op} \otimes_k A$-modules, we have natural isomorphisms $? \overset{L}{\otimes}_B X \simeq\ ? \overset{L}{\otimes}_B T$ and $\mathsf{RHom}_A(X, ?) \simeq \mathsf{RHom}_A(T, ?)$ ([10, Lemma 6.1 b)]). The desired result follows at once. $\surd$

**Remark.** a) *This result is due to Bazzoni* [2] *for* $n = 1$ *and Bazzoni–Mantese–Tonolo* [3] *for general* $n$ *for all rings* $A$.

b) *By Theorem 1, the left half of the recollement in the proof is induced from a dg homomorphism* $B \to C$. *For the case* $n = 1$ *and for all rings* $A$ *Chen–Xi obtained in* [6] *such a recollement with* $C$ *being an ordinary ring (so that the map* $B \to C$ *becomes a homomorphism of rings). They used some results in* [1] *and many other results such as the homological properties of the tilting module* $T$.

To state the next corollary, we need to introduce some notions. Let $\mathcal{A}$ be an *augmented* dg $k$-category ([10, Section 10.2]), *i.e.*

- distinct objects of $\mathcal{A}$ are non-isomorphic,
- for each $A \in \mathcal{A}$, a dg module $\bar{A}$ is given such that $H^0\bar{A}(A) \cong k$ and $H^n\bar{A}(A')$ whenever $n \neq 0$ or $A' \neq A$.

Let $(\mathcal{A}^*, X)$ be a standard lift of $\mathcal{U} = \{\bar{A} | A \in \mathcal{A}\} \subset \mathcal{D}\mathcal{A}$. By abuse of language, we call the dg $k$-category $\mathcal{A}^*$ the *Koszul dual* of $\mathcal{A}$. Assume that the condition (1) holds, *e.g.* this happens in the 'exterior' case in [10, Section 10.5].

**Corollary 2.** *Assume notations as above. There is a recollement of derived categories of dg $k$-categories*

$$\begin{tikzcd}
\mathcal{DC} \arrow[r] \arrow[r, bend left, <-] \arrow[r, bend right, <-] & \mathcal{DA}^* \arrow[r, "\mathbf{R}H_{X^T}\simeq\mathbf{L}T_X"] \arrow[r, bend left, <-, "\mathbf{L}T_{X^T}"] \arrow[r, bend right, <-, "\mathbf{R}H_X"'] & \mathcal{DA}\ .
\end{tikzcd}$$

*Proof.* This is a direct consequence of Theorem 1. $\surd$

## 2. The left term

As in the preceding section, we let $k$ be a field, $\mathcal{A}$ be a small dg $k$-category, $\mathcal{U}$ be a full small subcategory of the derived category $\mathcal{DA}$ such that $\mathsf{tria}\,\mathcal{U} \supseteq \mathsf{per}\,\mathcal{A}$, and let $(\mathcal{B}, X)$ be a standard lift of $\mathcal{U}$. Theorem 1 says that there is a recollement of $\mathcal{DB}$ in terms of $\mathcal{DA}$ and a third derived category $\mathcal{DC}$, where $\mathcal{C}$ is a dg $k$-category whose objects are in bijection with the objects of $\mathcal{U}$.

Let $\mathcal{V} = \{(X^T)^A | A \in \mathcal{A}\} \subset \mathcal{DB}$. From the proof of the theorem we obtain a commutative diagram

$$\begin{tikzcd}
\mathbf{R}H_X|_{\mathsf{tria}\,\mathcal{U}}: & \mathsf{tria}\,\mathcal{U} \arrow[r, "\sim"] & \mathsf{per}\,\mathcal{B} \\
\mathbf{R}H_X|_{\mathsf{per}\,\mathcal{A}}: & \mathsf{per}\,\mathcal{A} \arrow[r, "\sim"] \arrow[u, hook] & \mathsf{tria}\,\mathcal{V}. \arrow[u, hook]
\end{tikzcd}$$

Therefore $\mathrm{R}H_X$ induces a triangle equivalence between the triangle quotient categories

$$\mathsf{tria}\,\mathcal{U}/\,\mathsf{per}\,\mathcal{A} \xrightarrow{\sim} \mathsf{per}\,\mathcal{B}/\,\mathsf{tria}\,\mathcal{V}\ .$$

For a triangulated category $\mathcal{T}$, let $\mathcal{T}^c$ denote the subcategory of compact objects in $\mathcal{T}$. Let $\mathsf{Tria}\,\mathcal{V}$ be the localizing subcategory of $\mathcal{DB}$ generated by the objects in $\mathcal{V}$. We have $(\mathcal{DB})^c = \mathsf{per}\,\mathcal{B}$, and $(\mathsf{Tria}\,\mathcal{V})^c = \mathsf{tria}\,\mathcal{V}$. Thus by [16, Theorem 2.1], the category $(\mathcal{DB}/\,\mathsf{Tria}\,\mathcal{V})^c$ is triangle equivalent to the idempotent completion of $\mathsf{per}\,\mathcal{B}/\,\mathsf{tria}\,\mathcal{V}$. Since the essential image of $\mathbf{L}T_{X^T}$ is exactly $\mathsf{Tria}\,\mathcal{V}$, it follows that $\mathcal{DC}$ is triangle equivalent to the triangle quotient $\mathcal{DB}/\,\mathsf{Tria}\,\mathcal{V}$, and hence is an 'unbounded version' of $\mathsf{tria}\,\mathcal{U}/\,\mathsf{per}\,\mathcal{A} \cong \mathsf{per}\,\mathcal{B}/\,\mathsf{tria}\,\mathcal{V}$. Apparently, $\mathcal{DC}$ vanishes if and only if so does $\mathsf{tria}\,\mathcal{U}/\,\mathsf{per}\,\mathcal{A}$, in which case $\mathcal{U}$ consists of a set of compact generators for $\mathcal{DA}$.

In the following two special cases, we are able to identify $\mathcal{DC}$ with a certain known category (however, the dg category $\mathcal{C}$ is not easy to describe).

**Corollary 3.** *Let $(Q, W)$ be a quiver with potential. Let $\mathcal{A}_{(Q,W)}$ be the Kontsevich–Soibelman $A_\infty$-category (*[13, Section 3.3]*) (or its enveloping dg category), let $\widehat{\Gamma}_{(Q,W)}$ be the complete*

*Ginzburg dg category (*[7, Section 5]*), and let* $\widetilde{\mathcal{C}}_{(Q,W)}$ *be the 'unbounded version' of the generalized cluster category (*[11, Remark 4.1]*). Then there is a recollement of triangulated categories*

$$\begin{tikzcd} \widetilde{\mathcal{C}}_{(Q,W)} \arrow[r] & \mathcal{D}\widehat{\Gamma}_{(Q,W)} \arrow[r] \arrow[l, bend right] \arrow[l, bend left] & \mathcal{D}\mathcal{A}_{(Q,W)} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

*Proof.* Let $\mathcal{A} = \mathcal{A}_{(Q,W)}$ and let $\mathcal{U}$ be the category of simple $\mathcal{A}$-modules. Then condition (1) holds since $\mathcal{A}$ is finite-dimensional, and there is a standard lift $(\mathcal{B}, X)$ such that the dg category $\Gamma = \widehat{\Gamma}_{(Q,W)}$ (as the Koszul dual of $\mathcal{A}_{(Q,W)}$) is quasi-isomorphic to $\mathcal{B}$. By Corollary 2, there is a recollement with the middle term being $\mathcal{D}\Gamma$, the right term being $\mathcal{D}\mathcal{A}$, and the right upper functor being $\mathbf{L}T_{X^T}$. It remains to prove that the left term of this recollement is triangle equivalent to $\widetilde{\mathcal{C}}_{(Q,W)}$. Object sets of $\mathcal{A}$, of $\mathcal{U}$, and of $\Gamma$ can all be identified with the vertice set $Q_0$ of the quiver $Q$. For a vertex $i$ of $Q$, considered as an object of $\mathcal{A}$, the right dg $\Gamma$-module $(X^T)^i$ is isomorphic in $\mathcal{D}\Gamma$ to $\Sigma^{-3}S_i$, where $S_i$ the simple top of the free $\Gamma$-module $i^\wedge$. Thus the essential image of $\mathbf{L}T_{X^T}$ is the localizing subcategory $\mathcal{D}_0\Gamma = \mathsf{Tria}(S_i, i \in Q_0)$ of $\mathcal{D}\Gamma$ generated by the $S_i$, $i \in Q_0$. Thus the left term of the recollement is triangle equivalent to the triangle quotient $\mathcal{D}\Gamma/\mathcal{D}_0\Gamma$, which is by definition $\widetilde{\mathcal{C}}_{(Q,W)}$. √

Let $A$ be a finite-dimensional basic $k$-algebra. Let $S$ be the direct sum of the objects in a set of representatives of isomorphism classes of simple $A$-modules, and let $X$ be a projective resolution of $S$. Then $A^* = \mathsf{Dif}\, A(X, X)$ is the Koszul dual of $A$.

**Corollary 4** ([14])**.** *Let $A$ be a finite-dimensional basic self-injective $k$-algebra, and $A^*$ its Koszul dual. Let* $\underline{\mathsf{Mod}}A$ *be the stable category of the category* $\mathsf{Mod}\, A$ *of $A$-modules. Then there is a recollement of triangulated categories*

$$\begin{tikzcd} \underline{\mathsf{Mod}}A \arrow[r] & \mathcal{D}(A^*) \arrow[r] \arrow[l, bend right] \arrow[l, bend left] & \mathcal{D}(A) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

*Proof.* Let $\mathsf{mod}\, A$ be the category of finite-dimensional $A$-modules, and $\underline{\mathsf{mod}}A$ its stable category. As a triangulated subcategory of $\mathcal{D}(A)$, the bounded derived category $\mathcal{D}^b(\mathsf{mod}\, A)$ of $\mathsf{mod}\, A$ coincides with $\mathsf{tria}\, S$. Recall that the essential image of $? \overset{L}{\otimes}_A X^T$ is $\mathsf{Tria}\, X^T$. Consider the following commutative diagram

$$\begin{tikzcd}
\mathsf{Mod}\,A \arrow[r, hook] & \mathcal{D}(A) \arrow[r, "\mathsf{RHom}_A(X{,}?)"] & \mathcal{D}(A^*) \arrow[r] & \mathcal{D}(A^*)/\,\mathsf{Tria}\,X^T \\
\mathsf{mod}\,A \arrow[r, hook] \arrow[u, hook] & \mathcal{D}^b(\mathsf{mod}\,A) \arrow[r, "\sim"] \arrow[u, hook] & \mathsf{per}\,A^* \arrow[r] \arrow[u, hook] & \mathsf{per}\,A^*/\,\mathsf{tria}\,X^T \arrow[u, hook] \\
& \mathsf{per}\,A \arrow[r, "\sim"] \arrow[u, hook] & \mathsf{tria}\,X^T \arrow[u, hook] &
\end{tikzcd}$$

where the leftmost horizontal functors are the canonical embeddings, and the rightmost horizontal functors are the canonical projections. The restriction of $\mathsf{RHom}_A(X,?)$ on $\mathsf{Mod}\,A$ commutes with infinite direct sums, because $X$ can be chosen such that its component in each degree is a finitely generated projective $A$-module. Therefore the composition of the three functors in the first row, denoted by $F$, commutes with infinite direct sums. Since $\mathsf{RHom}_A(X,A) \cong X^T$ belongs to $\mathsf{Tria}\,X^T$, it follows that $F$ factors through the stable category $\underline{\mathsf{Mod}}A$. In this way, we obtain a triangle functor

$$\bar{F} : \underline{\mathsf{Mod}}A \to \mathcal{D}(A^*)/\,\mathsf{Tria}\,X^T,$$

which commutes with infinite direct sums. It is known that $\underline{\mathsf{Mod}}A$ is compactly generated by $\underline{\mathsf{mod}}A$ and $(\underline{\mathsf{Mod}}A)^c = \underline{\mathsf{mod}}A$. Moreover, the restriction $\bar{F}|_{\underline{\mathsf{mod}}A}$ is the composition of the following three functors

$$\begin{tikzcd}
\underline{\mathsf{mod}}A \arrow[r] & \mathcal{D}^b(\mathsf{mod}\,A)/\,\mathsf{per}\,A \arrow[r, "\sim"] & \mathsf{per}\,A^*/\,\mathsf{tria}\,X^T \arrow[r, hook] & \mathcal{D}(A^*)/\,\mathsf{Tria}\,X^T\ .
\end{tikzcd}$$

The first functor is also an equivalence ([19, Theorem 2.1]). Therefore $\bar{F}$ induces a triangle equivalence between $\underline{\mathsf{mod}}A = (\underline{\mathsf{Mod}}A)^c$ and $\mathsf{per}\,A^*/\,\mathsf{tria}\,X^T = (\mathcal{D}(A^*)/\,\mathsf{Tria}\,X^T)^c$. By [10, Lemma 4.2], $\bar{F}$ itself is an equivalence. Now applying Corollary 2 we obtain the desired recollement. √

**Remark.** *Let* $\mathcal{H}(\mathsf{Inj}\,A)$ *be the homotopy category of injective* $A$*-modules and* $\mathcal{H}_{ac}(\mathsf{Inj}\,A)$ *be its full subcategory of acyclic complexes. Applying a result of Krause* [14, Corollary 4.3] *to the algebra* $A$*, we obtain a recollement of* $\mathcal{H}(\mathsf{Inj}\,A)$ *in terms of* $\mathcal{D}(A)$ *and* $\mathcal{H}_{ac}(\mathsf{Inj}\,A)$ *with the right middle functor being the canonical projection* $Q : \mathcal{H}(\mathsf{Inj}\,A) \to \mathcal{D}(A)$*. We claim that this recollement is equivalent to the one in Corollary 4. Indeed, Krause proved in* [14] *that* $\mathcal{H}(\mathsf{Inj}\,A)$ *is compactly generated by (an injective resolution of) the* $A$*-module* $S$*, and that there is a triangle equivalence* $\Theta : \mathcal{H}(\mathsf{Inj}\,A) \to \mathcal{D}(A^*)$ *taking* $S$ *to* $A^*$*. Since both* $\Theta(?) \overset{L}{\otimes}_{A^*} X$ *and* $Q$ *commute with infinite direct sums and* $\Theta(S) \overset{L}{\otimes}_{A^*} X \cong X \cong S$*, it follows that they are isomorphic. Namely, the right middle parts of the two recollements are equivalent via the equivalence* $\Theta$*. Therefore the two recollements are equivalent.*

*Now let us construct the equivalence* $\Theta$ *by sketching the proof of the assertion that* $\mathcal{H}(\mathsf{Inj}\,A)$ *and* $\mathcal{D}(A^*)$ *are triangle equivalent. Let* $\mathsf{Dif}_{\mathsf{Inj}}\,A$ *be the full dg subcategory of* $\mathsf{Dif}\,A$ *consisting of complexes of injective* $A$*-modules, let* $\mathbf{i}S$ *be an injective resolution of the* $A$*-module* $S$*, and put* $B = \mathsf{Dif}\,A(\mathbf{i}S, \mathbf{i}S)$*. Then the dg* $B^{op} \otimes A^*$*-module* $\mathsf{Dif}\,A(X, \mathbf{i}S)$ *yields a triangle equivalence* $\Phi : \mathcal{D}(B) \to \mathcal{D}(A^*)$ *(see* [10, Section 7.3]*). Moreover,* $\mathsf{Dif}_{\mathsf{Inj}}\,A$ *is a dg enhancement of the triangulated category* $\mathcal{H}(\mathsf{Inj}\,A)$ *in the sense of Bondal–Kapranov* [5]*, and there is a dg functor* $\mathsf{Dif}_{\mathsf{Inj}}\,A(\mathbf{i}S, ?) : \mathsf{Dif}_{\mathsf{Inj}}\,A \to \mathsf{Dif}\,B$*. Taking zeroth comhomologies gives us a triangle functor* $\mathcal{H}(\mathsf{Inj}\,A) \to \mathcal{H}(B)$*, and composing it with the canonical projection* $\mathcal{H}(B) \to \mathcal{D}(B)$ *we obtain a triangle equivalence* $\Psi : \mathcal{H}(\mathsf{Inj}\,A) \to \mathcal{D}(B)$ *(*cf. *the proof of* [10, Theorem 4.3]*). Now the composition* $\Theta = \Phi \circ \Psi : \mathcal{H}(\mathsf{Inj}\,A) \to \mathcal{D}(A^*)$ *is a triangle equivalence, which takes* $\mathbf{i}S$ *to* $A^*$ *(up to isomorphism), as desired.*

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## References

[1] Lidia Angeleri Hügel, Steffen Koenig and Qunhua Liu, *Recollements and tilting objects*, J. Pure Appl. Algebra (2010), doi:10.1016/j.jpaa.2010.04.027.

[2] Silvana Bazzoni, *Equivalences induced by infinitely generated tilting modules*, Proc. Amer. Math. Soc. **138** (2010), no. 2, 533–544.

[3] Silvana Bazzoni, Francesca Mantese and Alberto Tonolo, *Derived equivalence induced by $n$-tilting modules*, arXiv:0905.3696.

[4] Alexander A. Beilinson, Joseph Bernstein and Pierre Deligne, *Analyse et topologie sur les espaces singuliers*, Astérisque, vol. 100, Soc. Math. France, 1982 (French).

[5] Aleksei I. Bondal, and Mikhail M. Kapranov, *Enhanced triangulated categories*, Mat. Sb. **181** (1990), no. 5, 669–683; translation in Math. USSR-Sb. **70** (1991) no. 1, 93–107.

[6] Hongxing Chen and Changchang Xi, *Recollements of derived module categories induced by infinitely generated good tilting modules*, preprint in preparation, 2010.

[7] Victor Ginzburg, *Calabi-Yau algebras*, arXiv:math/0612139v3 [math.AG].

[8] Dieter Happel, *Partial tilting modules and recollement*, Proceedings of the International Conference on Algebra, Part 2 (Novosibirsk, 1989), 345–361, Contemp. Math. **131**, Part 2, Amer. Math. Soc., Providence, RI, 1992.

[9] Peter Jørgensen, *Recollement for Differential Graded Algebras*, J. Algebra **299** (2006), 589–601,

[10] Bernhard Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1, 63–102.

[11] Bernhard Keller and Dong Yang, *Derived equivalences from mutations of quivers with potential*, Adv. Math. (2010), doi:10.1016/j.aim.2010.09.019.

[12] Steffen Koenig, *Tilting complexes, perpendicular categories and recollements of derived categories of rings*, J. Pure Appl. Algebra **73** (1991), 211–232.

[13] Maxim Kontsevich and Yan Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435.

[14] Henning Krause, *The stable derived category of a Noetherian scheme*, Compos. Math. **141** (2005), no. 5, 1128–1162.

[15] Jun-Ichi Miyachi, *Recollements and tilting complexes*, J. Pure Appl. Algebra **183** (2003), 245–273.

[16] Amnon Neeman, *The connection between the K–theory localisation theorem of Thomason, Trobaugh and Yao, and the smashing subcategories of Bousfield and Ravenel*, Ann. Sci. École Normale Supérieure **25** (1992), 547–566.

[17] Pedro Nicolás, and Manuel Saorín, *Parametrizing recollement data for triangulated categories*, J. Algebra, **322** (2009), no. 4, 1220–1250.

[18] Jeremy Rickard, *Morita theory for derived categories*, J. London Math. Soc. **39** (1989), 436–456.

[19] Jeremy Rickard, *Derived categories and stable equivalence*, J. Pure and Appl. Algebra **61** (1989), 303–317.

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