On holomorphic functions on a compact complex homogeneous supermanifold [1]

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Abstract. It is well-known that non-constant holomorphic functions do not exist on a compact complex manifold. This statement is false for a supermanifold with a compact reduction. In this paper we study the question under what conditions non-constant holomorphic functions do not exist on a compact homogeneous complex supermanifold. We describe also the vector bundles determined by split homogeneous complex supermanifolds.

As an application, we compute the algebra of holomorphic functions on the classical flag supermanifolds which were introduced in [10].

1. Preliminaries

1.1 Lie supergroups and homogeneous supermanifolds

We will use the word "supermanifold" in the sense of Berezin and Leites (see [3, 9]). All the time, we will be interested in the complex-analytic version of the theory. Let $(M, \mathcal{O}_M)$ be a supermanifold. The underlying complex manifold M is called the *reduction* of $(M, \mathcal{O}_M)$. The superalgebra $H^0(\mathcal{O}_M)$ is called the *superalgebra of (global) holomorphic functions* on $(M, \mathcal{O}_M)$. A function $f \in H^0(\mathcal{O}_M)$ is called *constant* if $f|_U$ does not depend on even and odd coordinates for every coordinate superdomain $(U, \mathcal{O}_M|_U) \subset (M, \mathcal{O}_M)$.

Example 1. Let $\mathcal{E}$ be a locally free sheaf on M. Then $(M, \bigwedge \mathcal{E})$ is a supermanifold. Let $U \subset M$ be a coordinate domain of M with coordinates (x_i). Assume that $\mathcal{E}|_U$ is free and (ξ_j) is a local basis. Then $(U, \bigwedge \mathcal{E}|_U)$ is a superdomain with coordinates (x_i, ξ_j). Note that any $f \in H^0(\bigwedge^p \mathcal{E})\backslash\{0\}$, where $p > 0$, is not constant. Suppose that M is compact. Obviously, it does not follow that $H^0(\bigwedge^p \mathcal{E}) = \{0\}$ for $p > 0$.

We denote by $\mathcal{J}_M \subset \mathcal{O}_M$ the subsheaf of ideals generated by odd elements of the structure sheaf. The sheaf $\mathcal{O}_M/\mathcal{J}_M$ is naturally identified with the structure sheaf $\mathcal{F}_M$ of M. The natural homomorphism $\mathcal{O}_M \to \mathcal{F}_M$ will be denoted by $f \mapsto f_{\mathrm{red}}$. A morphism $\phi : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ of supermanifolds will be denoted by $\phi = (\phi_{\mathrm{red}}, \phi^*)$, where $\phi_{\mathrm{red}} : M \to N$ is the corresponding mapping of the reductions and $\phi^* : \mathcal{O}_N \to (\phi_{\mathrm{red}})_*(\mathcal{O}_M)$ is the homomorphism of the structure sheaves. If $x \in M$ and $\mathfrak{m}_x$ is the maximal ideal of the local superalgebra $(\mathcal{O}_M)_x$, then the vector superspace $T_x(M, \mathcal{O}_M) = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$

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is the tangent space to $(M, \mathcal{O}_M)$ at $x \in M$. Denote by $\mathcal{T}_M$ the sheaf of derivations of the structure sheaf $\mathcal{O}_M$. It is a sheaf of Lie superalgebras with the Lie bracket $[X, Y] := X \circ Y - (-1)^{p(X)p(Y)} Y \circ X$, where $p(Z)$ is the parity of Z. We will use the following notation $\mathfrak{v}(M, \mathcal{O}_M) = H^0(\mathcal{T}_M)$ for the Lie superalgebra of vector fields on $(M, \mathcal{O}_M)$. From the inclusions $v(\mathfrak{m}_x) \subset (\mathcal{O}_M)_x$ and $v(\mathfrak{m}_x^2) \subset \mathfrak{m}_x$, where $v \in \mathfrak{v}(M, \mathcal{O}_M)$, it follows that v induces an even linear mapping $\mathrm{ev}_x(v) : \mathfrak{m}_x/\mathfrak{m}_x^2 \to (\mathcal{O}_M)_x/\mathfrak{m}_x \simeq \mathbb{C}$. In other words, $\mathrm{ev}_x(v) \in T_x(M, \mathcal{O}_M)$, and so we obtain an even linear map

$$\mathrm{ev}_x : \mathfrak{v}(M, \mathcal{O}_M) \to T_x(M, \mathcal{O}_M). \tag{1}$$

A *Lie supergroup* is a group object in the category of supermanifolds, i.e., a supermanifold $(G, \mathcal{O}_G)$, for which the following three morphisms are defined: $\mu : (G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \to (G, \mathcal{O}_G)$ (the multiplication morphism), $\iota : (G, \mathcal{O}_G) \to (G, \mathcal{O}_G)$ (the inversion morphism), $\varepsilon : (\mathrm{pt}, \mathbb{C}) \to (G, \mathcal{O}_G)$ (the identity morphism). Moreover, these morphisms should satisfy the usual conditions, modeling the group axioms. The underlying manifold G is a complex Lie group. The element $e = \varepsilon_{\mathrm{red}}(\mathrm{pt})$ is the identity element of G. We will denote by $\mathfrak{g}$ the Lie superalgebra of $(G, \mathcal{O}_G)$. By definition, $\mathfrak{g}$ is the subalgebra of $\mathfrak{v}(G, \mathcal{O}_G)$ consisting of all right invariant vector fields on $(G, \mathcal{O}_G)$. It is well known that any right invariant vector field Y has the form

$$Y = (X \otimes \mathrm{id}) \circ \mu^* \tag{2}$$

for a certain $X \in T_e(G, \mathcal{O}_G)$ and the map $X \mapsto (X \otimes \mathrm{id}) \circ \mu^*$ is an isomorphism of the vector space $T_e(G, \mathcal{O}_G)$ onto $\mathfrak{g}$, see [17], Theorem 7.1.1.

An *action of a Lie supergroup* $(G, \mathcal{O}_G)$ *on a supermanifold* $(M, \mathcal{O}_M)$ is a morphism $\nu : (G, \mathcal{O}_G) \times (M, \mathcal{O}_M) \to (M, \mathcal{O}_M)$ such that the following conditions hold:

- $\nu \circ (\mu \times \mathrm{id}) = \nu \circ (\mathrm{id} \times \nu)$;
- $\nu \circ (\varepsilon \times \mathrm{id}) = \mathrm{id}$.

In this case ν_{red} is the action of G on M.

Let $\nu : (G, \mathcal{O}_G) \times (M, \mathcal{O}_M) \to (M, \mathcal{O}_M)$ be an action. Then there is a homomorphism of the Lie superalgebras $\overline{\nu} : \mathfrak{g} \to \mathfrak{v}(M, \mathcal{O}_M)$ given by the formula

$$X \mapsto (X \otimes \mathrm{id}) \circ \nu^*. \tag{3}$$

As in [13], we use the following definition of a transitive action. An action ν is called *transitive* if ν_{red} is transitive and the mapping $\mathrm{ev}_x \circ \overline{\nu}$ is surjective for all $x \in M$. (The map ev_x is given by (1).) In this case the supermanifold

$(M, \mathcal{O}_M)$ is called $(G, \mathcal{O}_G)$-*homogeneous*. A supermanifold $(M, \mathcal{O}_M)$ is called *homogeneous* if it possesses a transitive action of a certain Lie supergroup.

Suppose that a closed Lie subsupergroup $(H, \mathcal{O}_H)$ of $(G, \mathcal{O}_G)$ (this means that the Lie subgroup H is closed in G) is given. Denote by j the inclusion of $(H, \mathcal{O}_H)$ into $(G, \mathcal{O}_G)$. Consider the corresponding coset superspace $(G/H, \mathcal{O}_{G/H})$, see [5, 7]. Denote by $\mu_{G\times H}$ the composition of the morphisms

$$(G, \mathcal{O}_G) \times (H, \mathcal{O}_H) \xrightarrow{\mathrm{id} \times j} (G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \xrightarrow{\mu} (G, \mathcal{O}_G),$$

by $\mathrm{pr}_1 : (G, \mathcal{O}_G) \times (H, \mathcal{O}_H) \to (G, \mathcal{O}_G)$ the projection onto the first factor, and by π the natural mapping $G \to G/H$, $g \mapsto gH$. Let us take $U \subset G/H$ open. Then

$$\mathcal{O}_{G/H}(U) = \{f \in \mathcal{O}_G(\pi^{-1}(U)) \mid (\mu_{G\times H})^*(f) = \mathrm{pr}_1^*(f)\}. \tag{4}$$

Denote by $\nu : (G, \mathcal{O}_G) \times (G/H, \mathcal{O}_{G/H}) \to (G/H, \mathcal{O}_{G/H})$ the natural action. It is given by $\nu^*(f) = \mu^*(f)$, where $f \in \mathcal{O}_{G/H}(U)$. Hence if $X \in \mathfrak{g}$, $f \in \mathcal{O}_{G/H}$, we have $\overline{\nu}(X)(f) = X(f)$. Sometimes we will denote the supermanifold $(G/H, \mathcal{O}_{G/H})$ also by $(G, \mathcal{O}_G)/(H, \mathcal{O}_H)$.

Example 2. Let $(G, \mathcal{O}_G)$ be a Lie supergroup, H a Lie subgroup of G. Then $(H, \mathcal{F}_H)$ is also a Lie subsupergroup of $(G, \mathcal{O}_G)$. It is well known that the sheaf $\mathcal{O}_{G/H}$ is isomorphic to $\mathcal{F}_{G/H} \otimes \bigwedge(\mathfrak{g}_{\bar{1}}^*)$, where $\mathfrak{g} = \mathrm{Lie}(G, \mathcal{O}_G)$, see, e.g., [18, Proposition 2]. If G/H is compact and connected, $H^0(\mathcal{O}_{G/H}) \simeq \bigwedge(\mathfrak{g}_{\bar{1}}^*)$. Hence there are compact homogeneous complex supermanifolds with non-constant holomorphic functions.

1.2 The Harish-Chandra pairs

The structure sheaf of a Lie supergroup and the supergroup morphisms can be explicitly described in terms of the corresponding Lie superalgebra using so-called (super) Harish-Chandra pairs, see [4]. A *Harish-Chandra pair* is a pair $(G, \mathfrak{g})$ that consists of a Lie group G and a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}}$ is the Lie algebra of G, provided with a representation α_G of G in $\mathfrak{g}$ such that

- α_G preserves the parity and induces the adjoint representation of G in $\mathfrak{g}_{\bar{0}}$,
- the differential $(\mathrm{d}\,\alpha_G)_e$ at the identity $e \in G$ coincides with the adjoint representation ad of $\mathfrak{g}_{\bar{0}}$ in $\mathfrak{g}$.

Super Harish-Chandra pairs form a category. (The definition of a morphism see in [4].) The following theorem was proved in [7].

Theorem 1. *The category of real Lie supergroups is equivalent to the category of real Harish-Chandra pairs.*

In the complex case, the equivalence of the categories was shown in [18].

If a Harish-Chandra pair $(G, \mathfrak{g})$ is given, it determines the Lie supergroup $(G, \widehat{\mathcal{O}}_G)$ in the following way, see [8]. Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping superalgebra of $\mathfrak{g}$. It is clear that $\mathfrak{U}(\mathfrak{g})$ is a $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$-module, where $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$ is the universal enveloping algebra of $\mathfrak{g}_{\bar{0}}$. The natural action of $\mathfrak{g}_{\bar{0}}$ on the sheaf $\mathcal{F}_G$ gives rise to a structure of $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$-module on $\mathcal{F}_G(U)$ for any open set $U \subset G$. Putting

$$\widehat{\mathcal{O}}_G(U) = \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})}(\mathfrak{U}(\mathfrak{g}), \mathcal{F}_G(U))$$

for every open $U \subset G$, we get a sheaf $\widehat{\mathcal{O}}_G$ of $\mathbb{Z}_2$-graded vector spaces (here we assume that the functions from $\mathcal{F}_G(U)$ are even). The enveloping superalgebra $\mathfrak{U}(\mathfrak{g})$ has a Hopf superalgebra structure (see [16]). Using this structure we can define the product of elements from $\widehat{\mathcal{O}}_G$ such that $\widehat{\mathcal{O}}_G$ becomes a sheaf of superlgebras. A supermanifold structure on $\widehat{\mathcal{O}}_G$ is determined by the isomorphism $\widehat{\mathcal{O}}_G \xrightarrow{\sim} \mathrm{Hom}(\bigwedge(\mathfrak{g}_{\bar{1}}), \mathcal{F}_G)$, $f \mapsto f \circ \gamma$, where

$$\gamma : \bigwedge(\mathfrak{g}_{\bar{1}}) \to \mathfrak{U}(\mathfrak{g}), \quad X_1 \wedge \cdots \wedge X_r \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} (-1)^{|\sigma|} X_{\sigma(1)} \cdots X_{\sigma(r)}. \tag{5}$$

The following formulas define the multiplication morphism, the inversion morphism and the identity morphism respectively (see [1]):

$$\begin{aligned} \mu^*(f)(X \otimes Y)(g,h) &= f(X \cdot \alpha_G(g)(Y))(gh); \\ \iota^*(f)(X)(g) &= f(\alpha_G(g^{-1})(S(X)))(g^{-1}); \\ \varepsilon^*(f) &= f(1)(e). \end{aligned} \tag{6}$$

Here $X, Y \in \mathfrak{U}(\mathfrak{g})$, $f \in \widehat{\mathcal{O}}_G$, g, $h \in G$ and S is the antipode map of $\mathfrak{U}(\mathfrak{g})$. Here we identify the enveloping superalgebra $\mathfrak{U}(\mathfrak{g} \oplus \mathfrak{g})$ with the tensor product $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$.

A Harish-Chandra pair $(H, \mathfrak{h})$ is called a *Harish-Chandra subpair* of a Harish-Chandra pair $(G, \mathfrak{g})$ if H is a Lie subgroup of G and $\mathfrak{h}$ is a Lie subsuperalgebra of $\mathfrak{g}$, s.t. $\mathfrak{h}_{\bar{0}} = \mathrm{Lie}\, H$ and $\alpha_H = \alpha_G|H$. There is a correspondence between Harish-Chandra subpairs of $(G, \mathfrak{g})$ and Lie subsupergroups of $(G, \mathcal{O}_G)$, see, e.g., [18]. (The Lie supergroup $(G, \mathcal{O}_G)$ corresponds to the Harish-Chandra pair $(G, \mathfrak{g})$.)

Let ν be an action of $(G, \mathcal{O}_G)$ on $(M, \mathcal{O}_M)$, $x \in M$ and $\delta_x : (\mathrm{pt}, \mathbb{C}) \to (M, \mathcal{O}_M)$ the morphism such that $\delta_x(\mathrm{pt}) = x$. Denote by ν_x the following composition:

$$(G, \mathcal{O}_G) \times (\mathrm{pt}, \mathbb{C}) \xrightarrow{\mathrm{id} \times \delta_x} (G, \mathcal{O}_G) \times (M, \mathcal{O}_M) \xrightarrow{\nu} (M, \mathcal{O}_M).$$

Consider the Harish-Chandra subpair $(G_x, \mathfrak{g}_x)$ of $(G, \mathfrak{g})$, $\mathfrak{g} = \mathrm{Lie}(G, \mathcal{O}_G)$, where $G_x \subset G$ is the stabilizer of x and $\mathfrak{g}_x = \mathrm{Ker}(\mathrm{d}\,\nu_x)_e$. A subsupergroup $(G_x, \mathcal{O}_{G_x})$ is called the *stabilizer* of x if it is determined by $(G_x, \mathfrak{g}_x)$.

Denote by l_g, $g \in G$, the following composition:

$$(G, \mathcal{O}_G) = (g, \mathbb{C}) \times (G, \mathcal{O}_G) \xrightarrow{\delta_g \times \mathrm{id}} (G, \mathcal{O}_G) \times (G, \mathcal{O}_G) \xrightarrow{\mu} (G, \mathcal{O}_G).$$

The morphism l_g is called the *left translation.* The *right translation* r_g, $g \in G$, can be defined similarly. Denote by $\bar{l}_g$ the following composition:

$$(M, \mathcal{O}_M) = (g, \mathbb{C}) \times (M, \mathcal{O}_M) \xrightarrow{\delta_g \times \mathrm{id}} (G, \mathcal{O}_G) \times (M, \mathcal{O}_M) \xrightarrow{\nu} (M, \mathcal{O}_M).$$

The representations of G_x in $T_x(M, \mathcal{O}_M)_{\bar{0}}$ and $T_x(M, \mathcal{O}_M)_{\bar{1}}$ given by $G_x \ni h \mapsto (\mathrm{d}\,\bar{l}_h)_x$ are called the *even and odd isotropy representation*, respectively.

Assume that $(M, \mathcal{O}_M)$ is $(G, \mathcal{O}_G)$-homogeneous, then $(\mathrm{d}\,\nu_x)_e$ is surjective. Hence, $T_x(M, \mathcal{O}_M) \simeq \mathfrak{g}/\mathfrak{g}_x$. Denote by Ad_G the adjoint representation of G on $\mathfrak{g}$. Recall that this representation is defined by $\mathrm{Ad}_G(g)(X) = (\mathrm{d}\,l_g \circ r_g)_e(X)$. Clearly, $\mathrm{Ad}_G(h)$ transforms $\mathfrak{g}_x$ into itself for all $h \in H$. It follows that there is a representation $\widehat{\mathrm{Ad}}_G$ of H in $\mathfrak{g}/\mathfrak{g}_x$ given by

$$\widehat{\mathrm{Ad}}_G(h)(X + \mathfrak{g}_x) = \mathrm{Ad}_G(h)(X) + \mathfrak{g}_x, \ X \in \mathfrak{g}, \ h \in H.$$

As in the classical case we have.

Lemma 1. *The representation $\widehat{\mathrm{Ad}}_G$ is equivalent to the isotropy representation and $(\mathrm{d}\,\nu_x)_e$ determines the corresponding equivalence. More precisely, $\widehat{\mathrm{Ad}}_G|_{(\mathfrak{g}_x)_{\bar{0}}}$ is equivalent to the even isotropy representation and $\widehat{\mathrm{Ad}}_G|_{(\mathfrak{g}_x)_{\bar{1}}}$ to the odd one.*

Proof. It is sufficient to check that for every $h \in H$, the following diagram is commutative:

$$\begin{array}{ccc} T_e(G, \mathcal{O}_G) & \xrightarrow{(\mathrm{d}\,\nu_x)_e} & T_x(M, \mathcal{O}_M) \\ {\scriptstyle \mathrm{Ad}_G(h)}\downarrow & & \downarrow{\scriptstyle (\mathrm{d}\,\bar{l}_h)_e} \\ T_e(G, \mathcal{O}_G) & \xrightarrow{(\mathrm{d}\,\nu_x)_e} & T_x(M, \mathcal{O}_M) \end{array}.$$

It is easy to see that

$$\nu_x \circ r_h = \nu_{hx} = \nu_x, \quad \nu_x \circ l_h = \bar{l}_h \circ \nu_x \quad \text{for all } \ h \in H.$$

Therefore,

$$\nu_x \circ r_h \circ l_h = \nu_x \circ l_h = \bar{l}_h \circ \nu_x. \quad \Box$$

Let $(M, \mathcal{O}_M)$ be a $(G, \mathcal{O}_G)$-homogeneous supermanifold, then G acts on $\mathcal{T}_M$ by

$$v \mapsto (\bar{l}_g^{-1})^* \circ v \circ (\bar{l}_g)^*. \tag{7}$$

1.3 Split supermanifolds

Let us describe the category SSM (split supermanifolds), which was introduced in [18]. Recall that a supermanifold $(M, \mathcal{O}_M)$ is called *split* if $\mathcal{O}_M \simeq \bigwedge_{\mathcal{F}_M} \mathcal{E}_M$ for a certain locally free sheaf $\mathcal{E}_M$ over M. We put

$$\text{Ob}\ \texttt{SSM} = \{(M, \textstyle\bigwedge \mathcal{E}_M) \mid \mathcal{E}_M \ \text{ is a locally free sheaf on } M\}.$$

Equivalently, we can say that Ob SSM consists of all split supermanifolds $(M, \mathcal{O}_M)$ with a fixed isomorphism $\mathcal{O}_M \simeq \bigwedge \mathcal{E}_M$ for a certain locally free sheaf $\mathcal{E}_M$ on M. Note that $\mathcal{O}_M$ is naturally $\mathbb{Z}$-graded by $(\mathcal{O}_M)_p \simeq \bigwedge^p \mathcal{E}_M$. All the time we will consider this $\mathbb{Z}$-grading for elements from Ob SSM. Further, if $X, Y \in$ Ob SSM, we put

$$\text{Hom}(X, Y) = \begin{array}{l} \text{all morphisms of } X \text{ to } Y \\ \text{preserving the } \mathbb{Z}\text{-gradings.} \end{array}$$

As in the category of supermanifolds, we can define in SSM a group object (split Lie supergroup), an action of a split Lie supergroup on a split supermanifold (split action) and a homogeneous split supermanifold.

There is a functor gr from the category of supermanifolds to the category of split supermanifolds. Let us briefly describe this construction (see, e.g., [10, 13]). Let $(M, \mathcal{O}_M)$ be a supermanifold. As above, denote by $\mathcal{J}_M \subset \mathcal{O}_M$ the subsheaf of ideals generated by odd elements of $\mathcal{O}_M$. Then by definition $\text{gr}(M, \mathcal{O}_M)$ is the split supermanifold $(M, \text{gr}\, \mathcal{O}_M)$, where

$$\text{gr}\, \mathcal{O}_M = \bigoplus_{p \geq 0} (\text{gr}\, \mathcal{O}_M)_p, \quad \mathcal{J}_M^0 := \mathcal{O}_M, \quad (\text{gr}\, \mathcal{O}_M)_p := \mathcal{J}_M^p / \mathcal{J}_M^{p+1}.$$

In this case $(\text{gr}\, \mathcal{O}_M)_1$ is a locally free sheaf and there is a natural isomorphism of $\text{gr}\, \mathcal{O}_M$ onto $\bigwedge (\text{gr}\, \mathcal{O}_M)_1$. If $\psi = (\psi_{\text{red}}, \psi^*) : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ is a morphism, then $\text{gr}(\psi) = (\psi_{\text{red}}, \text{gr}(\psi^*))$, where $\text{gr}(\psi^*) : \text{gr}\, \mathcal{O}_N \to \text{gr}\, \mathcal{O}_M$ is defined by

$$\text{gr}(\psi^*)(f + \mathcal{J}_N^p) := \psi^*(f) + \mathcal{J}_M^p \text{ for } f \in (\mathcal{J}_N)^{p-1}.$$

Recall that by definition every morphism ψ of supermanifolds is even and as a consequence sends $\mathcal{J}_N^p$ into $\mathcal{J}_M^p$.

Let $(G, \mathcal{O}_G)$ be a Lie supergroup with the group morphisms μ, ι and ε. Then it is easy to see that $\mathrm{gr}(G, \mathcal{O}_G)$ is a split Lie supergroup with the group morphisms $\mathrm{gr}(\mu)$, $\mathrm{gr}(\iota)$ and $\mathrm{gr}(\varepsilon)$. Similarly, an action $\nu : (G, \mathcal{O}_G) \times (M, \mathcal{O}_M) \to (M, \mathcal{O}_M)$ gives rise to the action $\mathrm{gr}(\nu) : \mathrm{gr}(G, \mathcal{O}_G) \times \mathrm{gr}(M, \mathcal{O}_M) \to \mathrm{gr}(M, \mathcal{O}_M)$.

Let $(M, \mathcal{O}_M) = (M, \bigwedge \mathcal{E}_M)$ be a split supermanifold. Then the sheaf $\mathcal{T}_M$ is a $\mathbb{Z}$-graded sheaf of Lie superalgebras. The $\mathbb{Z}$-grading is given by

$$(\mathcal{T}_M)_q = \{v \in \mathcal{T}_M \mid v(\bigwedge^p \mathcal{E}_M) \subset \bigwedge^{p+q} \mathcal{E}_M,\ p \geq 0\}. \tag{8}$$

The sheaf $(\mathcal{T}_M)_q$, $q \in \mathbb{Z}$, is a locally free sheaf of $\mathcal{F}_M$-modules. We will use the notation $\mathfrak{v}(M, \mathcal{O}_M)_q := H^0((\mathcal{T}_M)_q)$.

It was shown in [12] that $\mathcal{E}_M^* \simeq (\mathcal{T}_M)_{-1}$. This isomorphism identifies any sheaf homomorphism $\mathcal{E}_M \to \mathcal{F}_M$ with a derivation of degree -1 that is zero on $\mathcal{F}_M$. Denote by $\mathbb{E}$ the vector bundle corresponding to $\mathcal{E}_M$ and by $\mathbb{T}_{-1}$ the vector bundle corresponding to $(\mathcal{T}_M)_{-1}$. It is easy to see that $(\mathbb{T}_{-1})_x = T_x(M, \mathcal{O}_M)_{\bar{1}}$, $x \in M$.

Assume in addition that $(M, \mathcal{O}_M)$ is $(G, \mathcal{O}_G)$-homogeneous and the action of $(G, \mathcal{O}_G)$ on $(M, \mathcal{O}_M)$ is split. Then the action of G on $\bigwedge \mathcal{E}_M$ given by $g \mapsto \bar{l}_g$, $g \in G$, preserves the $\mathbb{Z}$-grading. Hence the vector bundles $\mathbb{E}$ and $\mathbb{E}^*$ are G-homogeneous. Furthermore, the corresponding action of G on $\mathcal{T}_M$ given by (7) preserves the Z-grading (8). The following Lemma is well-known.

Lemma 2. *Assume that $(M, \mathcal{O}_M)$ is $(G, \mathcal{O}_G)$-homogeneous and the action ν of $(G, \mathcal{O}_G)$ on $(M, \mathcal{O}_M)$ is split. Then $\mathbb{E} \simeq \mathbb{T}_{-1}^*$ as homogeneous vector bundles. In particular, $T_x(M, \mathcal{O}_M)_{\bar{1}} \simeq \mathbb{E}_x^*$ as G_x-modules, where G_x is the stabilizer of x by the action ν_{red} of G on M.*

2. Holomorphic functions on a complex homogeneous supermanifold with compact reduction

2.1 The retract of a homogeneous supermanifold

Let $\mathfrak{g}_{\bar{0}}$ be a Lie algebra and V a $\mathfrak{g}_{\bar{0}}$-module. Denote by $[\,,\,]_{\mathfrak{g}_{\bar{0}}}$ the Lie bracket in $\mathfrak{g}_{\bar{0}}$ and by $\cdot$ the module operation in V. We can construct a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ putting $\mathfrak{g}_{\bar{1}} = V$ and defining the Lie bracket by the following formula:

$$[X, Y] = \begin{cases} [X, Y]_{\mathfrak{g}_{\bar{0}}}, & \text{if } X, Y \in \mathfrak{g}_{\bar{0}}; \\ X \cdot Y, & \text{if } X \in \mathfrak{g}_{\bar{0}} \text{ and } Y \in V; \\ 0, & \text{if } X, Y \in V. \end{cases} \tag{9}$$

Let G be a Lie group, $\mathfrak{g}_{\bar 0} = \operatorname{Lie} G$, and V a G-module. Assume that $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus V$ is the Lie superalgebra with the Lie bracket given by (9). Using this data we can construct a Lie supergroup in the following way. Let us describe its Harish-Chandra pair. Denote by α_G the representation of G on $\mathfrak{g}$ given by:

$$\begin{aligned} \alpha_G|\mathfrak{g}_{\bar 0} &= \text{ adjoint representation of } G \text{ on } \mathfrak{g}_{\bar 0}, \\ (\alpha_G|\mathfrak{g}_{\bar 1})(g)(v) &:= g \cdot v,\ g \in G,\ v \in V \text{ (the given module operation).} \end{aligned} \tag{10}$$

Now the Harish-Chandra pair $(G, \mathfrak{g})$ is well-defined.

Let $(G, \mathcal{O}_G)$ be a Lie supergroup, $\mathfrak{g} = \operatorname{Lie}(G, \mathcal{O}_G)$. Denote by $\mathfrak{g}'$ the Lie superalgebra such that $\mathfrak{g}' \simeq \mathfrak{g}$ as vector superspaces, the Lie bracket is defined by (9) with $V := \mathfrak{g}_{\bar 1}$ and $X \cdot Y = [X, Y]$ for $X \in \mathfrak{g}_{\bar 0}$, $Y \in \mathfrak{g}_{\bar 1}$.

Theorem 2. 1. *The Lie supergroup* $\operatorname{gr}(G, \mathcal{O}_G)$ *is determined by the Harish-Chandra pair* $(G, \mathfrak{g}')$.

2. *If* $(H, \mathcal{O}_H)$ *is a closed subsupergroup of* $(G, \mathcal{O}_G)$, *then*

$$\operatorname{gr}((G, \mathcal{O}_G)/(H, \mathcal{O}_H)) \simeq \operatorname{gr}(G, \mathcal{O}_G)/\operatorname{gr}(H, \mathcal{O}_H).$$

3. *The supermanifold* $\operatorname{gr}(G, \mathcal{O}_G)/\operatorname{gr}(H, \mathcal{O}_H)$ *is split. If* $\mathbb{E}$ *is the corresponding homogeneous bundle, then it is determined by the* H*-module* $(\mathfrak{g}_{\bar 1}/\mathfrak{h}_{\bar 1})^*$.

Proof. To prove the first statement of the theorem, we have to prove that

$$[X, Y]_{\mathfrak{g}'} = \begin{cases} [X, Y]_{\mathfrak{g}}, & \text{if } X, Y \in \mathfrak{g}'_{\bar 0} \text{ or } X \in \mathfrak{g}'_{\bar 0},\ Y \in \mathfrak{g}'_{\bar 1}; \\ 0, & \text{if } X, Y \in \mathfrak{g}'_{\bar 1}. \end{cases} \tag{11}$$

Here $[\ ,\ ]_{\mathfrak{g}}$ and $[\ ,\ ]_{\mathfrak{g}'}$ are the Lie brackets in $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Let us take $X_e, Y_e \in T_e(\operatorname{gr}(G, \mathcal{O}_G)) = T_e(G, \mathcal{O}_G)$. We put

$$\begin{aligned} X &= (X_e \otimes \mathrm{id}) \circ \mu^*, & X' &= (X_e \otimes \mathrm{id}) \circ (\operatorname{gr} \mu)^*, \\ Y &= (Y_e \otimes \mathrm{id}) \circ \mu^*, & Y' &= (Y_e \otimes \mathrm{id}) \circ (\operatorname{gr} \mu)^*, \\ Z &= [X, Y]_{\mathfrak{g}}, & Z' &= [X', Y']_{\mathfrak{g}'}. \end{aligned}$$

To prove (11) it is enough to show that $\delta_e \circ Z = \delta_e \circ Z'$ if $X, Y \in \mathfrak{g}'_{\bar 0}$ or $X \in \mathfrak{g}'_{\bar 0}$, $Y \in \mathfrak{g}'_{\bar 1}$, and that $\delta_e \circ Z' = 0$ if $X, Y \in \mathfrak{g}'_{\bar 1}$.

Let us take $f \in (\mathcal{O}_G)_p$, then $\mu^*(f) = g_p + g_{p+2} + \ldots$ and $(\operatorname{gr} \mu)^*(f) = g_p$, where $g_i \in (\mathcal{O}_{G \times G})_i$. It is easy to see that $g_p \neq 0$ (it follows, for example, from the identity axiom of a Lie supergroup). Further, using (2) we get

$$\begin{aligned} \delta_e \circ Z &= \delta_e \circ ((-1)^{p(X)p(Y)}(Y_e \otimes X_e \otimes \mathrm{id}) - (X_e \otimes Y_e \otimes \mathrm{id})) \circ ((\mathrm{id} \times \mu) \circ \mu)^* \\ &= ((-1)^{p(X)p(Y)}(Y_e \otimes X_e) - (X_e \otimes Y_e)) \circ \mu^*. \end{aligned}$$

Similarly,

$$\delta_e \circ Z' = ((-1)^{p(X)p(Y)}(Y_e \otimes X_e) - (X_e \otimes Y_e)) \circ (\mathrm{gr}\,\mu)^*.$$

Assume that $X_e, Y_e \in T_e(G, \mathcal{O}_G)_{\bar{0}}$ and $f \in (\mathcal{O}_G)_p$. Then

$$\begin{gathered}\delta_e \circ Z(f) = ((Y_e \otimes X_e) - (X_e \otimes Y_e)) \circ \mu^*(f) = \\ ((Y_e \otimes X_e) - (X_e \otimes Y_e))(g_p + g_{p+2} + \ldots) = \\ \begin{cases} ((Y_e \otimes X_e) - (X_e \otimes Y_e))(g_0), & \text{if } p = 0; \\ 0, & \text{if } p \geq 1. \end{cases}\end{gathered}$$

Similarly,

$$\begin{gathered}\delta_e \circ Z'(f) = ((Y_e \otimes X_e) - (X_e \otimes Y_e)) \circ (\mathrm{gr}\,\mu)^*(f) = \\ ((Y_e \otimes X_e) - (X_e \otimes Y_e))(g_p) = \begin{cases} ((Y_e \otimes X_e) - (X_e \otimes Y_e))(g_0), & \text{if } p = 0; \\ 0, & \text{if } p \geq 1. \end{cases}\end{gathered}$$

Hence, in this case $\delta_e \circ Z = \delta_e \circ Z'$.

Assume that $X_e \in T_e(G, \mathcal{O}_G)_{\bar{0}}$, $Y_e \in T_e(G, \mathcal{O}_G)_{\bar{1}}$ and $f \in (\mathcal{O}_G)_p$. Then as above we get

$$\delta_e \circ Z(f) = \begin{cases} ((Y_e \otimes X_e) - (X_e \otimes Y_e))(g_1), & \text{if } p = 1; \\ 0, & \text{if } p = 0 \text{ or } p \geq 2. \end{cases}$$

and

$$\delta_e \circ Z'(f) = \begin{cases} ((Y_e \otimes X_e) - (X_e \otimes Y_e))(g_1), & \text{if } p = 1; \\ 0, & \text{if } p = 0 \text{ or } p \geq 2, \end{cases}$$

Hence, in this case $\delta_e \circ Z = \delta_e \circ Z'$ as well.

Assume that $X_e, Y_e \in T_e(G, \mathcal{O}_G)_{\bar{1}}$. Then

$$\begin{gathered}\delta_e \circ Z(f) = ((Y_e \otimes X_e) + (X_e \otimes Y_e)) \circ (\mathrm{gr}\,\mu)^*(f) = \\ ((Y_e \otimes X_e) - (X_e \otimes Y_e))(g_p) = 0, \;\; p \geq 0.\end{gathered}$$

The proof of (11) is complete.

To prove the second statement of the theorem, denote by ν the action of the Lie supergroup $(G, \mathcal{O}_G)$ on the supermanifold $(M, \mathcal{O}_M) := (G, \mathcal{O}_G)/(H, \mathcal{O}_H)$. It is easy to see that the action $\mathrm{gr}\,\nu$ is transitive on $\mathrm{gr}(M, \mathcal{O}_M)$ (see [18, Lemma 5]). Hence, it is enough to show that the stabilizer of the point $eH \in G/H$ is $\mathrm{gr}(H, \mathcal{O}_H)$. Note that H is the stabilizer of eH by the action $\nu_{\mathrm{red}} = (\mathrm{gr}\,\nu)_{\mathrm{red}}$ and $\mathrm{Ker}(\mathrm{d}\,\mathrm{gr}\,\nu_{eH})_e = \mathrm{Ker}(\mathrm{d}\,\nu_{eH})_e$ as vector spaces. Now this assertion follows from the first one.

To complete the proof, note that $\mathrm{gr}\,\mathrm{gr}(G, \mathcal{O}_G) = \mathrm{gr}(G, \mathcal{O}_G)$. The last assertion follows from Lemmas 1 and 2.□

The third part of this theorem was proved in a different way in [18, Theorem 2].

The description of the vector bundle determined by a split homogeneous supermanifold gives the following proposition.

Proposition 1. *Assume that $M = G/H$ is a complex compact homogeneous manifold and $(G, \mathcal{O}_G)$ is a Lie supergroup. Denote by $\mathbb{E}$ the vector bundle determined by a split $(G, \mathcal{O}_G)$-homogeneous supermanifold $(M, \bigwedge \mathcal{E})$. Then $\mathbb{E}$ is a homogeneous subbundle of a trivial homogeneous bundle $\mathbb{V}$ determined by a certain G-module V.*

Conversely, any homogeneous subbundle $\mathbb{E}$ of a trivial homogeneous bundle $\mathbb{V}$ determined by a G-module V corresponds to a certain split homogeneous supermanifold $(M, \bigwedge \mathcal{E})$.

Proof. Assume that $(M, \mathcal{O}_M)$ is $(G, \mathcal{O}_G)$-homogeneous. Then G acts on $(M, \mathcal{O}_M)$ by left translations $\overline{l}_g$, $g \in G$. Note that $\overline{l}_g$ is not an automorphism of the vector bundle $\mathbb{E}$. By Theorem 2 we have $(M, \mathcal{O}_M) \simeq \operatorname{gr}(G, \mathcal{O}_G)/\operatorname{gr}(H, \mathcal{O}_H)$. The left translations determined by the action of $\operatorname{gr}(G, \mathcal{O}_G)$ are automorphisms of $\mathbb{E}$. Hence, $\mathbb{E}$ is a G-homogeneous vector bundle. (The fact that the vector bundle determined by a split homogeneous supermanifold is homogeneous, was also noticed in [12].) Furthermore, the vector space $V := \mathfrak{v}(M, \mathcal{O}_M)^*_{-1}$ is a finite dimensional G-module because M is compact. Since $(M, \mathcal{O}_M)$ is homogeneous, the H-equivariant map

$$\operatorname{ev}_{eH} : \mathfrak{v}(M, \mathcal{O}_M)_{-1} \to T_{eH}(M, \mathcal{O}_M) \simeq \mathbb{E}^*_{eH}$$

is surjective. Hence, the dual map $\mathbb{E}_{eH} \to V$ is injective and $\mathbb{E}$ is a homogeneous subbundle of the trivial bundle $\mathbb{V}$ determined by the H-module V.

Conversely, we put $E = \mathbb{E}_{eH}$. This is an H-module. Denote by $(G, \mathcal{O}_G)$ the Lie supergroup determined by the Harish-Chandra pair $(G, \mathfrak{g})$, where $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus V^*$, $\mathfrak{g}_{\bar{0}} = \operatorname{Lie} G$ and the Lie bracket is given by (9). Let also $(H, \mathcal{O}_H)$ be the Lie subsupergroup of $(G, \mathcal{O}_G)$ determined by the Harish-Chandra subpair $(H, \mathfrak{h})$, where $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus E'$, $E' = \operatorname{Ker}(V^* \to E^*)$ and $\mathfrak{h}_{\bar{0}} = \operatorname{Lie} H$. Then by Theorem 2 the homogeneous supermanifold $(G/H, \mathcal{O}_{G/H})$ is split and the corresponding homogeneous vector bundle $\mathbb{E}$ is determined by the H-module $(V^*/E')^* \simeq E$. The proof is complete.□

Lemma 3. *Let G be a complex Lie group, $H \subset G$ a closed complex Lie subgroup, V a G-module and $E \subset V$ an H-submodule. Assume that G/H is compact and connected. Denote by $\mathbb{E}$ the homogeneous vector bundle that corresponds to E. The following conditions are equivalent:*

1) non-trivial G-modules W such that $W \subset E$ do not exist;

2) $\Gamma(\mathbb{E}) = \{0\}$.

Proof. 1) $\Rightarrow$ 2) Assume that $\Gamma(\mathbb{E}) \neq \{0\}$. Denote by $\mathbb{V}$ the homogeneous vector bundle determined by the H-module V. It is trivial and the evaluation map $\Gamma(\mathbb{V}) \to V$, $s \mapsto s_x$, $x = eH \in G/H$, is an isomorphism of G-modules. The bundle $\mathbb{E}$ is a subbundle of $\mathbb{V}$, hence there is an inclusion $\gamma : \Gamma(\mathbb{E}) \to \Gamma(\mathbb{V})$. Denote by W the image of $\gamma(\Gamma(\mathbb{E}))$ in V by the map $s \mapsto s_x$. It is a G-submodule in V and it is non-trivial by the assumption. Consider the commutative diagram:

$$\begin{array}{ccc} \Gamma(\mathbb{E}) & \longrightarrow & \Gamma(\mathbb{V}) \\ \big\downarrow & & \big\downarrow \\ E & \longrightarrow & V \end{array}$$

where the horizontal arrows are inclusions and the vertical arrows are evaluation maps at the point x. We see that the image of E contains W. We arrive at a contradiction.

2) $\Rightarrow$ 1) Assume that there is a non-trivial G-module W in E. Then there is a trivial subbundle $\mathbb{W}$ in $\mathbb{E}$, where $\mathbb{W}$ is the homogeneous vector bundle, determined by the H-module W. Hence $\Gamma(\mathbb{E}) \neq \{0\}$.$\Box$

2.2 Odd fundamental vector fields on a split homogeneous supermanifold.

Let $(G, \mathcal{O}_G)$ be a complex Lie supergroup. It was proved in [18] that it is isomorphic to the Lie supergroup $(G, \widehat{\mathcal{O}}_G)$ determined by the Harish-Chandra pair $(G, \mathfrak{g})$ using Koszul construction, see 1.3. The isomorphism is given by the following formula:

$$\Phi : \mathcal{O}_G \to \widehat{\mathcal{O}}_G, \;\; \Phi(f)(X)(g) = (-1)^{p(X)p(f)}(X(f))_{\mathrm{red}}(g). \tag{12}$$

We will identify the Lie supergroups $(G, \mathcal{O}_G)$ and $(G, \widehat{\mathcal{O}}_G)$ using this isomorphism. The sheaf $\mathcal{O}_G$ is $\mathbb{Z}$-graded, this $\mathbb{Z}$-grading is induced by the following $\mathbb{Z}$-grading:

$$\mathrm{Hom}(\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_G) = \bigoplus_{q \geq 0} \mathrm{Hom}(\bigwedge^q \mathfrak{g}_{\bar{1}}, \mathcal{F}_G). \tag{13}$$

In other words, the Lie supergroup $(G, \mathcal{O}_G)$ possesses a global odd coordinate system. Namely, let (ξ_i) be a basis of $\mathfrak{g}_{\bar{1}}$. Let $f^{\xi_i} \in \mathcal{O}_G$ so $f^{\xi_i} \circ \gamma \in \mathrm{Hom}(\mathfrak{g}_{\bar{1}}, \mathcal{F}_G)$ and $f^{\xi_i} \circ \gamma(\xi_j) = \delta_{ij}$. Then (f^{ξ_i}) is a global odd coordinate system on $(G, \mathcal{O}_G)$.

Our aim is now to describe right invariant vector fields in the chosen odd coordinates for a split Lie supergroup. Let us take $X \in \mathfrak{g}$, $Y \in \mathfrak{U}(\mathfrak{g})$ and $g \in G$. Using (12) we get

$$(X(f))(Y)(g) = (-1)^{p(Y)p(X(f))}(Y \cdot X(f))_{\mathrm{red}}(g) = (-1)^{p(X)} f(Y \cdot X)(g).$$

Hence, the right invariant vector field X is determined by

$$(X(f))(Y)(g) = (-1)^{p(X)} f(Y \cdot X)(g). \tag{14}$$

Assume in addition that $(G, \mathcal{O}_G)$ is a split Lie supergroup. From Theorem 2 it follows that this is equivalent to $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \{0\}$. In this case, the map γ from (5) is a homomorphism of algebras. Denote by X_{ξ_i} the right invariant vector field which corresponds to ξ_i. By (14) we get for $Y \in \bigwedge^p \mathfrak{g}_{\bar{1}}$ and $g \in G$,

$$(X_{\xi_i}(f^{\xi_j}) \circ \gamma)(Y)(g) = (X_{\xi_i}(f^{\xi_j}))(Y)(g) = -f^{\xi_j}(Y \cdot \xi_i) = \begin{cases} 0, & p \neq 0; \\ -\delta_{ij}, & p = 0. \end{cases} \tag{15}$$

Let μ be the multiplication morphism of $(G, \mathcal{O}_G)$. Since $\mathrm{gr}\,\mu = \mu$, we get that $X_{\xi_i} \in \mathfrak{v}(G, \mathcal{O}_G)_{-1}$ by (2). It follows that X_{ξ_i} is completely determined by (15) and has the form $-\frac{\partial}{\partial f^{\xi_i}}$ in the chosen odd coordinates. We have proved the following result:

Lemma 4. *Let $(G, \mathcal{O}_G)$ be a split Lie supergroup and (f^{ξ_i}) the global odd coordinate system described above. Then the vector fields $\frac{\partial}{\partial f^{\xi_i}}$, $i = 1, \ldots, \dim \mathfrak{g}_{\bar{1}}$ are right invariant.*

Now we are able to prove the following lemma.

Lemma 5. *Let $(M, \mathcal{O}_M)$ be a split homogeneous supermanifold and $\mathcal{O}_M \simeq \bigwedge \mathcal{E}$. If $H^0(\mathcal{E}) = 0$, then $H^0(\bigwedge^p \mathcal{E}) = 0$ for all $p > 0$.*

Proof. By Theorem 2 we may assume that $(M, \mathcal{O}_M)$ is a $(G, \mathcal{O}_G)$-homogeneous supermanifold, where $(G, \mathcal{O}_G)$ is a split Lie supergroup. Let $\mathfrak{g}$ be a Lie superalgebra of $(G, \mathcal{O}_G)$. Denote by $(H, \mathcal{O}_H)$ the stabilizer of a point $x \in M$. Then $(M, \mathcal{O}_M) \simeq (G/H, \mathcal{O}_{G/H})$ and $\mathcal{O}_{G/H} \subset \mathcal{O}_G$, see 1.1. In [18, Proposition 5] it was shown that if $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \{0\}$, then there is an isomorphism of sheaves $\bigwedge \mathcal{E} \to \mathcal{O}_{G/H}$ such that the composition $\bigwedge \mathcal{E} \to \mathcal{O}_{G/H} \hookrightarrow \mathcal{O}_G$ preserves the $\mathbb{Z}$-gradings of sheaves.

Assume that $H^0(\bigwedge^p \mathcal{E}) \neq 0$ for a certain $p > 0$ and let $f \in H^0(\bigwedge^p \mathcal{E})$, $f \neq 0$. Then $f \in H^0((\mathcal{O}_G)_p) \simeq H^0(\mathcal{F}_G) \otimes \bigwedge^p \mathfrak{g}_{\bar{1}}^*$. Let (f^{ξ_i}) be the global odd coordinate system described above. Then we can write f in the following form:

$$f = \sum_{i_1<\cdots<i_p} f_{i_1,\ldots,i_p} f^{\xi_{i_1}} \wedge \cdots \wedge f^{\xi_{i_p}}, \ \ f_{i_1,\ldots,i_p} \in H^0(\mathcal{F}_G).$$

Without loss of generality, we may assume that f^{ξ_1} occurs on the right-hand side of this equation. Hence, we can write $f = f^{\xi_1} g + h$, where $0 \neq g \in H^0(\mathcal{F}_G) \otimes \bigwedge^{p-1} \mathfrak{g}_{\bar{1}}^*$ and h does not depend on f^{ξ_1}. By Lemma 4, the vector field $\frac{\partial}{\partial f^{\xi_1}}$ is right invariant. It follows that

$$\frac{\partial}{\partial f^{\xi_1}}(f) = g \in (\mathcal{O}_{G/H})_{p-1} \simeq H^0(\bigwedge^{p-1} \mathcal{E}).$$

Hence, $H^0(\bigwedge^{p-1} \mathcal{E}) \neq 0$. By induction, the proof is complete.□

We need the following lemma.

Lemma 6. *Let $(M, \mathcal{O}_M)$ be a supermanifold. If $H^0(\operatorname{gr} \mathcal{O}_M) \simeq \mathbb{C}$ then $H^0(\mathcal{O}_M) \simeq \mathbb{C}$.*

Proof. If $H^0(\operatorname{gr} \mathcal{O}_M) \simeq \mathbb{C}$, then $H^0((\operatorname{gr} \mathcal{O}_M)_p) = \{0\}$ for all $p > 0$ and $H^0((\operatorname{gr} \mathcal{O}_M)_0) \simeq \mathbb{C}$. For each $p \geq 0$, we have the exact sequence

$$0 \to H^0(\mathcal{J}_M^{p+1}) \to H^0(\mathcal{J}_M^p) \to H^0((\operatorname{gr} \mathcal{O}_M)_p),$$

where $\mathcal{J}_M$ is the sheaf generated by odd elements of $\mathcal{O}_M$. We have $H^0(\mathcal{J}_M^p) = \{0\}$ if p is sufficiently large. Using induction, we see that $H^0(\mathcal{J}_M^p) = \{0\}$ for all $p > 0$. For $p = 0$, we have the exact sequence

$$0 \to H^0(\mathcal{J}_M^0) = H^0(\mathcal{O}_M) \to H^0((\operatorname{gr} \mathcal{O}_M)_0) \simeq \mathbb{C}.$$

Obviously, $H^0(\mathcal{O}_M) \supset \mathbb{C}$, since there are constant functions on every supermanifold. Hence, $H^0(M, \mathcal{O}_M) \simeq \mathbb{C}$.□

2.3 The main result

Theorem 3. *Let $(M, \mathcal{O}_M)$ be a $(G, \mathcal{O}_G)$-homogeneous supermanifold, M a compact connected manifold, $(H, \mathcal{O}_H)$ the stabilizer of a point $x \in M$, $\mathfrak{g} = \operatorname{Lie}(G, \mathcal{O}_G)$, $\mathfrak{h} = \operatorname{Lie}(H, \mathcal{O}_H)$. Consider the exact sequence of H-modules:*

$$0 \to \mathfrak{h}_{\bar{1}} \to \mathfrak{g}_{\bar{1}} \xrightarrow{\gamma} \mathfrak{g}_{\bar{1}}/\mathfrak{h}_{\bar{1}} \to 0.$$

If there do not exist non-trivial G-modules $W \subset \mathfrak{g}_{\bar{1}}^*$ such that $W \subset \operatorname{Im} \gamma^*$, then $H^0(\mathcal{O}_M) \simeq \mathbb{C}$. If in addition $(M, \mathcal{O}_M)$ is split, then the converse statement is also true.

Proof. Assume that $(M, \mathcal{O}_M)$ is split. By Theorem 2 we have $(M, \mathcal{O}_M) \simeq \operatorname{gr}(G, \mathcal{O}_G)/\operatorname{gr}(H, \mathcal{O}_H)$. Denote by $\mathbb{E}$ the vector bundle determined by $(M, \mathcal{O}_M)$. Put $\mathfrak{g}' = \operatorname{Lie}(gr(G, \mathcal{O}_G))$ and $\mathfrak{h}' = \operatorname{Lie}(gr(H, \mathcal{O}_H))$. By Lemmas 1 and 2, we see that $\mathbb{E}_x \simeq (\mathfrak{g}'_{\bar{1}}/\mathfrak{h}'_{\bar{1}})^*$ as H-modules. By Theorem 2, we have that $\mathfrak{g}'_{\bar{1}} = \mathfrak{g}_{\bar{1}}$ as G-modules and $\mathfrak{h}'_{\bar{1}} = \mathfrak{h}_{\bar{1}}$ as H-modules, hence by assumption non-trivial G-modules $W' \subset (\mathfrak{g}'_{\bar{1}})^*$ such that $W' \subset \operatorname{Im}\gamma'^*$, where $\gamma'^* : (\mathfrak{g}'_{\bar{1}}/\mathfrak{h}'_{\bar{1}})^* \to (\mathfrak{g}'_{\bar{1}})^*$, do not exist. It follows from Lemma 3 that $\Gamma(\mathbb{E}) = \{0\}$. Furthermore, by Lemma 5, we get $\Gamma(\bigwedge^p \mathbb{E}) = \{0\}$ for all $p > 0$. Hence, $H^0(\mathcal{O}_M) = H^0(\mathcal{F}_M) \simeq \mathbb{C}$.

Conversely, if $H^0(\mathcal{O}_M) \simeq \mathbb{C}$ then $H^0(\mathcal{F}_M) \simeq \mathbb{C}$ and $\Gamma(\bigwedge^p \mathbb{E}) = \{0\}$ for $p > 0$. It follows from Lemma 3 that non-trivial G-modules $W \subset \mathfrak{g}^*_{\bar{1}}$ such that $W \subset \operatorname{Im}\gamma^*$ do not exist.

For non-split supermanifolds the assertion follows from Lemma 6 and Theorem 2.□

Corollary. *Let $(M, \mathcal{O}_M)$ be a $(G, \mathcal{O}_G)$-homogeneous supermanifold, M a compact connected manifold, $(H, \mathcal{O}_H)$ the stabilizer of a point $x \in M$, $\mathfrak{g} = \operatorname{Lie}(G, \mathcal{O}_G)$, $\mathfrak{h} = \operatorname{Lie}(H, \mathcal{O}_H)$ and $\mathfrak{g}_{\bar{1}}$ an irreducible G-module. If the odd dimension of $(H, \mathcal{O}_H)$ is equal to 0, then $H^0(\mathcal{O}_M) \simeq \bigwedge(\mathfrak{g}^*_{\bar{1}})$. Otherwise, $H^0(\mathcal{O}_M) \simeq \mathbb{C}$.*

The following proposition can be useful for practical applications:

Proposition 2. *Let $(M, \mathcal{O}_M)$ be a $(G, \mathcal{O}_G)$-homogeneous supermanifold, M a compact connected manifold, $(H, \mathcal{O}_H)$ the stationary subsupergroup of a point $x \in M$, $\mathfrak{g} = \operatorname{Lie}(G, \mathcal{O}_G)$, $\mathfrak{h} = \operatorname{Lie}(H, \mathcal{O}_H)$. Assume that $\mathfrak{g}_{\bar{1}}$ is a completely reducible G-module. Consider the exact sequence of H-modules:*

$$0 \to \mathfrak{h}_{\bar{1}} \xrightarrow{\delta} \mathfrak{g}_{\bar{1}} \xrightarrow{\gamma} \mathfrak{g}_{\bar{1}}/\mathfrak{h}_{\bar{1}} \to 0.$$

Let $W \subset \operatorname{Im}\gamma^*$ be the maximal G-module and let $Y = \{y \in \mathfrak{g}_{\bar{1}} \mid W(y) = 0\}$. If $\delta(\mathfrak{h}_{\bar{1}}) \subset Y$, then $H^0(\mathcal{O}_M) \simeq \bigwedge W$. If in addition $(M, \mathcal{O}_M)$ is split, then $(M, \mathcal{O}_M) \simeq (N, \mathcal{O}_N) \times (\mathrm{pt}, \bigwedge W)$, where $(N, \mathcal{O}_N)$ is a homogeneous supermanifold such that $H^0(\mathcal{O}_N) \simeq \mathbb{C}$.

Let us first prove the following lemma.

Lemma 7. *Let $(G_i, \mathcal{O}_{G_i})$, $i = 1, 2$, be two Lie supergroups and $(H_i, \mathcal{O}_{H_i}) \subset (G_i, \mathcal{O}_{G_i})$ closed Lie subgroups. Then*

$$(G_1 \times G_2/H_1 \times H_2, \mathcal{O}_{G_1 \times G_2/H_1 \times H_2}) \simeq (G_1/H_1, \mathcal{O}_{G_1/H_1}) \times (G_2/H_2, \mathcal{O}_{G_2/H_2}).$$

Proof. Denote by ν_i the actions of $(G_i, \mathcal{O}_{G_i})$ on $(G_i/H_i, \mathcal{O}_{G_i/H_i})$. Then $\nu_1 \times \nu_2$ is the action of $(G_1, \mathcal{O}_{G_1}) \times (G_2, \mathcal{O}_{G_2})$ on $(G_1/H_1, \mathcal{O}_{G_1/H_1}) \times (G_2/H_2, \mathcal{O}_{G_2/H_2})$. Let us compute the stabilizer of the point $x = eH_1 \times eH_2$. Denote by $(H', \mathfrak{h}')$

the Harish-Chandra subpair determined by this stabilizer. Then H' is the stabilizer of x corresponding to the action $(\nu_1 \times \nu_2)_{\mathrm{red}} = (\nu_1)_{\mathrm{red}} \times (\nu_2)_{\mathrm{red}}$. Hence $H' = H_1 \times H_2$. Furthermore,

$$\mathfrak{h}' = \mathrm{Ker}(\mathrm{d}(\nu_1 \times \nu_2)_x)_e = \mathrm{Ker}(\mathrm{d}(\nu_1)_{eH_1})_e \oplus \mathrm{Ker}(\mathrm{d}(\nu_2)_{eH_2})_e = \mathfrak{h}_1 \oplus \mathfrak{h}_2,$$

where $\mathfrak{h}_i = \mathrm{Lie}(H_i, \mathcal{O}_{H_i})$. It follows that $(H', \mathfrak{h}')$ is the Harish-Chandra subpair of the Lie group $(H_1, \mathcal{O}_{H_1}) \times (H_2, \mathcal{O}_{H_2})$. The lemma follows.$\Box$

Proof of Proposition 2. If $(M, \mathcal{O}_M)$ is split, then by Theorem 2 we may assume that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \{0\}$. Let $V \subset \mathfrak{g}_{\bar{1}}^*$ be a G-submodule such that $\mathfrak{g}_{\bar{1}}^* = W \oplus V$. Put $X = \{x \in \mathfrak{g}_{\bar{1}} \mid V(x) = 0\}$. The subsuperspaces $\mathfrak{g}_{\bar{0}} \oplus Y$ and X are subsupralgebras of $\mathfrak{g}$. Denote by $(G_1, \mathcal{O}_{G_1})$ and by $(G_2, \mathcal{O}_{G_2})$ the Lie subsupergroups of $(G, \mathcal{O}_G)$ determined by the Harish-Chandra subpairs $(G, \mathfrak{g}_{\bar{0}} \oplus Y)$ and (e, X), respectively. Then $(G, \mathcal{O}_G) \simeq (G_1, \mathcal{O}_{G_1}) \times (G_2, \mathcal{O}_{G_2})$ and $(H, \mathcal{O}_H) \subset (G_1, \mathcal{O}_{G_1})$. By Lemma 7, we have

$$(M, \mathcal{O}_M) \simeq (G_1/H, \mathcal{O}_{G_1/H}) \times (G_2, \mathcal{O}_{G_2}) = (G_1/H, \mathcal{O}_{G_1/H}) \times (\mathrm{pt}, \bigwedge X^*).$$

Hence, it is enough to show that $H^0(\mathcal{O}_{G_1/H}) \simeq \mathbb{C}$. Let

$$\begin{aligned} &a : Y^* \to \mathfrak{g}_{\bar{1}}^*, \ \ f \mapsto f \circ \mathrm{pr}_Y, \\ &b : (Y/\mathfrak{h}_{\bar{1}})^* \to (\mathfrak{g}_{\bar{1}}/\mathfrak{h}_{\bar{1}})^*, \ \ f \mapsto f \circ \mathrm{pr}_{\gamma(Y)}. \end{aligned}$$

Then a is a homomorphism of G-modules, b is a homomorphism of H-modules, $a \circ (\gamma|_Y)^* = \gamma^* \circ b$ and $\mathrm{Im}\, a = V$. If there is a non-trivial G-module in $\mathrm{Im}(\gamma|_Y)^*$, then there is a non-trivial G-module in $\mathrm{Im}\, \gamma^* \cap V$. This contradicts the assumption that W is maximal. By Theorem 3, we get that $H^0(\mathcal{O}_{G_1/H}) \simeq \mathbb{C}$.

Assume now that $(M, \mathcal{O}_M)$ is not split. The subsuperspace $\mathfrak{g}_{\bar{0}} \oplus Y \subset \mathfrak{g}$ is again a Lie subsuperalgebra. Denote by $(G_1, \mathcal{O}_{G_1})$ the Lie subsupergroup of $(G, \mathcal{O}_G)$ determined by the Harish-Chandra subpair $(G, \mathfrak{g}_{\bar{0}} \oplus Y)$. Note that $(H, \mathcal{O}_H) \subset (G_1, \mathcal{O}_{G_1})$ and $(G/G_1, \mathcal{O}_{G/G_1}) \simeq (\mathrm{pt}, \bigwedge W)$. Denote by Φ the natural $(G, \mathcal{O}_G)$-equivariant morphism $(G/H, \mathcal{O}_{G/H}) \to (G/G_1, \mathcal{O}_{G/G_1})$. Note that Φ^* is injective, hence $\bigwedge W \simeq \Phi^*(H^0(\mathcal{O}_{G/G_1})) \subset H^0(\mathcal{O}_{G/H})$. Since $(M, \mathrm{gr}\, \mathcal{O}_M)$ is split, $H^0(\mathrm{gr}\, \mathcal{O}_M) \simeq \bigwedge W$. It is easy to see that $\dim H^0(\mathcal{O}_{G/H}) \leq \dim H^0(\mathrm{gr}\, \mathcal{O}_{G/H})$. It follows that $\mathcal{O}_{G/H} \simeq \bigwedge W$.$\Box$

2.4 An application of Theorem 3

Let $(M, \mathcal{O}_M)$, $(B, \mathcal{O}_B)$ and $(F, \mathcal{O}_F)$ be complex supermanifolds. The supermanifold $(M, \mathcal{O}_M)$ is called a *bundle with fiber* $(F, \mathcal{O}_F)$, *base space*

$(B, \mathcal{O}_B)$ *and with projection* $p : (M, \mathcal{O}_M) \to (B, \mathcal{O}_B)$ if the following condition holds: there is an open covering $\{U_i\}$ of the manifold B and isomorphisms $\psi_i : (p_1^{-1}(U_i), \mathcal{O}_M) \to (U_i, \mathcal{O}_B) \times (F, \mathcal{O}_F)$ such that the following diagram is commutative:

$$\begin{array}{ccc} (p_1^{-1}(U_i), \mathcal{O}_M) & \xrightarrow{\psi_i} & (U_i, \mathcal{O}_B) \times (F, \mathcal{O}_F) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ (U_i, \mathcal{O}_B) & = & (U_i, \mathcal{O}_B) \end{array} ,$$

where pr_1 is the projection onto the first factor.

Let $(M, \mathcal{O}_M)$ be a supermanifold, M a connected manifold and $\mathfrak{g}$ a complex (finite dimensional) Lie superalgebra. An action of $\mathfrak{g}$ on $(M, \mathcal{O}_M)$ is an arbitrary Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{v}(M, \mathcal{O}_M)$. Assume that $(M, \mathcal{O}_M)$ is a bundle with base $(B, \mathcal{O}_B)$ and projection map p. A natural question is under what conditions the action of $\mathfrak{g}$ on $(M, \mathcal{O}_M)$ induces an action of $\mathfrak{g}$ on $(B, \mathcal{O}_B)$.

Theorem 4. *Let* $p : (M, \mathcal{O}_M) \to (B, \mathcal{O}_B)$ *be the projection of a superbundle with fiber* $(F, \mathcal{O}_F)$*. If* $H^0(\mathcal{O}_F) \simeq \mathbb{C}$*, then any action of a Lie superalgebra is projectable with respect to* p*.*

This theorem was proved in [2] in the case when $p : (M, \mathcal{O}_M) = (B, \mathcal{O}_B) \times (F, \mathcal{O}_F) \to (B, \mathcal{O}_B)$ is the natural projection. Obviously, it can be generalized to bundles.

3. Holomorphic functions on classical flag supermanifolds

3.1 Classical flag supermanifolds

Yu.I. Manin [10] introduced four series of compact complex homogeneous supermanifolds corresponding to the following four series of classical linear complex Lie superalgebras:

1. $\mathfrak{gl}_{m|n}(\mathbb{C})$, the general linear Lie superalgebra of the vector superspace $\mathbb{C}^{m|n}$;

2. $\mathfrak{osp}_{m|n}(\mathbb{C})$, the orthosymplectic Lie superalgebra that annihilates a non-degenerate even symmetric bilinear form in $\mathbb{C}^{m|n}$, n even;

3. $\pi\mathfrak{sp}_{n|n}(\mathbb{C})$, the linear Lie superalgebra that annihilates a non-degenerate odd skew-symmetric bilinear form in $\mathbb{C}^{n|n}$;

4. $\mathfrak{q}_{n|n}(\mathbb{C})$, the linear Lie superalgebra that commutes with an odd involution π in $\mathbb{C}^{n|n}$.

These supermanifolds are called *supermanifolds of flags* in Case 1, *supermanifolds of isotropic flags* in Cases 2 and 3, and *supermanifolds of π-symmetric flags* in Case 4. We will call all of them *classical flag supermanifolds.* For further reading, see also [11, 14, 15].

Denote by $\mathbf{F}_k^m$ the usual manifold of flags of type $k = (k_1, \ldots, k_r)$ in $\mathbb{C}^m$, where $0 \leq k_r \leq \cdots \leq k_1 \leq m$. Let us describe an atlas on $\mathbf{F}_k^m$.

Let $\mathbb{C}^m \supset W_1 \supset \cdots \supset W_r$ be a flag of type $k_1, \ldots, k_r$. Choose a basis B_s in each W_s. Assume that $B_0 = (e_1, \ldots, e_m)$ is the standard basis of $\mathbb{C}^m$ and put $k_0 = m$. Then for any $s = 1, \ldots, r$ the matrix $X_s \in \mathrm{Mat}_{k_{s-1},k_s}(\mathbb{C})$ is defined in the following way: the columns of X_s are the coordinates of the vectors from B_s with respect to the basis B_{s-1}. Since $\mathrm{rk}\, X_s = k_s$, the matrix X_s contains a non-degenerate minor of size k_s.

For each $s = 1, \ldots, r$, let us fix a k_s-tuple $I_s \subset \{1, \ldots, k_{s-1}\}$. Put $I = (I_1, \ldots, I_r)$. Denote by U_I the set of flags f from $\mathbf{F}_k^m$ satisfying the following conditions: there exist bases B_s such that X_s contains the identity matrix of size k_s in the lines with numbers from I_s. It is easy to see that any flag from U_I is uniquely determined by those elements of X_s that are not contained in the identity matrix. Furthermore, any flag is contained in a certain U_I. The elements of X_s that are not contained in the identity matrix are the coordinates of a flag from U_I in the chart determined by I. Rename $X_{I_s} := X_s$. Hence the local coordinates in U_I are determined by r-tuple $(X_1, \ldots, X_r)$. If $J = (J_1, \ldots, J_s)$, where $J_s \subset \{1, \ldots, k_{s-1}\}$, $|J_s| = k_s$, then the transition functions between the charts U_I and U_J are given by:

$$X_{J_1} = X_{I_1} C_{I_1 J_1}^{-1}, \quad X_{J_s} = C_{I_{s-1} J_{s-1}} X_{I_s} C_{I_s J_s}^{-1}, \quad s \geq 2,$$

where $C_{I_1 J_1}$ is the submatrix of X_{I_1} formed by the lines with numbers from J_1 and $C_{I_s J_s}$, $s \geq 2$, is the submatrix of $C_{I_{s-1} J_{s-1}} X_{I_s}$ formed by lines with numbers from J_s.

Let us give an explicit description of classical flag supermanifolds in terms of atlases and local coordinates (see, [19, 20, 21]). (Note that in [10] such a description was given only for super-grassmannians.) Let us take $m, n \in \mathbb{N}$ and let $k = (k_1, \ldots, k_r)$ and $l = (l_1, \ldots, l_r)$ be two r-tuples such that $0 \leq k_r \leq \ldots \leq k_1 \leq m$, $0 \leq l_r \ldots \leq l_1 \leq n$ $\ 0 < k_r + l_r < \ldots < k_1 + l_1 < m + n$. Let us define the *supermanifold* $\mathbf{F}_{k|l}^{m|n}$ *of flags of type* $(k|l)$ in the superspace $V = \mathbb{C}^{m|n}$. The reduction of $\mathbf{F}_{k|l}^{m|n}$ will be the product $\mathbf{F}_k^m \times \mathbf{F}_l^n$ of two manifolds of flags of type k and l in $\mathbb{C}^m = V_{\bar 0}$ and $\mathbb{C}^n = V_{\bar 1}$.

For each $s = 1, \ldots, r$, let us fix k_s- and l_s-tuples of numbers $I_{s\bar 0} \subset \{1, \ldots, k_{s-1}\}$ and $I_{s\bar 1} \subset \{1, \ldots, l_{s-1}\}$, where $k_0 = m$, $l_0 = n$. We put

$I_s = (I_{s\bar{0}}, I_{s\bar{1}}),\ I = (I_1, \ldots, I_r)$. Our aim is now to construct a superdomain $\mathcal{W}_I$. To each I_s assign a matrix of size $(k_{s-1} + l_{s-1}) \times (k_s + l_s)$

$$Z_{I_s} = \begin{pmatrix} X_s & \Xi_s \\ \mathrm{H}_s & Y_s \end{pmatrix}, \quad s = 1, \ldots, r. \tag{16}$$

Suppose that the identity matrix $E_{k_s+l_s}$ is contained in the lines of Z_{I_s} with numbers $i \in I_{s\bar{0}}$ and $k_{s-1} + j,\ j \in I_{s\bar{1}}$. Here $X_s \in \mathrm{Mat}_{k_{s-1},k_s}(\mathbb{C}),\ Y_s \in \mathrm{Mat}_{l_{s-1},l_s}(\mathbb{C})$, where $\mathrm{Mat}_{a,b}(\mathbb{C})$ is the space of matrices of size $a \times b$ over $\mathbb{C}$. By definition, the entries of X_s and Y_s, $s = 1, \ldots, r$, that are not contained in the identity matrix form the even coordinate system of $\mathcal{W}_I$. The non-zero entries of Ξ_s and H_s form the odd coordinate system of $\mathcal{W}_I$.

Thus we have defined a set of superdomains on $\mathbf{F}^m_k \times \mathbf{F}^n_l$ indexed by I. Note that the reductions of these superdomains cover $\mathbf{F}^m_k \times \mathbf{F}^n_l$. The local coordinates of each superdomain are determined by the r-tuple of matrices $(Z_{I_1}, \ldots, Z_{I_r})$. Let us define the transition functions between two superdomains corresponding to $I = (I_s)$ and $J = (J_s)$ by the following formulas:

$$Z_{J_1} = Z_{I_1} C^{-1}_{I_1 J_1}, \quad Z_{J_s} = C_{I_{s-1} J_{s-1}} Z_{I_s} C^{-1}_{I_s J_s}, \quad s \geq 2, \tag{17}$$

where $C_{I_1 J_1}$ is the submatrix of Z_{I_1} that consists of the lines with numbers from J_1, and $C_{I_s J_s}$, $s \geq 2$, is the submatrix of $C_{I_{s-1}J_{s-1}} Z_{I_s}$ that consists of the lines with numbers from J_s. Gluing the superdomains $\mathcal{W}_I$, we define the *supermanifold of flags* $\mathbf{F}^{m|n}_{k|l}$. In the case $r = 1$, this supermanifold is called a *super-grassmannian*. In the literature the notation $\mathbf{Gr}_{m|n,k_1|l_1}$ is sometimes used.

The supermanifold $\mathbf{F}^{m|n}_{k|l}$ is $\mathrm{GL}_{m|n}(\mathbb{C})$-homogeneous. The action can be given by

$$\begin{aligned} &(L, (Z_{I_1}, \ldots, Z_{I_r})) \mapsto (\tilde{Z}_{J_1}, \ldots, \hat{Z}_{J_r}), \\ &\tilde{Z}_{J_1} = L Z_{I_1} C_1^{-1},\ \tilde{Z}_{J_s} = C_{s-1} Z_{I_s} C_s^{-1}. \end{aligned} \tag{18}$$

Here L is a coordinate matrix of $\mathrm{GL}_{m|n}(\mathbb{C})$, C_1 is the invertible submatrix of $L Z_{I_1}$ that consists of the lines with numbers from J_1, C_s, $s \geq 2$, is the invertible submatrix of $C_{s-1} Z_{I_s}$ that consists of the lines with numbers from J_s.

Let $\mathfrak{g}$ be one of the classical Lie superalgebras described in 3.1. Denote by $\mathbf{F}_{k|l}(\mathfrak{g})$ the flag supermanifold of type $(k|l)$ corresponding to $\mathfrak{g}$. We will also write $\mathbf{F}_{k|l}(\mathfrak{gl}_{m|n}(\mathbb{C})) = \mathbf{F}^{m|n}_{k|l}$. Let us describe $\mathbf{F}_{k|l}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{osp}_{m|n}(\mathbb{C})$, $\pi\mathfrak{sp}_{n|n}(\mathbb{C})$ or $\mathfrak{q}_{n|n}(\mathbb{C})$ in coordinates.

The subsupermanifold $\mathbf{F}_{k|l}(\mathfrak{osp}_{m|2n}(\mathbb{C}))$ of $\mathbf{F}^{m|2n}_{k|l}$ is given in coordinates (16) by the following equations:

$$\begin{pmatrix} X_1 & \Xi_1 \\ \mathrm{H}_1 & Y_1 \end{pmatrix}^{ST} \Gamma \begin{pmatrix} X_1 & \Xi_1 \\ \mathrm{H}_1 & Y_1 \end{pmatrix} = 0, \tag{19}$$

where

$$\Gamma = \begin{pmatrix} 0 & E_s & 0 & 0 \\ E_s & 0 & 0 & 0 \\ 0 & 0 & 0 & E_n \\ 0 & 0 & -E_n & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_s & 0 & 0 \\ 0 & E_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_n \\ 0 & 0 & 0 & -E_n & 0 \end{pmatrix}, \tag{20}$$

$m = 2s$ or $m = 2s + 1$ and

$$\begin{pmatrix} X & \Xi \\ \mathrm{H} & Y \end{pmatrix}^{ST} = \begin{pmatrix} X^T & \mathrm{H}^T \\ -\Xi^T & Y^T \end{pmatrix}$$

is the super-transposition.

The subsupermanifold $\mathbf{F}_{k|l}(\pi\mathfrak{sp}_n(\mathbb{C}))$ of $\mathbf{F}^{n|n}_{k|l}$ is given in coordinates (16) by the following equations:

$$\begin{pmatrix} X_1 & \Xi_1 \\ \mathrm{H}_1 & Y_1 \end{pmatrix}^{ST} \Upsilon \begin{pmatrix} X_1 & \Xi_1 \\ \mathrm{H}_1 & Y_1 \end{pmatrix} = 0, \tag{21}$$

where

$$\Upsilon = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix} \tag{22}$$

The subsupermanifold $\mathbf{F}_{k|l}(\mathfrak{q}_n(\mathbb{C}))$ of $\mathbf{F}^{n|n}_{k|k}$ is given in coordinates (16) by $X_s = Y_s$, $\Xi_s = \mathrm{H}_s$, $s = 1, \ldots, r$.

In [10] the action of $(G, \mathcal{O}_G) = \mathrm{OSp}_{m|2n}(\mathbb{C})$, $\Pi\mathrm{Sp}_n(\mathbb{C})$ or $\mathrm{Q}_n(\mathbb{C})$ on $\mathbf{F}_{k|l}(\mathfrak{g})$ was defined. In our coordinates this action is given by (18), where we assume that L is a cordinate matrix of $(G, \mathcal{O}_G)$.

3.2 Holomorphic functions on classical flag supermanifolds

To compute the algebra of holomorphic functions on $\mathbf{F}_{k|l}(\mathfrak{g})$ using Theorem 3 we need to know the Lie superalgebra $\mathfrak{p}_{\mathfrak{g}}$ of the stabilizer of a point $x \in (\mathbf{F}_{k|l}(\mathfrak{g}))_{\mathrm{red}}$ for the action (18). Such stabilizers are also called parabolic subsupergroups of $(G, \mathcal{O}_G) = \mathrm{OSp}_{m|2n}(\mathbb{C})$, $\Pi\mathrm{Sp}_n(\mathbb{C})$ or $\mathrm{Q}_n(\mathbb{C})$ (see [11, 15]).

We follow the approach of A.L. Onishchik [6]. We will use the following lemma.

Lemma 8. *In the conditions of Theorem 3, assume that $\mathfrak{g}_{\bar 1}$ is a completely reducible G-module. Let $W \subset \operatorname{Im}\gamma^*$ be a non-trivial G-module. Denote by X the following G-module*

$$X := \{v \in \mathfrak{g}_{\bar 1} \ \mid \ W(v) = \{0\}\}$$

and by Y a complement to X in $\mathfrak{g}_{\bar 1}$. Then $\gamma|_Y$ is injective.

Proof. Assume that $\gamma(v) = 0$ for some $v \in Y\backslash\{0\}$. Then there is an $f \in W$ such that $f(v) \neq 0$. (Otherwise $W(v) = \{0\}$ and $v \in X$.) Since there exists an $l \in (\mathfrak{g}_{\bar 1}/\mathfrak{h}_{\bar 1})^*$ such that $\gamma^*(l) = f$, we arrive at a contradiction.□

Case $\mathfrak{g} = \mathfrak{gl}_{m|n}(\mathbb{C})$. Let $e_1, \dots, e_m, f_1, \dots, f_n$ be the standard basis of $\mathbb{C}^{m|n}$. Consider the superdomain Z_I in $\mathbf{F}_{k|l}(\mathfrak{gl}_{m|n}(\mathbb{C}))$ corresponding to $I_{s\bar 0} = (1, \dots, k_s)$, $I_{s\bar 1} = (1, \dots, l_s)$. Denote by x the origin of Z_I. It is easy to see that $x = (V_1, \dots, V_r)$, where $V_i = \langle e_1, \dots, e_{k_i}\rangle \oplus \langle f_1, \dots, f_{l_i}\rangle$. Denote by $\mathfrak{p}(x)_{\mathfrak{gl}}$ the Lie superalgebra of the stabilizer of x for the action (18) of $\mathrm{GL}_{m|n}(\mathbb{C})$. It is easy to see that

$$\mathfrak{p}(x)_{\mathfrak{gl}} = \{X \in \mathfrak{gl}_{m|n}(\mathbb{C}) \ \mid \ X(V_i) \subset V_i\}.$$

The Lie superalgebra $\mathfrak{p}(x)_{\mathfrak{gl}}$ admits another description in terms of root systems, see [6], which we are going to describe now. Let us take a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{gl}_{m|n}(\mathbb{C})_{\bar 0}$ in the following form

$$\mathrm{diag}(x_1, \dots, x_m, y_1, \dots, y_n).$$

The corresponding root system $\Delta = \Delta_{\bar 0} \cup \Delta_{\bar 1}$ is given by

$$\Delta_{\bar 0} = \{x_i - x_j, y_i - y_j \ \mid \ i \neq j\}, \ \ \Delta_{\bar 1} = \{x_i - y_j, \, y_i - x_j\}.$$

Let us take an m-tuple $a = (a_1, \dots, a_m)$ and an n-tuple $b = (b_1, \dots, b_n)$ of real numbers such that

$$a_1 = \dots = a_{k_r} = b_1 = \dots = b_{l_r} > \dots > a_{k_2+1} = \dots = a_{k_1} = \\ b_{l_2+1} = \dots = b_{l_1} > a_{k_1+1} = \dots = a_m = b_{l_1+1} = \dots = b_n.$$

Then $(a, b) \in \mathfrak{t}(\mathbb{R})$. Let

$$\mathfrak{p}(a, b)_{\mathfrak{gl}} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta, \, \alpha(a,b) \geq 0} \mathfrak{gl}_{m|n}(\mathbb{C})_\alpha. \tag{23}$$

Note that $\mathfrak{p}(a,b)_{\mathfrak{gl}}$ depends only on the numbers k_i, l_i, $i=1,\dots,r$. From [6], Chapter 4, § 1, Proposition 1, it can be deduced that $\mathfrak{p}(a,b)_{\mathfrak{gl}} = \mathfrak{p}(x)_{\mathfrak{gl}}$. (This follows also from the direct calculation.)

Theorem 5. *If* $(k|l) \neq (m,\dots,m,k_{s+2},\dots,k_r)|(l_1,\dots,l_s,0,\dots,0)$ *and*

$$(k|l) \neq (k_1,\dots,k_s,0,\dots,0)|(n,\dots,n,l_{s+2},\dots,l_r),$$

then $H^0(\mathbf{F}_{k|l}(\mathfrak{gl}_{m|n}(\mathbb{C}))) \simeq \mathbb{C}$. *Otherwise*

$$\mathbf{F}_{k|l}(\mathfrak{gl}_{m|n}(\mathbb{C})) \simeq (\mathrm{pt}, \bigwedge(mn)) \times (\mathbf{F}_k \times \mathbf{F}_l)$$

and $H^0(\mathbf{F}_{k|l}(\mathfrak{gl}_{m|n}(\mathbb{C}))) \simeq \bigwedge(mn)$.

Proof. The odd part $\mathfrak{gl}_{m|n}(\mathbb{C})_{\bar 1}$ of the Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{C})$ for $m,n \geq 1$ is the direct sum of two irreducible $\mathfrak{gl}_{m|n}(\mathbb{C})_{\bar 0}$-submodules

$$V_1 = \left\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \ A \in \mathrm{Mat}_{m\times n}(\mathbb{C}) \right\}, V_2 = \left\{ \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, B \in \mathrm{Mat}_{n\times m}(\mathbb{C}) \right\},$$

and this decomposition is unique.

If $(k|l) = (m,\dots,m,k_{s+2},\dots,k_r)|(l_1,\dots,l_s,0,\dots,0)$ then the Lie superalgebra $\mathfrak{p}(x)_{\mathfrak{gl}} = \mathfrak{p}(a,b)_{\mathfrak{gl}}$ is determined by an m-tuple $a=(a_1,\dots,a_m)$ and an n-tuple $b=(b_1,\dots,b_n)$ such that

$$a_1 \geq \cdots \geq a_m > b_1 \geq \cdots \geq b_n.$$

Hence $\mathfrak{p}(x)_{\mathfrak{gl}} \supset V_1$. Consider the subalgebra $\mathfrak{g}' = \mathfrak{gl}_{m|n}(\mathbb{C})_{\bar 0} \oplus V_2$ in $\mathfrak{gl}_{m|n}(\mathbb{C})$. Denote by $(G', \mathcal{O}_{G'})$ the subsupergroup of $\mathrm{GL}_{m|n}(\mathbb{C})$ determined by the Harish-Chandra pair $(G, \mathfrak{g}')$. It is easy to see that $(G', \mathcal{O}_{G'})$ acts on $\mathbf{F}_{k|l}(\mathfrak{gl}_{m|n}(\mathbb{C}))$ transitively for this $(k|l)$ and the Lie superalgebra $\mathfrak{p}'_{\mathfrak{gl}}$ of the stabilizer of x for this action is $\mathfrak{p}(x)_{\mathfrak{gl}} \cap \mathfrak{g}' = (\mathfrak{p}(x)_{\mathfrak{gl}})_{\bar 0}$. It follows that the stabilizer is a usual Lie group. We get that $\mathbf{F}_{k|l}(\mathfrak{gl}_{m|n}(\mathbb{C}))$ is split and the structure sheaf of $\mathbf{F}_{k|l}(\mathfrak{gl}_{m|n}(\mathbb{C}))$ is isomorphic to $\mathcal{F}_M \otimes \bigwedge V_2^*$, where $M = (\mathbf{F}_{k|l}(\mathfrak{gl}_{m|n}(\mathbb{C})))_{\mathrm{red}}$, see Example 2. In particular,

$$H^0(\mathbf{F}_{k|l}(\mathfrak{gl}_{m|n}(\mathbb{C}))) \simeq \bigwedge V_2^* \simeq \bigwedge(mn).$$

In the case $(k|l) = (k_1,\dots,k_s,0,\dots,0)|(n,\dots,n,l_{s+2},\dots,l_r)$ the proof is similar.

Assume that

$(k|l) \neq (m,\dots,m,k_{s+2},\dots,k_r)|(l_1,\dots,l_s,0,\dots,0)$ and
$(k|l) \neq (k_1,\dots,k_s,0,\dots,0)|(n,\dots,n,l_{s+2},\dots,l_r)$.

Then there are i, j such that $a_i = b_j$ or there are i_1, j_1 and i_2, j_2 such that $a_{i_1} > b_{j_1}$ and $a_{i_2} < b_{j_2}$. In the first case, $\operatorname{Ker}\gamma$ contains the subspace $\langle E_{i,m+j}, E_{m+j,i}\rangle$. In the second case, $\operatorname{Ker}\gamma$ contains the subspace $\langle E_{i_1,m+j_1}, E_{m+j_2,i_2}\rangle$. Thus, $\gamma|_{V_k}$, $k = 1, 2$, cannot be injective. By Lemma 8 and Theorem 3, we get that $H^0(\mathbf{F}_{k|l}(\mathfrak{gl}_{m|n}(\mathbb{C}))) \simeq \mathbb{C}$.□

Case $\mathfrak{g} = \mathfrak{osp}_{m|n}(\mathbb{C})$. Since the manifold $\mathbf{F}_{k|l}(\mathfrak{osp}_{m|n}(\mathbb{C}))_{\mathrm{red}}$ consists of isotropic flags, it follows that $k_1 \leq p := [\frac{m}{2}]$ and $l_1 \leq q := \frac{n}{2}$. Let us fix a basis of $\mathbb{C}^{m|n}$ such that the matrix of the corresponding non-degenerate even symmetric bilinear form has the matrix Γ given by (20) and denote its elements as follows:

$$\begin{aligned} &e_1, \dots, e_{2p}, f_1, \dots f_n, \ \text{if } m \text{ is even},\\ &e_0, \dots, e_{2p}, f_1, \dots f_n, \ \text{if } m \text{ is odd}. \end{aligned}$$

Consider the superdomain Z_I in $\mathbf{F}^{m|n}_{k|l}$ corresponding to $I_{s\bar{0}} = (1, \dots, k_s)$ and $I_{s\bar{1}} = (1, \dots, l_s)$. Denote by x the origin of Z_I. It is easy to see that $x = (V_1, \dots, V_r)$, where $V_i = \langle e_1, \dots, e_{k_i}\rangle \oplus \langle f_1, \dots, f_{l_i}\rangle$, and x is isotropic. Denote by $\mathfrak{p}(x)_{\mathfrak{osp}}$ the Lie superalgebra of the stabilizer of x for the action (18) of $\mathrm{OSp}_{m|n}(\mathbb{C})$. It is easy to see that

$$\mathfrak{p}(x)_{\mathfrak{osp}} = \{X \in \mathfrak{osp}_{m|n}(\mathbb{C}) \mid X(V_i) \subset V_i\}.$$

The Lie superalgebra $\mathfrak{osp}_{m|n}(\mathbb{C})$ has the following forms for $m = 2p + 1$ or $m = 2p$, respectively:

$$\begin{pmatrix} 0 & -v^t & -u^t & w & w_1\\ u & A & B & U & U_1\\ v & C & -A^t & W & W_1\\ w_1^t & W_1^t & U_1^t & Y & Z\\ -w^t & -W^t & -U^t & T & -Y^t \end{pmatrix}, \quad \begin{pmatrix} A & B & U & U_1\\ C & -A^t & W & W_1\\ W_1^t & U_1^t & Y & Z\\ -W^t & -U^t & T & -Y^t \end{pmatrix},$$
$$B^t = -B,\ C^t = -C,\ Z^t = Z,\ T^t = T.$$

Here Y, Z, T are square matrices of order q, A, B, C are square matrices of order p, U, U_1, V, V_1 are $p \times q$-matrices, u, v are columns of height p, and w, w_1 are rows of length q. As a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{osp}_{m|n}(\mathbb{C})_{\bar{0}}$, one takes that of all diagonal matrices

$$\begin{aligned} &\operatorname{diag}(x_1, \dots, x_p, -x_1, \dots, -x_p, y_1, \dots, y_q, -y_1, \dots, -y_q), \ \text{for } m = 2p,\\ &\operatorname{diag}(0, x_1, \dots, x_p, -x_1, \dots, -x_p, y_1, \dots, y_q, -y_1, \dots, -y_q), \ \text{for } m = 2p + 1. \end{aligned}$$

The corresponding root system $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$ is given by

$$\Delta_{\bar{0}} = \begin{cases} \{\pm x_i \pm x_j,\ \pm y_i \pm y_j,\ \pm 2y_i \mid i \neq j\} & \text{for } m = 2p,\\ \{\pm x_i \pm x_j,\ \pm x_i,\ \pm y_i \pm y_j,\ \pm 2y_i \mid i \neq j\} & \text{for } m = 2p + 1, \end{cases}$$
$$\Delta_{\bar{1}} = \begin{cases} \{\pm x_i \pm y_j\} & \text{for } m = 2p,\\ \{\pm x_i \pm y_j,\ \pm y_i\} & \text{for } m = 2p + 1. \end{cases}$$

Let us take a p-tuple $a=(a_1,\dots,a_p)$ and a q-tuple $b=(b_1,\dots,b_q)$ of real numbers such that

$$a_1=\dots=a_{k_r}=b_1=\dots=b_{l_r}>\dots>a_{k_2+1}=\dots=a_{k_1}= \\ b_{l_2+1}=\dots=b_{l_1}>a_{k_1+1}=\dots=a_p=b_{l_1+1}=\dots=b_p=0.$$

Then $(a,-a,b,-b)\in\mathfrak{t}(\mathbb{R})$. Let

$$\mathfrak{p}(a,b)_{\mathfrak{osp}}=\mathfrak{t}\oplus\bigoplus_{\alpha\in\Delta,\,\alpha(a,-a,b,-b)\geq 0}\mathfrak{osp}_{m|n}(\mathbb{C})_\alpha. \tag{24}$$

Note that $\mathfrak{p}(a,b)_{\mathfrak{osp}}$ depends only on the numbers k_i, l_i, $i=1,\dots,r$. In [6], Chapter 4, § 2, Proposition 2, it was shown that $\mathfrak{p}(a,b)_{\mathfrak{osp}}=\mathfrak{p}(x)_{\mathfrak{osp}}$ if $m\geq 1$, $n\geq 2$.

Theorem 6. *Assume that $m\geq 1$, $n\geq 2$. If m is odd or m is even and $m>2$, then $H^0(\mathbf{F}_{k|l}(\mathfrak{osp}_{2|n}(\mathbb{C})))\simeq\mathbb{C}$.*

Suppose that $m=2$. If $k|l\neq(1,\dots,1|l_1,\dots,l_{r-1},0)$, then

$$H^0(\mathbf{F}_{k|l}(\mathfrak{osp}_{2|n}(\mathbb{C})))\simeq\mathbb{C}.$$

Suppose that $m=2$ and $k|l=(1,\dots,1|l_1,\dots,l_{r-1},0)$, then

$$\mathbf{F}_{k|l}(\mathfrak{osp}_{2|n}(\mathbb{C}))\simeq(\mathrm{pt},\textstyle\bigwedge(2q))\times M,$$

where $M=(\mathbf{F}_{k|l}(\mathfrak{osp}_{2|n}(\mathbb{C})))_{\mathrm{red}}$. In particular, $H^0(\mathbf{F}_{k|l}(\mathfrak{osp}_{2|n}(\mathbb{C})))\simeq\bigwedge(2q)$.

Proof. Assume that m is odd or m is even and $m>2$, then $\mathfrak{osp}_{m|n}(\mathbb{C})_{\bar 1}$ is an irreducible $\mathfrak{osp}_{m|n}(\mathbb{C})_{\bar 0}$-module. Hence by Lemma 8 it is sufficient to check that $(\mathfrak{p}(a,b)_{\mathfrak{osp}})_{\bar 1}\neq\{0\}$. Since $a_i,b_j\geq 0$, we get that $\mathfrak{osp}_{m|n}(\mathbb{C})_{x_i+y_j}\subset(\mathfrak{p}(a,b)_{\mathfrak{osp}})_{\bar 1}$ and $\mathfrak{osp}_{m|n}(\mathbb{C})_{y_j}\subset(\mathfrak{p}(a,b)_{\mathfrak{osp}})_{\bar 1}$.

Assume that $m=2$. In this case $\mathfrak{osp}_{m|n}(\mathbb{C})_{\bar 1}$ is a direct sum of two irreducible $\mathfrak{osp}_{m|n}(\mathbb{C})_{\bar 0}$-modules:

$$V_1=\begin{pmatrix}0&0&0&0\\0&0&W&W_1\\W_1^t&0&0&0\\-W^t&0&0&0\end{pmatrix},\quad V_2=\begin{pmatrix}0&0&U&U_1\\0&0&0&0\\0&U_1^y&0&0\\0&-U^t&0&0\end{pmatrix},$$

and this decomposition is unique. Assume that $k|l=(1,\dots,1|l_1,\dots,l_{r-1},0)$. Then the Lie superalgebra $\mathfrak{p}(x)_{\mathfrak{osp}}=\mathfrak{p}(a,b)_{\mathfrak{osp}}$ is determined by a 1-tuple $a=(a_1)$ and a q-tuple $b=(b_1,\dots,b_q)$ such that

$$a_1>b_1\geq\dots\geq b_q\geq 0.$$

In this case $(\mathfrak{p}(a,b)_{\mathfrak{osp}})_{\bar 1} = V_2$. Consider the subalgebra $\mathfrak{g}' = \mathfrak{osp}_{m|n}(\mathbb{C})_{\bar 0} \oplus V_1$ in $\mathfrak{osp}_{m|n}(\mathbb{C})$. Denote by $(G', \mathcal{O}_{G'})$ the subsupergroup of $\mathrm{OSp}_{m|n}(\mathbb{C})$ determined by the Harish-Chandra pair $(G, \mathfrak{g}')$. It is clear that $(G', \mathcal{O}_{G'})$ acts on $\mathbf{F}_{k|l}(\mathfrak{osp}_{m|n}(\mathbb{C}))$ transitively for this $(k|l)$ and the Lie superalgebra $\mathfrak{p}'_{\mathfrak{osp}}$ of the stabilizer of x for this action is $\mathfrak{p}(x)_{\mathfrak{osp}} \cap \mathfrak{g}' = (\mathfrak{p}(x)_{osp})_{\bar 0}$. It follows that the stabilizer is a usual Lie group. We get that $\mathbf{F}_{k|l}(\mathfrak{osp}_{m|n}(\mathbb{C}))$ is split and the structure sheaf of $\mathbf{F}_{k|l}(\mathfrak{osp}_{m|n}(\mathbb{C}))$ is isomorphic to $\mathcal{F}_M \otimes \bigwedge V_1^*$, where $M = (\mathbf{F}_{k|l}(\mathfrak{osp}_{m|n}(\mathbb{C})))_{\mathrm{red}}$, see Example 2. In particular,

$$H^0(\mathbf{F}_{k|l}(\mathfrak{osp}_{m|n}(\mathbb{C}))) \simeq \bigwedge V_1^* \simeq \bigwedge (mn).$$

Assume that $k|l \neq (1,\dots,1|l_1,\dots,l_{r-1},0)$. Then we have the following possibilities:

- there exists j such that $a_1 = b_j$,
- there exist i, j such that $a_1 > b_i$ and $a_1 < b_j$,
- $b_1 \geq \dots \geq b_n > a_1$

In the first case, $\gamma|_{V_k}$, $k = 1,2$, cannot be injective because $\mathfrak{osp}_{m|n}(\mathbb{C})_{x_1 - y_i} \subset V_2$, $\mathfrak{osp}_{m|n}(\mathbb{C})_{-x_1+y_i} \subset V_1$ and $\mathfrak{osp}_{m|n}(\mathbb{C})_{x_1-y_i} \oplus \mathfrak{osp}_{m|n}(\mathbb{C})_{-x_1+y_i} \subset (\mathfrak{p}(x)_{\mathfrak{osp}})_{\bar 1}$. In the second case, $\gamma|_{V_k}$, $k = 1,2$, cannot be injective because again

$$\mathfrak{osp}_{m|n}(\mathbb{C})_{x_1-y_i} \oplus \mathfrak{osp}_{m|n}(\mathbb{C})_{-x_1+y_j} \subset (\mathfrak{p}(x)_{\mathfrak{osp}})_{\bar 1}.$$

In the third case, $\gamma|_{V_k}$, $k = 1,2$, cannot be injective because

$$\mathfrak{osp}_{m|n}(\mathbb{C})_{x_1+y_i} \oplus \mathfrak{osp}_{m|n}(\mathbb{C})_{-x_1+y_j} \subset (\mathfrak{p}(x)_{\mathfrak{osp}})_{\bar 1}. \square$$

Case $\mathfrak{g} = \pi\mathfrak{sp}_{n|n}(\mathbb{C})$. The manifold $\mathbf{F}_{k|l}(\pi\mathfrak{sp}_{n|n}(\mathbb{C}))_{\mathrm{red}}$ consists of isotropic flags, so $k_1 + l_1 \leq n$. Let us fix a basis of $\mathbb{C}^{m|n}$ such that the matrix of the corresponding non-degenerate odd symmetric bilinear form has the matrix Υ, see (22), and denote its elements as follows:

$$e_1, \dots, e_n, f_1, \dots f_n.$$

Consider the superdomain Z_I in $\mathbf{F}^{n|n}_{k|l}$ corresponding to $I_{s\bar 0} = (1,\dots,k_s)$ and $I_{s\bar 1} = (n - l_s + 1, \dots, n)$. Denote by x the origin of Z_I. It is easy to see that $x = (V_1, \dots, V_r)$, where $V_i = \langle e_1, \dots, e_{k_i}\rangle \oplus \langle f_{n-l_i+1}, \dots, f_n\rangle$, and x is isotropic. Denote by $\mathfrak{p}(x)_{\pi\mathfrak{sp}}$ the Lie superalgebra of the stabilizer of x for the action (18) of $\Pi\mathrm{Sp}_{n|n}(\mathbb{C})$ on $\mathbf{F}_{k|l}(\pi\mathfrak{sp}_{n|n}(\mathbb{C}))$. Again

$$\mathfrak{p}(x)_{\pi\mathfrak{sp}} = \{X \in \pi\mathfrak{sp}_{n|n}(\mathbb{C}) \mid X(V_i) \subset V_i\}.$$

The Lie superalgebra $\pi\mathfrak{sp}_{n|n}(\mathbb{C})$ has the following form:

$$\begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix}, \quad X, Y, Z \in \mathfrak{gl}_n(\mathbb{C}),\ Y = -Y^t,\ Z = Z^t.$$

As a Cartan subalgebra $\mathfrak{t} \subset \pi\mathfrak{sp}_{n|n}(\mathbb{C})_{\bar{0}}$, one takes that of all diagonal matrices

$$\operatorname{diag}(x_1, \ldots, x_n, -x_1, \ldots, -x_n).$$

The corresponding root system $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$ is given by

$$\Delta_{\bar{0}} = \{x_i - x_j \mid i \neq j\},\ \ \Delta_{\bar{1}} = \{x_i + x_j \mid i < j\} \cup \{-x_i - x_j \mid i \leq j\}.$$

Let us take an n-tuple $a = (a_1, \ldots, a_n)$ of real numbers such that

$$\begin{gathered} |a_1| = \cdots = |a_{k_r}| = |a_{n-l_r+1}| = \cdots = |a_n| > \\ |a_{k_r+1}| = \cdots = |a_{k_{r-1}}| = |a_{n-l_{r-1}+1}| = \cdots = |a_{n-l_r}| > \cdots > \\ |a_{k_2} + 1| = \cdots = |a_{k_1}| = |a_{n-l_1+1}| = \cdots = |a_{n-l_2}| > \\ a_{k_1+1} = \cdots = a_{n-l_1} = 0, \\ a_i > 0,\ \text{if } i \in \{1, \ldots, k_1\}, \\ a_i < 0,\ \text{if } i \in \{n - l_1 + 1, \ldots, n\}. \end{gathered}$$

Then $(a, -a) \in \mathfrak{t}(\mathbb{R})$. Let

$$\mathfrak{p}(a)_{\pi\mathfrak{sp}} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta,\, \alpha(a,-a) \geq 0} \pi\mathfrak{sp}_{n|n}(\mathbb{C})_\alpha. \tag{25}$$

Note that $\mathfrak{p}(a)_{\pi\mathfrak{sp}}$ depends only on the numbers k_i, l_i, $i = 1, \ldots, r$. In [6], Chapter 4, § 3, Proposition 3, it was shown that $\mathfrak{p}(a)_{\pi\mathfrak{sp}} = \mathfrak{p}(x)_{\pi\mathfrak{sp}}$ if $n \geq 2$.

Theorem 7. 1. *Assume that* $n \geq 2$.

If $k|l = (n, k_2, \ldots, k_r|0, \ldots, 0)$, *then* $H^0(\mathbf{F}_{k|l}(\pi\mathfrak{sp}_{n|n}(\mathbb{C}))) \simeq \bigwedge((n+1)n/2)$.

If $k|l = (0, \ldots, 0|n, l_2, \ldots, l_r)$ *or* $(0, \ldots, 0|n-1, l_2, \ldots, l_r)$ *or* $(1, 0, \ldots, 0|n-1, n-1, l_3, \ldots, l_r)$, *then* $H^0(\mathbf{F}_{k|l}(\pi\mathfrak{sp}_{n|n}(\mathbb{C}))) \simeq \bigwedge((n-1)n/2)$.

For other $k|l$ *we have* $H^0(\mathbf{F}_{k|l}(\pi\mathfrak{sp}_{n|n}(\mathbb{C}))) \simeq \mathbb{C}$.

2. *Assume that* $n = 1$, *then* $k|l = (1|0)$ *or* $(0|1)$. *We have*

$$\begin{gathered} \mathbf{F}_{1|0}(\pi\mathfrak{sp}_{1|1}(\mathbb{C})) \simeq (\mathrm{pt}, \textstyle\bigwedge(1)) \ \textit{and}\ H^0(\mathbf{F}_{1|0}(\pi\mathfrak{sp}_{1|1}(\mathbb{C}))) \simeq \mathbb{C} \oplus \mathbb{C}, \\ \mathbf{F}_{0|1}(\pi\mathfrak{sp}_{1|1}(\mathbb{C})) \simeq (\mathrm{pt}, \mathbb{C}) \ \textit{and}\ H^0(\mathbf{F}_{0|1}(\pi\mathfrak{sp}_{1|1}(\mathbb{C}))) \simeq \mathbb{C}. \end{gathered}$$

Proof. Suppose that $n \geq 2$. Then $\pi\mathfrak{sp}_{n|n}(\mathbb{C})_{\bar{1}}$ is a direct sum of two irreducible $\pi\mathfrak{sp}_{n|n}(\mathbb{C})_{\bar{0}}$-modules

$$V_1 = \left\{ \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix} \right\}, \quad V_2 = \left\{ \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \right\},$$

and this decomposition is unique.

Assume that $\gamma|_{V_1}$ is injective. Then $a_i + a_j > 0$ for all $i \leq j$ and $k|l = (n, k_2, \dots, k_r|0, \dots, 0)$. Hence, $(\mathfrak{p}(a)_{\pi\mathfrak{sp}})_{\bar{1}} = V_2$. Consider the subsuperalgebra $\mathfrak{g}' = \pi\mathfrak{sp}_{n|n}(\mathbb{C})_{\bar{0}} \oplus V_1$ in $\pi\mathfrak{sp}_{n|n}(\mathbb{C})$. Denote by $(G', \mathcal{O}_{G'})$ the subsupergroup of $\Pi\,\mathrm{Sp}_{m|n}(\mathbb{C})$ determined by the Harish-Chandra pair $(G, \mathfrak{g}')$. It is clear that $(G', \mathcal{O}_{G'})$ acts on $\mathbf{F}_{k|l}(\pi\mathfrak{sp}_{n|n}(\mathbb{C}))$ transitively and the Lie superalgebra of the stabilizer of x for this action is $\mathfrak{p}(x)_{\pi\mathfrak{sp}_{n|n}(\mathbb{C})} \cap \mathfrak{g}' = (\mathfrak{p}(x)_{\pi\mathfrak{sp}_{n|n}(\mathbb{C})})_{\bar{0}}$. Since $(\mathfrak{p}(x)_{\pi\mathfrak{sp}_{n|n}(\mathbb{C})})_{\bar{0}}$ is a Lie algebra, we get that $\mathbf{F}_{k|l}(\pi\mathfrak{sp}_{n|n}(\mathbb{C}))$ is split, see Example 2, and the structure sheaf of $\mathbf{F}_{k|l}(\pi\mathfrak{sp}_{n|n}(\mathbb{C}))$ is isomorphic to $\mathcal{F}_M \otimes \bigwedge V_1^*$, where $M = (\mathbf{F}_{k|l}(\pi\mathfrak{sp}_{n|n}(\mathbb{C})))_{\mathrm{red}}$. In particular,

$$H^0(\mathbf{F}_{k|l}(\pi\mathfrak{sp}_{n|n}(\mathbb{C}))) \simeq \bigwedge V_1^* \simeq \bigwedge((n+1)n/2).$$

Assume that $\gamma|_{V_2}$ is injective. Then $a_i + a_j < 0$ for all $i < j$. Hence, $k|l = (0, \dots, 0|n, l_2, \dots, l_r)$ or $(0, \dots, 0|n-1, l_2, \dots, l_r)$ or $(1, 0, \dots, 0|n-1, n-1, l_3, \dots, l_r)$. In these cases $(\mathfrak{p}(x)_{\pi\mathfrak{sp}_{n|n}(\mathbb{C})})_{\bar{1}} \subset V_1$ and V_2^* is the maximal $\mathfrak{g}_{\bar{0}}$-module in $\mathrm{Im}\,\gamma^*$. By Proposition 2 it follows that $H^0(\mathbf{F}_{k|l}(\pi\mathfrak{sp}_{n|n}(\mathbb{C}))) \simeq \bigwedge V_2^*$.

If $\gamma|_{V_k}$, $k = 1, 2$, is not injective, then $H^0(\mathbf{F}_{k|l}(\pi\mathfrak{sp}_{n|n}(\mathbb{C}))) \simeq \mathbb{C}$ by Lemma 8 and Theorem 3.

Suppose that $n = 1$. Then

$$\pi\mathfrak{sp}_{1|1}(\mathbb{C}) = \left\{ \begin{pmatrix} x & 0 \\ z & -x \end{pmatrix},\ x, z \in \mathbb{C} \right\}$$

Since $k_1 + l_1 \leq n = 1$, we have $k|l = (1|0)$ or $(0|1)$. In the first case, $\mathfrak{p}(x)_{\pi\mathfrak{sp}_{1|1}(\mathbb{C})} = \mathrm{diag}(x, -x)$ and $\mathbf{F}_{1|0}(\pi\mathfrak{sp}_{1|1}(\mathbb{C})) \simeq (\mathrm{pt}, \bigwedge(1))$. In particular, $H^0(\mathbf{F}_{1|0}(\pi\mathfrak{sp}_{1|1}(\mathbb{C}))) \simeq \mathbb{C} \oplus \mathbb{C}$.

In the second case, $\mathfrak{p}(x)_{\pi\mathfrak{sp}_{1|1}(\mathbb{C})} = \pi\mathfrak{sp}_{1|1}(\mathbb{C}))$ and $\mathbf{F}_{0|1}(\pi\mathfrak{sp}_{1|1}(\mathbb{C})) \simeq (\mathrm{pt}, \mathbb{C})$ In particular, $H^0(\mathbf{F}_{0|1}(\pi\mathfrak{sp}_{1|1}(\mathbb{C}))) \simeq \mathbb{C}$.$\Box$

Case $\mathfrak{g} = \mathfrak{q}_{n|n}(\mathbb{C})$. Let $e_1, \dots, e_n, \pi(e_1), \dots, \pi(e_n)$ be a basis of $\mathbb{C}^{n|n}$ which agrees with π. Consider the superdomain Z_I in $\mathbf{F}_{k|k}(\mathfrak{q}_{n|n}(\mathbb{C}))$ corresponding to $I_{s\bar{0}} = I_{s\bar{1}} = (1, \dots, k_s)$. Denote by x the origin of Z_I. We see that $x = (V_1, \dots, V_r)$, where $V_i = \langle e_1, \dots, e_{k_i} \rangle \oplus \langle \pi(e_1), \dots, \pi(e_{k_i}) \rangle$. Denote by $\mathfrak{p}(x)_{\mathfrak{q}}$ the Lie superalgebra of the stabilizer of x for the action (18) of $\mathrm{Q}_{n|n}(\mathbb{C})$. It is easy to see that

$$\mathfrak{p}(x)_{\mathfrak{q}} = \{X \in \mathfrak{q}_{n|n}(\mathbb{C}) \mid X(V_i) \subset V_i\}.$$

Let us take a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{q}_{n|n}(\mathbb{C})_{\bar{0}}$ of the following form

$$\mathrm{diag}(x_1, \dots, x_n, x_1, \dots, x_n).$$

The corresponding root system $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$ is given by

$$\Delta_{\bar{0}} = \{x_i - x_j \mid i \neq j\}, \;\; \Delta_{\bar{1}} = \{x_i - x_j, \mid i \neq j\}.$$

Let us take an n-tuple $a = (a_1, \ldots, a_n)$ of real numbers such that

$$a_1 = \cdots = a_{k_r} > \cdots > a_{k_2+1} = \cdots = a_{k_1} > a_{k_1+1} = \cdots = a_n.$$

Then $(a, a) \in \mathfrak{t}$. Let

$$\mathfrak{p}(a)_{\mathfrak{q}} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta,\, \alpha(a,a) \geq 0} \mathfrak{q}_{n|n}(\mathbb{C})_{\alpha}. \tag{26}$$

Again $\mathfrak{p}(a)$ depends only on the numbers k_i, $i = 1, \ldots, r$. From [6], Chapter 4, § 4, Theorem 4.4, it can be deduced that $\mathfrak{p}(a)_{\mathfrak{q}} = \mathfrak{p}(x)_{\mathfrak{q}}$.

Theorem 8. *$H^0(\mathbf{F}_{k|k}(\mathfrak{q}_{n|n}(\mathbb{C}))) \simeq \mathbb{C}$.*

Proof. Since $V := \mathfrak{q}_{n|n}(\mathbb{C})_{\bar{1}}$ is an irreducible $\mathfrak{q}_{n|n}(\mathbb{C})_{\bar{0}}$-module and $\mathfrak{p}(a)_{\mathfrak{q}} \cap \mathfrak{q}_{n|n}(\mathbb{C})_{\bar{1}} \neq \{0\}$ for all a, the map $\gamma|_V$ cannot be injective. Now our assertion follows from Lemma 8 and Theorem 3.□

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