

On a tropical dual Nullstellensatz

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Abstract

Since a tropical Nullstellensatz fails even for tropical univariate polynomials we study a conjecture on a tropical *dual* Nullstellensatz for tropical polynomial systems in terms of solvability of a tropical linear system with the Cayley matrix associated to the tropical polynomial system. The conjecture on a tropical effective dual Nullstellensatz is proved for tropical univariate polynomials.

Keywords: dual Nullstellensatz, solving tropical polynomial systems

Introduction

Let T be a tropical semi-ring with operations \oplus, \otimes (see e. g. [2], [3], [9], [12]). Typically $\oplus = \min, \otimes = +$. Examples of T are \mathbb{Z} and $\mathbb{Z}_\infty = \mathbb{Z} \cup \{\infty\}$. A *tropical monomial* has a form $Q = a \otimes X_1^{\otimes i_1} \otimes \cdots \otimes X_n^{\otimes i_n}, a \in T$. The *tropical degree* $\text{trdeg}(Q) := i_1 + \cdots + i_n$. From the point of view of the classical algebra a tropical monomial is a linear function. A point $x = (x_1, \dots, x_n) \in T^n$ (with some of $x_i \neq \infty$) is a *tropical zero* of a *tropical polynomial* $f = \bigoplus_l Q_l$ if the minimum $\min_l \{Q_l(x)\}$ is attained for at least two different tropical monomials Q_l .

We study the issue of a tropical Nullstellensatz. Its direct formulation fails even for tropical univariate polynomials: for example, two tropical polynomials $X \oplus 0, X \oplus 1$ have no common tropical zero, while the generated by them tropical ideal does not contain 1 or any other tropical monomial. That is why we consider a *tropical "dual" Nullstellensatz*.

One can treat the (customary) Hilbert's Nullstellensatz as a reduction of solvability of a polynomial system to solvability of a suitable linear system. Namely, solvability of a polynomial system is equivalent to that the Cayley matrix C associated to the system does not contain the vector $(1, 0, \dots, 0)$ in the linear hull of its rows. In its turn it is equivalent to that the linear system $C \cdot (a_0, a_1, \dots) = 0$ has a solution with $a_0 \neq 0$ (cf. Section 1). The latter rephrasing of the Nullstellensatz we call the "dual" Nullstellensatz. It holds also for the *infinite* matrix C (we call it the infinite "dual" Nullstellensatz) unlike the

customary Nullstellensatz, and it holds for a finite submatrix of C with the size depending on n and on the degrees of the polynomials in the system (we call it the *effective* "dual" Nullstellensatz).

In Section 2 we formulate the conjecture on a tropical "dual" Nullstellensatz. In Section 3 we give a rephrasing of the conjecture in terms of the combinatorial convex geometry. Finally, in Section 4 we prove the tropical effective "dual" Nullstellensatz for univariate polynomials ($n = 1$).

Observe that the latter result in case of a system of two tropical polynomials f_1, f_2 follows from the approach of [11] which relies on the (classical) resultant of a pair of (classical) polynomials, so it fails for overdetermined systems in the tropical setting. We mention also that the problem of solvability of tropical polynomial systems is NP -complete even for tropical quadratic polynomials [12].

Solvability of tropical linear systems belongs to the complexity class $NP \cap co - NP$ [1], [5]. In [1], [5] two different algorithms for solving tropical linear systems were designed with the similar complexity bounds polynomial in $s \cdot M$, where s is the size of the tropical linear system (so, of its matrix) and M majorates the absolute values of the finite (integer) coefficients of the system. We note that the algorithm from [5] possesses an extra feature that it has also a complexity bound polynomial in $\exp(s), \log M$. The open question is whether it runs in fact, within complexity polynomial in $s, \log M$ (which would provide a polynomial complexity for the problem of solvability of tropical linear systems)?

In addition, the algorithm from [5] entails as a by-product the equivalence of solvability of a tropical linear system with the degeneration of its *tropical rank* and simultaneously with the degeneration of its *Kapranov rank*. The latter for systems with *finite* coefficients (say, from \mathbb{Z}) was shown in [3], also a part of this equivalence just for the tropical rank follows from [7].

Besides, we mention that in [6] the tropical (customary) Nullstellensatz was established for an introduced there a "ghost" tropical semi-ring. In [10] the radical of a tropical ideal was explicitly described.

1 "Dual" Nullstellensatz

Let $F_1, \dots, F_s \in K[X_1, \dots, X_n]$ be polynomials over an algebraically closed field K . Denote by $C := C(F_1, \dots, F_s)$ the (infinite size) Cayley matrix over K consisting of the coefficients of F_1, \dots, F_s . The columns of C correspond to all the monomials $X^I := X_1^{i_1} \dots X_n^{i_n}$, $I = (i_1, \dots, i_n)$, and the rows of C correspond to all the polynomials of the form $X^I \cdot F_j$, $1 \leq j \leq s$. Let the first column of C correspond to the monomial $X^0 = 1$. For an integer N denote by C_N the (finite size) submatrix of C formed by the rows $X^I \cdot F_j$, $1 \leq j \leq s$ with the degrees $\deg X^I = i_1 + \dots + i_n \leq N$ and the corresponding columns which contain a non-zero entry in at least one of these rows.

Nullstellensatz states that a polynomial system

$$F_1 = \dots = F_s = 0 \tag{1}$$

has a solution in K^n iff for any N the linear hull of the rows of C_N does not contain the vector $(1, 0, \dots, 0)$. An *effective* Nullstellensatz provides an upper bound on N for which the latter equivalence holds. The bound $N < (\max_j \{\deg(F_j)\})^{O(n)}$ close to optimal was obtained in [4], [8].

Thus, the effective Nullstellensatz is equivalent to the following. System (1) has a solution iff the linear system $C_N \cdot (y_1, y_2, \dots) = 0$ has a solution with $y_1 \neq 0$ for an appropriate N depending on n and on $\max_j \{\deg(F_j)\}$. We call the latter statement the *effective dual Nullstellensatz*. The equivalence that (1) has a solution iff the linear system $C_N \cdot (y_1, y_2, \dots) = 0$ has a solution with $y_1 \neq 0$ for any N , we call the *dual Nullstellensatz*. Finally, the statement (also equivalent to Nullstellensatz) that (1) has a solution iff the infinite linear system $C \cdot (y_1, y_2, \dots) = 0$ has a solution with $y_1 \neq 0$, we call the *infinite dual Nullstellensatz*. The latter infinite linear system makes sense because each row of C contains just a finite number of non-zero entries.

2 Conjecture on a tropical dual Nullstellensatz

Below we assume that the tropical semi-ring $T = \mathbb{R}_\infty$, but for the sake of simplifying the exposition we study tropical zeroes defined over \mathbb{R} (although, one could also consider zeroes defined over \mathbb{R}_∞). For each monomial $Q_l = a_l \otimes X_1^{\otimes i_{1,l}} \otimes \dots \otimes X_n^{\otimes i_{n,l}}$ of a tropical polynomial $f = \bigoplus_l Q_l$ we plot the point $(a_l, i_{1,l}, \dots, i_{n,l}) \in \mathbb{R} \times \mathbb{Z}^n \subset \mathbb{R}^{n+1}$. Then a point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is a tropical zero of f iff the linear function $(a, i_1, \dots, i_n) \rightarrow a + i_1 \cdot x_1 + \dots + i_n \cdot x_n$ attains its minimum at the plotted points at least twice.

Therefore, without changing the set of tropical zeroes of f one can replace the plotted points by their convex hull. Moreover, w.l.o.g. for any point $(a, b_1, \dots, b_n) \in \mathbb{R}^{n+1}$ from this convex hull one can add the ray $\{(b, b_1, \dots, b_n) : b \geq a\}$. The resulting convex set $P(f) \subset \mathbb{R}^{n+1}$ we call the (extended) *Newton polyhedron* of f . Thus, w.l.o.g. one can modify f replacing it by a tropical polynomial whose plotted points are just the points of the form $(a, i_1, \dots, i_n) \in (\mathbb{R} \times \mathbb{Z}^n) \cap P(f)$ with the minimal possible a . Finally, so modified tropical polynomial has the same set of tropical zeroes as f , and (in abuse of notations) we keep for it the same notation. We say that the modified tropical polynomial is in the *convex form*, and from now on we consider tropical polynomials only in the convex form. Observe that x is a tropical zero of f iff for the maximal $b \in \mathbb{R}$ such that the hyperplane $\{(z_1, \dots, z_{n+1}) : z_1 + x_1 \cdot z_2 + \dots + x_n \cdot z_{n+1} = b\} \subset \mathbb{R}^{n+1}$ has a non-empty intersection with $P(f)$, the hyperplane has at least two common points with $P(f)$.

Similarly to the classical algebra to a system of tropical polynomials

$$f_1, \dots, f_s \tag{2}$$

in n variables we associate the Cayley matrix $C := C(f_1, \dots, f_s)$ over \mathbb{R}_∞ consisting of the coefficients of (2). The columns of C correspond to the tropical monomials of the form $X^{\otimes I}$, $I \in \mathbb{Z}^n$, and the rows of C correspond to the tropical polynomials of the form $X^{\otimes I} \otimes f_j$, $I \in \mathbb{Z}^n$, $1 \leq j \leq s$. Note that unlike the classical algebra the tropical Cayley matrix is infinite in all 4 directions.

Conjecture 1 on a tropical infinite dual Nullstellensatz. System (2) has a tropical zero iff the matrix C has a tropical zero.

The latter statement is obvious in the direction that if (2) has a zero then C has a zero (the similar is true for two conjectures below as well).

Observe that being a particular case of tropical polynomials (of the tropical degree 1) matrix $C = (c_{i,I})$ (or in other words, a tropical linear system) has a tropical zero (\dots, y_I, \dots) if for every row i of C (in the language of classical algebra) the minimum $\min_I \{c_{i,I} + y_I\}$ is attained at least for two different coordinates I . Note that a tropical zero of C makes sense because every row of C contains just a finite number of finite (so, from \mathbb{R}) entries.

Similarly to the classical algebra for an integer N denote by C_N a (finite) submatrix of C formed by the rows $X^{\otimes I} \otimes f_j$, $I = (i_1, \dots, i_n) \in \mathbb{Z}^n$, $1 \leq j \leq s$ with $|i_1| + \dots + |i_n| \leq N$, and by the columns of C which contain at least one finite entry at one of these rows.

Conjecture 2 on a tropical dual Nullstellensatz. System (2) has a tropical zero iff for any N the matrix C_N has a tropical zero.

Conjecture 3 on a tropical effective dual Nullstellensatz. There is a function N on n and on $\text{trdeg}(f_j)$, $1 \leq j \leq s$ such that (2) has a tropical zero iff the matrix C_N has a tropical zero.

Clearly, Conjecture 3 implies Conjecture 2, which in its turn implies Conjecture 1.

3 Convex-geometric rephrasing of the tropical dual Nullstellensatz

In the present Section we give a rephrasing of Conjecture 1 (and similarly of Conjectures 2, 3) in terms of the convex geometry in \mathbb{R}^{n+1} . Thus, assume that Cayley matrix C has a tropical zero (\dots, y_I, \dots) , $I \in \mathbb{Z}^n$.

For any $I \in \mathbb{Z}^n$ consider the shift $P(f_j) + (0, I) \subset \mathbb{R}^{n+1}$, $1 \leq j \leq s$ of the Newton polyhedron. We say that a set $U \subset \mathbb{R}^{n+1}$ lies above (with respect to the first coordinate) a set $V \subset \mathbb{R}^{n+1}$ if for any pair of points $(u_1, w_1, \dots, w_n) \in U$, $(v_1, w_1, \dots, w_n) \in V$ we have $u_1 \geq v_1$.

Proposition 3.1 *The following statement is equivalent to Conjecture 1.*

For $I \in \mathbb{Z}^n$, $1 \leq j \leq s$ take the minimal $a \in \mathbb{R}$ such that the polyhedron $P(f_j) + (a, I)$ lies above the set $Y := \{(-y_J, J) : J \in \mathbb{Z}^n\}$. Assume that for any $I \in \mathbb{Z}^n$, $1 \leq j \leq s$ the polyhedron and Y have at least two common points. Then there exists a hyperplane $H \subset \mathbb{R}^{n+1}$ defined by a linear equation $z_1 + b_2 \cdot z_2 + \dots + b_n \cdot z_n = 0$ such that for each $1 \leq j \leq s$ for the minimal $b \in \mathbb{R}$ with the property that the polyhedron $P(f_j)$ lies above the hyperplane $H - (b, 0)$, the intersection of $P(f_j)$ and $H - (b, 0)$ has at least two points.

For an equivalent statement to Conjecture 2 one has for any N to consider all $I = (i_1, \dots, i_n)$ such that $|i_1| + \dots + |i_n| \leq N$. Respectively, for Conjecture 3 one has to take N as a suitable function in n and in $\text{trdeg}(f_j)$, $1 \leq j \leq s$.

4 Tropical effective dual Nullstellensatz for univariate polynomials

Now let $n = 1$. In this case for a pair of tropical polynomials f_1, f_2 a tropical effective dual Nullstellensatz follows from [11] with the bound $N \leq \text{trdeg}(f_1) + \text{trdeg}(f_2)$, but since this approach relies on the (classical) resultant of a pair of (classical) polynomials being *liftings* of f_1, f_2 , respectively, the approach fails for overdetermined tropical systems ($s \geq 3$).

Theorem 4.1 *A tropical effective dual Nullstellensatz for univariate tropical polynomials f_1, \dots, f_s holds with $N \leq 4 \cdot (\text{trdeg}(f_1) + \dots + \text{trdeg}(f_s))$.*

Proof. Fix $1 \leq j \leq s$ for the time being. For the convex polyhedron $P := P(f_j) \subset \mathbb{R}^2$ and $i \in \mathbb{Z}$ take the minimal $a_i \in \mathbb{R}$ such that the shifted polygon $P_i := P(f_j) + (a_i, i)$ lies above the set $Y = \{(-y_l, l) : l \in \mathbb{Z}\}$ (see Proposition 3.1). By the assumption for any $i \in \mathbb{Z}$ there exist at least two points $(u_1, l_1), (u_2, l_2) \in P_i \cap Y$, $l_1 < l_2$. Points from the latter intersection we call *extremal points* of P_i .

Lemma 4.2 *The function $i \rightarrow a_i$ is convex.*

Proof of Lemma 4.2. Suppose the contrary and let $2 \cdot a_i > a_{i-1} + a_{i+1}$ for a certain i . Let $(u_1, l_1), (u_2, l_2) \in P_i \cap Y$. Denote by

$$S = \{(v, w) : v - w \cdot (a_i - a_{i-1}) \leq u_1 - l_1 \cdot (a_i - a_{i-1}), v + w \cdot (a_i - a_{i+1}) \geq u_1 + l_1 \cdot (a_i - a_{i+1})\} \subset \mathbb{R}^2$$

the sector with the vertex at the point (u_1, l_1) between two rays $R_+ = (u_1, l_1) + \{\lambda \cdot (a_i - a_{i+1}, -1) : \lambda \geq 0\}$ and $R_- = (u_1, l_1) + \{\lambda \cdot (a_i - a_{i-1}, 1) : \lambda \geq 0\}$. We claim that $P_i \subset S$.

Indeed, consider a left adjacent to (u_1, l_1) point $(u_+, l_1 - 1) \in \partial P_i$ on the boundary of P_i (provided that such a point does exist). If $u_+ < u_1 + l_1 \cdot (a_i - a_{i+1})$ (in other words, the point $(u_+, l_1 - 1)$ lies strictly below the ray R_+ , cf. the description of S) then the point $(u_+, l_1 - 1) + (a_{i+1} - a_i, 1) \in P_{i+1}$ lies strictly below Y , the achieved contradiction implies that $(u_+, l_1 - 1) \in S$. In a similar way a right adjacent to (u_1, l_1) point $(u_-, l_1 + 1) \in \partial P_i$ (provided that it does exist) belongs to S , which justifies the claim.

By the same token the parallel shift $S + (u_2, l_2) - (u_1, l_1)$ of the sector S (with its vertex at the point (u_2, l_2)) also contains P_i . This contradicts to the convexity of P_i and completes the proof of Lemma 4.2. ■

Denote by $E := E(f_j) \subset \mathbb{R}^2$ the polygon with the vertices in the extremal points of P_i for all $i \in \mathbb{Z}$. Below we enumerate the (finite) edges of the polygon P from the left to the right. Denote by $(b_r, 1)$ the vector parallel to the r -th edge of P .

Lemma 4.3 *Let $(u_1, l_1), \dots, (u_t, l_t) \in P_i$, $l_1 < \dots < l_t$ be all the extremal points of P_i . Let the point $(u_t, l_t) - (a_i, i) \in P$ lie in the r -th (finite) edge of P (when the latter point belongs to the r -th and to the $(r + 1)$ -th edges we agree that the point lies in the r -th edge). Then $a_{i+1} - a_i \geq b_r$.*

For any extremal point (v, k) of P_{i+1} the point $(v, k) - (a_{i+1}, i + 1) \in P$ lying in the q -th edge of P either satisfies an inequality $q \geq r$ or $(v, k) - (a_{i+1}, i + 1)$ is the vertex of the $(r - 1)$ -th and r -th edges of P (in the latter case (v, k) is the leftmost extremal point of P_{i+1}). There exists an extremal point (v, k) for which either $q = r$ and $(v, k) - (a_{i+1}, i + 1)$ not being the vertex of the r -th and $(r + 1)$ -th edges of the polygon P or $(v, k) - (a_{i+1}, i + 1)$ is the vertex of the $(r - 1)$ -th and r -th edges of P iff $a_{i+1} - a_i = b_r$. Moreover, when $a_{i+1} - a_i = b_r$ any extremal point (u_m, l_m) of P_i with $(u_m, l_m) - (a_i, i)$ lying in the r -th edge of P is also an extremal point of P_{i+1} .

Proof of Lemma 4.3. Consider the point $(u_t, l_t) - (b_r, 1) \in P_i$. Then the point $((u_t, l_t) - (b_r, 1)) + (a_{i+1} - a_i, 1) \in P_{i+1}$ should lie above the extremal point (u_t, l_t) , this entails the inequality $a_{i+1} - a_i \geq b_r$.

Let (v, k) be an extremal point of P_{i+1} with $(v, k) - (a_{i+1}, i + 1)$ lying in the q -th edge of P . The point $(v, k) - (a_{i+1} - a_i, 1)$ lies in the q -th edge of the polygon P_i . If $q < r$ and the point $(v, k) - (a_{i+1}, i + 1)$ is not the vertex of the $(r - 1)$ -th and r -th edges of P then its shift $(v, k) = ((v, k) - (a_{i+1} - a_i, 1)) + (a_{i+1} - a_i, 1)$ lies *strictly* inside the polygon P_i , and therefore (v, k) can not be an extremal point. The achieved contradiction implies that either $q \geq r$ or $(v, k) - (a_{i+1}, i + 1)$ is the vertex of $(r - 1)$ -th and r -th edges of P .

When $a_{i+1} - a_i > b_r$ a similar argument shows that either $q > r$ or $(v, k) - (a_{i+1}, i + 1)$ is the vertex of the r -th and $(r + 1)$ -th edges of P . Finally, when $a_{i+1} - a_i = b_r$, for any extremal point (u_m, l_m) of P_i with $(u_m, l_m) - (a_i, i)$ lying in the r -th edge of P take the point $(u_m, l_m) - (a_{i+1} - a_i, 1) \in P_i$, then the point $(u_m, l_m) = ((u_m, l_m) - (a_{i+1} - a_i, 1)) + (a_{i+1} - a_i, 1) \in P_{i+1}$ is also an extremal point of P_{i+1} . ■

Remark 4.4 *Lemma 4.3 is formulated for the shifts passing from the polygon P_i to P_{i+1} (so, from the left to the right). By the same token a similar statement holds while passing from P_{i+1} to P_i (so, from the right to the left).*

Lemma 4.5 *The polygon E is convex.*

Proof of Lemma 4.5. We prove by induction on i that the union of the extremal points of P_0, \dots, P_i form a convex polygon E_i . At the inductive step we consider the polygon P_{i+1} (so, we move from the left to the right). By the same token one can alternatively consider the polygon P_{-1} (so, move from the right to the left). This would entail Lemma 4.5.

We say that a polygon with the vertices $\dots, w_{i-1}, w_i, w_{i+1}, \dots$ is convex at the vertex w_i if there is a line passing through w_i such that both points w_{i-1}, w_{i+1} lie above this line.

Let $(u'_1, l'_1), (u'_2, l'_2), \dots, l'_1 < l'_2 < \dots$ be all the extremal points of P_{i+1} (if they exist) being not extremal points of P_i . Lemma 4.3 implies that the point $(u'_1, l'_1) - (a_{i+1}, i + 1)$ either lies in the q -th edge of P with $q > r$ or it is the vertex of the r -th and $(r + 1)$ -th edges of P (we keep the notations from Lemma 4.3).

The inductive hypothesis and Lemma 4.3 entail that the polygon E_{i+1} is convex at all the vertices of E_i , perhaps, with the exception of the rightmost extremal point (u_t, l_t) of E_i (and simultaneously of P_i). The point (u_t, l_t) lies in the r -th edge of the polygon P_i , and both polygons P_i, P_{i+1} lie above the line L spanned by this edge (due to Lemma 4.3), whence E_{i+1} is convex at its vertex (u_t, l_t) as well.

Since the extremal points $(u'_1, l'_1), (u'_2, l'_2), \dots$ are located on the convex polygon P_{i+1} we get that E_{i+1} is convex at its vertices $(u'_2, l'_2), \dots$. Thus, it remains to verify that E_{i+1} is convex at its vertex (u'_1, l'_1) .

Denote the vector $w := (u'_2, l'_2) - (u'_1, l'_1)$. The points $p := (u'_1, l'_1) - (a_{i+1} - a_i, 1), (u'_2, l'_2) - (a_{i+1} - a_i, 1) \in P_i$. Therefore, the point p lies in a sector S_0 with the vertex (u_t, l_t) formed by the rays $(u_t, l_t) + \{\lambda \cdot (b_r, 1) : \lambda \geq 0\} \subset L$ and $(u_t, l_t) + \{\lambda \cdot w : \lambda \geq 0\}$. Now consider a sector $S_1 \subset S_0$ parallel to S_0 with the vertex p formed by the rays $p + \{\lambda \cdot (b_r, 1) : \lambda \geq 0\}$ and $p + \{\lambda \cdot w : \lambda \geq 0\}$. The point $(u'_1, l'_1) = p + (a_{i+1} - a_i, 1)$ is located in S_1 due to Lemma 4.3 and taking into the account that the point (u'_1, l'_1) is extremal in P_{i+1} and thereby, can not lie strictly inside P_i . Hence the polygon E_{i+1} is convex at its vertex (u'_1, l'_1) . ■

Remark 4.6 *The latter statement that E_{i+1} is convex at its vertex (u'_1, l'_1) becomes obvious when the point (u_t, l_t) is an extremal point of P_{i+1} , this is equivalent to the equality $a_{i+1} - a_i = b_r$ due to Lemma 4.3. In case when $a_{i+1} - a_i > b_r$ the polygon P_{i+1} has no common extremal points with E_i .*

Corollary 4.7 *Any edge $e = ((u, l), (u', l'))$ of the convex polygon E is one of the following three types:*

1) *either $(u, l), (u', l') \in P_i$ for a certain $i \in \mathbb{Z}$ where the point (u, l) lies in the r -th edge of P_i , the point (u', l') lies in the r' -th edge of P_i for some $r < r'$, except the case when (u, l) is the vertex of the $(r - 1)$ -th and r -th edges of P_i and (u', l') lies in the r -th edge of P_i (in the latter case e is parallel to the r -th edge of P_i , cf. 3) below);*

2) *either the point (u, l) lies in the r -th edge of P_i for a certain $i \in \mathbb{Z}$, the point (u', l') lies in the r' -th edge of P_{i+1} for some r, r' , and the point $(u', l') - (a_{i+1} - a_i, 1)$ lies in the r' -th edge of P_i , moreover either $r < r'$ or $(u', l') - (a_{i+1} - a_i, 1)$ is the vertex of the r -th and $(r + 1)$ -th edges of P_i . Case 2) occurs when $a_{i+1} - a_i > b_r$ (see Lemma 4.3 and Remark 4.6);*

3) *or e is parallel to an edge of P .*

The edges e of E of types 1), 2) we call *intermediate* and the edges of types 3) we call *r -principal* when e is parallel to the r -th edge of P . For an edge of the type either 1) or 2) we define its *projection* (to the second coordinate) as the interval $(l - i, l' - i)$ for the type 1) and as $(l - i, l' - i - 1)$ for the type 2).

Corollary 4.8 *i) For a pair of adjacent intermediate edges of E their projections are also adjacent (in the same order).*

ii) For each r all r -principal edges of E (if they exist) constitute an interval in E (parallel to the r -th edge of P). We call it r -interval. Among these intervals there are either two intervals infinite in one of directions or one interval infinite in both directions.

Let $e_- := ((u_-, l_-), (u'_-, l'_-))$ be an edge of E adjacent to the r -interval from the left (provided that the r -interval is not infinite to the left). Then e_- is intermediate. Assume for definiteness that $(u'_-, l'_-) \in P_i$ for a certain i , while either $(u_-, l_-) \in P_i$ in case of the type 1) (see Corollary 4.7) or $(u_-, l_-) \in P_{i-1}$ in case of the type 2). Then the point (u'_-, l'_-) lies in the r -th edge of P_i .

Similarly, let an edge $e_+ := ((u_+, l_+), (u'_+, l'_+))$ be an edge adjacent to the r -interval from the right (provided that the r -interval is not infinite to the right). Then e_+ is intermediate. Assume that $(u_+, l_+) \in P_i$ for a certain i and either $(u'_+, l'_+) \in P_i$ in case of the type 1) or $(u'_+, l'_+) \in P_{i+1}$ in case of the type 2). Then the point (u_+, l_+) either lies in the r -th edge of P_i or (u_+, l_+) is the vertex of the $(r-1)$ -th and r -th edges of P_i .

iii) Denote by $(d_1, k_1), (d_2, k_2)$ the endpoints of the r -th edge of P . Then for any pair of adjacent extremal points $(d'_1, k'_1), (d'_2, k'_2)$ in the r -interval of E we have $k'_2 - k'_1 \leq k_2 - k_1$.

Finally, we complete the proof of Theorem 4.1. So far, we studied the convex polygon $E(f_j)$ for a fixed $1 \leq j \leq s$ (see Lemma 4.5). Now we consider the intersection $\mathcal{E} := \bigcap_{1 \leq j \leq s} E(f_j)$. Every edge ϵ of the convex polygon \mathcal{E} is some subinterval of either an intermediate edge of $E(f_j)$ or an r -interval for certain $1 \leq j \leq s$ and r . The total sum of the lengths of the projections of the edges being subintervals of intermediate edges of $E(f_j)$, $1 \leq j \leq s$ does not exceed $3 \cdot \sum_{1 \leq j \leq s} \text{trdeg } f_j$ due to i), ii) of Corollary 4.8.

Observe that if ϵ is a subinterval of an r -interval of $E(f_j)$ for a certain $1 \leq j \leq s$ and not all the points of ϵ belong to all polygons $E(f_{j_1})$, $1 \leq j_1 \leq s$ (the latter is equivalent to that any strictly inside point of ϵ does not belong to all $E(f_{j_1})$, $1 \leq j_1 \leq s$) then ϵ can not contain extremal points strictly inside itself. Hence the total sum of the lengths of the projections of all the edges of \mathcal{E} being subintervals of some r -intervals of $E(f_j)$, $1 \leq j \leq s$ does not exceed $\sum_{1 \leq j \leq s} \text{trdeg } f_j$ by virtue of iii) of Corollary 4.8.

Thus, a truncation \mathcal{E}_N of \mathcal{E} with the length of the projection to the second coordinate equal N , where $N \geq 4 \cdot \sum_{1 \leq j \leq s} \text{trdeg } f_j$, contains an edge which is a common subinterval of r_j -intervals for appropriate r_j of all $E(f_j)$, $1 \leq j \leq s$. Taking into the account the Proposition 3.1 we conclude with Theorem 4.1. ■

It would be interesting to improve the factor 4 in Theorem 4.1.

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