Lines crossing a tetrahedron and the Bloch group

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According to B. Totaro ([7]), there is a hope that the Chow groups of a field k can be computed using a very small class of affine algebraic varieties (linear spaces in the right coordinates), whereas the current definition uses essentially all algebraic cycles in affine space. In this note we consider a simple modification of $\mathrm{CH}^2(\mathrm{Spec}(k), 3)$ using only linear subvarieties in affine spaces and show that it maps surjectively to the Bloch group $B(k)$ for any infinite field k. We also describe the kernel of this map.

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1 Lines crossing a tetrahedron

Let k be an arbitrary infinite field. Consider the projective spaces $\mathbb{P}^n(k)$ with fixed sets of homogenous coordinates $(t_0 : t_1 : \cdots : t_n) \in \mathbb{P}^n(k)$. We call a subspace $L \subset \mathbb{P}^n(k)$ of codimension r *admissible* if

$$\mathrm{codim}\Big(\mathrm{L} \cap \{\mathrm{t_{i_1}} = \cdots = \mathrm{t_{i_s}} = 0\}\Big) \;=\; \mathrm{r+s}$$

for every s and distinct $i_1, \ldots, i_s$. (Here $\mathrm{codim(X) > n}$ means $X = \emptyset$.) Let

$$\mathcal{C}_n^r \;=\; \mathbb{Z}\Big[\text{admissible } L \subset \mathbb{P}^n(k)\,,\ \mathrm{codim(L) = r}\Big]$$

be the free abelian group generated by all admissible subspaces of $\mathbb{P}^n(k)$ of codimension r. Then for every r we have a complex

$$\ldots \xrightarrow{d} \mathcal{C}_{r+2}^r \xrightarrow{d} \mathcal{C}_{r+1}^r \xrightarrow{d} \mathcal{C}_r^r \longrightarrow 0 \longrightarrow \ldots$$

(we assume that $\mathcal{C}_n^r = 0$ when $n < r$) with the differential

$$d[L] = \sum (-1)^i [L \cap \{t_i = 0\}] \tag{1}$$

where every $\{t_i = 0\} \subset \mathbb{P}^n(k)$ is naturally identified with $\mathbb{P}^{n-1}(k)$ by throwing away the coordinate t_i. We are interested in the homology groups of these complexes $H_n^r = H_n(\mathcal{C}_\bullet^r)$.

For example, one can easily see that $H_1^1 \cong k^*$. Indeed, a hyperplane $\{\sum \alpha_i t_i = 0\}$ is admissible whenever all the coefficients α_i are nonzero, and if we identify

$$\begin{aligned} \mathcal{C}_1^1 &\cong \mathbb{Z}[k^*] \qquad & [\{\alpha_0 t_0 + \alpha_1 t_1 = 0\}] &\longmapsto \left[\frac{\alpha_1}{\alpha_0}\right] \\ \mathcal{C}_2^1 &\cong \mathbb{Z}[k^* \times k^*] \qquad & [\{\alpha_0 t_0 + \alpha_1 t_1 + \alpha_2 t_2 = 0\}] &\longmapsto \left[\left(\frac{\alpha_1}{\alpha_0}, \frac{\alpha_2}{\alpha_1}\right)\right] \end{aligned} \tag{2}$$

then the differential $d : \mathcal{C}_2^1 \longrightarrow \mathcal{C}_1^1$ turns into

$$[(x, y)] \longmapsto [x] - [xy] + [y] .$$

(one can recognize Menelaus' theorem from plane geometry behind this simple computation). Hence we have

$$H_1^1 \cong \mathbb{Z}[k^*] \Big/ \big\{ [x] - [xy] + [y] \;:\; x, y \in k^* \big\} \cong k^* .$$

Continuing the identifications of (2), $C_\bullet^1$ turns into the bar complex for the group k^* (with the term of degree 0 thrown away) and therefore

$$H_n^1 = H_n(k^*, \mathbb{Z}) , \qquad n \geq 1 .$$

Now we switch to $r = 2$ and try to compute H_3^2. The four hyperplanes $\{t_i = 0\}$ form a tetrahedron Δ in the 3-dimensional projective space $\mathbb{P}^3(k)$ and the line ℓ is admissible if it

1) intersects every face of Δ transversely, i.e. at one point $P_i = \ell \cap \{t_i = 0\}$;

2) doesn't intersect edges $\{t_{i_1} = t_{i_2} = 0\}$ of Δ, i.e. all four points $P_0, \ldots, P_3 \in \ell$ are different .

Therefore it is natural to associate with ℓ a number, the cross-ratio of the four points $P_0, \ldots, P_3$ on ℓ. Namely, there is a unique way to identify ℓ with $\mathbb{P}^1(k)$ so that P_0, P_1 and P_2 become 0, ∞ and 1 respectively, and we denote the image of P_3 by $\lambda(\ell) \in \mathbb{P}^1(k) \smallsetminus \{0, \infty, 1\} = k^* \smallsetminus \{1\}$. We extend λ linearly to a map

$$\begin{aligned} \mathcal{C}_3^2 &\xrightarrow{\lambda} \mathbb{Z}[k^* \smallsetminus \{1\}] \\ \sum n_i [\ell_i] &\longmapsto \sum n_i [\lambda(\ell_i)] \end{aligned}$$

Theorem 1. *Let $\sigma : k^* \otimes k^* \longrightarrow k^* \otimes k^*$ be the involution $\sigma(x \otimes y) = -y \otimes x$.*

(i) If $d(\sum n_i [\ell_i]) = 0$ then $\sum n_i \lambda(\ell_i) \otimes (1 - \lambda(\ell_i)) = 0$ in $(k^* \otimes k^*)_\sigma$.

(ii) Let $L \subset \mathbb{P}^4(k)$ be an admissible plane and $\ell_i = L \cap \{t_i = 0\}$, $i = 0, \ldots, 4$. If we denote $x = \lambda(\ell_0)$ and $y = \lambda(\ell_1)$ then

$$\lambda(\ell_2) \;=\; \frac{y}{x} , \quad \lambda(\ell_3) \;=\; \frac{1 - x^{-1}}{1 - y^{-1}} \;\; \text{and} \;\; \lambda(\ell_4) \;=\; \frac{1 - x}{1 - y} .$$

(iii) The map induced by λ on homology

$$\lambda_* : H_3^2 \longrightarrow B(k) \tag{3}$$

is surjective, where

$$B(k) \;=\; \frac{\operatorname{Ker}\begin{pmatrix} \mathbb{Z}[k^* \smallsetminus \{1\}] \longrightarrow (k^*\otimes k^*)_\sigma \\ [a] \longmapsto a\otimes(1-a) \end{pmatrix}}{\left\langle [x]-[y]+\left[\frac{y}{x}\right]-\left[\frac{1-x^{-1}}{1-y^{-1}}\right]+\left[\frac{1-x}{1-y}\right],\ x\neq y\right\rangle}$$

is the Bloch group of k ([5]).

(iv) We have $H_3^2 \cong H_3(\mathrm{GL}_2(k))/H_3(k^*)$ and the kernel of (3)

$$K = \operatorname{Ker}\big(H_3^2 \xrightarrow{\lambda_*} B(k)\big)$$

fits into the exact sequence

$$0 \longrightarrow \operatorname{Tor}(k^*,k^*)^\sim \longrightarrow K/T(k) \longrightarrow k^*\otimes K_2(k) \longrightarrow K_3^M(k)/2 \longrightarrow 0\,, \quad (4)$$

where $\operatorname{Tor}(k^*,k^*)^\sim$ *is the unique nontrivial extension of* $\operatorname{Tor}(k^*,k^*)$ *by* $\mathbb{Z}/2$*, and $T(k)$ is a 2-torsion abelian group (conjectured to be trivial).*

We remark that $\operatorname{Tor}(k^*,k^*) = \operatorname{Tor}(\mu(k),\mu(k))$ is a finite abelian group if k is a finitely-generated field. Furthermore, it is proved in [5] that $B(k)$ has the following relation to $K_3(k)$: let $K_3^{\mathrm{ind}}(k)$ be the cokernel of the map from Milnor's K-theory $K_3^M(k) \longrightarrow K_3(k)$, then there is an exact sequence

$$0 \longrightarrow \operatorname{Tor}(k^*,k^*)^\sim \longrightarrow K_3^{\mathrm{ind}}(k) \longrightarrow B(k) \longrightarrow 0 \qquad (5)$$

In particular, if k is a number field then as a consequence of (5) and Borel's theorem ([1]) we have

$$\dim B(k)\otimes\mathbb{Q} \;=\; r_2\,,$$

where r_2 is the number of pairs of complex conjugate embeddings of k into $\mathbb{C}$.

Proof of (i) and (ii). One can check that the diagram

$$\begin{tikzcd} \mathcal{C}_3^2 \arrow[r, "d"] \arrow[d, "\lambda"] & \mathcal{C}_2^2 \arrow[d, "{[t_0:t_1:t_2]\longmapsto t_0\otimes(-t_1)+(-t_1)\otimes t_2+t_2\otimes t_0+t_0\otimes t_0}"] \\ \mathbb{Z}[k^*\smallsetminus\{1\}] \arrow[r, "{[a]\longmapsto a\otimes(1-a)}"] & (k^*\otimes k^*)_\sigma \end{tikzcd}$$

is commutative, and therefore (i) follows. It is another tedious computation to check (ii). $\square$

In the next section we will prove the remaining claims (iii) and (iv) and also show that

$$H_n^2 \cong H_n(\mathrm{GL}_2(k),\mathbb{Z})/H_n(k^*,\mathbb{Z}) \qquad n\geq 3\,. \qquad (6)$$

2 Complexes of configurations

We say that $n+1$ vectors $v_0, \dots, v_n \in k^r$ are *in general position* if every $\leq r$ of them are linearly independent. Let $C(r,n)$ be the free abelian group generated by $(n+1)$-tuples of vectors in k^r in general position. For fixed r we have a complex

$$\dots \xrightarrow{d} C(r,2) \xrightarrow{d} C(r,1) \xrightarrow{d} C(r,0)$$

with the differential

$$d[v_0, \dots, v_n] = \sum (-1)^i [v_0, \dots, \check{v}_i, \dots, v_n] \tag{7}$$

The augmented complex $C(r, \bullet) \longrightarrow \mathbb{Z} \longrightarrow 0$ is acyclic. Indeed, if

$$d\Big(\sum n_i [v_0^i, \dots, v_n^i]\Big) = 0$$

and $v \in k^r$ is such that all $(n+2)$-tuples $[v, v_0^i, \dots, v_n^i]$ are in general position (such vectors v exist since k is infinite) then

$$\sum n_i [v_0^i, \dots, v_n^i] \; = \; d\Big(\sum n_i [v, v_0^i, \dots, v_n^i]\Big)\,.$$

Lemma 1. *$\mathcal{C}_n^r \cong C(r,n)_{\mathrm{GL}_r(k)}$ for the diagonal action of $\mathrm{GL}_r(k)$ on tuples of vectors. Moreover, the complex $\mathcal{C}_\bullet^r$ is isomorphic to the truncated complex $C(r,\bullet)_{\mathrm{GL}_r(k), \bullet \geq r}$.*

Proof. For $n \geq r$ there is a bijective correspondence between subspaces of codimension r in $\mathbb{P}^n(k)$ and $\mathrm{GL}_r(k)$-orbits on $(n+1)$-tuples $[v_0, \dots, v_n]$ of vectors in k^r satisfying the condition that v_i span k^r. It is given by

$$\begin{aligned} L \subset \mathbb{P}^n \;&\longmapsto\; [v_0, \dots, v_n]\,,\; v_i = \text{image of } e_i \text{ in } k^{n+1}/\widetilde{L} \cong k^r \\ [v_0, \dots, v_n] \;&\longmapsto\; \widetilde{L} = \mathrm{Ker}[v_0, \dots, v_n]^T \subset k^{n+1} \end{aligned}$$

where $\widetilde{L}$ is the unique lift of L to a linear subspace in k^{n+1} and $e_0, \dots, e_n$ is a standard basis in k^{n+1}.

An admissible point in $\mathbb{P}^r(k)$ is a point which doesn't belong to any of the $r+1$ hyperplanes $\{t_i = 0\}$, and for the corresponding vectors $[v_0, \dots, v_r]$ it means that every r of them are linearly independent. For $n > r$ a subspace L of codimension r in $\mathbb{P}^n(k)$ is admissible whenever all the intersections $L \cap \{t_i = 0\}$ are admissible in $\mathbb{P}^{n-1}(k)$. Hence it follows by induction that admissible subspaces correspond exactly to $\mathrm{GL}_r(k)$-orbits of tuples "in general position". Obviously, differential (1) is precisely (7) for tuples. $\square$

The tuples of vectors in general position in k^r modulo the diagonal action of $\mathrm{GL}_r(k)$ are called *configurations*, so $C(r,n)_{\mathrm{GL}_r(k)}$ is the free abelian group generated by configurations of $n+1$ vectors in k^r.

Proof of (iii) in Theorem 1. For brevity we denote $C(2,n)$ by C_n and $\mathrm{GL}_2(k)$ by G. Since the complex of G-modules $C_\bullet$ is quasi-isomorphic to $\mathbb{Z}$ we have the hypercohomology spectral sequence with $E^1_{pq} = H_q(G, C_p) \Rightarrow H_{p+q}(G, \mathbb{Z})$. Since all modules C_p with $p > 0$ are free we have $E^1_{pq} = 0$ for $p, q > 0$ and $E^1_{p0} = (C_p)_G$. If $G_1 \subset G$ is the stabilizer of $\binom{1}{0}$ then $E^1_{0q} = H_q(G, \mathbb{Z}[G/G_1]) =$

$H_q(G_1,\mathbb{Z})$ by Shapiro's lemma. We have $k^* \subset G_1$ and $H_q(k^*,\mathbb{Z}) = H_q(G_1,\mathbb{Z})$ (see Section 1 in [6]), so $E^1_{0q} = H_q(k^*,\mathbb{Z})$. Further, $E^2_{p0} = H_p((C_\bullet)_G)$ and $E^2_{0q} = H_q(k^*,\mathbb{Z})$. This spectral sequence degenerates on the second term. Indeed, the embedding

$$k^* \hookrightarrow G$$
$$\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

is split by determinant, and therefore all maps $H_q(k^*,\mathbb{Z}) \longrightarrow H_q(G,\mathbb{Z})$ are injective. Consequently, $E^\infty_{pq} = E^2_{pq}$ and for every $n \geq 2$ we have a short exact sequence

$$0 \longrightarrow H_n(k^*,\mathbb{Z}) \longrightarrow H_n(G,\mathbb{Z}) \longrightarrow H_n\big((C_\bullet)_G\big) \longrightarrow 0\,.$$

It follows from Lemma 1 that

$$H^2_n \;=\; H_n\big((C_\bullet)_G\big) \;=\; H_n(G,\mathbb{Z})/H_n(k^*,\mathbb{Z})\,, \qquad n \geq 3\,.$$

Let D_n be the free abelian group generated by $(n+1)$-tuples of distinct points in $\mathbb{P}^1(k)$. Again we have the differential like (7) on $D_\bullet$ and the augmented complex $D_\bullet \longrightarrow \mathbb{Z} \longrightarrow 0$ is acyclic. We have a surjective map from $C_\bullet$ to $D_\bullet$ since a non-zero vector in k^2 defines a point in $P^1(k)$ and the group action agrees. The spectral sequence $\widetilde{E}^1_{pq} = H_q(G, D_p) \Rightarrow H_{p+q}(G,\mathbb{Z})$ was considered in [5]. In particular, $\widetilde{E}^1_{p0} \;=\; (D_p)_G$ is the free abelian group generated by $(p-2)$-tuples of different points since G-orbit of every $(p+1)$-tuple contains a unique element of the form $(0,\infty,1,x_1,\ldots,x_{p-2})$, and the differential $d^1 : \widetilde{E}^1_{04} \longrightarrow \widetilde{E}^1_{03}$ is given by

$$[x,y] \mapsto [x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right]. \tag{8}$$

According to [5], terms $\widetilde{E}^2_{pq}$ with small indices are

$$\begin{array}{cccc}
H_3(k^* \oplus k^*) & & & \\
H_2(k^*) \oplus (k^* \otimes k^*)_\sigma & (k^* \otimes k^*)^\sigma & & \\
k^* & 0 & 0 & \\
\mathbb{Z} & 0 & 0 & \mathfrak{p}(k)
\end{array}$$

where $\mathfrak{p}(k)$ is the quotient of $\mathbb{Z}[k^* \smallsetminus \{1\}]$ by all 5-term relations as in right-hand side of (8), and the only non-trivial differential starting from $\mathfrak{p}(k)$ is

$$\begin{aligned} d^3 : \mathfrak{p}(k) &\longrightarrow H_2(k^*) \oplus (k^* \otimes k^*)_\sigma = \Lambda^2(k^*) \oplus (k^* \otimes k^*)_\sigma \\ [x] &\mapsto\; x \wedge (1-x) \,-\, x \otimes (1-x) \end{aligned}$$

Therefore $\widetilde{E}^4_{30} = \widetilde{E}^\infty_{30} = B(k)$ and we have a commutative triangle

$$\begin{tikzcd}
H_3(G) \arrow[r, two heads] \arrow[dr, two heads] & E^\infty_{30} = H^2_3 \arrow[d] \\
& \widetilde{E}^\infty_{30} = B(k)
\end{tikzcd}$$

where both maps from $H_3(G)$ are surjective, hence the vertical arrow is also surjective. It remains to check that the vertical arrow coincides with λ_*. A line ℓ in $\mathbb{P}^3(k)$ is given by two linear equations and for an admissible line it is always possible to chose them in the form

$$\begin{cases} t_0 \qquad + x_1 t_2 + x_2 t_3 \; = \; 0\,, \\ \qquad t_1 + y_1 t_2 + y_2 t_3 \; = \; 0\,. \end{cases}$$

This line corresponds to the tuple of vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

which can be mapped to the points $0, \infty, 1, \frac{x_1 y_2}{y_1 x_2}$ in $\mathbb{P}^1(k)$, hence the vertical arrow maps it to $[\frac{x_1 y_2}{y_1 x_2}]$ (actually we need to consider a linear combination of lines which vanishes under d but for every line the result is given by this expression). On the other hand, four points of its intersection with the hyperplanes are

$$\begin{aligned} P_0 &= (0 : y_1 x_2 - y_2 x_1 : -x_2 : x_1) \\ P_1 &= (y_2 x_1 - y_1 x_2 : 0 : -y_2 : y_1) \\ P_2 &= (-x_2 : -y_2 : 0 : 1) \\ P_3 &= (-x_1 : -y_1 : 1 : 0) \end{aligned}$$

and if we represent every point on ℓ as $\alpha P_0 + \beta P_1$ then the corresponding ratios $\frac{\beta}{\alpha}$ will be $0, \infty, -\frac{x_2}{y_2}, -\frac{x_1}{y_1}$. Hence $\lambda(\ell) = \frac{x_1 y_2}{y_1 x_2}$ again and (iii) follows.

To prove (iv) we first observe that the Hochschild-Serre spectral sequence associated to

$$1 \longrightarrow \mathrm{SL}_2(k) \longrightarrow \mathrm{GL}_2(k) \stackrel{\det}{\longrightarrow} k^* \longrightarrow 1$$

gives a short exact sequence

$$\begin{aligned} 1 \longrightarrow H_0\Big(k^*, H_3(\mathrm{SL}_2(k), \mathbb{Z})\Big) \longrightarrow \mathrm{Ker}\Big(H_3(\mathrm{GL}_2(k), \mathbb{Z}) \stackrel{\det}{\longrightarrow} H_3(k^*, \mathbb{Z})\Big) \\ \longrightarrow H_1\Big(k^*, H_2(\mathrm{SL}_2(k), \mathbb{Z})\Big) \longrightarrow 1. \end{aligned} \tag{9}$$

The first term here maps surjectively to $K_3^{\mathrm{ind}}(k)$ (see the last section of [2]), and the map is conjectured by Suslin to be an isomorphism (see Sah [4]). It is known that its kernel is at worst 2-torsion (see Mirzaii [3]).

Thus we let

$$T(k) := \mathrm{Ker}\left(H_0(k^*, H_3(\mathrm{SL}_2(k), \mathbb{Z})) \longrightarrow K_3^{\mathrm{ind}}(k)\right).$$

By the preceeding remarks, this is a 2-torsion abelian group. Since the embedding $k^* \longrightarrow \mathrm{GL}_2(k)$ is split by the determinant, the middle term in (9) is isomorphic to H_3^2. Then applying the snake lemma to the diagram

$$\begin{tikzcd} 0 \arrow[r] & T(k) \arrow[r] \arrow[d, hook] & H_0\Big(k^*, H_3(\mathrm{SL}_2(k), \mathbb{Z})\Big) \arrow[r] \arrow[d, hook] & K_3^{\mathrm{ind}}(k) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \arrow[r] & H_2^3 \arrow[r] & B(k) \arrow[r] & 0 \end{tikzcd}$$

gives the short exact sequence

$$0 \longrightarrow \mathrm{Tor}(k^*, k^*)^{\sim} \longrightarrow K/T(k) \longrightarrow H_1\Big(k^*, H_2(\mathrm{SL}_2(k), \mathbb{Z})\Big) \longrightarrow 0.$$

Finally, it follows from [2] that there is a natural short exact sequence

$$0 \longrightarrow H_1\Big(k^*, H_2(\mathrm{SL}_2(k), \mathbb{Z})\Big) \longrightarrow k^* \otimes K_2^M(k) \longrightarrow K_3^M(k)/2 \longrightarrow 0\,.$$

This proves (4). $\square$

References

[1] A. Borel, *Cohomologie de SL_n et valeurs de fonctions zêta aux points entiers*, Ann. Scuola Norm. Sup. Pisa Cl. Sc., Sér. 4, no. 4 (1977), 613-636.

[2] K. Hutchinson, L. Tao, *The third homology of the special linear group of a field*, J. Pure. Appl. Algebra 213 (2009), no. 9, 1665–1680

[3] B. Mirzaii. Third homology of general linear groups. *J. Algebra*, 320(5) (2008), 1851–1877.

[4] Chih-Han Sah. Homology of classical Lie groups made discrete. III. *J. Pure Appl. Algebra*, 56(3):269–312, 1989.

[5] A.Suslin, *K_3 of a field and the Bloch group*, Proceedings of the Steklov Institute of Mathematics 1991, Issue 4

[6] A.Suslin, *The homology of* GL_n*, characteristic classes and Milnor K-theory*, Proceedings of the Steklov Institute of Mathematics 1985, Issue 3

[7] B.Totaro, *Milnor K-Theory is the Simplest Part of Algebraic K-theory*, K-theory 6 (1992), 177–189.