

QUADRATIC ALGEBRAS, YANG-BAXTER EQUATION, AND ARTIN-SCHELTER REGULARITY

TATIANA GATEVA-IVANOVA

ABSTRACT. We study quadratic algebras over a field \mathbf{k} . We show that an n -generated PBW algebra A has finite global dimension and polynomial growth iff its Hilbert series is $H_A(z) = 1/(1-z)^n$. Surprising amount can be said when the algebra A has *quantum binomial relations*, that is the defining relations are nondegenerate square-free binomials $xy - c_{xy}zt$ with non-zero coefficients $c_{xy} \in \mathbf{k}$. In this case various good algebraic and homological properties are closely related. The main result shows that for an n -generated quantum binomial algebra A the following conditions are equivalent: (i) A is a PBW algebra with finite global dimension; (ii) A is PBW and has polynomial growth; (iii) A is an Artin-Schelter regular PBW algebra; (iv) A is a Yang-Baxter algebra; (v) $H_A(z) = 1/(1-z)^n$; (vi) The dual $A^!$ is a quantum Grassman algebra; (vii) A is a binomial skew polynomial ring. So for quantum binomial algebras the problem of classification of Artin-Schelter regular PBW algebras of global dimension n is equivalent to the classification of square-free set-theoretic solutions of the Yang-Baxter equation (X, r) , on sets X of order n .

1. INTRODUCTION

We work with quadratic algebras A over a ground field \mathbf{k} . Following a classical tradition (and recent trend), we take a combinatorial approach to study A . The properties of A will be read off a presentation $A = \mathbf{k}\langle X \rangle / (\mathfrak{R})$, where X is a finite set of generators of degree 1, $|X| = n$, $\mathbf{k}\langle X \rangle$ is the unitary free associative algebra generated by X , and (\mathfrak{R}) is the two-sided ideal of relations, generated by a *finite* set \mathfrak{R} of homogeneous polynomials of degree two. Clearly A is a connected graded \mathbf{k} -algebra (naturally graded by length) $A = \bigoplus_{i \geq 0} A_i$, where $A_0 = \mathbf{k}$, A is generated by $A_1 = \text{Span}_{\mathbf{k}} X$, so each A_i is finite dimensional.

A quadratic algebra A is a *PBW algebra* if there exists an enumeration of X , $X = \{x_1, \dots, x_n\}$ such that the quadratic relations \mathfrak{R} form a (noncommutative) Gröbner basis with respect to the degree-lexicographic ordering on $\langle X \rangle$ induced from $x_1 < x_2 < \dots < x_n$. In this case the set of normal monomials (mod \mathfrak{R}) forms a \mathbf{k} -basis of A called a *PBW basis* and x_1, \dots, x_n (taken exactly with this enumeration) are called *PBW-generators of A* . The notion of a PBW algebra was introduced by Priddy, [38], his *PBW basis* is a generalization of the classical Poincaré-Birkhoff-Witt basis for the universal enveloping of a finite dimensional Lie algebra. PBW algebras form an important class of Koszul algebras. The interested

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reader can find information on PBW algebras and more references in [37]. One of the central problems that we consider is

the classification of Artin-Schelter regular PBW algebras.

It is far from its final resolution. The first question to be asked is

What can be said about PBW algebras with polynomial growth and finite global dimension?

We find that, surprisingly, the class \mathfrak{C}_n of n -generated PBW algebras with polynomial growth and finite global dimension is determined uniquely by its Hilbert series, this is in section 3.

Theorem 1.1. *Let $A = \mathbf{k}\langle X \rangle / (\mathfrak{R})$ be a quadratic PBW algebra, where $X = \{x_1, x_2, \dots, x_n\}$ is a set of PBW generators. The following are equivalent*

- (1) *A has polynomial growth and finite global dimension.*
- (2) *A has exactly $\binom{n}{2}$ relations and finite global dimension.*
- (3) *The Hilbert series of A is*

$$H_A(z) = \frac{1}{(1-z)^n}.$$

- (4) *There exists a permutation y_1, \dots, y_n of $x_1 \cdots x_n$, such that the set*

$$(1.1) \quad \mathcal{N} = \{y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} \mid \alpha_i \geq 0 \text{ for } 1 \leq i \leq n\}$$

is a \mathbf{k} -basis of A.

Furthermore the class \mathfrak{C}_n of all n -generated PBW algebras with polynomial growth and finite global dimension contains a unique (up to isomorphism) monomial algebra:

$$A^0 = \langle x_1, \dots, x_n \rangle / (x_j x_i \mid 1 \leq i < j \leq n).$$

Note that y_1, y_2, \dots, y_n is possibly a "new" enumeration of X , which induces a degree-lexicographic ordering \prec on $\langle X \rangle$ (with $y_1 \prec y_2 \prec \cdots \prec y_n$) different from the original ordering. The defining relations remain the same, but their leading terms w.r.t. \prec may be different from the original ones, and y_1, y_2, \dots, y_n are not necessarily PBW generators of A . In the terminology of Gröbner bases, \mathcal{N} is not necessarily a *normal basis* of A w.r.t. \prec .

A class of PBW Artin Schelter regular rings of arbitrarily high global dimension n , were introduced and studied in [15], [25], [18], [17]. These are *the binomial skew-polynomial rings*. It was shown in [25] that they are also closely related to the set-theoretic solutions of the Yang-Baxter equation. So we consider the so-called *quantum binomial algebras* introduced and studied in [17], [21]. These are quadratic algebras (not necessarily PBW) with square-free non-degenerate binomial relations, see Definition 2.7. The second question that we ask in the paper is

Which are the PBW Artin Schelter regular algebras in the class of quantum binomial algebras?

We prove that each quantum binomial PBW algebra with finite global dimension is a Yang-Baxter algebra, and therefore a binomial skew-polynomial ring. This implies that *in the class of quantum binomial algebras* the three notions: an Artin-Schelter regular PBW algebra, a binomial skew-polynomial ring, and a Yang-Baxter algebra (in the sense of Manin) are equivalent. The following result is proven in Section 5.

Theorem 1.2. *Let $A = \mathbf{k}\langle X \rangle / (\mathfrak{R})$ be a quantum binomial algebra. The following conditions are equivalent.*

- (1) *A is a PBW algebra with finite global dimension.*
- (2) *A is a PBW algebra with polynomial growth.*
- (3) *A is an Artin-Schelter regular PBW algebra.*
- (4) *A is a Yang-Baxter algebra, that is the set of relations \mathfrak{R} defines canonically a solution of the Yang-Baxter equation.*
- (5) *A is a binomial skew polynomial ring, with respect to some appropriate enumeration of X .*

(6)

$$\dim_{\mathbf{k}} A_3 = \binom{n+2}{3}, \quad \text{or equivalently} \quad \dim_{\mathbf{k}} A_3^! = \binom{n}{3}.$$

(7)

$$H_A(z) = \frac{1}{(1-z)^n}$$

(8) *The Koszul dual $A^!$ is a quantum Grassman algebra.*

Each of these condition implies that A is Koszul and a Noetherian domain.

It follows from Theorem 1.2 that

The problem of classification of Artin Schelter regular PBW algebras with quantum binomial relations and global dimension n is equivalent to the classification of square-free set-theoretic solutions of YBE, (X, r) , on sets X of order n .

Even under these strong restrictions on the shape of the relations, the problem is highly nontrivial. However for reasonably small n (say $n \leq 10$) the square-free solutions of YBE (X, r) are known. A possible classification for general n can be based on the so called *multipermutation level* of the solutions, see [23].

The paper is organized as follows.

In section 2 are recalled basic definitions, and some facts used throughout the paper. In section 3 we study the general case of PBW algebras with finite global dimension and polynomial growth and prove Theorem 1.1. The approach is combinatorial. To each PBW algebra A we associate two finite oriented graphs. The first is *the graph of normal words* $\Gamma_{\mathbf{N}}$ (this is a particular case of the Ufnarovski graph [42]), it determines the growth and the Hilbert series of A . The second is *the graph of obstructions*, $\Gamma_{\mathbf{W}}$, dual to $\Gamma_{\mathbf{N}}$. We define it via the set of obstructions (in the sense of Anick, [1], [2]), it gives a precise information about *the global dimension* of the algebra A . We prove that all algebras in the class \mathfrak{C}_n of n -generated PBW algebras determine a unique (up to isomorphism) graph of obstructions $\Gamma_{\mathbf{W}}$, which is the complete oriented graph K_n with no cycles. The two graphs play important role in the proof of Theorem 1.1. They can be used whenever PBW algebras are studied.

In section 4 we find some interesting combinatorial results on quantum binomial sets (X, r) and the corresponding quadratic algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$. We study the action of the infinite Dihedral group, $\mathcal{D} = \mathcal{D}(r)$, associated with r on X^3 and find some *counting formulae* for the \mathcal{D} -orbits. These are used to show that (X, r) is a set-theoretic solution of the Yang-Baxter equation *iff* $\dim_{\mathbf{k}} \mathcal{A}_3 = \binom{n+2}{3}$.

In section 5 we prove Theorem 1.2. The proof involves the results of sections 3, 4, and results on binomial skew polynomial rings and set-theoretic solutions of YBE from [25], [18], [17], [21].

2. PRELIMINARIES - SOME DEFINITIONS AND FACTS

In this section we recall basic notions and results which will be used in the paper. This paper is a natural continuation of [17]. We shall use the terminology, notation and results from our previous works [15, 25, 17, 18, 21]. The reader acquainted with these can proceed to the next section.

A connected graded algebra is called *Artin-Schelter regular* (or *AS regular*) if

- (i) A has *finite global dimension* d , that is, each graded A -module has a free resolution of length at most d .
- (ii) A has *finite Gelfand-Kirillov dimension*, meaning that the integer-valued function $i \mapsto \dim_{\mathbf{k}} A_i$ is bounded by a polynomial in i .
- (iii) A is *Gorenstein*, that is, $\text{Ext}_A^i(\mathbf{k}, A) = 0$ for $i \neq d$ and $\text{Ext}_A^d(\mathbf{k}, A) \cong \mathbf{k}$.

AS regular algebras were introduced and studied first in [3], [4], [5]. When $d \leq 3$ all regular algebras are classified. Since then AS regular algebras and their geometry are intensively studied. The problem of classification of regular rings is difficult and remains open even for regular rings of global dimension 4. The study of Artin-Schelter regular rings, their classification, and finding new classes of such rings is one of the basic problems for noncommutative geometry. Numerous works on this topic appeared during the last two decades, see for example [4, 5], [7], [28], [43], [30], et all.

A class of PBW AS regular algebras of global dimension n was introduced and studied in [15], [25] [18], [17]. These are *the binomial skew-polynomial rings*.

Definition 2.1. [15] A *binomial skew polynomial ring* is a quadratic algebra $A = \mathbf{k}\langle x_1, \dots, x_n \rangle / (\mathfrak{R})$ with precisely $\binom{n}{2}$ defining relations

$$\mathfrak{R} = \{x_j x_i - c_{ij} x_{i'} x_{j'}\}_{1 \leq i < j \leq n}$$

such that

- (a) $c_{ij} \in \mathbf{k}^\times$.
- (b) For every pair i, j , $1 \leq i < j \leq n$, the relation $x_j x_i - c_{ij} x_{i'} x_{j'} \in \mathfrak{R}$, satisfies $j > i'$, $i' < j'$;
- (c) Every ordered monomial $x_i x_j$, with $1 \leq i < j \leq n$ occurs in the right hand side of some relation in \mathfrak{R} ;
- (d) \mathfrak{R} is the *reduced Gröbner basis* of the two-sided ideal (\mathfrak{R}) , with respect to the order \prec on $\langle X \rangle$, or equivalently the ambiguities $x_k x_j x_i$, with $k > j > i$ do not give rise to new relations in A .

We say that \mathfrak{R} are *relations of skew-polynomial type* if conditions 2.1 (a), (b) and (c) are satisfied (we do not assume (d)).

By [6] condition 2.1 (d) may be rephrased by saying that *the set of ordered monomials*

$$\mathcal{N}_0 = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_n \geq 0 \text{ for } 1 \leq i \leq n\}$$

is a \mathbf{k} -basis of A .

Remark 2.2. In the terminology of this paper a *binomial skew polynomial ring* is a quadratic PBW algebra A with PBW generators $x_1 \cdots, x_n$ and relations of skew-polynomial type.

More generally, we will consider a class of quadratic algebras with binomial relations, called *quantum binomial algebras*, these are not necessarily PBW algebras.

We need to recall first the notions of a quadratic set and the associated with it quadratic algebras.

Definition 2.3. Let X be a nonempty set and let $r : X \times X \rightarrow X \times X$ be a bijective map. In this case we shall use notation (X, r) and refer to it as a *quadratic set*. We present the image of (x, y) under r as

$$(2.1) \quad r(x, y) = ({}^x y, x^y).$$

The formula (2.1) defines a “left action” $\mathcal{L} : X \times X \rightarrow X$, and a “right action” $\mathcal{R} : X \times X \rightarrow X$, on X as:

$$(2.2) \quad \mathcal{L}_x(y) = {}^x y, \quad \mathcal{R}_y(x) = x^y,$$

for all $x, y \in X$. r is *nondegenerate* if the maps \mathcal{L}_x and \mathcal{R}_x are bijective for each $x \in X$. r is *involutive* if $r^2 = id_{X \times X}$.

As a notational tool, we shall often identify the sets $X^{\times k}$ of ordered k -tuples, $k \geq 2$, and X^k , the set of all monomials of length k in the free monoid $\langle X \rangle$.

As in [18, 20, 21, 22, 23], to each quadratic set (X, r) we associate canonically algebraic objects (see Definition 2.4) generated by X and with quadratic defining relations $\mathfrak{R}_0 = \mathfrak{R}_0(r)$ naturally determined as

$$(2.3) \quad \begin{array}{l} xy = y'x' \in \mathfrak{R}_0(r), \quad \text{whenever} \\ r(x, y) = (y', x') \quad \text{and} \quad (x, y) \neq (y', x') \text{ hold in } X \times X. \end{array}$$

We can tell the precise number of defining relations, whenever (X, r) is nondegenerate and involutive, with $|X| = n$. In this case the nondegeneracy implies that $\mathfrak{R}(r)$, consists of precisely $\binom{n}{2}$ quadratic relations (see [19] Proposition 2.3).

Definition 2.4. [18, 21] Assume that $r : X^2 \rightarrow X^2$ is a bijective map.

(i) The monoid

$$S = S(X, r) = \langle X; \mathfrak{R}_0(r) \rangle,$$

with a set of generators X and a set of defining relations $\mathfrak{R}(r)$, is called *the monoid associated with (X, r)* . The *group $G = G(X, r)$ associated with (X, r)* is defined analogously.

(ii) For arbitrary fixed field \mathbf{k} , *the \mathbf{k} -algebra associated with (X, r)* is defined as

$$(2.4) \quad \mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) = \mathbf{k}\langle X; \mathfrak{R}_0(r) \rangle \simeq \mathbf{k}\langle X \rangle / (\mathfrak{R}),$$

where $\mathfrak{R} = \mathfrak{R}(r)$ is the set of quadratic *binomial relations*

$$(2.5) \quad \mathfrak{R} = \{xy - y'x' \mid xy = y'x' \in \mathfrak{R}_0(r)\}.$$

Clearly \mathcal{A} is a *quadratic algebra*, generated by X and with defining relations $\mathfrak{R}(r)$. Furthermore, \mathcal{A} is isomorphic to the monoid algebra $\mathbf{k}S(X, r)$. In many cases the associated algebra will be *standard finitely presented* with respect to the degree-lexicographic ordering induced by an appropriate enumeration of X , that is a PBW algebra. It is known in particular, that the algebra $\mathcal{A}(\mathbf{k}, X, r)$ has remarkable algebraic and homological properties when r is involutive, nondegenerate and obeys the braid or Yang-Baxter equation in $X \times X \times X$. Set-theoretic solutions were introduced in [11, 44] and have been under intensive study during the last decade. There are numerous works on set-theoretic solutions and related structures, of which a relevant selection for the interested reader is [44, 25, 39, 12, 29, 18, 17, 40, 45, 21, 22, 23].

Definition 2.5. Let (X, r) be a quadratic set.

- (1) (X, r) is said to be *square-free* if $r(x, x) = (x, x)$ for all $x \in X$.
- (2) (X, r) is called a *quantum binomial set* if it is nondegenerate, involutive and square-free.
- (3) (X, r) is a *set-theoretic solution of the Yang-Baxter equation* (YBE) if the braid relation

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$$

holds in $X \times X \times X$. In this case (X, r) is also called a *braided set*. If in addition r is involutive (X, r) is called a *symmetric set*.

Example 2.6. $X = \{x_1, x_2, x_3, x_4, x_5\}$ and r is defined via the actions $\mathcal{L}, \mathcal{R} = (\mathcal{L})^{-1} \in \text{Sym}(X)$ as

$$(2.6) \quad \begin{aligned} r(x, y) &= (\mathcal{L}_x(y), (\mathcal{L}_y)^{-1}(x)) \quad \text{where} \\ \mathcal{L}_{x_1} &= \mathcal{L}_{x_3} = (x_2x_4) \quad \mathcal{L}_{x_2} = \mathcal{L}_{x_4} = (x_1x_3) \\ \mathcal{L}_{x_5} &= (x_1x_2x_3x_4). \end{aligned}$$

This is a (square-free) symmetric set. (It has multipermutation level 3, see [23]). Presenting the solution (X, r) via the left and the right actions is an elegant and convenient way to express the corresponding $\binom{n}{2}$ quadratic relations $\mathfrak{R}(r)$ of the algebra $\mathcal{A}(\mathbf{k}, X, r)$, especially when n is large. We recommend the reader to write down explicitly the ten quadratic relations encoded in (2.6). Enumerated this way, x_1, \dots, x_5 are not PBW generators. If we reorder the generators as

$$y_1 = x_1 \prec y_2 = x_3 \prec y_3 = x_2 \prec y_4 = x_4 \prec y_5 = x_5,$$

then y_1, \dots, y_5 are PBW generators of \mathcal{A} , and \mathcal{A} is a binomial skew-polynomial ring w.r.t this new enumeration.

Definition 2.7. A quadratic algebra $A(\mathbf{k}, X, \mathfrak{R}) = \mathbf{k}\langle X \rangle / (\mathfrak{R})$ is a *quantum binomial algebra* if the relations \mathfrak{R} satisfy the following conditions

- (a) Each relation in \mathfrak{R} is of the shape

$$(2.7) \quad xy - c_{yx}y'x', \quad \text{where } x, y, x', y' \in X, \quad \text{and } c_{xy} \in \mathbf{k}^\times$$

(this is what we call a *binomial relation*).

- (b) Each $xy, x \neq y$ of length 2 occurs at most once in \mathfrak{R} .
- (c) Each relation is *square-free*, i.e. it does not contain a monomial of the shape $xx, x \in X$.
- (d) The relations \mathfrak{R} are *non degenerate*, i.e. the canonical bijection $r = r(\mathfrak{R}) : X \times X \rightarrow X \times X$, associated with \mathfrak{R} , is non degenerate.

Relations satisfying conditions (a)-(d) are called *quantum binomial relations*.

Clearly, each binomial skew-polynomial ring is a PBW quantum binomial algebra. The algebra $\mathcal{A}(\mathbf{k}, X, r)$ from the Example 2.6 is a concrete quantum binomial algebra. See more examples at the end of the section. We recall that (although this is not part of the definition) every n -generated quantum binomial algebra has exactly $\binom{n}{2}$ relations.

With each quantum binomial algebra we associate two maps determined canonically via its relations.

Definition 2.8. Let $\mathfrak{R} \subset \mathbf{k}\langle X \rangle$ be a set of quadratic binomial relations, satisfying conditions (a) and (b). Let $V = \text{Span}_{\mathbf{k}} X$.

The canonically *associated quadratic set* (X, r) , with $r = r(\mathfrak{R}) : X \times X \longrightarrow X \times X$ is defined as

$$r(x, y) = (y', x'), \text{ and } r(y', x') = (x, y),$$

if $xy - c_{xy}y'x' \in \mathfrak{R}$. If xy does not occur in any relation ($x = y$ is possible) then we set

$$r(x, y) = (x, y).$$

The *automorphism associated with* \mathfrak{R} , $R = R(\mathfrak{R}) : V^{\otimes 2} \longrightarrow V^{\otimes 2}$, is defined analogously. If $xy - c_{xy}y'x' \in \mathfrak{R}$, we set

$$R(x \otimes y) = c_{xy}y' \otimes x', \text{ and } R(y' \otimes x') = (c_{xy})^{-1}x \otimes y.$$

If xy does not occur in any relation we set

$$R(x \otimes y) = x \otimes y.$$

R is called *non-degenerate* if r is non-degenerate. In this case we shall also say that the defining relations \mathfrak{R} are *non degenerate binomial relations*.

Let V be a \mathbf{k} -vector space. A linear automorphism R of $V \otimes V$ is a *solution of the Yang-Baxter equation*, (YBE) if the equality

$$(2.8) \quad R^{12}R^{23}R^{12} = R^{23}R^{12}R^{23}$$

holds in the automorphism group of $V \otimes V \otimes V$, where R^{ij} means R acting on the i -th and j -th component.

A quantum binomial algebra $A = \mathbf{k}\langle X; \mathfrak{R} \rangle$, is a *Yang-Baxter algebra*, in the sense of Manin [35], if the associated map $R = R(\mathfrak{R})$ is a solution of the Yang-Baxter equation.

It was shown in [25] that each binomial skew polynomial ring is a Yang-Baxter algebra.

The results below can be extracted from [25], [15], and [17], Theorem B.

Fact 2.9. *Let $A = \mathbf{k}\langle X \rangle / (\mathfrak{R})$ be a quantum binomial algebra. Then the following two conditions are equivalent.*

- (1) *A is a binomial skew polynomial ring, with respect to some appropriate enumeration of X .*
- (2) *The automorphism $R = R(\mathfrak{R}) : V^{\otimes 2} \longrightarrow V^{\otimes 2}$ is a solution of the Yang-Baxter equation, so A is a Yang-Baxter algebra.*

Each of these conditions implies that A is an Artin-Schelter regular PBW algebra. Furthermore A is a left and right Noetherian domain.

We shall prove in Section 5 that conversely, in the class of quantum binomial algebras each Artin-Schelter regular PBW algebra defines canonically a solution of the YBE, and therefore is a Yang-Baxter algebra and a binomial skew-polynomial ring.

We end up the section with two concrete examples of quantum binomial algebras with 4 generators.

Example 2.10. Let $A = \mathbf{k}\langle x, y, z, t \rangle / (\mathfrak{R})$, where $X = \{x, y, z, t\}$, and

$$\mathfrak{R} = \{xy - zt, \quad ty - zx, \quad xz - yx, \quad tz - yt, \quad xt - tx, \quad yz - zy\}.$$

Clearly, the relations are square-free, a direct verification shows that they are non-degenerate. so A is a quantum binomial algebra. More sophisticated proof shows that the set of relations \mathfrak{R} is not a Gröbner basis w.r.t. deg-lex ordering coming

from any order (enumeration) of the set X . This example is studied with details in Section 4.

Example 2.11. Let $A = \mathbf{k}\langle X \rangle / (\mathfrak{R})$, where $X = \{x, y, z, t\}$, and

$$\mathfrak{R} = \{xy - zt, ty - zx, xz - yt, tz - yx, xt - tx, yz - zy\}.$$

We fix $t > x > z > y$, and take the corresponding deg-lex ordering on $\langle X \rangle$. Then direct verification shows that \mathfrak{R} is a Gröbner basis. To do this one has to show that the ambiguities txz , txy , tzy , xzy are solvable. In this case the set

$$\mathcal{N} = \{y^\alpha z^\beta x^\gamma t^\delta \mid \alpha, \beta, \gamma, \delta \geq 0\}$$

is the normal basis of A , (mod \mathfrak{R}).

Note that any order in which $\{t, x, \}$ $>$ $\{z, y, \}$, or $\{z, y, \}$ $>$ $\{t, x, \}$, makes A a PBW algebra, there are exactly eight such enumerations of X .

Furthermore, A is a Yang-Baxter algebra, and an AS-regular domain of global dimension 4.

3. PBW ALGEBRAS WITH POLYNOMIAL GROWTH AND FINITE GLOBAL DIMENSION

Let $X = \{x_1, \dots, x_n\}$. As usual, we fix the deg-lex ordering $<$ on $\langle X \rangle$. Each element $g \in \mathbf{k}\langle X \rangle$ has the shape $g = cu + h$, where $u \in \langle X \rangle$, $c \in \mathbf{k}^\times$, $h \in \langle X \rangle$, and either $h = 0$, or $h = \sum_\alpha c_\alpha u_\alpha$ is a linear combination of monomials $u_\alpha < u$. u is called the leading monomial of g (w.r.t. $<$) and denoted $LM(g)$. Every finitely presented graded algebra, $A = \mathbf{k}\langle X \rangle / I$, where I is a finitely generated ideal of $\mathbf{k}\langle X \rangle$ has an uniquely determined reduced Gröbner basis \mathbf{G} . In general, \mathbf{G} is infinite. Anick introduced *the set of obstructions* \mathbf{W} for a connected graded algebra, see [2]. It is easy to deduce from his definition that the set of obstructions \mathbf{W} is exactly *the set of leading monomials* of the elements of the reduced Gröbner basis \mathbf{G} , that is $\mathbf{W} = \{LM(g) \mid g \in \mathbf{G}\}$. The obstructions are used to construct a free resolution of the field \mathbf{k} considered as an A -module, [2].

Consider now a PBW algebra $A = \mathbf{k}\langle X \rangle / (\mathfrak{R})$ with PBW generators $X = \{x_1, \dots, x_n\}$. It follows from the definition of a PBW algebra that the relations \mathfrak{R} is exactly the reduced Gröbner basis of the ideal (\mathfrak{R}) . Hence, in this case, *the set of obstructions* \mathbf{W} is simply the set of leading monomials of the defining relations.

$$\mathbf{W} = \{LM(f) \mid f \in \mathfrak{R}\}.$$

Then $\mathbf{N} = X^2 \setminus \mathbf{W}$ is the set of *normal monomials* (mod \mathbf{W}) of length 2.

Notation 3.1. We set

$$\mathbf{N}^{(0)} = \{1\}, \quad \mathbf{N}^{(1)} = X,$$

$$\mathbf{N}^{(m)} = \{x_{i_1} x_{i_2} \cdots x_{i_m} \mid x_{i_k} x_{i_{k+1}} \in \mathbf{N}, \quad 1 \leq k \leq m-1\}, \quad m = 2, 3, \dots$$

$$\mathbf{N}^\infty = \bigcup_{m \geq 0} \mathbf{N}^{(m)}.$$

Note that the set $\mathbf{N}^{(m)}$, $m \geq 0$ is a \mathbf{k} -basis of A_m , so \mathbf{N}^∞ is the set of all normal (mod \mathbf{W}) words in $\langle X \rangle$. It is well-known that the set \mathbf{N}^∞ project to a basis of A . More precisely, the free associative algebra $\mathbf{k}\langle X \rangle$ splits into a direct sum of subspaces

$$\mathbf{k}\langle X \rangle \simeq \text{Span}_{\mathbf{k}} \mathbf{N}^\infty \oplus I.$$

So there are isomorphisms of vector spaces

$$A \simeq \text{Span}_{\mathbf{k}} \mathbf{N}^\infty$$

$$A_m \simeq \text{Span}_{\mathbf{k}} \mathbf{N}^{(m)}, \dim A_m = |\mathbf{N}^{(m)}|, m = 0, 1, 2, 3, \dots$$

For a PBW algebra A there is a canonically associated monomial algebra $A^0 = \mathbf{k}\langle X \rangle / (\mathbf{W})$. As a monomial algebra, A^0 is also PBW. Both algebras A and A^0 have the same set of obstructions \mathbf{W} and therefore they have the same *normal basis* \mathbf{N}^∞ , the same Hilbert series and the same growth. It follows from results of Anick that $gl.\dim A = gl.\dim A^0$. More generally, the sets of obstructions \mathbf{W} determines uniquely the Hilbert series, growth and the global dimension for the whole family of PBW algebras A sharing the same W . The binomial skew polynomial rings are an well known example of PBW algebras with polynomial growth and finite global dimension, moreover they are AS regular Noetherian domains, see [25].

Let $M \subset X^2$ be a set of monomials of length 2. We define *the graph* Γ_M *corresponding to* M as follows.

Definition 3.2. Γ_M is a directed graph with a set of vertices $V(\Gamma_M) = X$ and a set of directed edges (arrows) $E = E(\Gamma_M)$ defined as follows

$$x \longrightarrow y \in E \text{ iff } x, y \in X, \text{ and } xy \in M.$$

We recall that *the order* of a graph Γ is the number of its vertices, i.e. $|V(\Gamma)|$, so Γ_M is a graph of order $|X|$. A path of length $k - 1$ in Γ_M is a sequence of edges $v_1 \longrightarrow v_2 \longrightarrow \dots \longrightarrow v_k$, where $v_i \longrightarrow v_{i+1} \in E$.

A *cycle (of length k)* in Γ is a path of the shape $v_1 \longrightarrow v_2 \longrightarrow \dots \longrightarrow v_k \longrightarrow v_1$ where v_1, \dots, v_k are distinct vertices. A *loop* is a cycle of length 0, $x \longrightarrow x$. So the graph Γ_M contains a loop $x \longrightarrow x$ whenever $xx \in M$, and a cycle of length two $x \longrightarrow y \longrightarrow x$, whenever $xy, yx \in M$. In this case $x \longrightarrow y, y \longleftarrow x$ are called bidirected edges. Note that, following the terminology in graph theory, *we make difference between directed and oriented graphs*. A directed graph having no symmetric pair of directed edges (i.e., no pairs $x \longrightarrow y$ and $y \longrightarrow x$) is known as *an oriented graph*. An oriented graph with no cycles is called *an acyclic oriented graph*. In particular, such a graph has no loops.

Denote by \overline{M} the complement $X^2 \setminus M$. Then the graph $\Gamma_{\overline{M}}$ is *dual to* Γ_M in the sense that

$$x \longrightarrow y \in E(\Gamma_{\overline{M}}) \text{ iff } x \longrightarrow y \text{ is not an edge of } \Gamma_M.$$

Let A be a PBW algebra, let \mathbf{W} and \mathbf{N} be the set of obstructions, and the set of normal monomials of length 2, respectively. Then the graph $\Gamma_{\mathbf{N}}$ gives complete information about the growth of A , while the global dimension of A , can be read of $\Gamma_{\mathbf{W}}$.

The graph of normal words of A , $\Gamma_{\mathbf{N}}$ was introduced in more general context by Ufnarovski [42].

Note that, in general, $\Gamma_{\mathbf{N}}$ is a directed graph which may contain pairs of edges, $x \longrightarrow y, y \longrightarrow x$, so $\Gamma_{\mathbf{N}}$ is *not necessarily an oriented graph*.

The following is a particular case of a more general result of Ufnarovski.

Fact 3.3. [42] *For every $m \geq 1$ there is a one-to-one correspondence between the set $\mathbf{N}^{(m)}$ of normal words of length m and the set of paths of length $m - 1$ in the graph $\Gamma_{\mathbf{N}}$. The path $a_1 \longrightarrow a_2 \longrightarrow a_3 \longrightarrow \dots \longrightarrow a_m$ (these are not necessarily*

(distinct vertices) corresponds to the word $a_1 a_2 \cdots a_m \in \mathbf{N}^{(m)}$. The algebra A has polynomial growth of degree m iff

- (i) The graph $\Gamma_{\mathbf{N}}$ has no intersecting cycles, and
- (ii) m is the largest number of (oriented) cycles occurring in a path of $\Gamma_{\mathbf{N}}$.

Example 3.4. All binomial skew-polynomial algebras A with five PBW generators x_1, x_2, \dots, x_5 have the same graph $\Gamma_{\mathbf{N}}$ as in Figure 1. The graph of obstruction $\Gamma_{\mathbf{W}}$ for A can be seen in Figure 2. The Koszul dual $A^!$ has corresponding graph of normal words $\Gamma_{\mathbf{N}^!}$ represented in Figure 3. The graphs in Figure 2 and Figure 3 are acyclic tournaments, see Definition 3.6.

The graph $\Gamma_{\mathbf{W}}$ is dual to $\Gamma_{\mathbf{N}}$, i.e. $x \rightarrow y \in E(\mathbf{W})$ iff $x \rightarrow y$ is not an edge in $\Gamma_{\mathbf{N}}$. Similarly to $\Gamma_{\mathbf{N}}$, $\Gamma_{\mathbf{W}}$ is a directed graph and, in general, may contain pairs of edges, $x \rightarrow y, y \rightarrow x$ or loops $x \rightarrow x$.

It is straightforward from Anick's general definition of an m -chain, see [1, 2] that in the case of PBW algebras, each m -chain is a monomial of length $m + 1$, $y_{m+1} y_m \cdots y_1$, where $y_{i+1} y_i \in \mathbf{W}$, $1 \leq i \leq m$. (For completeness, the 0-chains are the elements of X , by definition). This implies that for every $m \geq 1$ there is a one-to-one correspondence between the set of m -chains, in the sense of Anick, and the set of paths of length m in the directed graph $\Gamma_{\mathbf{W}}$. The m -chain $y_{m+1} y_m \cdots y_1$, with $y_{i+1} y_i \in \mathbf{W}$, $1 \leq i \leq m$, corresponds to the path $y_{m+1} \rightarrow y_m \rightarrow \cdots \rightarrow y_1$ of length m in $\Gamma_{\mathbf{W}}$.

Note that Anick's resolution [2] [1] is minimal for PBW algebras, and for monomial algebras (not necessarily quadratic), and therefore a PBW algebra A has finite global dimension $d < \infty$ iff there is a $d - 1$ -chain, but there are no d -chains in $\langle X \rangle$. The following lemma is a "translation" of this in terms of the properties of $\Gamma_{\mathbf{W}}$.

Lemma 3.5. *gl. dim $A = d < \infty$ if and only if $\Gamma_{\mathbf{W}}$ is an acyclic oriented graph, and $d - 1$ is the maximal length of a path occurring in $\Gamma_{\mathbf{W}}$.*

All PBW algebras with the same set of PBW generators x_1, \dots, x_n and the same sets of obstructions \mathbf{W} , share the same graphs $\Gamma_{\mathbf{N}}$ and $\Gamma_{\mathbf{W}}$. In some cases it is convenient to study the corresponding monomial algebra A^0 instead of A .

Definition 3.6. A complete oriented graph Γ is called a *tournament or tour*. In other words, a tournament is a directed graph in which each pair of vertices is joined by a single edge having a unique direction. Clearly, a complete directed graph with no cycles (of any length) is an *acyclic tournament*.

The following is straightforward.

Remark 3.7. An acyclic oriented graph with n vertices is a tournament iff it has exactly $\binom{n}{2}$ (directed) edges.

We shall need the following proposition about oriented graphs.

Proposition 3.8. *Let Γ be an acyclic tournament of order n . Then the set of its vertices $V = V(\Gamma)$ can be labelled $V = \{y_1, y_2, \dots, y_n\}$, so that the set of edges is*

$$(3.1) \quad E(\Gamma) = \{y_j \rightarrow y_i \mid 1 \leq i < j \leq n\}.$$

Analogously, the vertices can be labelled $V = \{z_1, z_2, \dots, z_n\}$, so that

$$E(\Gamma) = \{z_i \rightarrow z_j \mid 1 \leq i < j \leq n\}.$$

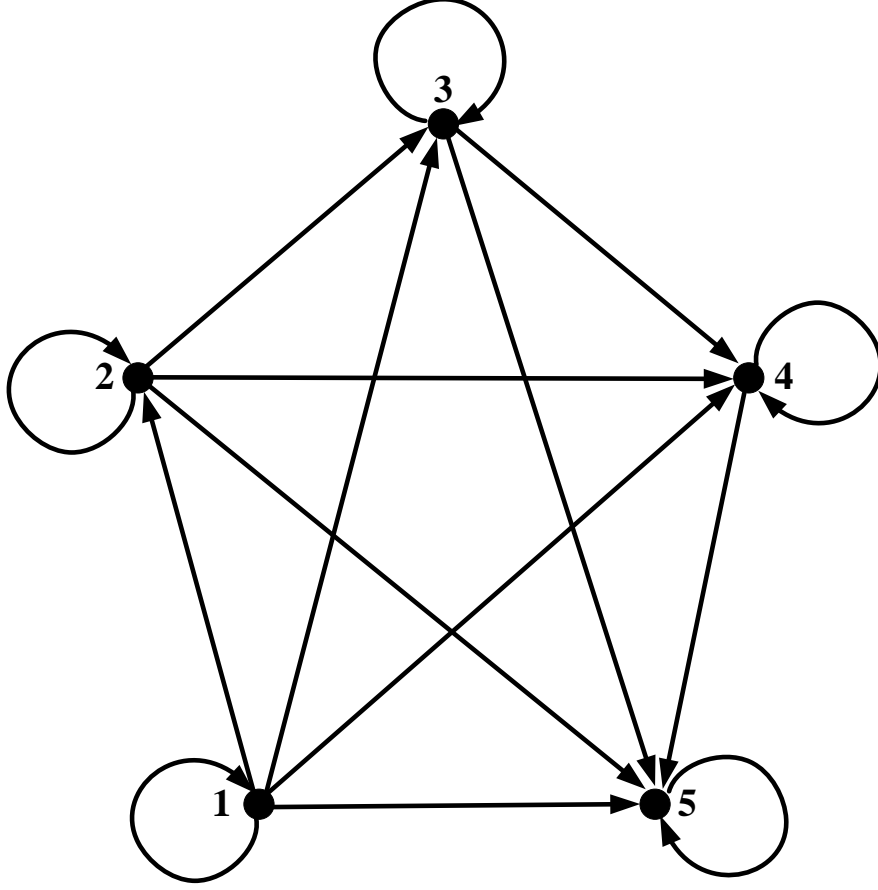


FIGURE 1. This is the graph of normal words $\Gamma_{\mathbf{N}}$ for a PBW algebra A with 5 generators, polynomial growth and finite global dimension.

Proof. We prove this by induction on the order of Γ .

The statement is obvious for $n = 2$. Assume the statement of proposition is true for graphs with $n - 1$ vertices. Let Γ be an acyclic tournament, of order n with vertices labelled $\{1, \dots, n\}$ and set of edges $E = E(\Gamma)$. Let Γ_{n-1} be the subgraph of Γ with set of vertices $V' = \{1, \dots, n - 1\}$ and set of edges

$$E' = \{i \rightarrow j \mid 1 \leq i, j \leq n - 1 \text{ } i \rightarrow j \in E(\Gamma)\}.$$

By the inductive assumption the set of vertices V' can be relabelled

$$V' = \{v_1, \dots, v_{n-1}\}$$

, s.t.

$$E' = \{v_j \rightarrow v_i \mid 1 \leq i < j \leq n - 1\}.$$

Denote by v the n -th vertex of Γ . Two cases are possible.

(a) $v_j \rightarrow v \in E(\Gamma)$, $\forall 1 \leq j \leq n - 1$. In this case the relabelling is clear, we set $y_1 = v$, and $y_{j+1} = v_j$, $1 \leq j \leq n - 1$. Then the labelling $V = \{y_1 \dots y_n\}$ agrees with 3.1.

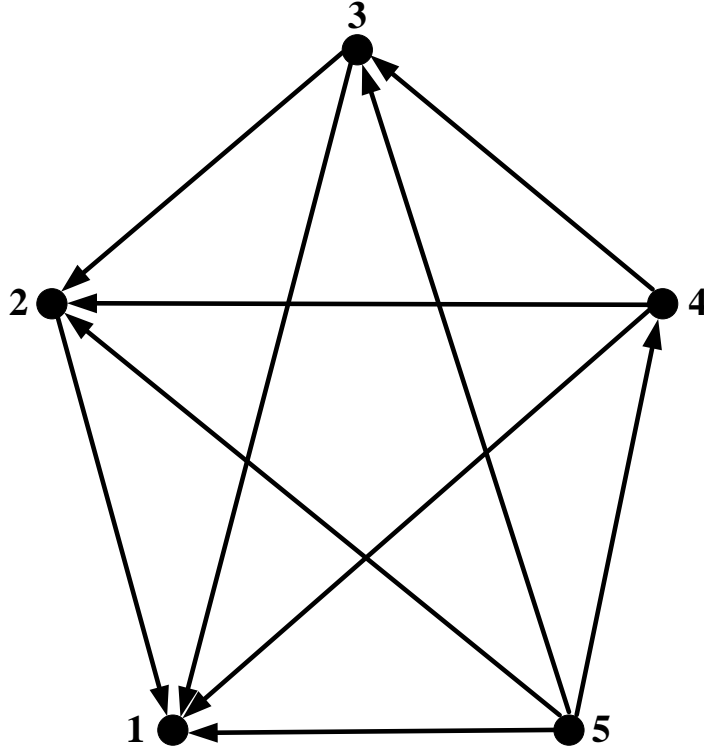


FIGURE 2. This is the graph of obstructions $\Gamma_{\mathbf{W}}$, dual to $\Gamma_{\mathbf{N}}$. It is **an acyclic tournament of order 5**, labelled "properly", as in Proposition 3.8.

(b) there exist a j , $1 \leq j \leq n - 1$, such that $v \rightarrow v_j \in E(\Gamma)$. Let j , $1 \leq j \leq n - 1$, be the maximal index with the property $v \rightarrow v_j \in E(\Gamma)$.

Assume $j > 1$, and let $1 \leq i < j$. We claim that $v \rightarrow v_i \in E(\Gamma)$. Assume the contrary. By assumption the vertices v, v_i are connected with a directed edge, so $v_i \rightarrow v \in E(\Gamma)$. Note that by the inductive assumption $v_j \rightarrow v_i \in E(\Gamma)$. So the graph Γ contains the cycle

$$v \rightarrow v_j \rightarrow v_i \rightarrow v,$$

but by hypothesis Γ is acyclic, a contradiction. Thus we have

$$v \rightarrow v_i \in E(\Gamma), \quad \forall i, 1 \leq i \leq j,$$

$$v_k \rightarrow v \in E(\Gamma), \quad \forall k, j < k \leq n - 1 \quad (\text{if } j < n - 1).$$

Three cases are possible

A. $j = 1$. In this case we set

$$y_1 = v_1, y_2 = v, y_{k+1} = v_k, 2 \leq k \leq n - 1.$$

B. $1 < j < n - 1$. In this case we set

$$y_k = v_k, 1 \leq k \leq j - 1, y_j = v, y_{k+1} = v_k, j \leq k \leq n - 1.$$

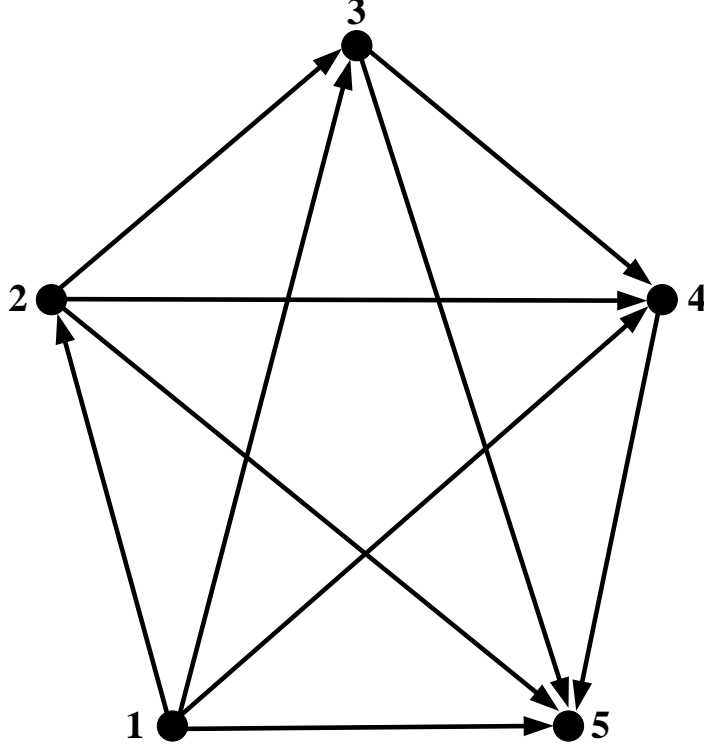


FIGURE 3. This is the graph of normal words $\Gamma^!$ for the Koszul dual $A^!$. It is an **acyclic tournament of order 5**

C. $j = n - 1$. In this case we set

$$y_k = v_k, 1 \leq k \leq n - 1, y_n = v.$$

□

A^0 is a *quadratic monomial algebra* if it has a presentation $A^0 = \mathbf{k}\langle X \rangle / (W)$, where W is a set of monomials of length 2. Any quadratic monomial algebra A^0 is a PBW algebra, furthermore any enumeration $x_1 \cdots, x_n$ of X gives PBW generators of A^0 .

Theorem 3.9. *Let $A^0 = \mathbf{k}\langle x_1 \cdots, x_n \rangle / (W)$ be a quadratic monomial algebra. The following conditions are equivalent*

- (1) A^0 has finite global dimension, and polynomial growth.
- (2) A^0 has finite global dimension, and $|\mathbf{W}| = \binom{n}{2}$.
- (3) A^0 has polynomial growth, $\mathbf{W} \cap \text{diag } X^2 = \emptyset$, and $|\mathbf{W}| = \binom{n}{2}$.
- (4) The graph $\Gamma_{\mathbf{W}}$ is an acyclic tournament.
- (5)

$$H_{A^0}(z) = \frac{1}{(1-z)^n}.$$

(6) *There is a permutation y_1, \dots, y_n of $x_1 \dots, x_n$ such that*
 (3.2)
$$\mathbf{N}^\infty = \{y_1^{\alpha_1} \dots y_n^{\alpha_n} \mid \alpha_i \geq 0, 1 \leq i \leq n\}.$$

(7) *There is a permutation y_1, \dots, y_n of $x_1 \dots, x_n$, such that*

$$\mathbf{W} = \{y_j y_i \mid 1 \leq i < j \leq n\}.$$

Furthermore, in this case

$$\text{gl. dim } A^0 = n = \text{the degree of polynomial growth of } A.$$

Proof. Condition (4) is central for our proof.

A. We will start with several easy implications.

Suppose (4) holds, so $\Gamma_{\mathbf{W}}$ is an acyclic tournament. By Proposition 3.8 the set of its vertices $V = V(\Gamma_{\mathbf{W}})$ can be relabelled $V = \{y_1, y_2, \dots, y_n\}$, so that

$$(3.3) \quad E(\Gamma_{\mathbf{W}}) = \{y_j \rightarrow y_i \mid 1 \leq i < j \leq n\}.$$

This clearly implies condition (7). The inverse implication is also clear.

So (4) \iff (7).

The following implications are straightforward

$$(6) \iff (7) \implies (5) \\ (7) \implies (3).$$

(4) \implies (2). Suppose (4) holds. As an acyclic tournament $\Gamma_{\mathbf{W}}$ contains exactly $\binom{n}{2}$ edges, and therefore $|\mathbf{W}| = \binom{n}{2}$. By (3.3) the set $E(\Gamma_{\mathbf{W}})$ contains the edges $y_j \rightarrow y_{j-1}, 2 \leq j \leq n$, so the graph $\Gamma_{\mathbf{W}}$ has a path $y_n \rightarrow y_{n-1} \rightarrow \dots \rightarrow y_1$ of length $n - 1$. Clearly, there are no longer paths in $\Gamma_{\mathbf{W}}$ thus $\text{gl. dim } A^0 = n$. This verifies (4) \implies (2).

It is also clear that (4) \implies (1).

(2) \implies (4). Note first that $|\mathbf{W}| = \binom{n}{2}$ implies that the graph has exactly $\binom{n}{2}$ edges. Next $\text{gl. dim } A^0 < \infty$ implies that $\Gamma_{\mathbf{W}}$ is an acyclic oriented graph, (see Lemma 3.5) so, by Remark 3.7, $\Gamma_{\mathbf{W}}$ is an acyclic tournament, we have shown

$$(2) \iff (4) \implies (1).$$

(3) \implies (4). Assume (3) holds. Then $\Gamma_{\mathbf{W}}$ has exactly $\binom{n}{2}$ edges, furthermore each edge has the shape $x \rightarrow y, x \neq y$. Therefore its dual graph $\Gamma_{\mathbf{N}}$ has a loop $x \rightarrow x$ at every vertex, and exactly $\binom{n}{2}$ edges $y \rightarrow x$, where $x \rightarrow y \in E(\Gamma_{\mathbf{W}})$. The polynomial growth of A^0 implies that $\Gamma_{\mathbf{N}}$ has no cycles of length ≥ 2 , and therefore every two vertices in $\Gamma_{\mathbf{N}}$ are connected with a single directed edge, so $\Gamma_{\mathbf{N}}$ is an oriented graph. It follows then that $\Gamma_{\mathbf{W}}$ is an acyclic oriented tournament, which verifies the implication (3) \implies (4). The inverse implication is clear. We have shown

$$(4) \iff (3).$$

It remains to show (1) \implies (4), and (5) \implies (4). The two implications are verified by similar argument.

B. (5) \implies (4). Note first that

$$(3.4) \quad H_{A^0}(z) = \frac{1}{(1-z)^n} = 1 + nz + \binom{n+1}{2} z^2 + \binom{n+2}{3} z^3 + \dots.$$

So

$$\dim A_2 = |\mathbf{N}| = \binom{n+1}{2}, \quad \text{which implies } |\mathbf{W}| = \binom{n}{2}.$$

Secondly, the special shape of Hilbert series $H_{A^0}(z)$ implies that A^0 has polynomial growth of degree n . Therefore by Fact 3.3 the graph $\Gamma_{\mathbf{N}}$ contains a path with n cycles. The only possibility for such a path is to have n distinct vertices and a loop at every vertex:

$$(3.5) \quad \begin{array}{c} \text{loop} \\ \curvearrowright \\ \bullet \\ a_1 \end{array} \longrightarrow \begin{array}{c} \text{loop} \\ \curvearrowright \\ \bullet \\ a_2 \end{array} \longrightarrow \begin{array}{c} \text{loop} \\ \curvearrowright \\ \bullet \\ a_3 \end{array} \longrightarrow \dots \longrightarrow \begin{array}{c} \text{loop} \\ \curvearrowright \\ \bullet \\ a_n \end{array}$$

Indeed $\Gamma_{\mathbf{N}}$ has exactly n vertices, and has no intersecting cycles. Each loop $x \rightarrow x$ in $\Gamma_{\mathbf{N}}$ implies $xx \in N$, so $\Delta_2 \subset \mathbf{N}$ ($\Delta_2 = \text{diag}(X^2)$). Then the complement $\mathbf{N} \setminus \Delta_2$ contains exactly $\binom{n}{2}$ monomials of the shape $xy, x \neq y$, or equivalently $\Gamma_{\mathbf{N}}$ has $\binom{n}{2}$ edges of the shape $x \rightarrow y, x \neq y$. Clearly, no pair $x \rightarrow y, y \rightarrow x$ belongs to $E(\Gamma_{\mathbf{N}})$, otherwise $\Gamma_{\mathbf{N}}$ will have two intersecting cycles $x \rightarrow x$ and $x \rightarrow y \rightarrow x$, but this is impossible since A^0 has polynomial growth. Therefore $\Gamma_{\mathbf{N}}$ is an oriented graph.

Consider now the dual graph $\Gamma_{\mathbf{W}}$. The properties of $\Gamma_{\mathbf{N}}$ imply that (a) $\Gamma_{\mathbf{W}}$ has no loops. (b) $\Gamma_{\mathbf{W}}$ has no cycles of length ≥ 2 . Each edge $x \rightarrow y$ in $E(\Gamma_{\mathbf{N}})$ has a corresponding edge $x \leftarrow y \in E(\Gamma_{\mathbf{W}})$. So $\Gamma_{\mathbf{W}}$ is an acyclic oriented graph with $\binom{n}{2}$ edges, and Remark 3.7 again implies that it is an acyclic tournament. This proves (5) \implies (4).

C. Finally we show (1) \implies (4).

Assume that A^0 has polynomial growth and finite global dimension. We shall use once more the nice balance between the dual graphs $\Gamma_{\mathbf{W}}$ and $\Gamma_{\mathbf{N}}$. Note first that $\Gamma_{\mathbf{N}}$ has no intersecting cycles, since otherwise A would have exponential growth. On the other hand $\Gamma_{\mathbf{W}}$ is acyclic, therefore it is an acyclic oriented graph. In particular, $\Gamma_{\mathbf{W}}$ has no loops, or equivalently \mathbf{W} does not contain monomials of the type xx . It follows then that the dual graph $\Gamma_{\mathbf{N}}$ has loops $x \rightarrow x$ at every vertex. Secondly, for each pair $x \neq y$ of vertices, there is exactly one edge $x \rightarrow y$, or $y \rightarrow x$, in $\Gamma_{\mathbf{N}}$. Indeed, $x \rightarrow y, y \rightarrow x \in E(\Gamma_{\mathbf{N}})$ would imply that $\Gamma_{\mathbf{N}}$ has intersecting cycles, which is impossible. Moreover if there is no edge connecting x and y in $\Gamma_{\mathbf{N}}$ this would imply that both $x \rightarrow y, y \rightarrow x$ are edges of $\Gamma_{\mathbf{W}}$, hence $\Gamma_{\mathbf{W}}$ has a cycle $x \rightarrow y \rightarrow x$, which contradicts $gl. \dim A < \infty$.

We have shown that $\Gamma_{\mathbf{W}}$ is an acyclic oriented graph with $\binom{n}{2}$ edges, and Remark 3.7 again implies that it is an acyclic tournament. \square

Remark 3.10. The implication (1) \implies (5) follows straightforwardly from a result of Anick, see [1] Theorem 6.

Proof of Theorem 1.1. Assume now that $A = \mathbf{k}\langle X \rangle / (\mathfrak{R})$ is a quadratic PBW algebra, with PBW generators $X = \{x_1, \dots, x_n\}$. Let \mathbf{W} be the set of obstructions, and let $A^0 = \mathbf{k}\langle X \rangle / (\mathbf{W})$ be the corresponding monomial algebra. The set \mathbf{N}^∞ , (Notation 3.1) is a \mathbf{k} -basis for both algebras A and A^0 . As we have noticed before, the two algebras have the same Hilbert series, equal degrees of growth, and by Lemma 3.5, there is an equality $gl. \dim A = gl. \dim A^0$.

(1) \implies (4). Suppose A has finite global dimension and polynomial growth. Then the same is valid for A^0 . By Theorem 3.9.6 there is a permutation y_1, \dots, y_n of $x_1 \dots, x_n$, such that

$$\mathbf{N}^\infty = \{y_1^{\alpha_1} \dots y_n^{\alpha_n} \mid \alpha_i \geq 0, 1 \leq i \leq n\},$$

so A has a \mathbf{k} -basis of the desired form. (In general, it is not true that \mathbf{N}^∞ is a *normal basis* for A w.r.t. the deg-lex ordering \prec defined via $y_1 \prec \cdots \prec y_n$).

(4) \implies (3) is clear.

(3) \implies (1) and (3) \implies (2).

Assume (3) holds. Then obviously A has polynomial growth of degree n . The equalities

$$H_{A^0}(z) = H_A(z) = \frac{1}{(1-z)^n}$$

and Theorem 3.9 imply that the monomial algebra A^0 has $gl.\dim A^0 = n$, and $|W| = \binom{n}{2}$. Clearly, then A has $\binom{n}{2}$ relations and global dimension n . This gives the implications (3) \implies (1), and (3) \implies (2)

Similarly, condition (2) is satisfied simultaneously by A and A^0 , so, by Theorem 3.9, (3) holds. We have shown (2) \iff (3).

The theorem has been proved.

4. COMBINATORICS IN QUANTUM BINOMIAL SETS

In this section (X, r) is a finite quantum binomial set.

When we study the monoid $S = S(X, r)$, or the algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) \simeq \mathbf{k}[S]$ associated with (X, r) (see Definition 2.4), it is convenient to use the action of the infinite groups, $\mathcal{D}_m(r)$, generated by maps associated with the quadratic relations, as follows. We consider the bijective maps

$$r^{ii+1} : X^m \longrightarrow X^m, \quad 1 \leq i \leq m-1, \quad \text{where } r^{ii+1} = Id_{X^{i-1}} \times r \times Id_{X^{m-i-1}}.$$

Note that these maps are elements of the symmetric group $\text{Sym}(X^m)$. Then the group $\mathcal{D}_m(r)$ generated by r^{ii+1} , $1 \leq i \leq m-1$, acts on X^m . r is involutive, so the bijective maps r^{ii+1} are involutive, as well, and $\mathcal{D}_m(r)$ is the infinite group

$$(4.1) \quad \mathcal{D}_m(r) = \text{gr} \langle r^{ii+1} \mid (r^{ii+1})^2 = e, \quad 1 \leq i \leq m-1 \rangle.$$

When $m = 3$, we use notation $\mathcal{D} = \mathcal{D}_3(r)$. Note that

$$\mathcal{D} = \text{gr} \langle r^{ii+1} \mid (r^{ii+1})^2 = e, \quad 1 \leq i \leq 2 \rangle$$

is simply *the infinite dihedral group*.

The problem of equality of words in the monoid $S = S(X, r)$ and in the quadratic algebra \mathcal{A} is solvable. Two elements $\omega, \omega' \in \langle X \rangle$ are equal in S *iff* they have the same length, $|\omega| = |\omega'| = m$ and belong to the same orbit of $\mathcal{D}_m(r)$ in X^m .

The action of the infinite dihedral group \mathcal{D} on X^3 is of particular importance in this section. Assuming that (X, r) is a quantum binomial set we will find some counting formulae and inequalities involving the orders of the \mathcal{D} -orbits in X^3 , and their number, see Proposition 4.4. These are used to find a necessary and sufficient conditions for (X, r) to be a symmetric set, Proposition 4.8, and to give upper bounds for $\dim A_3$ and $\dim A_3^!$ in the general case of quantum binomial algebra A , Corollary 4.11.

As usual, the orbit of a monomial $\omega \in X^3$ under the action of \mathcal{D} will be denoted by $\mathcal{O} = \mathcal{O}(\omega)$.

Denote by Δ_i the diagonal of $X^{\times i}$, $2 \leq i \leq 3$. One has $\Delta_3 = (\Delta_2 \times X) \cap (X \times \Delta_2)$.

Definition 4.1. We call a \mathcal{D} -orbit \mathcal{O} *square-free* if

$$\mathcal{O} \cap (\Delta_2 \times X \cup X \times \Delta_2) = \emptyset.$$

A monomial $\omega \in X^3$ is *square-free* in S if its orbit $\mathcal{O}(\omega)$ is square-free.

Remark 4.2. We recall that whenever (X, r) is a quadratic set, the left and the right "actions"

$${}^z\bullet : X \times X \longrightarrow X \quad \text{and} \quad \bullet^z : X \times X \longrightarrow X$$

induced by r reflect each property of r , see [21], and [19], Remark 2.1. We will need the following properties of the actions. Assume that r is square-free and nondegenerate, then

$$(4.2) \quad \begin{array}{l} {}^z t = {}^z u \implies t = u \iff t^z = u^z \\ {}^z t = z \iff t = z \iff t^z = z. \end{array}$$

Lemma 4.3. *Let (X, r) be a quantum binomial set, and let \mathcal{O} be a square-free \mathcal{D} -orbit in X^3 . Then $|\mathcal{O}| \geq 6$.*

Proof. Suppose $\mathcal{O} = \mathcal{O}(xyz)$ is a square-free orbit. Consider the set

$$O_1 = \{v_i \mid 1 \leq i \leq 6\} \subseteq \mathcal{O}$$

consisting of the first six elements of the "Yang-Baxter" diagram

$$(4.3) \quad \begin{array}{ccc} v_1 = xyz & \xrightarrow{r^{12}} & (xyx^y)z = v_2 \\ r^{23} \downarrow & & \downarrow r^{23} \\ v_3 = x(yzy^z) & & (x^y)(x^y z)(x^y)^z = v_5 \\ r^{12} \downarrow & & \downarrow r^{12} \\ v_4 = x(yz)(x^y z)(y^z) & & [{}^x y(x^y z)][(x^y)^z] = v_6. \end{array}$$

Clearly,

$$O_1 = U_1 \cup U_3 \cup U_5, \quad \text{where} \quad U_j = \{v_j, r^{12}(v_j) = v_{j+1}\}, \quad j = 1, 3, 5.$$

We claim that U_1, U_3, U_5 are pairwise disjoint sets, and each of them has order 2. Note first that since v_j is a square-free monomial, for each $j = 1, 3, 5$, one has $v_j \neq r^{12}(v_j) = v_{j+1}$, therefore

$$|U_j| = 2, \quad j = 1, 3, 5.$$

The monomials in each U_j have the same "tail". More precisely, $v_1 = (xy)z$, $v_2 = r(xy)z$, have a "tail" z , the tail of v_3 , and v_4 is y^z , and the tail of v_5 , and v_6 is $(x^y)^z$. It will be enough to show that the three elements $z, y^z, (x^y)^z \in X$ are pairwise distinct. But $\mathcal{O}(xyz)$ is square-free, so $y \neq z$ and by (4.2) $y^z \neq z$. Furthermore $v_2 = (x^y)(x^y)z \in \mathcal{O}(xyz)$ and therefore, $x^y \neq y$ and $x^y \neq z$. Now by (4.2) one has

$$\begin{array}{l} x^y \neq z \implies (x^y)^z \neq z \\ x^y \neq y \implies (x^y)^z \neq y^z. \end{array}$$

We have shown that the three elements $z, y^z, (x^y)^z \in X$ occurring as tails in U_1, U_3, U_5 , respectively, are pairwise distinct, so the three sets are pairwise disjoint. This implies $|O_1| = 6$, and therefore $|\mathcal{O}| \geq 6$. \square

Proposition 4.4. *Suppose (X, r) is a finite quantum binomial set. Let \mathcal{O} be a \mathcal{D} -orbit in X^3 , denote $\mathbf{E}(\mathcal{O}) = \mathcal{O} \cap ((\Delta_2 \times X \cup X \times \Delta_2) \setminus \Delta_3)$.*

(1) *The following implications hold.*

$$(i) \quad \mathcal{O} \cap \Delta_3 \neq \emptyset \implies |\mathcal{O}| = 1.$$

$$(ii) \quad \mathbf{E}(\mathcal{O}) \neq \emptyset \implies |\mathcal{O}| \geq 3 \quad \text{and} \quad |\mathbf{E}(\mathcal{O})| = 2.$$

In this case we say that \mathcal{O} is an orbit of type (ii)

$$(iii) \quad \mathcal{O} \cap (\Delta_2 \times X \cup X \times \Delta_2) = \emptyset \implies |\mathcal{O}| \geq 6.$$

(2) *There are exactly $n(n-1)$ orbits \mathcal{O} of type (ii) in X^3 .*

(3) *(X, r) satisfies the cyclic condition iff each orbit \mathcal{O} of type (ii) has order $|\mathcal{O}| = 3$. In this case*

$$x^y y = x y \quad \text{and} \quad x^x y = x^y, \quad \forall x, y \in X.$$

Proof. Clearly, the "fixed" points under the action of \mathcal{D} on X^3 are exactly the monomials xxx , $x \in X$, this gives (i).

Assume now that \mathcal{O} is of type (ii). Then it contains an element of the shape $\omega = xxy$, or $\omega = xyy$, where $x, y \in X, x \neq y$. Without loss of generality we can assume $\omega = xxy \in \mathcal{O}$.

The orbit $\mathcal{O}(\omega)$ can be obtained as follows. We fix as an initial element of the orbit $\omega = xxy$. Then there is a unique finite sequence $r^{23}, r^{12}, r^{23}, \dots$, which exhaust the whole orbit, and produces a "new" element at every step. r is involutive and square-free, thus in order to produce new elements at every step, the sequence must start with r^{23} and at every next step we have to alternate the actions r^{23} , and r^{12} .

We look at the "Yang-Baxter" diagram starting with ω and exhausting the whole orbit (without repetitions).

(4.4)

$$\omega = \omega_1 = xxy \xleftrightarrow{r^{23}} \omega_2 = x(xy)(x^y) \xleftrightarrow{r^{12}} \omega_3 = (x^2y)(x^xy)(x^y) \xleftrightarrow{\dots} \omega_m.$$

Note first that the first three elements $\omega_1, \omega_2, \omega_3$ are distinct monomials in X^3 . Indeed, $x \neq y$ implies $r(xy) \neq xy$ in X^2 , so $\omega_2 \neq \omega_1$. By assumption (X, r) is square-free, so $x^x = x$, but by the nondegeneracy $y \neq x$, also implies $x^y \neq x$. So $r(x(xy)) \neq x(xy)$, and therefore $\omega_3 \neq \omega_2$. Furthermore, $\omega_3 \neq \omega_1$. Indeed, if we assume $x = x^2y = x(xy)$ then by (4.2) one has $x^y = x$, and therefore $y = x$, a contradiction. We have obtained that $|\mathcal{O}| \geq 3$.

We claim now that the intersection $\mathbf{E} = \mathbf{E}(\mathcal{O})$ contains exactly two element. We analyze the diagram (4.4) looking from left to right.

Suppose we have made $k-1$ "steps" to the right obtaining new elements, so we have obtained

$$\omega_1 = xxy \xleftrightarrow{r^{23}} \omega_2 = x^x y x^y \xleftrightarrow{r^{12}} \omega_3 = x^2 y x^x y x^y \xleftrightarrow{\dots} \omega_{k-1} \xleftrightarrow{r^{ii+1}} \omega_k,$$

where $\omega_1, \omega_2, \dots, \omega_k$ are pairwise distinct. Note that all elements ω_s , $2 \leq s \leq k-1$, have the shape $\omega_s = a_s b_s c_s$, with $a_s \neq b_s$ and $b_s \neq c_s$. Two cases are possible.

(a) $\omega_k = a_k b_k c_k$, with $a_k \neq b_k$ and $b_k \neq c_k$, then applying r^{jj+1} (where $j = 2$ if $i = 1$ and $j = 1$ if $i = 2$), we obtain a new element ω_{k+1} of the orbit.

(b) $\omega_k = aac$, or $\omega_k = acc$, $a \neq c$. In this case r^{jj+1} with $j \neq i$ keeps ω_k fixed, so the process of obtaining new elements of the orbit stops at this step and the diagram is complete.

But our diagram is finite, so as a final step on the right it has to "reach" some $\omega_m = aac$, or $\omega_m = acc$, $a \neq c$ (we have already shown that $m \geq 3$). Note that $\omega_m \neq \omega_1$. Hence the intersection $\mathbf{E} = \mathbf{E}(\mathcal{O})$, contains exactly two elements.

Condition (iii) follows from Lemma 4.3. (1) has been verified.

We claim that there exists exactly $n(n-1)$ orbits of type (ii). Indeed, let $\mathcal{O}_1, \dots, \mathcal{O}_p$ be all orbits of type (ii). The intersections $E_i = E(\mathcal{O}_i)$, $1 \leq i \leq p$, are disjoint sets and each of them contains two elements. Now the equalities

$$\begin{aligned} \bigcup_{1 \leq i \leq p} E_i &= (\Delta_2 \times X \cup X \times \Delta_2) \setminus \Delta_3 \\ |E_i| &= 2, \quad |(\Delta_2 \times X \cup X \times \Delta_2) \setminus \Delta_3| = 2n(n-1) \end{aligned}$$

imply $p = n(n-1)$. This verifies (2)

Condition (3) follows straightforwardly from (4.4). \square

Example 4.5. Consider the quantum binomial algebra A given in example 2.10. Let (X, r) be the associated quadratic set, $S = S(X, r)$ the corresponding monoid. The relations are semigroup relations, so there is an algebra isomorphism $A = \mathcal{A}(\mathbf{k}, X, r) \simeq \mathbf{k}S$. We will find the corresponding \mathcal{D} orbits in X^3 . There are 12 orbits of type (ii). This agrees with Proposition 4.8.

$$\mathcal{O}_1 = xxy \xrightarrow{r^{23}} xzt \xrightarrow{r^{12}} yxt \xrightarrow{r^{23}} ytx \xrightarrow{r^{12}} tzx \xrightarrow{r^{23}} tty$$

$$\mathcal{O}_2 = xxz \xrightarrow{r^{23}} xyx \xrightarrow{r^{12}} ztx \xrightarrow{r^{23}} zxt \xrightarrow{r^{12}} tyt \xrightarrow{r^{23}} ttz$$

$$\mathcal{O}_3 = yxx \xrightarrow{r^{12}} xzx \xrightarrow{r^{23}} xty \xrightarrow{r^{12}} txy \xrightarrow{r^{23}} tzt \xrightarrow{r^{12}} ytt$$

$$\mathcal{O}_4 = zxx \xrightarrow{r^{12}} tyx \xrightarrow{r^{23}} txz \xrightarrow{r^{12}} xtz \xrightarrow{r^{23}} xyt \xrightarrow{r^{12}} ztt$$

$$\mathcal{O}_5 = xyy \xrightarrow{r^{12}} zty \xrightarrow{r^{23}} zxz; \quad \mathcal{O}_6 = xzz \xrightarrow{r^{12}} yxz \xrightarrow{r^{23}} yyx$$

$$\mathcal{O}_7 = tyx \xrightarrow{r^{12}} zxy \xrightarrow{r^{23}} zzt; \quad \mathcal{O}_8 = tzz \xrightarrow{r^{12}} ytz \xrightarrow{r^{23}} yyt$$

$$\mathcal{O}_9 = txx \xrightarrow{r^{12}} xtx \xrightarrow{r^{23}} xxt; \quad \mathcal{O}_{10} = xtt \xrightarrow{r^{12}} txt \xrightarrow{r^{23}} ttx$$

$$\mathcal{O}_{11} = yzz \xrightarrow{r^{12}} zyz \xrightarrow{r^{23}} zzy; \quad \mathcal{O}_{12} = zyy \xrightarrow{r^{12}} yzy \xrightarrow{r^{23}} yyz$$

There are only two square-free orbits, $\mathcal{O}^{(1)} = \mathcal{O}(xyz)$, and $\mathcal{O}^{(2)} = \mathcal{O}(tyz)$. Each of them has order 6.

$$(4.5) \quad \begin{array}{ccc} xyz & \xrightarrow{r^{12}} & ztz \\ r^{23} \downarrow & & \downarrow r^{23} \\ xzy & & zyt \\ r^{12} \downarrow & & \downarrow r^{12} \\ yxy & \xrightarrow{r^{12}} & yzt \end{array} \quad \begin{array}{ccc} tyz & \xrightarrow{r^{12}} & zxz \\ r^{23} \downarrow & & \downarrow r^{23} \\ tzy & & zyx \\ r^{12} \downarrow & & \downarrow r^{12} \\ yty & \xrightarrow{r^{12}} & yzx \end{array}$$

The one element orbits are $\{xxx\}, \{yyy\}, \{zzz\}, \{ttt\}$.

More detailed study of the orbits shows that A is *not PBW w.r.t. any enumeration of X* . Clearly, r does not satisfy the braid relation, so (X, r) is not a symmetric set.

Lemma 4.6. *A quantum binomial set (X, r) is symmetric iff the orders of \mathcal{D} -orbits \mathcal{O} in X^3 satisfy the following two conditions.*

$$(a) \quad \mathcal{O} \cap (\Delta_2 \times X \cup X \times \Delta_2) \setminus \Delta_3 \neq \emptyset \iff |\mathcal{O}| = 3.$$

$$(b) \quad \mathcal{O} \cap (\Delta_2 \times X \cup X \times \Delta_2) = \emptyset \iff |\mathcal{O}| = 6.$$

Proof. Look at the corresponding YBE diagrams □

Let (X, r) be a quantum binomial set, let \mathcal{D} be the infinite dihedral group acting on X^3 . We fix the following notation for the \mathcal{D} -orbits in X^3 .

Notation 4.7. We denote by $\mathcal{O}_i, 1 \leq i \leq n(n-1)$ the orbits of type (ii), and by $\mathcal{O}^{(j)}, 1 \leq j \leq q$ all square-free orbits in X^3 . The remaining \mathcal{D} -orbits in X^3 are the one-element orbits $\{xxx\}, x \in X$, their union is Δ_3 .

Proposition 4.8. *Let (X, r) be a finite quantum binomial set. Let $\mathcal{O}^{(j)}, 1 \leq j \leq q$, be the set of all (distinct) square-free \mathcal{D} -orbits in X^3 . Then*

- (1) $q \leq \binom{n}{3}$.
- (2) (X, r) is a symmetric set iff $q = \binom{n}{3}$.

Proof. Clearly, X^3 is a disjoint union

$$X^3 = \Delta_3 \cup \bigcup_{1 \leq i \leq n(n-1)} \mathcal{O}_i \cup \bigcup_{1 \leq j \leq q} \mathcal{O}^{(j)}.$$

Thus

$$(4.6) \quad |X^3| = |\Delta_3| + \sum_{1 \leq i \leq n(n-1)} |\mathcal{O}_i| + \sum_{1 \leq j \leq q} |\mathcal{O}^{(j)}|$$

Denote $m_i = |\mathcal{O}_i|, 1 \leq i \leq n(n-1), n_j = |\mathcal{O}^{(j)}|, 1 \leq j \leq q$. By Proposition 4.4 one has

$$m_i \geq 3, 1 \leq i \leq n(n-1), \quad \text{and} \quad n_j \geq 6, 1 \leq j \leq q.$$

We replace these inequalities in (4.6) and obtain

$$(4.7) \quad n^3 = n + \sum_{1 \leq i \leq n(n-1)} m_i + \sum_{1 \leq j \leq q} n_j \geq n + 3n(n-1) + 6q.$$

So

$$q \leq \frac{n^3 - 3n^2 + 2n}{6} = \binom{n}{3},$$

which verifies (1). Assume now $q = \binom{n}{3}$. Then (4.7) implies

$$n^3 = n + \sum_{1 \leq i \leq n(n-1)} m_i + \sum_{1 \leq j \leq \binom{n}{3}} n_j \geq n + 3n(n-1) + 6 \binom{n}{3} = n^3.$$

This is possible **iff** the following equalities hold

$$(4.8) \quad \begin{aligned} m_i &= |\mathcal{O}_i| = 3, 1 \leq i \leq n(n-1), \\ n_j &= |\mathcal{O}^{(j)}| = 6, 1 \leq j \leq q. \end{aligned}$$

By Lemma 4.6, the equalities (4.7) hold *iff* (X, r) is a symmetric set. \square

Corollary 4.9. *Let (X, r) be a finite quantum binomial set. (X, r) is a symmetric set iff the associated quadratic algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$ satisfies*

$$\dim \mathcal{A}_3 = \binom{n+2}{3}.$$

Proof. The distinct elements of the associated monoid $S = S(X, r)$, form a \mathbf{k} -basis of the monoidal algebra $\mathbf{k}S \simeq \mathcal{A}(\mathbf{k}, X, r)$. In particular $\dim \mathcal{A}_3$ equals the number of distinct monomials of length 3 in S which is exactly the number of \mathcal{D} -orbits in X^3 . \square

There is a close relation between Yang-Baxter monoids and a special class of *Garside monoids*, see [9], and [19]. Garside monoids and groups were introduced by Garside, [13]. The interested reader can find more information and references in [13, 10, 27], etall.

In [19], Definition 1.10. we introduce *the regular Garside monoids*. It follows from [19], the Main Theorem 1.16, that a finite quantum binomial set (X, r) is a solution of YBE *iff* the associated monoid $S = S(X, r)$ is a regular Garside monoid. This together with Corollary 4.9 imply the following.

Corollary 4.10. *Let (X, r) be a finite quantum binomial set. Let $S = S(X, r)$ be the associated monoid, and S^3 -the set of distinct elements of length 3 in S . Suppose the cardinality of S^3 is*

$$|S^3| = \binom{n+2}{3}.$$

Then S is a Garside monoid. Moreover, S is regular in the sense of [19].

Assume now that A is a quantum binomial algebra. We want to estimate the dimension $\dim \mathcal{A}_3$. Let (X, r) be the corresponding quantum binomial set, $S = S(X, r)$, $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$. We use Proposition 4.8 to find an upper bound for the number of distinct \mathcal{D} -orbits in X^3 , or equivalently, the order of S_3 , the set of (distinct) elements of length 3 in S . One has

$$|S_3| = n + n(n-1) + q \leq n + n(n-1) + \binom{n}{3} = \binom{n+2}{3}.$$

There is an isomorphism of vector spaces, $\mathcal{A}_3 \simeq \text{Span} S_3$, so

$$\dim \mathcal{A}_3 = |S_3| \leq \binom{n+2}{3}.$$

In the general case, a quantum binomial algebra, satisfies $\dim A_3 \leq \dim \mathcal{A}_3$, due to the coefficients c_{xy} appearing in the set of relations. We have proven the following corollary.

Corollary 4.11. *If A is a quantum binomial algebra, then*

$$\dim A_3 \leq \binom{n+2}{3}, \quad \dim A_3^! \leq \binom{n}{3}.$$

5. QUANTUM BINOMIAL ALGEBRAS. YANG-BAXTER EQUATION AND
ARTIN-SCHELTER REGULARITY

Definition 5.1. [35], [36] A graded algebra $A = \bigoplus_{i \geq 0} A_i$ is called a *Frobenius algebra of dimension n* , (or a *Frobenius quantum space of dimension n*) if

- (a) $\dim(A_n) = 1$, $A_i = 0$, for $i > n$.
- (b) For all $n \geq j \geq 0$ the multiplicative map $m : A_j \otimes A_{n-j} \rightarrow A_n$ is a perfect duality (nondegenerate pairing).

A Frobenius algebra A is called a *quantum Grassmann algebra* if in addition

- (c) $\dim_{\mathbf{k}} A_i = \binom{n}{i}$, for $1 \leq i \leq n$.

Lemma 5.2. Let $A = \mathbf{k}\langle X; \mathfrak{R} \rangle$ be a quantum binomial algebra, $|X| = n$, $A^!$ its Koszul dual. Let (X, r) , $r = r(\mathfrak{R})$ be the associated quantum binomial set (see Definition 2.8). Then each of the following three conditions implies that (X, r) is a symmetric set.

(1)

$$\dim A_3 = \binom{n+2}{3}.$$

(2)

$$\dim A_3^! = \binom{n}{3}.$$

(3) X can be enumerated $X = \{x_1 \cdots, x_n\}$, so that the set of ordered monomials of length 3

$$(5.1) \quad N_3 = \{x_{i_1} x_{i_2} x_{i_3} \mid 1 \leq i_1 \leq i_2 \leq i_3 \leq n\}$$

projects to a \mathbf{k} -basis of A_3

Proof. In the usual notation, $S = S(X, r)$ and $\mathcal{A} = \mathcal{A}(X, r)$ denote respectively be the associated monoid and quadratic algebra. We know that, in general $\dim A_3 \leq \dim \mathcal{A}_3$.

Assume (2) holds. Consider the relations

$$\binom{n+2}{3} = \dim A_3 \leq \dim \mathcal{A}_3 \leq \binom{n+2}{3},$$

where the right-hand side inequality follows from Corollary 4.9. This implies $\dim \mathcal{A}_3 = \binom{n+2}{3}$, or equivalently, $|S_3| = \binom{n+2}{3}$ and therefore there are exactly $q = \binom{n}{3}$ square-free \mathcal{D} -orbits in X^3 . Then Proposition 4.8 (2) implies that (X, r) is a symmetric set, which verifies (1) \implies (3). The converse (3) \implies (1) is straightforward. Finally, one has

$$\dim A_3 = \binom{n+2}{3} \iff \dim A_3^! = \binom{n}{3},$$

This can be proved directly using the \mathcal{D} -orbits in X^3 . It is also straightforward from the following formula for quadratic algebras, see [37], p 85.

$$\dim A_3^! = (\dim A_1)^3 - 2(\dim A_1)(\dim A_2) + \dim A_3.$$

□

Lemma 5.3. Let A be a quadratic algebra with relations of skew-polynomial type, let $A^!$ be its Koszul dual. Then the following conditions are equivalent.

$$(1) \quad \dim A_3 = \binom{n+2}{3}.$$

$$(2) \quad \dim A_3^! = \binom{n}{3}.$$

(3) *The set of defining relations \mathfrak{R} for A is a Gröbner basis, so A is a skew polynomial ring, and therefore a PBW algebra.*

(4) *The set of defining relations $\mathfrak{R}^!$ for $A^!$ is a Gröbner basis, so $A^!$ is a PBW algebra with PBW generators x_1, \dots, x_n .*

Sketch of the proof. As we already mentioned (1) \iff (2).

(1) \iff (3). Assume condition (1) holds. The set of monomials of length 3, which are normal mod the ideal (\mathfrak{R}) form a \mathbf{k} -basis of A_3 . The skew-polynomial shape of the relations implies that each monomial u which is normal mod the ideal (\mathfrak{R}) is in the set of ordered monomials

$$N_3 = \{x_{i_1}x_{i_2}x_{i_3} \mid 1 \leq i_1 \leq i_2 \leq i_3 \leq n\}.$$

There are equalities

$$|N_3| = \binom{n+2}{3} = \dim A_3,$$

hence all ordered monomials of length 3 are normal mod (\mathfrak{R}) . It follows then that all ambiguities $x_kx_jx_i, 1 \leq i < j < k \leq n$ are resolvable, and by Bergman's Diamond lemma [6], \mathfrak{R} is a Gröbner basis of the ideal (\mathfrak{R}) . Thus A is a PBW algebra, more precisely, A is a binomial skew-polynomial ring. This gives (1) \implies (3). The converse implication (1) \impliedby (3) is clear.

(2) \impliedby (4) is clear.

(2) \implies (4) is analogous to (1) \implies (3).

Theorem 5.4. *Let $A = \mathbf{k}\langle X \rangle / (\mathfrak{R})$ be a quantum binomial algebra, $|X| = n$, and let \mathfrak{R} be the associated automorphism $R = R(\mathfrak{R}) : V^{\otimes 2} \longrightarrow V^{\otimes 2}$, see Definition 2.8. Then the following three conditions are equivalent*

- (1) *$R = R(\mathfrak{R})$ is a solution of the Yang-Baxter equation.*
- (2) *A is a binomial skew-polynomial ring*
- (3)

$$\dim_{\mathbf{k}} A_3 = \binom{n+2}{3}.$$

Proof. We start with the implication (3) \implies (2).

Assume that $\dim_{\mathbf{k}} A_3 = \binom{n+2}{3}$. Consider the corresponding quadratic set (X, r) and the monoidal algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$. As before we conclude that $\dim_{\mathbf{k}} \mathcal{A}_3 = \binom{n+2}{3}$ and therefore, by Corollary 4.9 (X, r) is symmetric set.

By [18] Theorem 2.26 there exists an ordering on X , $X = \{x_1, x_2, \dots, x_n\}$ so that the algebra \mathcal{A} is a skew-polynomial ring and therefore a PBW algebra, with PBW generators x_1, x_2, \dots, x_n . It follows then that the relations \mathfrak{R} of A are relations of skew-polynomial type, that is condition (a), (b) and (c) of Definition 2.1 are satisfied.

By assumption $\dim_{\mathbf{k}} A_3 = \binom{n+2}{3}$ so Lemma 5.3 implies that the set of relations \mathfrak{R} is a Gröbner basis (A is PBW) and therefore A is a binomial skew polynomial ring. This verifies (3) \implies (2).

It follows from Definition 2.1 that if A is a binomial skew-polynomial ring then the set of monomials

$$\mathcal{N} = \{x_{i_1}x_{i_2}x_{i_3} \mid i_1 \leq x_{i_2} \leq x_{i_3}\}$$

is a \mathbf{k} -basis of A_3 , so $\dim_{\mathbf{k}} A_3 = \binom{n+2}{3}$, hence (2) \implies (3).

The equivalence (1) \iff (2) is proven in [17], Theorem B. \square

Proof of Theorem 1.2. The equivalence of conditions (4), (5) and (6) follows from Theorem 5.4.

The implication (7) \implies (6) is clear. The converse follows from (6) \implies (5) \implies (7).

It is straightforward that each binomial skew polynomial ring A is Koszul and satisfies (1) and (2). It is proven in [17] that the Koszul dual A^\dagger of a binomial skew polynomial ring is a quantum Grassman algebra. Thus (5) \implies (8).

Clearly, (8) \implies (6), and therefore (5) \iff (8).

It is also known that a Koszul algebra A is Gorenstein *iff* its dual A^\dagger is Frobenius. It follows then that (5) \implies (3).

The result that every binomial skew polynomial ring is AS regular follows also from the earlier work [25].

Finally we will show (1) \implies (5) and (2) \implies (5).

We know that a quantum binomial algebra A has exactly $\binom{n}{2}$ relations. Assume now that A is a PBW algebra. Then its set of obstructions \mathbf{W} has order $|\mathbf{W}| = |\mathfrak{R}| = \binom{n}{2}$. Consider now the corresponding monomial algebra $A^0 = \mathbf{k}\langle X \rangle / (\mathbf{W})$. Each of the conditions (1) and (2) is satisfied by A *iff* it is true for A^0 .

Assume first that A satisfies (1). Then the monomial algebra A^0 satisfies condition (2) of Theorem 3.9 and by the same theorem the Hilbert series of A^0 satisfies (7). The algebras A and A^0 have the same Hilbert series, and therefore

$$(1) \implies (7) \implies (5).$$

Similarly, if A satisfies (2), then the monomial algebra A^0 satisfies condition (3) of Theorem 3.9. (Note that the relations are square free, so

$$\mathbf{W} \cap \text{diag } X^2 = \emptyset.$$

Analogous to the previous implication we conclude (2) \implies (7) \implies (5). The equivalence of the conditions (1) \cdots (8) has been verified.

Each of these conditions imply that A is a binomial skew-polynomial ring and therefore it is a Noetherian domain, see [25], or Fact 2.9.

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INSTITUTE OF MATHEMATICS AND INFORMATICS, BULGARIAN ACADEMY OF SCIENCES, SOFIA 1113, BULGARIA, AND AMERICAN UNIVERSITY IN BULGARIA, 2700 BLAGOEVGRAD, BULGARIA
E-mail address: `tatianagateva@yahoo.com`, `tatyana@aubg.bg`