PROJECTIVE GEOMETRY FOR BLUEPRINTS

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ABSTRACT. In this note, we generalize the Proj–construction from usual schemes to blue schemes. This yields the definition of projective space and projective varieties over a blueprint. In particular, it is possible to descend closed subvarieties of a projective space to a canonical $\mathbb{F}_1$–model. We discuss this in case of the Grassmannian $\mathrm{Gr}(2,4)$.

1. INTRODUCTION

Blueprints are a common generalization of commutative (semi)rings and monoids. The associated geometric objects, blue schemes, are therefore a common generalization of usual scheme theory and $\mathbb{F}_1$–geometry (as considered by Kato [5], Deitmar [3] and Connes-Consani [2]). The possibility of forming semiring schemes allows us to talk about idempotent schemes and tropical schemes (cf. [11]). All this is worked out in [9].

It is known, though not covered in literature yet, that the Proj-construction from usual algebraic geometry has an analogue in $\mathbb{F}_1$-geometry (after Kato, Deitmar and Connes-Consani). In this note we describe a generalization of this to blueprints. In private communication, Koen Thas announced a treatment of Proj for monoidal schemes (see [13]).

We follow the notations and conventions of [10]. Namely, all blueprints that appear in this note are proper and with a zero. We remark that the following constructions can be carried out for the more general notion of a blueprint as considered in [9]; the reason that we restrict to proper blueprints with a zero is that this allows us to adopt a notation that is common in $\mathbb{F}_1$-geometry.

Namely, we denote by $\mathbb{A}^n_B$ the (blue) affine n-space $\mathrm{Spec}\big(B[T_1,\dots,T_n]\big)$ over a blueprint B. In case of a ring, this does not equal the usual affine n-space since $B[T_1,\dots,T_n]$ is not closed under addition. Therefore, we denote the usual affine n-space over a ring B by ${}^+\mathbb{A}^n_B = \mathrm{Spec}\big(B[T_1,\dots,T_n]^+\big)$. Similarly, we use a superscript "$+$" for the usual projective space ${}^+\mathbb{P}^n_B$ and the usual Grassmannian $\mathrm{Gr}(k,n)^+_B$ over a ring B.

2. GRADED BLUEPRINTS AND Proj

Let B be a blueprint and M a subset of B. We say that M is *additively closed in* B if for all additive relations $b \equiv \sum a_i$ with $a_i \in M$ also b is an element of M. Note that, in particular, 0 is an element of M. A *graded blueprint* is a blueprint B together with additively closed subsets B_i for $i \in \mathbb{N}$ such that $1 \in B_0$, such that for all $i,j \in \mathbb{N}$ and $a \in B_i$, $b \in B_j$, the product ab is an element of B_{i+j} and such that for every $b \in B$, there are a unique finite subset I of $\mathbb{N}$ and unique non-zero elements $a_i \in B_i$ for every $i \in I$ such that $b \equiv \sum a_i$. An element of $\bigcup_{i\geq 0} B_i$ is called *homogeneous*. If $a \in B_i$ is non-zero, then we say, more specifically, that a is *homogeneous of degree* i.

We collect some immediate facts for a graded blueprint B as above. The subset B_0 is multiplicatively closed, i.e. B_0 can be seen as a subblueprint of B. The subblueprint B_0 equals B if and only if for all $i > 0$, $B_i = \{0\}$. In this case we say that B is *trivially graded*. By the uniqueness of the decomposition into homogeneous elements, we have $B_i \cap B_j = \{0\}$ for $i \neq j$. This means that the union $\bigcup_{i\geq 0} B_i$ has the structure of a wedge product $\bigvee_{i\geq 0} B_i$. Since $\bigvee_{i\geq 0} B_i$ is multiplicatively closed, it can be seen as a subblueprint

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of B. We define $B_{\text{hom}} = \bigvee_{i\geq 0} B_i$ and call the subblueprint B_{hom} the *homogeneous part of* B.

Let S be a multiplicative subset of B. If b/s is an element of the localization $S^{-1}B$ where f is homogeneous of degree i and s is homogeneous of degree j, then we say that b/s is a homogeneous element of degree $i-j$. We define $S^{-1}B_0$ as the subset of homogeneous elements of degree 0. It is multiplicatively closed, and inherits thus a subblueprint structure from $S^{-1}B$. If S is the complement of a prime ideal $\mathfrak{p}$, then we write $B_{(\mathfrak{p})}$ for the subblueprint $(B_{\mathfrak{p}})_0$ of homogeneous elements of degree 0 in $B_{\mathfrak{p}}$.

An ideal I of a graded blueprint B is called *homogeneous* if it is generated by homogeneous elements, i.e. if for every $c \in I$, there are homogeneous elements $p_i, q_j \in I$ and elements $a_i, b_j \in B$ and an additive relation $\sum a_i p_i + c \equiv \sum b_j q_j$ in B.

Let B be a graded blueprint. Then we define $\operatorname{Proj} B$ as the set of all homogeneous prime ideals $\mathfrak{p}$ of B that do not contain $B^+_{\text{hom}} = \bigvee_{i>0} B_i$. The set $X = \operatorname{Proj} B$ comes together with the topology that is defined by the basis

$$U_h \quad = \quad \{\, \mathfrak{p} \in X \mid h \notin \mathfrak{p} \,\}$$

where h ranges through B_{hom} and with a structure sheaf $\mathcal{O}_X$ that is the sheafification of the association $U_h \mapsto B[h^{-1}]_0$ where $B[h^{-1}]$ is the localization of B at $S = \{h^i\}_{i\geq q0}$.

Note that if B is a ring, the above definitions yield the usual construction of $\operatorname{Proj} B$ for graded rings. In complete analogy to the case of graded rings, one proves the following theorem.

Theorem 2.1. *The space* $X = \operatorname{Proj} B$ *together with* $\mathcal{O}_X$ *is a blue scheme. The stalk at a point* $\mathfrak{p} \in \operatorname{Proj} B$ *is* $\mathcal{O}_{x,\mathfrak{p}} = B_{(\mathfrak{p})}$. *If* $h \in B^+_{\text{hom}}$, *then* $U_h \simeq \operatorname{Spec} B[h^{-1}]_0$. *The inclusions* $B_0 \hookrightarrow B[h^{-1}]_0$ *yield morphisms* $\operatorname{Spec} B[h^{-1}]_0 \to \operatorname{Spec} B_0$, *which glue to a structural morphism* $\operatorname{Proj} B \to \operatorname{Spec} B_0$. □

If B is a graded blueprint, then the associated semiring B^+ inherits a grading. Namely, let $B_{\text{hom}} = \bigvee_{i\geq 0} B_i$ the homogeneous part of B. Then we can define B_i^+ as the additive closure of B_i in B^+, i.e. as the set of all $b \in B$ such that there is an additive relation of the form $b \equiv \sum a_k$ in B with $a_k \in B_i$. Then $\bigvee B_i^+$ defines a grading of B^+. Similarly, the grading of B induces a grading on a tensor product $B \otimes_C D$ with respect to blueprint morphisms $C \to B$ and $C \to D$ under the assumption that the image of $C \to B$ is contained in B_0. Consequently, a grading of B implies a grading of $B_{\text{inv}} = B \otimes_{\mathbb{F}_1} \mathbb{F}_{1^2}$ and of the ring $B^+_{\mathbb{Z}} = B^+_{\text{inv}}$. Along the same lines, if both B and D are graded and the images of $C \to B$ and $C \to D$ lie in B_0 and C_0 respectively, then $B \otimes_C D$ inherits a grading obtained from the gradings of B and D.

3. Projective space

The functor Proj allows the definition of the projective space $\mathbb{P}^n_B$ over a blueprint B. Namely, the free blueprint $C = B[T_0, \dots, T_n]$ over B comes together with a natural grading (cf. [9, Section 1.12] for the definition of free blueprints). Namely, C_i consists of all monomials $bT_0^{e_0} \cdots T_n^{e_n}$ such that $e_0 + \cdots + e_n = i$ where $b \in B$. Note that $C_0 = B$ and $C_{\text{hom}} = C$. The projective space $\mathbb{P}^n_B$ is defined as $\operatorname{Proj} B[T_0, \dots, T_n]$. It comes together with a structure morphism $\mathbb{P}^n_B \to \operatorname{Spec} B$.

In case of $B = \mathbb{F}_1$, the projective space $\mathbb{P}^n_{\mathbb{F}_1}$ is the monoidal scheme that is known from $\mathbb{F}_1$-geometry (see [4], [1, Section 3.1.4]) and [10, Ex. 1.6]). The topological space of $\mathbb{P}^n_{\mathbb{F}_1}$ is finite. Its points correspond to the homogeneous prime ideals $(S_i)_{i\in I}$ of $\mathbb{F}_1[S_0, \dots, S_n]$ where I ranges through all proper subsets of $\{0, \dots, n\}$.

In case of a ring B, the projective space $\mathbb{P}^n_B$ does not coincide with the usual projective space since the free blueprint $B[S_0, \dots, S_n]$ is not a ring, but merely the blueprint of all monomials of the form $bS_0^{e_0} \cdots S_n^{e_n}$ with $b \in B$. However, the associated scheme ${}^+\mathbb{P}^n_B = (\mathbb{P}^n_B)^+$ coincides with the usual projective space over B, which equals $\operatorname{Proj} B[S_0, \dots, S_n]^+$.

4. Closed subschemes

Let $\mathcal{X}$ be a scheme of finite type. By an $\mathbb{F}_1$*-model of* $\mathcal{X}$ we mean a blue scheme X of finite type such that $X_{\mathbb{Z}}^+$ is isomorphic to $\mathcal{X}$. Since a finitely generated $\mathbb{Z}$-algebra is, by definition, generated by a finitely generated multiplicative subset as a $\mathbb{Z}$-module, every scheme of finite type has an $\mathbb{F}_1$-model. It is, on the contrary, true that a scheme of finite type possesses a large number of $\mathbb{F}_1$-models.

Given a scheme $\mathcal{X}$ with an $\mathbb{F}_1$-model X, we can associate to every closed subscheme $\mathcal{Y}$ of $\mathcal{X}$ the following closed subscheme Y of X, which is an $\mathbb{F}_1$-model of $\mathcal{Y}$. In case that $X = \operatorname{Spec} B$ is the spectrum of a blueprint $B = A/\!\!/\mathcal{R}$, and thus $\mathcal{X} \simeq \operatorname{Spec} B_{\mathbb{Z}}^+$ is an affine scheme, we can define Y as $\operatorname{Spec} C$ for $C = A /\!\!/ \mathcal{R}(Y)$ where $\mathcal{R}(Y)$ is the pre-addition that contains $\sum a_i \equiv \sum b_j$ whenever $\sum a_i = \sum b_j$ holds in the coordinate ring $\Gamma\mathcal{Y}$ of $\mathcal{Y}$. This is a process that we used already in [10, Section 3].

Since localizations commute with additive closures, i.e. $(S^{-1}B)_{\mathbb{Z}}^+ = S^{-1}(B_{\mathbb{Z}}^+)$ where S is a multiplicative subset of B, the above process is compatible with the restriction to affine opens $U \subset X$. This means that given $U = \operatorname{Spec}(S^{-1}B)$, which is an $\mathbb{F}_1$-model for $\mathcal{X}' = U_{\mathbb{Z}}^+$, then the $\mathbb{F}_1$–model Y' that is associated to the closed subscheme $\mathcal{Y}' = \mathcal{X}' \times_{\mathcal{X}} \mathcal{Y}$ of $\mathcal{X}'$ by the above process is the spectrum of the blueprint $S^{-1}C$. Consequently, we can associate with every closed subscheme $\mathcal{Y}$ of a scheme $\mathcal{X}$ with an $\mathbb{F}_1$-model X a closed subscheme Y of X, which is an $\mathbb{F}_1$–model of $\mathcal{Y}$; namely, we apply the above process to all affine open subschemes of $\mathcal{X}$ and glue them together, which is possible since additive closures commute with localizations.

In case of a projective variety, i.e. a closed subscheme $\mathcal{Y}$ of a projective space ${}^+\mathbb{P}_{\mathbb{Z}}^n$, we derive the following description of the associated $\mathbb{F}_1$-model Y in $\mathbb{P}_{\mathbb{F}_1}^n$ by homogeneous coordinate rings. Let C be the homogeneous coordinate ring of $\mathcal{Y}$, which is a quotient of $\mathbb{Z}[S_0, \dots, S_n]^+$ by a homogeneous ideal I. Let $\mathcal{R}$ be the pre-addition on $\mathbb{F}_1[S_0, \dots, S_n]$ that consists of all relations $\sum a_i \equiv \sum b_j$ such that $\sum a_i = \sum b_j$ in C. Then $B = \mathbb{F}_1[S_0, \dots, S_n] /\!\!/ \mathcal{R}$ inherits a grading from $\mathbb{F}_1[S_0, \dots, S_n]$ by defining B_i as the image of $\mathbb{F}_1[S_0, \dots, S_n]_i$ in B. Note that $B \subset C$ and that the sets B_i equal the intersections $B_i = C_i \cap B$ for $i \geq 0$ where C_i is the homogeneous part of degree i of C. Then the $\mathbb{F}_1$-model Y of $\mathcal{Y}$ equals $\operatorname{Proj} B$.

5. $\mathbb{F}_1$–models for Grassmannians

One of the simplest examples of projective varieties that is not a toric variety (and in particular, not a projective space) is the Grassmann variety $\operatorname{Gr}(2, 4)$. The problem of finding models over $\mathbb{F}_1$ for Grassmann varieties was originally posed by Soulè in [12], and solved by the authors by obtaining a torification from the Schubert cell decomposition (cf. [8, 7]).

In this note, we present $\mathbb{F}_1$-models for Grassmannians as projective varieties defined through (homogeneous) blueprints. The proposed construction for the Grassmannians fits within a more general framework for obtaining blueprints and totally positive blueprints from cluster data (cf. the forthcoming preprint [6]).

Classically, the homogeneous coordinate ring for the Grassmannian $\operatorname{Gr}(k, n)$ is obtained by quotienting out the homogeneous coordinate ring of the projective space $\mathbb{P}^{\binom{n}{k}-1}$ by the homogeneous ideal generated by the Plücker relations. A similar construction can be carried out using the framework of (graded) blueprints. In what follows, we make that construction explicit for the Grassmannian $\operatorname{Gr}(2, 4)$.

Define the blueprint $\mathcal{O}_{\mathbb{F}_1}(\operatorname{Gr}(2, 4)) = \mathbb{F}_1[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}] /\!\!/ \mathcal{R}$ where the congruence $\mathcal{R}$ is generated by the Plücker relation $x_{12}x_{34} + x_{14}x_{23} \equiv x_{13}x_{24}$ (the signs have been picked to ensure that the totally positive part of the Grassmannian is preserved, cf. [6]). Since $\mathcal{R}$ is generated by a homogeneous relation, $\mathcal{O}_{\mathbb{F}_1}(\operatorname{Gr}(2, 4))$ inherits a grading from the canonical morphism

$$\pi : \mathbb{F}_1[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}] \longrightarrow \mathbb{F}_1[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}] /\!\!/ \mathcal{R}.$$

Let $\mathrm{Gr}(2,4)_{\mathbb{F}_1} := \mathrm{Proj}(\mathcal{O}_{\mathbb{F}_1}(\mathrm{Gr}(2,4)))$. The base extension $\mathrm{Gr}(2,4)^+_{\mathbb{Z}}$ is the usual Grassmannian, and π defines a closed embedding of $\mathrm{Gr}(2,4)_{\mathbb{F}_1}$ into $\mathbb{P}^5_{\mathbb{F}_1}$, which extends to the classical Plücker embedding $\mathrm{Gr}(2,4)^+_{\mathbb{Z}} \hookrightarrow {}^+\mathbb{P}^5_{\mathbb{Z}}$.

Homogeneous prime ideals in $\mathcal{O}_{\mathbb{F}_1}(Gr(2,4))$ are described by their generators as the proper subsets $I \subsetneq \{x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{25}\}$ such that I is either contained in one of the sets $\{x_{12}, x_{34}\}$, $\{x_{14}, x_{23}\}$, $\{x_{13}, x_{24}\}$, or otherwise I has a nonempty intersection with all three of them. In other words, I cannot contain elements in two of the above sets without also containing an element of the third one.

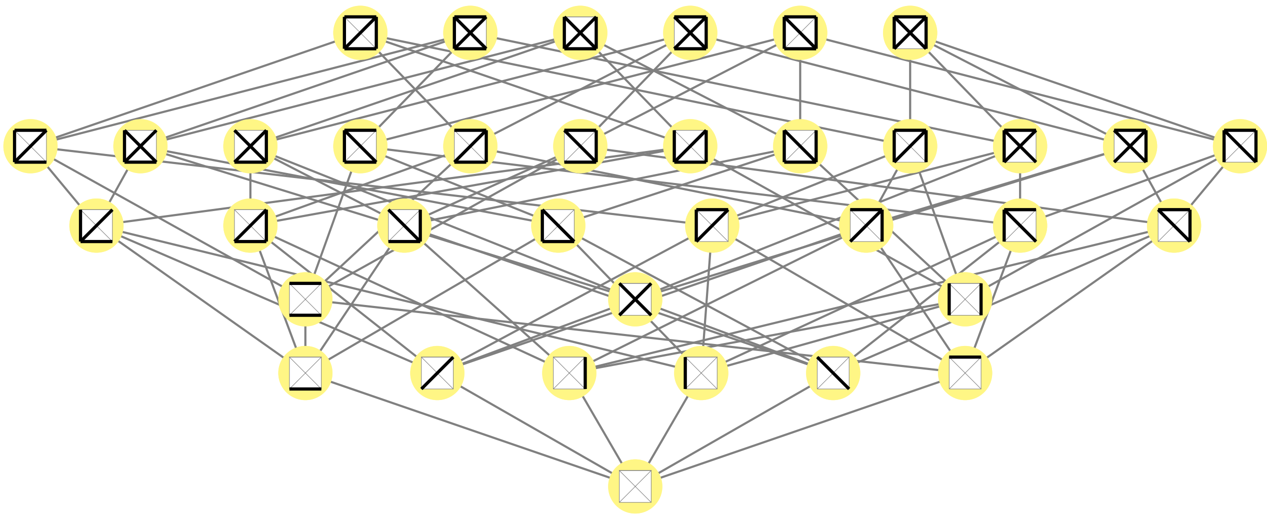

FIGURE 1. Points of the Grassmannian $\mathrm{Gr}(2,4)_{\mathbb{F}_1}$.

Generator x_{ij} belonging to an ideal is depicted as segment i–j in $\begin{smallmatrix} 4 & & 3 \\ & ⊠ & \\ 1 & & 2 \end{smallmatrix}$

The structure of the set of (homogeneous) prime ideals of $\mathcal{O}_{\mathbb{F}_1}(\mathrm{Gr}(2,4))$ is depicted in Figure 1. It consists of $6+12+11+6+1=36$ prime ideals of ranks 0, 1, 2, 3 and 4, respectively (cf. [10, Def. 2.3] for the definition of the rank of a prime ideal), thus resulting in a model essentially different to the one presented in [8] by means of torifications, which had $6+12+11+5+1=35$ points, in correspondence with the coefficients of the counting polynomial $N_{\mathrm{Gr}(2,4)}(q) = 6 + 12(q-1) + 11(q-1)^2 + 5(q-1)^3 + 1(q-1)^4$. It is worth noting that despite arising from different constructions, both $\mathbb{F}_1$-models for $\mathrm{Gr}(2,4)$ have $6 = \binom{4}{2}$ closed points, corresponding to the combinatorial interpretation of $\mathrm{Gr}(2,4)_{\mathbb{F}_1}$ as the set of all subsets with two elements inside a set with four elements. These six points correspond to the $\mathbb{F}_1$-rational Tits points of $\mathrm{Gr}(2,4)_{\mathbb{F}_1}$, which reflect the naive notion of $\mathbb{F}_1$-rational points of an $\mathbb{F}_1$-scheme (cf. [10, Section 2.2]).

Like in the classical geometrical setting, the Grassmannian $\mathrm{Gr}(2,4)_{\mathbb{F}_1}$ does admit a covering by six $\mathbb{F}_1$-models of affine 4-space, which correspond to the open subsets of $\mathrm{Gr}(2,4)_{\mathbb{F}_1}$ where one of x_{12}, x_{34}, x_{14}, x_{23}, x_{13} or x_{24} is non-zero. However, these $\mathbb{F}_1$-models of affine 4-space are not the standard model $\mathbb{A}^4_{\mathbb{F}_1} = \mathrm{Spec}\big(\mathbb{F}_1[a,b,c,d]\big)$, but the "$2\times 2$-matrices" $M_{2,\mathbb{F}_1} = \mathrm{Spec}\big(\mathbb{F}_1[a,b,c,d] /\!\!/ \langle ad \equiv bc + D\rangle\big)$ in case that one of x_{12}, x_{34}, x_{14} or x_{23} is non-zero, and the "twisted 2×2-matrices" $M^{\tau}_{2,\mathbb{F}_1} = \mathrm{Spec}\big(\mathbb{F}_1[a,b,c,d] /\!\!/ \langle ad + bc \equiv D\rangle\big)$ in case that one of x_{13} or x_{24} is non-zero.

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