Remarks on the Milnor conjecture over schemes

Asher Auel

Abstract.

The Milnor conjecture has been a driving force in the theory of quadratic forms over fields, guiding the development of the theory of cohomological invariants, ushering in the theory of motivic cohomology, and touching on questions ranging from sums of squares to the structure of absolute Galois groups. Here, we survey some recent work on generalizations of the Milnor conjecture to the context of schemes (mostly smooth varieties over fields of characteristic $\neq 2$). Surprisingly, a version of the Milnor conjecture fails to hold for certain smooth complete p-adic curves with no rational theta characteristic (this is the work of Parimala, Scharlau, and Sridharan). We explain how these examples fit into the larger context of the unramified Milnor question, offer a new approach to the question, and discuss new results in the case of curves over local fields and surfaces over finite fields.

The first cases of the (as of yet unnamed) Milnor conjecture were studied in Pfister's Habilitationsschrift [85] in the mid 1960s. As Pfister [86, p. 3] himself points out, "[the Milnor conjecture] stimulated research for quite some time." Indeed, it can be seen as one of the driving forces in the theory of quadratic forms since Milnor's original formulation [66] in the early 1970s.

The classical cohomological invariants of quadratic forms (rank, discriminant, and Clifford–Hasse–Witt invariant) have a deep connection with the history and development of the subject. In particular, they are used in the classification (Hasse–Minkowski local-global theorem) of quadratic forms over local and global fields. The first "higher invariant" was described in Arason's thesis [1], [3]. The celebrated results of Merkurjev [62] and Merkurjev–Suslin [64] settled special cases of the Milnor conjecture in the early 1980s, and served as a starting point for Voevodsky's development of the theory of motivic cohomology. Other special cases were settled by Arason–Elman–Jacob [5] and Jacob–Rost [46]. Voevodsky's motivic cohomology techniques [99] ultimately led to a complete solution of the Milnor conjecture, for which he was awarded the 2002 Fields Medal.

The consideration of quadratic forms over rings (more general than fields) has its roots in the number theoretic study of lattices (i.e. quadratic forms over $\mathbb{Z}$) by Gauss as well as the algebraic study of division algebras and hermitian forms (i.e. quadratic forms over algebras with involution) by

2010 *Mathematics Subject Classification.* Primary 11-02, 19-02; Secondary 11Exx, 19G12.

Albert. A general framework for the study of quadratic forms over rings was established by Bass [17], with the case of (semi)local rings treated in depth by Baeza [9]. Bilinear forms over Dedekind domains (i.e. unimodular lattices) were studied in a number theoretic context by Fröhlich [35], while the consideration of quadratic forms over algebraic curves (and their function fields) was initiated by Geyer, Harder, Knebusch, Scharlau [44], [39], [53], [55]. The theory of quadratic (and bilinear) forms over schemes was developed by Knebusch [54], [52], and utilized by Arason [2], Dietel [27], Parimala [76], [83], Fernández-Carmena [32], Sujatha [94], [82], Arason–Elman–Jacob [6], [7], and others. A theory of symmetric bilinear forms in additive and abelian categories was developed by Quebbemann–Scharlau–Schulte [87], [88]. Further enrichment came eventually from the triangulated category techniques of Balmer [10], [11], [12], and Walter [101]. This article will focus on progress in generalizing the Milnor conjecture to these contexts.

These remarks grew out of a lecture at the RIMS-Camp-Style seminar "Galois-theoretic Arithmetic Geometry" held October 19-23, 2010, in Kyoto, Japan. The author would like to thank the organizers for their wonderful hospitality during that time. He would also like to thank Stefan Gille, Moritz Kerz, R. Parimala, and V. Suresh for many useful conversations. The author acknowledges the generous support of the Max Plank Institute for Mathematics in Bonn, Germany where this article was written under excellent working conditions. This author is also partially supported by National Science Foundation grant MSPRF DMS-0903039.

Conventions. A graded abelian group or ring $\prod_{n\geq 0} M^n$ will be denoted by $M^\bullet$. If $0 \subset \cdots \subset N^2 \subset N^1 \subset N^0 = M$ is a decreasing filtration of a ring M by ideals, denote by $N^\bullet/N^{\bullet+1} = \prod_{n\geq 0} N^n/N^{n+1}$ the associated graded ring. Denote by ${}_2M$ the elements of order 2 in an abelian group M. All abelian groups will be written additively.

§1. The Milnor conjecture over fields

Let F be a field of characteristic $\neq 2$. The total Milnor K-ring $K_{\mathrm{M}}^\bullet(F) = T^\bullet(F^\times)/\langle a \otimes (1-a) \,:\, a \in F^\times \rangle$ was introduced in [66]. The total Galois cohomology ring $H^\bullet(F, \boldsymbol{\mu}_2^{\otimes \bullet}) = \bigoplus_{n\geq 2} H^n(F, \boldsymbol{\mu}_2^{\otimes n})$ is canonically isomorphic, under our hypothesis on the characteristic of F, to the total Galois cohomology ring $H^\bullet(F, \mathbb{Z}/2\mathbb{Z})$ with coefficients in the trivial Galois module $\mathbb{Z}/2\mathbb{Z}$. The Witt ring $W(F)$ of nondegenerate quadratic forms modulo hyperbolic forms has an decreasing filtration $0 \subset \cdots \subset I^1(F) \subset I^0(F) = W(F)$ generated by powers of the *fundamental ideal* $I(F)$ of even rank forms. The Milnor conjecture relates these three objects: Milnor K-theory, Galois cohomology, and quadratic forms.

The quotient map $K_{\mathrm{M}}^1(F) = F^\times \to F^\times/F^{\times 2} \cong H^1(X, \boldsymbol{\mu}_2)$ induces a graded ring homomorphism $h^\bullet : K_{\mathrm{M}}^\bullet(F)/2 \to H^\bullet(F, \boldsymbol{\mu}_2^{\otimes\bullet})$ called the *norm residue symbol* by Bass–Tate [16]. The *Pfister form* map $K_{\mathrm{M}}^1(F) = F^\times \to I(F)$ given by $a \mapsto \ll a \gg = < 1, -a >$ induces a group homomorphism $K_{\mathrm{M}}^1(F)/2 \to I^1(F)/I^2(F)$ (see Scharlau [91, 2 Lemma 12.10]), which extends to a surjective graded ring homomorphism $s^\bullet : K^\bullet_{\mathrm{M}}(F)/2 \to I^\bullet(F)/I^{\bullet+1}(F)$, see Milnor [66, Thm. 4.1].

Theorem 1 (Milnor conjecture)**.** *Let F be a field of characteristic $\neq 2$. There exists a graded ring homomorphism $e^\bullet : I^\bullet(F)/I^{\bullet+1}(F) \to H^\bullet(F, \boldsymbol{\mu}_2^{\otimes \bullet})$ called the* higher invariants *of quadratic forms, which fits into the following diagram*

$$\begin{tikzcd} K^\bullet_M(F)/2 \arrow[r, "h^\bullet"] \arrow[d, "s^\bullet"'] & H^\bullet(F, \boldsymbol{\mu}_2^{\otimes \bullet}) \\ I^\bullet(F)/I^{\bullet+1}(F) \arrow[ur, "e^\bullet"'] & \end{tikzcd}$$

of isomorphisms of graded rings.

Many excellent introductions to the Milnor conjecture and its proof exist in the literature. For example, see the surveys of Kahn [47], Friedlander–Rapoport–Suslin [34], Friedlander [33], Pfister [84], and Morel [68].

The conjecture breaks up naturally into three parts: the conjecture for the norm residue symbol $h^\bullet$, the conjecture for the Pfister form map $s^\bullet$, and the conjecture for the higher invariants $e^\bullet$. Milnor [66, Question 4.3, §6] originally made the conjecture for $h^\bullet$ and $s^\bullet$, which was already known for finite, local, global, and real closed fields, see [66, Lemma 6.2]. For general fields, the conjecture for h^1 follows from Hilbert's theorem 90, and for s^1 and e^1 by elementary arguments. The conjecture for s^2 is easy, see Pfister [85]. Merkurjev [62] proved the conjecture for h^2 (hence for e^2 as well), with alternate proofs given by Arason [4], Merkurjev [63], and Wadsworth [100]. The conjecture for h^3 was settled by Merkurjev–Suslin [64] (and independently by Rost [89]). The conjecture for $e^\bullet$ can be divided into two parts: to show the existence of maps $e^n : I^n(F) \to H^n(X, \boldsymbol{\mu}_2^{\otimes n})$ (which are *a priori* only defined on generators, the Pfister forms), and then to show they are surjective. The existence of e^3 was proved by Arason [1], [3]. The existence of e^4 was proved by Jacob–Rost [46] and independently Szyjewski [93]. Voevodsky [99] proved the conjecture for $h^\bullet$. Orlov–Vishik–Voevodsky [73] proved the conjecture for $s^\bullet$, with different proofs given by Morel [69] and Kahn–Sujatha [48].

1.1. Classical invariants of quadratic forms

The theory of quadratic forms over a general field has its genesis in Witt's famous paper [102]. Because of the assumption of characteristic $\neq 2$, we do not distinguish between quadratic and symmetric bilinear forms. The *orthogonal sum* $(V, b) \perp (V', b') = (V \oplus V', b + b')$ and *tensor product* $(V, b) \otimes (V', b') = (V \otimes V', b \otimes b')$ give a semiring structure on the set of isometry classes of symmetric bilinear forms over F. The *hyperbolic plane* is the symmetric bilinear form (H, h), where $H = F^2$ and $h((x, y), (x', y')) = xy' + x'y$. The *Witt ring* of symmetric bilinear forms is the quotient of the Grothendieck ring of nondegenerate symmetric bilinear forms over F with respect to $\perp$ and $\otimes$, modulo the ideal generated by the hyperbolic plane, see Scharlau [91, Ch. 2].

The *rank* of a bilinear form (V, b) is the F-vector space dimension of V. Since the hyperbolic plane has rank 2, the rank modulo 2 is a well defined invariant of an element of the Witt ring, and gives rise to a surjective ring homomorphism

$$e^0 : W(F) = I^0(F) \to \mathbb{Z}/2\mathbb{Z} = H^0(F, \mathbb{Z}/2\mathbb{Z})$$

whose kernel is the *fundamental ideal* $I(F)$.

The *signed discriminant* of a non-degenerate bilinear form (V, b) is defined as follows. Choosing an F-vector space basis $v_1, \ldots, v_r$ of V, we consider the Gram matrix M_b of b, i.e. the matrix given by $M_b = (b(v_i, v_j))$. Then b is given by the formula $b(v, w) = v^t M_b w$, where $v, w \in F^r \cong V$. The Gram matrix of b, with respect to a different basis for V with change of basis matrix T, is $T^t M_b T$. Thus $\det M_b \in F^\times$, which depends on the choice of basis, is only well-defined up to squares. For $a \in F^\times$, denote by (a) its class in the abelian group $F^\times/F^{\times 2}$. The signed discriminant of (V, b) is defined as $(-1)^{r(r-1)/2} \det M_b \in F^\times/F^{\times 2}$. Introducing the sign into the signed discriminant ensures its vanishing on the ideal of hyperbolic forms, hence it descents to the Witt group. While the signed discriminant is not additive on $W(F)$, its restriction to $I(F)$ gives rise to a group homomorphism

$$e^1 : I(F) \to F^\times/F^{\times 2} \cong H^1(F, \boldsymbol{\mu}_2)$$

which is easily seen to be surjective. It's then not difficult to check that its kernel coincides with the square $I^2(F)$ of the fundamental ideal. See Scharlau [91, §2.2] for more details.

The *Clifford invariant* of a non-degenerate symmetric bilinear form (V, b) is defined in terms of its Clifford algebra. The *Clifford algebra* $C(V, b)$ of (V, b) is the quotient of the tensor algebra $T(V) = \bigoplus_{r \geq 0} V^{\otimes r}$ by the two-sided ideal generated by $\{v \otimes w + w \otimes v - b(v, w) \ : \ v, w \in V\}$. If (V, b) has rank r, then $C(V, b) = C_0(V, b) \oplus C_1(V, b)$ inherits the structure of a $\mathbb{Z}/2\mathbb{Z}$-graded semisimple F-algebra of F-dimension 2^r, see Scharlau [91, §9.2]. By the structure theory of the Clifford algebra, $C(V, b)$ or $C_0(V, b)$ is a central simple F-algebra depending on whether (V, b) has even or odd rank, respectively. The *Clifford invariant* $c(V, b) \in \mathrm{Br}(F)$ is then the Brauer class of $C(V, b)$ or $C_0(V, b)$, respectively. Since the Clifford algebra and its even subalgebra carry canonical involutions of the first kind, their respective classes in the Brauer group are of order 2, see Knus [57, §IV.7.8]. While the Clifford invariant is not additive on $W(F)$, its restriction to $I^2(F)$ gives rise to a group homomorphism

$$e^2 : I^2(F) \to {}_2\mathrm{Br}(F) \cong H^2(F, \boldsymbol{\mu}_2) \cong H^2(F, \boldsymbol{\mu}_2^{\otimes 2}),$$

see Knus [57, IV Prop. 8.1.1].

Any symmetric bilinear form (V,b) over a field of characteristic $\neq 2$ can be diagonalized, i.e. a basis can be chosen for V so that the Gram matrix M_b is diagonal. For $a_1,\dots,a_r \in F^\times$, we write $<a_1,\dots,a_r>$ for the standard symmetric bilinear form with associated diagonal Gram matrix. For $a,b \in F^\times$, denote by $(a,b)_F$ the (quaternion) F-algebra generated by symbols x and y subject to the relations $x^2 = a$, $y^2 = b$, and $xy = -yx$. For example, $(-1,-1)_{\mathbb{R}}$ is Hamilton's ring of quaternions. Then the discriminant and Clifford invariant can be conveniently calculated in terms of a diagonalization. For $(V,b) \cong <a_1,\dots,a_r>$, we have

$$d_\pm(V,b) = ((-1)^{r(r-1)/2}a_1\cdots a_r) \in F^\times/F^{\times 2}$$

and

$$(1)\quad c(V,b) = \alpha(r)(-1,a_1\cdots a_r)_F + \beta(r)(-1,-1)_F + \sum_{i<j}(a_j,a_j)_F \in {}_2\mathrm{Br}(F)$$

where

$$\alpha(r) = \frac{(r-1)(r-2)}{2}, \qquad \beta(r) = \frac{(r+1)r(r-1)(r-2)}{24},$$

see Lam [59], Scharlau [91, II.12.7] or Esnault–Kahn–Levine–Viehweg [30, §1].

§2. Globalization of cohomology theories

Generalizations (what we will call *globalizations*) of the Milnor conjecture to the context of rings and schemes have emerged from many sources, see Parimala [75], Colliot-Thélène–Parimala [21], Parimala–Sridharan [78], Monnier [67], Pardon [74], Elbaz-Vincent–Müller-Stach [28], Gille [41], and Kerz [49]. To begin with, one must ask for appropriate globalizations of the objects in the conjecture: Milnor K-theory, Galois cohomology theory, and the Witt group with its fundamental filtration. While there are many possible choices of such globalizations, we will focus on two types: global and unramified.

2.1. Global globalization

Let F be a field of characteristic $\neq 2$. Let Field_F (resp. Ring_F) be the category of fields (resp. commutative unital rings) with an F-algebra structure together with F-algebra homomorphisms. Let Sch_F be the category of F-schemes, and Sm_F the category of smooth F-schemes. We will denote, by the same names, the associated (large) Zariski sites. Let Ab (resp. $\mathsf{Ab}^\bullet$) be the category of abelian groups (resp. graded abelian groups), we will always consider Ab as embedded in $\mathsf{Ab}^\bullet$ in degree 0.

Let $M^\bullet : \mathsf{Field}_F \to \mathsf{Ab}^\bullet$ be a functor. A *globalization* of $M^\bullet$ to rings (resp. schemes) is a functor $\tilde{M}^\bullet : \mathsf{Ring}_F \to \mathsf{Ab}^\bullet$ (resp. contravariant functor $\tilde{M}^\bullet : \mathsf{Sch}_F \to \mathsf{Ab}^\bullet$) extending $M^\bullet$. If $\tilde{M}^\bullet$ is a globalization of $M^\bullet$ to rings, then we can define a globalization to schemes by taking the sheaf $\mathcal{M}^\bullet$ associated to the presheaf $U \mapsto \tilde{M}^\bullet(\Gamma(U, \mathscr{O}_U))$ on Sch_F (always considered with the Zariski topology).

"Naïve" Milnor K-theory. For a commutative unital ring R, mimicking Milnor's tensorial construction (with the additional relation that $a \otimes (-a) = 0$, which is automatic for fields) yields a graded ring $K^\bullet_{\mathrm{M}}(R)$, which should be referred to as "naïve" Milnor K-theory. This already appears in Guin [43, §3] and later studied by Elbaz-Vincent–Müller-Stach [28]. Naïve Milnor K-theory has some bad properties when R has small finite residue fields, see Kerz [50] who also provides a improved Milnor K-theory repairing these defects. Thomason [96] has shown that there exists no globalization of Milnor K-theory to smooth schemes which satisfies $\mathbb{A}^1$-homotopy invariance and has a functorial homomorphism to algebraic K-theory.

Étale cohomology. Étale cohomology provides a natural globalization of Galois cohomology to schemes. We will thus consider the functor $X \mapsto H^\bullet_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes \bullet})$ on Sch_F.

Global Witt group. For a scheme X, the *global* Witt group $W(X)$ of regular symmetric bilinear forms introduced by Knebusch [52] provides a natural globalization of the Witt group to schemes. Other possible globalizations are obtained from the Witt groups of triangulated category with duality introduced by Balmer [10], [11], [12], [13]. These include: the *derived* Witt group of the bounded derived category of coherent locally free $\mathscr{O}_X$-modules; the *coherent* Witt group of the bounded derived category of quasicoherent $\mathscr{O}_X$-modules with coherent cohomology (assuming X has a dualizing complex, see Gille [40, §2.5], [41, §2]); the *perfect* Witt group of the derived category of perfect complexes of $\mathscr{O}_X$-modules. The global and derived Witt groups are canonically isomorphic by Balmer [12, Thm. 4.3]. All of the above Witt groups are isomorphic (though not necessarily canonically) if X is assumed to be regular.

Fundamental filtration and the classical invariants. Globalizations of the classical invariants of quadratic forms are defined as follows. Let $(\mathscr{E}, q)$ be a regular symmetric bilinear form of rank n on X.

The rank (modulo 2) of $\mathscr{E}$ gives rise to a functorial homomorphism

$$e^0 : W(X) \to \mathrm{Hom}_{\mathrm{cont}}(X, \mathbb{Z}/2\mathbb{Z}) = H^0_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}),$$

whose kernel $I^1(X)$ is called the *fundamental ideal* of $W(X)$.

The *signed discriminant form* $(\det \mathscr{E}, (-1)^{n(n-1)/2} \det q)$ gives rise to a functorial homomorphism

$$e^1 : I^1(X) \to H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$$

see Knus [57, III §4.2] and Milne [65, III §4]. Alternatively, the center of the (even) Clifford $\mathscr{O}_X$-algebra of $(\mathscr{E}, q)$ defines a class in $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ called the *Arf invariant*, which coincides with the signed discriminant under the canonical morphism $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \to H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$ (see Knus [57, IV Prop. 4.6.3] or Parimala–Srinivas [81, §2.2]). Denote the kernel of e^1 by $I^2(X)$, which is an ideal of $W(X)$. Note that $I^2(X)$ may not be the square of the ideal $I^1(X)$.

The Clifford $\mathscr{O}_X$-algebra $\mathscr{C}(\mathscr{E}, q)$ gives rise to a functorial homomorphism

$$e^2 : I^2(X) \to {}_2\mathrm{Br}(X)$$

called the *Clifford invariant*, see Knus–Ojanguren [58] and Parimala–Srinivas [81, §2]. Denote the kernel of e^2 by $I^3(X)$, which is an ideal of $W(X)$.

As Parimala–Srinivas [81, p. 223] point out, there is no functorial map $I^2(X) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$ lifting the Clifford invariant. Instead, we can work with Grothendieck–Witt groups. The rank (modulo 2) gives rise to a functorial homomorphism

$$ge^0 : GW(X) \to H^0_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$$

with kernel denoted by $GI(X)$. The signed discriminant gives rise to a functorial homomorphism

$$ge^1 : GI^1(X) \to H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$$

with kernel denoted by $GI^2(X)$. The class of the Clifford $\mathscr{O}_X$-algebra, together with it's canonical involution (via the "involutive" Brauer group construction of Parimala–Srinivas [81, §2]), gives rise to a functorial homomorphism

$$ge^2 : GI^2(X) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$$

also see Knus–Parimala–Sridharan [56]. Denote the kernel of ge^2 by $GI^3(X)$, which is an ideal of $GW(X)$.

Lemma 2.1. *Let X be a scheme satisfying ${}_2\mathrm{Br}(X) \cong {}_2H^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$. Then under the quotient map $GW(X) \to W(X)$, the image of the ideal $GI^n(X)$ is precisely the ideal $I^n(X)$, for $n \leq 3$.*

Proof. For $n = 1, 2$ this is a consequence of the following diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K_0(X) \arrow[r, equal] \arrow[d, "H"] & K_0(X) \arrow[d, "H"] & & \\
0 \arrow[r] & GI^n(X) \arrow[r] \arrow[d] & GI^{n-1}(X) \arrow[r, "ge^{n-1}"] \arrow[d] & H^{n-1}_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I^n(X) \arrow[r] \arrow[d] & I^{n-1}(X) \arrow[r, "e^{n-1}"] \arrow[d] & H^{n-1}_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

which is commutative since hyperbolic spaces have even rank and trivial signed discriminant. Here, $K_0(X)$ is the Grothendieck group of locally free

$\mathscr{O}_X$-modules of finite rank and H is the hyperbolic map $\mathscr{V} \mapsto H(\mathscr{V}) = \big(\mathscr{V} \oplus \mathscr{V}^\vee, ((v,f),(w,g)) \mapsto f(w) + g(x)\big)$.

For $n = 3$, we have the formula $ge^2(H(\mathscr{V})) = c_1(\mathscr{V}, \boldsymbol{\mu}_2)$ due to Esnault–Kahn–Viehweg [29, Prop. 5.5] (combined with (1)). Here $c_1(-, \boldsymbol{\mu}_2)$ is the 1st Chern class modulo 2, defined as the first coboundary map in the long-exact sequence in étale cohomology

$$\cdots \to \mathrm{Pic}(X) \xrightarrow{2} \mathrm{Pic}(X) \xrightarrow{c_1} H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \to H^2_{\text{ét}}(X, \mathbb{G}_\mathrm{m}) \xrightarrow{2} H^2_{\text{ét}}(X, \mathbb{G}_\mathrm{m}) \to \cdots$$

arising from the Kummer exact sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbb{G}_\mathrm{m} \xrightarrow{2} \mathbb{G}_\mathrm{m} \to 1,$$

see Grothendieck [42]. The claim then follows by a diagram chase through

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_0'(X) \arrow[r] \arrow[d, "H"] & K_0(X) \arrow[r, "\det"] \arrow[d, "H"] & \mathrm{Pic}(X)/2 \arrow[r] \arrow[d, "c_1"] & 0 \\
0 \arrow[r] & GI^3(X) \arrow[r] \arrow[d] & GI^2(X) \arrow[r, "ge^2"] \arrow[d] & H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \arrow[d] & \\
0 \arrow[r] & I^3(X) \arrow[r] \arrow[d] & I^2(X) \arrow[r, "e^2"] \arrow[d] & {}_2\mathrm{Br}(X) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the right vertical column arises from the Kummer sequence, and $K_0'(X)$ is the subgroup of $K_0(X)$ generated by locally free $\mathscr{O}_X$-modules whose determinant is a square. Q.E.D.

Remark 2.1. The hypothesis that ${}_2\mathrm{Br}(X) = {}_2H^2_{\text{ét}}(X, \mathbb{G}_\mathrm{m})$ is satisfied if X is a quasi-compact quasi-separated scheme admitting an ample invertible sheaf by de Jong's extension [26] (see also [61, Th. 2.2.2.1]) of a result of Gabber [36].

The existence of global globalizations of the higher invariants (e.g. a globalization of the Arason invariant) remains a mystery. Esnault–Kahn–Levine–Viehweg [30] have shown that for a regular symmetric bilinear form $(\mathscr{E}, q)$ that represents a class in $GI^3(X)$, the obstruction to having an Arason invariant in $H^3_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ is precisely the 2nd Chern class $c_2(\mathscr{E}) \in CH^2(X)/2$ in the Chow group modulo 2 (note that the invariant $c(\mathscr{E}) \in \mathrm{Pic}(X)/2$ of [30] is trivial if $(\mathscr{E}, q)$ represents a class in $GI^3(X)$). They also provide examples where this obstruction does not vanish. On the other hand, higher cohomological invariants always exist in unramified cohomology.

2.2. Unramified globalization

A functorial framework for the notion of "unramified element" is established in Colliot-Thélène [20, §2]. See also the survey by Zainoulline [103, §3]. Rost [90, Rem. 5.2] gives a different perspective in terms of cycle modules, also see Morel [69, §2]. Assume that X has finite Krull dimension and is equidimensional over a field F. For simplicity of exposition, assume that X is integral. Denote by $X^{(i)}$ its set of codimension i points.

Denote by Local_F the category of local F-algebras together with local F-algebra morphisms. Given a functor $M^\bullet : \mathsf{Local}_F \to \mathsf{Ab}^\bullet$, call

$$M^\bullet_{\mathrm{ur}}(X) = \bigcap_{x \in X^{(1)}} \mathrm{im}\big(M^\bullet(\mathscr{O}_{X,x}) \to M^\bullet(F(X))\big)$$

the group of *unramified elements* of $M^\bullet$ over X. Then $X \mapsto M^\bullet_{\mathrm{ur}}(X)$ is a globalization of $M^\bullet$ to schemes.

Given a functor $M^\bullet : \mathsf{Sch}_F \to \mathsf{Ab}^\bullet$, there is a natural map $M^\bullet(X) \to M^\bullet_{\mathrm{ur}}(X)$. If this map is injective, surjective, or bijective we say that the *injectivity*, *weak purity*, or *purity* property hold, respectively. Whether these properties hold for various functors $M^\bullet$ and schemes X is the subject of many conjectures and open problems, see Colliot-Thélène [20, §2.2] for examples.

Unramified Milnor K-theory. Define the *unramified Milnor K-theory (resp. modulo 2)* of X to be the graded ring of unramified elements $K^\bullet_{\mathrm{M,ur}}(X)$ (resp. $K^\bullet_{\mathrm{M,ur}}/2(X)$) of the "naïve" Milnor K-theory (resp. modulo 2) functor $K^\bullet_{\mathrm{M}}$ (resp. $K^\bullet_{\mathrm{M}}/2$) restricted to Local_F, see §2.1. Let $\mathcal{K}^\bullet_{\mathrm{M}}$ be the Zariski sheaf on Sch_F associated to "naïve" Milnor K-theory and $\mathcal{K}^\bullet_{\mathrm{M}}/2$ the associate sheaf quotient, which is the Zariski sheaf associated to the presheaf $U \mapsto K^\bullet_{\mathrm{M}}(U)/2$, see Morel [69, Lemma 2.7]. Then $K^\bullet_{\mathrm{M,ur}}(X) = \Gamma(X, \mathcal{K}^\bullet_{\mathrm{M}})$ and $K^\bullet_{\mathrm{M,ur}}/2(X) = \Gamma(X, \mathcal{K}^\bullet_{\mathrm{M}}/2)$ when X is smooth over an infinite field (compare with the remark in §2.1) by the Bloch–Ogus–Gabber theorem for Milnor K-theory, see Colliot-Thélène–Hoobler–Kahn [23, Cor. 5.1.11, §7.3(5)]. Also, see Kerz [49]. Note that the long exact sequence in Zariski cohomology yields a short exact sequence

$$0 \to K^\bullet_{\mathrm{M,ur}}(X)/2 \to K^\bullet_{\mathrm{M,ur}}/2(X) \to {}_2H^1(X, \mathcal{K}^\bullet_{\mathrm{M}}) \to 0$$

still assuming X is smooth over an infinite field.

Unramified cohomology. Define the *unramified étale cohomology (modulo 2)* of X to be the graded ring of unramified elements $H^\bullet_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes\bullet})$ of the functor $H^\bullet_{\text{ét}}(-, \boldsymbol{\mu}_2^{\otimes\bullet})$. Letting $\mathcal{H}^\bullet_{\text{ét}}$ be the Zariski sheaf on Sch_F associated to the functor $H^\bullet_{\text{ét}}(-, \boldsymbol{\mu}_2^{\otimes\bullet})$, then $\Gamma(X, \mathcal{H}^\bullet_{\text{ét}}) = H^\bullet_{\mathrm{ur}}(X, \mathbb{Z}/2\mathbb{Z})$ when X is smooth over a field of characteristic $\neq 2$ by the exactness of the Gersten complex for étale cohomology, see Bloch–Ogus [19, §2.1, Thm. 4.2].

Unramified fundamental filtration of the Witt group. Define the *unramified Witt group* of X to be the abelian group of unramified elements $W_{\mathrm{ur}}(X)$ of the global Witt group functor W. Letting $\mathcal{W}$ be the Zariski sheaf associated to the global Witt group functor, then $W_{\mathrm{ur}}(X) = \Gamma(X, \mathcal{W})$ when X

is regular over a field of characteristic $\neq 2$ by Ojanguren–Panin [72] (also see Morel [69, Thm. 2.2]). Writing $I^n_{\mathrm{ur}}(X) = I^n(F(X)) \cap W_{\mathrm{ur}}(X)$, then the functors $I^n_{\mathrm{ur}}(-)$ are Zariski sheaves (still assuming X is regular), denoted by $\mathcal{I}^n$, which form a filtration of $\mathcal{W}$, see Morel [69, Thm. 2.3].

Note that the long exact sequence in Zariski cohomology yields short exact sequences

$$0 \to I^n_{\mathrm{ur}}(X)/I^{n+1}_{\mathrm{ur}}(X) \to \mathcal{I}^n/\mathcal{I}^{n+1}(X) \to H^1(X, \mathcal{I}^{n+1})' \to 0$$

where $H^1(X, \mathcal{I}^{n+1})' = \ker\bigl(H^1(X, \mathcal{I}^n) \to H^1(X, \mathcal{I}^{n+1})\bigr)$ and we are still assuming X is regular over a field of characteristic $\neq 2$. If the obstruction group $H^1(X, \mathcal{I}^{n+!})'$ is nontrivial, then not every element of $\mathcal{I}^n/\mathcal{I}^{n+1}(X)$ is represented by a quadratic form on X. If X is the spectrum of a regular local ring, then $I^\bullet_{\mathrm{ur}}(X)/I^{\bullet+1}_{\mathrm{ur}}(X) = \mathcal{I}^\bullet/\mathcal{I}^{\bullet+1}(X)$, see Morel [69, Thm. 2.12].

Remark 2.2. As before, the notation $I^n_{\mathrm{ur}}(X)$ does not necessarily mean the nth power of $I_{\mathrm{ur}}(X)$. This is true, however, when X is the spectrum of a regular local ring containing an infinite field of characteristic $\neq 2$, see Kerz–Müller-Stach [51, Cor. 0.5].

2.3. Gersten complexes

Gersten complexes (Cousin complexes) exists in a very general framework, but for our purposes, we will only need the Gersten complex for Milnor K-theory, étale cohomology, and Witt groups.

Gersten complex for Milnor K-theory. Let X be a regular excellent integral F-scheme. Let $C(X, K^n_{\mathrm{M}})$ denote the Gersten complex for Milnor K-theory

$$0 \longrightarrow K^n_{\mathrm{M}}(F(X)) \xrightarrow{\partial^{K^n_{\mathrm{M}}}} \bigoplus_{x \in X^{(1)}} K^{n-1}_{\mathrm{M}}(F(x)) \xrightarrow{\partial^{K^{n-1}_{\mathrm{M}}}} \bigoplus_{y \in X^{(2)}} K^{n-2}_{\mathrm{M}}(F(y)) \longrightarrow \cdots$$

where $\partial^{K_{\mathrm{M}}}$ is the "tame symbol" homomorphism defined in Milnor [66, Lemma 2.1]. We have that $H^0(C(X, K^n_{\mathrm{M}})) = K^n_{\mathrm{M,ur}}(X)$. See Rost [90, §1] or Fasel [31, Ch. 2] for more details. We will also consider the Gersten complex $(C, K^n_{\mathrm{M}}/2)$ for Milnor K-theory modulo 2, for which we have that $H^0(C(X, K^n_{\mathrm{M}}/2)) = K^n_{\mathrm{M,ur}}/2(X)$.

Gersten complex for étale cohomology. Let X be a smooth integral F-scheme, with F of characteristic $\neq 2$. Let $C(X, H^n)$ denote the Gersten complex for étale cohomology

$$0 \longrightarrow H^n(F(X)) \xrightarrow{\partial^{H^n}} \bigoplus_{x \in X^{(1)}} H^{n-1}(F(x)) \xrightarrow{\partial^{H^{n-1}}} \bigoplus_{y \in X^{(2)}} H^{n-2}(F(y)) \longrightarrow \cdots$$

where $H^n(-) = H^n(-, \boldsymbol{\mu}_2^{\otimes n})$ and ∂^H is the homomorphism induced from the spectral sequence associated to the coniveau filtration, see Bloch–Ogus [19]. Then we have that $C(X, H^n)$ is a resolution of $H^n_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes n})$.

Gersten complex for Witt groups. Let X be a regular integral F-scheme of finite Krull dimension. Let $C(X, W)$ denote the Gersten–Witt complex

$$0 \longrightarrow W(F(X)) \xrightarrow{\partial^W} \bigoplus_{x\in X^{(1)}} W(F(x)) \xrightarrow{\partial^W} \bigoplus_{y\in X^{(2)}} W(F(y)) \longrightarrow \cdots$$

where ∂^W is the homomorphism induced from the second residue map for a set of choices of local parameters, see Balmer–Walter [15]. Because of these choices, $C(X, W)$ is only defined up to isomorphism, though there is a canonical complex defined in terms of Witt groups of finite length modules over the local rings of points. We have that $H^0(C(X, W)) = W_{\mathrm{ur}}(X)$.

Fundamental filtration. The filtration of the Gersten complex for Witt groups induced by the fundamental filtration was first studied methodically by Arason–Elman–Jacob [5], see also Parimala–Sridharan [78], Gille [41], and Fasel [31, §9].

The differentials of the Gersten complex for Witt groups respect the fundamental filtration as follows:

$$\partial^{I^n}\Big(\bigoplus_{x\in X^{(p)}} I^n(F(x))\Big) \subset \bigoplus_{y\in X^{(p+1)}} I^{n-1}(F(y)),$$

see Fasel [31, Thm. 9.2.4] and Gille [41]. Thus for all $n \geq 0$ we have complexes $C(X, I^n)$

$$0 \longrightarrow I^n(F(X)) \xrightarrow{\partial^{I^n}} \bigoplus_{x\in X^{(1)}} I^{n-1}(F(x)) \xrightarrow{\partial^{I^{n-1}}} \bigoplus_{y\in X^{(2)}} I^{n-2}(F(y)) \longrightarrow \cdots$$

which provide a filtration of $C(X, W)$ in the category of complexes of abelian groups. Here we write $I^n(-) = W(-)$ for $n \leq 0$. We have that $H^0(C(X, I^n)) = I^n_{\mathrm{ur}}(X)$.

The canonical inclusion $C(X, I^{n+1}) \to C(X, I^n)$ respects the differentials, and so defines a cokernel complex $C(X, I^n/I^{n+1})$

$$0 \longrightarrow I^n/I^{n+1}(F(X)) \longrightarrow \bigoplus_{x\in X^{(1)}} I^{n-1}/I^n(F(x)) \longrightarrow \bigoplus_{y\in X^{(2)}} I^{n-2}/I^{n-1}(F(y)) \longrightarrow \cdots$$

see Fasel [31, Déf. 9.2.10], where $I^n/I^{n+1}(L) = I^n(L)/I^{n+1}(L)$ for a field L. We have that $H^0(C(X, I^n/I^{n+1})) = \mathcal{I}^n/\mathcal{I}^{n+1}(X)$

Unramified norm residue symbol. The norm residue symbol for fields provides a morphism of complexes $h^n : C(X, K^n_{\mathrm{M}}/2) \to C(X, H^n)$, where the map on terms of degree j is h^{n-j}. By the Milnor conjecture for fields, this is an isomorphism of complexes. Upon restriction, we have an isomorphism $h^n_{\mathrm{ur}} : K^n_{\mathrm{M,ur}}/2(X) \to H^n_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes n})$. Upon further restriction, we have an injection $h^n_{\mathrm{ur}} : K^n_{\mathrm{M,ur}}(X)/2 \to H^n_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes n})$.

Unramified Pfister form map. The Pfister form map for fields provides a morphism of complexes $s^n : C(X, K^n_{\mathrm{M}}/2) \to C(X, I^n/I^{n+1})$, where the map on terms of degree j is s^{n-j}. Upon restriction, we have a homomorphism $s^n_{\mathrm{ur}} : K^n_{\mathrm{M,ur}}/2(X) \to \mathcal{I}^n/\mathcal{I}^{n+1}(X)$. See Fasel [31, Thm. 10.2.6].

Unramified higher cohomological invariants. By the Milnor conjecture for fields, there exists a higher cohomological invariant morphism of complexes $e^n : C(X, I^n) \to C(X, H^n)$, where the map on terms of degree j is e^{n-j}. Upon restriction, we have homomorphisms $e^n_{\mathrm{ur}} : I^n_{\mathrm{ur}}(X) \to H^n_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes n})$ factoring through to $e^n_{\mathrm{ur}} : I^n_{\mathrm{ur}}(X)/I^{n+1}_{\mathrm{ur}}(X) \to H^n_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes n})$.

Furthermore, on the level of complexes, the higher cohomological invariant morphisms factors through to a morphism of complexes $e^n : C(X, I^n/I^{n+1}) \to C(X, H^n)$, which by the Milnor conjecture over fields, is an isomorphism. Upon restriction, we have isomorphisms $e^n_{\mathrm{ur}} : \mathcal{I}^n/\mathcal{I}^{n+1}(X) \to H^n_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes n})$. Also see Morel [69, §2.3].

2.4. Motivic globalization

There is another important globalization of Milnor K-theory and Galois cohomology, but we only briefly mention it here. Conjectured to exist by Beĭlinson [18] and Lichtenbaum [60], and then constructed by Voevodsky [99], motivic complexes modulo 2 give rise to Zariski and étale motivic cohomology groups modulo 2 $H^n_{\mathrm{Zar}}(X, \mathbb{Z}/2\mathbb{Z}(m))$ and $H^n_{\mathrm{\acute{e}t}}(X, \mathbb{Z}/2\mathbb{Z}(m))$.

For a field F, Nesterenko–Suslin [70] and Totaro [97] establish a canonical isomorphism $H^n_{\mathrm{Zar}}(\operatorname{Spec} F, \mathbb{Z}/2\mathbb{Z}(n)) \cong K^2_{\mathrm{M}}(F)/2$ while the work of Bloch, Gabber, and Suslin (see the survey by Geisser [37, §1.3.1]) establishes an isomorphism $H^n_{\mathrm{\acute{e}t}}(\operatorname{Spec} F, \mathbb{Z}/2\mathbb{Z}(n)) \cong H^n(F, \boldsymbol{\mu}_2^{\otimes n})$ (for F of characteristic $\neq 2$). The natural pullback map

$$\varepsilon^* : H^n_{\mathrm{Zar}}(\operatorname{Spec} F, \mathbb{Z}/2\mathbb{Z}(n)) \to H^n_{\mathrm{\acute{e}t}}(\operatorname{Spec} F, \mathbb{Z}/2\mathbb{Z}(n))$$

induced from the change of site $\varepsilon : X_{\mathrm{\acute{e}t}} \to X_{\mathrm{Zar}}$ is then identified with the norm residue homomorphism. Thus $H^n_{\mathrm{Zar}}(-, \mathbb{Z}/2\mathbb{Z}(n))$ and $H^n_{\mathrm{\acute{e}t}}(-, \mathbb{Z}/2\mathbb{Z}(n))$ provide *motivic* globalizations of the mod 2 Milnor K-theory and Galois cohomology functors, respectively. On the other hand, there does not seem to exist a direct motivic globalization of the Witt group or its fundamental filtration.

§3. Globalization of the Milnor conjecture

Unramified. Let F be a field of characteristic $\neq 2$. Summarizing the results cited in §2.2–2.3, we have a commutative diagram of isomorphisms

$$\begin{tikzcd} \mathcal{K}^\bullet_{\mathrm{M}}/2 \arrow[r, "h^\bullet"] \arrow[d, "s^\bullet"'] & \mathcal{H}^\bullet_{\mathrm{\acute{e}t}} \\ \mathcal{I}^\bullet/\mathcal{I}^{\bullet+1} \arrow[ur, "e^\bullet"'] & \end{tikzcd}$$

of sheaves of graded abelian groups on Sm_F. In particular, we have such a commutative diagram of isomorphisms on the level of global sections. What we will consider as a globalization of the Milnor conjecture — the *unramified Milnor question* — is a refinement of this.

(S) Let X be a smooth scheme over a field of characteristic $\neq 2$. Then restricting $s^\bullet_{\mathrm{ur}}$ gives rise to a homomorphism $s^\bullet_{\mathrm{ur}} : K^\bullet_{\mathrm{M,ur}}(X)/2 \to I^\bullet_{\mathrm{ur}}(X)/I^{\bullet+1}_{\mathrm{ur}}(X)$.

Question 3.1 (Unramified Milnor question)**.** *Let X be a smooth scheme over a field of characteristic $\neq 2$. Consider the following diagram:*

$$\begin{tikzcd}
K^\bullet_{M,\mathrm{ur}}(X)/2 \arrow[r, hook, "i^\bullet_K"] \arrow[d, dashed, "?"'] & \mathcal{K}^\bullet_M/2(X) \arrow[r, "h^\bullet_{\mathrm{ur}}"] \arrow[d, "s^\bullet_{\mathrm{ur}}"'] & H^\bullet_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes \bullet}) \\
I^\bullet_{\mathrm{ur}}(X)/I^{\bullet+1}_{\mathrm{ur}}(X) \arrow[r, hook, "i^\bullet_I"] & \mathcal{I}^n/\mathcal{I}^{n+1}(X) \arrow[ur, "e^\bullet_{\mathrm{ur}}"'] &
\end{tikzcd}$$

(1) *Is the inclusion $i^\bullet_I$ surjective?*
(2) *Is the inclusion $i^\bullet_K$ surjective?*
(3) *Does the restriction of $s^\bullet_{\mathrm{ur}}$ to $K^\bullet_{M,\mathrm{ur}}(X)/2$ have image contained in $I^\bullet_{\mathrm{ur}}(X)/I^{\bullet+1}_{\mathrm{ur}}(X)$? If so, is it an isomorphism?*

Note that in degree n, Questions 3.1 (1), (2), and (3) can be rephrased in terms of the obstruction groups, respectively: does $H^1(X, \mathcal{I}^{n+1})'$ vanish; does ${}_2H^1(X, \mathcal{K}^n_{\mathrm{M}})$ vanish; and does the restriction of s^n_{ur} yield a map ${}_2H^1(X, \mathcal{K}^n_{\mathrm{M}}) \to H^1(X, \mathcal{I}^{n+1})'$ and is it an isomorphism?

From now on we shall focus mainly on the unramified Milnor question for quadratic forms (i.e. Question 3.1(1)), which was already explicitly asked by Parimala–Sridharan [78, Question Q].

Global Grothendieck–Witt. We mention a global globalization of the Milnor conjecture for quadratic forms. Because of the conditional definition of the global cohomological invariants, we restrict ourselves to the classical invariants on Grothendieck–Witt groups defined in §2.1.

Question 3.2 (Global Merkurjev question)**.** *Let X be a regular scheme with 2 invertible. For $n \leq 2$, consider the homomorphisms,*

$$ge^n : GI^n(X)/GI^{n+1}(X) \to H^n_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$$

induced from the (classical) cohomological invariants on Grothendieck–Witt groups. Are they surjective?

This can be viewed as a globalization of Merkurjev's theorem. Indeed, first note that the cases $n = 0, 1$ of Question 3.2 are easy. Next, a consequence of Lemma 2.1 is that $ge^2 : GI^2(X) \to H^2_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ is surjective if and only $e^2 : I^2(X) \to {}_2\mathrm{Br}(X)$ is surjective. This, in turn, is a consequence of a positive answer to Question 3.1(1) for any X satisfying weak purity for the Witt group (see §3.1 for examples).

Motivic. The globalization of the Milnor conjecture for Milnor K-theory using Zariski and étale motivic cohomology modulo 2 (see §2.4) is the (n, n) modulo 2 case of the Beĭlinson–Lichtenbaum conjecture: for a smooth variety X over a field, the canonical map $H^n_{\mathrm{Zar}}(X, \mathbb{Z}/2\mathbb{Z}(m)) \to H^n_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}(m))$ is an isomorphism for $n \leq m$. The combined work of Suslin–Voevodsky [95] and Geisser–Levine [38] show the Beĭlinson–Lichtenbaum conjecture to be a consequence of the Bloch–Kato conjecture, a proof of which has been announced by Rost and Voevodsky.

3.1. Some purity results

In this section we review some of the purity results (see §2.2) relating the global and unramified Witt groups and cohomology.

Purity results for Witt groups. For a survey on purity results for Witt groups, see Zainoulline [103]. Purity for the global Witt group means that the natural map $W(X) \to W_{\mathrm{ur}}(X)$ is an isomorphism.

Theorem 2. *Let X be a regular integral noetherian scheme with 2 invertible. Then purity holds for the global Witt group functor under the following hypotheses:*

(1) *X is dimension ≤ 3,*
(2) *X is the spectrum of a regular local ring of dimension ≤ 4,*
(3) *X is the spectrum of a regular local ring containing a field.*

For part (1), the case of dimension ≤ 2 is due to Colliot-Thélène–Sansuc [22, Cor. 2.5], the case of dimension 3 and X affine is due to Ojanguren–Parimala–Sridharan–Suresh [71], and for the general case (as well as (2)) see Balmer–Walter [15]. For (3), see Ojanguren–Panin [72].

As a consequence, for any scheme over which purity for the Witt group holds, the unramified Milnor question for e^n (with $n \leq 2$) is equivalent to the analogous global Milnor question. This is especially useful for the case of curves.

Purity results for étale cohomology. For X geometrically locally factorial and integral, the purity property holds for unramified cohomology in degree ≤ 1, i.e.

$$H^0_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) = H^0_{\mathrm{ur}}(X, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}, \quad \text{and} \quad H^1_{\text{ét}}(X, \boldsymbol{\mu}_2) = H^1_{\mathrm{ur}}(X, \boldsymbol{\mu}_2)$$

see Colliot-Thélène–Sansuc [24, Cor. 3.2, Prop. 4.1].

For X smooth over a field of characteristic $\neq 2$, we have a canonical identification ${}_2\mathrm{Br}(X) = H^2_{\mathrm{ur}}(X, \boldsymbol{\mu}_2)$ by Bloch–Ogus [19] such that the canonical map $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \to H^2_{\mathrm{ur}}(X, \boldsymbol{\mu}_2) = {}_2\mathrm{Br}(X)$ is the map arising from the Kummer exact sequence already considered in the proof of Lemma 2.1.

3.2. Positive results

We now survey some of the known positive cases of the unramified Milnor question in the literature.

Theorem 3 (Kerz–Müller-Stach [51, Cor. 0.8], Kerz [49, Thm. 1.2])**.** *Let R be a local ring with infinite residue field of characteristic $\neq 2$. Then the unramified Milnor question (all parts of Question 3.1) has a positive answer over* $\operatorname{Spec} R$*.*

Hoobler [45] had already proved this in degree 2.

The following result was communicated to us by Stefan Gille (who was inspired by Totaro [98]).

Theorem 4. *Let X be a proper smooth integral variety over a field of characteristic $\neq 2$. If X is F-rational then the unramified Milnor question (all parts of Question 3.1) has a positive answer over X.*

Proof. The groups $K^n_{\mathrm{M,ur}}(X)$, $H^n_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes n})$, and $I^n_{\mathrm{ur}}(X)$ are birational invariants of smooth proper F-varieties. To see this, one can use Colliot-Thélène [20, Prop. 2.1.8e] and the fact that the these functors satisfy weak purity for regular local rings (see Theorem 2). Another proof uses the fact that the complexes $C(X, K^n_{\mathrm{M}})$, $C(X, H^n)$, and $C(X, I^n)$ are cycle modules in the sense of Rost, see [90, Cor. 12.10]. In any case, by Colliot-Thélène [20, Prop. 2.1.9] the pullback induces isomorphisms $K^n_{\mathrm{M}}(F) \cong K^n_{\mathrm{M,ur}}(\mathbb{P}^m)$ (first proved by Milnor [66, Thm. 2.3] for $\mathbb{P}^1$), $H^n(F, \boldsymbol{\mu}_2^{\otimes n}) \cong H^n_{\mathrm{ur}}(\mathbb{P}^m, \boldsymbol{\mu}_2^{\otimes n})$, and $I^n_{\mathrm{ur}}(F) \cong I^n_{\mathrm{ur}}(F)(\mathbb{P}^m)$ for all $n \geq 0$ and $m \geq 1$. In particular, $K^n_{\mathrm{M}}(F)/2 \cong K^n_{\mathrm{M,ur}}(X)/2$ and $I^n_{\mathrm{ur}}(F)/I^{n+1}_{\mathrm{ur}}(F) \cong I^n_{\mathrm{ur}}(X)/I^{n+1}_{\mathrm{ur}}(X)$, and the theorem follows from the Milnor conjecture over fields. Q.E.D.

The following positive results are known for low dimensional schemes. Recall the notion of *cohomological dimension* $cd(F)$ of a field (see Serre [92, I §3.1]), *virtual cohomological 2-dimension* $vcd_2(F) = cd_2(F(\sqrt{-1}))$ and their 2-primary versions. Denoting by $d(F)$ any of these notions of dimension, note that if $d(F) \leq k$ and $\dim X \leq l$ then $d(F(X)) \leq k + l$.

Theorem 5 (Parimala–Sridharan [78], Monnier [67])**.** *Let X be a smooth integral curve over a field F of characteristic $\neq 2$. Then the unramified Milnor question for quadratic forms (Question 3.1(1)) has a positive answer over X in the following cases:*

(1) $cd_2(F) \leq 1$,
(2) $vcd(F) \leq 1$,
(3) $cd_2(F) = 2$ *and X is affine,*
(4) $vcd(F) = 2$ *and X is affine.*

Proof. For (1), this follows from Parimala–Sridharan [78, Lemma 4.1] and the fact that e^1 is always surjective. For (2), the case $vcd(F) = 0$ (i.e. F is real closed) is contained in Monnier [67, Cor. 3.2] and the case $vcd(F) = 1$ follows from a straightforward generalization to real closed fields of the results in [78, §5] for the real numbers. For (3), see [78, Lemma 4.2]. For (4), the statement follows from a generalization of [78, Thm. 6.1]. Q.E.D.

We wonder whether vcd can be replaced by vcd_2 in Theorem 5. Parimala–Sridharan [78, Rem. 4] ask whether there exist affine curves (over a well-chosen field) over which the unramified Milnor question has a negative answer.

For surfaces, there are positive results are in the case of $vcd(F) = 0$. If F is algebraically closed, then the unramified Milnor question for quadratic forms (Question 3.1(1)) has a positive answer by a direct computation, see Fernández-Carmena [32]. If F is real closed, one has the following result.

Theorem 6 (Monnier [67, Thm. 4.5]). *Let X be smooth integral surface over a real closed field F. If the number of connected components of $X(F)$ is ≤ 1 (i.e. in particular if $X(F) = \emptyset$), then the unramified Milnor question for quadratic forms (Question 3.1(1)) has a positive answer over X.*

Examples of surfaces with many connected components over a real closed field, and over which the unramified Milnor question still has a positive answer, are also given in Monnier [67].

Finally, as a consequence of [8, Cor. 3.4], the unramified Milnor question for quadratic forms (Question 3.1(1)) has a positive answer over any scheme X satisfying: ${}_2\mathrm{Br}(X)$ is generated by quaternion Azumaya algebras (i.e. index|period for 2-torsion classes); or ${}_2\mathrm{Br}(X)$ is generated by Azumaya algebras of degree dividing 4 (i.e. index|period^2 for 2-torsion classes) and $\mathrm{Pic}(X)$ is 2-divisible. In particular, this recovers the known cases of curves over finite fields (by class field theory) and surfaces over algebraically closed fields (by de Jong [25]).

§4. Negative results

Alex Hahn asked if there exists a ring R over which the global Merkurjev question (Question 3.2) has a negative answer, i.e. $e^2 : I^2(R) \to {}_2\mathrm{Br}(R)$ is not surjective. The results of Parimala, Scharlau, and Sridharan [77], [78], [79], show that there exist smooth complete curves X (over p-adic fields F) over which the unramified Milnor question (Question 3.1(1)) in degree 2 (and hence, by purity, the global Merkurjev question) has a negative answer.

Remark 4.1. The assertion (in Gille [41, §10.7] and Pardon [74, §5]) that the unramified Milnor question (Question 3.1(1)) has a positive answer over any smooth scheme (over a field of characteristic $\neq 2$) is incorrect. In these texts, the distinction between the groups $I^n_{\mathrm{ur}}(X)/I^{n+1}_{\mathrm{ur}}(X)$ and $\mathcal{I}^n/\mathcal{I}^{n+1}(X)$ is not made clear.

Definition 4.1 (Parimala–Sridharan [78]). A scheme X over a field F has the *extension property* for quadratic forms if there exists $x_0 \in X(F)$ such that every regular quadratic form on $X \setminus \{x_0\}$ extends to X.

Proposition 4.1 (Parimala–Sridharan [78, Lemma 4.3]). *Let F be a field of characteristic $\neq 2$ and with $cd_2 F \leq 2$ and X a smooth integral F-curve. Then the unramified Milnor question for quadratic forms (Question 3.1(1)) has a positive answer for X if and only if X has the extension property.*

The extension property is guaranteed when a residue theorem holds for the Witt group. The reside theorem for $X = \mathbb{P}^1$ is due to Milnor [66, §5]. For nonrational curves, the choice of local uniformizers inherent in defining the residue maps is eliminated by considering quadratic forms with values in the canonical bundle $\omega_{X/F}$.

Definition 4.2. Let X be a scheme and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. An *($\mathscr{L}$-valued) symmetric bilinear form* on X is a triple $(\mathscr{E}, b, \mathscr{L})$, where $\mathscr{E}$ is a locally free $\mathscr{O}_X$-module of finite rank and $b : S^2\mathscr{E} \to \mathscr{L}$ is an $\mathscr{O}_X$-module morphism.

Theorem 7 (Geyer–Harder–Knebusch–Scharlau [39])**.** *Let X be a smooth proper integral curve over a field F of characteristic $\neq 2$. Then there is a canonical complex (which is exact at the first two terms)*

$$0 \longrightarrow W(X, \omega_{X/F}) \longrightarrow W(F(X), \omega_{F(X)/F}) \xrightarrow{\partial^{\omega_X}} \bigoplus_{x \in X^{(1)}} W(F(x), \omega_{F(x)/F}) \xrightarrow{\mathrm{Tr}_{X/F}} W(F)$$

and thus in particular $W(X, \omega_{X/F})$ has a residue theorem.

Now any choice of isomorphism $\varphi : \mathscr{N}^{\otimes 2} \cong \omega_{X/F}$, induces a group isomorphism $W(X) \to W(X, \omega_{X/F})$ via $(\mathscr{E}, q) \mapsto (\mathscr{E} \otimes \mathscr{N}, \varphi \circ (q \otimes \mathrm{id}_{\mathscr{N}}), \omega_{X/F})$. Thus in particular, if $\omega_{X/F}$ is a square in $\mathrm{Pic}(X)$, then X has the extension property. Conversely:

Theorem 8 (Parimala–Sridharan [79, Thm. 3])**.** *Let F be a local field of characteristic $\neq 2$ and X a smooth integral hyperelliptic F-curve of genus ≥ 2 with $X(F) \neq \emptyset$. Then the unramified Milnor question for quadratic forms (Question 3.1(1)) holds over X if and only if $\omega_{X/k}$ is a square.*

Example 4.1 (Parimala–Sridharan [79, Rem. 3])**.** Let X be the smooth proper hyperelliptic curve over $\mathbb{Q}_3$ with affine model $y^2 = (x^2-3)(x^4+x^3+x^2+x+1)$. One can show using [77, Thm. 2.4] that $\omega_{X/F}$ is not a square. The point $(y, x) = (\sqrt{31}, 2)$ defines a $\mathbb{Q}_3$-rational point of X. Hence by Theorem 8, the unramified Milnor question has a negative answer over X.

Note that possible counter examples which are surfaces could be extracted from the following result.

Theorem 9 (Monnier [67, Thm. 4.5])**.** *Let X be a smooth integral surface over a real closed field F. Then the unramified Milnor question for quadratic forms (Question 3.1(1)) has a positive answer over X if and only if the cokernel of the mod 2 signature homomorphism is 4-torsion.*

§5. Line bundle-valued quadratic forms

Let X be a smooth F-scheme. Let $W(X, \mathscr{L})$ be the Witt group of $\mathscr{L}$-valued symmetric bilinear forms on X and $W_{\mathrm{tot}}(X) = \bigoplus_{\mathscr{L}} W(X, \mathscr{L})$ the *total* Witt group, where the sum is taken over a set of representatives of $\mathrm{Pic}(X)/2$. While this group is only defined up to non-canonical isomorphism depending on our choice of representatives, none of our cohomological invariants depend on such isomorphisms, see [8, §1.2]. Furthermore, we will not consider any ring structure on this group. Thus we will not need to descend into most of the important considerations of Balmer–Calmès [14].

Let e^0 be the usual rank modulo 2 map

$$e^0_{\mathrm{tot}} : W_{\mathrm{tot}}(X) \to \mathbb{Z}/2\mathbb{Z} = H^0_{\mathrm{ur}}(X, \mathbb{Z}/2\mathbb{Z})$$

and $I^1_{\mathrm{tot}}(X) = \oplus_{\mathscr{L}} I^1(X, \mathscr{L})$ its kernel. Then the signed discriminant (see [80]) defines a surjective map

$$e^1_{\mathrm{tot}} : I^1_{\mathrm{tot}}(X) \to H^1_{\text{ét}}(X, \boldsymbol{\mu}_2) = H^1_{\mathrm{ur}}(X, \boldsymbol{\mu}_2).$$

Finally, denote by $I^2(X, \mathscr{L}) \subset I^1(X, \mathscr{L})$ the subgroup generated by forms of trivial Arf invariant and $I^2_{\mathrm{tot}}(X) = \oplus_{\mathscr{L}} I^2(X, \mathscr{L})$. Then there exists a *total Clifford invariant* for line bundle-valued quadratic forms

$$e^2_{\mathrm{tot}} : I^2_{\mathrm{tot}}(X) \to {}_2\mathrm{Br}(X) = H^2_{\mathrm{ur}}(X, \boldsymbol{\mu}_2)$$

defined in [8, Prop. 1.4]. The surjectivity of the total Clifford invariant can be viewed as a version of the global Merkurjev question (Question 3.2) for line bundle-valued quadratic forms.

Theorem 10 ([8]). *Let X be a smooth proper integral curve over a local field F of characteristic $\neq 2$ or a smooth proper integral surface over a finite field F of characteristic $\neq 2$. Then the total Clifford invariant*

$$e^2_{\mathrm{tot}} : I^2_{\mathrm{tot}}(X) \to {}_2\mathrm{Br}(X)$$

is surjective.

The surjectivity of the total Clifford invariant can also be reinterpreted as the statement that while not every class in $\mathcal{I}^2/\mathcal{I}^3(X) = H^2_{\mathrm{ur}}(X)$ is represented by a quadratic form on X, every class is represented by a line bundle-valued quadratic form on X.

References

[1] J. Arason, *Quadratic forms and Galois-cohomology*, Séminaire de Théorie des Nombres, 1972–1973 (Univ. Bordeaux I, Talence), Exp. No. 22, Lab. Théorie des Nombres, Centre Nat. Recherche Sci., Talence, 1973, p. 5.

[2] J. Arason, *Der Wittring projektiver Räume*, Math. Ann. **253** (1980), 205–212.

[3] Jón Kr. Arason, *Cohomologische invarianten quadratischer Formen*, J. Algebra **36** (1975), no. 3, 448–491.

[4] ———, *A proof of Merkurjev's theorem*, Quadratic and Hermitian forms (Hamilton, Ont., 1983), CMS Conf. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 1984, pp. 121–130.

[5] Jón Kr. Arason, Richard Elman, and Bill Jacob, *The graded Witt ring and Galois cohomology. I*, Quadratic and Hermitian forms (Hamilton, Ont., 1983), CMS Conf. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 1984, pp. 17–50.

[6] ———, *On the Witt ring of elliptic curves*, K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), Proc. Sympos. Pure Math., vol. 58, Amer. Math. Soc., Providence, RI, 1995, pp. 1–25.

[7] ———, *The Witt ring of an elliptic curve over a local field*, Math. Ann. **306** (1996), no. 2, 391–418.

[8] Asher Auel, *Surjectivity of the total Clifford invariant and Brauer dimension*, arXiv:1108.5728v1, 2011.

[9] Ricardo Baeza, *Quadratic forms over semilocal rings*, Lecture Notes in Mathematics, Vol. 655, Springer-Verlag, Berlin, 1978.

[10] Paul Balmer, *Derived Witt groups of a scheme*, J. Pure Appl. Algebra **141** (1999), 101–129.

[11] ______, *Triangular Witt groups. I. The 12-term exact sequence*, K-Theory **19** (2000), 311–363.

[12] ______, *Triangular Witt groups. II. From usual to derived*, Math. Z. **236** (2001), 351–382.

[13] Paul Balmer, *Witt groups*, Handbook of K-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 539–576.

[14] Paul Balmer and Baptiste Calmès, *Bases of total Witt groups and lax-similitude*, preprint arXiv:1104.5051v1, April 2011.

[15] Paul Balmer and Charles Walter, *A Gersten-Witt spectral sequence for regular schemes*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 1, 127–152.

[16] H. Bass and J. Tate, *The Milnor ring of a global field*, Algebraic K-theory, II: “Classical” algebraic K-theory and connections with arithmetic (Proc. Conf., Seattle, Wash., Battelle Memorial Inst., 1972), Springer, Berlin, 1973, pp. 349–446. Lecture Notes in Math., Vol. 342.

[17] Hyman Bass, *Algebraic K-theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[18] A. A. Beĭlinson, *Height pairing between algebraic cycles*, K-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 1–25.

[19] Spencer Bloch and Arthur Ogus, *Gersten's conjecture and the homology of schemes*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 181–201 (1975). MR 0412191 (54 #318)

[20] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), Proc. Sympos. Pure Math., vol. 58, Amer. Math. Soc., Providence, RI, 1995, pp. 1–64.

[21] J.-L. Colliot-Thélène and R. Parimala, *Real components of algebraic varieties and étale cohomology*, Invent. Math. **101** (1990), no. 1, 81–99.

[22] J.-L. Colliot-Thélène and J.-J. Sansuc, *Fibrés quadratiques et composantes connexes réelles*, Math. Ann. **244** (1979), no. 2, 105–134.

[23] Jean-Louis Colliot-Thélène, Raymond T. Hoobler, and Bruno Kahn, *The Bloch-Ogus-Gabber theorem*, Algebraic K-theory (Toronto, ON, 1996), Fields Inst. Commun., vol. 16, Amer. Math. Soc., Providence, RI, 1997, pp. 31–94.

[24] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc, *Cohomologie des groupes de type multiplicatif sur les schémas réguliers*, C. R. Acad. Sci. Paris Sér. A-B **287** (1978), no. 6, A449–A452.

[25] A. J. de Jong, *The period-index problem for the Brauer group of an algebraic surface*, Duke Math. J. **123** (2004), no. 1, 71–94.

[26] A.J. de Jong, *A result of Gabber*, preprint, 2003.

[27] Gerhard Dietel, *Wittringe singulärer reeller Kurven. I, II*, Comm. Algebra **11** (1983), no. 21, 2393–2448, 2449–2494.

[28] Philippe Elbaz-Vincent and Stefan Müller-Stach, *Milnor K-theory of rings, higher Chow groups and applications*, Invent. Math. **148** (2002), no. 1, 177–206.

[29] Hélène Esnault, Bruno Kahn, and Eckart Viehweg, *Coverings with odd ramification and Stiefel-Whitney classes*, J. reine Angew. Math. **441** (1993), 145–188.

[30] Hélène Esnault, Bruno Kahn, Marc Levine, and Eckart Viehweg, *The Arason invariant and mod* 2 *algebraic cycles*, J. Amer. Math. Soc. **11** (1998), no. 1, 73–118.

[31] Jean Fasel, *Groupes de Chow-Witt*, Mém. Soc. Math. Fr. (N.S.) (2008), no. 113, viii+197.

[32] Fernando Fernández-Carmena, *On the injectivity of the map of the Witt group of a scheme into the Witt group of its function field*, Math. Ann. **277** (1987), no. 3, 453–468.

[33] Eric M. Friedlander, *Motivic complexes of Suslin and Voevodsky*, Astérisque (1997), no. 245, Exp. No. 833, 5, 355–378.

[34] Eric M. Friedlander, Michael Rapoport, and Andrei Suslin, *The mathematical work of the 2002 Fields medalists*, Notices Amer. Math. Soc. **50** (2003), no. 2, 212–217.

[35] A. Fröhlich, *On the K-theory of unimodular forms over rings of algebraic integers*, Quart. J. Math. Oxford Ser. (2) **22** (1971), 401–423.

[36] Ofer Gabber, *Some theorems on Azumaya algebras*, The Brauer group (Sem., Les Plans-sur-Bex, 1980), Lecture Notes in Math., vol. 844, Springer, Berlin, 1981, pp. 129–209.

[37] Thomas Geisser, *Motivic cohomology, K-theory and topological cyclic homology*, Handbook of K-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 193–234.

[38] Thomas Geisser and Marc Levine, *The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky*, J. Reine Angew. Math. **530** (2001), 55–103.

[39] W.-D. Geyer, G. Harder, M. Knebusch, and W. Scharlau, *Ein Residuensatz für symmetrische Bilinearformen*, Invent. Math. **11** (1970), 319–328.

[40] Stefan Gille, *On Witt groups with support*, Math. Ann. **322** (2002), no. 1, 103–137.

[41] ———, *A graded Gersten-Witt complex for schemes with a dualizing complex and the Chow group*, J. Pure Appl. Algebra **208** (2007), no. 2, 391–419.

[42] Alexander Grothendieck, *Dix exposés sur la cohomologie des schémas*, ch. VIII Classes de Chern et représentations linéaires des groupes discrets, pp. 215–305, North-Holland Publishing Company, Amsterdam; Masson & Cie, Éditeur, Paris, 1968.

[43] Daniel Guin, *Homologie du groupe linéaire et K-théorie de Milnor des anneaux*, J. Algebra **123** (1989), no. 1, 27–59.

[44] Günter Harder, *Halbeinfache Gruppenschemata über vollständigen Kurven*, Invent. Math. **6** (1968), 107–149.

[45] Raymond T. Hoobler, *The Merkuriev-Suslin theorem for any semi-local ring*, J. Pure Appl. Algebra **207** (2006), no. 3, 537–552.

[46] Bill Jacob and Markus Rost, *Degree four cohomological invariants for quadratic forms*, Invent. Math. **96** (1989), no. 3, 551–570.

[47] Bruno Kahn, *La conjecture de Milnor (d'après V. Voevodsky)*, Astérisque (1997), no. 245, Exp. No. 834, 5, 379–418, Séminaire Bourbaki, Vol. 1996/97.

[48] Bruno Kahn and R. Sujatha, *Motivic cohomology and unramified cohomology of quadrics*, J. Eur. Math. Soc. (JEMS) **2** (2000), no. 2, 145–177.

[49] Moritz Kerz, *The Gersten conjecture for Milnor K-theory*, Invent. Math. **175** (2009), no. 1, 1–33.

[50] ———, *Milnor K-theory of local rings with finite residue fields*, J. Algebraic Geom. **19** (2010), no. 1, 173–191. MR 2551760 (2010j:19006)

[51] Moritz Kerz and Stefan Müller-Stach, *The Milnor-Chow homomorphism revisited*, K-Theory **38** (2007), no. 1, 49–58.

[52] M. Knebusch, *Symmetric bilinear forms over algebraic varieties*, Conference on Quadratic Forms–1976 (Kingston, Ont.) (G. Orzech, ed.), Queen's Papers in Pure and Appl. Math., no. 46, Queen's Univ., 1977, pp. 103–283.

[53] M. Knebusch and W. Scharlau, *Quadratische Formen und quadratische Reziprozitätsgesetze über algebraischen Zahlkörpern*, Math. Z. **121** (1971), 346–368.

[54] Manfred Knebusch, *Grothendieck- und Wittringe von nichtausgearteten symmetrischen Bilinearformen*, S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl. **1969/70** (1969/1970), 93–157.

[55] ———, *On algebraic curves over real closed fields. II*, Math. Z. **151** (1976), no. 2, 189–205.

[56] M.-A. Knus, R. Parimala, and R. Sridharan, *On compositions and triality*, J. Reine Angew. Math. **457** (1994), 45–70.

[57] Max-Albert Knus, *Quadratic and hermitian forms over rings*, Springer-Verlag, Berlin, 1991.

[58] Max-Albert Knus and Manuel Ojanguren, *The Clifford algebra of a metabolic space*, Arch. Math. (Basel) **56** (1991), no. 5, 440–445.

[59] T. Y. Lam, *The algebraic theory of quadratic forms*, Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, Mass., 1980, Revised second printing, Mathematics Lecture Note Series.

[60] S. Lichtenbaum, *Values of zeta-functions at nonnegative integers*, Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983), Lecture Notes in Math., vol. 1068, Springer, Berlin, 1984, pp. 127–138.

[61] Max Lieblich, *Moduli of twisted sheaves*, Duke Math. J. **138** (2007), no. 1, 23–118.

[62] A. S. Merkur'ev, *On the norm residue symbol of degree* 2, Dokl. Akad. Nauk SSSR **261** (1981), no. 3, 542–547.

[63] Alexander Merkurjev, *On the norm residue homomorphism of degree two*, Proceedings of the St. Petersburg Mathematical Society. Vol. XII (Providence, RI), Amer. Math. Soc. Transl. Ser. 2, vol. 219, Amer. Math. Soc., 2006, pp. 103–124.

[64] A.S. Merkurjev and A.A. Suslin, *The norm residue homomorphism of degree 3*, Math. USSR Izv. **36** (1991), 349–368.

[65] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series, no. 33, Princeton University Press, Princeton, N.J., 1980.

[66] John Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1969/1970), 318–344.

[67] Jean-Philippe Monnier, *Unramified cohomology and quadratic forms*, Math. Z. **235** (2000), no. 3, 455–478.

[68] F. Morel, *Voevodsky's proof of Milnor's conjecture*, Bull. Amer. Math. Soc. (N.S.) **35** (1998), no. 2, 123–143.

[69] Fabien Morel, *Milnor's conjecture on quadratic forms and mod 2 motivic complexes*, Rend. Sem. Mat. Univ. Padova **114** (2005), 63–101 (2006).

[70] Yu. P. Nesterenko and A. A. Suslin, *Homology of the general linear group over a local ring, and Milnor's K-theory*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 121–146.

[71] M. Ojanguren, R. Parimala, R. Sridharan, and V. Suresh, *Witt groups of the punctured spectrum of a 3-dimensional regular local ring and a purity theorem*, J. London Math. Soc. (2) **59** (1999), no. 2, 521–540.

[72] Manuel Ojanguren and Ivan Panin, *A purity theorem for the Witt group*, Ann. Sci. École Norm. Sup. (4) **32** (1999), no. 1, 71–86.

[73] D. Orlov, A. Vishik, and V. Voevodsky, *An exact sequence for $K_*^M/2$ with applications to quadratic forms*, Ann. of Math. (2) **165** (2007), no. 1, 1–13.

[74] William Pardon, *The filtered Gersten–Witt resolution for regular schemes*, preprint, K-theory Preprint Archives, http://www.math.uiuc.edu/K-theory/0419/, May 2000.

[75] R. Parimala, *Witt groups of affine three-folds*, Duke Math. J. **57** (1988), no. 3, 947–954.

[76] ———, *Witt groups of conics, elliptic, and hyperelliptic curves*, J. Number Theory **28** (1988), no. 1, 69–93. MR 925609 (89a:14028)

[77] R. Parimala and W. Scharlau, *On the canonical class of a curve and the extension property for quadratic forms*, Recent advances in real algebraic geometry and quadratic forms (Berkeley, CA, 1990/1991; San Francisco, CA, 1991), Contemp. Math., vol. 155, Amer. Math. Soc., Providence, RI, 1994, pp. 339–350.

[78] R. Parimala and R. Sridharan, *Graded Witt ring and unramified cohomology*, *K*-Theory **6** (1992), no. 1, 29–44.

[79] ———, *Nonsurjectivity of the Clifford invariant map*, Proc. Indian Acad. Sci. Math. Sci. **104** (1994), no. 1, 49–56, K. G. Ramanathan memorial issue.

[80] R. Parimala and R. Sridharan, *Reduced norms and pfaffians via Brauer-Severi schemes*, Contemp. Math. **155** (1994), 351–363.

[81] R. Parimala and V. Srinivas, *Analogues of the Brauer group for algebras with involution*, Duke Math. J. **66** (1992), no. 2, 207–237.

[82] R. Parimala and R. Sujatha, *Witt group of hyperelliptic curves*, Comment. Math. Helv. **65** (1990), no. 4, 559–580.

[83] Raman Parimala, *Witt groups of curves over local fields*, Comm. Algebra **17** (1989), no. 11, 2857–2863.

[84] A. Pfister, *On the Milnor conjectures: history, influence, applications*, Jahresber. Deutsch. Math.-Verein. **102** (2000), no. 1, 15–41.

[85] Albrecht Pfister, *Quadratische Formen in beliebigen Körpern*, Invent. Math. **1** (1966), 116–132. MR 0200270 (34 #169)

[86] ———, *Some remarks on the historical development of the algebraic theory of quadratic forms*, Quadratic and Hermitian forms (Hamilton, Ont., 1983), CMS Conf. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 1984, pp. 1–16.

[87] H. G. Quebbemann, R. Scharlau, W. Scharlau, and M. Schulte, *Quadratische Formen in additiven Kategorien*, Bull. Soc. Math. France Suppl. Mem. (1976), no. 48, 93–101, Colloque sur les Formes Quadratiques (Montpellier, 1975).

[88] H.-G. Quebbemann, W. Scharlau, and M. Schulte, *Quadratic and Hermitian forms in additive and abelian categories*, J. Algebra **59** (1979), no. 2, 264–289.

[89] Markus Rost, *On Hilbert Satz 90 for k_3 for degree-two extensions*, preprint, May 1986.

[90] Markus Rost, *Chow groups with coefficients*, Doc. Math. **1** (1996), No. 16, 319–393 (electronic).

[91] Winfried Scharlau, *Quadratic and Hermitian forms*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 270, Springer-Verlag, Berlin, 1985.

[92] Jean-Pierre Serre, *Cohomologie galoisienne*, 5th ed., 2nd print., Lecture Notes in Math., vol. 5, Springer-Verlag, Berlin, 1997.

[93] M. Shyevski, *The fifth invariant of quadratic forms*, Dokl. Akad. Nauk SSSR **308** (1989), no. 3, 542–545.

[94] R. Sujatha, *Witt groups of real projective surfaces*, Math. Ann. **288** (1990), no. 1, 89–101.

[95] Andrei Suslin and Vladimir Voevodsky, *Bloch-Kato conjecture and motivic cohomology with finite coefficients*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci., vol. 548, Kluwer Acad. Publ., Dordrecht, 2000, pp. 117–189.

[96] R. W. Thomason, *Le principe de scindage et l'inexistence d'une K-theorie de Milnor globale*, Topology **31** (1992), no. 3, 571–588.

[97] Burt Totaro, *Milnor K-theory is the simplest part of algebraic K-theory*, K-Theory **6** (1992), no. 2, 177–189.

[98] ———, *Non-injectivity of the map from the Witt group of a variety to the Witt group of its function field*, J. Inst. Math. Jussieu **2** (2003), no. 3, 483–493.

[99] Vladimir Voevodsky, *Motivic cohomology with $\mathbf{Z}/2$-coefficients*, Publ. Math. Inst. Hautes Études Sci. (2003), no. 98, 59–104.

[100] Adrian R. Wadsworth, *Merkurjev's elementary proof of Merkurjev's theorem*, Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 741–776.

[101] Charles Walter, *Grothendieck-Witt groups of triangulated categories*, preprint, K-theory preprint archive, 2003.

[102] E. Witt, *Theorie der quadratischen Formen in beliebigen Körpern.*, J. reine angew. Math. **176** (1936), 31–44 (German).

[103] Kirill Zainoulline, *Witt groups of varieties and the purity problem*, Quadratic forms, linear algebraic groups, and cohomology, Dev. Math., vol. 18, Springer, New York, 2010, pp. 173–185.

Department of Mathematics & CS
Emory University
400 Dowman Drive NE W401
Atlanta, GA 30322
E-mail address: `auel@mathcs.emory.edu`