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The structure of a local embedding and Chern classes of weighted blow-ups

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Abstract. For a proper local embedding between two Deligne–Mumford stacks Y and X , we find, under certain mild conditions, a new (possibly non-separated) Deligne–Mumford stack X' , with an étale, surjective and universally closed map to the target X , and whose fiber product with the image of the local embedding is a finite union of stacks with corresponding étale, surjective and universally closed maps to Y . Moreover, a natural set of weights on the substacks of X' allows the construction of a universally closed push-forward, and thus a comparison between the Chow groups of X' and X . We apply the construction above to the computation of the Chern classes of a weighted blow-up along a regular local embedding via deformation to a weighted normal cone and localization. We describe the stack X' in the case when X is the moduli space of stable maps with local embeddings at the boundary. We apply the methods above to find the Chern classes of the stable map spaces.

Introduction

Local embeddings form an important class of morphisms of algebraic stacks. For instance, the morphisms from the components of the inertia stack of a Deligne–Mumford stack into the stack itself are in general local embeddings. As another fundamental example, the diagonal morphism of a stack is a local embedding and thus, the local study of this type of morphisms has led to a good definition of an intersection product on smooth Deligne–Mumford stacks by A. Vistoli [V], with a subsequent simplification by A. Kresch [K]. Their work relies on the existence, for any local embedding $Y \rightarrow X$, of an étale atlas $V \rightarrow Y$ such that the composition $V \rightarrow X$ can be factored into a closed embedding $V \rightarrow U$ followed by an étale atlas $U \rightarrow X$. Based on these covers, the normal cone of a local embedding ([V]) and a deformation of the ambient stack to the normal cone ([K]) can be constructed. However, these constructions are local in essence and as such, they fail to completely encode information on global invariants like Chern classes or Chow ring structures, and cannot be directly employed in Riemann–Roch type theorems like the Riemann–Roch without denominators as formulated for closed embeddings of schemes.

Under suitable assumptions, in this paper we replace the local construction above by one more suited to the purposes just mentioned, at the expense of working with non-

separated stacks rather than separated schemes. Let $g : Y \rightarrow X$ be a proper, local embedding of Noetherian stacks. Assume that Y is reduced and geometrically unibranch, that the morphism on the image $Y \rightarrow g(Y)$ is equidimensional, and that its degree is equal to a fixed number d at all generic points. Theorems 1.20 and 1.30 highlight the existence, for every such map $g : Y \rightarrow X$, of an étale cover by a stack $X' \rightarrow X$ such that $Y' := g(Y) \times_X X'$, a finite union of étale covers Y'_i of Y , is embedded in X' , and such that the morphisms $Y' \rightarrow Y$ and $X' \rightarrow X$ are universally closed. Moreover, the morphism $X' \rightarrow X$ is an isomorphism outside the image of Y . Thus the study of proper local embeddings can be reduced for practical purposes to that of closed embeddings of stacks.

While the map $p : X' \rightarrow X$ is forcefully not proper, it is universally closed. A weight function w on the set of substacks of X' is naturally attached to the map p , referring to the number of possible extensions of maps involved in the valuative criterion of properness. This “probabilistic weight” contributes to a good definition of push-forward p_* between the Chow groups of X' and X , an extension of the usual definition of proper push-forward to this type of universally closed maps. In effect, the Chow group $A(X)$ can be recovered from $A(X')$ via the universally closed push-forward p_* .

The definition of the étale lift X' associated to the proper local embedding $g : Y \rightarrow X$ and the subsequent constructions are based on a network of local embeddings associated to g . This network in turn depends on the choice of a suitable atlas U of X and a partition of $Y \times_X U$, with properties specified in Proposition 1.11. The dependence is only partial: while the number of spaces which are nodes in the network may vary, the spaces themselves are intrinsically associated to g . For example, by replacing U with a number of copies of itself, one increases the number of components in $Y \times_X X'$. In view of this, one could enquire on the existence of a “minimal” choice of étale atlas for X that would yield a canonical étale lift $X' \rightarrow X$. This problem is addressed by the authors in [MM4]. However, there are contexts where other factors, like, e.g., a moduli problem, may determine the naturalness of a (non-minimal) choice of étale atlas, and thus of an étale lift X' . This is the case for the moduli spaces of stable maps with their boundary, which we study in the third part of this article.

One useful feature of our construction comes from the fact that for a suitable étale atlas U of X , we take into account the entire pull-back of that atlas to the locally embedded Y . Locally, in the neighborhoods of some points in U , this pull-back may consist of a number of intersecting components. Their intersections contribute essentially to the structure of the morphism $Y \rightarrow X$; in a first instance, to the associated flat stratification of X . For this reason we encode them in a network of morphisms of stacks, a stack version of the configuration schemes of [L]. In [MM1], we have defined the extended Chow ring of such a network. In Theorem 1.34 we prove that this extended Chow ring is isomorphic to $A(X')$. In particular, in the case of the moduli space of stable maps and the local embeddings of its boundary divisors, the extended Chow ring has been calculated in [MM1]. In Theorem 3.12 we now identify the corresponding stack $\overline{M}_{0,m}(\mathbb{P}^n, d)$ and formulate its moduli problem in terms of stable maps with marked components.

Our approach is relevant, for instance, when considering a blow-up along a local embedding. Let us return to the initial picture of a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

where the top map is an embedding. Then simply taking the blow-up of U along V does not lead to an étale atlas of a complete blow-up of X along Y , due to a break in symmetry at the level of relations. Indeed, in small enough neighborhoods around each point in U , one needs to consider blow-up along each of the components of $Y \times_X U$ before defining the étale atlas for $\text{Bly } X$. In Section 2, we show how this can be done for smooth stacks Y and X .

Weighted blow-ups form a class of morphisms with a variety of applications. They come up, for example, when considering variation of GIT. As algebraic stacks provide a natural context for the study of weighted blow-ups, this class may be extended to weighted blow-ups along regular local embeddings. Moduli spaces of (weighted) curves and (weighted) stable maps are examples for which this type of morphisms comes up naturally.

As an application to the universally closed étale cover construction, blowing up Chern classes along local embeddings of smooth stacks is reduced to the case of smooth embeddings. The basic idea of this computation, like in the case of schemes, is to retrieve the Chern classes from their pull-back to the exceptional divisor, for example via the Riemann–Roch without denominators formula ([F]). However, when weights are considered, a less standard approach is necessary for the retrieval step. By a deformation to a “weighted normal cone”, we reduce the problem to the case when both the blow-up locus and the exceptional divisor are fixed loci for \mathbb{C}^* -actions making the blow-down morphism equivariant. The Atiyah–Bott localization theorem then means that the class of the exceptional divisor can be inverted, allowing us to retrieve a class on the blow-up from its pull-back to the exceptional divisor.

The paper is organized as follows: In the first section we construct the universally closed morphism which turns a proper local embedding $Y \rightarrow X$ into an embedding. The first step is the case when the local embedding is étale on its image. The general case is reduced to this situation by flat stratification. A network of local embeddings depending on the étale structure of the strata is highlighted in Section 1.2, and X' is constructed by induction on strata, such that all the local embeddings in the network are replaced by embeddings. In Section 1.3, universally closed morphisms and push-forwards associated to probabilistic weights are defined. The existence of an isomorphism between the Chow rings of X' and of the corresponding network, as introduced in [MM1], is proven, and the relation between the Chow rings of X and X' is discussed. The second section contains the calculation of Chern classes for weighted blow-ups. The third section is dedicated to the example of stable map spaces and intermediate weighted stable map spaces. The Appendix discusses the Euler sequence of a weighted projective bundle. Although this sequence is most likely known, we could not find a proof in the literature, and so decided to

carefully trace the sequence through the groupoid presentation of the weighted projective bundle.

1. The universal lift of a local embedding

The stacks in this article are assumed to be algebraic in the sense of Deligne–Mumford, Noetherian, and all morphisms considered between them are of finite type.

1.1. The lift of a local embedding étale on its image

Definition 1.1. Following [V], we will let a *local embedding* be any representable unramified morphism of finite type of stacks. A *regular local embedding* is a local embedding which is also locally a complete intersection.

Given a proper local embedding $Y \rightarrow X$, there is a diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{g_1} & U \\ p_1 \downarrow & & \downarrow p \\ Y & \xrightarrow{g} & X \end{array}$$

the vertical morphisms being étale atlases and g_1 being a closed embedding of schemes (Lemma 1.19 in [V]). Let V be the scheme representing the fiber product $Y \times_X U$, with the induced map $g_U : V \rightarrow U$, and the image of g_U denoted by W . As $V_1 \rightarrow Y$ and $V \rightarrow Y$ are étale, then so must be the induced morphism $i_1 : V_1 \rightarrow V$. Since $V_1 \rightarrow U$ is an embedding, so must be $V_1 \rightarrow V$. We write $V = V_1 \cup V_2$, where V_2 is the closure in V of $V \setminus V_1$. In fact, V_1 and V_2 are disjoint, as the map $V = V_1 \cup V_2 \rightarrow Y$ is étale. Denote by p_i the restriction $p_i : V_i \rightarrow Y$, and by W_i the image of V_i in U .

Furthermore, if the morphism g is étale on its image, then by the étale lifting property ([G, I, Proposition 8.1]), V_1 and U can be chosen such that

$$V_1 = g(Y) \times_X U,$$

and so there is an étale morphism $h : V \rightarrow V_1$ with $h \circ i_1 = \text{id}_{V_1}$.

We recall the étale groupoid presentation $[R \rightrightarrows U]$ of the stack X , given by the two projection morphisms $R := U \times_X U \rightrightarrows U$, together with canonical morphisms e, m , and i . Here the identity e is the diagonal morphism $e : U \rightarrow U \times_X U$, the multiplication m is

$$m := \pi_{13} : U \times_X U \times_X U \cong (U \times_X U) \times_U (U \times_X U) \rightarrow U \times_X U,$$

and the inverse morphism $i : U \times_X U \rightarrow U \times_X U$ switches the two terms of the product.

Notation. Let

$$S_{ij} := \text{Im}(\phi_{ij} : V_i \times_Y V_j \rightarrow U \times_X U),$$

for the map ϕ_{ij} given as the composition

$$V_i \times_Y V_j \hookrightarrow V \times_Y V = V \times_Y (Y \times_X U) \cong V \times_X U \rightarrow U \times_X U.$$

We denote by R' the subset

$$R' := R \setminus (S_{12} \cup S_{21} \cup (S_{22} \setminus S_{11})) \cup \text{Im } e.$$

From now on let $g : Y \rightarrow X$ be a proper local embedding. Then S_{ij} are closed subschemes of R . In general $S_{22} \setminus S_{11}$ might not be closed, but it is so in the case when g is étale on its image, for dimension reasons. As U is an étale atlas of X , the diagonal $e : U \rightarrow U \times_X U$ is an open embedding, and thus, under the above condition on g , the subscheme R' of R is open.

Proposition 1.2. *Let $g : Y \rightarrow X$ be a proper morphism, étale on its image. The restrictions $s_1, s_2 : R' \rightarrow U$ of the two projection morphisms $R \rightarrow U$, together with e , and with the restriction of m to $R' \times_U R'$ and of i to R' , form the groupoid presentation of a Deligne–Mumford stack X_Y .*

Proof. First note that $S_{12} \cap S_{11} = \emptyset$ and $S_{21} \cap S_{11} = \emptyset$. Indeed,

$$U \times_X U \leftarrow V_1 \times_X U \cong V_1 \times_Y \times (Y \times_X U) = V_1 \times_Y (V_1 \sqcup V_2) = (V_1 \times_Y V_1) \sqcup (V_1 \times_Y V_2).$$

In fact R' may be written alternatively as the difference

$$R' := R \setminus (S_{12} \cup S_{21} \cup S_{22} \setminus S_{11}).$$

Indeed, $S_{12} \cap \text{Im } e \subseteq S_{12} \cap S_{11} = \emptyset$, and symmetrically $S_{12} \cap \text{Im } e = S_{21} \cap \text{Im } e = \emptyset$. Furthermore, there is a sequence of consecutive Cartesian diagrams

$$\begin{array}{ccccc} V_2 & \hookrightarrow & V & \twoheadrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ V_2 \times_Y V & \hookrightarrow & V \times_Y V & \twoheadrightarrow & U \times_X U \end{array}$$

where the vertical maps are diagonal morphisms, with $V_2 \times_Y V_2 \subseteq V_2 \times_Y V$ and $V \times_Y V \cong V \times_X U$. This implies that $S_{22} \cap \text{Im } e = \text{Im } e_2$, where for each $i \in \{1, 2\}$, e_i is the composition of the diagonal $V_i \rightarrow V_i \times_Y V_i$ with the morphism $\phi_{ii} : V_i \times_Y V_i \rightarrow U \times_X U$. But $\text{Im } e_2 = \text{Im } e_1 \subseteq S_{11}$, due to the commutative diagram

$$\begin{array}{ccccccc} V_2 & \longrightarrow & V & \xrightarrow{h} & V_1 & \xrightarrow{g_1} & U \\ \downarrow & & \downarrow & & \searrow^{e_1} & & \downarrow e \\ V_2 \times_Y V_2 & \longrightarrow & V \times_Y V & \longrightarrow & & \longrightarrow & U \times_X U \end{array}$$

As R' is both symmetric and reflexive by construction, the existence of a groupoid structure $[R' \rightrightarrows U]$ reduces to checking the closedness of R' under multiplication. We note that for $k, l \in \{1, 2\}$,

$$\pi_{13}^{-1}(S_{kl}) \subseteq W \times_X U \times_X W \cong (W \times_X U) \times_U (U \times_X W).$$

But $W \times_X U$ is the image of $V \times_X U \cong \bigsqcup_{i,j \in \{1,2\}} V_i \times_Y V_j$, namely $\bigcup_{i,j \in \{1,2\}} S_{ij}$.

The injectivity of $g_1 : V_1 \rightarrow U$ directly implies $\pi_{13}(S_{11} \times_U S_{11}) \subseteq S_{11}$. Thus the multiplication m is well defined on R' . \square

Example 1.3. Let $Z \hookrightarrow X$ be a closed embedding of stacks and let $Y := \bigsqcup_{i=1}^n Z_i$, where $Z_i \cong Z$, with the natural morphism to X identifying all copies of Z . Then X_Y is obtained by gluing n copies of X along $X \setminus Z$.

With the notation from the previous proposition, the following also holds.

Proposition 1.4. *There exist a natural embedding $Y \hookrightarrow X_Y$ and an étale map $X_Y \rightarrow X$ making the following diagram Cartesian:*

$$\begin{array}{ccc} Y & \longrightarrow & X_Y \\ \downarrow & & \downarrow \\ g(Y) & \longrightarrow & X \end{array}$$

Proof. The composition of étale morphisms $R' \hookrightarrow R \rightrightarrows U$ being étale, the natural morphism of groupoids

$$\begin{array}{ccc} R' & \longrightarrow & R \\ \Downarrow & & \Downarrow \\ U & \longrightarrow & U \end{array}$$

induces an étale morphism of stacks $X_Y \rightarrow X$. Next we check that $V_1 = Y \times_{X_Y} U$, namely that the diagram defining a groupoid morphism

$$\begin{array}{ccc} V_1 \times_Y V_1 & \longrightarrow & R' \\ \Downarrow & & \Downarrow \\ V_1 & \longrightarrow & U \end{array}$$

is Cartesian. Indeed,

$$V \times_Y V \cong V \times_X U \cong V \times_U (U \times_X U) = V \times_U R$$

and therefore

$$V \times_U R' \cong (V \times_Y V) \times_R R' \cong (V_1 \times_Y V_1) \sqcup \phi_{22}^{-1}(S_{11}),$$

and by restriction to V_1 ,

$$V_1 \times_U R' \cong V_1 \times_Y V_1.$$

This proves the existence of a natural embedding of Y in X_Y whose composition with the étale map $X_Y \rightarrow X$ yields the local embedding of Y into X .

At the beginning of this section, the cover V_1 of Y was constructed as a fiber product $V_1 = g(Y) \times_X U$, and thus $V_1 \times_Y V_1 \cong V_1 \times_U R' \cong g(Y) \times_X R'$, forming an isomorphism of groupoids

$$[V_1 \times_Y V_1 \rightrightarrows V_1] \cong [g(Y) \times_X R' \rightrightarrows g(Y) \times_X U],$$

which induces an isomorphism $Y \cong g(Y) \times_X X_Y$. □

Although the groupoid presentation of the stack X_Y depended on particular choices of covers for X and Y , the stack X_Y is uniquely defined by a universality property, which can be expressed in terms of moduli problems as follows:

Theorem 1.5. *There is an equivalence of categories from the category of morphisms $Z \rightarrow X_Y$ defined on stacks Z of finite type to that of morphisms $Z \rightarrow X$ endowed with a section*

$$s : g(Y) \times_X Z \rightarrow Y \times_X Z$$

for the étale map $Y \times_X Z \rightarrow g(Y) \times_X Z$.

Proof. Indeed, at the level of objects, given a map $Z \rightarrow X_Y$ from a scheme Z of finite type to X_Y , and its composition with the étale map $X_Y \rightarrow X$, there is an induced isomorphism

$$g(Y) \times_X Z \cong (g(Y) \times_X X_Y) \times_{X_Y} Z \cong Y \times_{X_Y} Z,$$

and a natural map $Y \times_{X_Y} Z \rightarrow Y \times_X Z$ which, when composed with the étale map $g_Z : Y \times_X Z \rightarrow g(Y) \times_X Z \cong Y \times_{X_Y} Z$, yields the identity map.

Consider now a scheme of finite type Z and a morphism $Z \rightarrow X$ represented by a morphism of groupoids

$$\begin{array}{ccc} R_Z & \longrightarrow & R \\ \Downarrow & & \Downarrow \\ U_Z & \longrightarrow & U \end{array}$$

Assume that there is a section $s : g(Y) \times_X Z \rightarrow Y \times_X Z$ of $g_Z : Y \times_X Z \rightarrow g(Y) \times_X Z$, represented by the groupoid map

$$\begin{array}{ccc} V_1 \times_U R_Z & \longrightarrow & V \times_U R_Z \\ \Downarrow & & \Downarrow \\ V_1 \times_U U_Z & \longrightarrow & V \times_U U_Z \end{array}$$

Then the map $R_Z \rightarrow R$ factors through $R_Z \rightarrow R' \rightarrow R$. Indeed, as the restrictions of R and R' to $U \setminus \text{Im } V_1$ coincide, it is enough to show the factorization of groupoid maps

$$\begin{array}{ccccc} V_1 \times_U R_Z & \longrightarrow & V_1 \times_U R' & \longrightarrow & V_1 \times_U R \\ \Downarrow & & \Downarrow & & \Downarrow \\ V_1 \times_U U_Z & \longrightarrow & V_1 & \longrightarrow & V_1 \end{array}$$

The map $V_1 \times_U U_Z \rightarrow V_1$ is the composition of $V_1 \times_U U_Z \rightarrow V \times_U U_Z$ above with the natural maps $V \times_U U_Z \rightarrow V \rightarrow V_1$, while $V_1 \times_U R_Z \rightarrow V_1 \times_U R'$ is the composition

$$\begin{aligned} V_1 \times_U R_Z &\rightarrow V \times_U R_Z \rightarrow V \times_U R \cong V \times_Y V \\ &\rightarrow V_1 \times_Y V \cong V \times_U R' \rightarrow V_1 \times_U R' \cong V_1 \times_Y V_1. \end{aligned}$$

Together these maps induce the desired morphism $Z \rightarrow X_Y$. A direct check shows that the constructions above are functorial, and that the two functors constructed between the category $\text{Hom}(Z, X_Y)$ and that of morphisms $Z \rightarrow X$ with the extra property specified in the hypothesis are inverse to each other. \square

Remark 1.6. Although the embedding $V_1 \hookrightarrow V$ does induce morphisms $V_1 \times_U U_Z \rightarrow V \times_U U_Z$ and $V_1 \times_U R_Z \rightarrow V \times_U R_Z$, together these induced morphisms do not in general form a morphism of groupoids, as they are not necessarily compatible with multiplication on R_Z . In particular, when $Z = X$, our construction does not imply the existence of a section for $Y \rightarrow g(Y)$, or for $X_Y \rightarrow X$.

Corollary 1.7. (a) *Let $g : Y \rightarrow X$ be a morphism étale on its image, and let $Z \hookrightarrow Y$ be a closed embedding such that $g|_Z$ is proper and $Z \cong g(Z) \times_X Y$. Then there exists a natural morphism $g' : Y \rightarrow X_Z$ étale on its image.*
 (b) *Furthermore, if g is proper, then g' is proper as well and $(X_Z)_Y \cong X_Y$.*

Proof. A canonical section

$$s : g(Z) \times_X Y \rightarrow Z \times_X Y$$

is given by the diagonal morphism $Z \rightarrow Z \times_X Z$ via the isomorphisms $g(Z) \times_X Y \cong Z$ and $Z \times_X Z \cong Z \times_X Y$. By the previous theorem, this induces a morphism $g' : Y \rightarrow X_Z$, making the triangles in the following diagram commutative:

$$\begin{array}{ccc} & X_Z & \longrightarrow X \\ & \uparrow g' & \nearrow g \\ Z & \longrightarrow Y & \end{array}$$

Also, $g'(Y) \cong X_Z \times_X g(Y)$. Indeed, by Proposition 1.4, $g(Z) \times_X X_Z \cong Z \cong g'(Z)$, while $X \setminus g(Z) \cong X_Z \setminus g'(Z)$.

(b) Assume g is proper. As a consequence of the relation $g'(Y) \cong X_Z \times_X g(Y)$, g' is closed. It is separated, because g is. Moreover, g' satisfies the valuative criterion

of properness ([V]). Indeed, consider a valuation ring R' with fraction field K' , and a commutative diagram

$$\begin{array}{ccc} \text{Spec } K' & \longrightarrow & \text{Spec } R' \\ \downarrow & & \downarrow r' \\ Y & \xrightarrow{g'} & X_Z \end{array}$$

Since the composition $Y \rightarrow X_Z \rightarrow X$ is a proper morphism, there exists a finite extension K of K' such that, for the integral closure R of R' in K , the composition morphism $\text{Spec } K \rightarrow Y$ extends to $\phi : \text{Spec } R \rightarrow Y$. Let r denote the composition $\text{Spec } R \rightarrow \text{Spec } R' \rightarrow X_Z$. It remains to check that $g' \circ \phi = r$, or, equivalently, via Theorem 1.5, that the section

$$g(Z) \times_X \text{Spec } R \rightarrow Z \times_X \text{Spec } R$$

induced by ϕ is obtained from the canonical section s , after fiber product with $\text{Spec } R$ over Y via r :

$$g(Z) \times_X Y \times_Y \text{Spec } R \cong g(Z) \times_X \text{Spec } R \rightarrow Z \times_X Y \times_Y \text{Spec } R \cong Z \times_X \text{Spec } R.$$

Indeed, this is the case as both morphisms $g' \circ \phi$ and r give the same morphism when composed with the projection $X_Z \rightarrow X$. This proves the properness of g' . Furthermore, as g is étale on its image and $X_Z \rightarrow X$ is étale, g' will be étale on the image as well. Thus we can construct $(X_Z)_Y$ as in Proposition 1.2.

Note that, by the same argument as above with $\text{Spec } R$ replaced by $Y \times_X X_Z$, the triangles in the following diagram are commutative:

$$\begin{array}{ccc} Y \times_X X_Z & \xrightarrow{p_1} & X_Z \\ p_2 \downarrow & \nearrow g' & \downarrow \\ Y & \longrightarrow & X \end{array}$$

We apply this to show that the functors of $(X_Z)_Y$ and X_Y are equivalent. Indeed, any map $T \rightarrow X_Y$ induces a map $T \rightarrow X$ and a section

$$t : g(Y) \times_X T \rightarrow Y \times_X T,$$

which by restriction yields a section $g(Z) \times_X T \rightarrow Z \times_X T$, and thus a map $T \rightarrow X_Z$. For the existence of a morphism $T \rightarrow (X_Z)_Y$, a section $t' : g'(Y) \times_{X_Z} T \rightarrow Y \times_{X_Z} T$ is necessary and sufficient. But

$$\begin{aligned} g'(Y) \times_{X_Z} T &\cong (g(Y) \times_X X_Z) \times_{X_Z} T \cong g(Y) \times_X T, \\ Y \times_X T &\cong (Y \times_X X_Z) \times_{X_Z} T \cong Y \times_{X_Z} T, \end{aligned}$$

due to the two commutative triangles above, so via these isomorphisms we can identify t' with t . Conversely, any map $T \rightarrow (X_Z)_Y$ induces a map $T \rightarrow X_Z$ and a section $t' : g'(Y) \times_{X_Z} T \rightarrow Y \times_{X_Z} T$ which yields the section t canonically via the above isomorphisms. □

Corollary 1.8. *Consider any morphism of Noetherian stacks $f : X_1 \rightarrow X_2$ and any proper map $g_2 : Y_2 \rightarrow X_2$ étale on its image, and let $Y_1 := Y_2 \times_{X_2} X_1$. Then there exists a morphism $f' : (X_1)_{Y_1} \rightarrow (X_2)_{Y_2}$ making the following diagram Cartesian:*

$$\begin{array}{ccc} (X_1)_{Y_1} & \xrightarrow{f'} & (X_2)_{Y_2} \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

In particular, if f is proper, then f' is proper as well.

Proof. The induced map $g_1 : Y_1 \rightarrow X_1$ is also étale on its image so that $(X_1)_{Y_1}$ satisfying the functorial property in Theorem 1.5 exists. Also, for any stack Z with a morphism $Z \rightarrow X_1$, there are isomorphisms

$$\begin{aligned} Y_2 \times_{X_2} Z &\cong Y_2 \times_{X_2} (X_1 \times_{X_1} Z) \cong (Y_2 \times_{X_2} X_1) \times_{X_1} Z \cong Y_1 \times_{X_1} Z, \\ g_2(Y_2) \times_{X_2} Z &\cong g_2(Y_2) \times_{X_2} (X_1 \times_{X_1} Z) \cong (g_2(Y_2) \times_{X_2} X_1) \times_{X_1} Z \cong g_1(Y_1) \times_{X_1} Z. \end{aligned}$$

Thus any morphism $Z \rightarrow (X_1)_{Y_1}$, corresponding to a section $g_1(Y_1) \times_{X_1} Z \rightarrow Y_1 \times_{X_1} Z$ by Theorem 1.5, induces a section $g_2(Y_2) \times_{X_2} Z \rightarrow Y_2 \times_{X_2} Z$ and thus a morphism $Z \rightarrow (X_2)_{Y_2}$. Conversely, any two morphisms $Z \rightarrow (X_2)_{Y_2}$ and $Z \rightarrow X_1$ making the diagram

$$\begin{array}{ccc} Z & \longrightarrow & (X_2)_{Y_2} \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

commutative, factor through a unique morphism $Z \rightarrow (X_1)_{Y_1}$. □

1.2. The étale structure of a local embedding

Under certain assumptions, we can find a more precise description for the local étale structure of a local embedding. We recall the following definitions which we will need for our assumptions.

Definition 1.9. A morphism of schemes $g : Y \rightarrow X$ is called *equidimensional* if the following conditions hold:

- (1) g is a morphism of finite type.
- (2) The function $\dim_y g^{-1}(g(y))$ is constant for all $y \in Y$.
- (3) Any irreducible component of Y dominates an irreducible component of X .

As these conditions are stable under étale base change ([EGA, 13.3.8]), the notion of equidimensional morphism extends canonically to morphisms of stacks.

Definition 1.10. Let Y be a Noetherian stack. It is called *geometrically unibranch* if it has an étale atlas V which is geometrically unibranch, meaning that for any point v of V , the scheme $\text{Spec } \mathcal{O}_{v,V}^{\text{sh}}$ is irreducible, where $\mathcal{O}_{v,V}^{\text{sh}}$ denotes the strict henselization of the local ring of v in V .

Notation. Consider a proper local embedding of Noetherian stacks $g : Y \rightarrow X$, with Y reduced. Let Y_1 denote the locus of points in Y where the map g is not étale on its image.

Proposition 1.11. *Let $g : Y \rightarrow X$ be a proper, local embedding of Noetherian stacks. Assume that Y is reduced and geometrically unibranch, that the morphism on the image $Y \rightarrow g(Y)$ is equidimensional, and that its degree is equal to a fixed number d at all points of $g(Y \setminus Y_1)$.*

There exist an étale atlas U of X and closed, isomorphic subschemes $\{W_l\}_{l \in L}$ of U , with intersections $W_I = \bigcap_{l \in I} W_l$ for each $I \in \mathcal{P}(L)$, as well as isomorphic subschemes $\{V_l^a\}_{l \in L, a \in A_l := \{1, \dots, d\}}$ of $Y \times_X U$, each mapping onto Y , with isomorphisms $V_l^a \rightarrow W_l$ forming commutative diagrams with $g : Y \rightarrow g(Y)$, such that, at the level of supports,

$$g(Y) \times_X U = \bigcup_{l \in L} W_l, \quad Y \times_X U = \bigsqcup_{l \in L, a \in A_l} V_l^a, \quad g(Y_1) \times_X U = \bigcup_{l, j \in L, l \neq j} W_{lj}.$$

In particular $Y_1 = g^{-1}(\{p \in X; \deg g_p \geq 2d\})$.

More generally, for integers $k > 1$, each $Y_{k-1} := g^{-1}(\{p \in X; \deg g_p \geq dk\})$ satisfies

$$g(Y_{k-1}) \times_X U = \bigcup_{I \in \mathcal{P}_k} W_I \quad \text{and} \quad Y_{k-1} \times_X U = \bigcup_{I \in \mathcal{P}_k, a \in A_I} V_I^a,$$

at the level of supports, where $\mathcal{P}_k \subset \mathcal{P}(L)$ is the set of cardinality k subsets of L ; for each $I \in \mathcal{P}_k$ as above, $A_I := \bigsqcup_{l \in I} A_l$; and for $l \in I$ and $a \in A_l$, the scheme V_I^a is the preimage of W_I in V_l^a .

Proof. For any quasi-finite map f , we denote by $\text{md}(f)$ the maximum degree of f :

$$\text{md}(f) := \max_{p \in \text{Im } f} \deg f_p.$$

The degree function $\deg g_p$ is upper semicontinuous. Let Z_m denote the locus of points in Y where the degree of g reaches its maximum, so that, unless g is étale on its image,

$$\text{md}(g|_{Z_m}) > \deg(g|_{Y \setminus Y_1}) = d. \tag{1.1}$$

We will first focus on proving the existence of an étale atlas U^0 of X and $\{W_l^0\}_{l \in L}$, $\{V_l^{0a}\}_{l \in L, a \in A_l}$ such that

$$Y \times_X U^0 = \bigsqcup_{l, a \in A_l} V_l^{0a},$$

and such that each V_l^{0a} is mapped isomorphically to $W_l^0 \subseteq g(Y) \times_X U^0$, and étale surjectively to Y by natural morphisms. We note that once such an atlas U^0 has been

found, any other scheme U' mapping étale surjectively to U^0 will work as well, as we can construct W'_l and V'_l as pull-backs of W_l^{0a} and V_l^{0a} , respectively.

We proceed by induction on $\text{md}(g)$. If $\text{md}(g) = 1$, then g is an embedding and any étale atlas U^0 of X would work, with $V^0 = Y \times_X U^0$ étale atlas of Y , embedded in U_0 . Assume now that $\text{md}(g) \geq 2$, and that the statement above can be proven for morphisms of smaller maximum degree.

Consider a commutative diagram as in Lemma 1.19 of [V],

$$\begin{array}{ccc} V_1 & \xrightarrow{g_1} & U \\ \pi_{V_1} \downarrow & & \downarrow \pi_U \\ Y & \xrightarrow{g} & X \end{array}$$

such that the vertical morphisms are étale and surjective, and g_1 is a closed embedding. The induced morphism $V_1 \rightarrow Y \times_X U$ is a closed embedding, and étale, and so V_1 is a union of connected components of $Y \times_X U$. Let g_U denote the morphism $Y \times_X U \rightarrow U$. The fibered product $Y \times_X U$ can be split into a disjoint union

$$Y \times_X U = V_1 \sqcup V' \sqcup V'',$$

where V'' is the union of all components $V_0 \not\subseteq V_1$ with $g_U(V_0) \subseteq g_U(V_1)$. We note that $Y \times_X U$ is geometrically unibranch, and so its connected components are also irreducible. Let $W_1 := g_U(V_1)$, $W' := g_U(V')$ and $W'' := g_U(V'')$. By construction, W' and W_1 are closed subsets of U and none contains an irreducible component of the other.

First we notice that

$$g(Y_1) = \pi_U(W_1 \cap W'). \tag{1.2}$$

In other words, for any closed point y of Y and for any $v \in V_1$ such that $\pi_{V_1}(v) = y$, the map g is étale on its image at y if and only if $w := g_U(v) \in W_1 \setminus W'$. Indeed, if g is étale on its image at y , then the composition of the natural morphisms $W_1 \cong V_1 \rightarrow Y \rightarrow g(Y)$, as well as the projection $W_1 \cup W' = g(Y) \times_X U \rightarrow g(Y)$ are étale at v , which makes the inclusion $W_1 \rightarrow W_1 \cup W'$ étale at $w = g_U(v)$, so $w \in W_1 \setminus W'$. Conversely, in the natural Cartesian square

$$\begin{array}{ccc} Y \times_X (W_1 \setminus W') & \longrightarrow & W_1 \setminus W' \\ \downarrow & \nearrow f_1 & \downarrow \\ Y & \xrightarrow{\quad} & g(Y) \end{array}$$

both the vertical morphisms and f_1 are étale, making the upper horizontal map étale as well. Thus the restriction of g to $g^{-1}(\pi_U(W_1 \setminus W'))$ must be étale. Here f_1 is the composition $W_1 \setminus W' \rightarrow W_1 \cong V_1 \rightarrow Y$.

Both $g_{U|_{V''}}$ and $g_{U|_{V'}}$ are proper local embeddings. Clearly the definition of V'' implies that the maximum degrees satisfy

$$\text{md}(g_{U|_{V''}}) < \text{md}(g_U). \tag{1.3}$$

Assuming $V'' \neq \emptyset$, we note that $g_U(V'') = W_1$. This is true as $g_U(V'')$ is closed in U due to the properness of g_U , and $W_1 \setminus W' \subseteq g_U(V'')$. Indeed, as $W_1 \cap W'$ has positive codimension in W_1 , while $\dim g_U(V'') = \dim W_1$, it follows that $g_U(V'') \cap (W_1 \setminus W') \neq \emptyset$. Due to the definition of V'' , the degree of g_U on this set is at least 2. But (1.2) and the condition that $\deg_{g|_{Y \setminus Y_1}}$ is constant imply that $\deg g_U$ is constant on $W_1 \setminus W'$, and as such, at least 2 at every point there. This can only happen if $W_1 \setminus W' \subseteq g_U(V'')$, and therefore if $g_U(V'') = W_1$.

Next we show that for appropriate choices of U , we also have

$$\text{md}(g_{U|_{V'}}) < \text{md}(g_U). \tag{1.4}$$

Consider E a connected, closed subscheme of $g(Y) \times_X U$, and assume that the restriction of g_U on $g_U^{-1}(E)$ is étale on its image. Then we can assume that either $E \cap W_1 = \emptyset$ or $E \subseteq W_1$. Furthermore, in the first case, U can be replaced by $U \setminus E$ without changing V_1 , or any of the properties of U above.

Indeed, the restriction of g_U on $g_U^{-1}(E)$ can be split as

$$(g_U^{-1}(E) \cap (V_1 \sqcup V'')) \sqcup (g_U^{-1}(E) \cap V') \rightarrow (E \cap W_1) \cup (E \cap W') = E,$$

and thus either $E \cap W_1 = \emptyset$, or $E \cap W' = \emptyset$, or $E \subseteq W_1 \cap W'$.

In particular, taking $E = \pi_U^{-1}(g(Z_m))$, for an appropriate choice of U we may assume that

$$\pi_U^{-1}(g(Z_m)) \subseteq W_1.$$

This, together with (1.1), implies that $\text{md}(g_{U|_{V'}}) < \text{md}(g_U)$.

Moreover, the degrees of $g_{U|_{V'}}$ and $g_{U|_{V''}}$ are constant over all points in $W' \setminus W_1$ and $W_1 \setminus W'$, respectively. Due to (1.3) and (1.4), we can now apply the induction hypothesis to $g_{U|_{V'}}$ and $g_{U|_{V''}}$, obtaining two surjective, étale morphisms $U' \rightarrow U$ and $U'' \rightarrow U$ and $\{V_l^{a'}\}_{l \in L'}$, $\{V_l^{a''}\}_{l \in L''}$ such that

$$V' \times_U U' = \bigsqcup_{l', a'} V_l^{a'}, \quad V'' \times_U U'' = \bigsqcup_{l'', a''} V_l^{a''},$$

the maps $V_l^{a'} \rightarrow U'$ and $V_l^{a''} \rightarrow U''$ are closed embeddings, while $V_l^{a'} \rightarrow V'$ and $V_l^{a''} \rightarrow V''$ are surjective, étale. Consider now $U^0 = U' \times_U U''$, as well as $V_l^{0a'} := V_l^{a'} \times_{U'} U^0$ and $V_l^{0a''} := V_l^{a''} \times_{U''} U^0$. Finally, let $V_1^0 = V_1 \times_U U^0$ and $L = L' \sqcup L'' \sqcup \{1\}$. The schemes $\{V_l^{0a}\}_{l \in L}$ thus obtained satisfy

$$Y \times_X U^0 = \bigsqcup_{l, a \in A_l} V_l^{0a}$$

and each V_l^{0a} is naturally embedded into U^0 .

The surjectivity of the maps $V_l^{0a} \rightarrow Y$ remains to be established. Since $V_l^{0a'} \rightarrow V'$ and $V_l^{0a''} \rightarrow V''$ are known to be surjective from the above, this reduces to showing that both V' and V'' map surjectively to Y (unless $V'' = \emptyset$).

Now, if for any closed point y_1 of $Y \setminus Y_1$ we have $(\{y_1\} \times_X U) \cap V' = \emptyset$, then consider $v_1 \in (\{y_1\} \times_X U) \cap V_1$ and replace U by $U \sqcup U_{g_U(v_1)}$, where $U_{g_U(v_1)} \subset U \setminus W'$ is an open neighborhood of $g_U(v_1)$ in U . Thus V' is replaced by $V' \sqcup g_U^{-1}(U_{g_U(v_1)})$, while the maximum degree of g_U remains unchanged. Since Y is Noetherian, we may thus assume that all points in $Y \setminus Y_1$ have preimages in V' . They have preimages in V'' as well, as long as $V'' \neq \emptyset$. Indeed, if $\deg g_p \geq 2$ for any $p \in g(Y)$, then for any y_1 closed point in $Y \setminus Y_1$, choose $y_2 \in Y$ such that $g(y_1) = g(y_2)$, and $v_2 \in (\{y_2\} \times_X U) \cap V_1$. Then since $g_{U|_{V_1}}$ is injective, $\emptyset \neq \{y_1\} \times_X \{g_U(v_2)\} \subset V''$.

If $y_1 \in Y_1$, we can prove $(\{y_1\} \times_X U) \cap V' \neq \emptyset$ due to the structure of $g(Y) \times_X U$ around $\{g(y_1)\} \times_X U$. For this, first let $y_2 \in Y_1$ be such that $g(y_2) = g(y_1)$, and $v_2 \in (\{y_2\} \times_X U) \cap V'$ (cf. (1.2)). Consider a valuation ring R_1 and a map $T_1 = \text{Spec } R_1 \rightarrow Y$, taking the closed point to y_1 and the generic point (0) to $z_1 \in Y \setminus Y_1$. Construct another map $T_2 = \text{Spec } R_2 \rightarrow Y$ taking the closed point of the valuation ring R_2 to y_2 and the generic point to $z_2 \in Y \setminus Y_1$ such that $g(z_1) = g(z_2)$ (such a map exists because g is proper), as well as $T'_2 = \text{Spec } R'_2 \rightarrow V'$ taking the closed point to v_2 (since $V' \rightarrow Y$ is étale). Let C_2 denote its image in U . Then the generic point of $\{z_1\} \times_X C_2$ must be in V' and by specialization, we get a point in V' whose image in Y is y_1 .

The same argument works for V'' as well. With this, the proof that V' and V'' map surjectively to Y is complete.

Consider now the étale atlas U^0 of X constructed above and let W_l^0 denote the image of $V_l^{0a} \rightarrow U$ for each l . The arguments in the proof of (1.2), when applied successively to g_{U^0} and its restrictions to each $Y_k \times_X U_0$, yield

$$g(Y_k) \times_X U^0 = \bigcup_{l \in \mathcal{P}_k} W_l^0.$$

However, each W_l^0 may not necessarily map surjectively to $g(Y_k)$. To adjust this, we consider the permutation group S_L of L and relabel

$$U := \bigsqcup_{\sigma \in S_L} U^0, \quad \text{while} \quad W_l := \bigsqcup_{\sigma \in S_L} W_{\sigma(l)}^0 \quad \text{and} \quad V_l^a := \bigsqcup_{\sigma \in S_L} V_{\sigma(l)}^{0a}$$

for each $l \in L$ and $a \in A_l$ (for fixed bijections between the sets A_l). Then U satisfies all the conditions required in the proposition. Moreover, we note that now $W_l \cong W_j$ for any l and j as above. □

Definition 1.12. Consider a proper local embedding of Noetherian stacks $g : Y \rightarrow X$. Assume that there exists an étale atlas U of X with all the properties listed in Proposition 1.11. With the notation from the same proposition, let

$$V := Y \times_X U = \bigsqcup_{i,a} V_i^a \quad \text{and} \quad V_i := \bigsqcup_a V_i^a$$

for each $i \in L$, where $a \in A_i$. For any $i, j \in L$, $a \in A_i$ and $b \in A_j$, we denote

$$S_{ij}^{ab} := \text{Im}(V_i^a \times_Y V_j^b \rightarrow W_i \times_X W_j).$$

For fixed $i \in L$ and $a \in A_i$, we define $R_{Z,i} := \bigsqcup_{b \in A_i} S_{ii}^{ab}$.

Lemma 1.13. *With the notation from Definition 1.12, $R_{Z,i}$ does not depend on the choice of a .*

Proof. For any $i, j \in L, a, c \in A_i$ and $b \in A_j$, the sequences of Cartesian diagrams

$$\begin{array}{ccccc} (V_i^a \times_Y V_j^b) \times_{U \times_X U} (V_i^c \times_Y V) & \longrightarrow & V_i^c \times_Y V & \longrightarrow & V_i^c \\ \downarrow & & \downarrow & & \downarrow \\ V_i^a \times_Y V_j^b & \longrightarrow & U \times_X U & \longrightarrow & U \end{array}$$

yield a canonical isomorphism $F : V_j^b \times_Y V_i^a \times_U V_i^c \cong (V_i^a \times_Y V_j^b) \times_{U \times_X U} (V_i^c \times_Y V)$. These spaces are also isomorphic to $(V_i^a \times_U V_i^c) \times_{Y \times_X Y} (V_j^b \times_U V)$ over V , e.g. due to the sequence of Cartesian diagrams

$$\begin{array}{ccccc} (V_i^a \times_U V_i^c) \times_{Y \times_X Y} (V_j^b \times_U V) & \longrightarrow & V_j^b \times_U V & \longrightarrow & V_j^b \\ \downarrow & & \downarrow & & \downarrow \\ V_i^a \times_U V_i^c & \longrightarrow & Y \times_X Y & \longrightarrow & Y \end{array}$$

Moreover,

$$(V_i^a \times_Y V_j^b) \times_{U \times_X U} (V_i^c \times_Y V_j) \cong (V_i^a \times_U V_i^c) \times_{Y \times_X Y} (V_j^b \times_U V_j) \tag{1.5}$$

contains the image of $(V_j^b \setminus \bigcup_{k \neq j} V_{jk}^b) \times_Y V_i^a \times_U V_i^c$ through F and, as the codimension of $\bigcup_{k \neq j} V_{jk}^b$ in V_j^b is at least 1, F can also be understood as

$$V_j^b \times_Y V_i^a \times_U V_i^c \cong (V_i^a \times_Y V_j^b) \times_{U \times_X U} (V_i^c \times_Y V_j). \tag{1.6}$$

In particular, when $i = j$, this yields $\bigsqcup_{b \in A_i} S_{ii}^{ab} = \bigsqcup_{b \in A_i} S_{ii}^{cb}$ for any $a, c \in A_i$. \square

Lemma 1.14. *Let $g : Y \rightarrow X$ be a proper local embedding of Noetherian stacks, and assume that the degree of g is the same at all points of $g(Y \setminus Y_1)$. Assume that there exists an étale atlas U of X satisfying the conditions in Proposition 1.11. Then there exists a stack Z , an étale morphism $f : Y \rightarrow Z$ and a proper local embedding $h : Z \rightarrow X$ of generic degree 1 such that $g = h \circ f$.*

Moreover, with the notation from Proposition 1.11,

$$\bigsqcup_{l \in L} W_l \cong Z \times_X U.$$

Proof. The restrictions to $R_{Z,i}$ of the two natural projections on $W \times_X W \rightrightarrows W$ yield a groupoid scheme $R_{Z,i} \rightrightarrows W_i$. Indeed, the identity $e : W_i \rightarrow S_{ii}^{aa} \subset R_{Z,i}$ is induced by the diagonal $V_i^a \rightarrow V_i^a \times_Y V_i^a$, and there are natural inverse and multiplication maps, built from $i : S_{ii}^{ab} \rightarrow S_{ii}^{ba}$ and $m : S_{ii}^{ab} \times_U S_{ii}^{bc} \rightarrow S_{ii}^{ac}$, the last of which can be identified

with the projection on the first and third factors $V_i^a \times_Y V_i^b \times_Y V_i^c \rightarrow V_i^a \times_Y V_i^c$. We will denote by Z the Deligne–Mumford stack with groupoid presentation $[R_{Z,i} \rightrightarrows W_i]$.

Due to (1.6), the following is a Cartesian diagram of groupoid schemes:

$$\begin{array}{ccc} \bigsqcup_{c,d} V_i^c \times_Y V_i^d & \longrightarrow & R_{Z,i} \\ \Downarrow & & \Downarrow \\ \bigsqcup_c V_i^c & \longrightarrow & W_i \end{array}$$

proving the existence of an étale morphism $f : Y \rightarrow Z$.

There is also a natural morphism of groupoid schemes from $R_{Z,i} \rightrightarrows W_i$ to $W_i \times_X W_i \rightrightarrows W_i$, which is moreover an isomorphism over $W_i \setminus \bigcup_{j \neq i} W_j$. This yields $g : Z \rightarrow g(Y) \hookrightarrow X$, generically one-to-one.

It remains to construct an étale map $\bigsqcup_j W_j \rightarrow Z$ and prove that

$$Z \times_X U \cong Z \times_{g(Y)} (g(Y) \times_X U) \cong Z \times_{g(Y)} \bigcup_j W_j \cong \bigsqcup_j W_j.$$

Let $W := \bigcup_j W_j$. As $W_i \times_X W \cong V_i^a \times_X U \cong V_i^a \times_Y (Y \times_X U) \cong \bigsqcup_{j,b} V_i^a \times_Y V_j^b$, the projection from $W_i \times_X W$ to the second term factors through $p_2 : W_i \times_X W \rightarrow \bigsqcup_j W_j$. Moreover, p_2 is étale and surjective. Next, we check the existence of a canonical isomorphism of groupoids

$$\begin{array}{ccc} (W_i \times_X W) \times_{\bigsqcup_j W_j} (W_i \times_X W) & \longrightarrow & \bigsqcup_b S_{ii}^{ab} \times_X W \\ \Downarrow & & \Downarrow \\ W_i \times_X W & \longrightarrow & W_i \times_X W \end{array} \tag{1.7}$$

which, after descent, will induce the map $\bigsqcup_j W_j \rightarrow Z$. (1.7) follows from a sequence of isomorphisms

$$\begin{aligned} (W_i \times_X W) \times_{\bigsqcup_j W_j} (W_i \times_X W) &\cong (V_i^a \times_Y V) \times_{\bigsqcup_j W_j} (V_i^e \times_Y V) \\ &\cong \bigsqcup_{j,c,d} (V_i^a \times_Y V_j^c) \times_U (V_i^e \times_Y V_j^d) \cong \bigsqcup_{j,c,d} V_i^a \times_Y ((V_j^c \times_U V_j^d) \times_{Y \times_X Y} (V_i^e \times_U V_i)) \\ &\cong \bigsqcup_{j,c} V_i^a \times_Y ((V_j^c \times_U V) \times_{Y \times_X Y} (V_i^e \times_U V_i)) \cong \bigsqcup_{j,c} V_i^a \times_Y (V_j^c \times_Y (V_i^e \times_U V_i)) \end{aligned}$$

(based on isomorphisms (1.5) and (1.6)), and furthermore

$$\cong \bigsqcup_{j,c} V_i^a \times_Y (V_j^c \times_Y (V_i^e \times_{W_i} V_i)) \cong V_i^a \times_Y V \times_Y V_i \cong \bigsqcup_b S_{ii}^{ab} \times_X W.$$

Finally, we note that $(\bigsqcup_j W_j) \times_X W \cong (\bigsqcup_j W_j) \times_U (U \times_X U) \rightrightarrows \bigsqcup_j W_j$ is the pull-back on $\bigsqcup_j W_j$ of the groupoid presentation of X , and that the fibered product via p_2 on the

left, and then $p_1 : W_i \times_X W \rightarrow W_i \rightarrow \bigsqcup W_j$ on the right end,

$$\begin{aligned} & (W_i \times_X W) \times_{\bigsqcup_j W_j} \left(\bigsqcup_j W_j \times_X W \right) \times_{\bigsqcup_j W_j} (W_i \times_X W) \\ & \cong W \times_X (W_i \times_X W) \times_{\bigsqcup_j W_j} (W_i \times_X W) \\ & \cong W \times_X \left(\bigsqcup_b S_{ii}^{ab} \right) \times_X W \cong (W_i \times_X W) \times_{W_i} \left(\bigsqcup_b S_{ii}^{ab} \right) \times_{W_i} (W_i \times_X W) \end{aligned}$$

give alternative groupoid presentations of Z , and so $Z \times_X U \cong \bigsqcup_j W_j$. □

Definition 1.15. Let $g : Y \rightarrow X$ be a proper local embedding of Noetherian stacks, and assume that the degree of g is the same at all points of $g(Y \setminus Y_1)$. Assume that there exists an étale atlas $U \rightarrow X$ with the properties from Proposition 1.11. The *network of local embeddings* of g and U is a set of stacks $\{Y_I\}_{I \in \mathcal{P}(L)}$ and morphisms $\phi_J^I : Y_J \rightarrow Y_I$ for each pair $I \subseteq J$, $I \in \mathcal{P}_i$ and $J \in \mathcal{P}_j$, constructed as follows:

- (1) If g factors through $f : Y \rightarrow Z$ étale and $h : Z \rightarrow X$ of generic degree one on its image, as in the previous lemma, then we define $Y_I := Z_I \times_Z Y$, where the network $\{Z_I, \phi_J^I\}$ of Z is constructed as below. The morphisms $\phi_J^I : Y_J \rightarrow Y_I$ are also obtained by pull-back from the network of Z .
- (2) If g is generically one-to-one on its image, then:

Let $Y_\emptyset := X$ with the given presentation $[R_\emptyset := U \times_X U \rightrightarrows U]$. If $I = \{i\} \in \mathcal{P}_1$, then $Y_i \cong Y$, having thus a groupoid presentation $R_i \rightrightarrows V_i^a$ where

$$R_i = V_i^a \times_Y V_i^a \cong W_i \times_X W_i \setminus \bigcup_{j \neq i} S_{ij}^{ab},$$

for $S_{ij}^{ab} = \text{Im}(V_i^a \times_Y V_j^b \rightarrow W_i \times_X W_j)$. (In this case the indices a and b are uniquely associated to i and j , respectively, but we keep employing upper indices as for $i, j \in I$, we can thus discriminate between $V_i^a \supseteq V_i^a \neq V_i^b \subseteq V_j^b$).

For any $I \in \mathcal{P}_k$, define Y_I as the stack of groupoid presentation

$$\left[R_I = \left(\prod_{i \in I} \right)_{R_\emptyset} R_i \rightrightarrows V_I^a \cong W_I = \left(\prod_{i \in I} \right)_U W_i \right],$$

where $(\prod)_{R_\emptyset}$ denotes the fiber product over R_\emptyset , the groupoid structure is induced from $[R_i \rightrightarrows V_i^a \cong W_i]$, and a is the index associated to some $i \in I$.

For any K, J there is a natural isomorphism $R_{J \cup K} \cong R_J \times_{R_{J \cap K}} R_K$. By induction we find that for a fixed index a ,

$$R_I = V_I^a \times_{Y_I} V_I^a \cong W_I \times_X W_I \setminus \bigcup_{j \neq i \in I} S_{ij}^{ab}. \tag{1.8}$$

The morphisms $\phi_J^I : Y_J \rightarrow Y_I$ for $J \supseteq I$ correspond to the natural morphism between the groupoid presentations $[R_J \rightrightarrows V_J^a]$ and $[R_I \rightrightarrows V_I^a]$. In particular, $\phi_J^I = \text{id}_{Y_I}$.

Notation. If I is a set and $h \notin I$, we will write $Ih := I \cup \{h\}$.

Alternatively, the objects Y_I and the morphisms $\phi_J^I : Y_J \rightarrow Y_I$ are uniquely defined by the following lemma.

Lemma 1.16. For any $I, K \in \mathcal{P}_k$ and $b \in A_K$, there exists a natural étale map $V_K^b \rightarrow Y_I$ and

$$Y_{Ih} \times_{Y_I} V_K^b \cong \bigsqcup_{j \in L \setminus K} V_{Kj}^b. \tag{1.9}$$

More generally, for all $J \in \mathcal{P}_l$ and $I \in \mathcal{P}_k$ such that $J \supset I$,

$$Y_J \times_{Y_I} V_I^a \cong \bigsqcup_{K=K_I \supset K_{I-1} \supset \dots \supset K_{k+1} \supset K_k=I} V_K^a \tag{1.10}$$

for all chains $K_l \supset K_{l-1} \supset \dots \supset K_{k+1} \supset K_k = I$ with $K_s \in \mathcal{P}_s$ for each s . When $J = \emptyset$,

$$Y_J \times_X U \cong \bigsqcup_{a, K=K_k \supset K_{k-1} \supset \dots \supset K_1 \supset K_0=\emptyset} V_K^a \tag{1.11}$$

for $a \in K_1$.

Proof. It is enough to prove the lemma in the case when g is generically one-to-one, due to Lemma 1.14 and the construction of the network. Once (1.9) is known for all $I \in \mathcal{P}_{k-1}$, the isomorphisms (1.10) and (1.11) follow for all $I \in \mathcal{P}_k$ by successive applications of (1.9). We will prove (1.9) by induction on k . The case $k = 1$ results from the definition of Y_i 's. Assume that (1.9) holds for all $I \in \mathcal{P}_{k-1}$. Choose I and $K \in \mathcal{P}_k$.

We will denote $V := Y \times_X U$ and $V_I := \bigsqcup_{a, K=K_k \supset K_{k-1} \supset \dots \supset K_1 \supset K_0=\emptyset} V_K^a \cong Y_I \times_X U$ (from the induction hypothesis). We will construct an étale, surjective morphism $\bigsqcup_{j \in L} V_{Kj}^b \rightarrow Y_{Ih}$. For this, recall the groupoid presentation $[R_{Ih} \rightrightarrows V_{Ih}^a]$ of Y_{Ih} . We note that the first projection $p_1 : V_{Ih}^a \times_{Y_I} V_K^b \rightarrow V_{Ih}^a$ is étale and surjective. We will prove the existence of another étale, surjective morphism

$$p_2 : V_{Ih}^a \times_{Y_I} V_K^b = V_{Ih}^a \times_{Y_I} \bigcup_j V_{Kj}^b \rightarrow \bigsqcup_{j \in L} V_{Kj}^b, \tag{1.12}$$

and a morphism of groupoid schemes

$$\begin{array}{ccc} (V_{Ih}^a \times_{Y_I} V_K^b) \times \bigsqcup_j V_{Kj}^b & \longrightarrow & R_{Ih} \\ \Downarrow & & \Downarrow \\ V_{Ih}^a \times_{Y_I} V_K^b & \longrightarrow & V_{Ih}^a \end{array} \tag{1.13}$$

Indeed, $V_{Ih}^a \cong W_{Ih} \cong V_I^a \times_U V_h^c$, inducing an isomorphism

$$\varphi : V_{Ih}^a \times_{Y_I} V_K^b \rightarrow (V_I^a \times_U V_h^c) \times_{Y_I} V_K^b.$$

As $V_K^b \times_U V \cong V_K^b \times_U (Y \times_X U) \cong V_K^b \times_X Y$, there is a sequence of Cartesian diagrams

$$\begin{array}{ccccc} (V_I^a \times_U V_h^c) \times_{Y_I \times_X Y} (V_K^b \times_U V) & \longrightarrow & V_K^b \times_U V & \longrightarrow & V_K^b \\ \downarrow \pi_2 & & \downarrow & & \downarrow \\ V_I^a \times_U V_h^c & \longrightarrow & Y_I \times_X Y & \longrightarrow & Y_I \end{array}$$

giving a canonical isomorphism $F : (V_I^a \times_U V_h^c) \times_{Y_I} V_K^b \cong (V_I^a \times_U V_h^c) \times_{Y_I \times_X Y} (V_K^b \times_U V)$. Via the isomorphisms φ and $V_K^b \times_U V_j^d \cong V_{K_j}^b$ for any j , we can now define $p_2 : V_{Ih}^a \times_{Y_I} V_K^b \rightarrow \bigsqcup_{j \in L} V_{K_j}^b$ as π_2 , which is moreover étale.

Via the isomorphism F , the fiber product space $(V_{Ih}^a \times_{Y_I} V_K^b) \times_{\bigsqcup_j V_{K_j}^b} (V_{Ih}^a \times_{Y_I} V_K^b)$ is isomorphic to

$$\begin{aligned} (V_I^a \times_U V_h^c) \times_{Y_I \times_X Y} (V_K^b \times_U V) \times_{\bigsqcup_j V_{K_j}^b} (V_K^b \times_U V) \times_{Y_I \times_X Y} (V_I^a \times_U V_h^c) \\ \cong (V_I^a \times_U V_h^c) \times_{Y_I \times_X Y} (V_K^b \times_U V) \times_{Y_I \times_X Y} (V_I^a \times_U V_h^c) \\ \cong (V_I^a \times_U V_h^c) \times_{Y_I \times_X Y} (V_I^a \times_U V_h^c) \times_{Y_I} V_K^b. \end{aligned}$$

On the other hand, as

$$R_{Ih} \cong (V_I^a \times_{Y_I} V_I^a) \times_{U \times_X U} (V_h^c \times_Y V_h^c) \cong (V_I^a \times_U V_h^c) \times_{Y_I \times_X Y} (V_I^a \times_U V_h^c),$$

there is a natural projection

$$(V_{Ih}^a \times_{Y_I} V_K^b) \times_{\bigsqcup_j V_{K_j}^b} (V_{Ih}^a \times_{Y_I} V_K^b) \rightarrow R_{Ih}$$

making (1.13) into a morphism of groupoid schemes. Finally, due to the isomorphisms above, the étale atlas $V_{Ih}^a \times_{Y_I} V_K^b$ of Y_{Ih} yields a groupoid presentation with relations

$$\begin{aligned} (V_{Ih}^a \times_{Y_I} V_K^b) \times_{Y_{Ih}} (V_{Ih}^a \times_{Y_I} V_K^b) &\cong (V_{Ih}^a \times_{Y_I} V_K^b) \times_{V_{Ih}^a} R_{Ih} \times_{V_{Ih}^a} (V_{Ih}^a \times_{Y_I} V_K^b) \\ &\cong V_K^b \times_{Y_I} (V_I^a \times_U V_h^c) \times_{Y_I \times_X Y} (V_I^a \times_U V_h^c) \times_{Y_I} V_K^b \\ &\cong V_K^b \times_{Y_I} (V_{Ih}^a \times_{Y_I} V_K^b) \times_{\bigsqcup_j V_{K_j}^b} (V_{Ih}^a \times_{Y_I} V_K^b) \\ &\cong (V_{Ih}^a \times_{Y_I} V_K^b) \times_{\bigsqcup_j V_{K_j}^b} \left(\bigsqcup_j V_{K_j}^b \times_{Y_I} V_K^b \right) \times_{\bigsqcup_j V_{K_j}^b} (V_K^b \times_{Y_I} V_{Ih}^a), \end{aligned}$$

while $(\bigsqcup_j V_{K_j}^b \times_{Y_I} V_K^b) \rightrightarrows \bigsqcup_j V_{K_j}^b$ is the pull-back of $V_K^b \times_{Y_I} V_K^b \rightrightarrows V_K^b$. This completes the proof of isomorphism (1.9). \square

Corollary 1.17. *The morphisms $\phi_J^I : Y_J \rightarrow Y_I$ for $J \supset I$ are proper local embeddings. Let $J = I \cup \{j\}$ for some $j \notin I$. If U is an étale atlas of X with the properties listed in Proposition 1.11 for the morphism $Y \rightarrow X$, then $W_I \cong V_I^a$ is an étale atlas of Y_I satisfying the same set of properties for the morphism $\phi_J^I : Y_J \rightarrow Y_I$, and the network of local embeddings associated to the local embedding ϕ_J^I with the étale atlas W_I consists of $\{\phi_K^H : Y_K \rightarrow Y_H\}_{K \supset H \supseteq J}$.*

Proof. The properness of the morphisms $\phi_J^I : Y_J \rightarrow Y_I$ is a direct consequence of (1.10) and the fact that $g : Y \rightarrow X$ is proper. Formulas (1.9) and (1.10) also show that $W_I \cong V_I^a$ has the properties listed in Proposition 1.11. Thus for $J = I \cup \{j\}$, the network of local embeddings associated to ϕ_J^I with the étale atlas V_I^a is made up of stacks with étale atlases given by W_K for some $K \supseteq J$, and relations given by

$$\left(\prod_{k \in K \setminus I} \right)_{R_I} R_{Ik} \cong R_K,$$

due to Definition 1.15. The stacks in the network are thus $\{Y_K\}_{K \supseteq J}$. Accordingly, the morphisms of the network are exactly ϕ_K^H with $K \supset H \supseteq J$. \square

Although the stacks Y_I are not fibered products of stacks Y_i with $i \in I$, the above arguments show that they can be constructed intrinsically from a succession of fibered products, after removing the diagonal components. In particular, there exists a closed embedding

$$Y_I \hookrightarrow \left(\prod_{i \in I} \right)_X Y_i.$$

More precisely:

Corollary 1.18. *Each $\iota_{Ih} := (\phi_{Ih}^I, \phi_{Ih}^h) : Y_{Ih} \hookrightarrow Y_I \times_X Y$ is a closed embedding, and*

$$Y_I \times_X Y \cong \left(\bigsqcup_{h \notin I, b \in A_h} \text{Im } \iota_{Ih} \right) \sqcup \left(\bigsqcup_{i \in I} \text{Im}(\phi_i^I, \phi_i^i) \right),$$

with the diagonal morphisms $(\phi_i^I, \phi_i^i) : Y_i \rightarrow Y_I \times_X Y$ yielding the higher-dimensional components of $Y_I \times_X Y$.

Proof. This is also a direct consequence of Lemma 1.16, due to the Cartesian diagrams

$$\begin{array}{ccccc} \bigsqcup_{j \notin I} V_{Ij}^a & \longrightarrow & V_I^a \times_U V & \longrightarrow & V_I^a \\ \downarrow & & \downarrow & & \downarrow \\ Y_{Ih} & \longrightarrow & Y_I \times_X Y & \longrightarrow & Y_I \end{array}$$

where $V = \bigsqcup_{j,b} V_j^b$, while $V_{Ij}^a \cong V_I^a \times_U V_j^b$ for fixed $b \in A_j$. \square

Remark 1.19. By the previous corollary applied successively to each index I , the objects of the network of local embeddings associated to the proper local embedding $g : Y \rightarrow X$ and the étale atlas U are in fact independent of the choice of atlas with the properties listed in Proposition 1.11. The choice of U determines only the number of copies of each Y_I contained in the associated network. For example, by replacing the étale atlas U of X with $U \sqcup U$ and keeping the choice of étale atlases for Y unchanged, we obtain a network which, apart from the morphisms $Y_i \rightarrow X$, is a disjoint union of two copies of the network for U .

Notation. Let $g : Y \rightarrow X$ be a proper local embedding of Noetherian stacks satisfying the assumptions from Proposition 1.11. For any two generic points ξ_J and ξ_K of Y_J and Y_K respectively, such that $K \supset J$ and such that ξ_J specializes to $\varphi_K^J(\xi_K)$, we denote by $[\xi_K \rightarrow \xi_J]$ the degree of φ_K^J at $\varphi_K^J(\xi_K)$.

For $K \in \mathcal{P}_k$ and $J \in \mathcal{P}_j$ as above, the following relation follows directly from Lemma 1.16:

$$[\xi_K \rightarrow \xi_J] = N(K, J)v(\xi_K, \xi_J),$$

where $N(K, J)$ is the number of all maximal chains $K = K_k \supset K_{k-1} \supset \dots \supset K_{j+1} \supset K_j = J$ and

$$v(\xi_K, \xi_J) = |\{K' \in \mathcal{P}_k; \exists \xi_{K'} \text{ generic point of } Y_{K'} \text{ such that } \varphi_{K'}^J(\xi_{K'}) = \varphi_K^J(\xi_K)\}|.$$

Thus if $K \in \mathcal{P}_k, J \in \mathcal{P}_j$ and $I \in \mathcal{P}_i$ satisfy $K \supset J \supset I$, then by a count of chains

$$\frac{[\xi_K \rightarrow \xi_J][\xi_J \rightarrow \xi_I]}{[\xi_K \rightarrow \xi_I]} = \frac{|\{K_k \in \mathcal{P}_k; K_k \supset I\}|}{|\{K_k \in \mathcal{P}_k; K_k \supset J\}| |\{K_j \in \mathcal{P}_j; K_j \supset I\}|} \tag{1.14}$$

for any generic points ξ_I, ξ_J and ξ_K of Y_I, Y_J and Y_K respectively, such that ξ_J specializes to $\varphi_K^J(\xi_K)$ and ξ_I to $\varphi_J^I(\xi_J)$.

Theorem 1.20. Consider a network of proper local embeddings $\phi_J^I : Y_J \rightarrow Y_I$ for $I \subseteq J, I \in \mathcal{P}_i$ and $J \in \mathcal{P}_j$, associated to a proper local embedding $Y \rightarrow X$ by Definition 1.15 under the assumptions of Proposition 1.11, where by convention $Y_\emptyset = X$. For each such morphism ϕ_J^I , there exists a closed embedding of stacks $\phi_J^I : Y'_J \hookrightarrow Y'_I$, together with étale surjective morphisms $p_J : Y'_J \rightarrow Y_J$ and $p_I : Y'_I \rightarrow Y_I$ making the diagram

$$\begin{array}{ccc} Y'_J & \xrightarrow{\phi_J^I} & Y'_I \\ p_J \downarrow & & \downarrow p_I \\ Y_J & \xrightarrow{\phi_J^I} & Y_I \end{array}$$

commutative, and such that

$$Y_J \times_{Y_I} Y'_I = \bigsqcup_{J' \in \mathcal{P}_j, J' \supseteq I} Y'_{J'} \quad \text{for } J \in \mathcal{P}_j.$$

Proof. We construct Y'_I in decreasing order of I , with $Y'_I = Y_I$ for I maximal. For $I \in \mathcal{P}_{n-1}$, the stack $N_I := \bigsqcup_{J \supseteq I} Y_J$ comes with a natural map $n_I : N_I \rightarrow Y_I$ étale on its image, and the étale atlas

$$\bigsqcup_{J \supseteq I} W_J \cong n_I(N_I) \times_{Y_I} W_I.$$

Y'_I is constructed as in Proposition 1.2, so that it admits

- (1) a surjective étale morphism $p_I : Y'_I \rightarrow Y_I$,
- (2) an embedding $N_I \hookrightarrow Y'_I$ whose composition with p_I is n_I ,

- (3) an étale atlas W_I satisfying $Y'_J \times_{Y'_I} W_I \cong W_J$, and
- (4) a groupoid presentation $R'_I \rightrightarrows W_I$ given by

$$R'_I \cong W_I \times_X W_I \setminus \bigcup_{i \neq j, i \in I} S_{ij}^{ab} \setminus \bigcup_{k \neq l, k, l \notin I} S_{lk}^{ac}.$$

Given any $k < n - 1$, $I \in \mathcal{P}_k$ and assuming the stacks Y'_J with the above properties constructed for all $J \supset I$, the network stack N_I is constructed out of all stacks $Y'_{I \cup \{h\}}$, by gluing each pair $Y'_{I \cup \{h\}}$ and $Y'_{I \cup \{h'\}}$ along $Y'_{I \cup \{h, h'\}}$ as in [AGV, Proposition A.1.1 and Corollary A.1.2]. Accordingly, N_I admits a natural groupoid presentation $[\bigcup_{J \supset I} R'_J \rightrightarrows \bigcup_{J \supset I} W_J]$, where the unions are considered inside $U \times_X U$ and U , respectively. Next, the composition maps $Y'_{I \cup \{h\}} \rightarrow Y_{I \cup \{h\}} \rightarrow Y_I$ glue together to a morphism $n_I : N_I \rightarrow Y_I$ which is étale on its image, as noted from the groupoid presentations of N_I and Y_I . Moreover, for every $J \supseteq I$ there exists a canonically defined closed embedding $Y'_J \hookrightarrow N_I$. If n_I were proper, the construction in Proposition 1.2 applied to n_I would yield the stack $Y'_I := (Y_I)_{N_I}$ with the desired properties (1)–(4). However, n_I is not necessarily proper, so we will obtain the same construction indirectly. We consider a canonical stratification

$$N_I^n \hookrightarrow N_I^{n-1} \hookrightarrow \dots \hookrightarrow N_I^{k+1} = N_I$$

and a sequence of lifts $(Y_I)_{N_I^l}$, for $0 \leq k \leq l \leq n - 1$, where $N_I^n = \bigsqcup_{J \supset I, J \in \mathcal{P}_n} Y_J$ has a proper map into Y_I , and $N_I^{k+1} = N_I$ has a proper map into $(Y_I)_{N_I^{k+2}}$. This is discussed in the next lemma.

Although the process described in the lemma is indirect, the groupoid presentation of the resulting space can be constructed directly as in Proposition 1.2, as indicated by property (2) in the lemma. Moreover, the étale atlas $\bigcup_{J \supset I} W_J$ of N_I is embedded in the étale atlas W_I of Y_I , and so the relations R'_I defining $(Y_I)_{N_I}$ are obtained, via Proposition 1.2, by restricting the preimage of $\bigcup_{J \supset I} W_J$ in R_I so that $R'_I|_{\bigcup_{J \supset I} W_J}$ coincides with the image of $\bigcup_{J \supset I} R'_J$ in R_I . This directly yields the presentation given at point (4) above.

At the final step, the stack X' admits a groupoid presentation $[R' \rightrightarrows U]$, with

$$R' = U \times_X U \setminus \bigcup_{i \neq j} S_{ij}^{ab}. \tag{1.15}$$

□

Lemma 1.21. *With the notation above, let k and l be any integers such that $0 \leq k \leq l \leq n - 1$, and let $I \in \mathcal{P}_k$. Define*

$$N_I^l := \text{Im} \left(\bigsqcup_{J \supseteq I, J \in \mathcal{P}_l} Y'_J \rightarrow N_I \right).$$

Then there exists a sequence of étale, surjective morphisms

$$Y'_I = (Y_I)_{N_I^{k+1}} \rightarrow (Y_I)_{N_I^{k+2}} \rightarrow \dots \rightarrow (Y_I)_{N_I^{n-1}} \rightarrow (Y_I)_{N_I^n} \rightarrow Y_I,$$

and morphisms $n_I^l : N_I = N_I^{k+1} \rightarrow (Y_I)_{N_I^l}$ for each l , étale on their images, such that the following properties hold:

- (1) The restriction of n_I^{l+1} to N_I^l is a proper morphism. In particular, for $l = k + 1$, the morphism $n_I : N_I \rightarrow (Y_I)_{N_I^{k+2}}$ is proper.
- (2) With the notation from Proposition 1.2, $((Y_I)_{N_I^{l+1}})_{N_I^l} \cong (Y_I)_{N_I^l}$.
- (3) For each $J \supset I$, $J \in \mathcal{P}_s$ and $l > s$, there exist canonical Cartesian diagrams

$$\begin{array}{ccc} N_J^l & \longrightarrow & N_I^l \\ \downarrow & & \downarrow \\ (Y_J)_{N_J^{l+1}} & \longrightarrow & (Y_I)_{N_I^{l+1}} \end{array}$$

Proof. The construction of $(Y_I)_{N_I^l}$ and the proofs of properties (1)–(3) are conducted by decreasing induction on l . When $l = n$, consider $N_I^{n+1} = \emptyset$ and $(Y_I)_{N_I^{n+1}} = Y_I$. The map

$$n_I^n|_{N_I^n} : N_I^n = \bigsqcup_{J \supset I, J \in \mathcal{P}_n} Y_J \rightarrow Y_I$$

is proper and étale on its image, so $(Y_I)_{N_I^n}$ is constructed as in Proposition 1.2. As $n_I : N_I \rightarrow Y_I$ is étale on its image and N_I^n has a closed embedding in N_I satisfying the conditions of Corollary 1.7(a), the map n_I lifts to $n_I^n : N_I \rightarrow (Y_I)_{N_I^n}$. Furthermore, for all $J \in \mathcal{P}_l$, $l \geq k$, the Cartesian diagram

$$\begin{array}{ccc} N_J^n & \longrightarrow & N_I^n \\ \downarrow & & \downarrow \\ Y_J & \longrightarrow & Y_I \end{array}$$

induces a proper morphism $(Y_J)_{N_J^n} \rightarrow (Y_I)_{N_I^n}$ as in Corollary 1.8. Gluing all Y_J 's with $J \supset I$, $J \in \mathcal{P}_{n-1}$ gives N_I^{n-1} with a proper morphism to $(Y_I)_{N_I^n}$, the restriction of n_I^n . This leads to the next step of induction, with the construction of $(Y_I)_{N_I^{n-1}} := ((Y_I)_{N_I^n})_{N_I^{n-1}}$. Assume now the constructions of $(Y_J)_{N_J^{l'}}$, $n_J^{l'}$ and properties (1)–(3) are known for all $l' > l > s - 1$, $J \supseteq I$, $J \in \mathcal{P}_s$. Then by property (3) and Corollaries 1.8, 1.7, there is a proper morphism

$$(Y_K)_{N_K^l} := ((Y_K)_{N_K^{l+1}})_{N_K^l} \rightarrow ((Y_J)_{N_J^{l+1}})_{N_J^l} =: (Y_J)_{N_J^l}$$

for any $K \supset J \supseteq I$. Gluing the stacks $Y'_K = (Y_K)_{N_K^l}$ for all such $K \in \mathcal{P}_{l-1}$ gives N_J^{l-1} with a proper morphism to $(Y_J)_{N_J^l}$. This is the restriction of the morphism $n_J^l : N_J \rightarrow (Y_J)_{N_J^l}$ étale on its image, which was obtained from n_J^{l+1} by Corollary 1.7. Finally, these proper morphisms fit together in Cartesian diagrams as in (3) with l replaced by $l - 1$,

since the diagrams

$$\begin{array}{ccc}
 \bigsqcup_{K \supset J, K \in \mathcal{P}_{l-1}} (Y_K)_{N_K^s} & \longrightarrow & \bigsqcup_{K \supset I, K \in \mathcal{P}_{l-1}} (Y_K)_{N_K^s} \\
 \downarrow & & \downarrow \\
 (Y_J)_{N_J^s} & \longrightarrow & (Y_I)_{N_I^s}
 \end{array}$$

are Cartesian for all $s \geq l$, due to Corollary 1.8 and decreasing induction on s . \square

Definition 1.22. For a proper local embedding $g : Y \rightarrow X$ and an étale atlas U , the morphism $p = p_\emptyset : X' \rightarrow X$ introduced in Theorem 1.20 (in the particular case when $I = \emptyset$) will be called the *étale lift* of g with respect to the étale atlas U .

Remark 1.23. We note that with this terminology, the étale morphisms $p_I : Y'_I \rightarrow Y_I$ introduced in Theorem 1.20 are the étale lifts of ϕ_J^I with respect to the étale atlas W_I for any $J = I \cup \{j\}$ and $j \notin I$. Indeed, this is a direct consequence of Corollary 1.17, as all the stacks N_J^s and Y'_J for $J \supseteq I$ constructed in the course of the proof of Theorem 1.20 depend only on the network $\{\phi_K^H : Y_K \rightarrow Y_H\}_{K \supset H \supseteq J}$.

Example 1.24. Consider a projective curve X whose singular locus consists of a simple node x , and let $g : Y \rightarrow X$ be its normalization, with $g^{-1}(x) = \{y_1, y_2\}$. Then X' is the union of two copies Y^1 and Y^2 of Y , glued together along $g^{-1}(x)$ such that $y_1^1 = y_2^2$ and $y_2^1 = y_1^2$.

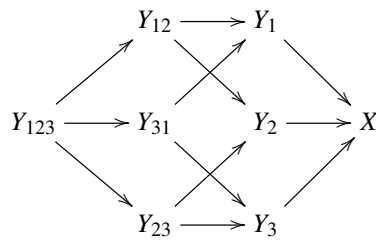
Example 1.25. Let $U^0 := \text{Spec } k[x_1, x_2, x_3]$ and consider the action of $\mathbb{Z}_3 \cong A_3$ on U_0 which permutes the coordinates,

$$\sigma([x_1, x_2, x_3]) := [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}].$$

Let $V^0 := \text{Spec } k[x_1, x_2] \sqcup \text{Spec } k[x_2, x_3] \sqcup \text{Spec } k[x_1, x_3]$, with the natural local embedding $g^0 : V^0 \rightarrow (x_1 x_2 x_3 = 0) \hookrightarrow U^0$, and the natural action of \mathbb{Z}_3 on V^0 which is compatible with g^0 . Taking quotients yields a local embedding

$$g : \mathbb{A}^2 \rightarrow [\mathbb{A}^3 / \mathbb{Z}_3].$$

We denote $Y := \mathbb{A}^2$, $X := [\mathbb{A}^3 / \mathbb{Z}_3]$. Then $U := \bigsqcup_{\alpha \in S_3} U^\alpha$ is an étale atlas of X satisfying all the properties listed in Proposition 1.11, where $Y \times_X U = V_1 \sqcup V_2 \sqcup V_3$ and $V_i \cong W_i = \bigsqcup_{\alpha \in S_3} (x_{\alpha(i)} = 0) \hookrightarrow \bigsqcup_{\alpha \in S_3} \mathbb{A}^3$. The associated network of local embeddings will thus be of the form



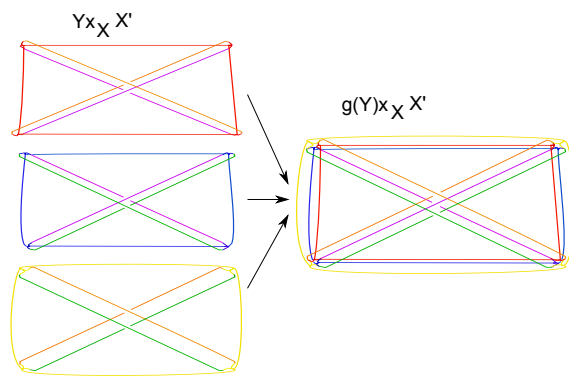


Fig. 1. Example 1.25.

where $Y_1 \cong Y_2 \cong Y_3 \cong Y = \mathbb{A}^2$ with the map $g : Y \rightarrow X$. Moreover $Y_{12} \cong Y_{31} \cong Y_{23} = \mathbb{A}^1 \sqcup \mathbb{A}^1$, and $\phi_{12}^1 : Y_{12} \rightarrow Y_1 = \text{Spec } k[x_1, x_2]$ maps each copy of \mathbb{A}^1 into $(x_1 = 0)$ and $(x_2 = 0)$, respectively, while $\phi_{12}^2 = \phi_{12}^1 \circ \tau$, where τ is the transposition switching the two copies of \mathbb{A}^1 , and similarly for the other morphisms ϕ_{ij}^i . Finally, Y_{123} is a copy of two points, each mapped to zero on one of the lines of Y_{ij} , respectively. Thus, with the notation from Theorem 1.20, $Y'_i = (Y_i)_{N_i}$ is obtained by gluing two copies of \mathbb{A}^2 outside the union of two lines, and N is the union of Y'_i 's for $i \in \{1, 2, 3\}$, glued along the Y_{ij} 's as indicated by the arrows in the network. The colors in Figure 1 (see the pdf file) show which pairs of lines are identified in N . Thus $X' = X_N$ is isomorphic to X outside N , while

$$N = g(Y) \times_X X' \quad \text{and} \quad Y \times_X X' = \bigsqcup Y'_i,$$

as shown in the figure.

1.3. Chow rings and universally closed push-forwards

Let $g : Y \rightarrow X$ be a proper, local embedding of Noetherian stacks. Assume that Y is reduced and geometrically unibranch, that the morphism on the image $Y \rightarrow g(Y)$ is equidimensional, and that its degree is equal to a fixed number d at all generic points. The stacks Y'_i constructed in the previous sections are in general non-separated. However, they do satisfy the existence part in the valuative criterion of properness.

Definition 1.26. A morphism of stacks $f : F \rightarrow G$ will be called *universally closed* if it is of finite type and, for any complete valuation ring R with field of fractions K and any commutative diagram

$$\begin{array}{ccc} \text{Spec } K & \xrightarrow{u} & F \\ \downarrow & & \downarrow f \\ \text{Spec } R & \longrightarrow & G \end{array}$$

there exist finite extensions K' of K such that, for the integral closure R' of R in K' , the composition morphism $\text{Spec } K' \rightarrow F$ extends to $\text{Spec } R'$. As in [V], we are mostly interested in stacks having coarse moduli schemes. Then by Proposition 2.6 and Definition 2.1 in [V], the image of such a morphism of stacks is defined and the definition above is a natural extension to stacks of the notion of universally closed morphism of schemes.

Definition 1.27. Consider a universally closed morphism of stacks $f : F \rightarrow G$. A *probabilistic weight* w of f is a map defined on the set of all integral substacks of F , with values in the interval $[0, 1]$, such that the weight of the generic point of F is 1 and, for any commutative diagram as in Definition 1.26,

$$w(D) = \sum_i w(P_i),$$

where D is the image in F of the unique point in $\text{Spec } K$, and P_i are the images of the closed point in $\text{Spec } R'$ for all map extensions $\text{Spec } R' \rightarrow F$ as above.

Definition 1.28. Given a universally closed morphism of integral stacks $f : F \rightarrow G$ with probabilistic weight w , let

$$f_*[V] = w(V) \deg(V/W)[W]$$

for any closed integral substack V of F , where $W = f(V)$ and $\deg(V/W)$ is as in [V, Definition 1.15]. A homomorphism $f_* : Z_k(F) \rightarrow Z_k(G)$ is then defined by linear extension.

Proposition 1.29. *The homomorphism $f_* : Z_k(F) \rightarrow Z_k(G)$ induces a well defined universally closed push-forward homomorphism $f_* : A_k(F) \rightarrow A_k(G)$.*

Proof. Propositions 3.7 in [V] and Proposition 1.4 of [F] deal with the proper push-forward in the cases of stacks and schemes, respectively. The difference for universally closed maps lies in the proof of the latter, in the case when $\dim F = \dim G$. The case when f is finite is identical to Case 2 in Proposition 1.4 of [F]. Following [F] closely, we take normalizations of the source and target, and the problem is thus reduced to the case of a universally closed morphism of normal varieties. Let W be a codimension one subvariety of G , let A be the local ring of W on G , and B the integral closure of A in the field $k(F)$ of rational functions on F , such that B is a discrete valuation ring. Then by Definition 1.26, for each maximal ideal m_i of B there are a finite number of codimension one subvarieties V_i^l of F such that B dominates, and is therefore equal to, the local ring of each V_i^l in X . Then for any $r \in k(F)^*$,

$$\sum_i \text{ord}_{V_i^l}(r) \deg(V_i^l/W) = \text{ord}_W(N(r)),$$

where for each i a choice of the index l has been fixed, and $N(r)$ is the determinant of the $k(G)$ -linear endomorphism of $k(F)$ given by multiplication by r . Finally,

$$f_*[\text{div}(r)] = \sum_{V_i^l} \text{ord}_{V_i^l}(r) f_*[V_i^l] = \sum_{V_i^l} \text{ord}_{V_i^l}(r) w(V_i^l) \deg(V_i^l/W)[W].$$

For i fixed, $\text{ord}_{V_i^l}(r) \deg(V_i^l/W)$ is constant and $\sum_l w(V_i^l) = w(F) = 1$ from Definition 1.27. Thus

$$f_*[\text{div}(r)] = \sum_W \text{ord}_W(N(r))[W]$$

as in the case of proper morphisms. This finishes the part of the proof specific to the universal closedness of f . \square

Universally closed push-forwards enjoy the usual properties of their proper relatives: for example, they commute with flat pull-backs, and the usual projection formula holds for f universally closed and flat.

Theorem 1.30. *Let $g : Y \rightarrow X$ be a proper, local embedding of Noetherian stacks. Assume that Y is reduced and geometrically unibranch, that the morphism on the image $Y \rightarrow g(Y)$ is equidimensional, and that its degree is equal to a fixed number d at all points of $g(Y \setminus Y_1)$. There exists a Deligne–Mumford stack X' with a surjective étale morphism to X , such that the fiber product $Y' = g(Y) \times_X X'$ is a finite union of stacks Y'_i mapping étale onto Y , and such that the maps $Y'_i \rightarrow Y$ and $p : X' \rightarrow X$ are universally closed. Moreover, p admits a probabilistic weight w .*

Proof. Consider a network of local embeddings for $g : Y \rightarrow X$ as in Definition 1.15. Let X' be the corresponding lift of X constructed in Theorem 1.20. Let ζ_I denote a generic point of Y'_I , with ζ_\emptyset the generic point of X' which specializes to the image of ζ_I . With the notation from (1.14), define

$$w(\zeta_I) := \frac{1}{|\mathcal{P}_k[\xi_I \rightarrow \xi_\emptyset]|}$$

for any $I \in \mathcal{P}_k$. Here for any K , we let ξ_K denote the image of ζ_K in Y_K . Then (1.14) becomes

$$w(\zeta_I) = \sum [\xi_J \rightarrow \xi_I] w(\zeta_J), \tag{1.16}$$

where the sum is taken over all $J \in \mathcal{P}_j, J \supset I$, such that ξ_I specializes to $\phi_J^I(\xi_J)$. As $\{Y'_I \setminus \bigcup_{J \supset I} Y'_J\}_I$ forms a locally closed stratification of X' , the generic point of any integral substack D of X' will be found in exactly one of the above strata. We extend w to a function on all points of X' by identifying $w(D)$ with the weight of the generic point of its associated stratum which specializes to it.

The universal closedness property will result during the proof that w is a probabilistic weight of $p : X' \rightarrow X$. Consider a complete discrete valuation ring R with field of fractions K , a commutative diagram

$$\begin{array}{ccc} \text{Spec } K & \xrightarrow{u} & X' \\ \downarrow & & \downarrow f \\ \text{Spec } R & \xrightarrow{v} & X \end{array}$$

such that the image q_0 in X' of the generic point of $\text{Spec } R$ lies in a stratum $Y'_I \setminus \bigcup_{K \supset I} Y'_K$, and the image q_1 in X of the closed point of $\text{Spec } R$ lies in $\text{Im}(Y_J \setminus \bigcup_{K \supset J} Y_K \rightarrow X)$ for some $J \supseteq I$. Since $Y'_I \setminus \bigcup_{K \supset I} Y'_K \cong Y_I \setminus \bigcup_{K \supset I} Y_K$ and the map $Y_I \rightarrow X$ is proper, there is a unique extension $v' : \text{Spec } R' \rightarrow Y'_I$ of the composition $\text{Spec } K' \rightarrow \text{Spec } K \rightarrow Y'_I \rightarrow Y_I$, with the notation of Definition 1.26. Let $q \in \text{Im}(Y_J \rightarrow Y_I)$ be the lift of q_1 through this extension. Then through each point in the preimage of q in $Y'_I \subset X'$ there is a unique lift $\text{Spec } R' \rightarrow Y'_I \hookrightarrow X'$ of the map v' . The generic point of each such lift has to be q_0 , because the map $p_i : Y'_I \rightarrow Y_I$ restricts to the above mentioned isomorphism $Y'_I \setminus \bigcup_{K \supset I} Y'_K \cong Y_I \setminus \bigcup_{K \supset I} Y_K$. Let y_i be the images of the closed points of these lifts. They all have the same weight $w(\xi_J)$. Then by (1.16),

$$w(q_0) = \sum_i w(y_i),$$

which proves that w is a probabilistic weight for the universally closed morphism p . \square

Corollary 1.31. *For each $i \in \{0, \dots, n\}$ and $I \in \mathcal{P}_i$, there is a universally closed push-forward map $p_{I*} : A_k(Y'_I) \rightarrow A_k(Y_I)$ such that for any connected component Z of Y_I , the restriction of the map*

$$p_{I*} \circ p_I^* : A_k(Y_I) \rightarrow A_k(Y_I)$$

to $A_k(Z)$ is $d \cdot \text{id}_{A_k(Z)}$, where d is the degree of the morphism $p_I^{-1}(Z) \rightarrow Z$. In particular, the flat pull-back p_I^ is injective.*

We note that all maps p_I are universally closed due to Theorem 1.30 in conjunction with Remark 1.23.

By convention, $X' = Y'_\emptyset$, and thus the corollary shows how $A(X)$ can be regarded as a subgroup of $A(X')$, and how classes in $A(X)$ can be recovered from $A(X')$ via push-forward.

In addition to the assumptions of Theorem 1.30, for the remainder of this section we will assume that X is smooth, and that the morphisms $\phi_I^J : Y_I \rightarrow Y_J$ are local regular embeddings. An extended Chow ring of the network $\{\phi_I^J : Y_I \rightarrow Y_J\}_{I,J}$ was introduced in Definition 3.6 of [MM1]. We recall this definition with a slight variation that does without the action of a symmetry group on $\mathcal{P} = \bigcup \mathcal{P}_k$.

Notation. Fix I and $I \cup \{h\} \in \mathcal{P}$. For any cycle $\alpha = [V] \in Z_l(\text{Im}(Y_{I \cup \{h\}} \rightarrow Y_I))$, let $\alpha_h \in Z_l(Y_{I \cup \{h\}})$ be defined as follows:

$$\alpha_h = \frac{\sum_i [V_h^i]}{\text{deg}((\phi_{I \cup \{h\}}^I)^{-1}(V)/V)}$$

where $\{V_h^i\}_i$ are the l -dimensional components of $(\phi_{I \cup \{h\}}^I)^{-1}(V)$.

Definition 1.32. The vector spaces $A_l(\{Y_I\}_{I \in \mathcal{P}}; \mathbb{Q})$ are defined by

$$A_l(\{Y_I\}_{I \in \mathcal{P}}; \mathbb{Q}) := \bigoplus_I Z^{l - \text{codim}_Z Y_I}(Y_I) / \sim,$$

the sum taken over all $I \in \mathcal{P}$ with $\text{codim}_X Y_I \leq l$. The equivalence relation \sim is generated by rational equivalence together with relations of the type

$$\alpha \sim \sum_{h'} \alpha_{h'}$$

for any cycle $\alpha = \phi_{I \cup \{h\}}^I \alpha_h \in Z_l(\text{Im}(Y_{I \cup \{h\}} \rightarrow Y_I))$.

The field of coefficients \mathbb{Q} will be omitted in the notation throughout the rest of the text.

Definition 1.33. Multiplication is defined as

$$\alpha \cdot_r \beta := \phi_{I \cup J}^{I*}(\alpha) \cdot \phi_{I \cup J}^{J*}(\beta) \cdot \frac{c_{\text{top}}(\phi_{I \cup J}^{I*} \mathcal{N}_{Y_I|X}) c_{\text{top}}(\phi_{I \cup J}^{J*} \mathcal{N}_{Y_J|X})}{c_{\text{top}}(\mathcal{N}_{Y_{I \cup J}|X})}$$

in $A(Y_{I \cup J})$, for any two classes $\alpha \in A(Y_I)$ and $\beta \in A(Y_J)$. Here $\phi_{I \cup J}^{I*}, \phi_{I \cup J}^{J*}$ are the (generalized) Gysin homomorphisms, as defined in [V], while $c_{\text{top}}(\mathcal{N}_{Y_K|X})$ denotes the highest Chern class of the normal bundle $\mathcal{N}_{Y_K|X}$.

Theorem 1.34. *The following rings are isomorphic:*

$$A(X') \cong A(\{Y_I\}_{I \in \mathcal{P}}).$$

Proof. Compositions of the flat pull-backs p_I^* with the push-forward of embeddings ϕ_{I*}^{\emptyset} add up to a morphism $\bigoplus_{I \in \mathcal{P}} A(Y_I) \rightarrow A(X')$, which moreover factors through $F : A(\{Y_I\}_{I \in \mathcal{P}}) \rightarrow A(X')$. The compatibility of the group morphism F with the product operations is a direct consequence of the excess intersection formula for the embeddings of Y'_I and Y'_J into $Y'_{I \cap J}$, with intersection $Y'_{I \cup J}$.

Construct an inverse for F as follows. For each $J \in \mathcal{P}$, let U_J denote the complement in Y'_J of all the images of Y'_I , with $J \subset I$. (To define this complement one can work with supports of the corresponding coarse moduli schemes, but there is a canonical stack structure on U_J .) Note that $U_J = Y_J \setminus \bigcup_{I \supset J} \text{Im}(Y_I \rightarrow Y_J)$ as well. Working with the commutative diagram of open/closed exact sequences

$$\begin{array}{ccccccc} \bigoplus_{k \notin J} A(Y_{J \cup \{k\}}) & \longrightarrow & A(Y_J) & \longrightarrow & A(U_J) & \longrightarrow & 0 \\ p_{J \cup \{k\}}^* \downarrow & & p_J^* \downarrow & & = \downarrow & & \\ \bigoplus_{k \notin J} A(Y'_{J \cup \{k\}}) & \longrightarrow & A(Y'_J) & \longrightarrow & A(U_J) & \longrightarrow & 0 \end{array}$$

one finds, for each $\alpha'_J \in A(Y'_J)$, classes $\alpha_J \in A(Y_J)$ and $\alpha'_{J \cup \{k\}} \in A(Y'_{J \cup \{k\}})$ such that

$$\alpha'_J = p_J^* \alpha_J + \sum_{k \notin J} \phi_{J \cup \{k\}}^{J*} \alpha'_{J \cup \{k\}}.$$

Reiterating this argument one finds a collection of classes $[\alpha_I] \in \bigoplus_I A(Y_I)$ for all I containing J such that

$$\alpha'_J = \sum_{I \supseteq J} p_J^* \phi_{J*}^I \alpha_I.$$

The choice of α_I is unique up to the equivalence relation \sim . Indeed, if $\sum_{I \supseteq J} p_J^* \phi_{J*}^I \beta_I = 0$ then $\sum_{I \supseteq J} \phi_{J*}^I \beta_I = 0$ by Corollary 1.31, or equivalently, $\beta_J = -\sum_{I \supset J} \phi_{J*}^I \beta_I$. After applying the equivalence relation to each $\phi_{J*}^I \beta_I$, β_J can be replaced by a sum of classes from $A(Y_I)$ with $I \supset J$, and induction on the index set can then be applied by Lemma 3.13 in [MM1]. When $J = \emptyset$, the collection $[\alpha_I] \in \bigoplus_I A(Y_I)$ defines the desired inverse of F . \square

2. The Chern classes of a weighted projective blow-up

In this section we extend the notion of a blow-up along a closed embedding to the case of a proper local embedding $g : Y \rightarrow X$ of Noetherian stacks. We assume that Y is reduced and geometrically unibranch, that the morphism on the image $Y \rightarrow g(Y)$ is equidimensional, and that its degree is equal to a fixed number d at all generic points.

We note that for practical purposes, it is often enough to work with just $\text{Bl}_{Y'_i} X'$, for universally closed étale lifts $p : X' \rightarrow X$, and a corresponding étale morphism $Y'_i \rightarrow Y$ constructed as in Section 1. For example, this is the case when one is interested in intersection theory on a smooth Deligne–Mumford stack. However, in other contexts, a stack \tilde{X} with a proper morphism $f : \tilde{X} \rightarrow X$ is required. A natural construction of \tilde{X} follows. For special morphisms $g : Y \rightarrow X$, the stack \tilde{X} was defined in [MM1].

Definition 2.1. Let $g : Y \hookrightarrow X$ be a proper local embedding of smooth stacks, and consider a groupoid presentation $[R \rightrightarrows U]$ of X such that $Y \times_X U = \bigsqcup_{i,a} V_i^a$ has all the properties listed in Proposition 1.11. The *blow-up* $\tilde{X} = \text{Bl}_Y X$ of X along Y is defined as the Deligne–Mumford stack of étale groupoid presentation $\tilde{X} = [\tilde{R} \rightrightarrows \tilde{U}]$, where \tilde{U} is the fibered product over U of the blow-ups $\text{Bl}_{W_i} U$ for all i , and \tilde{R} is the fibered product over R of the blow-ups $\text{Bl}_{S_{ij}^{ab}} R$ for all i, j, b , where a is fixed.

The natural morphisms $\tilde{s}, \tilde{t} : \tilde{R} \rightrightarrows \tilde{U}$, as well as $\tilde{e} : \tilde{U} \rightarrow \tilde{R}$, $\tilde{i} : \tilde{R} \rightarrow \tilde{R}$ and $\tilde{m} : \tilde{R} \times_{\tilde{U}} \tilde{R} \rightarrow \tilde{R}$ making up the groupoid structure are induced from the groupoid morphisms s, t, e, i, m of $[R \rightrightarrows U]$ by the universal property of the blow-up. Indeed, this is due to the following lemma.

Lemma 2.2. *With the notation from Proposition 1.11, the following hold:*

- (1) $V_i^a \times_U R \cong \bigsqcup_{j,b} S_{ij}^{ab}$ via each of the étale morphisms s, t .
- (2) $\text{Bl}_{\bigsqcup_{j,b} S_{ij}^{ab}} R \cong (\text{Bl}_{W_i} U) \times_U R$ via each of the étale morphisms s, t .
- (3) $e^{-1} \mathcal{I}_{V_i^a \times_U R} = \mathcal{I}_{W_i}$, while $i^{-1} \mathcal{I}_{S_{ij}^{ab}} = \mathcal{I}_{S_{ji}^{ba}}$ and $m^{-1} \mathcal{I}_{S_{ij}^{ab}} = \mathcal{I}_{\bigsqcup_{k,c} S_{ik}^{ac} \times_U S_{kj}^{cb}}$.
- (4) $\tilde{R} \times_{\tilde{U}} \tilde{R} \cong (\prod_{i,a,j,b,k,c} R \times_U R) \text{Bl}_{S_{ik}^{ac} \times_U S_{kj}^{cb}} (R \times_U R)$.

Proof. We will identify V_i^a with its image W_i in U . Statements (1) and (3) follow from the definitions of V_i^a and S_{ij}^{ab} . Statement (2) is a consequence of (1), together with the observation that $\mathcal{T}_{\bigsqcup_{j,b} S_{ij}^{ab}}^n \cong (s^{-1}\mathcal{T}_{W_i})^n \cong s^*\mathcal{T}_{W_i}^n$ and $\mathcal{T}_{\bigsqcup_{j,b} S_{ij}^{ab}}^n \cong (t^{-1}\mathcal{T}_{W_i})^n \cong t^*\mathcal{T}_{W_i}^n$ due to the fact that s, t are étale. Statement (4) follows in the same way, due to (3) and the fact that m is étale. \square

In particular, statement (2) leads to

Corollary 2.3. *Let $g : Y \hookrightarrow X$ be a proper local embedding of smooth stacks, and consider a groupoid presentation $[R \rightrightarrows U]$ of X such that $Y \times_X U = \bigsqcup_{i,a} V_i^a$ has all the properties listed in Proposition 1.11. Then the blow-up morphism $f : \tilde{X} \rightarrow X$ is proper, and there exists a Cartesian diagram*

$$\begin{array}{ccc} \tilde{X}' & \xrightarrow{f'} & X' \\ \tilde{p} \downarrow & & \downarrow p \\ \tilde{X} & \xrightarrow{f} & X \end{array}$$

where $p : X' \rightarrow X$ is the universally closed étale cover constructed in Section 1, and with the notation from Section 1,

$$\tilde{X}' := \left(\prod_i \right)_{X'} \text{Bl}_{Y'_i} X'.$$

Proof. With the notation from Definition 2.1, we have $\tilde{U} \cong \tilde{X} \times_X U$, due to Lemma 2.2(2). Thus f is proper since $\tilde{U} \rightarrow U$ is. On the other hand, U is also an étale atlas of X' , and so \tilde{U} is also an étale atlas of \tilde{X}' . It remains to study the morphisms induced at the level of relations. Due to (1.15), we have $\bigsqcup_{j,b} S_{ij}^{ab} \cap R' = \bigsqcup_c S_{ii}^{ac} \cap R'$, which is the relation space for Y'_i (Definition 1.15). This is enough to deduce that the diagram of stacks is Cartesian. \square

The blow-up \tilde{X} is independent of the choice of étale atlas U . This follows by standard functorial arguments.

For the remainder of this article, whenever we talk about the Chern class of \tilde{X} , we will assume that X and Y are smooth stacks, and that the images W_i of V_i^a in U for all i intersect each other, as well as their intersections, transversely, so that \tilde{X} is a smooth stack. In this case,

$$p^*\mathcal{T}_X \cong \mathcal{T}_{X'}, \quad \tilde{p}^*\mathcal{T}_{\tilde{X}} \cong \mathcal{T}_{\tilde{X}'}, \quad p_i^*\mathcal{N}_{Y|X} \cong \mathcal{N}_{Y'_i|X'}, \quad q_i^*\mathcal{N}_{\tilde{Y}|\tilde{X}} \cong \mathcal{N}_{\tilde{Y}'_i|\tilde{X}'}$$

for $p_i : Y'_i \rightarrow Y$ and the map $q_i : \tilde{Y}'_i \rightarrow \tilde{Y}$ between exceptional divisors. Thus calculating the Chern invariants of \tilde{X} reduces to the calculation for \tilde{X}' due to Corollary 1.31. We can thus reduce the problem for a local embedding of smooth stacks to that for a succession of smooth embeddings. If, moreover, the ideals \mathcal{I}_i of Y'_i have compatible filtrations with weights such that weighted blow-ups can be defined, then the above reasoning applies to weighted blow-ups as well.

Remark 2.4. If X and Y are smooth, but the images W_i of V_i^a in U do not intersect each other or their intersections transversely for all i , then in general \tilde{X} and \tilde{X}' are not smooth stacks. However, even in this case they come equipped with a Chow class suitable enough for intersection theory. Indeed, in this case $\tilde{X}' := (\prod_i)_{X'} \text{Bl}_{Y'_i} X'$ is still a fibered product of smooth stacks over a smooth stack, and the Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\Delta} & \prod_i X' \\ \uparrow & & \uparrow \\ \tilde{X}' & \longrightarrow & \prod_i \text{Bl}_{Y'_i} X' \end{array}$$

produces a Chow class $\Delta^1([\prod_i \text{Bl}_{Y'_i} X'])$ on \tilde{X}' , invariant under deformations of the map g to X .

2.1. *Weighted projective blow-up and locally trivial weighted projective fibration*

In this section we will work with stacks over \mathbb{C} . Let Y be a smooth substack of a smooth stack X . Consider an increasing filtration $\{\mathcal{I}_n\}_{n \geq 0}$ of the ideal \mathcal{I}_Y of Y in X such that $\mathcal{I}_0 = \mathcal{O}_X$, $\mathcal{I}_1 = \mathcal{I}_Y$ and $\mathcal{I}_n \mathcal{I}_m \subseteq \mathcal{I}_{m+n}$ for all $m, n \geq 0$.

Lemma 2.5. *Assume that $\{\mathcal{I}_n\}_{n \geq 0}$ has the following properties ([MM1, Section 3]):*

- (1) $\mathcal{I}_k \cap \mathcal{I}_Y^2 = \sum_{j=1}^{k-1} \mathcal{I}_j \mathcal{I}_{k-j}$,
- (2) $\mathcal{I}_k / (\mathcal{I}_k \cap \mathcal{I}_Y^2)$ is a subbundle of the conormal bundle $\mathcal{I}_Y / \mathcal{I}_Y^2$.

Then $\text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_n)$ has only quotient singularities. This implies the existence of a natural desingularization \tilde{X} of $\text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_n)$ (constructed as in [V, Proposition 2.8]) such that locally in the étale topology the morphism $f : \tilde{X} \rightarrow \text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_n)$ is of the form

$$[W/H] \rightarrow W/H,$$

where W is a scheme and H is a finite group acting on it.

Proof. Consider an étale atlas of X made up of affine schemes $U = \text{Spec } R$ such that $V := Y \times_X U$ is complete intersection in U , and such that there exists a set of generators $\{x_{ni}\}_{n,i}$ of $\mathcal{I}(U)$ with $x_{ni} \in \mathcal{I}_n(U) \setminus \mathcal{I}_{n+1}(U)$, and with the property that the images of $\{x_{ni}\}_{n \leq k,i}$ in $\mathcal{I}_k(U) / (\mathcal{I}_k(U) \cap \mathcal{I}_Y^2(U))$ form a basis for $\mathcal{I}_k(U) / (\mathcal{I}_k(U) \cap \mathcal{I}_Y^2(U))$. Let

$$R_1 := R[\{y_{ni}\}_{n,i}] / \langle \{y_{ni}^n - x_{ni}\}_{n,i} \rangle,$$

and let $Y_1 \cong V$ be the zero locus of $I_1 := \{y_{ni}\}_{n,i}$ in $U_1 := \text{Spec } R_1$. Then U_1 is smooth and Y_1 is smooth too, because Y is. The finite group $G \cong \bigoplus_{n,i} \mathbb{Z}_n$ has a natural action on R_1 . Due to condition (1),

$$(I_1^n)^G = \sum_{\sum_k k a_k = n} \prod_k \mathcal{I}_k^{a_k}(U) = \mathcal{I}_n(U). \tag{2.1}$$

Here a_k are non-negative integers.

If $\bigsqcup_R \text{Spec } R$ is an étale atlas for X , then $\bigsqcup_R (\text{Bl}_{Y_1} U_1/G)$ is an étale atlas for the stack $\text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_n)$, where $\text{Bl}_{Y_1} U_1$ represents the blow-up of U_1 along Y_1 , with the natural action of G induced from the action on R_1 .

As $\bigsqcup_R (\text{Bl}_{Y_1} U_1/G)$ has only quotient singularities, we can construct a natural desingularization of $\text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_n)$ by following [V, Proposition 2.8]: We can choose $T = \bigsqcup_{R,x} \tilde{U}_x/M$, for finitely many choices of $x \in \text{Bl}_{Y_1} U_1$ and suitable neighborhoods \tilde{U}_x in $\text{Bl}_{Y_1} U_1$, where M is the largest small subgroup of the stabilizer for x (which ensures that T is smooth). Then the normalization R_T of $T \times_{\text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_n)} T$, with the morphisms induced by the two projections on T , defines an étale groupoid structure on T . Indeed, the multiplicative structure comes naturally via the isomorphism of $R_T \times_T R_T$ with the normalization of

$$(T \times_{\text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_n)} T) \times_T (T \times_{\text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_n)} T) \cong T \times_{\text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_n)} T \times_{\text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_n)} T.$$

(by the Purity of the Branch Locus).

By (2.1), the quotient $W/(G/M)$ of the smooth scheme $W := \tilde{U}_x/M$ is open in $\text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_n|_U)$. Moreover,

$$W \times (G/M) \cong \overline{W \times_{\tilde{U}_x/G} W},$$

where $\overline{W \times_{\tilde{U}_x/G} W}$ denotes the normalization of $W \times_{\tilde{U}_x/G} W$ (by the Purity of the Branch Locus). The construction of \tilde{X} now implies that the following diagram is Cartesian:

$$\begin{array}{ccc} [W/(G/M)] & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ W/(G/M) & \longrightarrow & \text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_n) \end{array}$$

and the horizontal arrows are étale. □

Definition 2.6. Let Y be a smooth substack of a smooth stack X , satisfying conditions (1) and (2) in Lemma 2.5. Then \tilde{X} constructed above, with the natural morphism $\pi : \tilde{X} \rightarrow X$, will be called the *weighted blow-up* of X along Y , with the filtration $\{\mathcal{I}_n\}_{n \geq 0}$.

Let $X^\# := \text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_n)$, with the morphism $\pi^\# : X^\# \rightarrow X$. By Lemma 3.1 in [MM1], the reduced structure of $X^\# \times_X Y$ is $Y^\# := \text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_n/\mathcal{I}_{n+1})$, and $\mathcal{I}_n = \pi_* \mathcal{I}_{Y^\#}^n$.

Recall the morphism $f : \tilde{X} \rightarrow X^\#$. The closure in \tilde{X} of $f^{-1}(Y^\# \setminus \text{Sing}(X^\#))$ will be called the *exceptional divisor* of π , and denoted by \tilde{Y} . With the notation from the proof of Lemma 2.5, \tilde{Y} is given locally in the étale topology by $[(\tilde{Y}_{1,x}/M)/(G/M)]$, where $\tilde{Y}_{1,x} \subset \tilde{U}_x$ is the restriction of the exceptional divisor \tilde{Y}_1 in $\text{Bl}_{Y_1} \text{Spec } R_1$ to the open set \tilde{U}_x .

Lemma 2.5 also implies $f_* \mathcal{O}_{\tilde{X}} \cong \mathcal{O}_{X^\#}$ and $f_* \mathcal{I}_{\tilde{Y}}^n \cong \mathcal{I}_{Y^\#}^n$, as \tilde{X} and $X^\#$ coincide outside a codimension two locus. Thus $\pi_* \mathcal{I}_{\tilde{Y}}^n \cong \mathcal{I}_n$ for any n .

Definition 2.7. Consider the weighted blow-up $\tilde{X} \rightarrow X$ constructed in Lemma 2.5. Define

$$A := \text{Spec} \left(\bigoplus_{n \geq 0} \mathcal{I}_n / \mathcal{I}_{n+1} \right).$$

Suppose $A \rightarrow Y$ is a \mathbb{C}^* -equivariant affine fibration, with a trivialization (locally in the Zariski topology) on which \mathbb{C}^* acts linearly on the fibers. Thus the fixed point locus of A with respect to \mathbb{C}^* is a section of $A \rightarrow Y$. We will call it the *zero section* of the fibration, and identify it with Y .

Denote $\mathcal{L} := \mathcal{O}_{\tilde{Y}} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{\tilde{X}}(-\tilde{Y}) = \mathcal{I}_{\tilde{Y}} / \mathcal{I}_{\tilde{Y}}^2$, the conormal bundle of the regularly embedded \tilde{Y} in \tilde{X} .

Lemma 2.8. Let $\pi : \tilde{X} \rightarrow X$ be a weighted blow-up of X along $Y \hookrightarrow X$, with the filtration $\{\mathcal{I}_n\}_{n \geq 0}$, and let \tilde{Y} denote its exceptional divisor. With the notation set up above, $\pi_* \mathcal{L}^n \cong \mathcal{I}_n / \mathcal{I}_{n+1}$, and

$$\tilde{A} = \text{Spec} \left(\bigoplus_n \mathcal{L}^n \right) \rightarrow A = \text{Spec} \left(\bigoplus_n \pi_* \mathcal{L}^n \right)$$

is the weighted blow-up of A along its zero section Y , for the filtration $\{\mathcal{J}_n\}_n$ of the ideal $\mathcal{J} = \bigoplus_{n \geq 0} \mathcal{I}_n / \mathcal{I}_{n+1}$ given by $\mathcal{J}_n := \bigoplus_{k \geq n} \mathcal{I}_k / \mathcal{I}_{k+1}$. Then $\tilde{Y} = \text{Proj}(\bigoplus_n \mathcal{L}^n)$ is also the exceptional divisor of \tilde{A} .

Proof. We noted that $f_* \mathcal{I}_{\tilde{Y}}^n \cong \mathcal{I}_{Y^\#}^n$ and thus also $f_*(\mathcal{I}_{\tilde{Y}}^n / \mathcal{I}_{\tilde{Y}}^{n+1}) \cong \mathcal{I}_{Y^\#}^n / \mathcal{I}_{Y^\#}^{n+1}$ as f has finite-dimensional fibers. This implies $\pi_* \mathcal{L}^n \cong \mathcal{I}_n / \mathcal{I}_{n+1}$ due to [MM1, proof of Lemma 3.1].

Consider an étale atlas of X made up of affine schemes $U = \text{Spec } R$, as in the proof of Lemma 2.5. We construct the weighted blow-up $\tilde{A} \rightarrow A$ by the method outlined in the above mentioned proof. We will keep the notation found there throughout this proof as well. Let $V = \text{Spec } S := Y \times_X U$. Then an étale atlas of A is made up of affine schemes $\text{Spec } S[\{x_{ni}\}_{n,i}]$. Consider covers $T_1 := \text{Spec } S[\{y_{ni}\}_{n,i}]$ with $y_{ni}^n = x_{ni}$, let $Z_1 \cong V$ be the zero locus of $J_1 := \{y_{ni}\}_{n,i}$ in T_1 , and consider the blow-up $\tilde{T}_1 = \text{Bl}_{Z_1} T_1$, with exceptional divisor \tilde{Z}_1 . The group $G \cong \bigoplus_{n,i} \mathbb{Z}_n$ acts on T_1 and its blow-up.

With these data, $Z_1 \cong Y_1 \cong V$. Moreover, there is an isomorphism of normal bundles $\mathcal{N}_{Z_1|T_1} \cong \mathcal{N}_{Y_1|U_1}$, compatible with the action of G on them. Thus $\tilde{Z}_1 \cong \tilde{Y}_1$, as well as $\mathcal{N}_{\tilde{Z}_1|\tilde{T}_1} \cong \mathcal{N}_{\tilde{Y}_1|\tilde{U}_1}$, and the largest small subgroups of the stabilizer for corresponding points in the exceptional divisors coincide as well (while for points not in the exceptional divisor, the stabilizer itself is a small group). An étale atlas of \tilde{A} is $\bigsqcup_x \tilde{T}_{1x} / M$, for finitely many choices of $x \in \tilde{Z}_1$, with the corresponding small subgroup M and suitable open neighborhoods $\tilde{T}_{1x} \subset \tilde{T}_1$. Then

$$\tilde{T}_1 \cong \text{Spec} \left(\bigoplus_{n \geq 0} \mathcal{I}_{\tilde{Z}_1}^n / \mathcal{I}_{\tilde{Z}_1}^{n+1} \right) \cong \text{Spec} \left(\bigoplus_{n \geq 0} \mathcal{I}_{\tilde{Y}_1}^n / \mathcal{I}_{\tilde{Y}_1}^{n+1} \right),$$

so that

$$\tilde{T}_1 / M \cong \text{Spec} \left(\bigoplus_{n \geq 0} \mathcal{I}_{\tilde{Z}_1}^n / \mathcal{I}_{\tilde{Z}_1}^{n+1} \right)^M \cong \text{Spec} \left(\bigoplus_{n \geq 0} (\mathcal{I}_{\tilde{Y}_1}^n / \mathcal{I}_{\tilde{Y}_1}^{n+1})^M \right),$$

thus $\tilde{T}_{1x} / M \cong \text{Spec}(\bigoplus_{n \geq 0} \mathcal{I}_{\tilde{Y}_{1x}/M}^n / \mathcal{I}_{\tilde{Y}_{1x}/M}^{n+1})$. This proves $\tilde{A} = \text{Spec}(\bigoplus_n \mathcal{L}^n)$. □

Conversely, given a morphism $p : P \rightarrow Y$ with a sheaf \mathcal{L} on P such that $P \cong \text{Proj}(\bigoplus_n \pi_* \mathcal{L}^n)$, then P can be understood as the exceptional divisor of the following weighted blow-up:

$$\text{Spec}\left(\bigoplus_n \mathcal{L}^n\right) \rightarrow \text{Spec}\left(\bigoplus_n \pi_* \mathcal{L}^n\right).$$

Lemma 2.9. *Let $A \rightarrow Y$ be a \mathbb{C}^* -equivariant affine fibration with a trivialization (locally in the Zariski topology) on which \mathbb{C}^* acts linearly on the fibers, as in Definition 2.7. Consider the natural filtration of the ideal \mathcal{I} of the zero section Y in A induced by the weights of the \mathbb{C}^* -action, and let $\tilde{A} \rightarrow A$ be the corresponding weighted blow-up, with exceptional divisor \tilde{Y} . Then*

$$\tilde{Y} \cong [(A \setminus Y)/\mathbb{C}^*].$$

Proof. Here we will employ the same notation as in the proof of Lemma 2.8. For $x \in \tilde{Z}_1$, with the corresponding small subgroup M and suitable open neighborhood $\tilde{T}_{1x} \subset \tilde{T}_1$, let $\tilde{Z}_{1x} := \tilde{T}_{1x} \cap \tilde{Z}_1$, and let T_{1x} be the preimage of \tilde{Z}_{1x} in the \mathbb{C}^* -bundle $T_1 \setminus Z_1 \rightarrow \tilde{Z}_1$. Consider the commutative diagram of GIT quotients

$$\begin{array}{ccc} T_{1x} & \longrightarrow & T_{1x}/M \\ \downarrow / \mathbb{C}^* & & \downarrow / \mathbb{C}^* \\ \tilde{Z}_{1x} & \longrightarrow & \tilde{Z}_{1x}/M \end{array}$$

Let O_x denote the \mathbb{C}^* -orbit parametrized by x . We claim that M keeps O_x pointwise fixed, so \mathbb{C}^* acts freely on $O_x/M \cong O_x$, and thus for suitable choice of T_{1x} we have

$$\tilde{Z}_{1x}/M \cong [(T_{1x}/M)/\mathbb{C}^*]. \tag{2.2}$$

Indeed, any element in $\sigma_i \in M$ is a reflection, meaning that its fixed point locus is a divisor \tilde{D} in \tilde{T}_{1x} . Moreover, from the definition of the G -actions, the divisor \tilde{D} is a strict transform of a divisor $D = (y_{ni} = 0)$ fixed by σ_i . But $x \in \tilde{D} \Leftrightarrow O_x \subseteq D$, and thus $M = \text{Stab}_q$ for any $q \in O_x$.

From (2.2) it follows that

$$\begin{aligned} [(\tilde{Z}_{1x}/M)/(G/M)] &\cong [(T_{1x}/M)/\mathbb{C}^*]/(G/M) \cong [(T_{1x}/M)/(G/M)]/\mathbb{C}^* \\ &\cong [(T_{1x}/G)/\mathbb{C}^*], \end{aligned}$$

where T_{1x}/G is an open in $A \setminus Y$. □

Notation. For any \mathbb{C}^* -equivariant affine fibration $A \rightarrow Y$ with a trivialization on which \mathbb{C}^* acts linearly on the fibers, we denote

$$\mathcal{P}^w(A) := [(A \setminus Y)/\mathbb{C}^*].$$

We will say that $\mathcal{P}^w(A)$ is a *weighted projective fibration*.

We return now to the weighted blow-up $\pi : \tilde{X} \rightarrow X$ of X along Y , with filtration $\{\mathcal{I}_n\}_{n \geq 0}$. We saw that the relation $\pi_* \mathcal{I}_Y^n \cong \mathcal{I}_n$ holds for any n . The proof of Lemma 2.5 also implies isomorphisms between Chow groups:

$$A(\tilde{X}) \cong A(X^\#) \quad \text{and} \quad A(\tilde{Y}) \cong A(Y^\#),$$

as \tilde{X} and $X^\#$ have the same coarse moduli space. With this, we have the following description of the Chow ring of \tilde{Y} (Lemma 3.2 in [MM1]).

Lemma 2.10. (a) *The normal bundle in $A = \text{Spec}(\bigoplus_{n \geq 0} \mathcal{I}_n / \mathcal{I}_{n+1})$ of the fixed locus Y under the natural \mathbb{C}^* -action on A is*

$$\mathcal{N}_{Y|A} = \bigoplus_{n \geq 1} \mathcal{N}_n / \mathcal{N}_{n+1},$$

where $\{\mathcal{N}_n\}_n$ is the filtration of the normal bundle $\mathcal{N}_{Y|X}$ dual to the filtration $\{\mathcal{I}_n / (\mathcal{I}_n \cap \mathcal{I}_Y^2)\}_n$ of $\mathcal{I}_Y / \mathcal{I}_Y^2$.

(b) *There is a ring isomorphism*

$$A(\tilde{Y}; \mathbb{Q}) \cong A(Y; \mathbb{Q})[\tau] / \langle P_{Y|X}(\tau) \rangle,$$

where $P_{Y|X}(t)$ is the top equivariant Chern class of the bundle $\mathcal{N}_{Y|A}$. In particular, the free term of $P_{Y|X}(t)$ is the top Chern class of $\mathcal{N}_{Y|X}$. Here τ is the first Chern class of $\mathcal{O}_{\tilde{Y}}(1) := \mathcal{N}_{\tilde{Y}|\tilde{X}}^\vee$.

Lemma 2.10 sets up the context for calculating the Chern classes of the locally trivial weighted projective fibration $p : \tilde{Y} \rightarrow Y$ by deforming \tilde{Y} to a weighted projective bundle on Y and applying the Euler sequence from the Appendix. Here by a weighted projective bundle on Y we mean a stacky quotient $[(N \setminus Y) / \mathbb{C}^*]$ where $N \rightarrow Y$ is a vector bundle with a linear \mathbb{C}^* -action.

Indeed, with the notation above, consider the standard deformation

$$D = \text{Bl}_{Y \times \{\infty\}}(A \times \mathbb{P}^1) \setminus \tilde{A}$$

of A to the total space $N_{Y|A}$ of the normal bundle $\mathcal{N}_{Y|A}$, where $\tilde{A} = \text{Bl}_Y A$ is one of the components of the fiber over ∞ of $\text{Bl}_{Y \times \{\infty\}}(A \times \mathbb{P}^1) \rightarrow \mathbb{P}^1$. Thus the fiber over ∞ of $D \rightarrow \mathbb{P}^1$ is $N_{Y|A}$, and the action of \mathbb{C}^* over $A \times \mathbb{P}^1$, with fixed locus $Y \times \mathbb{P}^1$, induces a natural \mathbb{C}^* -action on D . Moreover, due to Lemma 2.10(a), the quotient of $N_{Y|A} \setminus Y$ by \mathbb{C}^* is a weighted projective bundle, where we have identified the zero section in $N_{Y|A}$ with Y . Push-forward by the composition

$$[(N_{Y|A} \setminus Y) / \mathbb{C}^*] \hookrightarrow [(D \setminus Z) / \mathbb{C}^*] \rightarrow \tilde{Y} \times \mathbb{P}^1 \rightarrow \tilde{Y}$$

(where Z is the fixed locus in D) induces an isomorphism between the Chow rings of \tilde{Y} and of the weighted projective bundle $[(N_{Y|A} \setminus Y) / \mathbb{C}^*]$. We obtain the following

Proposition 2.11. *Let $p : \tilde{Y} \rightarrow Y$ be a weighted projective fibration as above, and $Q_n := \mathcal{N}_{w_n}/\mathcal{N}_{w_{n+1}}$, for all indices w_n such that $\mathcal{N}_{w_n} \neq \mathcal{N}_{w_{n+1}}$, on which \mathbb{C}^* acts with weight w_n . Then the total Chern class of \tilde{C} is*

$$c(\tilde{Y}) = p^*c(Y) \prod_n c(Q_n \otimes \mathcal{L}^{\otimes w_n}),$$

with $\mathcal{L} := \mathcal{O}_{\tilde{Y}}(1)$ and $c(Q_n \otimes \mathcal{L}^{\otimes w_n}) := \prod_i (1 + a_i + w_n c_1(\mathcal{L}))$ where a_i are the Chern roots of Q_n .

Proof. Let $i_\infty : [(N_{Y|A} \setminus Y)/\mathbb{C}^*] \hookrightarrow [(D \setminus Z)/\mathbb{C}^*]$ and $i_0 : \tilde{Y} \rightarrow [(D \setminus Z)/\mathbb{C}^*]$ be embeddings of fibers in the flat family $[(D \setminus Z)/\mathbb{C}^*] \rightarrow \mathbb{P}^1$ and let $q : [(D \setminus Z)/\mathbb{C}^*] \rightarrow \tilde{Y}$ be the natural projection obtained after taking quotients of $D \rightarrow A$, such that $q \circ i_0 = \text{id}_{\tilde{Y}}$. Then by the projection formula and the rational equivalence of fibers,

$$i_{0*}c(\mathcal{T}_{\tilde{Y}}) = i_{0*}i_0^*c(\mathcal{T}_{[(D \setminus Z)/\mathbb{C}^*]}) = i_{\infty*}i_\infty^*c(\mathcal{T}_{[(D \setminus Z)/\mathbb{C}^*]}) = i_{\infty*}c([(N_{Y|A} \setminus Y)/\mathbb{C}^*]),$$

and thus after composing with q_* ,

$$c(\mathcal{T}_{\tilde{Y}}) = q_*i_{\infty*}c([(N_{Y|A} \setminus Y)/\mathbb{C}^*]),$$

which by the Appendix and Lemma 2.10 is of the form described in this proposition. \square

2.2. Model for a weighted blow-up

We start our Chern class calculations with the most approachable type of weighted blow-ups: when the blow-up locus is the fixed locus of a \mathbb{C}^* -action on the entire space. In this case, equivariant cohomology techniques permit the recovery of Chern classes of the blow-up from their pull-backs to the exceptional divisor, which in turn are easily computable.

Let Y be a stack, A a \mathbb{C}^* -equivariant affine fibration on Y as in Definition 2.7, such that the \mathbb{C}^* -action on A induces a decomposition of the normal bundle of the fixed locus Y

$$\mathcal{N}_{Y|A} = \bigoplus_h Q_n$$

with weights $\{w_n\}_n$ and $\text{rk } Q_n = k_n$. Let \mathcal{O}_Y denote the trivial line bundle on Y . Consider the torus $T := \mathbb{C}^* \times \mathbb{C}^*$ action on $A \times \mathcal{O}_Y$ coming from the individual action of the first \mathbb{C}^* on A and the second on \mathcal{O}_Y . In this subsection we will denote by $X := \mathcal{P}^w(A \oplus \mathcal{O}_Y)$ the locally trivial weighted projective fibration obtained as a quotient of $A \times \mathcal{O}_Y \setminus Z$ by \mathbb{C}^* embedded diagonally in T (where Z denotes the fixed locus), and we set $\tilde{Y} := \mathcal{P}^w(A)$ and $\tilde{X} := \mathcal{P}(\mathcal{O}_{\tilde{Y}}(-1) \oplus \mathcal{O}_{\tilde{Y}})$.

We obtain a blow-up diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{j} & \tilde{X} \\ g \downarrow & & \downarrow f \\ Y & \xrightarrow{i} & X \end{array}$$

where $Y = \mathbb{P}(\mathcal{O}_Y) \hookrightarrow \mathcal{P}^w(A \oplus \mathcal{O}_Y)$, and similarly $j : \tilde{Y} \hookrightarrow \tilde{X}$ is the embedding $\tilde{Y} = \mathcal{P}(\mathcal{O}_{\tilde{Y}}) \hookrightarrow \mathcal{P}(\mathcal{O}_{\tilde{Y}}(-1) \oplus \mathcal{O}_{\tilde{Y}})$, the exceptional divisor in the weighted projective blow-up $f : \tilde{X} \rightarrow X$.

Proposition 2.12. *Keeping notation from the above, assume that for each n , the total Chern class $c(\mathcal{Q}_n) = c_{k_n}(\mathcal{Q}_n) + \dots + c_1(\mathcal{Q}_n) + 1$ can be written as the pull-back of a class $p(\mathcal{Q}_n) = p_{k_n}(\mathcal{Q}_n) + \dots + p_1(\mathcal{Q}_n) + 1 \in A(X; \mathbb{Q})$. Then*

$$c(\tilde{X}) = f^*c(X) \frac{(E + 1) \prod_{n=1}^l p(\mathcal{Q}_n(-w_n E))}{\prod_{n=1}^l p(\mathcal{Q}_n)},$$

where E is the class of the exceptional divisor and $p(\mathcal{Q}_n(w_n s)) := \prod_{i=1}^{k_n} (a_i + w_n s + 1)$ where the pull-backs of a_i on Y are the Chern roots of \mathcal{Q}_n .

Proof. The morphism f is equivariant with respect to the natural $\mathbb{C}^* := T/\mathbb{C}^*$ -actions on \tilde{X} and X with the weights specified above, such that \tilde{Y} and the section at infinity are the fixed loci for the \mathbb{C}^* -action on \tilde{X} , and Y and the section at infinity are the fixed loci for the \mathbb{C}^* -action on X . Thus the diagram above yields another weighted blow-up diagram

$$\begin{array}{ccc} \tilde{Y}_{\mathbb{C}^*} & \xrightarrow{j_{\mathbb{C}^*}} & \tilde{X}_{\mathbb{C}^*} \\ g_{\mathbb{C}^*} \downarrow & & \downarrow f_{\mathbb{C}^*} \\ Y_{\mathbb{C}^*} & \xrightarrow{i_{\mathbb{C}^*}} & X_{\mathbb{C}^*} \end{array}$$

where $Z_{\mathbb{C}^*} := Z \times_{\mathbb{C}^*} E\mathbb{C}^*$ for each stack Z , and $B\mathbb{C}^*$ is the classifying space of \mathbb{C}^* , with universal family $E\mathbb{C}^*$. As noticed above, $Y_{\mathbb{C}^*} \cong Y$ and $\tilde{Y}_{\mathbb{C}^*} \cong \tilde{Y}$.

In the following, for ease of notation, we will drop the subscript \mathbb{C}^* for maps, and only employ it to denote equivariant classes.

In the equivariant Chow ring $A_{\mathbb{C}^*}(\tilde{Y})$,

$$e^{\mathbb{C}^*}(j^*(\mathcal{T}_{\tilde{X}})) = e^{\mathbb{C}^*}(\tilde{Y})e^{\mathbb{C}^*}(\mathcal{N}_{\tilde{Y}|\tilde{X}}), e^{\mathbb{C}^*}(i^*(\mathcal{T}_X)) = e^{\mathbb{C}^*}(Y)e^{\mathbb{C}^*}(\mathcal{N}_{Y|X}),$$

so

$$\frac{j^*e^{\mathbb{C}^*}(\tilde{X})}{j^*f^*e^{\mathbb{C}^*}(X)} = \frac{e^{\mathbb{C}^*}(\mathcal{N}_{\tilde{Y}|\tilde{X}})}{g^*e^{\mathbb{C}^*}(\mathcal{N}_{Y|X})} \frac{e^{\mathbb{C}^*}(\tilde{Y})}{g^*e^{\mathbb{C}^*}(Y)} = \frac{(-\xi + t + 1) \prod_{n=1}^l c(\mathcal{Q}_n(w_n \xi))}{\prod_{n=1}^l c(\mathcal{Q}_n(w_n t))},$$

where ξ is the first Chern class of $\mathcal{O}_{\tilde{Y}}(1)$.

Let

$$\alpha := e^{\mathbb{C}^*}(\tilde{X})/f^*e^{\mathbb{C}^*}(X) - 1 \quad \text{and} \quad P(t) := \frac{(-\xi + t + 1) \prod_{n=1}^l p(\mathcal{Q}_n(w_n \xi))}{\prod_{n=1}^l p(\mathcal{Q}_n(w_n t))} - 1.$$

We note that $P(\xi) = 0$. Thus $\beta := P(t)/(t - \xi)$ is well defined and by the self-intersection formula on the exceptional divisor,

$$j^*\alpha = j^*j_*\beta. \tag{2.3}$$

By the Atiyah–Bott localization theorem,

$$\alpha = j_* \frac{j^* \alpha}{e^{\mathbb{C}^*}(\mathcal{N}_{\tilde{Y}|\tilde{X}})} + l_* \frac{l^* \alpha}{e^{\mathbb{C}^*}(\mathcal{N}_{E_\infty|\tilde{X}})}, \tag{2.4}$$

where $j : \tilde{Y} \hookrightarrow \tilde{X}$ and $l : E_\infty \hookrightarrow \tilde{X}$ are the two fixed point loci of \tilde{X} under the \mathbb{C}^* -action. On the one hand, by (2.3),

$$j_* \frac{j^* \alpha}{e^{\mathbb{C}^*}(\mathcal{N}_{\tilde{Y}|\tilde{X}})} = j_* \frac{j^* j_* \beta}{e^{\mathbb{C}^*}(\mathcal{N}_{\tilde{Y}|\tilde{X}})} = j_* \beta,$$

and on the other hand, as the section at infinity $E_\infty = \mathcal{P}^w(N)$ and \tilde{Y} are disjoint, $l^* \mathcal{T}_{\tilde{X}} \cong l^* f^* \mathcal{T}_X$ and thus the second term on the right-hand side of (2.4) is zero. Thus in fact $\alpha = j_* \beta$, which implies

$$e^{\mathbb{C}^*}(\tilde{X}) = f^* e^{\mathbb{C}^*}(X) \frac{(E + t + 1) \prod_{n=1}^l p(\mathcal{Q}_n(-w_n E))}{\prod_{n=1}^l p(\mathcal{Q}_n(w_n t))}.$$

The classical limit $t \rightarrow 0$ yields the desired relation. □

2.3. Deformation to the weighted normal cone

Returning to the general case, let Y be a smooth substack of a smooth stack X . Let $\pi : \tilde{X} \rightarrow X$ be the weighted blow-up of X along Y for an increasing filtration $\{\mathcal{I}_n\}_{n \geq 0}$ of the ideal \mathcal{I}_Y , satisfying properties (1) and (2) in Lemma 2.5. The ideal sheaf \mathcal{J} of $Y \times \{\infty\}$ in $X \times \mathbb{P}^1$ admits a filtration formed by the sheaves $\mathcal{J}_n := \sum_{k=0}^n \mathcal{I}_k \mathcal{K}^{n-k}$ where \mathcal{K} is the ideal of ∞ in \mathbb{P}^1 pulled back to $X \times \mathbb{P}^1$. Let M be the weighted projective blow-up of $X \times \mathbb{P}^1$ along $Y \times \{\infty\}$ with the filtration $\{\mathcal{J}_n\}_n$.

The usual properties of the deformation to the normal cone carry out for this construction with the suitable changes in weights:

- (1) There is a natural closed regular embedding $J : Y \times \mathbb{P}^1 \hookrightarrow M$.
- (2) The composition $\rho = p_2 \circ \Pi$ of the blow-up map $\Pi : M \rightarrow X \times \mathbb{P}^1$ with the projection $p_2 : X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is a flat morphism of stacks, and the following diagram commutes:

$$\begin{array}{ccc} Y \times \mathbb{P}^1 & \xrightarrow{J} & M \\ p_2 \downarrow & \swarrow \rho & \\ \mathbb{P}^1 & & \end{array}$$

- (3) Over $\mathbb{P}^1 \setminus \{\infty\}$, $\rho^{-1}(\mathbb{A}^1) = X \times \mathbb{A}^1$ and J is the trivial embedding.
- (4) As a Cartier divisor,

$$M_\infty := \rho^{-1}(\infty) = P + \tilde{X},$$

where $P = \mathcal{P}^w(\text{Spec}(\bigoplus_{n \geq 0} \mathcal{J}_n / \mathcal{J}_{n+1}))$ is a locally trivial weighted projective fibration and \tilde{X} is the weighted blow-up of X along the locus Y , with the filtration $\{\mathcal{I}_n\}$. Both P and \tilde{X} are Cartier divisors of M , intersecting in $\tilde{Y} :=$

$\mathcal{P}^w(\text{Spec}((\bigoplus_{n \geq 0} \mathcal{I}_n/\mathcal{I}_{n+1})))$, which is embedded as the section at infinity in P and as the exceptional divisor in \tilde{X} . On the other hand, $Y_\infty = Y \times \{\infty\}$ embeds in M_∞ as the zero section in P and is thus disjoint from $\tilde{X} \hookrightarrow M_\infty$.

The proof is analogous to [F, (5.1)].

Theorem 2.13. *Consider the weighted blow-up $f : \tilde{X} \rightarrow X$ of a smooth stack X along a smooth substack Y with an increasing filtration $\{\mathcal{I}_n\}_{n \geq 0}$ of the ideal \mathcal{I}_Y , satisfying conditions (1)–(2) in 2.1. Let $\{\mathcal{N}_n\}_n$ denote the filtration of the normal bundle $\mathcal{N}_{Y|X}$ dual to the filtration $\{\mathcal{I}_n/(\mathcal{I}_n \cap \mathcal{I}_Y^2)\}_n$ of $\mathcal{I}_Y/\mathcal{I}_Y^2$. Let $\mathcal{Q}_n = \mathcal{N}_{w_n}/\mathcal{N}_{w_{n+1}}$ for all indices w_n such that $\mathcal{N}_{w_n} \neq \mathcal{N}_{w_{n+1}}$. Then \mathbb{C}^* acts with weight w_n on \mathcal{Q}_n .*

Assume that for each n , the total Chern class $c(\mathcal{Q}_n) = c_{k_n}(\mathcal{Q}_n) + \dots + c_1(\mathcal{Q}_n) + 1$ can be written as the pull-back of a class $p(\mathcal{Q}_n) = p_{k_n}(\mathcal{Q}_n) + \dots + p_1(\mathcal{Q}_n) + 1 \in A(X; \mathbb{Q})$. Then

$$c(\tilde{X}) = f^*c(X) \frac{(E + 1) \prod_{n=1}^l p(\mathcal{Q}_n(-w_n E))}{\prod_{n=1}^l p(\mathcal{Q}_n)},$$

where E is the class of the exceptional divisor in \tilde{X} and $p(\mathcal{Q}_n(w_n s)) := \prod_{i=1}^{k_n} (a_i + w_n s + 1)$ where the pull-backs of a_i on Y are the Chern roots of \mathcal{Q}_n .

Proof. Let \tilde{M} be the weighted blow-up of M along $Y \times \mathbb{P}^1$ with the filtration $r^*\mathcal{I}_n$, where $r = p_1 \circ \Pi$ is the composition $M \rightarrow X \times \mathbb{P}^1 \rightarrow X$. Looking at the fibers over 0 and $\infty \in \mathbb{P}^1$,

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{\tilde{j}_0} & \tilde{M} & \xleftarrow{\tilde{j}_\infty} & \tilde{P} + \tilde{X} \\ f_0 \downarrow & & \downarrow F & & f_\infty \downarrow \\ X & \xrightarrow{j_0} & M & \xleftarrow{j_\infty} & P + \tilde{X} \\ \downarrow & & \downarrow \rho & & \downarrow \\ 0 & \longrightarrow & \mathbb{P}^1 & \longleftarrow & \infty \end{array}$$

The embeddings \tilde{j}_∞ , respectively j_∞ , split into $\tilde{k} : \tilde{P} \rightarrow \tilde{M}, \tilde{l} : \tilde{X} \rightarrow \tilde{M}$, respectively $k : P \rightarrow M, l : \tilde{X} \rightarrow M$, where $F \circ \tilde{l}$ and l can be naturally identified. Pull-backs of the quotient sheaf $\mathcal{G}_M := \mathcal{T}_{\tilde{M}}/F^*\mathcal{T}_M$ yield $\tilde{j}_0^*\mathcal{G}_M = \mathcal{G}_0 := \mathcal{T}_{\tilde{X}}/f_0^*\mathcal{T}_X$ and $\tilde{j}_\infty^*\mathcal{G}_M = \mathcal{G}_\infty$ which on \tilde{P} is the quotient $\mathcal{T}_{\tilde{P}}/f^*\mathcal{T}_P$, and on \tilde{X} is the zero sheaf. We note that the maps of locally free sheaves on stacks

$$\mathcal{T}_{\tilde{M}} \rightarrow F^*\mathcal{T}_M, \quad \mathcal{T}_{\tilde{X}} \rightarrow f_0^*\mathcal{T}_X \quad \text{and} \quad \mathcal{T}_{\tilde{P}} \rightarrow f_\infty^*\mathcal{T}_P$$

are monomorphisms, as \tilde{M} and M are isomorphic outside the exceptional divisor and its image, etc. Moreover,

$$\tilde{j}_{0*}c(\mathcal{G}_0) = \tilde{j}_{0*}\tilde{j}_0^*c(\mathcal{G}_M) = c(\mathcal{G}_M) \cdot [\tilde{M}_0],$$

which by rational equivalence is identified with

$$c(\mathcal{G}_M) \cdot [\tilde{M}_\infty] = \tilde{j}_{\infty*}c(\mathcal{G}_\infty) = k_*c(\mathcal{G}_{\infty|\tilde{P}}) + [\tilde{X}],$$

as $\tilde{l}^*c(\mathcal{G}_M) = c(\tilde{l}^*\mathcal{G}_M) = 1$ on \tilde{X} . On the one hand, the formula for $c(\mathcal{G}_{\infty|\tilde{P}})$ is given by Proposition 2.11.

On the other hand, \tilde{M} is isomorphic to the blow-up of $\tilde{X} \times \mathbb{P}^1$ along $\tilde{Y} \times \{\infty\}$, where \tilde{Y} is the exceptional divisor of \tilde{X} . Hence, there is a map $\tilde{\Pi} : \tilde{M} \rightarrow \tilde{X} \times \mathbb{P}^1$ whose composition q with the projection $\tilde{X} \times \mathbb{P}^1 \rightarrow \tilde{X}$ satisfies $q \circ \tilde{j}_0 = \text{id}_{\tilde{X}}$. From this and the model case in Proposition 2.12 we recover the general formula in the theorem. \square

3. The moduli space $\overline{M}'_{0,m}(\mathbb{P}^n, d)$

Let m, n, d be nonnegative integers. In this section we apply the general theory for weighted blow-ups along local embeddings of smooth stacks to calculate the total Chern class of the moduli space of stable maps $\overline{M}_{0,m}(\mathbb{P}^n, d)$. We first set up the context by briefly recalling the blow-up constructions of $\overline{M}_{0,m}(\mathbb{P}^n, d)$ from [MM1] (for $m = 1$), [MM2] (for $m > 1$), and [MM3] (for $m = 0$). They pertain to a family of smooth Deligne–Mumford stacks $\overline{M}_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$, and of weighted blow-ups

$$\overline{M}_{0,\mathcal{A}}(\mathbb{P}^n, d, a) \rightarrow \overline{M}_{0,\mathcal{A}'}(\mathbb{P}^n, d, a'),$$

where $a, a' \in \mathbb{Q}$, $\mathcal{A} = (a_1, \dots, a_m)$, $\mathcal{A}' = (a'_1, \dots, a'_m) \in \mathbb{Q}^m$ such that

$$\sum_{i=1}^m a_i + da > 2, \quad \sum_{i=1}^m a'_i + da' > 2, \quad 1 \geq a \geq a' > 0, \quad 1 \geq a_j \geq a'_j \geq 0$$

for all $j = 1, \dots, m$. Here $\overline{M}_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$ is the stack of (\mathcal{A}, a) -weighted stable maps as defined in [MM2], parametrizing (\mathcal{A}, a) -stable maps.

Definition 3.1. An (\mathcal{A}, a) -stable map consists of a family of rational curves $\pi : C \rightarrow S$, whose fibers are either smooth or with nodes as singularities, with m marked sections not intersecting the nodes of the fibers, with a line bundle \mathcal{L} on C of degree d on each fiber C_s , and a morphism $e : \mathcal{O}_C^{n+1} \rightarrow \mathcal{L}$ (specified up to isomorphisms of the target) satisfying a series of stability conditions:

- (1) $\omega_{C|S}(\sum_{i=1}^m a_i p_i) \otimes \mathcal{L}^a$ is relatively ample over S ,
- (2) $\mathcal{G} := \text{Coker } e$, restricted over each fiber C_s , is a skyscraper sheaf supported only on smooth points of C_s , and
- (3) for any $s \in S$ and $p \in C_s$ and for any $I \subseteq \{1, \dots, m\}$ (possibly empty) such that $p = p_i$ for all $i \in I$ we have

$$\sum_{i \in I} a_i + a \dim \mathcal{G}_p \leq 1.$$

Let C be a curve. A *tail* of C is a closed connected subcurve C' of C with the property that $C \setminus C'$ is connected.

We will now consider the case when $\mathcal{A} = \{1\}$. The basic definitions for this case are introduced in [MM1]. All the other cases of weights \mathcal{A} which will be of interest to us can be deduced from this case ([MM2], [MM3]).

Definition 3.2. Let $D = \{1, \dots, d\}$. We say $I \subset \mathcal{P}(D) \setminus \{\emptyset, D\}$ is a *nested set* if, for any two $h, h' \in I$, the intersection $h \cap h'$ is either h, h' or \emptyset .

For any number $l > 0$, we denote $I_{\leq l} := \{h \in I; |h| \leq l\}$ and $I_{> l} := \{h \in I; |h| > l\}$.

Fix a positive number $a < 1$ and a nested set $I \subset \mathcal{P} \setminus \{\emptyset, D\}$ such that $h \cap h' = \emptyset$ for any distinct $h, h' \in I_{\leq 1/a}$. In [MM1, Proposition 2.3], a boundary map $\overline{M}_I^a \rightarrow \overline{M}_{0,1}(\mathbb{P}^n, d, a)$ was described.

Definition 3.3. With our notation, \overline{M}_I^a is the stack of I -type, a -stable, degree d maps from a rational curve into \mathbb{P}^n , i.e.

$$(C, p_1, \{p_h\}_{h \in I_{\leq 1/a}}, \{C_h\}_{h \in I_{> 1/a}}, \mathcal{L}, e)$$

made up of an a -stable, degree d pointed map (C, p_1, \mathcal{L}, e) , together with marked points $\{p_h\}_{h \in I_{\leq 1/a}}$ and connected subcurves $\{C_h\}_{h \in I_{> 1/a}}$ satisfying the following properties:

- (1) for all $h \in I_{> 1/a}$, $p_1 \notin C_h \subset C$ and $\deg \mathcal{L}|_{C_h} = |h|$;
- (2) for all $h \in I_{\leq 1/a}$, $\dim \text{Coker } e_{p_h} = |h|$;
- (3) compatibility of incidence relations:
 - for all $h \in I_{\leq 1/a}$ and $h' \in I_{> 1/a}$, $h \subset h'$ iff $p_h \in C_{h'}$;
 - for all $h, h' \in I_{> 1/a}$, if $h' \subset h$ then $C_{h'} \subset C_h$, if $h \subset h'$ then $C_h \subset C_{h'}$, otherwise $C_h \cap C_{h'} = \emptyset$.

Notation. By convention, $\overline{M}_{\emptyset}^a = \overline{M}_{0,1}(\mathbb{P}^n, d, a)$. When $I = \{h\}$, we will denote \overline{M}_I^a simply by \overline{M}_h^a .

For each nested set I , we let $G_I \subset S_d$ be the largest subgroup that keeps each $h \in I$ invariant. In particular, $|G_h| = |h|!(d - |h|)!$.

When $|h| > 1/a$, the stack \overline{M}_h^a maps to a codimension 1 substack of $\overline{M}_{0,1}(\mathbb{P}^n, d, a)$, and its generic point represents a map whose source is split as a union of two curves, one containing the marked point and the other of degree $|h|$. If $|h| \leq 1/a$, the stack \overline{M}_h^a maps to a substack of higher codimension in $\overline{M}_{0,1}(\mathbb{P}^n, d, a)$, and its generic point represents a map whose source contains the marked point p_1 and another marked point of weight $|h|a$.

There is a special étale atlas for $\overline{M}_{0,1}(\mathbb{P}^n, d, a)$. Its definition is based on the notion of \bar{t} -rigid weighted stable maps, for any system \bar{t} of homogeneous coordinates on \mathbb{P}^n . These are an adaptation of the rigid stable maps introduced in [FP], and were discussed in more detail in [MM1].

Definition 3.4. A \bar{t} -rigid stable map is given by the same data as a point of the moduli space $\overline{M}_{0,1}(\mathbb{P}^n, d, a)$, together with an extra set of sections $\{q_{i,j}\}_{0 \leq i \leq n, 1 \leq j \leq d}$ of weights $a_{i,j} = a/(n+1)$, and satisfying, via e ,

$$(\bar{t}_i) = \sum_{j=1}^d q_{i,j}$$

for each homogeneous coordinate \bar{t}_i in \bar{t} .

The moduli space of $\overline{M}_{0,1}(\mathbb{P}^n, d, a, \bar{t})$ of \bar{t} -rigid weighted stable maps is represented by a torus bundle over an open subset of $\overline{M}_{0,\mathcal{A}'}$, Hassett’s moduli spaces of weighted stable curves. There is a natural action on $\overline{M}_{0,1}(\mathbb{P}^n, d, a, \bar{t})$ by the finite group $(S_d)^{n+1}$, which permutes the extra marked points $q_{i,j}$ defined above.

Lemma 3.5 ([MM1, Proposition 1.11], and in more generality [MM2, Proposition 1.7]). *An étale atlas for the stack $\overline{M}_{0,1}(\mathbb{P}^n, d, a)$ is given by*

$$U := \bigsqcup_{x, \bar{t}} U_x(\bar{t}),$$

where for suitable choices of finitely many coordinate systems \bar{t} and of finitely many points x in $\overline{M}_{0,1}(\mathbb{P}^n, d, a, \bar{t})$, the smooth scheme $U_x(\bar{t})$ represents the quotient of an appropriately small affine neighborhood of x by the largest small subgroup H_x of the stabilizer $\text{Stab}_x \subseteq (S_d)^{n+1}$.

This construction is based on [V, proof of Proposition 2.8].

Notation. We choose k to be the integer in $\{2, \dots, d\}$ with $1/k \leq a < 1/(k-1)$, or $k = 1$ if $a = 1$. (We note that if $0 < a < 1/d$, then the space $\overline{M}_{0,1}(\mathbb{P}^n, d, a)$ is empty.)

With this notation, the following holds:

Lemma 3.6. *For each nested set I whose elements h satisfy $|h| \geq k$, the map $\phi_I : \overline{M}_I^a \rightarrow \overline{M}_{0,1}(\mathbb{P}^n, d, a)$ is a proper local embedding. Moreover, for choices of $U_x(\bar{t})$ appropriately small, the étale atlas U for the stack $\overline{M}_{0,1}(\mathbb{P}^n, d, a)$ defined in Lemma 3.5 satisfies the properties listed in Proposition 1.11 for each of these local embeddings ϕ_I . Thus*

$$U \times_{\overline{M}_{0,1}(\mathbb{P}^n, d, a)} \overline{M}_I^a = \bigsqcup_{g \in S_d/G_I} V_{g(I)} \quad \text{and, more generally,} \quad V_I \times_{\overline{M}_I^a} \overline{M}_J^a = \bigsqcup_{g \in G_I/G_J} V_{g(J)},$$

for $J \supset I$ whose elements also satisfy $|h| \geq k$. Here $V_{g(J)}$ are étale atlases of \overline{M}_J^a naturally embedded in U , and $V_J \hookrightarrow V_I$ whenever $J \supset I$.

Moreover, $V_J \cap V_K = V_{J \cup K}$ is a transverse intersection in $V_{J \cap K}$ for all nested sets J, K whose elements h satisfy $|h| \geq k$.

Let $V_\emptyset := U$. For each nested set I and $h \notin I$ with $|h| \geq k$, the network of local embeddings associated to $\phi_{Ih}^I : \overline{M}_{Ih}^a \rightarrow \overline{M}_I^a$ and the étale atlas V_I consists of morphisms $\phi_K^J : \overline{M}_K^a \rightarrow \overline{M}_J^a$, where K and J are nested sets with $K \supset J \supseteq I$, whose elements h' satisfy $|h'| = |h|$.

Proof. We recall the relevant construction from [MM1, proof of Proposition 2.3]. There exists a sequence of smooth varieties and morphisms

$$\overline{M}_{0,1}(\mathbb{P}^n, d, a, \bar{t}) \hookrightarrow \overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^1, (d, 1), (a, 1), \bar{t}) \xrightarrow{f_{\bar{t}}} \mathbb{P}_d^n(\bar{t}) \xrightarrow{p_{\bar{t}}} (\mathbb{P}^1)^{d(n+1)},$$

where $\mathbb{P}_d^n(\bar{t})$ is a $(\mathbb{C}^*)^n$ -torus over an open subset of $(\mathbb{P}^1)^{d(n+1)}$, and $f_{\bar{t}}$ is a composition of blow-ups along some diagonals and their strict transforms, in a suitable order.

Let $N = \{0, \dots, n\}$, let $\Delta_{N \times h}$ denote the diagonal in $(\mathbb{P}^1)^{d(n+1)}$ where the coordinates corresponding to the set $N \times h \subset N \times \{1, \dots, d\}$ agree. More precisely, the blow-ups are along the (strict transforms of) diagonals $\Delta_{N \times h}$ with $|h| \geq k$, in increasing order of dimension. At each blow-up step l , all (strict transforms of) diagonals with $|h| \geq l$ will intersect each other, and intersections of other (strict transforms of) diagonals, transversely ([M]).

$D_{N \times h}^a(\bar{t}) \subset \overline{M}_{0,1}(\mathbb{P}^n, d, a, \bar{t})$ is constructed from $\Delta_{N \times h}^0(\bar{t}) = p_{\bar{t}}^{-1}(\Delta_{N \times h})$ by taking its strict transforms through the successive blow-ups, respectively the exceptional divisor at the $(d - |h|)$ -th step, and finally intersecting the resulting space with $\overline{M}_{0,1}(\mathbb{P}^n, d, a, \bar{t})$. We then let $D_{N \times I}^a(\bar{t}) := \bigcap_{h \in I} D_{N \times h}^a(\bar{t})$ for any nested set $I \subset \mathcal{P}$.

Define

$$V_I := \bigsqcup_{x, \bar{t}} \bigsqcup_{[g'] \in (S_d/G_I)^n} (U_x(\bar{t}) \cap D_{g'(N \times I)}^a(\bar{t}))/H'_x, \tag{3.1}$$

for finitely many x, \bar{t} also employed in the construction of U , where the set of orbits $(S_d/G_I)^n$ corresponds to permutations on $(N \setminus \{0\}) \times I$, and H'_x is the largest small subgroup of the stabilizer of x for the action of G_I^{n+1} on $D_{g'(N \times I)}^a(\bar{t})$ (via conjugation by g').

With the notation from the previous lemma, $H'_x = H_x$ whenever all the elements $h \in I$ satisfy $a|h| \geq 1$. Then each such V_I embeds in U and is an étale atlas of \overline{M}_I^a , canonically constructed as in [V, proof of Proposition 2.8]. Moreover, from the construction of V_I ,

$$U \times_{\overline{M}_{0,1}(\mathbb{P}^n, d, a)} \overline{M}_I^a = \bigsqcup_{g \in S_d/G_I} V_{g(I)}; \quad \text{as well,} \quad V_I \times_{\overline{M}_I^a} \overline{M}_J^a = \bigsqcup_{g \in G_I/G_J} V_{g(J)},$$

for $J \supset I$ whose elements also satisfy $a|h| \geq 1$. These formulae are equivalent to Lemma 1.16 for the spaces \overline{M}_J^a , which uniquely defines networks of local embeddings. Accordingly, for each nested set I and $h \notin I$ with $|h| \geq k$, the morphisms $\phi_K^J : \overline{M}_K^a \rightarrow \overline{M}_J^a$, where K and J are nested sets with $K \supset J \supseteq I$, whose elements h' satisfy $|h'| = |h|$, form the network of local embeddings associated to $\phi_{Ih}^I : \overline{M}_{Ih}^a \rightarrow \overline{M}_I^a$ and the étale atlas V_I .

Furthermore, since the diagonals and their transforms intersect each other, and intersections of other diagonals, transversely, it follows that $V_J \cap V_K = V_{J \cup K}$ is a transverse intersection in $V_{J \cap K}$ for all nested sets J, K whose elements h satisfy $a|h| \geq 1$. \square

Let d be a positive integer. As described in [MM2], contractions $\overline{M}_{0,m}(\mathbb{P}^n, d, a)$ of the space $\overline{M}_{0,m}(\mathbb{P}^n, d)$ can be thought of as locally embedded in $\overline{M}_{0,1}(\mathbb{P}^n, d+m-1, a)$ when $a < 1$. More precisely, after choosing a privileged point 1_M among the m marked points

and adjoining the rest of marked points $M = \{2_M, \dots, m_M\}$ to the set $\{1_D, \dots, d_D\}$ to form $D' = \{1_D, \dots, d_D, 2_M, \dots, m_M\}$, one can write

$$\overline{M}_{0,\mathcal{A}}(\mathbb{P}^n, d, a) \cong \overline{M}_J^a$$

for $J = \{\{2_M\}, \{3_M\}, \dots, \{d_M\}\}$ and the m -tuple $\mathcal{A} = (1, a, \dots, a)$.

For this reason we introduce the following notation.

Notation. From now on, we will let \mathcal{A} denote the m -tuple $(1, a, \dots, a)$, and work with subsets h of $D' = \{1_D, \dots, d'_D, 2_M, \dots, m_M\}$, and all nested subsets will be elements in $\mathcal{P}(D')$. We let $d' := d + m - 1$. As before, we choose $k > 1$ to be the integer with $1/k \leq a < 1/(k - 1)$, or $k = 1$ if $a = 1$.

We note that while for the generic curve parametrized by \overline{M}_J^a , all tails and base locus points are marked by elements of I , other curves, represented by points in the boundary of \overline{M}_J^a , may have unmarked tails and base points. It is due to such points that the maps $\overline{M}_J^a \rightarrow \overline{M}_I^a$ are in general only local embeddings. This suggests that marking all components and base points of curves and their maps will result in moduli stacks $\overline{M}_I^{a'}$ embedded in $\overline{M}'_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$, constructed as in Theorem 1.20.

Definition 3.7. Consider the moduli functor from schemes to sets, associating to any scheme S the set of (\mathcal{A}, a) -stable, degree d pointed maps $(C \rightarrow S, \{p_i\}_{i \in \{1, \dots, m\}}, \mathcal{L}, e)$, together with a collection $\{P_s\}_{s \in S}$ of partitions

$$D = \bigsqcup_{\alpha \in N_s} B_\alpha$$

of the set $D = \{1, \dots, d\}$, one for each $s \in S$, such that for every $s \in S$, the set

$$N_s \cong \{\text{irreducible components of the curve } C_s\},$$

and

- (1) the partition is compatible with the structure of the map given by (\mathcal{L}, e) : if $\alpha \in N_s$ corresponds to the component $C \subset C_s$ then $|B_\alpha| = \deg \mathcal{L}|_C$;
- (2) the partition is compatible with specialization in S : if $s_1, s_2 \in S$ then $s_1 \in \overline{\{s_2\}} \Leftrightarrow P_{s_1}$ is a refinement of P_{s_2} .

A set of data as above will be called a *semi-rigid (\mathcal{A}, a) -stable map* over S . An isomorphism of semi-rigid (\mathcal{A}, a) -stable maps is an isomorphism of (\mathcal{A}, a) -stable maps which also preserves the partitions of D .

Definition 3.8. Let $\overline{M}'_{0,\mathcal{A}}(\mathbb{P}^n, d, a) \rightarrow \overline{M}_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$ be the étale surjective morphism constructed inductively as follows:

$$\overline{M}_{0,\mathcal{A}}^{(d')}(\mathbb{P}^n, d, a) := \overline{M}_{0,\mathcal{A}}(\mathbb{P}^n, d, a), \quad \overline{M}_h^{a,(d')} := \overline{M}_h^a,$$

while for $l \in \{k, \dots, d' - 1\}$,

$$\overline{M}_{0,\mathcal{A}}^{(l)}(\mathbb{P}^n, d, a) \rightarrow \overline{M}_{0,\mathcal{A}}^{(l+1)}(\mathbb{P}^n, d, a)$$

is the étale lift for the proper local embedding $\overline{M}_h^{a,(l+1)} \rightarrow \overline{M}_{0,\mathcal{A}}^{(l+1)}(\mathbb{P}^n, d, a)$, for some h satisfying $|h| = l$, with the étale atlas U from Lemmas 3.5 and 3.6, while for each nested set I such that $h \notin I$,

$$\overline{M}_I^{a,(l)} \rightarrow \overline{M}_I^{a,(l+1)}$$

is the étale lift for the proper local embedding $\overline{M}_{hI}^{a,(l+1)} \rightarrow \overline{M}_I^{a,(l+1)}$ with the étale atlas V_I . Finally,

$$\overline{M}'_{0,\mathcal{A}}(\mathbb{P}^n, d, a) := \overline{M}_{0,\mathcal{A}}^{(k)}(\mathbb{P}^n, d, a).$$

The following lemma ensures that Theorem 1.20 can be applied at each step in the definition above.

Lemma 3.9. *The following properties hold for the spaces $\overline{M}_I^{a,(l)}$:*

- (1) *There exist natural proper local embeddings $\phi_J^{I,(l)} : \overline{M}_J^{a,(l)} \rightarrow \overline{M}_I^{a,(l)}$.*
- (2) *For $h \notin I$, the maps $\phi_K^{J,(l)} : \overline{M}_K^{a,(l)} \rightarrow \overline{M}_J^{a,(l)}$, with $K \supset J \supseteq I$ such that the elements $h' \in K \setminus J$ satisfy $|h'| = |h|$, form the network of local embeddings associated to the proper local embedding $\phi_{Ih}^{I,(l)}$ with the étale atlas V_I defined by (3.1).*
- (3) *The following diagram is Cartesian:*

$$\begin{array}{ccc} \overline{M}_{J \cup K}^{a,(l)} & \longrightarrow & \overline{M}_J^{a,(l)} \\ \downarrow & & \downarrow \\ \overline{M}_K^{a,(l)} & \longrightarrow & \overline{M}_{K \cap J}^{a,(l)} \end{array}$$

- (4) *For $J \supset I$ and $l > k$, the following diagram is Cartesian:*

$$\begin{array}{ccc} \bigsqcup_{g \in G_{J_{l-1}/G_J}} \overline{M}_{g(J)}^{a,(l-1)} & \longrightarrow & \overline{M}_I^{a,(l-1)} \\ \downarrow & & \downarrow \\ \overline{M}_J^{a,(l)} & \longrightarrow & \overline{M}_I^{a,(l)} \end{array}$$

where $J_{l-1} = \{h \in J; |h| = l - 1 \text{ or } h \in I\}$.

All properties hold as long as all the elements h of $J \setminus I$, $K \setminus I$ satisfy $|h| \geq k$. Here, according to our convention, $\overline{M}_\emptyset^{a,(l)} = \overline{M}_{0,\mathcal{A}}^{(l)}(\mathbb{P}^n, d, a)$.

Proof. Properties (1)–(3) follow by decreasing induction on l . The first step, when $l = d'$, is true due to Lemma 3.6. If (1)–(3) hold for $l > k$, then by (3), Corollary 1.8 can be applied at each step in the proof of Theorem 1.20, yielding the Cartesian diagram

$$\begin{array}{ccc} \overline{M}_J^{a,(l-1)} & \longrightarrow & \overline{M}_I^{a,(l-1)} \\ \downarrow & & \downarrow \\ \overline{M}_J^{a,(l)} & \longrightarrow & \overline{M}_I^{a,(l)} \end{array}$$

for $J \supset I$ such that $J \setminus I$ contains no elements h with $|h| = l - 1$, and $l > k$, while for $J \supset I$ such that $J \setminus I$ contains only elements h with $|h| = l - 1$, Theorem 1.20 implies that the following diagram is Cartesian:

$$\begin{CD} \bigsqcup_{g \in G_I/G_J} \overline{M}_{g(J)}^{a,(l-1)} @>>> \overline{M}_I^{a,(l-1)} \\ @VVV @VVV \\ \overline{M}_J^{a,(l)} @>>> \overline{M}_I^{a,(l)} \end{CD}$$

In general, splitting $J = J_{l-1} \sqcup (J \setminus J_{l-1})$ yields (4). As well, at the level of étale atlases,

$$\overline{M}_J^{a,(l)} \times_{\overline{M}_I^{a,(l)}} V_I \cong \bigsqcup_{g \in G_{J_{l-1}}/G_J} V_{g(J)}.$$

This ensures that (1)–(3) are true when l is replaced by $l - 1$ as well. Indeed, (1) and (3) follow directly from the Cartesian diagrams above, while (2) follows from the relation between étale atlases, which is equivalent to Lemma 1.16 for our spaces. \square

Remark 3.10. Alternatively, $\overline{M}'_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$ can be constructed by applying the construction steps in the proof of Theorem 1.20 directly to the network $\{\phi_J^I : \overline{M}_J^a \rightarrow \overline{M}_I^a\}$, where I and J are nested sets whose elements h satisfy $|h| \geq k$. Indeed, the étale atlases do not change throughout the construction, while the groupoid relations of $\overline{M}'_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$, obtained by applying (1.15) to the étale atlases of Lemma 3.6, are independent of the order in which the étale lifts above are performed.

Remark 3.11. One could ask if simply applying Proposition 1.11 and Theorem 1.20 directly to the local embedding

$$Y := \bigsqcup_h \overline{M}_h^a \rightarrow X := \overline{M}_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$$

(when the union is taken over one copy of h for each cardinality $|h| \geq 1/a$) would not yield the same outcome as in Definition 3.8. However we note that for the map above, the étale atlas U introduced in Lemma 3.5 does admit a partition of $Y \times_X U$ with all properties listed in Proposition 1.11. Moreover, the étale lift associated to the map above would contain the same number of copies of boundary divisors mapping onto \overline{M}_h^a irrespectively of $|h|$, which is different from the case of $\overline{M}'_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$ constructed by us.

We also note, in view of Remark 1.19, that while the étale atlas U from Lemma 3.5 is suitable for all the local embeddings

$$\overline{M}_h^a \rightarrow \overline{M}_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$$

(with $|h| \geq 1/a$) at once, it is not minimal for each map taken separately. For example, if $|h| > d/2$ then \overline{M}_h^a is embedded in $\overline{M}_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$, and yet $\overline{M}'_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$ will still contain $\binom{d}{|h|}$ copies of \overline{M}_h^a .

Theorem 3.12. *The moduli problem of semi-rigid (\mathcal{A}, a) -stable maps is finely represented by the Deligne–Mumford stack $\overline{M}'_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$.*

Proof. Let $p : \overline{M}'_{0,\mathcal{A}}(\mathbb{P}^n, d, a) \rightarrow \overline{M}_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$ denote the universally closed, étale map resulting from the construction in Definition 3.8, and let \mathcal{U} denote the universal family on $\overline{M}_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$. For each $s \in \overline{M}'_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$, we will define a partition P_s of D compatible with $\mathcal{U}_{p(s)}$ in the sense of Definition 3.7. Then \mathcal{U} together with the family of partitions $\{P_s\}_s$ will form the universal family \mathcal{U}' on $\overline{M}'_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$.

Indeed, for every $s \in \overline{M}'_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$ there exists a uniquely associated nested set I (possibly $I = \emptyset$) such that $s \in (\overline{M}'_I)^a \setminus \bigcup_{J \supset I} (\overline{M}'_J)^a$, and if I_1 and I_2 are associated to s_1, s_2 , respectively, then

$$s_1 \in \overline{\{s_2\}} \Rightarrow I_1 \supseteq I_2.$$

There exists a correspondence between nested sets $I \in \mathcal{P}(D)$ and partitions P_I of D such that

$$J \supseteq I \Leftrightarrow P_J \text{ is a refinement of } P_I,$$

namely the elements of the partition P_I are the sets $h \setminus \bigcup_{h'' \in I, h'' \neq h} h''$ for all $h \in I$, together with $D \setminus \bigcup_{h'' \in I} h''$.

Conversely, given any semi-rigid (\mathcal{A}, a) -stable map $(C \rightarrow S, \{p_i\}_i, \mathcal{L}, e)$, then for each point $s \in S$, the associated partition P_s of D uniquely defines a nested set $I(s)$: the elements h of $I(s)$ correspond to chains of components of C , of length at least two, which start from the component containing the special marked point p_1 and end with a tail. Then $h = \bigcup_{\alpha} B_{\alpha}$, where the union is taken over all the components C_{α} in the chain with the exception of the first one. This leads to a stratification of S indexed by nested sets, with locally closed strata

$$\{s \in S; I(s) = I\}.$$

Let S_I denote the closure of the above set in S . Then by condition (2) in Definition 3.7,

$$S_I = \{s \in S; I(s) \supseteq I\}.$$

If J and K are nested sets such that $J \cup K$ is nested as well, then $S_J \cap S_K = S_{J \cup K}$. For each nested set I , there exists a natural map $f_I : S_I \rightarrow \overline{M}'_I^a$, obtained by forgetting the partitions P_s but remembering the associated nested sets. When $I = \emptyset$ we get $f : S \rightarrow \overline{M}'_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$. If $J \supset I$, then the diagram

$$\begin{array}{ccc} S_J & \xrightarrow{f_J} & \overline{M}'_J^a \\ \downarrow \subset & & \downarrow \phi'_J \\ S_I & \xrightarrow{f_I} & \overline{M}'_I^a \end{array}$$

is commutative. Moreover, if $G_I \subset S_d$ is the largest subgroup that keeps each $h \in I$ fixed,

$$S_I \times_{\overline{M}_I^a} \phi_J^I(\overline{M}_J^a) \cong \bigcup_{g \in G_I/G_J} S_{g(J)}. \tag{3.2}$$

These conditions are sufficient to define $f' : S \rightarrow \overline{M}'_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$. Indeed, lifts $f'_K : S_K \rightarrow (\overline{M}'_K)^a$ can be constructed by decreasing induction on K . For the largest sets $K \supset I$, relation (3.2) implies $S_I \times_{\overline{M}_I^a} \phi_K^I(\overline{M}_K^a) \cong \bigsqcup_{g \in G_I/G_K} S_{g(K)} \rightarrow \bigsqcup_{g \in G_I/G_K} (\overline{M}'_{g(K)})^a$, which by Theorem 1.5 induces a morphism from S_I to the étale lift $(\overline{M}'_I)^{a(K)}$ of ϕ_K^I ; the commutative diagram above lifts as well. Now for any nested set K , moving on to the step when $(\overline{M}'_I)^a$ have been constructed for all $I \supset K$, then by [AGV, Appendix 1], the lifts of the commutative diagram above glue to a morphism

$$S_K \times_{\overline{M}_K^a} \phi_I^K(\overline{M}_I^a) \cong \bigcup_{g \in G_I/G_K} S_{g(I)} \rightarrow N_K,$$

where N_K is obtained by gluing all $(\overline{M}'_{g(I)})^a$ along $(\overline{M}'_{g(I) \cup g'(I)})^a$. We note that $\phi_I^K(\overline{M}_I^a)$ is the image of N_K in \overline{M}_K^a . Now the lift $(\overline{M}'_K)^a$ is constructed so that $N_K \hookrightarrow (\overline{M}'_K)^a$. Again by Theorem 1.5, one obtains a lift $S_K \rightarrow (\overline{M}'_K)^a$.

We note that $(C \rightarrow S, \{p_i\}_i, \mathcal{L}, e)$ is the pull-back through f of the universal family \mathcal{U} . As for every $s \in S$, the partition P_s on C_s is completely determined by the nested set I such that $s \in S_I \setminus \bigcup_{J \supset I} S_J$, it follows that the partition P_s is also inherited by pull-back through f' from the universal family on $\overline{M}'_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$. \square

Notation. For each integer $k \in \{2, \dots, d\}$, let a_k be any real number with $1/k \leq a_k < 1/(k - 1)$. Let $a_1 = 1$.

By [MM2], a sequence of birational contractions of $\overline{M}_{0,m}(\mathbb{P}^n, d)$ is given by the spaces $\overline{M}_{0,\mathcal{A}_k}(\mathbb{P}^n, d, a_k)$ with $\mathcal{A}_k = (1, a_k, \dots, a_k)$ for $d > k > 0$.

Lemma 3.13. *For each integer l with $d' \geq l > k$, the morphism $f_I^{k,(l)} : \overline{M}_I^{a_{k-1},(l)} \rightarrow \overline{M}_I^{a_k,(l)}$ is a weighted blow-up along the local embedding $\phi_{Ih}^{I,(l)} : \overline{M}_{Ih}^{a_k,(l)} \rightarrow \overline{M}_I^{a_k,(l)}$ with $h \notin I$ such that $|h| = k$.*

If $k = l$, then $\phi_{Ih}^{I,(l)}$ is an embedding and $f_I^{k,(l)}$ is the weighted blow-up along all such embeddings with $|h| = l$.

Proof. The statement follows by decreasing induction on l . The paragraph above the lemma sets up the initial step $l = d$. Assume that the lemma holds for $l > k$. Consider h and h' such that $|h| = k$ and $|h'| = l - 1$. By the induction hypothesis, both $f_I^{k,(l)}$ and $f_{Ih'}^{k,(l)}$ are weighted blow-ups along the local embeddings, and the corresponding local embeddings are connected by a Cartesian diagram due to Lemma 3.9(3) applied to $K = I \cup \{h\}$ and $J = I \cup \{h'\}$. Thus the natural diagram containing $f_I^{k,(l)}$ and $f_{Ih'}^{k,(l)}$ is Cartesian, following the definition of blow-ups along the local embeddings and the natural étale

atlases described in the proof of Lemma 3.9. We can now apply Corollary 1.8 at each step in the construction of the $(l - 1)$ -étale lifts, leading to the following diagram being Cartesian:

$$\begin{array}{ccc} \overline{M}_I^{a_{k-1},(l-1)} & \xrightarrow{f_I^{k,(l-1)}} & \overline{M}_I^{a_k,(l-1)} \\ \downarrow & & \downarrow \\ \overline{M}_I^{a_{k-1},(l)} & \xrightarrow{f_I^{k,(l)}} & \overline{M}_I^{a_k,(l)} \end{array}$$

If $l - 1 > k$, the following diagram is Cartesian both when $j = k$ and when $j = k - 1$ by Lemma 3.9(4):

$$\begin{array}{ccc} \overline{M}_{Ih}^{a_j,(l-1)} & \xrightarrow{\phi_{Ih}^{I,(l-1)}} & \overline{M}_I^{a_j,(l-1)} \\ \downarrow & & \downarrow \\ \overline{M}_{Ih}^{a_j,(l)} & \xrightarrow{\phi_{Ih}^{I,(l)}} & \overline{M}_I^{a_j,(l)} \end{array}$$

Since $f_I^{k,(l)}$ was assumed to be a weighted blow-up along the local embedding $\phi_{Ih}^{I,(l)}$, from the above it follows that $f_I^{k,(l-1)}$ is a weighted blow-up along the local embedding $\phi_{Ih}^{I,(l-1)}$. Finally, when $l - 1 = k$, Lemma 3.9(4) implies that the following diagrams are Cartesian:

$$\begin{array}{ccc} \bigsqcup_{|h''|=k} \overline{M}_{Ih''}^{a_j,(l-1)} & \xrightarrow{\sqcup \phi_{Ih''}^{I,(l-1)}} & \overline{M}_I^{a_j,(l-1)} \\ \downarrow & & \downarrow \\ \overline{M}_{Ih}^{a_j,(l)} & \xrightarrow{\phi_{Ih}^{I,(l)}} & \overline{M}_I^{a_j,(l)} \end{array}$$

and $f_I^{k,(k)}$ is the weighted blow-up along $\bigsqcup_{|h''|=k} \overline{M}_{Ih''}^{a_j,(k)}$. □

Theorem 3.14. Consider the normalization $\overline{M}_I^{a_k}$ of a boundary stratum in $\overline{M}_\emptyset^{a_k} := \overline{M}_{0,1}(\mathbb{P}^n, d, a_k)$. The total Chern class of $\overline{M}_I^{a_k}$ is written in $A(\overline{M}_\emptyset^{a_k,(k)})$ as

$$\begin{aligned} c(\overline{M}_I^{a_k}) &= (1 + H)^{n+1} (1 + \psi)^{s_I-1} \prod_{i=1}^{d-l_I} (1 + H + i\psi)^{n+1} \\ &\cdot \prod_{h, |h|>k} \frac{(1 + D_h)(1 + \psi_h)^{|I_h|-1} \prod_{j=1}^{|h \setminus \cup_{h' \in I} h'|} (1 + H_{I;h} + j\psi_h)^{n+1}}{(1 + \psi_h^0)^{|I_h|-1} \prod_{j=1}^{|h \setminus \cup_{h' \in I} h'|} (1 + H_{I;h} + j\psi_h^0)^{n+1}} \end{aligned}$$

where the product is taken after all h such that $I \cup \{h\}$ is still a nested set. Here $l_I := |\cup_{h \in I} h|$ and s_I is the number of maximal elements of I . Moreover H denotes

the pull-back of the hyperplane divisor from \mathbb{P}^n , ψ denotes the cotangent line class, and D_h represents the class of the divisor $\overline{M}_{(h)}^{a_k}$, while

$$\begin{aligned} \psi_h &:= \psi - \sum_{h' \supseteq h} D_{h'} \quad \text{and} \quad \psi_h^0 := \psi - \sum_{h' \supset h} D_{h'}, \\ H_{I;h} &:= H + \left(d - \left| h \cup \left(\bigcup_{h' \in I} h' \right) \right| \right) \psi - \sum_{h' \supset h} \left| h' \setminus \left(h \cup \left(\bigcup_{h'' \in I} h'' \right) \right) \right| D_{h'}. \end{aligned}$$

Proof. We proceed by induction on k . The last member in the sequence of contractions of $\overline{M}_{0,1}(\mathbb{P}^n, d)$ is $\overline{M}_{0,1}(\mathbb{P}^n, d, a_{d-1})$, a weighted projective fibration $\mathcal{P}(A)$ over \mathbb{P}^n described in [MM1, Lemma 3.3]. The normal bundle of the zero section in A splits by weights as $\mathcal{N}_{\mathbb{P}^n|A} = \bigoplus_I \mathcal{N}_I / \mathcal{N}_{I+1}$, where

$$\mathcal{N}_I = \left(\bigoplus_{i=1}^{(n+1)(d-l)} \mathcal{O}_{\mathbb{P}^n}(1) \right) / \mathcal{O}_{\mathbb{P}^n}.$$

Thus by Proposition 2.11, the total Chern class is

$$c(\overline{M}_{0,1}(\mathbb{P}^n, d, a_{d-1})) = (1 + H)^{n+1} (1 + \psi)^{-1} \prod_{l=1}^d (1 + H + l\psi)^{n+1}.$$

More generally, for any $I \in \mathcal{P}(D)$, denote $l_I := |\bigcup_{h \in I} h|$ and let s_I be the number of maximal elements of I , i.e. elements $h \in I$ such that there is no $h' \in I$ with $h \subset h'$.

The normal bundle

$$\mathcal{N}_{\overline{M}_I^{a_{d-1}} | \overline{M}^{a_{d-1}}} = \mathcal{N}_{\mathbb{P}^1 \times \mathbb{P}^n | (\mathbb{P}^1)^{s_I} \times \mathbb{P}_{d-l_I}^n} |_{\mathbb{P}^n} = \left(\mathcal{O}_{\mathbb{P}^n}^{s_I} \oplus \bigoplus_{i=1}^{(n+1)(d-l_I)} \mathcal{O}_{\mathbb{P}^n}(1) \right) / \mathcal{O}_{\mathbb{P}^n}$$

admits a natural filtration $\{\mathcal{N}_I\}_I$ described in Lemma 3.2 of [MM1]. Here

$$\mathcal{N}_I = \left(\mathcal{O}_{\mathbb{P}^n}^{s_I} \oplus \bigoplus_{i=1}^{(n+1)(d-l_I-l)} \mathcal{O}_{\mathbb{P}^n}(1) \right) / \mathcal{O}_{\mathbb{P}^n}.$$

\mathbb{C}^* acts on the bundle $\mathcal{N} = \bigoplus_I \mathcal{N}_I / \mathcal{N}_{I+1}$ with weights $(1, \dots, d - l_I)$. Thus by Proposition 2.11, the total Chern class of the weighted projective fibration $\overline{M}_I^{a_{d-1}} \rightarrow \mathbb{P}^n$ is

$$c(\overline{M}_I^{a_{d-1}}) = (1 + H)^{n+1} (1 + \psi)^{s_I-1} \prod_{i=1}^{d-l_I} (1 + H + i\psi)^{n+1}.$$

The morphism $f_I^{a_k} : \overline{M}_I^{a_{k-1}} \rightarrow \overline{M}_I^{a_k}$ is a weighted blow-up along the local embeddings $\overline{M}_{Ih}^{a_k} \rightarrow \overline{M}_I^{a_k}$ for $h \notin I$ with $|h| = k$. By Lemma 3.13, the k -th étale lift $\overline{M}_I^{(a_{k-1},(k))} \rightarrow \overline{M}_I^{(a_k,(k))}$ is obtained by successive weighted blow-ups of all embeddings $\overline{M}_{Ih}^{(a_k,(k))} \rightarrow \overline{M}_I^{(a_k,(k))}$ where $h \notin I$ with $|h| = k$. As the étale covers are preserved through étale lifts, with the notation from Lemma 3.6, the étale atlas V_I of $\overline{M}_I^{(a_k,(k))}$ satisfies $V_I \times_{\overline{M}_I^{(a_k,(k))}}$

$\overline{M}_{I_h}^{(a_k, (k))} = V_{I_h}$ by construction, while by Lemma 3.6, all strata V_{I_h} with $|h| = k$ intersect each other, and their intersections, transversely. The normal bundle of the blow-up locus for $\overline{M}_I^{a_{k-1}} \rightarrow \overline{M}_I^{a_k}$ when $k \in \{1, \dots, d-1\}$ and the weights of the appropriate \mathbb{C}^* -action on it have been calculated in [MM1, Lemma 3.21]. Let $h \in \mathcal{P}(D)$ be such that $|h| = k$ and denote by $I_h \subseteq I$ the set of all $h' \in I$ such that $h' \subset h$.

The top equivariant Chern class of the normal bundle $\mathcal{N}_{\overline{M}_{I \cup \{h\}}^{a_k} / \overline{M}_I^{a_k}}$ when evaluated at $t = D_h$ is

$$(1 + \psi_h)^{|I_h|-1} \prod_{j=1}^{|h \setminus \bigcup_{h' \in I_h} h'|} (1 + H_{I;h} + j\psi_h)^{n+1},$$

and thus formula (2.1) reads in this case

$$\begin{aligned} c(\overline{M}_I^{a_{k-1}}) &= f_I^{a_k*} c(\overline{M}_I^{a_k}) \prod_{h, |h|=k} \frac{(1 + D_h)(1 + \psi_h)^{|I_h|-1} \prod_{j=1}^{|h \setminus \bigcup_{h' \in I} h'|} (1 + H_{I;h} + j\psi_h)^{n+1}}{(1 + \psi_h^0)^{|I_h|-1} \prod_{j=1}^{|h \setminus \bigcup_{h' \in I} h'|} (1 + H_{I;h} + j\psi_h^0)^{n+1}}. \end{aligned}$$

Iterating the formula above for all $k \in \{1, \dots, d-1\}$, we obtain the relation stated in the theorem. □

In particular,

$$\begin{aligned} c(\overline{M}_{0,1}(\mathbb{P}^n, d)) &= (1 + H)^{n+1} \frac{\prod_{i=1}^d (1 + H + i\psi)^{n+1}}{1 + \psi} \prod_h \frac{(1 + D_h)(1 + \psi_h^0) \prod_{j=1}^{|h|} (1 + H_h + j\psi_h)^{n+1}}{(1 + \psi_h) \prod_{j=1}^{|h|} (1 + H_h + j\psi_h^0)^{n+1}}, \end{aligned}$$

where $H_h = H + (d - |h|)\psi - \sum_{h' \supset h} |h' \setminus h| D_{h'} = H_{\emptyset;h}$ from above. We recall that the spaces $\overline{M}_{0,m}(\mathbb{P}^n, d, a_k)$ with $k \in \{1, \dots, d-1\}$ can be thought of as normalized strata of $\overline{M}_{0,1}(\mathbb{P}^n, d + m - 1, a_k)$. Indeed, as mentioned before, one can write

$$\overline{M}_{0,m}(\mathbb{P}^n, d, a_k) \cong \overline{M}_I^{a_k}$$

for $I = \{2_M, 3_M, \dots, d_M\} \subset \mathcal{P}(\{1_D, \dots, d_D, 2_M, \dots, m_M\})$. Thus

$$\begin{aligned} c(\overline{M}_{0,m}(\mathbb{P}^n, d)) &= (1 + H)^{n+1} (1 + \psi)^{m-2} \prod_{i=1}^d (1 + H + i\psi)^{n+1} \\ &\cdot \prod_{h \in \mathcal{P}(D) \setminus \{\emptyset, \{2_M\}, \dots, \{m_M\}\}} \frac{(1 + D_h)(1 + \psi_h)^{|h_M|-1} \prod_{j=1}^{|h \cap D|} (1 + H_{m;h} + j\psi_h)^{n+1}}{(1 + \psi_h^0)^{|h_M|-1} \prod_{j=1}^{|h \cap D|} (1 + H_{m;h} + j\psi_h^0)^{n+1}}, \end{aligned}$$

where $h_M = h \cap \{2_M, \dots, m_M\}$ and

$$H_{m;h} = H + |D \setminus h|\psi - \sum_{h' \supset h} |D \cap h' \setminus h| D_{h'}.$$

In particular, one recovers the formulae for the first Chern classes as calculated in [P] and in a more general setup in [DJS].

Let $f : \overline{M}_{0,1}(\mathbb{P}^n, d) \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d)$ be the forgetful morphism. In [MM3] this has been split into a sequence of simple blow-ups and a relatively simpler fibration $\overline{U}^{\lfloor (d-1)/2 \rfloor} \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d)$.

Choose $I := \{h \subset \{1, \dots, d\}; |h| > d/2\}$ if d is odd, and let I additionally contain half of the sets h with $|h| = d/2$ if d is even, under the condition that no two sets h, \bar{h} are simultaneously in I . The class

$$\psi'_I := \psi - \sum_{h \in I} D_h$$

can be considered as the $\mathcal{O}(1)$ -line bundle for the fibration $\overline{U}^{\lfloor (d-1)/2 \rfloor} \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d)$ above. When d is odd, ψ'_I is the pull-back of the relative cotangent class for the morphism $\overline{U}^{\lfloor (d-1)/2 \rfloor} \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d)$. The classes D_h with $h \in I$ are pull-backs of the classes of the exceptional divisors on $\overline{M}_{0,1}(\mathbb{P}^n, d)$. From the sequence of blow-ups mentioned above we obtain a formula comparing the total Chern classes of moduli spaces with/without marked points:

$$f^*(c(\overline{M}_{0,0}(\mathbb{P}^n, d))) = c(\overline{M}_{0,1}(\mathbb{P}^n, d)) \frac{\prod_{h \in I} (1 + D_h) P_h(1 - D_h)}{P(1 + \psi_I) \prod_{h \in I} P_h(1)},$$

where $P(\psi_I)$ and $P_h(-D_h)$ are the quadratic expressions in [MM2, Theorem 3.3(1)–(2)].

Proposition 2.1 in [MM3] shows how to recover the class $c(\overline{M}_{0,0}(\mathbb{P}^n, d))$ from its pull-back.

Appendix. Euler’s sequence for a weighted projective bundle

Let $g : P \rightarrow Y$ be a weighted projective bundle and a smooth morphism of stacks. With the notation from Section 2, consider the splitting $\mathcal{N}_{Y|A} = \bigoplus_i L_i$ such that for all n with $\mathcal{N}_n \neq \mathcal{N}_{n+1}$ there is a unique index i with $L_i = \mathcal{N}_n/\mathcal{N}_{n+1}$. Denote by $w_i = n$ the weight of the naturally induced \mathbb{C}^* -action on L_i .

Let $\mathcal{T}_{P|Y} = \ker(\mathcal{T}_P \rightarrow g^*\mathcal{T}_Y)$ denote the relative tangent bundle of $g : P \rightarrow Y$.

Lemma 3.15. *There is an exact sequence of vector bundles on P*

$$0 \rightarrow \mathcal{O}_P \xrightarrow{\sigma} \bigoplus_i g^*L_i \otimes \mathcal{O}_P(w_i) \xrightarrow{e} \mathcal{T}_{P|Y} \rightarrow 0.$$

Proof. The weighted projective bundle P is locally trivial over Y , i.e. $P|_U \cong \mathbb{P}[w_0 : \dots : w_n] \times U$ for each open set U in some open cover of Y . We first discuss briefly the case of the weighted projective space $\mathcal{P}[w_0 : \dots : w_n] = [\text{Spec } \mathbb{C}[x_0, \dots, x_n] \setminus \{0\}]/\mathbb{C}^*$, where \mathbb{C}^* acts on x_i with weight w_i .

An étale presentation of the Deligne–Mumford stack $\mathcal{P}[w_0 : \dots : w_n]$ is given by

$$\bigsqcup_{i,j \in \{0, \dots, n\}} V_{ij} \rightrightarrows \bigsqcup_{i \in \{0, \dots, n\}} V_i$$

where $V_i = \mathbb{C}^n$ with coordinates $\{u_i^k\}_{k \in \{0, \dots, n\} \setminus \{i\}}$ and $V_{ij} = V_{ji} = \mathbb{C}^{n-1} \times \mathbb{C}^*$ with coordinates $\{v_{ij}^k\}_{k \in \{0, \dots, n\} \setminus \{i\}}$ such that $v_{ij}^j \neq 0$, and the étale maps $\phi_{ij}^i : V_{ij} \rightarrow V_i$ given by

$$\phi_{ij}^i(\{v_{ij}^k\}_k) = (v_{ij}^0, \dots, (v_{ij}^j)^{w_j}, \dots, v_{ij}^n),$$

with $(v_{ij}^j)^{w_j}$ in the j -th position, and the change of coordinates

$$v_{ji}^i = 1/v_{ij}^j, \quad v_{ji}^k = v_{ij}^k / (v_{ij}^j)^{w_k} \quad \text{for } k \neq i, j$$

on V_{ij} .

If $\{u_i^k\}_{k \in \{0, \dots, n\} \setminus \{i\}}$ are coordinates on V_i , then the map to the coarse moduli space $\mathcal{P}[w_0 : \dots : w_n] \rightarrow \mathbb{P}[w_0 : \dots : w_n]$ sends $(u_i^k)_{k \in \{0, \dots, n\} \setminus \{i\}} \in V_i$ to $[u_i^0 : \dots : 1 : \dots : u_i^n]$ with 1 in the i -th position.

The line bundle $\mathcal{O}_{\mathcal{P}[w_0 : \dots : w_n]}(1)$ is determined by the trivializations $L_i \cong V_i \times \mathbb{C}$, with gluing maps

$$\phi_{ij}^{i*} L_i \rightarrow \phi_{ij}^{j*} L_j$$

given by $\phi_{ij}^{j*} s_j = v_{ij}^j \phi_{ij}^{i*} s_i$, where s_i and s_j are the unitary sections on L_i and L_j . Thus the total space of $\mathcal{O}_{\mathcal{P}[w_0 : \dots : w_n]}(1)$ is

$$\mathcal{P}[w_0 : \dots : w_n : 1] \setminus \{[0 : \dots : 0 : 1]\} \rightarrow \mathcal{P}[w_0 : \dots : w_n].$$

Each weighted projective coordinate x_i gives rise to a global section of $\mathcal{O}_{\mathcal{P}[w_0 : \dots : w_n]}(w_i)$, which will be denoted by x_i as well.

The tangent bundle to $\mathcal{P}[w_0 : \dots : w_n]$ is determined by the vector bundles $\mathcal{T}_{V_i} \cong V_i \times \mathbb{C}^n$ with isomorphisms

$$\phi_{ij}^{i*} \mathcal{T}_{V_i} \cong \phi_{ij}^{j*} \mathcal{T}_{V_j},$$

identifying

$$\phi_{ij}^{j*} \left(\frac{\partial}{\partial u_j^k} \right) = (v_{ij}^j)^{w_k} \phi_{ij}^{i*} \left(\frac{\partial}{\partial u_i^k} \right), \quad \phi_{ij}^{j*} \left(\frac{\partial}{\partial u_j^i} \right) = (v_{ij}^j)^{w_i} \phi_{ij}^{i*} \left(- \sum_{k \neq i} \frac{w_k}{w_i} v_{ij}^k \frac{\partial}{\partial u_i^k} \right).$$

Indeed,

$$\phi_{ij}^{i*} \mathcal{T}_{V_i} = \left\langle \left\{ \phi_{ij}^{i*} \left(\frac{\partial}{\partial u_i^k} \right) = \frac{\partial}{\partial v_{ij}^k} \right\}_{k \in \{0, \dots, n\} \setminus \{i\}}, \phi_{ij}^{i*} \left(\frac{\partial}{\partial u_j^i} \right) = \frac{1}{w_j} (v_{ij}^j)^{w_j-1} \frac{\partial}{\partial v_{ij}^i} \right\rangle,$$

and the identifications above are naturally derived from the change of coordinates v_{ij}^k into v_{ji}^k .

The short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{P}[w_0 : \dots : w_n]} \xrightarrow{\sigma} \bigoplus_i \mathcal{O}_{\mathcal{P}[w_0 : \dots : w_n]}(w_i) \xrightarrow{\ell} \mathcal{T}_{\mathcal{P}[w_0 : \dots : w_n]} \rightarrow 0$$

is defined locally by

$$\sigma(1) := (w_0x_0, \dots, w_nx_n), \quad e(s_i^{\otimes a_k}) := \frac{\partial}{\partial u_i^k}, \quad e(s_i^{\otimes a_i}) := - \sum_{k \neq i} \frac{w_k}{w_i} u_i^k \frac{\partial}{\partial u_i^k},$$

and by the presentation of the bundles above this is well defined globally.

Given a weighted projective bundle $g : P = [A/\mathbb{C}^*] \rightarrow Y$ with weights (w_0, \dots, w_n) , consider two open embeddings $U, U' \hookrightarrow Y$ and trivializations $x_U : A|_U \cong U \times \mathbb{C}^{n+1}$ and $x_{U'} : A|_{U'} \cong U' \times \mathbb{C}^{n+1}$, with gluing morphisms over $U \times_Y U'$

$$\varphi = (\varphi_0, \dots, \varphi_n) : (U \times_Y U') \times \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1},$$

such that φ_i is homogeneous of degree w_i with respect to the given weights. For each trivialization above there is an étale presentation

$$\bigsqcup_{i,j \in \{0, \dots, n\}} (V_{ij} \times U) \rightrightarrows \bigsqcup_{i \in \{0, \dots, n\}} (V_i \times U) \quad \text{for } P|_{g^{-1}(U)},$$

respectively

$$\bigsqcup_{i,j \in \{0, \dots, n\}} (V'_{ij} \times U') \rightrightarrows \bigsqcup_{i \in \{0, \dots, n\}} (V'_i \times U') \quad \text{for } P|_{g^{-1}(U')}.$$

Let g_U denote the restriction of g to $P|_{g^{-1}(U)}$, and similarly for U' . Pull-back of the trivialization x_U to $V_i \times_P g_U^*(A|_U)$, of $x_{U'}$ to $V_i \times_P g_{U'}^*(A|_{U'})$ and of the gluing morphism φ to $((V_i \times U) \times_P (V'_i \times U')) \times \mathbb{C}^{n+1}$ amounts to choosing a root φ_i^{1/w_i} of φ_i such that a point $(V_i \times U) \times_P (V'_i \times U')$ admits a change of coordinates

$$u_i'^j = \varphi_j(g(u), p_i(u)) \varphi_i^{-w_j/w_i}(g(u), p_i(u)),$$

where $u = (g(u), (u_i^j)_{j \neq i})$ and $p_i(u) = (u_i^0, \dots, 1, \dots, u_i^n) \in \mathbb{C}^{n+1}$, with 1 in the i -th position. Thus differentiating,

$$\frac{\partial}{\partial u_i^j} = \sum_k \varphi_{kj} \frac{\partial}{\partial u_i^k} (\varphi_i^{-w_k/w_i}) - \varphi_{ij} \left(\sum_k \frac{w_k}{w_i} u_i^k \frac{\partial}{\partial u_i^k} \right) (\varphi_i^{-1}),$$

for $j \neq i$, where the functions $\varphi_{kj} := \partial \varphi_k / \partial x_j$ define a local change of basis for the pull-back of the normal bundle $\mathcal{N}_{Y|A}$ and φ_i^{-1/w_i} defines a local change of basis for $\mathcal{O}_P(1)$. Furthermore, as P is a weighted projective bundle, $\varphi_{kj} = 0$ unless $w_k = w_j$.

This proves that the trivial extensions of the exact sequence (3.1) to $P|_{g^{-1}(U)}$ and $P|_{g^{-1}(U')}$ glue to a restriction of the sequence (3.2) on $P|_{g^{-1}(U \times_Y U')}$. \square

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