

GEODESIC FLOW FOR CAT(0)-GROUPS

ARTHUR BARTELS AND WOLFGANG LÜCK

ABSTRACT. We associate to a CAT(0)-space a flow space that can be used as the replacement for the geodesic flow on the sphere tangent bundle of a Riemannian manifold. We use this flow space to prove that CAT(0)-groups are transfer reducible over the family of virtually cyclic groups. This result is an important ingredient in our proof of the Farrell-Jones Conjecture for these groups.

INTRODUCTION

In [1] we introduced the concept of *transfer reducible groups* with respect to a family of subgroups. This definition is somewhat technical and recalled as Definition 0.4 below. We showed that groups that are transfer reducible over the family of virtually cyclic subgroups satisfy the Farrell-Jones Conjecture with coefficients in an additive category. For further explanations about the Farrell-Jones Conjecture we refer for instance to [1, 3, 8], where more information about the applications, history, literature and status is given.

By a CAT(0)-group we mean a group G that admits a cocompact proper action by isometries on a finite dimensional CAT(0)-space. The following is our main result in this paper and has already been used in [1].

Main Theorem. *Every CAT(0)-group is transfer reducible over the family of virtually cyclic subgroups.*

A similar result for hyperbolic groups has been proven in [1, Proposition 2.1] using the technical paper [2], where an important input is the flow space for hyperbolic groups due to Mineyev [9]. The methods for hyperbolic groups cannot be transferred directly to CAT(0)-groups, but the general program is the same and carried out in this paper.

An important step in the proof of the theorem above is the construction of a flow space $FS(X)$ associated to CAT(0)-spaces X , which is a replacement for the geodesic flow on the sphere tangent bundle of a Riemannian manifold of non-positive curvature. In particular, the dynamic of the flow on $FS(X)$ is similar to the geodesic flow. As in the hyperbolic case, the flow space $FS(X)$ is not a bundle over X .

In Section 1 we assign to any metric space X its flow space $FS(X)$ (see Definition 1.2). Elements in $FS(X)$ are generalized geodesics, i.e., continuous maps $c: \mathbb{R} \rightarrow X$ such that either c is constant or there exists $c_-, c_+ \in \mathbb{R} = \mathbb{R} \amalg \{-\infty, \infty\}$ with $c_- < c_+$ such that c is locally constant outside the interval (c_-, c_+) and its restriction to (c_-, c_+) is an isometric embedding. The flow on $FS(X)$ is given by $\Phi_\tau(c)(t) := c(t + \tau)$. The topology on $FS(X)$ is the topology of uniform convergence on compact subsets; this is also the topology associated to a natural metric on $FS(X)$. Many properties of X can be transported to $FS(X)$. For example, if

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a group G acts by isometries on X , then there is an induced isometric action on $FS(X)$. If the action on X is cocompact, then the induced action is also cocompact.

In Sections 2, 4 and 3 we study properties of $FS(X)$ under the assumption that X is a CAT(0)-space. The main observation in Section 2 is that the endpoint evaluation maps $c \mapsto c(\pm\infty)$ from $FS(X)$ to the bordification of \overline{X} of X are continuous on the complement of the subspace $FS(X)^{\mathbb{R}}$ of constant generalized geodesics. These are used in Proposition 2.6 to give coordinates on $FS(X) - FS(X)^{\mathbb{R}}$ and allow a detailed study of the topology of $FS(X) - FS(X)^{\mathbb{R}}$. We also discuss in Subsection 2.5 the case where X is a non-positively curved manifold. In Section 3 we prove our main flow estimates for $FS(X)$. These are crucial for our main result and differ from the corresponding estimates in the hyperbolic case. In the hyperbolic case the flow acts contracting on geodesics that determine the same point at infinity. This is not true in the CAT(0)-situation, for instance on flats the flow acts as an isometry. This problem is overcome by using a variant of the focal transfer (formulated as a homotopy action) from [7]. In Section 4 we assume that G acts properly and by isometries on X and study the periodic orbits of $FS(X)$ with respect to the induced action. In Theorem 4.2 we construct certain open covers for the subspace $FS(X)_{\leq \gamma}$ of $FS(X)$ consisting of all geodesics of period $\leq \gamma$. The dimension of this cover is uniformly bounded and the cover is long in the sense that for every $c \in FS(X)_{\leq \gamma}$ there is a member U of this cover that contains $\Phi_{[-\gamma, \gamma]}(c)$. (In fact, U will contain even $\Phi_{\mathbb{R}}(c)$.) This result is much harder than the corresponding result in the hyperbolic case, because in the CAT(0)-case the periodic orbits are no longer discrete, but appear in continuous families.

Section 5 contains the final preparation for the proof of our main theorem. We show in Proposition 5.11 that the existence of a suitable flow space for a group G implies that G is transfer reducible over a given family. This result depends very much on the long thin covers for flow spaces from [2].

In Section 6 we put our previous results together and prove our main theorem. It is only here that we assume that the action of G on the CAT(0)-space X is cocompact. All previous results are formulated without this assumption. This forces for example the appearance of a compact set K in Theorem 4.2 and in Definition 5.5. There are of course prominent groups, for instance $SL_n(\mathbb{Z})$, that are naturally equipped with an isometric proper action on a CAT(0)-space, where the action is not cocompact. We hope that the level of generality in Sections 1 to 5 will be useful to prove the Farrell-Jones Conjecture for some of these groups.

Conventions. Let H be a (discrete) group that acts on a space Z . We will say that the action is *proper*, if for any $x \in X$ there is an open neighborhood U such that $\{h \in H \mid h \cdot U \cap U \neq \emptyset\}$ is finite. If Z is locally compact, this is equivalent to the condition that for any compact subset $K \subset Z$ the set $\{h \in H \mid h \cdot K \cap K \neq \emptyset\}$ is finite.

We will say that the action is *cocompact* if the quotient space $H \backslash Z$ is compact. If Z is locally compact, this is equivalent to the existence of a compact subset $L \subseteq Z$ such that $G \cdot L = Z$.

Let X be a topological space. Let \mathcal{U} be an open covering. Its *dimension* $\dim(\mathcal{U}) \in \{0, 1, 2, \dots\} \amalg \{\infty\}$ is the infimum over all integers $d \geq 0$ such that for any collection U_0, U_1, \dots, U_{d+1} of pairwise distinct elements in \mathcal{U} the intersection $\bigcap_{i=0}^{d+1} U_i$ is empty. An open covering \mathcal{V} is a refinement of \mathcal{U} if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ with $V \subseteq U$. The (*topological*) *dimension* (sometimes also called *covering dimension*) of a topological space X

$$\dim(X) \in \{0, 1, 2, \dots\} \amalg \{\infty\}$$

is the infimum over all integers $d \geq 0$ such that any open covering \mathcal{U} possesses a refinement \mathcal{V} with $\dim(\mathcal{V}) \leq d$.

For a metric space Z , a subset $A \subseteq Z$ and $\epsilon > 0$, we set

$$\begin{aligned} B_\epsilon(A) &:= \{z \in Z \mid \exists z' \in A \text{ with } d_Z(z, z') < \epsilon\}. \\ \overline{B}_\epsilon(A) &:= \{z \in Z \mid \exists z' \in A \text{ with } d_Z(z, z') \leq \epsilon\}. \end{aligned}$$

We abbreviate $B_\epsilon(z) = B_\epsilon(\{z\})$ and $\overline{B}_\epsilon(z) = \overline{B}_\epsilon(\{z\})$. A metric space Z is called *proper* if for every $R > 0$ and every point $z \in Z$ the closed ball $\overline{B}_R(z)$ of radius R around z is compact. A map is called *proper* if the preimage of any compact subset is again compact.

A *family of subgroups of a group G* is a set of subgroups closed under conjugation and taking subgroups. Denote by \mathcal{VCyc} the family of virtually cyclic subgroups.

Homotopy actions and S -long covers. Next we explain the notion of transfer reducible. In [1] we introduced the following definitions in order to formulate conditions on groups that imply that a group satisfies the Farrell-Jones conjecture in K - and L -theory.

Definition 0.1 (Homotopy S -action). Let S be a finite subset of a group G . Assume that S contains the trivial element $e \in G$. Let X be a space.

- (i) A *homotopy S -action* (φ, H) on X consists of continuous maps $\varphi_g: X \rightarrow X$ for $g \in S$ and homotopies $H_{g,h}: X \times [0, 1] \rightarrow X$ for $g, h \in S$ with $gh \in S$ such that $H_{g,h}(-, 0) = \varphi_g \circ \varphi_h$ and $H_{g,h}(-, 1) = \varphi_{gh}$ holds for $g, h \in S$ with $gh \in S$. Moreover, we require that $H_{e,e}(-, t) = \varphi_e = \text{id}_X$ for all $t \in [0, 1]$;
- (ii) Let (φ, H) be a homotopy S -action on X . For $g \in S$ let $F_g(\varphi, H)$ be the set of all maps $X \rightarrow X$ of the form $x \mapsto H_{r,s}(x, t)$ where $t \in [0, 1]$ and $r, s \in S$ with $rs = g$;
- (iii) Let (φ, H) be a homotopy S -action on X . For $(g, x) \in G \times X$ and $n \in \mathbb{N}$, let $S_{\varphi, H}^n(g, x)$ be the subset of $G \times X$ consisting of all (h, y) with the following property: There are $x_0, \dots, x_n \in X$, $a_1, b_1, \dots, a_n, b_n \in S$, $f_1, \hat{f}_1, \dots, f_n, \hat{f}_n: X \rightarrow X$, such that $x_0 = x$, $x_n = y$, $f_i \in F_{a_i}(\varphi, H)$, $\hat{f}_i \in F_{b_i}(\varphi, H)$, $f_i(x_{i-1}) = \hat{f}_i(x_i)$ and $h = ga_1^{-1}b_1 \dots a_n^{-1}b_n$;
- (iv) Let (φ, H) be a homotopy S -action on X and \mathcal{U} be an open cover of $G \times X$. We say that \mathcal{U} is *S -long with respect to (φ, H)* if for every $(g, x) \in G \times X$ there is $U \in \mathcal{U}$ containing $S_{\varphi, H}^{|S|}(g, x)$ where $|S|$ is the cardinality of S .

Definition 0.2 (N -dominated space). Let X be a metric space and $N \in \mathbb{N}$. We say that X is *controlled N -dominated* if for every $\epsilon > 0$ there is a finite CW -complex K of dimension at most N , maps $i: X \rightarrow K$, $p: K \rightarrow X$ and a homotopy $H: X \times [0, 1] \rightarrow X$ between $p \circ i$ and id_X such that for every $x \in X$ the diameter of $\{H(x, t) \mid t \in [0, 1]\}$ is at most ϵ .

Definition 0.3 (Open \mathcal{F} -cover). Let Y be a G -space. Let \mathcal{F} be a family of subgroups of G . A subset $U \subseteq Y$ is called an *\mathcal{F} -subset* if

- (i) For $g \in G$ and $U \in \mathcal{U}$ we have $g(U) = U$ or $U \cap g(U) = \emptyset$, where $g(U) := \{gx \mid x \in U\}$;
- (ii) The subgroup $G_U := \{g \in G \mid g(U) = U\}$ lies in \mathcal{F} .

An *open \mathcal{F} -cover* of Y is a collection \mathcal{U} of open \mathcal{F} -subsets of Y such that the following conditions are satisfied:

- (i) $Y = \bigcup_{U \in \mathcal{U}} U$;
- (ii) For $g \in G$, $U \in \mathcal{U}$ the set $g(U)$ belongs to \mathcal{U} .

Definition 0.4 (Transfer reducible). Let G be a group and \mathcal{F} be a family of subgroups. We will say that G is *transfer reducible* over \mathcal{F} if there is a number N with the following property:

For every finite subset S of G there are

- a contractible compact controlled N -dominated metric space X ;
- a homotopy S -action (φ, H) on X ;
- a cover \mathcal{U} of $G \times X$ by open sets,

such that the following holds for the G -action on $G \times X$ given by $g \cdot (h, x) = (gh, x)$:

- (i) $\dim \mathcal{U} \leq N$;
- (ii) \mathcal{U} is S -long with respect to (φ, H) ;
- (iii) \mathcal{U} is an open \mathcal{F} -covering.

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1. A FLOW SPACE ASSOCIATED TO A METRIC SPACE

Summary. In this section we introduce the flow space $FS(X)$ for arbitrary metric spaces. We show that $FS(X)$ is a proper metric space if X is a proper metric space (see Proposition 1.9). If X comes with a proper cocompact isometric G -action, then $FS(X)$ inherits a proper cocompact isometric G -action (see Proposition 1.11).

Definition 1.1. Let X be a metric space. A continuous map $c: \mathbb{R} \rightarrow X$ is called a *generalized geodesic* if there are $c_-, c_+ \in \overline{\mathbb{R}} := \mathbb{R} \amalg \{-\infty, \infty\}$ satisfying

$$c_- \leq c_+, \quad c_- \neq \infty, \quad c_+ \neq -\infty,$$

such that c is locally constant on the complement of the interval $I_c := (c_-, c_+)$ and restricts to an isometry on I_c .

The numbers c_- and c_+ are uniquely determined by c , provided that c is not constant.

Definition 1.2. Let (X, d_X) be a metric space. Let $FS = FS(X)$ be the set of all generalized geodesics in X . We define a metric on $FS(X)$ by

$$d_{FS(X)}(c, d) := \int_{\mathbb{R}} \frac{d_X(c(t), d(t))}{2e^{|t|}} dt.$$

Define a flow

$$\Phi: FS(X) \times \mathbb{R} \rightarrow FS(X)$$

by $\Phi_\tau(c)(t) = c(t + \tau)$ for $\tau \in \mathbb{R}$, $c \in FS(X)$ and $t \in \mathbb{R}$.

The integral $\int_{-\infty}^{+\infty} \frac{d_X(c(t), d(t))}{2e^{|t|}} dt$ exists since $d_X(c(t), d(t)) \leq 2|t| + d_X(c(0), d(0))$ by the triangle inequality. Obviously $\Phi_\tau(c)$ is a generalized geodesic with

$$\begin{aligned} \Phi_\tau(c)_- &= c_- - \tau; \\ \Phi_\tau(c)_+ &= c_+ - \tau, \end{aligned}$$

where $-\infty - \tau := -\infty$ and $\infty - \tau := \infty$.

We note that any isometry $(X, d_X) \rightarrow (Y, d_Y)$ induces an isometry $FS(X) \rightarrow FS(Y)$ by composition. In particular, the isometry group of (X, d_X) acts canonically on $FS(X)$. Moreover, this action commutes with the flow.

For a general metric space X all generalized geodesics may be constant. Later we will consider the case where X is a CAT(0)-space, but in the remainder of this section we will consider properties of $FS(X)$ that do not depend on the CAT(0)-condition.

Lemma 1.3. *Let (X, d_X) be a metric space. The map Φ is a continuous flow and we have for $c, d \in FS(X)$ and $\tau, \sigma \in \mathbb{R}$*

$$d_{FS(X)}(\Phi_\tau(c), \Phi_\sigma(d)) \leq e^{|\tau|} \cdot d_{FS(X)}(c, d) + |\sigma - \tau|.$$

Proof. Obviously $\Phi_\tau \circ \Phi_\sigma = \Phi_{\tau+\sigma}$ for $\tau, \sigma \in \mathbb{R}$ and $\Phi_0 = \text{id}_{FS(X)}$. The main task is to show that $\Phi: FS(X) \times \mathbb{R} \rightarrow FS(X)$ is continuous.

We estimate for $c \in FS(X)$ and $\tau \in \mathbb{R}$

$$\begin{aligned} d_{FS(X)}(c, \Phi_\tau(c)) &= \int_{\mathbb{R}} \frac{d_X(c(t), c(t+\tau))}{2e^{|\tau|}} dt \\ &\leq \int_{\mathbb{R}} \frac{|\tau|}{2e^{|\tau|}} dt \\ &= |\tau| \cdot \int_{\mathbb{R}} \frac{1}{2e^{|\tau|}} dt \\ &= |\tau|. \end{aligned}$$

We estimate for $c, d \in FS(X)$ and $\tau \in \mathbb{R}$

$$\begin{aligned} d_{FS(X)}(\Phi_\tau(c), \Phi_\tau(d)) &= \int_{\mathbb{R}} \frac{d_X(c(t+\tau), d(t+\tau))}{2e^{|\tau|}} dt \\ &= \int_{\mathbb{R}} \frac{d_X(c(t), d(t))}{2e^{|\tau-t|}} dt \\ &\leq \int_{\mathbb{R}} \frac{d_X(c(t), d(t))}{2e^{|\tau|-|\tau-t|}} dt \\ &= e^{|\tau|} \cdot \int_{\mathbb{R}} \frac{d_X(c(t), d(t))}{2e^{|\tau|}} dt \\ &= e^{|\tau|} \cdot d_{FS(X)}(c, d). \end{aligned}$$

The two inequalities above together with the triangle inequality imply for $c, d \in FS(X)$ and $\tau, \sigma \in \mathbb{R}$

$$\begin{aligned} d_{FS(X)}(\Phi_\tau(c), \Phi_\sigma(d)) &= d_{FS(X)}(\Phi_\tau(c), \Phi_{\sigma-\tau} \circ \Phi_\tau(d)) \\ &\leq d_{FS(X)}(\Phi_\tau(c), \Phi_\tau(d)) + d_{FS(X)}(\Phi_\tau(d), \Phi_{\sigma-\tau} \circ \Phi_\tau(d)) \\ &\leq e^{|\tau|} \cdot d_{FS(X)}(c, d) + |\sigma - \tau|. \end{aligned}$$

This implies that Φ is continuous at (c, τ) . □

The following lemma relates distance in X to distance in $FS(X)$.

Lemma 1.4. *Let $c, d: \mathbb{R} \rightarrow X$ be generalized geodesics. Consider $t_0 \in \mathbb{R}$.*

- (i) $d_X(c(t_0), d(t_0)) \leq e^{|t_0|} \cdot d_{FS}(c, d) + 2;$
- (ii) *If $d_{FS}(c, d) \leq 2e^{-|t_0|-1}$, then*

$$d_X(c(t_0), d(t_0)) \leq \sqrt{4e^{|t_0|+1}} \cdot \sqrt{d_{FS}(c, d)}.$$

In particular, $c \mapsto c(t_0)$ defines a uniform continuous map $FS(X) \rightarrow X$.

Proof. We abbreviate $D := d_X(c(t_0), d(t_0))$. Since c and d are generalized geodesics, we conclude using the triangle inequality for $t \in \mathbb{R}$

$$d_X(c(t), d(t)) \geq D - d_X(c(t_0), c(t)) - d_X(d(t_0), d(t)) \geq D - 2 \cdot |t - t_0|.$$

This implies

$$\begin{aligned}
d_{FS(X)}(c, d) &= \int_{-\infty}^{+\infty} \frac{d_X(c(t), d(t))}{2e^{|t|}} dt \\
&\geq \int_{-D/2+t_0}^{D/2+t_0} \frac{D - 2 \cdot |t - t_0|}{2e^{|t|}} dt \\
&= \int_{-D/2}^{D/2} \frac{D - 2 \cdot |t|}{2e^{|t+t_0|}} dt \\
&\geq \int_{-D/2}^{D/2} \frac{D - 2 \cdot |t|}{2e^{|t|+|t_0|}} dt \\
&= e^{-|t_0|} \cdot \int_{-D/2}^{D/2} \frac{D - 2 \cdot |t|}{2e^{|t|}} dt \\
&= e^{-|t_0|} \cdot \int_0^{D/2} (D - 2t) \cdot e^{-t} dt \\
&= e^{-|t_0|} \cdot [(-D + 2 + 2t) \cdot e^{-t}]_0^{D/2} \\
(1.5) \quad &= e^{-|t_0|} \cdot (2 \cdot e^{-D/2} + D - 2).
\end{aligned}$$

Since $e^{-|t_0|} \cdot (2 \cdot e^{-D/2} + D - 2) \geq e^{-|t_0|} \cdot (D - 2)$, assertion (i) follows. It remains to prove assertion (ii).

Consider the function $f(x) = e^{-x} + x - 1 - \frac{x^2}{2e}$. We have $f'(x) = -e^{-x} + 1 - \frac{x}{e}$ and $f''(x) = e^{-x} - e^{-1}$. Hence $f''(x) \geq 0$ for $x \in [0, 1]$. Since $f'(0) = 0$, this implies $f'(x) \geq 0$ for $x \in [0, 1]$. Since $f(0) = 0$, this implies $f(x) \geq 0$ for $x \in [0, 1]$. Setting $x = D/2$ we obtain $\frac{(D/2)^2}{2e} \leq e^{-D/2} + D/2 - 1$ for $D \in [0, 2]$. From (1.5) we get $d_{FS}(c, d) \geq 2e^{-|t_0|} \cdot (e^{-D/2} + D/2 - 1)$. Therefore

$$\frac{D^2}{4e^{|t_0|+1}} \leq d_{FS(X)}(c, d) \quad \text{if } D \leq 2.$$

Consider the function $g(x) = e^{-x} + x - 1$. Since $g'(x) = -e^{-x} + 1 > 0$ for $x \geq 1$, we conclude $g(x) > g(1) = e^{-1}$ for $x > 1$. Hence (1.5) implies

$$d_{FS(X)}(c, d) > 2e^{-|t_0|-1} \quad \text{if } D > 2.$$

Hence we have

$$d_X(c(t_0), d(t_0)) = D \leq \sqrt{4e^{|t_0|+1}} \cdot \sqrt{d_{FS}(c, d)}.$$

if $d_{FS}(c, d) \leq 2 \cdot e^{-|t_0|-1}$. □

Lemma 1.6. *Let (X, d_X) be a metric space. The maps*

$$\begin{aligned}
FS(X) - FS(X)^{\mathbb{R}} &\rightarrow \overline{\mathbb{R}}, & c &\mapsto c_-; \\
FS(X) - FS(X)^{\mathbb{R}} &\rightarrow \overline{\mathbb{R}}, & c &\mapsto c_+,
\end{aligned}$$

are continuous.

Proof. By an obvious symmetry it suffices to consider the second map. Let $c \in FS(X) - FS(X)^{\mathbb{R}}$. Let $\alpha_0 \in \mathbb{R}$ with $\alpha_0 < c_+$. We will first show that there is ε_0 such that $d_+ > \alpha_0$ if $d_{FS}(c, d) < \varepsilon_0$. Pick $s_0, t_0 \in \mathbb{R}$ such that $\alpha_0 < s_0 < t_0 < c_+$

and $c(s_0) \neq c(t_0)$. By Lemma 1.4 (ii) there is $\varepsilon_0 > 0$ such that $d_{FS}(c, d) < \varepsilon_0$ implies

$$\max \{d_X(c(s_0), d(s_0)), d_X(c(t_0), d(t_0))\} < \frac{d_X(c(s_0), c(t_0))}{3}.$$

For d with $d_{FS}(c, d) < \varepsilon_0$ we have $d_X(d(s_0), d(t_0)) > d_X(c(s_0), c(t_0))/3$ by the triangular inequality and in particular, $d(s_0) \neq d(t_0)$. This implies $d_+ > s_0$ and therefore $d_+ > \alpha_0$. If $c_+ = +\infty$, then this shows that the second map is continuous at c .

If $c_+ < \infty$, then we need to show in addition that for a given $\alpha_1 > c_+$, there is $\varepsilon_1 > 0$ such that $d_+ < \alpha_1$ for all d with $d_{FS}(c, d) < \varepsilon_1$. Note that the previous argument also implied that $d_- < t_0 < c_+$ if $d_{FS}(c, d) < \varepsilon_0$ (because then $d(s_0) \neq d(t_0)$). Pick now $s_1, t_1 \in \mathbb{R}$ with $c_+ < s_1 < t_1 < \alpha_1$. By Lemma 1.4 (ii) there is $\varepsilon_1 \in \mathbb{R}$ satisfying $0 < \varepsilon_1 < \varepsilon_0$ such that $d_{FS}(c, d) < \varepsilon_1$ implies

$$\max \{d_X(c(s_1), d(s_1)), d_X(c(t_1), d(t_1))\} < \frac{t_1 - s_1}{2}.$$

Because $c(s_1) = c(t_1)$ we get $d_X(d(s_1), d(t_1)) < t_1 - s_1$ for d with $d_{FS}(c, d) < \varepsilon_1$. This implies $d_+ < t_1$ or $d_- > s_1$, because otherwise $d_X(d(s_1), d(t_1)) = t_1 - s_1$. However, $d_- < c_+ < s_1$ because $\varepsilon_1 < \varepsilon_0$. Thus $d_+ < t_1 < \alpha_1$. \square

Proposition 1.7. *Let (X, d_X) be a metric space. Let $(c_n)_{n \in \mathbb{N}}$ be a sequence in $FS(X)$. Then it converges uniformly on compact subsets to $c \in FS(X)$ if and only if it converges to c with respect to $d_{FS}(X)$.*

Proof. From Lemma 1.4 (ii) we conclude that convergence with respect to $d_{FS}(X)$ implies uniform convergence on compact subsets.

Let $(c_n)_{n \in \mathbb{N}}$ be a sequence in $FS(X)$ that converges uniformly on compact subsets to $c \in FS(X)$. Let $\varepsilon > 0$. Pick $\alpha \geq 1$ such that $\int_{\alpha}^{\infty} \frac{2t+\varepsilon}{e^t} dt < \varepsilon$. Because of uniform convergence on $[-\alpha, \alpha]$, there is n_0 such that $d_X(c_n(t), c(t)) \leq \varepsilon/\alpha$ for all $n \geq n_0$, $t \in [-\alpha, \alpha]$. In particular, $d_X(c_n(t), c(t)) \leq \varepsilon + 2|t|$ for all t , provided $n \geq n_0$. Thus for $n \geq n_0$

$$\begin{aligned} d_{FS}(c_n, c) &= \int_{-\infty}^{\infty} \frac{d_X(c_n(t), c(t))}{2e^{|t|}} dt \\ &\leq \int_{-\alpha}^{\alpha} \frac{\varepsilon/\alpha}{2e^{|t|}} dt + 2 \int_{\alpha}^{\infty} \frac{\varepsilon + 2t}{2e^{|t|}} dt \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

This shows $c_n \rightarrow c$ with respect to d_{FS} , because ε was arbitrary. \square

Lemma 1.8. *Let (X, d_X) be a metric space. The flow space $FS(X)$ is sequentially closed in the space of all maps $\mathbb{R} \rightarrow X$ with respect to the topology of uniform convergence on compact subsets.*

Proof. Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of generalized geodesics that converges uniformly on compact subsets to $f: \mathbb{R} \rightarrow X$. We have to show that f is a generalized geodesic. By passing to a subsequence we can assume that either $c_n \in FS^{\mathbb{R}}$ for all n , or $c_n \notin FS^{\mathbb{R}}$ for all n . In the first case $f \in FS^{\mathbb{R}}$. Thus it remains to treat the second case. In this case we have well-defined sequences $(c_n)_-$ and $(c_n)_+$. After passing to a further subsequence we can assume that these sequences converge in $[-\infty, \infty]$. Thus there are $\alpha_-, \alpha_+ \in [-\infty, \infty]$ such that $(c_n)_{\pm} \rightarrow \alpha_{\pm}$ as $n \rightarrow \infty$. We will show that f is a generalized geodesic with $f_{\pm} = \alpha_{\pm}$ or $f \in FS^{\mathbb{R}}$. Clearly, $d_X(f(s), f(t)) \leq |t - s|$ for all s, t .

If $\alpha_- > -\infty$, then we have to show that $f(s) = f(t)$ for all $s < t \leq \alpha_-$. Pick $\varepsilon > 0$. There is n_0 such that $|f(\tau) - c_n(\tau)| < \varepsilon$ for all $\tau \in [s - \varepsilon, t]$, $n \geq n_0$. Since $(c_n)_- \rightarrow \alpha_-$, there is $k \geq n_0$ such that $s - \varepsilon, t - \varepsilon \leq (c_k)_-$. Thus

$$\begin{aligned} d_X(f(s), f(t)) &\leq d_X(f(s - \varepsilon), f(t - \varepsilon)) + 2\varepsilon \\ &\leq d_X(c_k(s - \varepsilon), c_k(t - \varepsilon)) + 4\varepsilon \\ &= 4\varepsilon. \end{aligned}$$

Because ε is arbitrary, we conclude $f(s) = f(t)$. If $\alpha_+ < \infty$, then a similar argument shows that $f(s) = f(t)$ for all $s, t \geq \alpha_+$. If $\alpha_- = \alpha_+$, then $f \in FS(X)^\mathbb{R}$ and we are done. It remains to treat the case $\alpha_- < \alpha_+$. We have to show that $d_X(f(s), f(t)) = t - s$ for all $s, t \in \mathbb{R}$ with $\alpha_- \leq s < t \leq \alpha_+$. Pick $\varepsilon > 0$, such that $2\varepsilon < t - s$. There is n_0 such that $|f(\tau) - c_n(\tau)| < \varepsilon$ for all $\tau \in [s, t]$, $n \geq n_0$. Since $(c_n)_\pm \rightarrow \alpha_\pm$, there is $k \geq n_0$ such that $(c_k)_- \leq s + \varepsilon < t - \varepsilon \leq (c_k)_+$. Thus

$$\begin{aligned} d_X(f(s), f(t)) &\geq d_X(f(s + \varepsilon), f(t - \varepsilon)) - 2\varepsilon \\ &\geq d_X(c_k(s + \varepsilon), c_k(t - \varepsilon)) - 4\varepsilon \\ &= ((t - \varepsilon) - (s + \varepsilon)) - 4\varepsilon \\ &= t - s - 6\varepsilon. \end{aligned}$$

This implies $d_X(f(s), f(t)) = t - s$, because ε was arbitrarily small. \square

Proposition 1.9. *If (X, d_X) is a proper metric space, then $(FS(X), d_{FS(X)})$ is a proper metric space.*

Proof. Let $R > 0$ and $c \in FS(X)$. It suffices to show that the closed ball $\overline{B}_R(c)$ in $FS(X)$ is sequentially compact since any metric space satisfies the second countability axiom. Let $(c_n)_n \in \mathbb{N}$ be a sequence in $\overline{B}_R(c)$. By Lemma 1.4 (i) there is $R' > 0$ such that $c_n(0) \in \overline{B}_{R'}(c(0))$. By assumption $\overline{B}_{R'}(c(0))$ is compact. Now we can apply the Arzelà-Ascoli theorem (see for example [4, I.3.10, p.36]). Thus after passing to a subsequence there is $d: \mathbb{R} \rightarrow X$ such that $c_n \rightarrow d$ uniformly on compact subsets. Lemma 1.8 implies $d \in FS(X)$. \square

Lemma 1.10. *Let (X, d_X) be a proper metric space and $t_0 \in \mathbb{R}$. Then the evaluation map $FS(X) \rightarrow X$ defined by $c \mapsto c(t_0)$ is uniformly continuous and proper.*

Proof. The map is uniformly continuous by Lemma 1.4 (ii). To show that it is also proper, it suffices by Proposition 1.9 to show that preimages of closed balls have finite diameter. If $d_X(c(t_0), d(t_0)) \leq r$, then $d_X(c(t), d(t)) \leq r + 2|t - t_0|$. Thus

$$d_{FS}(c, d) \leq \int_{\mathbb{R}} \frac{r + 2|t - t_0|}{2e^{|t|}} dt \quad \text{provided} \quad d_X(c(t_0), d(t_0)) \leq r.$$

\square

Proposition 1.11. *Let G act isometrically and proper on the proper metric space (X, d_X) . Then the action of G on $(FS(X), d_{FS})$ is also isometric and proper. If the action of G on X is in addition cocompact, then this is also true for the action on $FS(X)$.*

Proof. The action of G on $FS(X)$ is isometric. The map $FS(X) \rightarrow X$ defined by $c \mapsto c(0)$ is G -equivariant, continuous and proper by Lemma 1.10. The existence of such a map implies that the G -action on $FS(X)$ is also proper. This also implies that the action of G on $FS(X)$ is cocompact, provided that the action on X is cocompact. \square

Lemma 1.12. *Let (X, d_X) be a metric space. Then $FS(X)^\mathbb{R}$ is closed in $FS(X)$.*

Proof. Note that $FS(X)^\mathbb{R}$ is the space of constant generalized geodesics. Let $c \in FS(X) - FS(X)^\mathbb{R}$. Pick $t_0, t_1 \in \mathbb{R}$ such that $c(t_0) \neq c(t_1)$. Set $\delta := d_X(c(t_0), c(t_1))/2$. For $x \in X$ then $d_X(x, c(t_0)) \geq \delta$ or $d_X(x, c(t_1)) \geq \delta$. Denote by c_x the constant generalized geodesic at x . If $d_X(x, c(t_0)) \geq \delta$, then $d_X(x, c(t)) \geq \delta/2$ if $t \in [t_0 - \delta/2, t_0 + \delta/2]$. Thus in this case

$$d_{FS}(c_x, c) \geq \int_{t_0 - \delta/2}^{t_0 + \delta/2} \frac{\delta/2}{2e^{|t|}} dt =: \varepsilon_0.$$

Similarly,

$$d_{FS}(c_x, c) \geq \int_{t_1 - \delta/2}^{t_1 + \delta/2} \frac{\delta/2}{2e^{|t|}} dt =: \varepsilon_1,$$

if $d_X(x, c(t_1)) \geq \delta/2$. Hence $B_\varepsilon(c) \cap FS(X)^\mathbb{R} = \emptyset$ if we set $\varepsilon := \min\{\varepsilon_0/2, \varepsilon_1/2\}$. \square

Notation 1.13. Let X be a metric space. For $c \in FS(X)$ and $T \in [0, \infty]$, define $c|_{[-T, T]} \in FS(X)$ by

$$c|_{[-T, T]}(t) := \begin{cases} c(-T) & \text{if } t \leq -T; \\ c(t) & \text{if } -T \leq t \leq T; \\ c(T) & \text{if } t \geq T. \end{cases}$$

Obviously $c|_{[-\infty, \infty]} = c$ and if $c \notin FS(X)^\mathbb{R}$ and $(-T, T) \cap (c_-, c_+) \neq \emptyset$ then $(c|_{[-T, T]})_- = \max\{c_-, -T\}$ and $(c|_{[-T, T]})_+ = \min\{c_+, T\}$.

We denote by

$$FS(X)_f := \{c \in FS(X) - FS(X)^\mathbb{R} \mid c_- > -\infty, c_+ < \infty\} \cup FS(X)^\mathbb{R}$$

the subspace of finite geodesics.

Lemma 1.14. *Let (X, d_X) be a metric space. The map $H: FS(X) \times [0, 1] \rightarrow FS(X)$ defined by $H_\tau(c) := c|_{[\ln(\tau), -\ln(\tau)]}$ is continuous and satisfies $H_0 = \text{id}_{FS(X)}$ and $H_\tau(c) \in FS(X)_f$ for $\tau > 0$.*

Proof. Observe that for $c \in FS(X)$ and $T, T' \geq 0$

$$d_X(c|_{[-T, T]}(t), c|_{[-T', T']}(t)) \leq |T - T'| \quad \text{for all } t \in \mathbb{R}.$$

Recall from Proposition 1.7 that the topology on $FS(X)$ is the topology of uniform convergence on compact subsets. Let $c_n \rightarrow c$ uniformly on compact subsets, and $\tau_n \rightarrow \tau$. Let $\alpha > 0$. We need to show that $c_n|_{[\ln(\tau_n), -\ln(\tau)]} \rightarrow c|_{[\ln(\tau), -\ln(\tau)]}$ uniformly on $[-\alpha, \alpha]$.

Consider first the case $\tau = 0$. Then $c = c|_{[\ln(\tau), -\ln(\tau)]}$. Moreover, $-\ln(\tau_n) > \alpha$ for sufficient large n . Thus $c_n(t) = c_n|_{[\ln(\tau_n), -\ln(\tau)]}(t)$ for such n and $t \in [-\alpha, \alpha]$. This implies the claim for $\tau = 0$.

Next consider the case $\tau \in (0, 1]$. Let $\varepsilon > 0$. There is $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned} d_X(c_n(t), c(t)) &\leq \varepsilon & \text{if } n \geq n_0, t \in [-\alpha, \alpha]; \\ |\ln(\tau) - \ln(\tau_n)| &\leq \varepsilon & \text{if } n \geq n_0. \end{aligned}$$

Then for $t \in [-\alpha, \alpha]$, $n \geq n_0$,

$$\begin{aligned} &d_X(c_n|_{[\ln(\tau_n), -\ln(\tau_n)]}(t), c|_{[\ln(\tau), -\ln(\tau)]}(t)) \\ &\leq d_X(c_n|_{[\ln(\tau_n), -\ln(\tau_n)]}(t), c_n|_{[\ln(\tau), -\ln(\tau)]}(t)) \\ &\quad + d_X(c_n|_{[\ln(\tau), -\ln(\tau)]}(t), c|_{[\ln(\tau), -\ln(\tau)]}(t)) \\ &\leq 2\varepsilon. \end{aligned}$$

This implies the claim in the second case because ε was arbitrary. \square

2. THE FLOW SPACE ASSOCIATED TO A CAT(0)- SPACE

Summary. In this section we study $FS(X)$ further in the case where X is a CAT(0)-space. Let \overline{X} be the bordification of X . We construct an injective continuous map from $FS(X) - FS(X)^{\mathbb{R}}$ to $\overline{\mathbb{R}} \times \overline{X} \times X \times \overline{X} \times \overline{\mathbb{R}}$ which is a homeomorphism onto its image (see Proposition 2.6). It sends a generalized geodesic c to $(c_-, c(-\infty), c(0), c(\infty), c_+)$, where $c(-\infty)$ and $c(\infty)$ are the two endpoints of c . This is used to show that $FS(X) - FS(X)^{\mathbb{R}}$ has finite dimension if X has (see Proposition 2.9), and that $FS(X) - FS(X)^{\mathbb{R}}$ is locally connected (see Proposition 2.10). We will relate our construction to the geodesic flow on the sphere tangent bundle of a simply connected Riemannian manifold with non-positive sectional curvature in Subsection 2.5.

For the definition of a CAT(0)-space we refer to [4, II.1.1 on p. 158], namely to be a geodesic space all of whose geodesic triangles satisfy the CAT(0)-inequality. We will follow the notation and the description of the *bordification* $\overline{X} = X \cup \partial X$ of a CAT(0)-space X given in [4, Chapter II.8]. The definition of the topology of this bordification is briefly reviewed in Remark 2.3. In this section we will use the following convention.

Convention 2.1. *Let*

- X be a complete CAT(0)-space;
- $\overline{X} := X \cup \partial X$ be the bordification of X , see [4, Chapter II.8].

2.1. Evaluation of generalized geodesics at infinity.

Definition 2.2. For $c \in FS(X)$ we set $c(\pm\infty) := \lim_{t \rightarrow \pm\infty} c(t)$, where the limit is taken in \overline{X} .

Since X is by assumption a CAT(0)-space, we can find for $x_- \in X$ and $x_+ \in \overline{X}$ a generalized geodesic $c: \mathbb{R} \rightarrow X$ with $c(\pm\infty) = x_{\pm}$ (see [4, II.8.2 on p. 261]). It is not true in general that for two different points x_- and x_+ in ∂X there is a geodesic c with $c(-\infty) = x_-$ and $c(\infty) = x_+$.

Remark 2.3 (Cone topology on \overline{X} .) A *generalized geodesic ray* is a generalized geodesic c that is either a constant generalized geodesic or a non-constant generalized geodesic with $c_- = 0$. Fix a base point $x_0 \in X$. For every $x \in \overline{X}$, there is a unique generalized geodesic ray c_x such that $c(0) = x_0$ and $c(\infty) = x$, see [4, II.8.2 on p.261]. Define for $r > 0$

$$\rho_r = \rho_{r, x_0}: \overline{X} \rightarrow \overline{B}_r(x_0)$$

by $\rho_r(x) := c_x(r)$. The sets $(\rho_r)^{-1}(V)$ with $r > 0$, V an open subset of $\overline{B}_r(x_0)$ are a basis for the cone topology on \overline{X} , see [4, II.8.6 on p.263]. A map f whose target is \overline{X} is continuous if and only if $\rho_r \circ f$ is continuous for all r . The cone topology is independent of the choice of base point, see [4, II.8.8 on p.264].

Lemma 2.4. *The maps*

$$\begin{aligned} FS(X) - FS(X)^{\mathbb{R}} &\rightarrow \overline{X}, & c &\mapsto c(-\infty); \\ FS(X) - FS(X)^{\mathbb{R}} &\rightarrow \overline{X}, & c &\mapsto c(\infty), \end{aligned}$$

are continuous.

The proof of this Lemma depends on the following result.

Lemma 2.5. *Given $\varepsilon > 0$, $a > 0$ and $s > 0$, there exists a constant $T = T(\varepsilon, a, s) > 0$ such that the following is true: if $x, x' \in X$ with $d_X(x, x') \leq a$, if $c: \mathbb{R} \rightarrow X$ is a generalized geodesic ray with $c(0) = x$, and if $\sigma_t: [0, d(x', c(t))] \rightarrow X$ is the geodesic from x' to $c(t)$, then $d_X(\sigma_t(s), \sigma_{t+t'}(s)) < \varepsilon$ for all $t \geq T$ and all $t' \geq 0$.*

Proof. In [4, II.8.3 on p.261] this is proven under the additional assumptions that c is a geodesic ray and that $d_X(x, x') = a$. But the proof given in [4] can be adapted as follows to give our more general result.

The argument given in [4] can be applied without change to show that there is T such that $d_X(\sigma_t(s), \sigma_{t+t'}(s)) < \varepsilon$ for all $t \geq T$, $t' \geq 0$ provided that $t+t' \leq c_+$. (This is needed to deduce that $|t-a| \leq d_X(x', c(t))$ and that $|t+t'-a| \leq d_X(x', c(t+t'))$.)

It remains to treat the case where $t \geq T$, $t' \geq 0$ and $t+t' \geq c_+$. If $t \geq c_+$, then $\sigma_t = \sigma_{t+t'}$ (because $c(t+t') = c(t) = c(c_+)$) and there is nothing to show. Thus we can assume $t \leq c_+$. Set $t'' := c_+ - t$. Then $t+t'' \leq c_+$, $t'' \geq 0$ and $\sigma_{t+t'} = \sigma_{t+t''}$ (because $c(t+t') = c(t+t'') = c(c_+)$). Thus $d_X(\sigma_t(s), \sigma_{t+t'}(s)) = d_X(\sigma_t(s), \sigma_{t+t''}(s)) < \varepsilon$. \square

Proof of Lemma 2.4. By an obvious symmetry it suffices to consider the second map. Let $c \in FS(X) - FS(X)^\mathbb{R}$. Set $s_0 := \max\{0, c_-\} + 1$ and $x_0 := c(s_0)$. Then

$$\tilde{c}: \mathbb{R} \rightarrow X, \quad t \mapsto \begin{cases} c(t+s_0) & t \geq 0; \\ c(s_0) & t \leq 0, \end{cases}$$

is a generalized geodesic ray starting at x_0 . We need to show that the map $f_r: FS(X) \rightarrow \overline{B}_r(x_0)$ defined by $f_r(d) := \rho_r(d(\infty))$ is for all $r > 0$ continuous at c , see Remark 2.3. Note that $f_r(c) = \tilde{c}(r) = c(s_0 + r)$.

Let $\varepsilon > 0$ be given. By Lemma 1.6 there is δ_0 such that $d_- < s_0$ for all d with $d_{FS}(c, d) < \delta_0$. In particular, we obtain for any such generalized geodesic ray d a generalized geodesic ray \tilde{d} by putting $\tilde{d}(t) = d(t+s_0)$ for $t \geq 0$ and $\tilde{d}(t) = d(s_0)$ for $t \leq 0$.

For $t > 0$ and $d \in FS$ with $d_{FS}(c, d) < \delta_0$ denote by $\sigma_t^d: [0, d_X(x_0, d(s_0+t))] \rightarrow X$ the geodesic from x_0 to $d(s_0+t)$. By Lemma 2.5 there is a number T (not depending on d !) such that $d_X(\sigma_t^d(r), \sigma_{t+t'}^d(r)) < \varepsilon$ for all $t' > 0$, $t \geq T$, provided that $d_X(d(s_0), x_0) \leq 1$. We extend σ_t^d to a generalized geodesic ray by setting $\sigma_t^d(s) := d(s_0+t)$ for $s > d_X(x_0, d(s_0+t))$ and $\sigma_t^d(s) := x_0$ for $s < 0$. The unique generalized geodesic ray $c_{d(\infty)}$ from x_0 to $d(\infty)$ can be constructed as the limit of the σ_t^d , see [4, Proof of (8.2) on p.262]. It follows that $\sigma_t^d(r) \rightarrow c_{d(\infty)}(r)$ as $t \rightarrow \infty$. By definition of ρ_r we have $c_{d(\infty)}(r) = f_r(d)$. Therefore $d_X(\sigma_T(r), f_r(d)) \leq \varepsilon$, provided that $d_X(d(s_0), c(s_0)) \leq 1$.

By Lemma 1.4 (ii) there exists $0 < \delta_1 < \delta_0$ such that

$$d_X(c(s_0), d(s_0)) < 1 \quad \text{and} \quad d_X(c(s_0+T), d(s_0+T)) < \varepsilon$$

if $d_{FS}(c, d) < \delta_1$. Consider the triangle whose vertices are $x_0 = c(s_0)$, $c(s_0+T)$ and $d(s_0+T)$. Recall that σ_T^d is side of this triangle that connects x_0 to $d(s_0+T)$. Using the CAT(0)-condition in this triangle it can be deduced that if $d_{FS}(c, d) < \delta_1$ then

$$d_X(c(s_0+r), \sigma_T^d(r)) < 2\varepsilon.$$

Therefore for $d_{FS}(c, d) < \delta_1$ we conclude

$$\begin{aligned} d_X(f_r(c), f_r(d)) &= d_X(c(r+s_0), c_{d(\infty)}(r)) \\ &\leq d_X(c(r+s_0), \sigma_T^d(r)) + d_X(\sigma_T^d(r), c_{d(\infty)}(r)) \\ &< 3\varepsilon. \end{aligned}$$

Because ε was arbitrary this implies that f_r is continuous at c . \square

2.2. Embeddings of the flow space.

Proposition 2.6. *If X is proper as a metric space, then the map*

$$E: FS(X) - FS(X)^\mathbb{R} \rightarrow \overline{R} \times \overline{X} \times X \times \overline{X} \times \overline{R}$$

defined by $E(c) := (c_-, c(-\infty), c(0), c(\infty), c_+)$ is injective and continuous. It is a homeomorphism onto its image.

Proof. Lemmas 1.4 (ii), 1.6 and 2.4 imply that E is continuous.

Next we show that E is injective. Let $c \in FS(X) - FS(X)^\mathbb{R}$. If $t_0 \in [c_-, c_+]$, then $t \mapsto c(t+t_0)$, $t \geq 0$ is the unique [4, II.8.2 on p.261] (generalized) geodesic ray from $c(t_0)$ to $c(\infty)$, and similarly $t \mapsto c(t_0-t)$, $t \geq 0$ is the unique (generalized) geodesic ray from $c(t_0)$ to $c(-\infty)$. Let $c, d \in FS(X) - FS(X)^\mathbb{R}$, with $c(\pm\infty) = d(\pm\infty)$, $c_\pm = d_\pm$. Then c and d will agree if and only if $c(t) = d(t)$ for some $t \in [c_-, c_+]$, $t \neq \pm\infty$. If $E(c) = E(d)$, then there is such a t : if one of c_- and c_+ is real (not $\pm\infty$), then it can be used, otherwise $t = 0$ works.

It remains to show that the inverse E^{-1} to E defined on the image of E is continuous. Let $(c_n)_{n \in \mathbb{N}}$ be a sequence in $FS(X) - FS(X)^\mathbb{R}$ and $c \in FS(X) - FS(X)^\mathbb{R}$ such that $E(c_n) \rightarrow E(c)$ as $n \rightarrow \infty$. We have to show that $c_n \rightarrow c$ as $n \rightarrow \infty$. We proceed by contradiction and assume this fails. Then there is a subsequence c_{n_k} and $\rho > 0$ such that $d_{FS}(c, c_{n_k}) > \rho$ for all k . We can pass to this subsequence and assume $d_{FS}(c, c_n) > \rho$ for all n . We have $c_n(0) \rightarrow c(0)$ as $n \rightarrow \infty$. The evaluation at $t = 0$ is a proper map $FS(X) \rightarrow X$ by Lemma 1.10. Thus we can pass to a further subsequence and assume that $c_n \rightarrow d$ as $n \rightarrow \infty$ with $d \in FS(X)$.

We claim that $d \notin FS(X)^\mathbb{R}$. Because $c \notin FS(X)^\mathbb{R}$ we have either $c(-\infty) \neq c(0)$ or $c(\infty) \neq c(0)$. By symmetry we may assume $c(-\infty) \neq c(0)$. We consider now two different cases.

First case: $c_- \neq -\infty$. Then $c_- \in \mathbb{R}$ and we can consider the evaluation at c_- . We have $(c_n)_- \rightarrow c_-$, $c_n(-\infty) \rightarrow c(-\infty)$ and $c_n(0) \rightarrow c(0)$ since $E(c_n) \rightarrow E(c)$ as $n \rightarrow \infty$. Moreover $c_n(c_-) \rightarrow d(c_-)$ and $c_n(0) \rightarrow d(0)$ since $c_n \rightarrow d$ as $n \rightarrow \infty$. Therefore $c(0) = d(0)$ and we get

$$\begin{aligned} d_X(d(c_-), c(c_-)) &\leq d_X(d(c_-), c_n(c_-)) + d_X(c_n(c_-), c_n((c_n)_-)) \\ &\quad + d_X(c_n((c_n)_-), c(c_-)) \\ &\leq d_X(d(c_-), c_n(c_-)) + |c_- - (c_n)_-| + d_X(c_n(-\infty), c(-\infty)) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus $d(c_-) = c(c_-) = c(-\infty) \neq c(0) = d(0)$. Therefore $d \notin FS(X)^\mathbb{R}$.

Second case: $c_- = -\infty$. Because $(c_n)_+ \rightarrow c_+ \neq -\infty$, there is $K > 0$ such that $-K < (c_n)_+$ for all n . Since $(c_n)_- \rightarrow c_- = -\infty$ we have $(c_n)_- < -2K$ for all sufficiently large n . Then

$$\begin{aligned} d_X(c_n(-2K), c_n(0)) &= d_X(c_n(-2K), c_n(-K)) + d_X(c_n(-K), c_n(0)) \\ &\geq d_X(c_n(-2K), c_n(-K)) \\ &= K \end{aligned}$$

for sufficiently large n . Using Lemma 1.10 we conclude $d_X(d(-2K), d(0)) \geq K$. Therefore $d \notin FS(X)^\mathbb{R}$. This finishes the proof of the claim.

Because $d \in FS(X) - FS(X)^\mathbb{R}$ we can apply E to d and deduce $E(d) = E(c)$ from continuity of E . Thus $c = d$ because E is injective. This contradicts $d_{FS}(c, c_n) > \rho$ for all n and finishes the proof. \square

Recall that $FS(X)_f$ is the subspace of finite geodesics, see Notation 1.13.

Proposition 2.7. *Assume that X is proper as a metric space. Then the map*

$$E_f: FS(X)_f - FS(X)^\mathbb{R} \rightarrow \mathbb{R} \times X \times X$$

defined by $E_f(c) = (c_-, c(-\infty), c(\infty))$ is a homeomorphism onto its image

$$\text{im } E_f = \{(r, x, y) \mid x \neq y\}.$$

In particular, $FS(X)_f - FS(X)^\mathbb{R}$ is locally path connected.

Proof. Lemmas 1.6 and 2.4 imply that E_f is continuous. The map E_f is injective with the stated image because of existence and uniqueness of geodesics between points in X , see [4, II.1.4 on p.160],

Next we show that the induced map

$$E_f: FS(X)_f - FS(X)^{\mathbb{R}} \rightarrow \{(r, x, y) \mid x \neq y\}$$

is proper. Let $K \subset \{(r, x, y) \mid x \neq y\}$ be compact. We will show that $E_f^{-1}(K)$ is sequentially compact. Let $(c_n)_{n \in \mathbb{N}}$ be a sequence in $E_f^{-1}(K)$. After passing to a subsequence, we may assume that $E_f(c_n)$ converges in K . Thus $(c_n)_- \rightarrow t_0 \in \mathbb{R}$, $c_n(-\infty) \rightarrow x_- \in X$, $c_n(\infty) \rightarrow x_+ \in X$, and $x_- \neq x_+$. We have

$$\begin{aligned} d_X(c_n(t_0), x_-) &\leq d_X(c_n(t_0), c_n(-\infty)) + d_X(c_n(-\infty), x_-) \\ &\leq |t_0 - (c_n)_-| + d_X(c_n(-\infty), x_-). \end{aligned}$$

Thus $c_n(t_0) \rightarrow x_-$ as $n \rightarrow \infty$. Using Lemma 1.10 we deduce that $(c_n)_{n \in \mathbb{N}}$ has a convergent subsequence in $FS(X)$, that we will again denote by c_n . So now $c_n \rightarrow c$ in $FS(X)$ for some $c \in FS(X)$.

We will show next that $c \notin FS^{\mathbb{R}}$. We have $(c_n)_+ - (c_n)_- = d_X(c_n(\infty), c_n(-\infty)) \rightarrow d_X(x_-, x_+)$, as $n \rightarrow \infty$. Thus $(c_n)_+ \rightarrow t_1 := t_0 + d_X(x_-, x_+)$. We have

$$d_X(c_n(t_1), x_+) \leq d_X(c_n(t_1), c_n(\infty)) + d_X(c_n(\infty), x_+) \leq |t_1 - (c_n)_+| + d_X(c_n(\infty), x_+).$$

Thus $c_n(t_1) \rightarrow x_+$ as $n \rightarrow \infty$. From Lemma 1.10 we conclude

$$c(t_1) = \lim c_n(t_1) = x_+ \neq x_- = c(t_0).$$

Thus $c(t_1) \neq c(t_0)$ and $c \in FS - FS^{\mathbb{R}}$.

Now $E_f(c_n) \rightarrow E_f(c)$ as E_f is continuous. Therefore $c_- = t_0$, $c(-\infty) = x_-$, $c(\infty) = x_+$. Thus $c \in E_f^{-1}(K)$. Hence $E_f: FS(X)_f - FS(X)^{\mathbb{R}} \rightarrow \{(r, x, y) \mid x \neq y\}$ is an injective continuous proper maps of metric spaces. This implies that it is a homeomorphism (see [12, 2.2 and 2.7]).

By the existence of geodesics the image of E_f is locally path connected. Hence $FS(X)_f - FS(X)^{\mathbb{R}}$ is locally path connected. \square

2.3. Covering dimension of the flow space. We will need the following elementary fact.

Lemma 2.8. *If X is proper as a metric space and its covering dimension $\dim X$ is $\leq N$, then $\dim \overline{X} \leq N$.*

Proof. Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open covering of \overline{X} . Recall from Remark 2.3 that a basis for the topology on \overline{X} is given by sets of the form $\rho_r^{-1}(W)$ for $r \geq 0$ and open $W \subseteq \overline{B}_r(x_0)$, where we fix a base point x_0 .) Thus for every $x \in \overline{X}$ there are $r_x, W_x \subseteq \overline{B}_{r_x}(x_0)$ and $U_x \in \mathcal{U}$ such that $x \in \rho_{r_x}^{-1}(W_x) \subset U_x$. Because \overline{X} is compact a finite number of the sets $\rho_{r_x}^{-1}(W_x)$ cover \overline{X} . Note that $\rho_r = \rho_r|_{\overline{B}_{r'}(x_0)} \circ \rho_{r'}$, if $r' > r$. Therefore we can refine \mathcal{U} to a finite cover \mathcal{V} , such that there is r and a finite cover \mathcal{W} of $\overline{B}_r(x_0)$ such that

$$\mathcal{V} = \rho_r^{-1}(\mathcal{W}) := \{\rho_r^{-1}(W) \mid W \in \mathcal{W}\}.$$

The result follows because $\overline{B}_r(x_0)$ is closed in X and thus $\dim \overline{B}_r(x_0) \leq \dim X$. \square

Proposition 2.9. *Assume that X is proper and that $\dim X \leq N$. Then*

$$\dim(FS(X) - FS(X)^{\mathbb{R}}) \leq 3N + 2.$$

Proof. The image of any compact subset under a continuous map is compact and a bijective continuous map with a compact subset as source and Hausdorff space as target is a homeomorphism. Hence every compact subset K of $FS(X) - FS(X)^{\mathbb{R}}$ is homeomorphic to a compact subset of $\overline{R} \times \overline{X} \times X \times \overline{X} \times \overline{R}$ by Proposition 2.6 and

hence its topological dimension satisfies because of Lemma 2.8 Since $FS(X)$ is a proper metric space by Lemma 1.9, it is locally compact and can be written as the countable union of compact subspaces and hence contains a countable dense subset. This implies that $FS(X)$ has a countable basis for its topology. Since $FS(X) - FS(X)^{\mathbb{R}}$ is an open subset of $FS(X)$, the topological space $FS(X) - FS(X)^{\mathbb{R}}$ is locally compact and has a countable basis for its topology. Now $\dim(FS(X) - FS(X)^{\mathbb{R}}) \leq 3N + 2$ follows from [10, Exercise 9 in Chapter 7.9 on page 315]. \square

2.4. The flow space is locally connected. A topological space Y is called *semi-locally path-connected* if for any $y \in Y$ and neighborhood V of y there is an open neighborhood U of y such that for every $z \in U$ there is a path w in V from y to z . Recall that Y is called *locally connected* or *locally path-connected* if any neighborhood V of any point $y \in Y$ contains an open neighborhood U of y such that U itself is connected or path-connected respectively. Suppose that Y is semi-locally path-connected. Then any open subset of Y is again semi-locally path-connected and each component of any open subset of Y is an open subset of Y . The latter is equivalent to the condition that Y is locally connected. Hence semi-locally path-connected implies locally connected. The notion of semi-locally path-connected is weaker than the notion of locally path-connected.

Proposition 2.10. *Assume that X is proper as a metric space. Then $FS(X) - FS(X)^{\mathbb{R}}$ is semi-locally path-connected. In particular, $FS(X) - FS(X)^{\mathbb{R}}$ is locally connected.*

Proof. Consider $c \in FS(X) - FS(X)^{\mathbb{R}}$ and a neighborhood $V \subseteq FS(X) - FS(X)^{\mathbb{R}}$ of c . By Lemma 1.14 there is a homotopy $H_t: FS(X) \rightarrow FS(X)$ such that $H_0 = \text{id}$ and $H_t(FS(X)) \subseteq FS(X)_f$ for all $t > 0$. Lemma 1.12 implies that V is also open as a subset of $FS(X)$. Since H is continuous, there is $\delta > 0$ and an open neighborhood $U_1 \subseteq V$ of c such that $H_t(U_1) \subseteq V$ for all $t \in [0, \delta]$. For any $d \in U_1$, $\omega_d(t) := H_{t\delta}(d)$ defines a path in V from d to $H_\delta(d)$. We have $H_\delta(c) \in FS(X)_f$. From Proposition 2.6 we conclude that $FS(X)_f - FS(X)^{\mathbb{R}}$ is open in $FS(X) - FS(X)^{\mathbb{R}}$. By Proposition 2.7 we can find a path-connected neighborhood $W \subseteq V \cap (FS(X)_f - FS(X)^{\mathbb{R}})$ of $H_\delta(c)$. Set now $U := U_1 \cap (H_\delta)^{-1}(W)$.

Consider $d \in U$. Then $H_\delta(c)$ and $H_\delta(d)$ both lie in W . Thus there is a path in W from $H_\delta(c)$ to $H_\delta(d)$. This is in particular a path in V , since $W \subseteq V$. Then $\omega := \omega_c * v * \omega_d$ is a path in V from c to d . Hence $FS(X) - FS(X)^{\mathbb{R}}$ is semi-locally path connected. \square

2.5. The example of a complete Riemannian manifold with non-positive sectional curvature. Let M be a simply connected complete Riemannian manifold with non-positive sectional curvature. It is a CAT(0)-space with respect to the metric coming from the Riemannian metric (see [4, I.A.6 on page 173]). Let STM be its sphere tangent bundle. For every $x \in M$ and $v \in ST_x M$ there is precisely one geodesic $c_v: \mathbb{R} \rightarrow M$ for which $c_v(0) = x$ and $c'_v(0) = v$ holds. Given a geodesic $c: \mathbb{R} \rightarrow M$ in M and $a_-, a_+ \in \overline{\mathbb{R}}$ with $a_- \leq a_+$, define the generalized geodesic $c_{[a_-, a_+]}: \mathbb{R} \rightarrow M$ by sending t to $c(a_-)$ if $t \leq a_-$, to $c(t)$ if $a_- \leq t \leq a_+$, and to $c(a_+)$ if $t \geq a_+$. Obviously $c_{[-\infty, \infty]} = c$. Let $d: \mathbb{R} \rightarrow M$ be a generalized geodesic with $d_- < d_+$. Then there is precisely one geodesic $\hat{d}: \mathbb{R} \rightarrow M$ with $\hat{d}_{[d_-, d_+]} = d$.

Define maps

$$\begin{aligned} \alpha: STM \times \{(a_i, a_+) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a_- < a_+\} &\rightarrow FS(M), & (v, a_i, a_+) &\mapsto c_v|_{[a_-, a_+]}; \\ \beta: FS(M) &\rightarrow STM \times \{(a_i, a_+) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a_- < a_+\}, & c &\mapsto (\hat{c}'(0), c_-, c_+). \end{aligned}$$

Then α and β are to another inverse homeomorphisms. They are compatible with the flow on $FS(M)$ of Definition 1.2, if one uses on $STM \times \{(a_i, a_+) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid$

$a_- < a_+$ } the product flow given by the geodesic flow on STM and the flow on \overline{R} which is at time t given by the homeomorphism $\overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ sending $s \in \mathbb{R}$ to $s - t$, $-\infty$ to $-\infty$, and ∞ to ∞ .

3. DYNAMIC PROPERTIES OF THE FLOW SPACE

Summary. In Definition 3.2 introduce the homotopy action that we will use to show that CAT(0)-groups are transfer reducible over \mathcal{VCyc} . It will act on a large ball in X . (The action of G on the bordification \overline{X} is not suitable, because it has to large isotropy groups.) In Propositions 3.5 and 3.8 we study the dynamics of the flow with respect to the homotopy action. In the language of Section 5 this shows that $FS(X)$ admits contracting transfers.

Throughout this section we fix the following convention.

Convention 3.1. *Let*

- (X, d_X) be a CAT(0)-space which is proper as a metric space;
- $x_0 \in X$ be a fixed base point;
- G be a group with a proper isometric action on (X, d_X) .

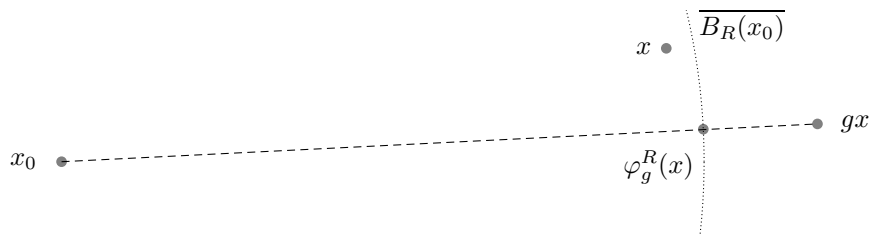
For $x, y \in X$ and $t \in [0, 1]$ we will denote by $t \cdot x + (1 - t) \cdot y$ the unique point z on the geodesic from x to y such that $d_X(x, z) = td_X(x, y)$ and $d_X(z, y) = (1 - t)d_X(x, y)$. For $x, y \in X$ we will denote by $c_{x,y}$ the generalized geodesic determined by $(c_{x,y})_- = 0$, $c(-\infty) = x$ and $c(\infty) = y$. (By [4, II.1.4(1) on p.160] and Lemma 1.7, $(x, y) \mapsto c_{x,y}$ defines a continuous map $X \times X \rightarrow FS(X)$). Note that $g \cdot c_{x,y} = c_{gx,gy}$.

3.1. The homotopy action on $\overline{B}_R(x)$. Recall that for $r > 0$ and $z \in X$ we denote by $\rho_{r,z}: X \rightarrow \overline{B}_r(z)$ the canonical projection along geodesics, i.e., $\rho_{r,z}(x) = c_{z,x}(r)$, see also Remark 2.3. Note that $g \cdot \rho_{r,z}(x) = \rho_{r,gz}(gx)$ for $x, z \in X$ and $g \in G$.

Definition 3.2 (The homotopy S -action on $\overline{B}_R(x_0)$). Let $S \subseteq G$ be a finite subset of G with $e \in G$ and $R > 0$. Define a homotopy S -action (φ^R, H^R) on $\overline{B}_R(x)$ in the sense of Definition 0.1 (i) as follows. For $g \in S$, we define the map

$$\varphi_g^R: \overline{B}_R(x_0) \rightarrow \overline{B}_R(x_0)$$

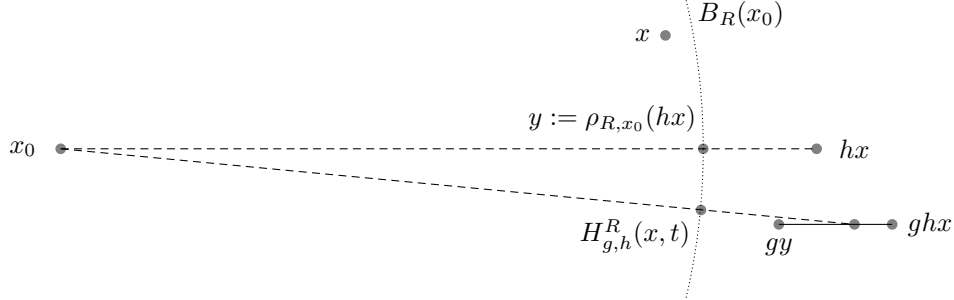
by $\varphi_g^R(x) := \rho_{R,x_0}(gx)$.



For $g, h \in S$ with $gh \in S$ we define the homotopy

$$H_{g,h}^R: \varphi_g^R \circ \varphi_h^R \simeq \varphi_{gh}^R$$

by $H_{g,h}^R(x,t) := \rho_{R,x_0}(t \cdot (ghx) + (1-t) \cdot (g \cdot \rho_{R,x_0}(hx)))$.



Remark 3.3. Notice that $H_{g,h}^R$ is indeed a homotopy from $\varphi_g^R \circ \varphi_h^R$ to φ_{gh} since

$$\begin{aligned} H_{g,h}^R(x,0) &= \rho_{R,x_0}(0 \cdot (ghx) + 1 \cdot (g \cdot \rho_{R,x_0}(hx))) \\ &= \rho_{R,x_0}(g \cdot \rho_{R,x_0}(hx)) \\ &= \varphi_g^R \circ \varphi_h^R(x), \end{aligned}$$

and

$$\begin{aligned} H_{g,h}^R(x,1) &= \rho_{R,x_0}(1 \cdot (ghx) + 0 \cdot (g \cdot \rho_{R,x_0}(hx))) \\ &= \rho_{R,x_0}(ghx) \\ &= \varphi_{gh}^R(x). \end{aligned}$$

It turns out that the more obvious homotopy given by convex combination $(x,t) \mapsto t \cdot \varphi_{gh}^R(x) + (1-t) \cdot \varphi_g^R \circ \varphi_h^R(x)$ is not appropriate for our purposes.

Definition 3.4 (The map ι). Define the map

$$\iota: G \times X \rightarrow FS(X)$$

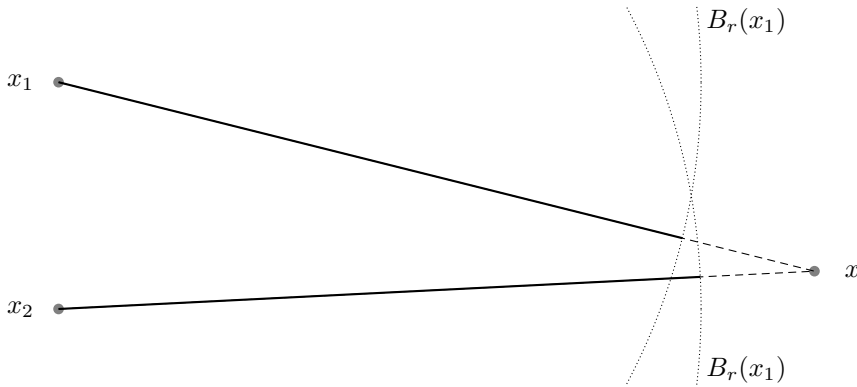
as follows. For $(g,x) \in G \times X$ let $\iota(g,x) := c_{gx_0,gx}$.

The map ι is G -equivariant for the action on $G \times X$ defined by $g \cdot (h,x) = (gh,x)$.

3.2. The flow estimate.

Proposition 3.5. *Let $\beta, L > 0$. For all $\delta > 0$ there are $T, r > 0$ such that for $x_1, x_2 \in X$ with $d_X(x_1, x_2) \leq \beta$, $x \in \overline{B}_{r+L}(x_1)$ there is $\tau \in [-\beta, \beta]$ such that*

$$d_{FS}(\Phi_T(c_{x_1, \rho_{r,x_1}(x)}), \Phi_{T+\tau}(c_{x_2, \rho_{r,x_2}(x)})) \leq \delta.$$



The proof requires some preparation.

Lemma 3.6. *Let $r', L, \beta > 0$, $r'' > \beta$. Set $T := r'' + r'$, $r := r'' + 2r' + \beta$. Let $x_1, x_2 \in X$ such that $d_X(x_1, x_2) \leq \beta$. Let $x \in \overline{B}_{r+L}(x_1)$. Set $\tau := d_X(x_2, x) - d_X(x_1, x)$. Then for all $t \in [T - r', T + r']$*

- (i) $d_X(c_{x_1, x}(t), c_{x_2, x}(t + \tau)) \leq \frac{2 \cdot \beta \cdot (L + 2r' + \beta)}{r''}$;
- (ii) $c_{x_1, \rho_{r, x_1}(x)}(t) = c_{x_1, x}(t)$ and $c_{x_2, \rho_{r, x_2}(x)}(t + \tau) = c_{x_2, x}(t + \tau)$.

Proof. (i) Let $t \in [T - r', T + r']$. Note that $|\tau| \leq \beta$. From $T - r' = r'' > \beta$ we conclude $t, t + \tau > 0$. If $t \geq d_X(x, x_1)$, then $c_{x_1, x}(t) = x = c_{x_2, x}(t + \tau)$ and the assertion follows in this case. Hence we can assume $0 < t < d_X(x, x_1)$. A straight forward computation shows that $0 < t + \tau < d_X(x, x_2)$ and $d_X(c_{x_1, x}(t), x) = d_X(c_{x_2, x}(t + \tau), x)$. We get $r'' = T - r' \leq t < d_X(x, x_1)$. Applying the CAT(0)-condition to the triangle Δ_{x, x_1, x_2} we deduce that

$$d_X(c_{x_1, x}(t), c_{x_2, x}(t + \tau)) \leq \frac{2 \cdot d_X(x_1, x_2) \cdot (d_X(x, x_1) - t)}{d_X(x, x_1)} \leq \frac{2 \cdot \beta \cdot (d_X(x, x_1) - t)}{d_X(x, x_1)}.$$

Combining this with $d_X(x, x_1) > r''$ and $d_X(x, x_1) - t \leq (r + L) - (T - r') = r'' + 2r' + \beta + L - r'' - r' + r' = 2r' + \beta + L$ we obtain the asserted inequality.

(ii) We have $t \leq T + r' = 2r' + r'' = r - \beta$ and $t \geq T - r' = r'' > \beta$. Thus $t, t + \tau \in [0, r]$. Thus $c_{x_1, \rho_{r, x_1}(x)}(t) = c_{x_1, x}(t)$ and $c_{x_2, \rho_{r, x_2}(x)}(t + \tau) = c_{x_2, x}(t + \tau)$. \square

Proof of Proposition 3.5. Let $\delta > 0$ be given. Pick $r' > 0$, $r'' > \beta$, $1 > \delta' > 0$ such that

$$\int_{-\infty}^{-r'} \frac{1 + |t|}{e^{|t|}} dt \leq \frac{\delta}{3}, \quad \int_{-r'}^{r'} \frac{\delta'}{2e^{|t|}} dt \leq \frac{\delta}{3}$$

and

$$\frac{2 \cdot \beta(L + 2r' + \beta)}{r''} \leq \delta'.$$

Set $r := 2r' + r'' + \beta$ and $T := r' + r''$. Let $x_1, x_2 \in X$ with $d_X(x_1, x_2) \leq \beta$. Let $x \in \overline{B}_r(x_1)$ be given. Set $\tau := d_X(x_2, x) - d_X(x_1, x)$. Then $|\tau| \leq d_X(x_2, x_1) \leq \beta$. Using Lemma 3.6 we conclude that for all $t \in [-r', r']$

$$\begin{aligned} & d_X(c_{x_1, \rho_{r, x_1}(x)}(T + t), c_{x_2, \rho_{r, x_2}(x)}(T + t + \tau)) \\ &= d_X(c_{x_1, x}(T + t), c_{x_2, x}(T + t + \tau)) \\ &\leq \frac{2 \cdot \beta(L + 2r' + \beta)}{r''} \leq \delta'. \end{aligned}$$

Thus

$$\begin{aligned} & d_{FS}(\Phi_T(c_{x_1, \rho_{r, x_1}(x)}), \Phi_{T+\tau}(c_{x_2, \rho_{r, x_2}(x)})) \\ &= \int_{-\infty}^{\infty} \frac{d_X(c_{x_1, \rho_{r, x_1}(x)}(T + t), c_{x_2, \rho_{r, x_2}(x)}(T + t + \tau))}{2e^{|t|}} dt \\ &\leq \int_{-\infty}^{-r'} \frac{2|t| + \delta'}{2e^{|t|}} dt + \int_{-r'}^{r'} \frac{\delta'}{2e^{|t|}} dt + \int_{r'}^{\infty} \frac{\delta' + 2|t|}{2e^{|t|}} dt \\ &\leq \int_{-\infty}^{-r'} \frac{|t| + 1}{e^{|t|}} dt + \int_{-r'}^{r'} \frac{\delta'}{2e^{|t|}} dt + \int_{r'}^{\infty} \frac{1 + |t|}{e^{|t|}} dt \\ &\leq \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta. \end{aligned}$$

\square

Lemma 3.7. *Let $\varepsilon > 0$, $\beta > 0$. Then there is $\delta > 0$ such that for all $|\tau| \leq \beta$*

$$d_{FS}(c_0, c_1) \leq \delta \implies d_{FS}(\Phi_\tau(c_0), \Phi_\tau(c_1)) \leq \varepsilon$$

for $c_0, c_1 \in FS(X)$.

Proof. This follows from Lemma 1.3. \square

Proposition 3.8. *Let S be a finite subset of G (containing e). Then there is $\beta > 0$ such that the following holds:*

For all $\delta > 0$ there are $T, R > 0$ such that for every $(a, x) \in G \times \overline{B}_R(X)$, $s \in S$, $f \in F_s(\varphi^R, H^R)$ there is $\tau \in [-\beta, \beta]$ such that

$$d_{FS}(\Phi_T(\iota(a, x)), \Phi_{T+\tau}(\iota(as^{-1}, f(x)))) \leq \delta.$$

Proof. Pick β such that $\frac{\beta}{2} \geq d_X(sx_0, x_0)$ for all $s \in S$. Let $L := \beta$. Let $\delta > 0$ be given. By Lemma 3.7 there is $\frac{\delta}{2} > \delta_0 > 0$ such that for $|\tau'| \leq \beta$

$$(3.9) \quad d_{FS}(c_0, c_1) \leq \delta_0 \implies d_{FS}(\Phi_{\tau'}(c_0), \Phi_{\tau'}(c_1)) \leq \frac{\delta}{2}$$

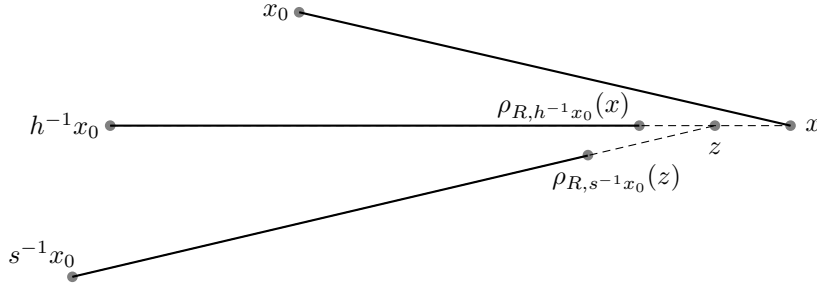
for $c_0, c_1 \in FS(X)$. By Proposition 3.5 there are $T, R > 0$ such that for $x, x_1, x_2 \in X$ with $d_X(x_1, x_2) \leq \beta$ and $d_X(x, x_1) \leq R + L$ there is $\tau = \tau(x, x_1, x_2) \in [-\frac{\beta}{2}, \frac{\beta}{2}]$ such that

$$d_{FS}(\Phi_T(c_{x_1, \rho_{R, x_1}(x)}), \Phi_{T+\tau}(c_{x_2, \rho_{R, x_2}(x)})) \leq \delta_0.$$

Let $(a, x) \in G \times \overline{B}_R(X)$, $s \in S$, $f \in F_s(\varphi^R, H^R)$. Note that

$$d_{FS}(\Phi_T(\iota(a, x)), \Phi_{T+\tau}(\iota(as^{-1}, f(x)))) = d_{FS}(\Phi_T(\iota(e, x)), \Phi_{T+\tau}(\iota(s^{-1}, f(x))))$$

because ι and Φ are G -equivariant and d_{FS} is G -invariant. Thus it suffices to consider the case $a = e$. Then there are $t \in [0, 1]$ and $g, h \in S$ such that $s = gh$ and $f(x) = H_{g, h}^R(x, t) = \rho_{R, x_0}(t \cdot (ghx) + (1-t) \cdot (g \cdot \rho_{R, x_0}(hx)))$. Therefore $s^{-1}f(x) = \rho_{R, s^{-1}x_0}(t \cdot x + (1-t) \cdot \rho_{R, h^{-1}x_0}(x))$. Set $z := t \cdot x + (1-t) \cdot \rho_{R, h^{-1}x_0}(x)$. Then $\iota(s^{-1}, f(x)) = c_{s^{-1}x_0, \rho_{R, s^{-1}x_0}(z)}$.



We have $d_X(x, x_0) \leq R$. Moreover, $d_X(z, h^{-1}x_0) \leq d_X(x, h^{-1}x_0) \leq d_X(x, x_0) + d_X(x_0, h^{-1}x_0) \leq R + L$. Therefore, we can set $\tau_1 := \tau(x, x_0, h^{-1}x_0)$, $\tau_2 := \tau(z, h^{-1}x_0, s^{-1}x_0)$ and $\tau := \tau_1 + \tau_2$. Note that $|\tau| \leq \beta$, since $|\tau_i| \leq \frac{\beta}{2}$. We have

$$d_{FS}(\Phi_T(c_{h^{-1}x_0, \rho_{R, h^{-1}x_0}(z)}), \Phi_{T+\tau_2}(c_{s^{-1}x_0, \rho_{R, s^{-1}x_0}(z)})) \leq \delta_0$$

and therefore by (3.9)

$$d_{FS}(\Phi_{T+\tau_1}(c_{h^{-1}x_0, \rho_{R, h^{-1}x_0}(z)}), \Phi_{T+\tau_1+\tau_2}(c_{s^{-1}x_0, \rho_{R, s^{-1}x_0}(z)})) \leq \frac{\delta}{2}.$$

Thus

$$\begin{aligned} & d_{FS}(\Phi_T(\iota(e, x)), \Phi_{T+\tau}(\iota(s^{-1}, f(x)))) \\ &= d_{FS}(\Phi_T(c_{x_0, x}), \Phi_{T+\tau}(c_{s^{-1}x_0, \rho_{R, s^{-1}x_0}(z)})) \\ &\leq d_{FS}(\Phi_T(c_{x_0, x}), \Phi_{T+\tau_1}(c_{h^{-1}x_0, \rho_{R, h^{-1}x_0}(x)})) \\ &\quad + d_{FS}(\Phi_{T+\tau_1}(c_{h^{-1}x_0, \rho_{R, h^{-1}x_0}(z)}), \Phi_{T+\tau_1+\tau_2}(c_{s^{-1}x_0, \rho_{R, s^{-1}x_0}(z)})) \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

□

4. ORBITS WITH BOUNDED G -PERIOD

Summary. Let $FS(X)_{\leq \gamma}$ be the part of $FS(X)$ that contains of in some sense periodic orbits, namely, those generalized geodesics for which there exists for every $\epsilon > 0$ an element $\tau \in (0, \gamma + \epsilon]$ and $g \in G$ such that $g \cdot c = \Phi_\tau(c)$ holds (see (5.4)). Our main result here is Theorem 4.2 that asserts that there is a cover of uniformly bounded dimension for $FS(X)_{\leq \gamma}$ that is long in the direction of the flow. To this end we study hyperbolic elements in G and their axis. These come in parallel families (called FS_a below) that project to convex subspaces of X . We construct the desired cover first for the FS_a by considering the quotient Y_a of FS_a by the flow. One difficulty here is that the group G_a that naturally acts on Y_a , does so with infinite isotropy. The isotropy groups here are virtually cyclic and this forces the appearance of the family \mathcal{VCyc} in Theorem 4.2 and in our main result.

Throughout this section we fix the following convention.

Convention 4.1. *Let*

- (X, d_X) be a CAT(0)-space which is proper as a metric space and has finite covering dimension;
- G be a group with a proper isometric action on (X, d_X) ;
- $K \subseteq X$ be a compact subset.

The following is the main result of this section.

Theorem 4.2. *There is a natural number M such that for every $\gamma > 0$ there exists a collection \mathcal{V} of subsets of $FS(X)$ satisfying:*

- (i) *Each element $V \in \mathcal{V}$ is an open \mathcal{VCyc} -subsets of the G -space $FS(X)$ (see Definition 0.3);*
- (ii) *\mathcal{V} is G -invariant; i.e., for $g \in G$ and $V \in \mathcal{V}$ we have $g \cdot V \in \mathcal{V}$;*
- (iii) *$G \backslash \mathcal{V}$ is finite;*
- (iv) *We have $\dim \mathcal{V} \leq M$;*
- (v) *There is $\epsilon > 0$ with the following property: for $c \in FS_{\leq \gamma}$ such that $c(t) \in G \cdot K$ for some $t \in \mathbb{R}$ there is $V \in \mathcal{V}$ such that $B_\epsilon(\Phi_{[-\gamma, \gamma]}(c)) \subseteq V$.*

4.1. Hyperbolic isometries of spaces. We recall some basic facts about isometries of a CAT(0)-space from [4, Chapter II.6]. Let $\gamma: X \rightarrow X$ be an isometry. The *displacement function* of γ is defined by

$$d_\gamma: X \rightarrow [0, \infty), \quad x \mapsto d_X(\gamma x, x).$$

The *translation length* of γ is defined by

$$l(\gamma) := \inf\{d_\gamma(x) \mid x \in X\}.$$

Define a subspace of X by

$$\text{Min}(\gamma) := \{x \in X \mid d_\gamma(x) = l(\gamma)\}.$$

We call γ *elliptic* if γ has a fixed point and *hyperbolic* if the displacement function d_γ attains a strictly positive minimum, or, equivalently, $l(\gamma) > 0$ and $\text{Min}(\gamma) \neq \emptyset$.

Lemma 4.3. *Let $\gamma: X \rightarrow X$ be an isometry.*

- (i) *If $\alpha: X \rightarrow X$ is an isometry, then $l(\alpha\gamma\alpha^{-1}) = l(\gamma)$ and $\text{Min}(\alpha\gamma\alpha^{-1}) = \alpha(\text{Min}(\gamma))$;*
- (ii) *$\text{Min}(\gamma)$ is a closed convex set;*
- (iii) *The isometry γ is hyperbolic if and only if it possesses an axis, i.e., there is a geodesic $c: \mathbb{R} \rightarrow X$ and $\tau > 0$ such that $\gamma \circ c(t) = c(t + \tau)$ holds for $t \in \mathbb{R}$. In this case $\tau = l(\gamma)$;*

- (iv) Two axes c and d for the hyperbolic isometry γ are parallel, i.e., $d_X(c(t), d(t))$ is constant in $t \in \mathbb{R}$. The union of the images $c(\mathbb{R})$ of all axes c for γ is $\text{Min}(\gamma)$.

Proof. See [4, II.6.2(2) on p.229] for (i). See [4, II.6.2(3) on p.229] for (ii). See [4, II.6.8(1) on p.231] for (iii). See [4, II.6.8(3) on p.231] for (iv). \square

We emphasize that an axis for a hyperbolic element γ is a (parametrized) geodesic $c: \mathbb{R} \rightarrow X$ and not only $c(\mathbb{R})$. So two hyperbolic elements γ_1 and γ_2 have a common axis if there exists a geodesic $c: \mathbb{R} \rightarrow X$ such that $\gamma_1 \cdot c(t) = c(t + l(\gamma_1))$ and $\gamma_2 \cdot c(t) = c(t + l(\gamma_2))$ holds for all $t \in \mathbb{R}$. We denote by $l(g)$ the translation length of the isometry $X \rightarrow X$ given by multiplication with $g \in G$. We will say that g is hyperbolic if this isometry is hyperbolic.

Lemma 4.4. *Let $c: \mathbb{R} \rightarrow X$ be a geodesic. Put*

$$\begin{aligned} G_{\Phi_{\mathbb{R}}(c)} &= \{g \in G \mid g(\Phi_{\mathbb{R}}(c)) = \Phi_{\mathbb{R}}(c)\} \\ &= \{g \in G \mid \exists \tau \in \mathbb{R} \text{ with } gc(t) = c(t + \tau) \text{ for all } t \in \mathbb{R}\} \end{aligned}$$

and

$$G_{c(\mathbb{R})} := \{g \in G \mid g \cdot c(\mathbb{R}) = c(\mathbb{R})\}.$$

Then $G_{\Phi_{\mathbb{R}}(c)}$ is virtually cyclic of type I and $G_{c(\mathbb{R})}$ is virtually cyclic.

Proof. The group $G_{c(\mathbb{R})}$ acts properly and isometrically on \mathbb{R} since $c(\mathbb{R})$ is isometric to \mathbb{R} . The isometry group of \mathbb{R} fits into the exact sequence $1 \rightarrow \mathbb{R} \xrightarrow{i} \text{Isom}(\mathbb{R}) \xrightarrow{p} \{\pm 1\} \rightarrow 1$, where i sends a real number r to the isometry $t \mapsto t + r$ and p sends an isometry to 1 if it is strictly monotone increasing and to -1 otherwise. Since $G_{c(\mathbb{R})}$ acts properly on \mathbb{R} , the obvious homomorphism $G \rightarrow \text{Isom}(\mathbb{R})$ has a finite kernel and its image is a discrete subgroup of $\text{Isom}(\mathbb{R})$. This implies that $G_{c(\mathbb{R})}$ is virtually cyclic.

Since the action of $G_{\Phi_{\mathbb{R}}(c)}$ on c is by translations, the same argument shows that $G_{\Phi_{\mathbb{R}}(c)}$ is virtually cyclic of type I. \square

4.2. Axes in the flow space.

Notation 4.5. Let $\gamma > 0$. Let K be the compact subset from Convention 4.1.

- (i) Let

$$G_{\leq \gamma}^{hyp} \subseteq G$$

be the set of all hyperbolic $g \in G$ of translation length $l(g) \leq \gamma$ such that some axis c for g intersects $G \cdot K$.

Consider the equivalence relation \sim on $G_{\leq \gamma}^{hyp}$ for which $g \sim g'$ if and only if there exists parallel axes c_g and $c_{g'}$ for g and g' . (This relation is transitive by Lemma 4.3 (iv).) Put

$$A_{\leq \gamma} := G_{\leq \gamma}^{hyp} / \sim.$$

The conjugation action of G on G restricts to an action on $G_{\leq \gamma}^{hyp}$ and descends to an action of G on $A_{\leq \gamma}$, see Lemma 4.3 (i). For $a \in A_{\leq \gamma}$ we set

$$G_a := \{g \in G \mid g \cdot a = a\}.$$

- (ii) For $a \in A_{\leq \gamma}$ let

$$FS_a \subseteq FS(X)$$

denote the subspace of $FS(X)$ that consists of all geodesics $c: \mathbb{R} \rightarrow X$, that are an axis for some $g \in a$ and intersect $G \cdot K$. We remark that

$c \in FS(X)$ is an axis for g if and only if $\Phi_\tau(c) = gc$ for some $\tau > 0$ and in this case $\tau = l(g)$, see Lemma 4.3 (iii). Define

$$p_a: FS_a \rightarrow X, \quad c \mapsto c(0).$$

We denote by

$$Y_a := FS_a / \Phi$$

the quotient of FS_a by the action of the flow Φ . Let

$$q_a: FS_a \rightarrow Y_a$$

be the canonical projection. The action of G on $FS(X)$ restricts to an action of G_a on FS_a . Because p_a is G_a -equivariant and because the flow Φ commutes with the G -action on $FS(X)$ we obtain an action of G_a on Y_a and q_a is G_a -equivariant for this action.

Lemma 4.6. *Let (Z, d_Z) be a proper metric space with a proper isometric action of a group H . If $(z_n)_{n \in \mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}}$ are sequences in Z and H such that $z_n \rightarrow z$ and $h_n z_n \rightarrow z'$ converge in Z , then $\{h_n \mid n \in \mathbb{N}\}$ is finite and for every $h \in H$ such that $h_n = h$ for infinitely many n we have $hz = z'$.*

Proof. Let $n_0 > 0$ such that $d_Z(z, z_n) < 1$ and $d_Z(z', h_n z_n) < 1$ for all $n \geq n_0$. Thus $d_Z(h_n z, z') < 2$ for all $n \geq n_0$. Thus $d_Z((h_{n_0})^{-1} h_n z, z) < 4$ for all $n \geq n_0$. Thus $\{(h_{n_0})^{-1} h_n \mid n \geq n_0\}$ is finite, because the action is proper. Therefore $\{h_n \mid n \in \mathbb{N}\}$ is finite. If $h_n = h$ for infinitely many $n \in \mathbb{N}$, then $hz = \lim_{n \rightarrow \infty} h z_n = \lim_{n \rightarrow \infty} h_n z_n = z'$. \square

Corollary 4.7. *Let (Z, d_Z) be a proper metric space with a proper isometric action of a group H . If $L \subset Z$ is compact, then $H \cdot L \subset Z$ is closed.*

Proof. Let $h_n z_n \rightarrow z$ with $h_n \in H$ and $z_n \in L$. After passing to a subsequence we have $z_n \rightarrow z'$. Lemma 4.6 implies that we can pass to a further subsequence for which $h_n = h$ is constant. Thus $z \in h \cdot L \subset H \cdot L$. \square

Lemma 4.8. *There is a compact subset $K_\gamma \subseteq X$ such that $c(0) \in G \cdot K_\gamma$ for all $c \in FS_a$, $a \in A_{\leq \gamma}$.*

Proof. Set $K_\gamma := \overline{B}_\gamma(K)$. This is compact because K is compact and X is proper as a metric space. For $a \in A_{\leq \gamma}$, $c \in FS_a$ there are $g \in a$, $t \in [0, \gamma]$ such that $c(s+t) = gc(s)$ for all $s \in \mathbb{R}$. Because c intersects $G \cdot K$, this implies that there is $s_0 \in [0, \gamma]$ such that $c(s_0) \in G \cdot K$. Thus $c(0) \in G \cdot \overline{B}_\gamma(K) = G \cdot K_\gamma$. \square

Lemma 4.9. *Let $\gamma > 0$. Let $(c_n)_{n \in \mathbb{N}}$ be a sequence in $\bigcup_{a \in A_{\leq \gamma}} FS_a$ that converges to c in $FS(X)$. Then there are $g \in G_{\leq \gamma}^{\text{hyp}}$ and an infinite subset $I \subseteq \mathbb{N}$ such that c and all c_i , $i \in I$ are axes for g and intersect $G \cdot K$, where K is the compact subset from Convention 4.1. In particular, $c \in FS_a$ and $c_i \in FS_a$ for all $i \in I$, where $a \in A_{\leq \gamma}$ is the class of g .*

Proof. There are $g_n \in G$, $t_n \in [0, \gamma]$ such that $g_n c_n = \Phi_{t_n}(c_n)$. We can pass to a subsequence and assume that $t_n \rightarrow t_0$. Then $g_n c_n = \Phi_{t_n}(c_n) \rightarrow \Phi_{t_0}(c)$ (using Lemma 1.3). Since $FS(X)$ is proper and G acts properly on $FS(X)$ (see Propositions 1.9 and 1.11) we can apply Lemma 4.6 and assume after passing to a further subsequence that $g_n = g$ is constant. Then $gc = \lim g_n c_n = \lim \Phi_{t_n} c_n = \Phi_{t_0}(c)$. It remains to show that c intersects $G \cdot K$.

For each n there is $s_n \in \mathbb{R}$ such that $c_n(s_n) \in G \cdot K$. Since $g_n c_n(s) = c_n(s + t_n)$ and $t_n \in [0, \gamma]$, we can arrange $s_n \in [0, \gamma]$ for all $n \geq 0$. By passing to a subsequence we can arrange that $s_n \rightarrow s_0$ for $n \rightarrow \infty$ for some $s_0 \in [0, \gamma]$. Then $c_n(s_n) \rightarrow c(s_0)$ (using Proposition 1.7). By Corollary 4.7 $G \cdot K$ is closed. Thus $c(s) \in G \cdot K$. \square

Lemma 4.10. *Let $\gamma > 0$. Then*

- (i) $G \setminus A_{\leq \gamma}$ is finite;
- (ii) $G \cdot FS_a \subseteq FS(X)$ is closed for all $a \in A_{\leq \gamma}$; there is $K_a \subseteq G \cdot FS_a$ compact such that $G \cdot K_a = G \cdot FS_a$;
- (iii) there is $\varepsilon > 0$ such that $d_X(FS_a, FS_b) > \varepsilon$ for all $a \neq b \in A_{\leq \gamma}$;
- (iv) Consider $c \in FS(X)_{\leq \gamma} - FS(X)^{\mathbb{R}}$ such that $c(t) \in K$ for some $t \in \mathbb{R}$, where K is the compact subset from Convention 4.1. Then there are $a \in A_{\leq \gamma}$ and $y \in Y_a$ such that $\Phi_{\mathbb{R}}(c) = q_a^{-1}(y)$.

Proof. (i) We proceed by contradiction and assume that there are a_1, a_2, \dots in $A_{\leq \gamma}$ such that $G a_i \cap G a_j = \emptyset$ if $i \neq j$. Then there are $g_i \in a_i$ and $c_i \in FS(X)_{a_i}$ such that c_i is an axis for g_i . After replacing a_i by $h_i a_i$, g_i by $h_i g_i h_i^{-1}$ and c_i by $h_i c_i$ for suitable $h_i \in G$ we can assume that $c_i(0) \in K_\gamma$, where K_γ is the compact subset of X from Lemma 4.8. Lemma 1.10 implies that $\hat{K}_\gamma := \{c \in FS(X) \mid c(0) \in K_\gamma\}$ is also compact. Thus we can pass to a subsequence and arrange that $c_i \rightarrow c \in FS(X)$. Lemma 4.9 yields a contradiction.

(ii) By Lemma 4.9 $G \cdot FS_a \subseteq FS(X)$ is closed. Thus $K_a := \hat{K}_\gamma \cap G \cdot FS_a$ is compact. Now we get $G \cdot FS_a = G \cdot \hat{K}_\gamma \cap G \cdot FS_a = G \cdot (\hat{K}_\gamma \cap G \cdot FS_a) = G \cdot K_a$ using Lemma 4.8.

(iii) We proceed by contradiction and assume that for every n , there are $a_n \neq b_n \in A_{\leq \gamma}$ and $c_n \in FS_{a_n}$, $d_n \in FS_{b_n}$ such that $d_{FS}(c_n, d_n) < 1/n$. Because of (ii) there are $h_n \in G$ such that a subsequence of $h_n c_n$ converges to some $c \in FS(X)$. After replacing a_n by $h_n a_n$, b_n by $h_n b_n$, c_n by $h_n c_n$, d_n by $h_n d_n$ and after passing to a suitable subsequence we can thus assume that $c_n \rightarrow c \in FS(X)$. Then also $d_n \rightarrow c$. Using Lemma 4.9 for the sequence $(c_n)_{n \in \mathbb{N}}$ we can after passing to a subsequence assume that $a_n = a$ is constant and that $c \in FS_a$. Using Lemma 4.9 for the sequence $(d_n)_{n \in \mathbb{N}}$ we can after passing to a further subsequence assume that $b_n = b$ is constant and $c \in FS_b$. Thus $FS_b \cap FS_a \neq \emptyset$. From the definition of $A_{\leq \gamma}$ it is immediate that this implies $a = b$, contradicting $a_n \neq b_n$.

(iv) Let $c \in FS(X)_{\leq \gamma} - FS(X)^{\mathbb{R}}$. Then we can find for each natural number n a real number τ_n and $g_n \in G$ satisfying $\gamma + \frac{1}{n} > \tau_n > 0$ and $g_n c(t) = c(t + \tau_n)$. In particular, c is a geodesic that is an axis for each g_n and $l(g_n) = \tau_n$. After passing to a subsequence we can assume that $\tau_n \rightarrow \tau_0$. Thus, $d_X(g_n c(0), c(\tau_0)) \rightarrow 0$. Because of Lemma 4.6 $l(g_n) = \tau_n = \tau_0$ for infinitely many n . For such an n we have $g_n \in G_{\leq \gamma}^{\text{hyp}}$. Now assume additionally that $c(t) \in K$ for some t . If a is the equivalence class of such a g_n , then $c \in FS_a$ and $\Phi_{\mathbb{R}}(c) = q_a^{-1}(q_a(c))$. \square

Proposition 4.11. *Let $\gamma > 0$ and $a \in A_{\leq \gamma}$. Then*

- (i) $p_a: FS_a \rightarrow X$ is an isometric embedding with closed image;
- (ii) there is a G_a -invariant metric d_a on Y_a that generates the topology; with this metric Y_a is a proper metric space;
- (iii) there is $\tau_a: FS_a \rightarrow \mathbb{R}$, such that $c \mapsto (q_a(c), \tau_a(c))$ defines an isometry $FS_a \rightarrow Y_a \times \mathbb{R}$ which is compatible with the flow, i.e., $\tau_a(\Phi_t(c)) = \tau_a(c) + t$ for $t \in \mathbb{R}$, $c \in FS_a$;
- (iv) for $y \in Y_a$, $G_y := \{g \in G_a \mid gy = y\}$ is virtually cyclic of type I and $G_a y \subseteq Y_a$ is discrete;

Proof. (i) If c and d are parallel, then $d_X(c(t), d(t))$ is constant by definition. An easy computation shows $d_X(c(0), d(0)) = d_{FS}(c, d)$. Thus p_a is an isometry. It remains to show that $p_a(FS_a)$ is closed. Let $c_n \in FS_a$ such that $c_n(0) \rightarrow x \in X$. Because $c \mapsto c(0)$ is proper (Lemma 1.10) we can pass to a subsequence and assume that $c_n \rightarrow c$ in $FS(X)$. Then $c \in FS_a$ by Lemma 4.9 and $c(0) = x$.

(ii) and (iii) Let FS_a^+ be the subset of all d that are parallel to some (and therefore

all) $c \in FS_a$. Define

$$p_a^+ : FS_a^+ \rightarrow X, \quad c \mapsto c(0).$$

By the argument from the proof of assertion (i) $p_a^+ : FS_a^+ \rightarrow X$ is an isometric embedding. It follows from [4, II.2.14 on p.183] that there is convex subspace Y_a^+ of X (which is therefore a CAT(0)-space) and an isometry $\psi : Y_a^+ \times \mathbb{R} \rightarrow FS_a^+$ such that $\Phi_t(\psi(y, s)) = \psi(y, s + t)$ for $s, t \in \mathbb{R}$ and $y \in Y_a^+$. This identifies Y_a with a subspace of Y_a^+ and provides the metric on Y_a . If $g \in G_a$, then the action of g on X permutes the images of geodesics $c \in FS_a^+$. It follows from [4, I.5.3(4) on p.56] that the induced action of g on Y_a^+ (and therefore also the induced action on Y_a) is isometric. Since X is proper by assumption, FS_a is proper by assertion (i). Since FS_a is isometric to $Y_a \times \mathbb{R}$, the metric space Y_a is proper.

(iv) By Lemma 4.4 G_y is virtually cyclic of type I. We proceed by contradiction to show that $G_a y$ is discrete. Assume that there are $g_n \in G_a$, $n \in \mathbb{N}$ such that $g_n y \neq g_m y$ if $n \neq m$ and $g_n y \rightarrow y_0$. Pick $c \in FS_a$ such that $q_a(c) = y$ and $\tau_a(c) = 0$. Pick $g \in a \subseteq G_{\leq \gamma}^{hyp}$ such that c is an axis for g . Then $gc = \Phi_{l(g)}(c)$, see Lemma 4.3 (iii). Thus $\tau_a(g_n g g_n^{-1}(g_n c)) = l(g) + \tau_a(g_n c)$ and $(g_n g g_n^{-1})g_n y = g_n y$. By replacing g_n by $(g_n g g_n^{-1})^{l_n} g_n$ for some $l_n \in \mathbb{Z}$ we can arrange that $\tau_a(g_n c) \in [0, l(g)]$. By passing to a subsequence we can arrange that the sequence $\tau_a(g_n c)$ converges. Since $q_a(g_n c) = g_n c$ converges to y_0 , we conclude from (iii) that $g_n c \rightarrow d$ as $n \rightarrow \infty$ for some $d \in FS(X)$. Now Lemma 4.6 implies that $g_n c = g_m c$ for infinitely many n, m . This contradicts $g_n y \neq g_m y$ for $n \neq m$. \square

In the proofs of the next two results we will denote by $\pi_a : Y_a \rightarrow G_a \backslash Y_a$ the quotient map. We point out that π_a is open, since for any open subset $V \subseteq Y_a$ the subset $\pi_a^{-1}(\pi_a(V)) = \bigcup_{g \in G_a} g \cdot V$ is open.

Lemma 4.12. *We have*

$$\dim(G_a \backslash Y_a) \leq \dim(X).$$

Proof. Let $K_a \subseteq G \cdot FS_a$ be compact such that $G \cdot K_a = G \cdot FS_a$, see Lemma 4.10 (ii). Using Lemma 4.10 (iii) we conclude that there is a compact subset $K'_a \subseteq FS_a$ such that $FS_a = G_a \cdot K'_a$. Let $K''_a \subseteq Y_a$ be the compact subset $q_a(K'_a)$. Define $U := B_1(K''_a)$. Then U is an open subset of Y_a with $Y_a = G_a \cdot U$. Let $i : U \rightarrow Y_a$ be the inclusion. Since U is open, i is open. As pointed out above, $\pi_a : Y_a \rightarrow G_a \backslash Y_a$ is open. Hence the composite $\pi_a \circ i : U \rightarrow G_a \backslash Y_a$ is open and surjective. Since Y_a is a proper metric space by Proposition 4.11 (ii) and $K''_a \subseteq Y_a$ is compact, the set $\overline{B}_1(K''_a)$ is compact subset of Y_a . Since for every $y \in Y$ the orbit $G_a y \subseteq Y_a$ is discrete by Proposition 4.11 (iv), the intersection $\overline{B}_1(K''_a) \cap G_a y$ and hence also the intersection $U \cap G_a y$ is finite. Hence the composite $\pi_a \circ i : U \rightarrow G_a \backslash Y_a$ is finite-to-one. Its source is a metric as U is a subspace of the metric space Y_a . Since G_a acts isometrically on Y_a and for every $y \in Y$ the orbit $G_a y \subseteq Y_a$ is discrete by Proposition 4.11 (iv), the quotient $G_a \backslash Y_a$ is also a metric space. Since every metric space is paracompact by Stone's Theorem (see [10, Theorem 4.3 in Chapter VI on p. 256], the composite $\pi_a \circ i : U \rightarrow G_a \backslash Y_a$ is a finite-to-one open surjective map of paracompact spaces. Hence we conclude from [11, 4.1 on p.35]

$$\dim(U) = \dim(G_a \backslash Y_a).$$

Since Y_a is a proper metric space by Lemma 4.11 (ii) it is locally compact and can be written as the countable union of compact subspaces and hence contains a countable dense subset. Hence the open subset U is locally compact and has a countable basis for its topology. Since any compact subset $L \subset U$ is a closed subset of X and satisfies $\dim(L) \leq \dim(X)$, we conclude $\dim(U) \leq \dim(X)$ from [10, Exercise 9 in Chapter 7.9 on p. 315]. This finishes the proof of Lemma 4.12. \square

Proposition 4.13. *Let $\gamma > 0$ and $a \in A_{\leq \gamma}$. There is an open \mathcal{VCyc} -cover \mathcal{V}_a of Y_a such that*

- (i) $\dim \mathcal{V}_a \leq \dim X$;
- (ii) \mathcal{V}_a is G_a -invariant, i.e., $g \cdot V \in \mathcal{V}_a$ if $g \in G_a$ and $V \in \mathcal{V}_a$;
- (iii) $G_a \backslash \mathcal{V}_a$ is finite.

Proof. Because of Proposition 4.11 (ii) and (iv) for any $y \in Y_a$, the open ball of sufficiently small radius is \mathcal{VCyc} -neighborhood for y . Pick for each y such a ball V_y . Because $\pi_a: Y_a \rightarrow G_a \backslash Y_a$ is open, $\{\pi_a(V_y) \mid y \in Y_a\}$ is an open cover of $G_a \backslash Y_a$. By Lemma 4.12 it has a refinement \mathcal{W} whose dimension is bounded by $\dim X$. The G_a -action on Y_a is cocompact because it is cocompact on FS_a , see Lemma 4.10 (ii). Therefore $G_a \backslash Y_a$ is compact. Thus we may assume that \mathcal{W} is finite. For any $W \in \mathcal{W}$ pick $y_W \in FS_a$ such that $W \subseteq \pi_a(V_{y_W})$. Now define $\mathcal{V}_a := \{\pi_a^{-1}(W) \cap gV_{y_W} \mid W \in \mathcal{W}, g \in G_a\}$. This is an open \mathcal{VCyc} -cover because each V_y is an open \mathcal{VCyc} -set. Its dimension is bounded by $\dim X$, because the dimension of \mathcal{W} is bounded by $\dim X$ and because for $g \in G_a$ and $y \in Y$ we have either $V_y = gV_{y_W}$ or $V_y \cap gV_{y_W} = \emptyset$. It is G_a -invariant because each $\pi_a^{-1}(W)$ is G_a -invariant. Finally, $G_a \backslash \mathcal{V}_a$ is finite because \mathcal{W} is finite. \square

4.3. The cover \mathcal{V} .

Lemma 4.14. *Let (Z, d_Z) be a metric space with an action of a group H by isometries. Let A be a H -invariant subspace. For $\emptyset \neq U \subsetneq A$, we define*

$$Z(U) := \{z \in Z \mid d_Z(z, U) < d_Z(z, A - U)\}$$

and set $Z(A) := Z$, $Z(\emptyset) := \emptyset$. Then for $U, V \subseteq A$,

- (i) $Z(U)$ is open in Z ;
- (ii) $Z(U \cap V) = Z(U) \cap Z(V)$;
- (iii) $Z(U) \cap A = U$ holds if and only if U is open in A ;
- (iv) for all $g \in H$ we have $Z(gU) = gZ(U)$.

Proof. (i) Either $Z(U)$ is \emptyset or Z or can be written as the preimage of $(0, \infty)$ for a continuous function on Z . Hence $Z(U)$ is open for every $U \subseteq Z$.

(ii) This is obviously true if $U = A$, $U = \emptyset$, $V = \emptyset$ or $V = A$ holds, so we can assume without loss of generality $\emptyset \neq U \neq A$ and $\emptyset \neq V \neq A$ in the sequel. Recall that $d_Z(z, U) := \inf\{d_Z(z, u) \mid u \in U\}$ for $z \in Z$ and $\emptyset \neq U \subseteq Z$. One easily checks that $d_Z(z, U), d_Z(z, V) \leq d_Z(z, U \cap V)$ and $d_Z(z, A - (U \cap V)) = \min\{d_Z(z, A - U), d_Z(z, A - V)\}$ hold for $z \in Z$ and open subsets $U, V \subseteq A$. This implies $Z(U \cap V) \subseteq Z(U) \cap Z(V)$.

It remains to show $Z(U) \cap Z(V) \subseteq Z(U \cap V)$. Let $z \in Z(U) \cap Z(V)$. Because of $d_Z(z, U) < d_Z(z, A - U)$ there is $u \in U$ with $d_Z(z, u) < d_Z(z, A - U)$. Because of $d_Z(z, V) < d_Z(z, A - V)$ there is $v \in V$ with $d_Z(z, v) < d_Z(z, A - V)$. If $u \notin V$ then $d_Z(z, v) < d_Z(z, u)$. If $v \notin U$ then $d_Z(z, u) < d_Z(z, v)$. In particular we have $u \in V$ or $v \in U$.

Suppose that $u \notin V$. Then $v \in U$ and $d_Z(z, v) < d_Z(z, u) < d_Z(z, A - U)$. Thus

$$\begin{aligned} d_Z(z, V \cap U) &\leq d_Z(z, v) < \min\{d_Z(z, A - U), d_Z(z, A - V)\} \\ &= d_Z(z, A - (U \cap V)) \end{aligned}$$

and $z \in Z(U \cap V)$. Analogously one shows $z \in Z(U \cap V)$ if $v \notin U$. It remains to treat the case, where $u \in V$ and $v \in U$, or, equivalently, where $u, v \in U \cap V$. We may assume without loss of generality $d_Z(z, v) \leq d_Z(z, u)$. Then $d_Z(z, v) \leq$

$d_Z(z, u) < d_Z(z, A - U)$. Thus

$$\begin{aligned} d_Z(z, V \cap U) &\leq d_Z(z, v) < \min\{d_Z(z, A - U), d_Z(z, A - V)\} \\ &= d_Z(z, A - (U \cap V)) \end{aligned}$$

and $z \in Z(U \cap V)$. This proves $Z(U \cap V) = Z(U) \cap Z(V)$.

(iii) If $U = \emptyset$ or $U = A$, this is obvious so that it suffices to treat the case $\emptyset \neq U \neq A$. If $Z(U) \cap A = U$, then U is open in A , because $Z(U)$ is open in Z by (i).

Assume that U is open in A . Consider $z \in U$. Since $U \subseteq A$ is open, there exists $\epsilon > 0$ such that $\{y \in Z \mid d_Z(z, y) < \epsilon\} \cap A \subseteq U$. Hence $d_Z(x, z) \geq \epsilon$ for $x \in A - U$ and hence $d_Z(z, A - U) \geq \epsilon > 0 = d_Z(z, U)$. Therefore $z \in Z(U) \cap A$. Consider $z \in Z(U) \cap A$. Since $d_Z(z, U) < d_Z(z, A - U)$ implies $d_Z(z, A - U) > 0$ and hence $z \notin A - U$, we conclude $z \in U$. This proves $Z(U) \cap A = U$.

(iv) This is obvious as H acts by isometries. \square

Lemma 4.15. *There is $\epsilon_{\mathbb{R}} > 0$ and a G -invariant cofinite collection $\mathcal{V}_{\mathbb{R}}$ of open \mathcal{F} in-subsets of $FS(X)$ such that*

- (i) $\dim \mathcal{V}_{\mathbb{R}} < \infty$;
- (ii) *if the image of $c \in FS(X)^{\mathbb{R}}$ as a generalized geodesic in X (which is a point) intersects (and is therefore contained in) $G \cdot K$, then there is $U \in \mathcal{V}_{\mathbb{R}}$ such that $B_{\epsilon_{\mathbb{R}}}(c) = B_{\epsilon_{\mathbb{R}}}(\Phi_{\mathbb{R}}(c)) \subseteq U$.*

Proof. By Lemma 1.10 the evaluation map $FS(X) \rightarrow X$ defined by $c \mapsto c(0)$ is proper. As $FS(X)^{\mathbb{R}}$ is closed, this implies that there is a compact subset $K_{\mathbb{R}} \subseteq FS(X)^{\mathbb{R}}$ such that $G \cdot K_{\mathbb{R}}$ contains all $c \in FS(X)^{\mathbb{R}}$ whose image lies in $G \cdot K$. For any $c \in K_{\mathbb{R}}$ there is an open \mathcal{F} in-subset V_c of $FS(X)$ containing c , because the action of G on $FS(X)$ is proper (see Proposition 1.11). Because $K_{\mathbb{R}}$ is compact there is $\epsilon_{\mathbb{R}} > 0$ and a finite subset Λ of $K_{\mathbb{R}}$ such that for any $c \in K_{\mathbb{R}}$ there is $\lambda_c \in \Lambda$ such that $B_{\epsilon_{\mathbb{R}}}(c) \subseteq V_{\lambda_c}$. Now set $\mathcal{V}_{\mathbb{R}} := \{gV_{\lambda} \mid g \in G, \lambda \in \Lambda\}$. This is a finite dimensional collection, because the gV_{λ} are \mathcal{F} in-sets and Λ is finite. \square

Proof of Theorem 4.2. Let $R \subseteq A_{\leq \gamma}$ be a subset that contains exactly one element from each orbit of the G -action. Then R is finite by Lemma 4.10 (i). For each $a \in R$, let \mathcal{V}_a be a covering of Y_a satisfying the assertions from Proposition 4.13. Let $\mathcal{W}_a := (q_a)^{-1}\mathcal{V}_a = \{q_a^{-1}(V) \mid V \in \mathcal{V}_a\}$. For $b \in A_{\leq \gamma}$ pick $a \in R$ and $g \in G$, such that $ga = b$ and set $\mathcal{W}_b := g(\mathcal{W}_a) = \{gW \mid W \in \mathcal{W}_a\}$. (This does not depend on the choice of g , as q_a is G_a -equivariant and \mathcal{V}_a is G_a -invariant.) By Lemma 4.10 (iii) there is $\delta > 0$ such that, if we set $U_a := B_{\delta}(FS_a)$, then $U_a \cap U_b = \emptyset$ for $a \neq b \in A_{\leq \gamma}$. We now use Lemma 4.14 to extend the $W \in \mathcal{W}_a$ to open subsets of $FS(X)$ and define the collection \mathcal{U} by

$$\mathcal{U} := \bigcup_{a \in A_{\leq \gamma}} \{Z_a(W) \cap U_a \mid W \in \mathcal{W}_a\},$$

where $Z_a(W) := \{c \in FS(X) \mid d_{FS}(c, FS_a) < d_{FS}(c, FS(X) - FS_a)\}$. Define the desired collection of open subsets of $FS(X)$ by

$$\mathcal{V} := \mathcal{U} \cup \mathcal{V}_{\mathbb{R}},$$

where $\mathcal{V}_{\mathbb{R}}$ is from Lemma 4.15. It remains to show, that \mathcal{V} has the desired properties.

(i) The members of each \mathcal{V}_a are open \mathcal{V} Cyc-sets with respect to the G_a -action by Proposition 4.13. The q_a are continuous and G_a -equivariant by construction. Thus the members of each \mathcal{W}_a are also open \mathcal{V} Cyc-sets with respect to the G_a -action. For each a , FS_a is a G_a -set. Because the U_a are mutually disjoint, each U_a is an G_a -set as well. By Lemma 4.14 each $Z_a(W) \cap U_a$ with $W \in \mathcal{W}_a$ is an open \mathcal{V} Cyc-subset of

$FS(X)$ with respect to the G -action.

(ii) Each \mathcal{V}_a is G_a -invariant. The union of the \mathcal{W}_a is G -invariant, because the q_a are G_a -equivariant. Thus by Lemma 4.14 the collection of all $Z_a(W)$ is G -invariant. The collection of the U_a is G -invariant. Therefore \mathcal{U} is G -invariant. Since $\mathcal{U}_{\mathbb{R}}$ is G -invariant this implies that \mathcal{V} is G -invariant.

(iii) Each $G_a \backslash \mathcal{V}_a$ is finite by Proposition 4.13 (iii). Therefore $G \backslash \mathcal{U}$ is finite. Since $G \backslash \mathcal{U}_{\mathbb{R}}$ is finite, $G \backslash \mathcal{V}$ is finite.

(iv) The U_a are mutually disjoint. For each a , $\dim \mathcal{W}_a = \dim \mathcal{V}_a$. Using Proposition 4.13 (i) and Lemma 4.14 we get therefore

$$\begin{aligned} \dim \mathcal{U} &= \max_{a \in A} \dim \{Z_a(W) \cap U_a \mid W \in \mathcal{W}_a\} \\ &= \max_{a \in R} \dim \mathcal{V}_a \leq \dim X. \end{aligned}$$

Put $M := 1 + \dim(\mathcal{V}_{\mathbb{R}}) + \dim(X)$. This number is independent of γ and

$$\dim(\mathcal{V}) \leq 1 + \dim(\mathcal{V}_{\mathbb{R}}) + \dim(\mathcal{U}) = M.$$

(v) Let $c \in FS(X)$ such that c intersects $G \cdot K$. If $c \in FS(X)^{\mathbb{R}}$, then $B_{\varepsilon_{\mathbb{R}}}(\Phi_{\mathbb{R}}(c)) = B_{\varepsilon_{\mathbb{R}}}(c) \subseteq U$ for some $U \in \mathcal{V}_{\mathbb{R}}$, where $\varepsilon_{\mathbb{R}}$ is from Lemma 4.15. Hence it remains to show that there is $\varepsilon_{\mathcal{U}}$ such that for any $c \in FS(X)_{\leq \gamma} - FS(X)^{\mathbb{R}}$ such that c intersects $G \cdot K$, there is $U \in \mathcal{U}$ satisfying $B_{\varepsilon_{\mathcal{U}}}(\Phi_{[-\gamma, \gamma]}(c)) \subseteq U$ because then we can take $\varepsilon = \min\{\varepsilon_{\mathbb{R}}, \varepsilon_{\mathcal{U}}\}$.

Now suppose that the desired $\varepsilon_{\mathcal{U}}$ does not exist, i.e., there are $c_n \in FS(X)_{\leq \gamma} - FS(X)^{\mathbb{R}}$, $d_n \in FS(X)$ and $t_n \in \mathbb{R}$ for $n \geq 1$ such that c_n intersects $G \cdot K$, $d_{FS}(\Phi_{t_n}(c_n), d_n) < 1/n$ and $d_n \notin U$ for all $U \in \mathcal{U}$ that contain $\Phi_{[-\gamma, \gamma]}(c_n)$. Choose $a_n \in R$ with $c_n \in G \cdot FS_{a_n}$. As R is finite, we arrange by passing to a subsequence that there is $a \in R$ with $a_n = a$ for all n .

Because of Lemma 4.10 (ii) there are $g_n \in G_a$ such that all $g_n c_n$ are contained in a compact set $K_a \subset G \cdot FS_a$. After passing to a subsequence and replacing c_n by $g_n c_n$ and d_n by $g_n d_n$ we can assume that $c_n \rightarrow c$ for some $c \in K_a \subset G \cdot FS_a$. Choose $g \in G$ with $g^{-1}c \in FS_a$. Choose $V \in \mathcal{V}_a$ with $q_a(g^{-1}c) \in V$. Then $g^{-1}c \in q_a^{-1}(V)$ and hence $\Phi_{\mathbb{R}}(g^{-1}c) \subset q_a^{-1}(V)$. Thus $\Phi_{\mathbb{R}}(c) \subseteq U$ for some $U \in \mathcal{U}$, namely for $U = g \cdot Z_a(W) \cap U_{g_a}$, where we set $W := q_a^{-1}(V)$.

Passing to a further subsequence we can arrange that $t_n \rightarrow t_0$ for some $t_0 \in [-\gamma, \gamma]$. Then also $d_n \rightarrow \Phi_{t_0}(c)$. Hence there is n_0 such that $d_n \in U$ for $n \geq n_0$, because $d_n \rightarrow \Phi_{t_0}(c) \in U$.

The set $\Phi_{[-\gamma, \gamma]}(c)$ is a compact subset of the open set U . Hence we can find $\delta > 0$ such that $B_{\delta}(\Phi_{[-\gamma, \gamma]}(c)) \subseteq U$. We have $d_X(\Phi_t(c), \Phi_t(c_n)) \leq e^{\tau} \cdot d_{FS(X)}(c_n, c)$ for $n \geq 1$ and $t \in [-\tau, \tau]$ by Lemma 1.3. Since there exists n_1 such that $d_{FS(X)}(c_n, c) < e^{-\tau} \cdot \delta$ for $n \geq n_1$, we get $\Phi_{[-\gamma, \gamma]}(c_n) \subseteq U$ for $n \geq n_1$, a contradiction. This finishes the proof of Theorem 4.2. \square

5. FLOW SPACES AND S -LONG COVERS

Summary. In Definitions 5.5 and 5.9 we formulate two conditions for a flow space FS with an action of a group G . The first condition asks for the existence of long covers of uniformly bounded dimension for a subset of FS that contains the periodic orbits of the flow and is large in the sense that its complement is cocompact for the action of G . (In our application later the action of G on FS will be cocompact and we will get the second part of this condition for free; we expect however that this condition can also be verified in situations where the action is not cocompact.) The second condition concerns the dynamic of the flow with respect to a suitable

homotopy action. Our main result is Proposition 5.11 which asserts that G is transfer reducible, provided the two conditions are satisfied.

In this section we fix the following convention, compare [2, Convention 1.3].

Convention 5.1. *Let*

- G be a group;
- \mathcal{F} be a family of subgroups of G ;
- (FS, d_{FS}) be a locally compact metric space with a proper isometric G -action;
- $\Phi: FS \times \mathbb{R} \rightarrow FS$ be a flow.

We assume that the following conditions are satisfied:

- Φ is G -equivariant;
- $FS - FS^{\mathbb{R}}$ is locally connected;
- $k_G := \sup\{|H| \mid H \subseteq G \text{ subgroup with finite order } |H|\} < \infty$;
- $\dim(FS - FS^{\mathbb{R}}) < \infty$;
- the flow is uniformly continuous in the following sense: for $\alpha > 0$ and $\varepsilon > 0$ there is $\delta > 0$ such that

$$(5.2) \quad d_{FS}(z, z') \leq \delta, \tau \in [-\alpha, \alpha] \implies d_{FS}(\Phi_\tau(z), \Phi_\tau(z')) \leq \varepsilon.$$

For a subset $I \subseteq \mathbb{R}$ we set $\Phi_I(z) := \{\Phi_t(z) \mid t \in I\}$. For $z \in FS$ we define its G -period

$$\text{per}_{\Phi}^G(z) = \inf\{\tau \mid \tau > 0, \exists g \in G \text{ with } \Phi_\tau(z) = gz\} \in [0, \infty],$$

where the infimum over the empty set is defined to be ∞ . Obviously $\text{per}_{\Phi}^G(z) = 0$ if and only if $z \in FS^{\mathbb{R}}$. If $L \subseteq FS$ is an orbit of the flow Φ , define its G -period by

$$\text{per}_{\Phi}^G(L) := \text{per}_{\Phi}^G(z)$$

for any choice of $z \in FS$ with $L = \Phi_{\mathbb{R}}(z)$. For $\gamma \geq 0$ put

$$(5.3) \quad FS_{>\gamma} := \{z \in FS \mid \text{per}_{\Phi}^G(z) > \gamma\};$$

$$(5.4) \quad FS_{\leq\gamma} := \{z \in FS \mid \text{per}_{\Phi}^G(z) \leq \gamma\}.$$

Definition 5.5. We will say that FS admits long \mathcal{F} -covers at infinity and periodic flow lines if the following holds:

There is $M > 0$ such that for every $\gamma > 0$ there is a collection \mathcal{V} of open \mathcal{F} -subsets of FS and $\varepsilon > 0$ satisfying:

- (i) \mathcal{V} is G -invariant: $g \in G, V \in \mathcal{V} \implies gV \in \mathcal{V}$;
- (ii) $\dim \mathcal{V} \leq M$;
- (iii) there is a compact subset $K \subseteq FS$ such that
 - $FS_{\leq\gamma} \cap G \cdot K = \emptyset$;
 - for $z \in FS - G \cdot K$ there is $V \in \mathcal{V}$ such that $B_\varepsilon(\Phi_{[-\gamma, \gamma]}(z)) \subset V$.

Theorem 5.6. *There is $M \in \mathbb{N}$ such that the following holds:*

For any $\alpha > 0$ there is $\gamma > 0$ such that for any compact subset K of $FS_{>\gamma}$ there is a collection of open \mathcal{VCyc} -subsets of FS such that

- (i) \mathcal{V} is G -invariant: $g \in G, V \in \mathcal{V} \implies gV \in \mathcal{V}$;
- (ii) $\dim \mathcal{V} \leq M$;
- (iii) $G \setminus \mathcal{V}$ is finite;
- (iv) for every $z \in G \cdot K$ there is $V \in \mathcal{V}$ such that $\Phi_{[-\alpha, \alpha]}(z) \subset V$.

Proof. This follows from the techniques used and developed in Sections 2-5 of [2], but is unfortunately not stated in precisely this form.

The main input is Proposition 4.1 of [2]. In this reference it is assumed that G acts cocompactly on FS , but this is not used in its proof, mainly because the

statement concerns only a cocompact part of the flow space. (In [2] cocompactness of the action is used to conclude that the flow space is locally compact; note that we assumed this in Convention 5.1.) Therefore Proposition 4.1 of [2] is valid in the present situation as well. Theorem 5.6 can be deduced from this using the argument given on p.1848 of [2]. \square

Theorem 5.7. *Assume that FS admits long \mathcal{F} -covers at infinity and periodic flow lines and that \mathcal{F} contains the family \mathcal{VCyc} of virtually cyclic subgroups.*

Then there is $\widehat{N} \in \mathbb{N}$ such that for every $\alpha > 0$ there exists an open \mathcal{F} -cover \mathcal{U} of FS of dimension at most \widehat{N} and $\epsilon > 0$ (depending on α) such that the following holds:

- (i) *For every $z \in FS$ there is $U \in \mathcal{U}$ such that $B_\epsilon(\Phi_{[-\alpha, \alpha]}(z)) \subseteq U$.*
- (ii) *\mathcal{U}/G is finite.*

Proof. Let M_0 be the number M appearing in Definition 5.5 and M_1 be the number M appearing in Theorem 5.6. We set $\widehat{N} := M_0 + M_1 + 1$. Theorem 5.6 also provides a number γ depending on α . We can assume that $\gamma \geq \alpha$. Since FS admits long \mathcal{F} -covers at infinity and periodic flow lines we can find a collection \mathcal{V}_0 of open \mathcal{F} -subsets of FS and $\epsilon_0 > 0$ such that (i) to (iii) from Definition 5.5 hold.

Next we apply Theorem 5.6 and obtain a collection \mathcal{V}_1 of open \mathcal{VCyc} -subsets of FS and a compact subset $K \subseteq FS$ such that (i) to (iv) from Theorem 5.6 hold. A simple compactness argument, provided in Lemma 5.8 below, shows that there is $\epsilon_1 > 0$ such that for every $z \in G \cdot K$ there is $V \in \mathcal{V}_1$ such that

$$B_{\epsilon_1}(\Phi_{[-\alpha, \alpha]}(z)) \subset V.$$

Now set $\epsilon := \min\{\epsilon_0, \epsilon_1\}$ and $\mathcal{U} := \mathcal{V}_0 \cup \mathcal{V}_1$. \square

Lemma 5.8. *Assertion (iv) in Theorem 5.6 can be strengthened to*

- (iv') *There is $\epsilon > 0$ such that for any $z \in G \cdot K$ there is $V \in \mathcal{V}$ such that $B_\epsilon(\Phi_{[-\alpha, \alpha]}(z)) \subset V$.*

Proof. Suppose that there is no such ϵ . Then we can find a sequence $(z_n)_{n \geq 1}$ of points in $G \cdot K$ such that $B_{1/n}(\Phi_{[-\alpha, \alpha]}(z_n)) \not\subseteq U$ holds for every $n \geq 1$ and every $U \in \mathcal{V}$. Because \mathcal{V} is G -invariant and Φ is G -equivariant, we can assume without loss of generality that $z_n \in K$ for all n . After passing to a subsequence we can assume that $z_n \rightarrow z$ as $n \rightarrow \infty$. Choose $V \in \mathcal{V}$ with $\Phi_{[-\alpha, \alpha]}(z) \subseteq V$. Since $\Phi_{[-\alpha, \alpha]}(z)$ is compact and V is open, we can find $\mu > 0$ with $B_\mu(\Phi_{[-\alpha, \alpha]}(z)) \subseteq V$. Because of the uniform continuity of the flow, we can find $\delta > 0$ such that $d_{FS}(\Phi_\tau(z), \Phi_\tau(z')) < \mu/2$ holds for all $\tau \in [-\alpha, \alpha]$ provided that $d_{FS}(z, z') \leq \delta$ is true. Since $\lim_{n \rightarrow \infty} z_n = z$, we can find $n \geq 1$ such that $d_{FS}(z, z_n) < \delta$ and $1/n < \mu/2$ hold. This implies $d_{FS}(\Phi_\tau(z), \Phi_\tau(z_n)) < \mu/2$ for all $\tau \in [-\alpha, \alpha]$. Thus

$$B_{1/n}(\Phi_{[-\alpha, \alpha]}(z_n)) \subseteq B_\mu(\Phi_{[-\alpha, \alpha]}(z)) \subseteq V,$$

contradicting the assumption. \square

Definition 5.9. We will say that FS admits *contracting transfers* if for every finite subset S of G (containing e) there exists $\beta > 0$ and $\widetilde{N} \in \mathbb{N}$ such that the following holds:

For every $\delta > 0$ there is

- (i) $T > 0$;
- (ii) a contractible compact controlled \widetilde{N} -dominated space X ;
- (iii) a homotopy S -action (φ, H) on X ;
- (iv) a G -equivariant map $\iota: G \times X \rightarrow FS$ (where we use the left action $g \cdot (h, x) = (gh, x)$ on $G \times X$),

such that

(5.10) for every $(g, x) \in G \times X$, $s \in S$, $f \in F_s(\varphi, H)$ there is $\tau \in [-\beta, \beta]$ such that

$$d_{FS}(\Phi_T(\iota(g, x)), \Phi_{T+\tau}(\iota(gs^{-1}, f(x)))) \leq \delta.$$

Proposition 5.11. *If FS satisfies the assumptions appearing in Convention 5.1, admits long covers at infinity and periodic orbits (see Definition 5.5), and admits contracting transfers (see Definition 5.9), then G is transfer reducible over the family \mathcal{F} in the sense of Definition 0.4.*

The proof of Proposition 5.11 will use the following the lemma.

Lemma 5.12. *Let $\beta > 0$, $T > 0$ and $\epsilon > 0$. Let S be a finite subset of G (containing e). Set $n := |S|$ and $\alpha := 2n\beta$. Pick δ such that (5.2) holds. Let (φ, H) be a homotopy S -action on a compact metric space X and let $\iota: G \times X \rightarrow FS$ be a G -equivariant map. Assume (5.10) holds.*

Then

(5.13) for every $(g, x) \in G \times X$ and $(h, y) \in S_{\varphi, H}^n(g, x)$ (see Definition 0.1) there is $\tau \in [-\alpha, \alpha]$ such that

$$d_{FS}(\Phi_T(\iota(g, x)), \Phi_{T+\tau}(\iota(h, y))) \leq 2n\epsilon.$$

Proof. If $(h, y) \in S_{\varphi, H}^n(g, x)$, then there are $x_0, \dots, x_n \in X$, $a_1, b_1, \dots, a_n, b_n \in S$, $f_1, \tilde{f}_1, \dots, f_n, \tilde{f}_n: X \rightarrow X$, such that $x_0 = x$, $x_n = y$, $f_i \in F_{a_i}(\varphi, H)$, $\tilde{f}_i \in F_{b_i}(\varphi, H)$, $f_i(x_{i-1}) = \tilde{f}_i(x_i)$ and $h = ga_1^{-1}b_1 \dots a_n^{-1}b_n$. Set $g_i := ga_1^{-1}b_1 \dots a_i^{-1}b_i$. By (5.10) there are $\tau_1, \tilde{\tau}_1, \dots, \tau_n, \tilde{\tau}_n \in [-\beta, \beta]$ such that

$$\begin{aligned} d_{FS}(\Phi_T(\iota(g_{i-1}, x_{i-1})), \Phi_{T+\tau_i}(\iota(g_{i-1}a_i^{-1}, f_i(x_{i-1})))) &\leq \delta; \\ d_{FS}(\Phi_T(\iota(g_i, x_i)), \Phi_{T+\tilde{\tau}_i}(\iota(g_i b_i^{-1}, \tilde{f}_i(x_i)))) &\leq \delta \end{aligned}$$

for $i = 1, \dots, n$. Put $\sigma_0 = 0$ and $\sigma_i := (-\tilde{\tau}_1 + \tau_1) + \dots + (-\tilde{\tau}_i + \tau_i)$ for $i = 1, 2, \dots, n$. Since $\sigma_i \in [-\alpha, \alpha]$, we conclude for $i = 1, \dots, n$ from (5.2)

$$\begin{aligned} d_{FS}(\Phi_{T+\sigma_{i-1}}(\iota(g_{i-1}, x_{i-1})), \Phi_{T+\tau_i+\sigma_{i-1}}(\iota(g_{i-1}a_i^{-1}, f_i(x_{i-1})))) &\leq \epsilon; \\ d_{FS}(\Phi_{T+\sigma_i}(\iota(g_i, x_i)), \Phi_{T+\tilde{\tau}_i+\sigma_i}(\iota(g_i b_i^{-1}, \tilde{f}_i(x_i)))) &\leq \epsilon. \end{aligned}$$

Since $g_{i-1}a_i^{-1} = g_i b_i^{-1}$, $f_i(x_{i-1}) = \tilde{f}_i(x_i)$ and $T + \tau_i + \sigma_{i-1} = T + \tilde{\tau}_i + \sigma_i$ we conclude for $i = 1, \dots, n$ from the triangle inequality

$$d_{FS}(\Phi_{T+\sigma_{i-1}}(\iota(g_{i-1}, x_{i-1})), \Phi_{T+\sigma_i}(\iota(g_i, x_i))) \leq 2\epsilon.$$

Using the triangle inequality we obtain

$$\begin{aligned} &d_{FS}(\Phi_T(\iota(g, x)), \Phi_{T+\sigma_n}(\iota(h, y))) \\ &= d_{FS}(\Phi_T(\iota(g_0, x_0)), \Phi_{T+\sigma_n}(\iota(g_n, x_n))) \\ &\leq \sum_{i=1}^n d_{FS}(\Phi_{T+\sigma_{i-1}}(\iota(g_{i-1}, x_{i-1})), \Phi_{T+\sigma_i}(\iota(g_i, x_i))) \\ &\leq \sum_{i=1}^n 2\epsilon \\ &\leq 2n\epsilon. \end{aligned}$$

□

Proof of Proposition 5.11. Let S be a finite subset of G . Let \hat{N} be the number from Theorem 5.7. Let \tilde{N} and β be the numbers (depending on S) appearing in Definition 5.9. Put $\alpha := 2\beta|S|$. By Theorem 5.7 there is an open \mathcal{F} -cover \mathcal{U} of FS

of dimension at most \widehat{N} and $\epsilon_0 > 0$ with the property that for every $z \in FS$ there is $U_z \in \mathcal{U}$ such that

$$B_{\epsilon_0}(\Phi_{[-\alpha, \alpha]}(z)) \subseteq U_z.$$

Put $\epsilon := \frac{\epsilon_0}{2|S|}$. Pick $\delta > 0$ such that (5.2) holds. By assumption (see Definition 5.9), there are $T > 0$, a contractible compact controlled \widetilde{N} -dominated space X , a G -equivariant map $\iota: G \times X \rightarrow FS$ and a homotopy S -action (φ, H) on X such that (5.10) holds. Using Lemma 5.12 we conclude that for every $(g, x) \in G \times X$ we have

$$\Phi_T(\iota(h, y)) \in B_{\epsilon_0}(\Phi_{[-\alpha, \alpha]}(\Phi_T(\iota(g, x))))$$

for all $(h, y) \in S_{\varphi, H}^n(g, x)$, $n \leq |S|$. Hence we get $\Phi_T(\iota(h, y)) \in U_{\Phi_T(\iota(g, x))}$ for all $(h, y) \in S_{\varphi, H}^n(g, x)$, $n \leq |S|$. This implies that $\mathcal{V} := \{(\Phi_T \circ \iota)^{-1}(U) \mid U \in \mathcal{U}\}$ is S -long with respect to (φ, H) . Finally, $\dim \mathcal{V} \leq \dim \mathcal{U} \leq \widehat{N}$. Thus G is transfer reducible over \mathcal{F} , where we use $N := \max\{\widehat{N}, \widetilde{N}\}$. \square

6. NON-POSITIVELY CURVED GROUPS ARE TRANSFER REDUCIBLE

In this section we prove our Main Theorem as stated in the introduction. Let G be a group with an isometric cocompact proper action on a finite dimensional CAT(0)-space X . We need to show that G is transfer reducible over the family \mathcal{VCyc} of virtually cyclic subgroups, see Definition 0.4. To this end we will show that Proposition 5.11 applies, where $\mathcal{F} = \mathcal{VCyc}$.

6.1. $\overline{B}_R(x)$ is $(2 \cdot \dim(X) + 1)$ -dominated. Fix a base point $x_0 \in X$. For $r \geq 0$ let

$$\rho_{r, x_0}: \overline{X} \rightarrow \overline{B}_r(x_0)$$

be the natural projection introduced in Remark 2.3. The map ρ_{r, x_0} is the identity on $\overline{B}_r(x_0)$. If $x \in \overline{X}$ with $x \notin B_r(x_0)$, then $\rho_{r, x_0}(x) = c_{x_0, x}(r)$, where $c_{x_0, x}: \mathbb{R} \rightarrow X$ is the generalized geodesic uniquely determined by $c_- = 0$, $c(-\infty) = x_0$ and $c(\infty) = x$ (see Subsection 3.1).

Lemma 6.1. *The space X is a Euclidean neighborhood retract, i.e., there is a natural number N , a closed subset $A \subseteq \mathbb{R}^N$, an open neighborhood U of A in \mathbb{R}^N and a map $r: U \rightarrow A$ such that $r|_A = \text{id}_A$ and X is homeomorphic to A . The number N can be chosen to be $2 \cdot \dim(X) + 1$.*

Proof. Since X is proper as metric space, it is locally compact. Since any two points in X can be joined by a unique geodesic, X is connected and locally contractible. Hence X has a countable basis for its topology (see [10, Exercise 2 in Chapter 6.5 on page 261]). Obviously X is Hausdorff. By assumption $\dim(X) < \infty$. We conclude from [10, Exercise 10 in Chapter 7.9 on page 315] that X is homeomorphic to a closed subset A of \mathbb{R}^N for $N = 2 \cdot \dim(X) + 1$. Now apply [5, Proposition 8.12 in Chapter IV.8 on p. 83]. \square

Lemma 6.2. *The space $\overline{B}_R(x_0)$ is a compact contractible metric space which is controlled $(2 \cdot \dim(X) + 1)$ -dominated (in the sense of Definition 0.2).*

Proof. Since X is proper as metric space by assumption, the closed ball $\overline{B}_R(x_0)$ is compact. The space $\overline{B}_R(x_0)$ inherits from X a metric and is contractible.

Because of Lemma 6.1 we can find an open subset $U \subset \mathbb{R}^{2 \cdot \dim(X) + 1}$ and maps $i: X \rightarrow U$ and $r: U \rightarrow X$ with $r \circ i = \text{id}_X$. Since U is a smooth manifold, it can be triangulated and hence is a simplicial complex of dimension $(2 \cdot \dim(X) + 1)$. Since $\overline{B}_R(x_0)$ is compact, $i(\overline{B}_R(x_0))$ is compact and hence contained in a finite

subcomplex $K \subseteq U$. Since K and hence $r(K)$ are compact, we can find $C > 0$ with $r(K) \subseteq \overline{B_{R+C}}(x_0)$. Let

$$i': \overline{B_R}(x_0) \rightarrow K$$

be the map defined by i . Let

$$r': K \rightarrow \overline{B_R}(x_0)$$

be the composite

$$K \xrightarrow{r|_K} \overline{B_{R+C}}(x_0) \xrightarrow{\rho_{R,x_0}|_{\overline{B_{R+C}}(x_0)}} \overline{B_R}(x_0).$$

Then $r' \circ i' = \text{id}_{\overline{B_R}(x_0)}$ and K is a finite $(2 \cdot \dim(X) + 1)$ -dimensional simplicial complex. This implies that $\overline{B_R}(x_0)$ is controlled $(2 \cdot \dim(X) + 1)$ -dominated. \square

6.2. Convention 5.1 applies to $FS(X)$. Let $FS(X)$ be the flow space for X from Definition 1.2. We will show that this flow space satisfies the conditions from Convention 5.1. By Proposition 1.11 the action of G on $FS(X)$ is isometric, proper and cocompact. In particular, the flow space $FS(X)$ is locally compact. By Proposition 2.10 the flow space $FS(X)$ is locally connected. By [4, II.2.8(2) on p.179] there is $k_G < \infty$ such that finite subgroups of G have order at most k_G . By Proposition 2.9 $\dim FS(X) - FS(X)^{\mathbb{R}}$ is finite. The uniform continuity of the flow follows from Lemma 1.3.

6.3. Long \mathcal{VCyc} -covers for $FS(X)$ at periodic flow lines. We need to show that the flow space $FS(X)$ admits long \mathcal{VCyc} -covers at infinity and periodic flow lines in the sense of Definition 5.5. We take for M the number appearing in Theorem 4.2. Let $\gamma > 0$ be given. Because the action of G on $FS(X)$ is cocompact we conclude from Theorem 4.2 that there is $\varepsilon > 0$ and a G -invariant cofinite collection \mathcal{V} of open \mathcal{VCyc} -subsets of $FS(X)$ such that

- $\dim \mathcal{V} \leq M$;
- for any $c \in FS(X)_{\leq \gamma}$ there is $V \in \mathcal{V}$ such that $B_{2\varepsilon}(\Phi_{[-\gamma, \gamma]}(c)) \subseteq V$.

Put $S := \{c \in FS(X) \mid \exists U \in \mathcal{V} \text{ such that } \overline{B_\varepsilon}(\Phi_{[-\gamma, \gamma]}(c)) \subseteq U\}$. Note that S is G -invariant, because \mathcal{V} is. Moreover, $FS(X)_{\leq \gamma} \subseteq S$. It remains to show, that there is a compact subset $K \subseteq FS(X)$ such that $FS(X) - S = G \cdot K$. Because the action of G on $FS(X)$ is cocompact (Proposition 1.11) and $FS(X)$ is locally compact (Proposition 1.9) it suffices to show that S is open.

Choose $U \in \mathcal{V}$ with $\overline{B_\varepsilon}(\Phi_{[-\gamma, \gamma]}(c_0)) \subseteq U$. It suffices to show that there is $\delta > 0$ such that $\overline{B_\varepsilon}(\Phi_{[-\gamma, \gamma]}(c)) \subseteq U$ for all $c \in FS(X)$ with $d_{FS}(c, c_0) < \delta$. We proceed by contradiction and assume that there is no such δ . Then there are $c_n, d_n \in FS(X)$, $t_n \in [-\gamma, \gamma]$ such that $d_{FS}(c_n, c_0) < 1/n$, $d_{FS}(d_n, \Phi_{t_n}(c_n)) \leq \varepsilon$ but $d_n \notin U$. In particular $c_n \rightarrow c_0$. By passing to a subsequence we can assume that $t_n \rightarrow t \in [-\gamma, \gamma]$. Thus $\Phi_{t_n}(c_n) \rightarrow \Phi_t(c_0)$. It follows that $d_{FS}(d_n, \Phi_t(c_0))$ is bounded independent of n . Because $FS(X)$ is a proper metric space (Proposition 1.9) we can pass to a further subsequence and assume that $d_n \rightarrow d$. Then $d_{FS}(d, \Phi_t(c_0)) \leq \varepsilon$. Thus $d \in U$. But this implies $d_n \in U$ for sufficiently large n , a contradiction.

This shows that the flow space $FS(X)$ admits long \mathcal{VCyc} -covers at infinity and periodic flow lines.

6.4. Contracting transfers for $FS(X)$. Finally we need to show that $FS(X)$ admits contracting transfers in sense of Definition 5.9. Let S be a finite subset of G (containing e). Let β be the number appearing in Proposition 3.8. Set $\tilde{N} := 2 \dim X + 1$. Let $\delta > 0$ be given. Let $T, R > 0$ be the numbers coming from Proposition 3.8. Then $\overline{B_R}(x_0)$ is compact, contractible and controlled \tilde{N} -dominated by Lemma 6.2. We use the homotopy S action (φ, H) on $B_R(x_0)$ from Definition 3.2

and the map $\iota: G \times B_r(x_0) \rightarrow FS(X)$ obtained by restriction from the map from Definition 3.4. It follows from Proposition 3.8 that (5.10) holds.

Thus Proposition 5.11 applies and we conclude that G is transfer reducible over \mathcal{VCyc} as claimed in our main Theorem in the introduction.

Remark 6.3. Our argument proves a little more than is stated in our main Theorem. Namely, the cover we construct is in addition cofinite for the action of G , i.e., $G \backslash \mathcal{U}$ is finite. This follows from Theorem 4.2 (iii) and Theorem 5.6 (iii).

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WESTFÄLISCHE WILHELMS-UNIVERSITÄT MÜNSTER, MATHEMATISCHES INSTITUT, EINSTEINSTR. 62, D-48149 MÜNSTER, GERMANY

E-mail address: bartelsa@math.uni-muenster.de

URL: <http://www.math.uni-muenster.de/u/bartelsa>

E-mail address: lueck@math.uni-muenster.de

URL: <http://www.math.uni-muenster.de/u/lueck>