Generalized triangle inequalities in thick Euclidean buildings of rank 2

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Abstract

We give the generalized triangle inequalities which determine the possible Δ-valued side lengths of n-gons in thick Euclidean buildings of rank 2.

1 Introduction

Let X be a symmetric space of noncompact type or a thick Euclidean building. We are interested in the following geometric question:

Which are the possible side lengths of polygons in X?

In this context the appropriate notion of *length* of an oriented geodesic segment is given by a vector in the Euclidean Weyl chamber Δ_{euc} associated to X. If $X = G/K$ is a symmetric space, the full invariant of a segment modulo the action of G is precisely this vector-valued length since we can identify $X \times X/G \cong \Delta_{euc}$ (cf. [KLM09a]). For X a Euclidean building the same notion of vector-valued length can be defined (cf. [KLM09b]). We denote by $\mathcal{P}_n(X) \subset \Delta_{euc}^n$ the set of all possible Δ_{euc}-valued side lengths of n-gons in X.

An algebraic question (the so-called *Eigenvalue Problem*), which goes back to 1912 when it was already studied by H. Weyl, is closely related to a special case of the geometric question above, namely, for the symmetric space $X = SL(m, \mathbb{C})/SU(m)$. It is one of the motivations for considering this geometric problem. The Eigenvalue Problem asks:

How are the eigenvalues of two Hermitian matrices related to the eigenvalues of their sum?

We refer to [KLM09a] for more information on the relation between these two questions and [Fu00] for more history on this problem.

In [KLM09a] and [KLM09b] it is shown that the set $\mathcal{P}_n(X)$ depends only on the spherical Coxeter complex associated to X (i.e. on the spherical Weyl chamber $\triangle_{sph}$). We will therefore sometimes refer to $\mathcal{P}_n(\triangle_{sph})$ as the set of side lengths of n-gons in X a symmetric space or a Euclidean building with $\triangle_{sph}$ as spherical Weyl chamber.

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For a symmetric space $X = G/K$ the set of possible side lengths has been completely determined in [KLM09a]: $\mathcal{P}_n(X)$ is a finite sided convex polyhedral cone and it can be described as the solution set of a finite set of homogeneous linear inequalities in terms of the Schubert calculus in the homology of the *generalized Grassmannian manifolds* associated to the symmetric space G/K. It follows, that for a Euclidean building X' with the same associated spherical Weyl chamber $\triangle_{sph}$ as X, the set $\mathcal{P}_n(X')$ is also a finite sided convex polyhedral cone determined by the same inequalities as $\mathcal{P}_n(X) = \mathcal{P}_n(\triangle_{sph})$.

As already pointed out in [KLM09b] for the case of *exotic* spherical Coxeter complexes (i.e. when it is the Coxeter complex of a Euclidean building but it does not occur for a symmetric space) the structure of the set $\mathcal{P}_n(\triangle_{sph})$ cannot be described with this method, since we do not have a Schubert calculus for these Coxeter complexes. Thus, the structure of $\mathcal{P}_n(\triangle_{sph})$ for these Coxeter complexes and even its convexity were unknown. It is clear that we can restrict our attention to irreducible Coxeter complexes. By a result of Tits [Ti77], exotic irreducible Coxeter complexes occur only in rank 2. Our main result is the description of $\mathcal{P}_n(X)$ in this case (compare with Theorem 6.14).

Theorem 1.1. *For a Euclidean building X of rank 2, the space $\mathcal{P}_n(X)$ is a finite sided convex polyhedral cone. The set of inequalities defining $\mathcal{P}_n(X)$ can be given in terms of the combinatorics of the spherical Coxeter complex associated to X.*

The inequalities given in our main theorem coincide with the so-called *weak triangle inequalities* (cf. [KLM09a, Sec. 3.8]). Moreover, our arguments also work (see Remark 6.12) to prove the weak triangle inequalities for buildings of arbitrary rank (cf. [KLM09a, Thm. 3.34]). For symmetric spaces, these inequalities correspond to specially simple intersections of Schubert cells in the description of $\mathcal{P}_n(X)$ given in [KLM09a]. Their description depend only in the Weyl group of X and therefore, they can be defined for arbitrary Coxeter complexes.

Consider the *side length map* $\sigma : Pol_n(X) = X^n \to \Delta^n_{euc}$. The set $\mathcal{P}_n(X)$ which we are interested in is nothing else than the image of σ. We use a direct geometric approach to describe this image. Our main idea is to study the singular values of σ by deforming the sides of a given polygon in X. This strategy was already used for the case of symmetric spaces by B. Leeb in [Le] to give a simple proof of the Thompson Conjecture (cf. [KLM09a, Theorem 1.1]). In this paper we adapt this *variational* method to the case of Euclidean buildings and use it to describe the space $\mathcal{P}_n(X)$.

Throughout this paper we state the results, whenever possible, in such a way that they apply to Euclidean buildings of arbitrary rank. In particular, Sections 4, 5 and 6.1 (except Lemma 6.6 and Proposition 6.7) do not use the assumption on the rank of the building. And when we do use the assumption, we indicate it explicitly in the statement of the corresponding result.

The set of inequalities obtained in Theorem 1.1 constitute an irredundant system defining the polyhedral cone $\mathcal{P}_n(X)$. The inequalities given by Schubert calculus in [KLM09a] are known to be irredundant for the cases of type A_n (see [KTW04]), however, these seem to be the only cases. A smaller set of inequalities is given in [BK06] by defining a new product in the cohomology of flag varieties. The irredundancy of this set has been recently shown in [Re10].

After a first version of this paper was written, the author learned about a recent related paper of Berenstein and Kapovich [BKa10], where the generalized triangle inequalities for rank 2 are also determined by a different approach.

Acknowledgments. I would like to thank Bernhard Leeb for bringing this problem to my attention and sharing his ideas in the case of symmetric spaces with me.

Contents

2 Preliminaries

A very good introduction to the concepts used in this paper is the work [KL98, ch. 2-4]. We refer also to [BH99] for more information on metric spaces with upper curvature bounds and to [KLM09b, ch. 2-3] for the different concepts of *length* in Euclidean buildings.

2.1 CAT(0) spaces

Recall that a complete geodesic metric space X is said to be $CAT(0)$ if the geodesic triangles in X are not *thicker* that the corresponding triangles in the Euclidean space.

For two points $x, y \in X$ we denote with xy the geodesic segment between them. The link $\Sigma_x X$ is the completion of the space of directions at x with the angle metric. $\overrightarrow{xy} \in \Sigma_x X$ denotes the direction of the segment xy at x.

Two complete geodesic lines γ_1, γ_2 are said to be *parallel* if they have finite Hausdorff distance, or equivalently, if the functions $d(\cdot, \gamma_i)|_{\gamma_{3-i}}$ are constant. The *parallel set* P_γ is defined as the union of all geodesic lines parallel to γ. It is a closed convex set that splits as a metric product $P_\gamma \cong \mathbb{R} \times Y$, where Y is also a $CAT(0)$ space.

For a *polygon* p, or more precisely, an *n-gon* in X we mean the union of n oriented geodesic segments $x_0x_1, \ldots, x_{n-1}x_n$ with $x_n = x_0$. Since geodesic segments in $CAT(0)$ spaces between two given points are unique, we can also describe p by its *vertices*. We write $p = (x_0, \ldots, x_{n-1})$. The union q of n oriented geodesic segments $x_0x_1, \ldots, x_{n-1}x_n$ where $x_n \neq x_0$ will be called a *polyhedral path* and we write $q = (x_0, \ldots, x_n)$.

2.2 Coxeter complex

A *spherical Coxeter complex* is a pair (S, W) consisting of a unit sphere S with its usual metric and a finite group W of isometries, the *Weyl group*, generated by reflections on total geodesic spheres of codimension one. A *Weyl chamber* in S is a fundamental domain of the action $W \curvearrowright S$. The *model Weyl chamber* is defined as $\Delta_{sph} := S/W$. We say that two points in S have the same *W-type* (or just *type*) if they belong to the same W-orbit.

A *Euclidean Coxeter complex* is a pair (E, W_{aff}) consisting of a Euclidean space E and a group of isometries W_{aff}, the *affine Weyl group*, generated by reflections on hyperplanes and such that its rotational part $W := rot(W_{aff})$ is finite. The set of fixed points of reflections in W_{aff} are called *walls* of (E, W_{aff}). We define the *W_{aff}-type* of a point in E as above. To (E, W_{aff}), we can associate the spherical Coxeter complex (S, W), where $S := \partial_\infty E$ is the Tits boundary of E. The *Euclidean model Weyl chamber* Δ_{euc} is the complete Euclidean cone over Δ_{sph}.

The link $\Sigma_x E$ of a point $x \in E$ is naturally a spherical Coxeter complex with Weyl group $Stab_{W_{aff}}(x)$. We will also use another structure on $\Sigma_x E$ as a Coxeter complex with Weyl group W. This will be given by the natural identification $\Sigma_x E \cong \partial_\infty E$.

The *refined length* of the oriented geodesic segment $xy \subset E$ is defined as the image of (x, y) under the projection $E \times E \to (E \times E)/W_{aff}$. The *$\Delta$-valued length*, or just *length*, is the image of the refined length under the natural *forgetful* map $(E \times E)/W_{aff} \to \Delta_{euc}$. We denote with σ the *length map* assigning to a segment its Δ-valued length.

We can also define the *refined length* of an oriented segment xy in the spherical Coxeter complex (S, W) analogously as the image of (x, y) under the projection $S \times S \to (S \times S)/W$.

2.3 Buildings

For an introduction to spherical and Euclidean buildings from the point of view of metric geometry, we refer to [KL98].

Let X be a thick Euclidean building modelled in the Euclidean Coxeter complex (E, W_{aff}). The concepts of refined length and Δ-valued length of an oriented geodesic segment $xy \subset X$ can be also defined naturally by identifying an apartment containing xy with the Coxeter complex (E, W_{aff}).

For a polygon $p = (x_0, \dots, x_{n-1})$ in X, we write $\sigma(p) = (\sigma(x_0x_1), \dots, \sigma(x_{n-1}x_0)) \in \Delta_{euc}^n$ and call $\sigma : X^n \to \Delta_{euc}^n$ the *side length map.* The space $\mathcal{P}_n(X) := \sigma(X^n)$ is the set of possible Δ-valued side lengths of n-gons in X. We say that a polygon in X is *regular* if all its sides are regular, that is if their Δ-valued lengths lie in the interior of Δ_{euc}. The space of regular polygons is an open dense subset of X^n.

We will use following result from [KLM09b] concerning the refined side lengths of polygons in X. We reproduce here its statement for the convenience of the reader.

Theorem 2.1 (Transfer theorem). *Let X and X' be thick Euclidean buildings modelled on the same Euclidean Coxeter complex (E, W_{aff}). Let $p = (x_0, \dots, x_{n-1})$ be a polygon in X and let $x'_0x'_1$ be a segment in X' with the same refined length as x_0x_1. Then there exists a polygon $p' = (x'_0, x'_1, \dots, x'_{n-1})$ in X' with the same refined side lengths as of p.*

3 The set of functionals $\mathcal{L}_n$

We fix a vertex v_0 of (E, W_{aff}) with $Stab_{W_{aff}}(v_0) \cong W$. We obtain in this way an identification $E \cong \mathbb{R}^{\dim E}$. By fixing v_0 we get an embedding $W \hookrightarrow W_{aff}$ and also the (coarser) structure (E, W) as Euclidean Coxeter complex. We will think of the Euclidean Weyl chamber $\Delta_{euc} \cong E/W$ as embedded in E, such that Δ_{euc} is a fundamental domain of the action $W \curvearrowright E$. Hence, the cone point of Δ_{euc} corresponds to v_0.

Let $\eta \in E$ be a maximal singular unit vector, i.e. $\overrightarrow{v_0\eta}$ is a vertex of $(\Sigma_{v_0}E, W)$. We define the following linear functional:

$$\begin{aligned} l_\eta : \Delta_{euc} &\to \mathbb{R} \\ v &\mapsto \langle v, \eta \rangle \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^{\dim E}$. We denote with $\mathcal{L}_n$ the finite set of functionals on Δ_{euc}^n of the form $L(e_1, \dots, e_n) = l_{\eta_1}(e_1) + \dots + l_{\eta_n}(e_n)$ where all the η_i have the same W-type. We write $L = (l_{\eta_1}, \dots, l_{\eta_n})$ for such a functional.

Let H_L denote the hyperplane $L^{-1}(0) \cap \Delta_{euc}^n$ for $L \in \mathcal{L}_n$. We call H_L a *wall* in Δ_{euc}^n. The set of walls H_L divide Δ_{euc}^n in finitely many convex polyhedral cones. We denote with $\mathcal{C}_n$ the family of the interiors of these cones, i.e. $\mathcal{C}_n$ is the set of the connected components of $\text{int}(\Delta_{euc}^n) \setminus \bigcup_{L \in \mathcal{L}_n} H_L$.

4 Polygons

4.1 Holonomy map

Let $p = (x_0, \dots, x_{n-1})$ be an n-gon in X. We say that a n-tuple $\mathcal{F} = (F_1, \dots, F_n)$ of apartments in X *supports the polygon* p if $e_i := x_{i-1}x_i \subset F_i$ and the convex set $F_i \cap F_{i+1}$ is top dimensional and contains x_i in its interior.

Remark 4.1. If p is a regular polygon then there always exists an n-tuple $\mathcal{F}$ supporting p. $\mathcal{F}$ can be constructed as follows: Let $A \in \Sigma_{x_0}X$ be an apartment containing $\overrightarrow{x_0x_1}$ and $\overrightarrow{x_0x_{n-1}}$ and take $v \in A$ antipodal to $\overrightarrow{x_0x_1}$. Extend the segment x_0x_1 a little further than x_0 in direction of v to a segment $x_0'x_1$. Inductively for $i = 1, \dots, n-1$ choose $F_i \in X$ to be an apartment containing $x_{i-1}'x_i$ and an initial part of x_ix_{i+1} and extend x_ix_{i+1} in F_i a little further than x_i to a segment $x_i'x_{i+1}$. Finally choose F_n to contain $x_{n-1}'x_0$ and an initial part of x_0x_1 and x_0x_0'. This last step is possible because of our first choice of x_0'. The polyhedron $F_i \cap F_{i+1}$ contains a regular segment with x_i in its interior. In particular $F_i \cap F_{i+1}$ is top dimensional.

Let now p be a polygon and $\mathcal{F}$ an n-tuple supporting it. Notice that the convex set $F_i \cap F_{i+1}$ is a neighborhood of x_i in F_i and F_{i+1}. Therefore we have:

$$S_i := \Sigma_{x_i} F_i = \Sigma_{x_i} F_{i+1} = \Sigma_{x_i}(F_i \cap F_{i+1}).$$

So we have a natural map $\phi_i : S_i \to S_{i+1}$ (just take *parallel transport* in F_{i+1} along the side e_{i+1}) and an associated *holonomy map* $\phi_p : S_i \to S_i$ defined as the composition $\phi_p = \phi_{i+n-1} \circ \cdots \circ \phi_{i+1} \circ \phi_i$. We introduce also the following notation:

$$\phi_i^k := \phi_{i+k-1} \circ \cdots \circ \phi_i : S_i \to S_{i+k}$$

If we identify S_i with $\partial_\infty F_i$ in the natural way, we obtain a structure of spherical Coxeter complex on S_i with Weyl group W. With this structure the maps ϕ_i are isomorphisms of Coxeter complexes and the holonomy map ϕ_p is an element of the Weyl group W. In particular the set of fixed points of ϕ_p is a singular sphere in (S_i, W). Notice that the holonomy map (and therefore also its fixed points set) depends on the choice of the n-tuple $\mathcal{F}$ supporting p. We will make use of this flexibility later.

4.2 Opening a polygon in an apartment

Let $p = (x_0, \dots, x_{n-1})$ be a n-gon in X and let $\mathcal{F}$ be an n-tuple supporting it. We construct points $x_i' \in F_1$, $i = 1, \dots, n$ inductively as follows: for $i = 0, 1$ just set $x_0' = x_0$ and $x_1' = x_1$ and suppose we have already constructed x_i'. For each $x \in F_1$ we can identify naturally $\Sigma_x F_1$ with $\partial_\infty F_1$ thus giving it a structure of spherical Coxeter complex with Weyl group W. Let $\psi_i : S_i \to \Sigma_{x_i'} F$ be a isomorphism of spherical Coxeter complexes such that $\psi_i(\overrightarrow{x_ix_{i-1}}) = \overrightarrow{x_i'x_{i-1}'}$. Notice that if p is regular such an isomorphism is unique. Let now x_{i+1}' be the point in F_1 such that $d(x_i', x_{i+1}') = d(x_i, x_{i+1})$ and $\overrightarrow{x_i', x_{i+1}'} = \psi_i(\overrightarrow{x_i, x_{i+1}})$ (see Fig. 1). We remark that in general $x_n' \neq x_0'$ and $(x_0', \dots, x_n')$ is a polygonal path hence the expression "opening a polygon". We can continue this process and define $x_j' \in F_1$ for $j > n$.

The isomorphisms ψ_i can be chosen (and we do so) so that the induced automorphisms of (S_1, W)

$$\begin{array}{ccc} S_i & \xrightarrow{\phi_i} & S_{i+1} \\ \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle \psi_{i+1}} \\ \Sigma_{x_i'} F \cong S_1 & \longrightarrow & \Sigma_{x_{i+1}'} F \cong S_1 \end{array}$$

are just the identity map.

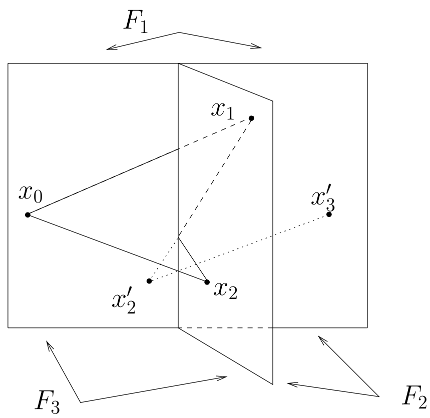

Figure 1: Opening a triangle

4.3 Folding a polygon into an apartment

This construction was first considered in [KLM08, Sec. 6.1].

For simplicity on the notation, suppose $p = (x_0, x_1, x_2)$ is a triangle in X. There is a partition $y_1 = x_1, y_1, \dots, y_k = x_2$ of the segment x_1x_2 such that the triangles (x_0, y_i, y_{i+1}) for $i = 1, \dots, k-1$ are contained in an apartment A_i. We define points $\hat{y}_i$ in the apartment A_1 inductively as follows: for $i = 1$ set $\hat{y}_1 = y_1 = x_1$ and suppose we have already defined $\hat{y}_i$. Let $\beta_i : A_i \to A_1$ be an isomorphism of Euclidean Coxeter complexes, such that $\beta(x_0y_i) = x_0\hat{y}_i$. We define $\hat{y}_{i+1} := \beta(y_{i+1})$. We say that the polygon $\hat{p} = (x_0, \hat{y}_1, \dots, \hat{y}_k)$ is the result of *folding* the triangle p into A_1. We say that the points $\hat{y}_i$ for $i = 2, \dots, k-1$ are the *break points* of the folded polygon $\hat{p}$. Notice that the segments x_0x_1 and x_0x_2 have the same refined side lengths as the segments $x_0\hat{y}_1$ and $x_0\hat{y}_k$ respectively. Write $y_0 = x_0$ and define $\zeta_i := \overrightarrow{y_iy_{i-1}}$ and $\xi_i := \overrightarrow{y_iy_{i+1}}$, analogously $\hat{\zeta}_i := \overrightarrow{\hat{y}_i\hat{y}_{i-1}}$ and $\hat{\xi}_i := \overrightarrow{\hat{y}_i\hat{y}_{i+1}}$.

A *billiard triangle* is a polygon $\hat{p} = (x_0, \hat{y}_1, \dots, \hat{y}_k)$ in an apartment A_1 such that for $i = 2, \dots, k-1$ the directions $\hat{\zeta}_i$ and $\hat{\xi}_i$ are antipodal in the spherical Coxeter complex $(\Sigma_{\hat{y}_i}A_1, Stab_{W_{aff}}(\hat{y}_i))$ modulo the action of the Weyl group $Stab_{W_{aff}}(\hat{y}_i)$. Clearly, a folded triangle is a billiard triangle. Conversely, the next condition is necessary and sufficient for a billiard triangle to be a folded triangle.

For $i = 2, \dots, k-1$ there is a triangle $(\zeta_i', \xi_i', \tau_i')$ in the spherical building $\Sigma_{\hat{y}_i}X$ such that $d(\zeta_i', \xi_i') = \pi$ and the refined lengths of $\zeta_i'\tau_i'$ and $\xi_i'\tau_i'$ are the same as of $\hat{\zeta}_i\overrightarrow{\hat{y}_ix_0}$ and $\hat{\xi}_i\overrightarrow{\hat{y}_ix_0}$ respectively.

We investigate now the relation between the constructions of opening and folding a polygon. Let $p = (x_0, x_1, x_2)$ be a triangle in X and let $\mathcal{F}$ be a triple supporting p. Observe that we can choose $A_1 = F_1$. Let $\hat{p} = (x_0, \hat{y}_1, \dots, \hat{y}_k)$ be the folded triangle. Again we identify naturally $\Sigma_xF_j \cong S_j$ with $\partial_\infty F_j$ for each $x \in F_j$ and give the structure of spherical Coxeter complex with Weyl group W.

For $i = 1, \dots, k-1$. Let $\alpha_i : S_2 \cong \Sigma_{y_i}F_2 \to S_1 \cong \Sigma_{\hat{y}_i}F_1$ be an isomorphism of spherical Coxeter complexes so that $\alpha_i(\zeta_i) = \hat{\zeta}_i$. Notice that for $i = 1$ we just have $\alpha_1 = \phi_1^{-1}$. Analogously, let $\alpha_k : S_3 \cong \Sigma_{x_2}F_3 \to S_1$ be an isomorphism so that $\alpha_k(\zeta_k = \overrightarrow{x_2x_1}) = \hat{\zeta}_k$ and let $\alpha_0 : S_1 \cong \Sigma_{x_0}F_1 \to S_1$ be so that $\alpha_0(\overrightarrow{x_0x_2}) = \overrightarrow{\hat{y}_0\hat{y}_k} = \hat{\zeta}_0$. Observe that if p is regular, then the α_i are unique.

Since $\hat{p} = (x_0, \hat{y}_1, \ldots, \hat{y}_k)$ is a billiard triangle, there are isometries μ_i of F_1 in the affine Weyl group W_{aff} for $i = 0, \ldots, k$ such that $\hat{\zeta}_i \overrightarrow{\hat{y}_i \mu_i(\hat{y}_{i+1})}$ has the same refined length as $\zeta_i \xi_i$. In particular for $i = 2, \ldots, k-1$ the points $\hat{y}_{i-1}$, $\hat{y}_i$, $\mu_i(\hat{y}_{i+1})$ lie on a geodesic segment. Hence, we call the μ_i the *straightening* isometries. It holds:

$$\mu_1 \circ \cdots \circ \mu_k \circ \mu_0(\hat{y}_1) = x'_{n+1}$$

where x'_{n+1} is constructed as in Section 4.2. Consider the natural action of μ_i on S_1. The straightening isometries can be chosen (if p is regular then they are unique) such that

$$\begin{array}{ll} \alpha_i = \mu_i \circ \alpha_{i+1} & \text{for } i = 1, \ldots, k-2 \\ \alpha_i = \mu_i \circ \alpha_{i+1} \circ \phi_{i-1} & \text{for } i = k-1, k, 0. \end{array}$$

It follows that

$$\mu_0^{-1} \circ \mu_k^{-1} \circ \cdots \circ \mu_1^{-1} = \phi_0 \circ \phi_2 \circ \phi_1 = \phi_p : S_1 \to S_1$$

is the holonomy map at x_1.

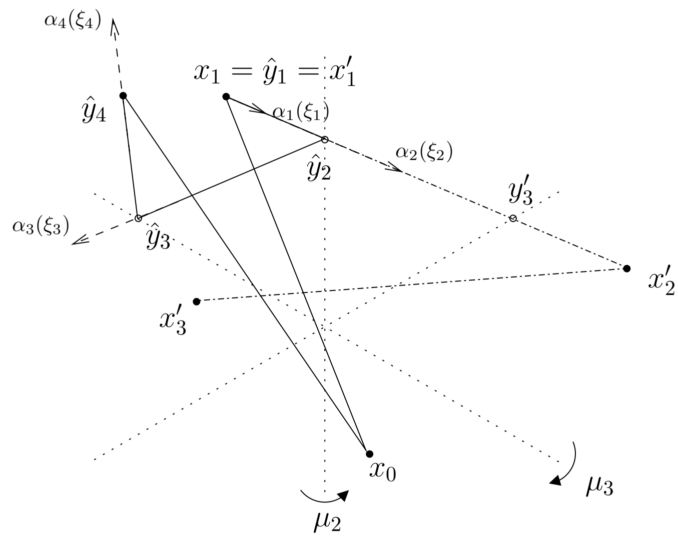

Figure 2: Folding and opening a triangle

The constructions for n-gons ($n > 3$) are analogous.

5 Critical values of the side length map σ

For a *regular* value of the side length map σ we mean a value $s \in \mathcal{P}_n(X)$ for which there is a polygon p with $\sigma(p) = s$ and such that σ is an open map at p. First we give a sufficient condition in terms of the holonomy map for $\sigma(p)$ being a *regular* value of σ.

Proposition 5.1. *Let p be an n-gon in X and $\mathcal{F}$ an n-tuple supporting p. Suppose that the holonomy map ϕ_p has no fixed points, then the space $\mathcal{P}_n(X)$ is a neighborhood of $\sigma(p)$ in Δ^n_{euc}.*

Proof. Choose $\epsilon > 0$ so that $B_{x_i}(n\epsilon) \subset F_i \cap F_{i+1}$ for all i. For $v \in S_i$ and for $0 < t < n\epsilon$ we write $\exp(tv)$ to denote the point $x \in F_i \cap F_{i+1}$ with $d(x, x_i) = t$ and $\overrightarrow{x_i x} = v$.

We want to *vary* the polygon p *along* $v \in S_i$ to a polygon $p_v = (x_0^v, \ldots, x_{n-1}^v)$ with side lengths $\sigma(e_j^v) = \sigma(e_j)$ for $j \neq i$. For this, let $t < \epsilon$ and define $x_{i+k}^v := \exp(t\,\phi_i^k(v))$ for $k = 0, \ldots, n-1$ where the subindices are considered modulo n. Notice that for $j \neq i$ the segment $e_j^v = x_{j-1}^v x_j^v$ is just a translation in the apartment F_j of the segment e_j. Hence the condition on the side lengths above is clearly fulfilled. But since $\phi_i^n(v) = \phi_p(v) \neq v$ we get (see Fig. 3)

$$\sigma(e_i^v) = \sigma(\exp(d(x_{i-1}, x_i)\,\overrightarrow{x_{i-1}x_i} - t\,(\phi_p(v) - v))).$$

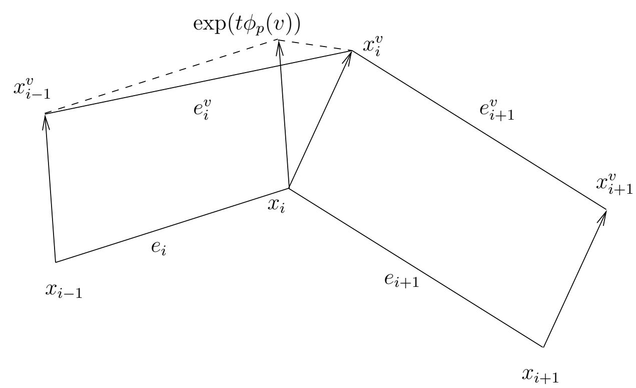

Figure 3: Variation of the side e_i

Since ϕ_p has no fixed points the set $\{\sigma(e_i^v) \mid v \in S_i,\ 0 \leq t < \epsilon\}$ is a neighborhood of $\sigma(e_i)$ in Δ_{euc}. This means that we can deform every side length of p independently, thus $\mathcal{P}_n(X)$ is a neighborhood of $\sigma(p)$ in Δ_{euc}^n. □

The next proposition says that for a building with only one vertex the critical values of σ must lie in the walls H_L.

Proposition 5.2. *Let p be an n-gon in a thick Euclidean building X which has only one vertex. Let $\mathcal{F}$ be an n-tuple supporting p. Suppose that the holonomy map ϕ_p fixes a maximal singular direction. Then there exists a functional $L \in \mathcal{L}_n$, such that $L(\sigma(p)) = 0$.*

Proof. First observe that we have a natural identification of any apartment with $\mathbb{R}^{\dim X}$ since we assumed that X has only one vertex. This gives us also an identification $W_{aff} = W$. Let $\eta \in S_1$ be a maximal singular direction fixed by $\phi_p : S_1 \to S_1$. Let $v \in F_1$ be a unit vector with direction $\eta \in S_1$. Now open the polygon $p = (x_0, \ldots, x_n)$ in the apartment F_1 to the polygonal path $p' = (x_1', \ldots, x_{n+1}')$. We can also fold p into F_1 and obtain the straightening isometry $\mu := \mu_0^{-1} \circ \mu_k^{-1} \circ \cdots \circ \mu_1^{-1}$. Recall that $\mu(x_{n+1}') = x_1'$ and $\mu(v) = v$ since μ induces the holonomy map. It then follows that $\langle x_1', v\rangle = \langle x_{n+1}', v\rangle$.

Now let $\eta_i \in E$ be a maximal singular unit vector of the same W-type as η, such that $l_{\eta_i}(\sigma(x_{i-1}x_i)) = \langle x_i' - x_{i-1}', v\rangle$. Set $L = (l_{\eta_1}, \ldots, l_{\eta_n})$, then

$$L(\sigma(p)) = \int_{p'} \langle \cdot, v\rangle = \langle x_{n+1}', v\rangle - \langle x_1', v\rangle = 0.$$

□

We use next the result in [KLM09b] that $\mathcal{P}_n(X)$ depends only on the spherical Coxeter complex to transfer the result above to arbitrary buildings.

Corollary 5.3. *Let $s \in \mathcal{P}_n(X) \cap \operatorname{int} \Delta^n_{euc}$ and suppose that $L(s) \neq 0$ for all functionals $L \in \mathcal{L}_n$. Then $\mathcal{P}_n(X)$ is a neighborhood of s in Δ^n_{euc}.*

Proof. By [KLM09b] we may assume that X has only one vertex. Let p be a regular polygon with $\sigma(p) = s$ and let $\mathcal{F}$ be an n-tuple supporting p. By Proposition 5.2 the holonomy map has no fixed points. The result now follows from Proposition 5.1. □

Lemma 5.4. *Let p_k be a sequence of regular n-gons in X such that $\sigma(p_k) \to s$ in Δ^n_{euc}, then there exists an n-gon p in X such that $\sigma(p) = s$.*

Proof. We assume again that X has only one vertex. Let $p_k = (x^k_0, \dots, x^k_{n-1})$ and let $\mathcal{F}_k = (F^k_1, \dots, F^k_n)$ be n-tuples supporting p_k. After transferring the polygons p_k (cf. Theorem 2.1) we may assume that the sides $x^k_0 x^k_1$ lie in the same apartment F and that x^k_0 lie in the same Euclidean Weyl chamber $\Delta_{euc} \subset F$. After a small perturbation of the polygons we may also suppose that x^k_0 lie in the interior of Δ_{euc}. We open now the polygons p_k in the apartment F to polygonal paths $p'_k = ({x^k_0}', \dots, {x^k_n}')$.

If ${x^k_0}' \to \infty$ in F, then for k big enough p'_k must be completely contained in the interior of Δ_{euc}. In particular, folding the polygon p_k into F cannot have break points. This implies that p_k is contained in the apartment F for k big enough. Since $\sigma(p_k) \to s$, then it is clear that the polygons p_k subconverge in F modulo translations in F to a polygon p with $\sigma(p) = s$.

Suppose now that ${x^k_0}'$ stay in a bounded region. Then after taking a subsequence we can assume that the polygonal paths p'_k converge to a polygonal path $p' = (x'_0, \dots, x'_n)$ with Δ-valued side lengths s. We want now to lift this polygonal path near the polygons p_k. Let $\rho^k_i : F^k_i \to F$ be the isomorphisms of Euclidean Coxeter complexes that send $x^k_{i-1} x^k_i$ to ${x^k_{i-1}}' {x^k_i}'$. So we have ${x^k_i}' \in \rho^k_i(F^k_i \cap F^k_{i+1}) = \rho^k_{i+1}(F^k_i \cap F^k_{i+1})$. Hence, for k big enough we have $x'_i \in \rho^k_i(F^k_i \cap F^k_{i+1}) = \rho^k_{i+1}(F^k_i \cap F^k_{i+1})$ and we can define $z^k_i := (\rho^k_i)^{-1}(x'_i) = (\rho^k_{i+1})^{-1}(x'_i) \in F^k_i \cap F^k_{i+1}$. Then $q_k := (z^k_0, \dots, z^k_n)$ is a polygonal path with the same side lengths as p', i.e. $\sigma(q_k) = s$. However q_k may still not be a closed polygon.

Notice that $d(z^k_0, x^k_0) = d(x'_0, {x^k_0}')$ and $d(z^k_n, x^k_0) = d(x'_n, {x^k_n}')$, thus $d(z^k_0, z^k_n) \leq d(x'_0, {x^k_0}') + d(x'_n, {x^k_n}') \to 0$. On the other hand, observe that ${x^k_0}'$ and ${x^k_n}'$ have the same W_{aff}-type and therefore also z^k_0 and z^k_n have the same type. But W_{aff} is finite, so $d(z^k_0, z^k_n)$ can only take finitely many values. It follows that for k big enough $z^k_0 = z^k_n$ and q_k is a closed polygon with Δ-valued side lengths s. □

Corollary 5.5. *For any open cone $C \in \mathcal{C}_n$ the intersection $\mathcal{P}_n(X) \cap C$ is empty or C. Moreover, if $C \subset \mathcal{P}_n(X)$, then $\bar{C} \subset \mathcal{P}_n(X)$.*

Proof. The intersection $\mathcal{P}_n(X) \cap C$ is open by Corollary 5.3 and closed by Lemma 5.4. □

6 The generalized triangle inequalities

6.1 Crossing the walls H_L

Suppose p is a polygon in X with $\sigma(p) = s \in H_L$ for some functional $L \in \mathcal{L}_n$. Considering Corollary 5.5 the natural question is if there is a cone $C \in \mathcal{C}_n$ such that $s \in \bar{C} \subset \mathcal{P}_n(X)$. We would also like to describe all cones in $\mathcal{C}_n$ with this property. With this in mind we investigate in this section following question. When can we find polygons p' with Δ-valued side lengths near s and such that $L \circ \sigma(p) > 0$ (or < 0)? For this we might try to study the side lengths of small perturbations of p. However since a Euclidean building has dimension equal to his rank, we do not have much flexibility to perturbate the polygon. Thus we must be more compliant with the variations of p that we want to admit. Therefore we will often have to translate the polygon to other place in X where we can perform the perturbations.

Let $L = (l_{\eta_1}, \dots, l_{\eta_n})$ be a functional in $\mathcal{L}_n$. For the rest of this section $p = (x_0, \dots, x_{n-1})$ will be always a regular n-gon such that $\sigma(p) \in H_L$.

Let $\mathcal{F}$ be a n-tuple of apartments supporting p. Let $v_i, w_i \in S_i$ be maximal singular directions (in the structure coming from $S_i \cong \partial_\infty F_i$ with Weyl group W) such that if $y_i \in F_i$, $z_i \in F_{i+1}$ are unit vectors with base point x_i and directions v_i and w_i respectively, then $l_{\eta_i}(\sigma(e_i)) = \langle e_i, y_i \rangle$ and $l_{\eta_{i+1}}(\sigma(e_{i+1})) = \langle e_{i+1}, z_i \rangle$. Observe that v_i, w_i are of the same W-type as $\eta_i, \dots, \eta_n$ We will therefore sometimes write $l_{\eta_i} \circ \sigma = \langle \cdot, v_i \rangle$ and $l_{\eta_{i+1}} \circ \sigma = \langle \cdot, w_i \rangle$. Notice that y_i is just the parallel translation along e_i in F_i of z_{i-1}, that is $v_i = \phi_{i-1}(w_{i-1})$.

Lemma 6.1. *If in the notation above $v_i \neq w_i$ for some i, then for any neighborhood U of $\sigma(p)$ in Δ^n_{euc} there exist n-gons p_1, p_2 in X with $\sigma(p_i) \in U$ and $L \circ \sigma(p_1) > 0 > L \circ \sigma(p_2)$.*

Proof. The proof is similar to the one of Proposition 5.1. For $\epsilon > 0$ small, let $x_i' := \exp(\epsilon v_i)$. Consider the polygon $p_1 := (x_0, \dots, x_i', \dots, x_{n-1})$, then

$$\begin{aligned} L(\sigma(p_1)) &= l_{\eta_1}(\sigma(x_0x_1)) + \dots + \langle x_{i-1}x_i + \epsilon\, y_i, y_i \rangle + \langle x_i x_{i+1} - \epsilon\, y_i, z_i \rangle + \dots + l_{\eta_n}(\sigma(x_{n-1}x_0)) \\ &= L(\sigma(p)) + \epsilon(\langle y_i, y_i \rangle - \langle y_i, z_i \rangle) \quad > \quad L(\sigma(p)) = 0. \end{aligned}$$

Analogously for $p_2 := (x_1, \dots, \exp(\epsilon w_i), \dots, x_{n-1})$ we have $L(\sigma(p_2)) < L(\sigma(p)) = 0$. ☐

Assume now that $v_i = w_i \in S_i$ for all i. In particular, the holonomy map $\phi_p : S_i \to S_i$ has the fixed point v_i. Let γ_i (resp. λ_i) be the line (i.e. complete geodesic) in F_i (resp. F_{i+1}) with $x_i = \gamma_i(0) = \lambda_i(0)$ and $v_i = \dot{\gamma}_i(0) = \dot{\lambda}_i(0)$. If $\gamma_i = \lambda_i$ for all i, then the polygon p is contained in a parallel set, namely the set P_{γ_0} of all lines parallel to γ_0.

Lemma 6.2. *Suppose p is not contained in any parallel set P_γ, where γ is a geodesic line with $\eta = \gamma(\infty)$ such that $v_i = \overrightarrow{x_i\eta}$ for all i. Then for any neighborhood U of $\sigma(p)$ in Δ^n_{euc} there exist n-gons p_1, p_2 in X with $\sigma(p_i) \in U$ and $L \circ \sigma(p_1) > 0 > L \circ \sigma(p_2)$.*

Proof. Let $P = (\nu_0, \dots, \nu_{n-1})$ be an n-tuple of geodesic segments $\nu_i : [s^-, s^+] \to X$ with $\nu_i(0) = x_i$, $\dot{\nu}_i = v_i$. and such that the convex hull $CH(\nu_i, \nu_{i+1})$ is a (2-dimensional) flat quadrilateral. Such a P exists, just take the initial parts of the geodesics $\gamma_i \cap \lambda_i$. Suppose now that P is maximal, i.e. the segments ν_i cannot be extended. If $|s^\pm| = \infty$, then the ν_i are parallel geodesic lines and $p \subset P_{\nu_0}$. Hence at least one of s^+ or $-s^-$ must be $< \infty$. Suppose $s = s^+ < \infty$ (the other case is analogous).

Now we want to displace p along ν_i to the region, where it does not look locally like a parallel set anymore: set $p' = (x'_0, \ldots, x'_{n-1}) = (\nu_0(s), \ldots, \nu_{n-1}(s))$. Then p' is an n-gon with $\sigma(p') = \sigma(p)$. Choose apartments A_i containing the convex sets $CH(\nu_{i-1}, \nu_i)$. Let $u_i := -\dot{\nu}_i(s) \in \Sigma_{x'_i}(A_i \cap A_{i+1})$ and let $v'_i \in \Sigma_{x'_i} A_i$, $w'_i \in \Sigma_{x'_i} A_{i+1}$ be the antipodes of u_i in $\Sigma_{x'_i} A_i$ and $\Sigma_{x'_i} A_{i+1}$ respectively.

If $v'_i = w'_i$ for all i, then we can extend the ν_i inside $A_i \cap A_{i+1}$ contradicting the maximality of P. Hence, there is a j such that $v'_j \neq w'_j$.

Moreover, if it holds for all i that $d(\overrightarrow{x'_i x'_{i+1}}, v'_i) = d(\overrightarrow{x'_i x'_{i+1}}, w'_i)$, then $u_i \overrightarrow{x'_i x'_{i+1}} v'_i$ is a geodesic segment in $\Sigma_{x'_i} X$ of length π. Let $z_{i+1} \in A_{i+1}$ be a point near x'_{i+1} with $\overrightarrow{x'_{i+1} z_{i+1}} = v'_{i+1}$. We can choose z_{i+1} close enough to x'_{i+1}, so that $\overrightarrow{x'_i z_{i+1}}$ is a regular point in the same Weyl chamber as $\overrightarrow{x'_i x'_{i+1}}$. It follows that $\overrightarrow{x'_i z_{i+1}}$ lies in the intersection of the segments $u_i \overrightarrow{x'_i x'_{i+1}} v'_i$ and $u_i \overrightarrow{x'_i x'_{i+1}} w'_i$. Thus $u_i \overrightarrow{x'_i z_{i+1}} v'_i$ is a geodesic segment of length π. Let now $z_i \in A_i$ be a point with $\overrightarrow{x'_i z_i} = v'_i$ and so that $CH(x'_i, z_i, z_{i+1})$ is a flat triangle. It follows that the union of the (2-dimensional) flat convex sets $CH(x_i, x_{i+1}, x'_{i+1}, x'_i)$, $CH(x'_i, x_{i+1'}, z_{i+1})$ and $CH(x'_i, z_{i+1}, z_i)$ is a flat convex quadrilateral. (See Figure 4.) Notice also that $\nu_i(s^-) z_i$ are extensions of the geodesic segments $\nu_i(s^-)\nu_i(s^+)$. Thus this contradicts as well the maximality of P. Hence, there is a j such that $d(\overrightarrow{x'_j x'_{j+1}}, v'_j) > d(\overrightarrow{x'_j x'_{j+1}}, w'_j)$.

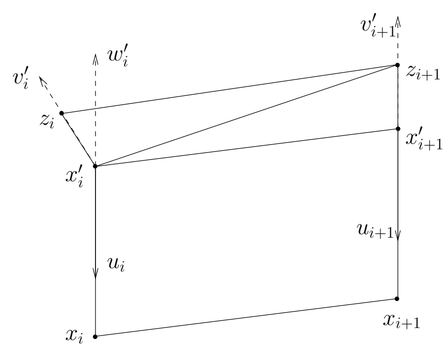

Figure 4: Extending the geodesics ν_i

Let $\tilde{x}_j := \exp(\epsilon v'_j)$ in A_j for some small $\epsilon > 0$. Then $\sigma(\tilde{x}_j x'_{j+1}) = \sigma(x'_j x'_{j+1}) - \epsilon\tilde{\eta} = \sigma(x_j x_{j+1}) - \epsilon\tilde{\eta}$ for some unit vector $\tilde{\eta} \in E$ of the same type as η_{j+1}. By the above consideration we must have $\tilde{\eta} \neq \eta_{j+1}$, otherwise $d(\overrightarrow{x'_j x'_{j+1}}, v'_j) = d(\overrightarrow{x'_j x'_{j+1}}, w'_j)$. In particular $l_{\eta_{j+1}}(\sigma(\tilde{x}_j x'_{j+1})) = \langle \sigma(x_j x_{j+1}) - \epsilon\tilde{\eta}\,,\ \eta_{j+1} \rangle > l_{\eta_{j+1}}(\sigma(x_j x_{j+1})) - \epsilon$. On the other hand, $l_{\eta_j}(\sigma(x'_{j-1}\tilde{x}_j)) = \langle x'_{j-1} x'_j + \epsilon v'_j\,,\ v'_j \rangle = l_{\eta_j}(\sigma(x_{j-1} x_j)) + \epsilon$. Thus, for $p_1 := (x'_1, \ldots, \tilde{x}_j, \ldots, x'_{n-1})$ we have $L(\sigma(p_1)) > L(\sigma(p)) = 0$.

Analogously for $p_2 := (x'_1, \ldots, \exp(\epsilon w'_j), \ldots, x'_{n-1})$ we get $L(\sigma(p_2)) < L(\sigma(p)) = 0$. $\square$

The next question is what happens when p is contained in such a parallel set P_γ. In this last situation we cannot always get the same conclusion as in Lemmata 6.1 and 6.2. For instance, if the wall H_L lies in the boundary of $\mathcal{P}_n(X)$, then we can cross H_L in one direction but not in the opposite one.

Remark 6.3. Suppose that p is contained in P_γ. Let $b_{\eta_-}: X \to \mathbb{R}$ be a Busemann function associated to $\eta_- = \gamma(-\infty)$ (see e.g. [KLM09b, Sec. 2.2] for a definition). Then by considering an apartment parallel to γ containing the side $x_{i-1}x_i$, we see that $l_{\eta_i}(\sigma(x_{i-1}x_i)) = b_{\eta_-}(x_i) - b_{\eta_-}(x_{i-1})$. In particular

$$L(\sigma(p)) = l_{\eta_1}(\sigma(x_0x_1)) + \cdots + l_{\eta_n}(\sigma(x_{n-1}x_0)) = \sum_{i=1}^{n}(b_{\eta_-}(x_i) - b_{\eta_-}(x_{i-1})) = 0.$$

Thus, if p' is the result of a small variation of the polygon p within the parallel set P_γ, it still holds $L(\sigma(p')) = 0$.

The next lemma gives a contition that let us cross the wall H_L in the positive direction.

Suppose p is contained in P_γ. Assume also that there are vertices x_i, x_j, x_{j+1} of p with the following property. Let A_0, A_1 be apartments in P_γ containing the segment x_jx_{j+1} and an initial part of the segment x_jx_i and $x_{j+1}x_i$ respectively. Let $y_k \in A_k$ for $k = 0, 1$ be points in the initial parts of the segments x_jx_i and $x_{j+1}x_i$ respectively. Thus $x_jx_{j+1}y_k$ are flat triangles in A_k. Suppose that for some $k = 0, 1$ there is a singular hyperplane $w_k \subset A_k$ such that the directions $\eta = \gamma(\infty)$, $\overrightarrow{x_jx_{j+1}}$ and $(-1)^k\overrightarrow{y_kx_{j+k}}$ lie in the same open half space determined by w_k (after the natural identification of $\partial_\infty A_i$ and $\Sigma_x A_i$ for $x \in A_i$). (See Figure 5.)

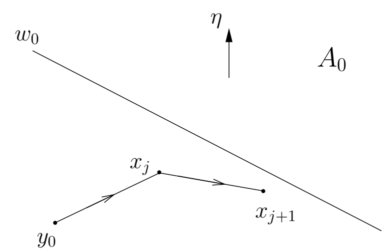

Figure 5: Setting of Lemma 6.4

Lemma 6.4. *Under the assumptions above, for any neighborhood U of $\sigma(p)$ in Δ^n_{euc} there is an n-gon $\bar{p}$ in X with $\sigma(\bar{p}) \in U$ and $L \circ \sigma(\bar{p}) > 0$.*

Proof. We show the lemma when the singular hyperplane w_k exists for $k = 0$. The other case $k = 1$ is analogous.

Denote $h^\pm$ the open half space of A_0 determined by w_0 containing the direction $\gamma(\pm\infty)$. Let $\epsilon > 0$ be small. First we displace the polygon p along γ such that x_j lies in h^+ and $d(x_j, w_0) < \epsilon$. Let A_0' be an apartment in X such that $A_0 \cap A_0' = \overline{h^-}$. Let $x_j' \in A_0'$ be the point such that $d(y_0, x_j') = d(y_0, x_j)$ and $\overrightarrow{y_0x_j'} = \overrightarrow{y_0x_j}$. Let $z \in A_0$ be the reflection of x_j in the hyperplane w_0.

Observe that $x_j' \notin A_0$ and $x_{j+1} \notin A_0'$. It follows that $\sigma(x_j'x_{j+1}) = \sigma(zx_{j+1})$. In particular

$$l_{\eta_{j+1}}(\sigma(x_j'x_{j+1})) = l_{\eta_{j+1}}(\sigma(zx_{j+1})) = \langle zx_{j+1}, \eta\rangle = l_{\eta_{j+1}}(\sigma(x_jx_{j+1})) + \langle zx_j, \eta\rangle > l_{\eta_{j+1}}(\sigma(x_jx_{j+1})).$$

Notice that the refined length of $x_j'x_i$ is the same as of x_jx_i. Hence, by Theorem 2.1 we can transfer the polygon $(x_i, x_{i+1}, \ldots, x_j)$ to a polygon $(x_i', x_{i+1}', \ldots, x_j')$ with the same Δ-valued side lengths. The n-gon $\bar{p} = (x_i', x_{i+1}', \ldots, x_j', x_{j+1}, \ldots, x_{i-1})$ satisfy the conclusion of the lemma. ☐

If the polygon p is completely contained in an apartment in P_γ, then the condition for the lemma above can be stated more easily.

Corollary 6.5. *Suppose p is contained in an apartment $A \subset P_\gamma$. Suppose there are two sides $x_i x_{i+1}$, $x_j x_{j+1}$ of p and a singular hyperplane $w \subset A$, such that the directions $\eta = \gamma(\infty)$, $\overrightarrow{x_i x_{i+1}}$ and $\overrightarrow{x_j x_{j+1}}$ lie in the same open half space determined by w. Then for any neighborhood U of $\sigma(p)$ in Δ^n_{euc} there is an n-gon $\bar{p}$ in X with $\sigma(\bar{p}) \in U$ and $L \circ \sigma(\bar{p}) > 0$.*

Proof. Consider the segments $d_1 = x_i x_j$ and $d_2 = x_j x_i$. After a small variation of the polygon p inside of the apartment A, we may assume that d_1 (and therefore also d_2) is regular. Then for one $k = 1, 2$, it must hold, that d_k and η lie in the same open half space determined by w. Suppose w.l.o.g. $k = 1$. Then Lemma 6.4 applies for the vertices x_i, x_j, x_{j+1}. □

Let us assume now that the building X has rank 2. We explain another method special for this case to cross the wall H_L.

Let $p = (x_0, x_1, x_2)$ be a regular triangle contained in P_γ but not contained in any apartment. It is easy to see, that when we fold p into an apartment A, it has exactly one break point. After relabeling the vertices we can assume that the break point y lies in the side $x_1 x_2$ and that the sides of the folded triangle $\hat{p} = (\hat{x}_0 = x_0, \hat{x}_1 = x_1, y, \hat{x}_2)$ do not intersect in their interiors (see Figure 6). After displacing $\hat{p}$ along γ we can assume that y is a vertex of X. We can take γ to be contained in A and go through y.

Lemma 6.6. *We use the setting above (in particular, rank$(X) = 2$). Suppose that the Weyl chamber containing $\overrightarrow{yx_1}$ is not adjacent to $\Sigma_y \gamma$. Then for any neighborhood U of $\sigma(p)$ in Δ^3_{euc} there are triangles p_1, p_2 in X with $\sigma(p_i) \in U$ and $L \circ \sigma(p_1) > 0 > L \circ \sigma(p_2)$.*

Proof. We identify A with $\mathbb{R}^2$ by taking y to the origin. For a unit vector $a \in A$ we write $h_a^\pm := \{\pm\langle \cdot, a\rangle > 0\}$. Let $\ell \subset A$ be the singular line through y such that $\Sigma_y \ell$ is adjacent to the simplicial convex hull of $\overrightarrow{yx_1}\overrightarrow{y\hat{x}_2}$ and the directions $\eta = \gamma(\infty)$, $\overrightarrow{yx_1}$ and $\overrightarrow{y\hat{x}_2}$ are in the same open half plane determined by ℓ. It exists by the assumptions of the lemma. Let $\ell' \subset A$ be the reflection of ℓ in γ. Let u, v, v' be unit vectors orthogonal to γ, ℓ and ℓ' respectively and such that $x_0 \in h_u^-$ and $\eta = \gamma(\infty) \in h_v^+ \cap h_{v'}^+$. Then the simplicial convex hull of $\overrightarrow{yx_1}\overrightarrow{y\hat{x}_2}$ is $\Sigma_y(\overline{h_v^+} \cap \overline{h_{v'}^-})$. (See Figure 6.)

Let A_3 be an apartment in X such that $A \cap A_3 = \overline{h_u^-} \cap \overline{h_v^-}$. Let $x_2' \in A_3$ be the point so that $d(x_0, x_2') = d(x_0, \hat{x}_2)$ and $\overrightarrow{x_0 x_2'} = \overrightarrow{x_0 \hat{x}_2}$. Notice that $\hat{x}_2 \notin A_3$, thus, $x_2' \neq \hat{x}_2$. Observe also that $x_2' \notin P_\gamma$, hence, $x_2' \neq x_2$.

Let $\zeta := \gamma(-\infty)$. The concatenation of the segments $\overrightarrow{yx_1}\overrightarrow{y\zeta} \in \Sigma_y A$ and $\overrightarrow{y\zeta}\overrightarrow{yx_2'} \in \Sigma_y A_3$ gives a segment in $\Sigma_y X$ of length π (see Figure 7). Therefore $x_1 y x_2$ is a geodesic segment and the triangle $p' = (x_0, x_1, x_2') =: (z_0, z_1, z_2)$ has the same side lengths as p. Set $A_1 := A$ and let A_2 be an apartment in X containing the segment $z_1 z_2$.

Let ν_i be the geodesic rays with $\nu_i(0) = z_i$ and $\nu_i(-\infty) = \zeta$. Then $CH(\nu_i, \nu_{i+1})$ are (2-dimensional) flat stripes. We want to see that the ν_i cannot be extended to parallel geodesic lines. Suppose then the contrary: there are parallel geodesic lines ν_i' containing ν_i. Set $\eta' := \nu_i'(\infty)$. Then $p' \subset Y := P_{\nu_0'}$ and in particular, $\overrightarrow{y\zeta}, \overrightarrow{yz_i} \in \Sigma_y Y$. Since $\overrightarrow{yz_1}, \overrightarrow{yz_2} \in \Sigma_y A_2$ are antipodal regular points, the apartment containing them is unique. Therefore $\Sigma_y A_2 \subset \Sigma_y Y$

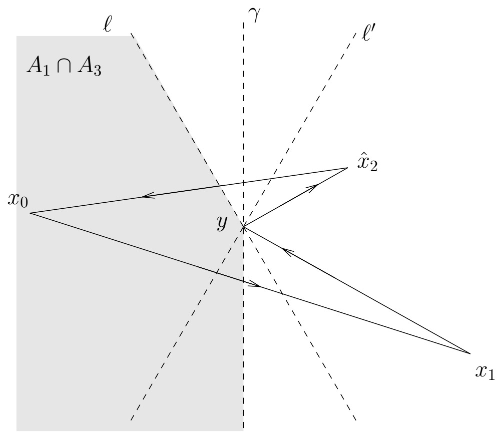

Figure 6: The folded triangle $\hat{p}$

and in particular, $\overrightarrow{y\eta'} \in \Sigma_y A_2$.

Let $k \in \{1, 2\}$ so that the Weyl chamber containing $\overrightarrow{y\hat{x}_k}$ is adjacent to $\Sigma_y \ell'$. Let $\sigma_k \subset \Sigma_y A_{2k-1}$ be the Weyl chamber containing $\overrightarrow{yz_k}$ and let $\hat{\sigma}_k \in \Sigma_y(A_1 \cap A_3)$ be the antipodal chamber to σ_k. (See Figure 7 for $k = 2$.) Notice that $\overrightarrow{y\zeta}\overrightarrow{yz_0}$ intersects $\hat{\sigma}_k$ in its interior. In particular $\hat{\sigma}_k \subset \Sigma_y Y$. It follows that the unique apartment containing σ_k and $\hat{\sigma}_k$ is contained in $\Sigma_y Y$, i.e. $\Sigma_y A_{2k-1} \subset \Sigma_y Y$.

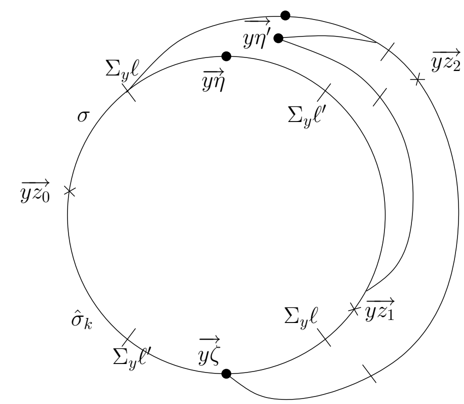

Figure 7: $\Sigma_y X$

Let $\sigma \subset \Sigma_y(A_1 \cap A_3) \subset \Sigma_y Y$ be the Weyl chamber adjacent to ℓ. The Weyl chamber containing $\overrightarrow{yz_{3-k}}$ is antipodal to σ. Hence, the unique apartment containing σ and $\overrightarrow{yz_{3-k}}$ is contained in $\Sigma_y Y$, i.e. $\Sigma_y A_{5-2k} \subset \Sigma_y Y$.

We have conclude that $A_1, A_3 \subset \Sigma_y Y = \Sigma_y P_{\nu_0'}$, but this is not possible because of the construction of A_3. Therefore the geodesic rays ν_i cannot be extended to complete parallel geodesic lines. The lemma now follows from Lemma 6.2 and its proof. □

We can show now that for rank 2 the space $\mathcal{P}_n(X)$ is a polyhedral cone. Its convexity will be shown in the next section.

Proposition 6.7. *If X has rank 2, then $\mathcal{P}_n(X)$ is a union of the closures of polyhedral cones in $\mathcal{C}_n$.*

Proof. We have already seen in Corollary 5.5 that if for $C \in \mathcal{C}_n$ holds $\mathcal{P}_n(X) \cap C \neq \emptyset$, then $\overline{C} \subset \mathcal{P}_n(X)$. Now let $p = (x_0, \dots, x_{n-1})$ be a polygon in X. We want to show that $\sigma(p)$ is contained in $\overline{C}$ for some $C \in \mathcal{C}_n$. Since any polygon can be approximated by regular polygons, we may assume that p is regular. Suppose now $s := \sigma(p) \in H_L$. If for any neighborhood U of s we can find polygons with side lengths in $U \setminus H_L$, then we are done. Indeed, in this case, there is an open cone $C \in \mathcal{C}_n$ such that $\mathcal{P}_n(X) \cap C \neq \emptyset$ and $s \in \overline{C}$.

Suppose then that for some neighborhood U of $\sigma(p)$ we cannot find polygons p' with side lengths in U and $L \circ \sigma(p') \neq 0$. Lemmata 6.1 and 6.2 implies that p lies in a parallel set P_γ and the functional L is given in p by taking scalar product with the direction of $\eta = \gamma(\infty)$. Suppose first that the triangle $t = (x_0, x_1, x_2)$ lies in an apartment parallel to γ. Then it is easy to see that Lemma 6.4 must apply for one of the functionals $L' = (l_{\eta_1}, l_{\eta_2}, l_{\eta'})$ or $-L'$, where η' is so that $l_{\eta'}(\sigma(x_2x_0)) = \langle x_2x_0, \eta \rangle$. If t is not contained in an apartment, then we fold it into an apartment as in the setting of Lemma 6.6. Then, either Lemma 6.6 applies or the Weyl chamber containing the direction $\overrightarrow{x_ix_{i+1}}$ of the side of t with the break point must be adjacent to γ. If the last occurs, it is again easy to see, that Lemma 6.4 must apply for L' or $-L'$. In either case, we find a triangle $t' = (x'_0, x'_1, x'_2)$ with $L'(\sigma(t)) \neq 0$ and such that (modulo displacement along γ) the refined side lengths of t' are as near as we want to the ones of t. After a small variation of the polygon $(x_0, x_2, \dots, x_{n-1})$ inside the parallel set P_γ and displacing it along γ, we obtain a polygon $q = (x''_0, x''_2, \dots, x''_{n-1})$ so that the refined side length of $x'_0x'_2$ is the same as of $x''_0x''_2$. Then by the Transfer Theorem 2.1 we can glue t and q along $x'_0x'_2$ and $x''_0x''_2$ to a polygon p' with Δ-valued side lengths near s and $L(\sigma(p)) \neq 0$. ☐

Remark 6.8. Proposition 6.7 is also true in rank > 2 by the results of [KLM09a] and [KLM09b]. However our proof here uses Lemma 6.6, which we only showed in rank 2.

6.2 The boundary of $\mathcal{P}_n(X)$

We have seen in the previous section different methods which allows to cross certain walls H_L within the space $\mathcal{P}_n(X)$. We will show in this section that for the case of buildings of rank 2 the walls where this method cannot be applied are precisely the walls that determine the boundary of $\mathcal{P}_n(X)$. That is, if a wall cannot be crossed with the methods of Section 6.1, it is because that wall cannot be crossed at all.

First we characterize the walls H_L that cannot be crossed with the methods above in terms of the combinatorics of the associated spherical Coxeter complex (S, W). Let $\eta \in \Delta_{euc} \subset E$ be a maximal singular unit vector (we use the same notation as in Section 3). We define the following set of singular hyperplanes of E through v_0 (i.e. walls of (E, W)):

$$T_\eta := \{w \subset E \mid w \text{ is a wall of } (E, W) \text{ not containing } \eta\}.$$

For each element $\omega \in W \cong Stab_{W_{aff}}(v_0)$ we define the subset of T_η

$$T^\omega_\eta := \{w \in T \mid \eta \text{ and } \omega\Delta_{euc} \text{ lie in the same half space determined by } w\}.$$

Finally define B_η as the set of n-tuples $(\eta_1, \dots, \eta_n) \in (W\eta)^n$ such that for $i = 1, \dots, n$ there are $\omega_i \in W$ with $\omega_i\eta_i = \eta$ and with the following properties:

(∗) $T_\eta^{\omega_i} \cap T_\eta^{\omega_j} = \emptyset$ for all $i \neq j$,

(∗∗) $\bigcup_{i=1}^{n} T_\eta^{\omega_i} = T_\eta$.

For $\bar\eta = (\eta_1, \ldots, \eta_n) \in (W\eta)^n$ write $L_{\bar\eta} = (l_{\eta_1}, \ldots, l_{\eta_n})$. Let $\mathcal{B}_n \subset \mathcal{L}_n$ be the union of the sets $\{L_{\bar\eta} \mid \bar\eta \in B_\eta\}$ for all maximal singular unit vectors $\eta \in \Delta_{euc}$.

We will see in Lemma 6.13 below that the walls H_L that cannot be crossed with our previous methods are precisely the ones of the form $L_{\bar\eta}$ with $\bar\eta = (\eta_1, \ldots, \eta_n)$ satisfying the property (∗). A motivation for this property (∗) can already be seen in Corollary 6.5. The property (∗∗) is introduced to avoid later obvious redundancies in the set of generalized triangle inequalities. This can be seen in the Proposition 6.11.

Lemma 6.9. *If (E, W) has rank 2, then $\bar\eta \in B_\eta$ if and only if for $i = 1, \ldots, n$ we can find $\omega_i \in W$ with $\omega_i\eta_i = \eta$ such that there exist $j, j' \in \{1, \ldots, n\}$, $j \neq j'$ with $\omega_j\Delta_{euc}$ antipodal to $\omega_{j'}\Delta_{euc}$ and $\omega_i\Delta_{euc}$ adjacent to $-\eta$ for $i \neq j, j'$.*

Proof. (⇐). $\omega_j\Delta_{euc}$ is antipodal to $\omega_{j'}\Delta_{euc}$ if and only if $T_\eta^{\omega_j} = T_\eta \setminus T_\eta^{\omega_{j'}}$. On the other hand, $\omega_i\Delta_{euc}$ is adjacent to $-\eta$ if and only if $T_\eta^{\omega_i} = \emptyset$.

(⇒). By property (∗∗), there is a j with $T_\eta^{\omega_j} \neq \emptyset$. If $T_\eta^{\omega_j} = T_\eta$, then $\omega_j\Delta_{euc} = \Delta_{euc}$ and the assertion is clear. Otherwise let $\ell_1 \in T_\eta^{\omega_j}$ be the singular line adjacent to $\omega_j\Delta_{euc}$. Let ℓ_2 be the other singular line adjacent to $\omega_j\Delta_{euc}$. Then $\ell_2 \notin T_\eta^{\omega_j}$. Let j' be such that $\ell_2 \in T_\eta^{\omega_{j'}}$, it follows that $\ell_1 \notin T_\eta^{\omega_{j'}}$ and $\omega_{j'}\Delta_{euc}$ must be antipodal to $\omega_j\Delta_{euc}$. The rest follows as in the first part. □

Remark 6.10. The ⇐ direction in Lemma 6.9 holds for arbitrary rank. Let $\mathcal{B}_n^w \subset \mathcal{L}_n$ be the set of functionals $L_{\bar\eta}$ for $\bar\eta$ with this property (the assumption in the ⇐ direction). The inequalities $L \leq 0$ for $L \in \mathcal{B}_n^w$ are the so-called *weak triangle inequalities* (cf. [KLM09a, Section 3.8]). Thus, Lemma 6.9 states that $\mathcal{B}_n^w \subset \mathcal{B}_n$ and for rank 2 also holds $\mathcal{B}_n = \mathcal{B}_n^w$.

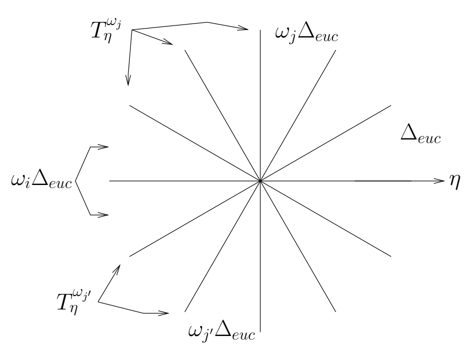

Figure 8: $\mathcal{B}_n^w$: weak triangle inequalities

Proposition 6.11. *Suppose X has rank 2. For any n-gon p in X and any functional $L \in \mathcal{B}_n$ holds $L \circ \sigma(p) \leq 0$. That is,*

$$\mathcal{P}_n(X) \subset \bigcap_{L \in \mathcal{B}_n} \{L \leq 0\}.$$

Moreover, if $\bar{\eta} \in (W\eta)^n$ satisfies the property $(*)$ but not the property $(**)$, then there is a $\bar{\eta}' \in B_\eta$ so that $L_{\bar{\eta}} \circ \sigma(p) \leq L_{\bar{\eta}'} \circ \sigma(p)$ for all n-gons p in X. If p is regular, then the strict inequality holds.

Proof. Let $p = (x_0, \ldots, x_{n-1})$ be an n-gon in X. For the functional $L = (l_{\eta_1}, \ldots, l_{\eta_n}) \in \mathcal{B}_n$, let $\omega_i \in W$ and j, j' be as in Lemma 6.9. Notice that since for $i \neq j, j'$, $\omega_i \Delta_{euc}$ is adjacent to $-\eta$ and $\omega_i \eta_i = \eta$ then we have $l_{\eta_i} \leq l_{\eta'}$ in Δ_{euc} for all η' of the same type as η. That is, l_{η_i} is the smallest functional of the same type as η. After shifting the subindices of the polygon and the functional we can assume that $j = 1$.

Suppose first that $j' = j - 1$, that is $j' = n$. Fold the polygon p into an apartment A, so that the broken sides are $x_1x_2, \ldots, x_{n-2}x_{n-1}$. Let $\rho : A \to E$ be an isometry that sends x_0 to the vertex of $\Delta_{euc} \subset E$, induces an isomorphism of the Coxeter complexes $(\partial_\infty A, W)$ and (E, W) and so that $\rho(x_0x_1) \subset \omega_1 \Delta_{euc}$. Notice that ρ is not necessarily an isomorphism of Coxeter complexes with the Weyl group W_{aff}. Denote with q the image under ρ of the folded polygon. By folding E onto the Euclidean Weyl chamber $\omega_1 \Delta_{euc}$ with the natural "accordion" map, we obtain a further folded polygon $q' = (y_0, \ldots, y_k)$ where y_0 is the vertex of Δ_{euc} and the Δ-valued side lengths of $y_0y_1, y_ky_0 \subset \omega_1 \Delta_{euc}$ are the same as for x_0x_1 and $x_{n-1}x_0$ respectively. Observe that q' is not necessarily a billiard polygon in (E, W_{aff}), but if the side x_rx_{r+1} of p is broken in q' to the sides $y_sy_{s+1}, y_{s+1}y_{s+2}, \ldots, y_{t-1}y_t$, then the vectors $\sigma(y_sy_{s+1}), \ldots, \sigma(y_{t-1}y_t)$ are just multiples of $\sigma(x_rx_{r+1})$. This means, that if W'_{aff} is the group generated by W_{aff} and the whole translation group of E, then q' is a billiard polygon in (E, W'_{aff}). Notice also that for $r \neq 1, n$ holds $l_{\eta_r}(\sigma(y_ly_{l+1})) \leq \langle y_{l+1}, \eta\rangle - \langle y_l, \eta\rangle$ because of the observation at the beginning of the proof. It follows that

$$l_{\eta_2}(\sigma(x_1x_2)) + \cdots + l_{\eta_{n-1}}(\sigma(x_{n-2}x_{n-1})) \leq \langle y_k, \eta\rangle - \langle y_1, \eta\rangle.$$

On the other hand, since $y_0y_1, y_ky_0 \subset \omega_1\Delta_{euc}$ and $\omega_n\Delta_{euc}$ is antipodal to $\omega_1\Delta_{euc}$ it follows that $l_{\eta_1}(\sigma(x_0x_1)) = l_{\eta_1}(\sigma(y_0y_1)) = \langle y_1, \eta\rangle - \langle y_0, \eta\rangle$ and $l_{\eta_n}(\sigma(x_{n-1}x_0)) = l_{\eta_n}(\sigma(y_ky_0)) = \langle y_0, \eta\rangle - \langle y_k, \eta\rangle$. Hence, $L(\sigma(p)) \leq \langle y_k, \eta\rangle - \langle y_1, \eta\rangle + \langle y_1, \eta\rangle - \langle y_0, \eta\rangle + \langle y_0, \eta\rangle - \langle y_k, \eta\rangle = 0$.

The general case now follows from the special case above by considering the polygons $p_1 = (x_{j'-1}, x_{j-1}, x_j, \ldots, x_{j'-2})$, i.e. p_1 is the polygon p with the vertices $x_{j'}, x_{j'+1}, \ldots, x_{j-2}$ deleted, and $p_2 = (x_{j-1}, x_{j'-1}, x_{j'}, \ldots, x_{j-2})$ with the functionals $(l_{\eta_{j'}}, l_{\eta_j}, l_{\eta_{j+1}}, \ldots, l_{\eta_{j'-1}})$ respectively $(l_{\eta_j}, l_{\eta'_j}, l_{\eta_{j'+1}}, \ldots, l_{\eta_{j-1}})$. Indeed, notice that since $\omega_j\Delta_{euc}$ and $\omega_{j'}\Delta_{euc}$ are antipodal, it follows $l_{\eta_{j'}}(\sigma(x_{j'-1}x_{j-1})) = -l_{\eta_j}(\sigma(x_{j-1}x_{j'-1}))$.

For the second assertion, let $\omega_i\eta_i = \eta$ satisfy the property $(*)$. It is easy to see that in rank 2 at most for two indices i can hold $T_\eta^{\omega_i} \neq \emptyset$. Let $j \neq j'$ be so that $T_\eta^{\omega_i} = \emptyset$ for all $i \neq j, j'$. If $\bar{\eta}$ does not satisfy the property $(**)$, then $\omega_{j'}\Delta_{euc}$ is not antipodal to $\omega_j\Delta_{euc}$. Let $\hat{\omega}_j \in W$ be so that $\hat{\omega}_j\Delta_{euc}$ is antipodal to $\omega_j\Delta_{euc}$. From the property $(*)$ follows that for $\hat{\eta}_j := \hat{\omega}_j^{-1}\eta$ holds $l_{\eta_{j'}} \leq l_{\hat{\eta}_j}$ and since $\hat{\eta}_j \neq \eta_{j'}$ the strict inequality holds for regular segments. □

Remark 6.12. The same proof as for the first assertion of Proposition 6.11 works for buildings of arbitrary rank to prove the weak triangle inequalities (see Remark 6.10). That is,

$$\mathcal{P}_n(X) \subset \bigcap_{L \in \mathcal{B}_n^w} \{L \leq 0\}.$$

Lemma 6.13. *Suppose X has rank 2 and let p be a regular n-gon in X. Suppose that $\sigma(p) \in H_L$ for some functional L with $L, -L \in \mathcal{L}_n \setminus \mathcal{B}_n$. Then for any neighborhood U of $\sigma(p)$ in Δ^n_{euc} there exist n-gons p_1, p_2 in X with $\sigma(p_i) \in U$ and $L \circ \sigma(p_1) > 0 > L \circ \sigma(p_2)$.*

Proof. Suppose that for a neighborhood U of $\sigma(p)$ in Δ^n_{euc}, we cannot find a polygon p_1 in X with $\sigma(p_1) \in U$ and $L \circ \sigma(p_1) > 0$. (The other inequality follows considering the functional $-L$.) It follows from Lemmata 6.1 and 6.2 that p lies in a parallel set P_γ and the functional L in p is just given by taking scalar product with the direction of $\gamma(\infty)$. Fold the polygon in an apartment $A \subset P_\gamma$ so that the broken sides are $x_1x_2, \dots, x_{n-2}x_{n-1}$. Let $\rho : A \to E$ be an isomorphism that sends $\eta = \gamma(\infty)$ to the singular direction in Δ_{euc} of the same type. By abusing the notation, we write also η to denote the unit vector in Δ_{euc} with direction $\rho(\eta)$.

Suppose X has only one vertex and γ goes through it. Then the break points of the folded polygon all lie on γ. We may assume that the folded polygon has only one break point because any two consecutive break points can be simultaneously *unfolded*. Let k be so that the break point y lies on the side x_kx_{k+1} (if there is no break point we take $k = n-1$). Then the folded polygon has the form $p' = (x_0, x_1, \dots, x_k, y, \hat{x}_{k+1}, \dots, \hat{x}_{n-1})$. Let $\omega_i \in W$ be so that $\omega_i\Delta_{euc}$ contains the direction $\rho(\overrightarrow{x_{i-1}x_i})$ for $1 \le i \le k$, $\rho(\overrightarrow{x_ky})$ for $i = k+1$, $\rho(\overrightarrow{\hat{x}_{i-1}\hat{x}_i})$ for $k+2 \le i \le n-1$, and $\rho(\overrightarrow{\hat{x}_{n-1}x_0})$ for $i = n$, respectively. Then the functional L is just given by $(l_{\eta_1}, \dots, l_{\eta_n})$ for $\eta_i = \omega_i^{-1}\eta$. After a small variation inside the parallel set P_γ we may assume that the segments x_0x_k and $x_0\hat{x}_{k+1}$ are regular. Let $\alpha, \beta \in W$ be so that $\alpha\Delta_{euc}$ contains the direction $\rho(\overrightarrow{x_0x_k})$ and $\beta\Delta_{euc}$ contains $\rho(\overrightarrow{\hat{x}_{k+1}x_0})$. Let $\delta \in W$ be such that Δ_{euc} and $\delta\Delta_{euc}$ are antipodal. For $\omega \in W$ set $\tilde{\omega} := \delta\omega$.

Consider the regular polygon $q = (x_0, \dots, x_k) \subset A$ and the functional $L' = (l_{\eta_1}, \dots, l_{\eta_k}, l_{\eta'})$ for $\eta' := \tilde{\alpha}^{-1}\eta$. That is, L' is the functional given in q by taking scalar product with the direction η. Hence $L'(\sigma(q)) = 0$. Set $(\tau_1, \dots, \tau_k, \tau_{k+1}) := (\omega_1, \dots, \omega_k, \tilde{\alpha})$. Suppose that there are $1 \le i < j \le k+1$ such that $T^{\tau_i}_\eta \cap T^{\tau_j}_\eta \neq \emptyset$. Corollary 6.5 and its proof imply that there is a polygon $q' = (z_0, \dots, z_k)$ with $L'(\sigma(q')) > 0$ and with refined side lengths as near as we want to those of q modulo displacement along γ. We can then choose $x'_k \in P_\gamma$ near x_k such that $x_0x'_k$ has the same refined side length (again modulo displacement along γ) as z_0z_k. The functional $(-l_{\eta'}, l_{\eta_{k+1}}, \dots, l_{\eta_n})$ applied to the polygon $(x_0, x'_k, x_{k+1}, \dots, x_{n-1})$ is 0 because it is contained in the parallel set P_γ. After displacing the polygon $(x_0, x'_k, x_{k+1}, \dots, x_{n-1})$ along γ we can glue it together to q' and obtain a polygon p_1 with Δ-valued side lengths as near as we want to those of p and with $L(\sigma(p_1)) > 0$ (compare with the proof of Proposition 6.7). This contradicts the assumption at the beginning of the proof. Thus, $T^{\tau_i}_\eta \cap T^{\tau_j}_\eta = \emptyset$ for all $1 \le i < j \le k+1$. Since q is a regular polygon with $L(\sigma(q)) = 0$, then by the second claim in Proposition 6.11 we must also have $T^{\tilde{\alpha}}_\eta = T_\eta \setminus \bigcup_{i=1}^{k} T^{\omega_i}_\eta$, or equivalently, $T^{\alpha}_\eta = \bigcup_{i=1}^{k} T^{\omega_i}_\eta$.

Analogously, considering the polygon $(x_0, \hat{x}_{k+1}, \dots, \hat{x}_{n-1})$ we obtain $T^{\omega_i}_\eta \cap T^{\omega_j}_\eta = \emptyset$ for all $k+2 \le i < j \le n$ and $T^{\beta}_\eta = \bigcup_{i=k+2}^{n} T^{\omega_i}_\eta$.

Consider now the triangle $t = (x_0, x_k, x_{k+1})$ with the functional $L'' = (l_{\alpha^{-1}\eta}, l_{\eta_{k+1}}, l_{\beta^{-1}\eta})$. Let $\omega'_{k+1} \in W$ be so that $\omega'_{k+1}\Delta_{euc}$ contains the direction $\rho(\overrightarrow{y\hat{x}_{k+1}})$. Then $\omega'_{k+1}\eta_{k+1} = \omega_{k+1}\eta_{k+1} = \eta$. We want to show that $(\alpha, \omega_{k+1}, \beta)$ or $(\alpha, \omega'_{k+1}, \beta)$ have the property $(*)$. By Lemma 6.4 applied to the side x_0x_k we get $T^{\alpha}_\eta \cap T^{\beta}_\eta = \emptyset = T^{\alpha}_\eta \cap T^{\omega_{k+1}}_\eta$. Again by Lemma 6.4 now applied to the side $x_{k+1}x_0$ we obtain $T_\eta^\beta \cap T_\eta^{\omega'_{k+1}} = \emptyset$. Therefore if T_η^α or $T_\eta^\beta = \emptyset$, then we are done, so suppose both are nonempty. Now by Lemma 6.6 one of $\alpha\Delta_{euc}$, $\beta\Delta_{euc}$ or $\omega_{k+1}\Delta_{euc}$ must be adjacent to $\rho(\gamma)$. Notice that for $\omega \in W$, $\omega\Delta_{euc}$ is adjacent to $\rho(\gamma)$ if and only if $T_\eta^\omega \in \{\emptyset, T_\eta\}$. This and $T_\eta^\alpha \cap T_\eta^\beta = \emptyset$ imply that $\omega_{k+1}\Delta_{euc}$ must be adjacent to $\rho(\gamma)$. $T_\eta^\alpha \cap T_\eta^{\omega_{k+1}} = \emptyset$ implies that $T_\eta^{\omega_{k+1}}$ must be empty and we are also done in this case.

So we have conclude that $\bar{\eta} = (\eta_1, \dots, \eta_n)$ has the property $(*)$ and since p is a regular polygon with $L(\sigma(p)) = 0$, it follows from Proposition 6.11 that $L \in \mathcal{B}_n$. $\square$

Now we are ready to prove our main theorem.

Theorem 6.14. *Let X be a building of rank 2. $\mathcal{P}_n(X)$ is a convex polyhedral cone determined by the inequalities $\{L \leq 0\}$ for $L \in \mathcal{B}_n$. That is,*

$$\mathcal{P}_n(X) = \bigcap_{L \in \mathcal{B}_n} \{L \leq 0\}.$$

This inequalities constitute an irredundant set of inequalities.

Proof. Let $Q \subset \mathcal{C}_n$ be the subset of open cones such that $\bigcap_{L \in \mathcal{B}_n} \{L \leq 0\} = \bigcup_{C \in Q} \bar{C}$. Analogously, let $Q' \subset \mathcal{C}_n$ be the subset of open cones such that $\mathcal{P}_n(X) = \bigcup_{C \in Q'} \bar{C}$ (this can be done by Proposition 6.7). We have shown in Proposition 6.11 that $Q' \subset Q$. Let $C_0 \in Q'$ and $C \in Q$. Take a chain $C_0, C_1, \dots, C_k = C \in Q$ such that $\overline{C_i} \cap \overline{C_{i+1}}$ is a face of codimension one. We prove now inductively that $C_i \in Q'$. Suppose then that $C_i \in Q'$ and take a regular polygon p with $\sigma(p)$ in the interior of the face $\overline{C_i} \cap \overline{C_{i+1}}$. Since $\overline{C_i} \cap \overline{C_{i+1}}$ is not in the boundary of $\bigcap_{L \in \mathcal{B}_n} \{L \leq 0\}$, it lies in a wall H_L with neither $L, -L$ in $\mathcal{B}_n$. It follows from Lemma 6.13 that $\mathcal{P}_n(X) \cap C_{i+1}$ is not empty and therefore $C_{i+1} \subset \mathcal{P}_n(X)$. Thus $C \in Q'$, and $Q = Q'$.

For $L \in \mathcal{B}_n$ it is clear that we can find a regular polygon p in an apartment A and $\gamma \subset A$ a maximal singular line, such that the functional L in p is given by taking scalar product with the direction of $\eta = \gamma(\infty)$. In particular, $L(\sigma(p)) = 0$. It is also clear that we can find a regular polygon p' in P_γ but not contained in any apartment and such that the functional L in p' is also given by taking scalar product with the direction of η. It follows from Lemmata 6.1 and 6.2 that L is the only functional in $\mathcal{B}_n$ for which it can hold $L(\sigma(p')) = 0$. Thus the inequalities $\{L \leq 0\}$ with $L \in \mathcal{B}_n$ are irredundant. $\square$

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