Chiral Differential Operators on Supermanifolds

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Abstract

The first part of this paper provides a new description of chiral differential operators (CDOs) in terms of global geometric quantities. The main result is a recipe to define all sheaves of CDOs on a smooth cs-manifold; its ingredients consist of an affine connection ∇ and an even 3-form that trivializes $p_1(\nabla)$. With ∇ fixed, two suitable 3-forms define isomorphic sheaves of CDOs if and only if their difference is exact. Moreover, conformal structures are in one-to-one correspondence with even 1-forms that trivialize $c_1(\nabla)$.

Applying our work in the first part, we construct what may be called "chiral Dolbeault complexes" of a complex manifold M, and analyze conditions under which these differential vertex superalgebras admit compatible conformal structures or extra gradings (fermion numbers). When M is compact, their cohomology computes (in various cases) the Witten genus, the two-variable elliptic genus and a spin^c version of the Witten genus. This part contains some new results as well as provides a geometric formulation of certain known facts from the study of holomorphic CDOs and σ-models.

§1. Introduction

In physics, the study of a type of quantum field theory called *σ-models* has inspired many important insights in topology and geometry. The theory of elliptic genera is an example. In particular, associated to any compact, string manifold [1] M is a σ-model whose "partition function" equals, up to a constant factor, the formal power series

$$W(M) = \int_M \hat{A}(TM)\, ch\Big(\bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TM\otimes\mathbb{C})\Big)\cdot \prod_{n=1}^{\infty}(1-q^n)^{\dim M}$$

known as the *Witten genus* of M. [Wit87, Wit88] Similarly, associated to any compact, spin manifold M is another σ-model, which gives rise to the formal power series

$$\mathrm{Och}(M) = \int_M L(TM)\, ch\Big(\bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TM\otimes\mathbb{C})\otimes \bigotimes_{n=1}^{\infty}\wedge_{q^n}(TM\otimes\mathbb{C})\Big)\cdot \prod_{n=1}^{\infty}\Big(\frac{1-q^n}{1+q^n}\Big)^{\dim M}$$

known as the *Ochanine elliptic genus* of M. [Och87, Wit87] The physical interpretation of these topological invariants have led to predictions that are not immediately clear from the mathematical point of view. Even though many of them have since been verified, e.g. [Zag88, BT89], a complete, geometric understanding of elliptic genera has yet to emerge. The latter probably requires to some extent a mathematical framework for σ-models.

Sheaves of vertex algebras provide a mathematical approach to σ-models. Important constructions along this line include the *chiral de Rham complex* and, more generally, sheaves of *chiral differential operators*, or *CDOs*. [MSV99, GMS00] In particular, a complex manifold M admits a sheaf of holomorphic

[1] Let $\lambda \in H^4(B\mathrm{Spin};\mathbb{Z}) \cong \mathbb{Z}$ be the generator such that $2\lambda = p_1$. This defines a characteristic class $\lambda(\cdot)$ for spin vector bundles. A spin manifold M is said to be *string* if $\lambda(TM) = 0$. Moreover, a *string structure* on M is a "trivialization of $\lambda(TM)$", i.e. a homotopy class of liftings of the classifying map $M \to B\mathrm{Spin}$ along the homotopy fiber of $\lambda : B\mathrm{Spin} \to K(\mathbb{Z}, 4)$.

CDOs $\mathcal{D}_M^{\mathrm{ch}}$ with a conformal structure if and only if $c_1^{\mathrm{hol}}(TM) = c_2^{\mathrm{hol}}(TM) = 0$; [2] notice that M as a spin^c manifold admits a string structure if and only if $c_1(TM) = c_2(TM) = 0$. Furthermore, if M is compact

$$\operatorname{char} H^*(M, \mathcal{D}_M^{\mathrm{ch}}) = W(M) \cdot (\text{a constant factor})$$

suggesting a connection between $\mathcal{D}_M^{\mathrm{ch}}$ and the σ-model underlying the Witten genus. In fact, physicists have recognized a connection between CDOs and σ-models of various flavors. [Kap06, Wit07, Tan06] More recently, a new construction of the Witten genus has been given under a systematic mathematical framework for perturbative quantum field theory. [Cos10]

The first goal of this paper is to provide a new description of CDOs using global geometric quantities and the language of cs-manifolds, i.e. supermanifolds equipped with $\mathbb{C}$-valued functions. The algebra of *smooth* CDOs on $\mathbb{R}^{p|q}$ is the smooth analogue of the conformal vertex superalgebra $(\beta\gamma)^{\otimes p} \otimes (bc)^{\otimes q}$ (§2.1, Proposition 2.2); its behavior under a change of coordinates, first computed in [GMS00], are restated here in more geometric terms (§2.3, Proposition 2.4). The notions of a sheaf of CDOs and its conformal structures are then generalized from $\mathbb{R}^{p|q}$ to a general cs-manifold $\mathbf{M}$ in a natural way (Definition 2.5). After dealing with some technical issues (Lemmas 2.6, 2.7), we prove the main result on the global construction of CDOs (Theorem 2.8). Namely, given an affine connection ∇ and an even 3-form H that satisfies

$$dH = \operatorname{Str}(R \wedge R)$$

where $R = \operatorname{curv}(\nabla)$, there is a recipe to define a sheaf of CDOs $\mathcal{D}_{\mathbf{M},\nabla,H}^{\mathrm{ch}}$, and this recipe yields essentially all sheaves of CDOs on $\mathbf{M}$. Moreover, conformal structures on $\mathcal{D}_{\mathbf{M},\nabla,H}^{\mathrm{ch}}$ are in one-to-one correspondence with even 1-forms ω that satisfy

$$d\omega = \operatorname{Str} R.$$

To classify these objects, we also prove that, with ∇ fixed, two suitable 3-forms H, H' define isomorphic sheaves of CDOs if and only if $H - H'$ is exact (Theorem 2.11). In contrast to [GMS00], our description of CDOs does not rely on a choice of coordinate charts or other local data. For the special case of the chiral de Rham complex, in which both H and ω are trivial (Example 2.13), an invariant description has also been given in [BHS08]. The formulation of CDOs developed here has been applied e.g. to study how to lift a Lie group action on a manifold to a "formal loop group action" on CDOs. [Che11]

In the rest of the paper, we apply our work in the first part to construct what may be called "chiral Dolbeault complexes." Let M be a complex manifold and $E \to M$ a holomorphic vector bundle. The Dolbeault complex of M valued in $\wedge^* E^\vee$ is identified with the smooth functions on the cs-manifold

$$\mathbf{M} = \Pi(\overline{TM} \oplus E)$$

under the action of an odd vector field Q that satisfies $Q^2 = 0$ (§3.1). This motivates us to construct a sheaf of CDOs $\mathcal{D}_{\mathbf{M},\nabla,H}^{\mathrm{ch}}$ on $\mathbf{M}$ (§3.2), and study the condition under which the supersymmetry Q lifts to one on CDOs, i.e. an odd derivation $\hat{Q}$ on $\mathcal{D}_{\mathbf{M},\nabla,H}^{\mathrm{ch}}$ that satisfies $\hat{Q}^2 = 0$ (Theorem 3.3, Proposition 3.5). At the same time we also analyze the condition for $\hat{Q}$ to respect a conformal structure. Moreover, if one or both of the line bundles $\det TM$, $\det E$ are flat, $\hat{Q}$ is compatible with certain gradings on $\mathcal{D}_{\mathbf{M},\nabla,H}^{\mathrm{ch}}$ called fermion numbers (§3.6, Propositions 3.7, 3.8). The sheaf of differential vertex superalgebras

$$(\mathcal{D}_{\mathbf{M},\nabla,H}^{\mathrm{ch}}, \hat{Q})$$

may be thought of as a Dolbeault resolution of holomorphic CDOs on ΠE, as well as a particular limit of a σ-model. [Kap06] When M is compact, its cohomology computes various elliptic genera (Theorem 3.10), including the Witten genus in the case $E = 0$ (Example 3.13), a two-variable generalization of the Ochanine genus in the case $E = TM$ (Example 3.14), and a spin^c version of the Witten genus [CHZ10] in the cases $E = \det TM$ and $E = (\det TM)^{\otimes 2} - \det TM$ (Examples 3.15, 3.16). Most of the results in this

[2] See Definition 3.4.

part are similar to and consistent with what is known from the study of holomorphic CDOs and σ-models, but our formulation may provide a new geometric point of view. On the other hand, the last two examples seem to be new.

The first appendix reviews the notion of vertex algebroids (first introduced in [GMS04]), their relation with vertex algebras, and gives some examples. Despite the rather complicated-looking definition, vertex algebroids and their super analogues provide a convenient tool in our study of CDOs. In the second appendix, we construct affine connections on cs-manifolds and obtain formulae that are needed in various calculations with CDOs.

Conventions. For the definition of a vertex superalgebra, see [Kac98, FB04]. In this paper, every vertex superalgebra V is graded by non-negative integers called weights. The notation V_k means its component of weight k, and L_0 denotes the weight operator, so that $L_0|_{V_k} = k$.

For the definition of a cs-manifold, see [DM99]. Given a smooth cs-manifold $\mathbf{M}$, we always denote by $C^\infty_{\mathbf{M}}$, $\mathcal{T}_{\mathbf{M}}$ and $\Omega^n_{\mathbf{M}}$ its sheaves of *smooth* functions, vector fields and n-forms; when "$\mathbf{M}$" appears in parentheses instead of the subscript, it means the corresponding spaces of global sections. Restricting $C^\infty_{\mathbf{M}}$ to an open subset $U \subset \mathbf{M}^{\text{red}}$ defines a new cs-manifold, denoted by $\mathbf{M}|_U$. Square brackets are used for supercommutators between operators of any parities, while "Str" stands for the supertrace. Notice that $\mathbb{R}^{p|q}$ is regarded as a cs-manifold in this paper, namely

$$C^\infty_{\mathbb{R}^{p|q}} = C^\infty_{\mathbb{R}^p} \otimes \wedge^*(\mathbb{R}^q) \otimes \mathbb{C}.$$

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§2. Chiral Differential Operators

Sheaves of CDOs on a manifold were first studied in [GMS00]. This section provides an alternative construction of the smooth version using global geometric quantities.

§ 2.1. The sheaf of CDOs on $\mathbb{R}^{p|q}$. Let $b^1, \ldots, b^p$ and $b^{p+1}, \ldots, b^{p+q}$ be respectively the even and odd coordinates of $\mathbb{R}^{p|q}$. The following notations are used

$$\partial_i = \frac{\partial}{\partial b^i}\,, \qquad |\cdot| = \text{parity}, \quad \epsilon_i = (-1)^{|b^i|}, \quad \epsilon_{ij} = (-1)^{|b^i||b^j|}$$

and repeated indices are summed over (but not counting those from $\epsilon_i, \epsilon_{ij}$). Regard $\mathbb{R}^{p|q}$ as a smooth cs-manifold, namely

$$C^\infty(\mathbb{R}^{p|q}) = C^\infty(\mathbb{R}^p) \otimes \bigwedge(b^{p+1}, \ldots, b^{p+q}) \otimes \mathbb{C}.$$

Given an open set $W \subset \mathbb{R}^p$, let $\mathbf{W} = (\mathbb{R}^{p|q})|_W$. Consider the vertex superalgebra $\mathcal{D}^{\mathrm{ch}}(\mathbf{W})$ constructed in §A.15. It is freely generated by a vertex superalgebroid

$$\big(C^\infty(\mathbf{W}), \Omega^1(\mathbf{W}), \mathcal{T}(\mathbf{W}), *^c, \{\ \}^c, \{\ \}^c_\Omega\big)$$

and, by the following result, equipped with a family of conformal elements

$$\nu^\omega := \epsilon_i \partial_{i,-1} db^i + \frac{1}{2}\omega_{-2}\mathbf{1}, \qquad \omega \in \Omega^1(\mathbf{W}),\ |\omega| = \bar{0},\ d\omega = 0. \tag{2.1}$$

The assignment $W \mapsto \mathcal{D}^{\mathrm{ch}}_{p|q}(W) := \mathcal{D}^{\mathrm{ch}}(\mathbf{W})$ defines a sheaf of conformal vertex superalgebras on $\mathbb{R}^p$.

Proposition 2.2. *The elements ν^ω in (2.1) are conformal in $\mathcal{D}^{\mathrm{ch}}(\mathbf{W})$ of central charge $2(p-q)$.*

Proof. First consider $\nu := \epsilon_i \partial_{i,-1} db^i$. Let us show that

$$\text{(i)}\quad \nu_{(0)} = T,\ \nu_{(1)} = L_0 \text{ on } C^\infty(\mathbf{W}) \cup \{\partial_k\}_{k=1}^{p+q} \qquad \text{(ii)}\quad \nu_{(3)}\nu = p - q$$

The operators $\partial_{i,n}$ for $i = 1, \ldots, p+q$ and $n \in \mathbb{Z}$ commute with each other, because

$$[\partial_{i,n}, \partial_{j,m}] = [\partial_i, \partial_j]_{n+m} + \{\partial_i, \partial_j\}^c_{\Omega, n+m} + n\{\partial_i, \partial_j\}^c_{n+m} = 0.$$

Keeping this in mind, we compute the following for $f \in C^\infty(\mathbf{W})$ and $k = 1, \ldots, p+q$

$$\begin{aligned}
\nu_{(1)} f &= (db^i)_0 \partial_{i,0} f = 0 \\
\nu_{(0)} f &= \epsilon_i \partial_{i,-1} (db^i)_0 f + (db^i)_{-1} \partial_{i,0} f = 0 + db^i \cdot \partial_i f = df = Tf \\
\nu_{(2)} \partial_k &= (db^i)_1 \partial_{i,0} \partial_k + (db^i)_0 \partial_{i,1} \partial_k = 0 \\
\nu_{(1)} \partial_k &= \epsilon_i \partial_{i,-1} (db^i)_1 \partial_k + (db^i)_0 \partial_{i,0} \partial_k + (db^i)_{-1} \partial_{i,1} \partial_k = \epsilon_i \partial_{i,-1} db^i(\partial_k) + 0 + 0 = \partial_k \\
\nu_{(0)} \partial_k &= \epsilon_i \partial_{i,-2} (db^i)_1 \partial_k + \epsilon_i \partial_{i,-1} (db^i)_0 \partial_k + (db^i)_{-1} \partial_{i,0} \partial_k + (db^i)_{-2} \partial_{i,1} \partial_k \\
&= \epsilon_i \partial_{i,-2} db^i(\partial_k) + 0 + 0 + 0 = \partial_{k,-2} \mathbf{1} = T\partial_k \\
\nu_{(3)} \nu &= \epsilon_i [\nu_{(3)}, \partial_{i,-1}] db^i = \epsilon_i (\nu_{(0)} \partial_i)_{(2)} db^i + 3\epsilon_i (\nu_{(1)} \partial_i)_{(1)} db^i + 3\epsilon_i (\nu_{(2)} \partial_i)_{(0)} db^i \\
&= -2\epsilon_i \partial_{i,1} db^i + 3\epsilon_i \partial_{i,1} db^i + 0 = db^i(\partial_i) = p - q
\end{aligned}$$

This proves (i) and (ii). Now notice that $[\nu_{(1)}, f_0] = (\nu_{(0)} f)_{(0)} + (\nu_{(1)} f)_{(-1)} = 0$, and also that both $\nu_{(0)}, T$ are vertex superalgebra derivations commuting with T. Then compute for $\alpha \in \Omega^1(\mathbf{W})$, $X \in \mathcal{T}(\mathbf{W})$

$$\begin{aligned}
\nu_{(1)} \alpha &= \nu_{(1)} (\alpha_k db^k) = \nu_{(1)} \alpha_{k,0} T b^k = [\nu_{(1)}, \alpha_{k,0}] T b^k + \alpha_{k,0} [\nu_{(1)}, T] b^k + \alpha_{k,0} T \nu_{(1)} b^k \\
&= 0 + \alpha_{k,0} \nu_{(0)} b^k + 0 = \alpha_k db^k = \alpha \\
\nu_{(0)} \alpha &= \nu_{(0)} (\alpha_k db^k) = \nu_{(0)} \alpha_{k,0} T b^k = T \alpha_{k,0} T b^k = T\alpha \\
\nu_{(1)} X &= \nu_{(1)} (X^k \partial_k) = \nu_{(1)} X^k_0 \partial_k - \nu_{(1)} (X^k *^c \partial_k) = [\nu_{(1)}, X^k_0] \partial_k + X^k_0 \nu_{(1)} \partial_k - X^k *^c \partial_k \\
&= 0 + X^k_0 \partial_k - X^k *^c \partial_k = X^k \partial_k = X \\
\nu_{(0)} X &= \nu_{(0)} (X^k \partial_k) = \nu_{(0)} X^k_0 \partial_k - \nu_{(0)} (X^k *^c \partial_k) = T X^k_0 \partial_k - T(X^k *^c \partial_k) = TX
\end{aligned}$$

Hence (i) implies that $\nu_{(0)} = T$, $\nu_{(1)} = L_0$ also hold on $\Omega^1(\mathbf{W})$ and $\mathcal{T}(\mathbf{W})$. This yields the commutation relations:

$$\begin{array}{lll} [\nu_{(0)}, f_n] = (1-n)f_{n-1} & [\nu_{(0)}, \alpha_n] = -n\alpha_{n-1} & [\nu_{(0)}, X_n] = -nX_{n-1} \\ [\nu_{(1)}, f_n] = -nf_n & [\nu_{(1)}, \alpha_n] = -n\alpha_n & [\nu_{(1)}, X_n] = -nX_n \end{array}$$

Since we also have $\nu_{(0)}\mathbf{1} = 0 = \nu_{(1)}\mathbf{1}$, the operators $\nu_{(0)}, \nu_{(1)}$ satisfy respectively the defining relations of T and L_0, i.e. $\nu_{(0)} = T$, $\nu_{(1)} = L_0$ on the entire vertex superalgebra $\mathcal{D}^{\mathrm{ch}}(\mathbf{W})$. By Lemma 3.4.5 of [FB04], this together with (ii) proves the proposition for ν.

Let δ denote an even element of $\mathcal{D}^{\mathrm{ch}}(\mathbf{W})_2$. Replacing ν by $\nu + \delta$ in the above arguments shows that $\nu + \delta$ is also conformal of the same central charge if

$$\text{(i)}' \quad \delta_{(0)} = \delta_{(1)} = 0 \text{ on } C^\infty(\mathbf{W}) \cup \{\partial_k\}_{k=1}^{p+q} \qquad\qquad \text{(ii)}' \quad \nu_{(3)}\delta + \delta_{(3)}\nu + \delta_{(3)}\delta = 0$$

Suppose $\delta = \omega_{-2}\mathbf{1} = T\omega$, where $\omega \in \Omega^1(\mathbf{W})$. Then $\delta_{(n)} = -n\omega_{n-1}$. For $f \in C^\infty(\mathbf{W})$, $k = 1, \ldots, p+q$

$$\begin{aligned} &\delta_{(0)} f = 0 = \delta_{(0)}\partial_k \\ &\delta_{(1)} f = -\omega_0 f = 0 \\ &\delta_{(1)}\partial_k = -\omega_0\partial_k = [\partial_{k,-1}, \omega_0]\mathbf{1} = L_{\partial_k}\omega - d\omega(\partial_k) = \iota_{\partial_k} d\omega \end{aligned}$$

Hence (i)$'$ is satisfied if $d\omega = 0$. On the other hand, $[\nu_{(2)}, f_0] = (\nu_{(0)} f)_{(1)} + 2(\nu_{(1)} f)_{(0)} = -f_1$ implies

$$\begin{aligned} & \nu_{(2)}\omega = \nu_{(2)}(\omega_k db^k) = \nu_{(2)}\omega_{k,0}Tb^k = -\omega_{k,1}Tb^k + \omega_{k,0}\nu_{(2)}Tb^k = 0 + 2\omega_{k,0}\nu_{(1)}b^k = 0 \\ \Rightarrow \qquad & \nu_{(3)}\delta + \delta_{(3)}\nu + \delta_{(3)}\delta = \nu_{(3)}\omega_{-2}\mathbf{1} - 3\omega_2\nu - 3\omega_2\omega_{-2}\mathbf{1} \\ & \qquad = [\nu_{(3)}, \omega_{-2}]\mathbf{1} + 3[\nu_{(-1)}, \omega_2]\mathbf{1} - 3[\omega_2, \omega_{-2}]\mathbf{1} \\ & \qquad = 4(\nu_{(0)}\omega)_{(1)}\mathbf{1} + 6\nu_{(2)}\omega = 0 \end{aligned}$$

so that (ii)$'$ holds. This completes the proof of the proposition. □

Remark. In the above proof, full details are shown in order to demonstrate the type of arguments involved in similar calculations. Subsequent proofs will be given more briefly.

§ 2.3. Coordinate transformations of CDOs on $\mathbb{R}^{p|q}$. Let $\mathbf{W}$, $\mathbf{W}'$, $\mathbf{W}''$ be restrictions of $\mathbb{R}^{p|q}$ (as a cs-manifold) to open sets in $\mathbb{R}^p$. Suppose $\boldsymbol{\varphi} : \mathbf{W} \to \mathbf{W}'$ is a diffeomorphism of cs-manifolds. Recall the notations in §A.16 and Theorem A.17. Given an even 2-form ξ on $\mathbf{W}$ with $d\xi = WZ_{\boldsymbol{\varphi}}$, there is a corresponding isomorphism of vertex superalgebras

$$\boldsymbol{\varphi}_\xi^* : \mathcal{D}^{\mathrm{ch}}(\mathbf{W}') \to \mathcal{D}^{\mathrm{ch}}(\mathbf{W}).$$

For $f \in C^\infty(\mathbf{W}')$, $\alpha \in \Omega^1(\mathbf{W}')$ and $X \in \mathcal{T}(\mathbf{W}')$, we have

$$\boldsymbol{\varphi}_\xi^*(f) = \boldsymbol{\varphi}^* f, \qquad \boldsymbol{\varphi}_\xi^*(\alpha) = \boldsymbol{\varphi}^*\alpha, \qquad \boldsymbol{\varphi}_\xi^*(X) = \boldsymbol{\varphi}^* X + \Delta_{\boldsymbol{\varphi},\xi}(X). \tag{2.2}$$

All isomorphisms between $\mathcal{D}^{\mathrm{ch}}(\mathbf{W}')$ and $\mathcal{D}^{\mathrm{ch}}(\mathbf{W})$ are of this form. According to the result below, $\boldsymbol{\varphi}_\xi^*$ permutes the conformal elements (2.1). If $\boldsymbol{\varphi}' : \mathbf{W}' \to \mathbf{W}''$ is another diffeomorphism of cs-manifolds and ξ' is an even 2-form on $\mathbf{W}'$ with $d\xi' = WZ_{\boldsymbol{\varphi}'}$, then

$$\boldsymbol{\varphi}_\xi^* \circ \boldsymbol{\varphi}'^*_{\xi'} = (\boldsymbol{\varphi}'\boldsymbol{\varphi})_\eta^*, \qquad \eta = \xi + \boldsymbol{\varphi}^*\xi' + \sigma_{\boldsymbol{\varphi}',\boldsymbol{\varphi}}.$$

Proposition 2.4. *Consider the isomorphism $\boldsymbol{\varphi}_\xi^* : \mathcal{D}^{\mathrm{ch}}(\mathbf{W}') \to \mathcal{D}^{\mathrm{ch}}(\mathbf{W})$ described above. For even closed 1-forms ω on $\mathbf{W}'$, we have*

$$\boldsymbol{\varphi}_\xi^*(\nu^\omega) = \nu^{\boldsymbol{\varphi}^*\omega - \mathrm{Str}\,\theta_{\boldsymbol{\varphi}}}.$$

Remark. Notice that $d\theta_{\boldsymbol{\varphi}} = -\theta_{\boldsymbol{\varphi}} \wedge \theta_{\boldsymbol{\varphi}}$ implies $\mathrm{Str}\,\theta_{\boldsymbol{\varphi}}$ is closed. Also, as a consistency check, it follows from $\theta_{\boldsymbol{\varphi}'\boldsymbol{\varphi}} = \theta_{\boldsymbol{\varphi}} + g_{\boldsymbol{\varphi}}^{-1} \cdot \boldsymbol{\varphi}^*\theta_{\boldsymbol{\varphi}'} \cdot g_{\boldsymbol{\varphi}}$ that $\mathrm{Str}\,\theta_{\boldsymbol{\varphi}'\boldsymbol{\varphi}} = \mathrm{Str}\,\theta_{\boldsymbol{\varphi}} + \boldsymbol{\varphi}^*\mathrm{Str}\,\theta_{\boldsymbol{\varphi}'}$.

Proof of Proposition 2.4. It suffices to consider the case $\omega = 0$. To simplify notations, let us write $g = g_{\boldsymbol{\varphi}}$, $h = g^{-1}$, $\theta = \theta_{\boldsymbol{\varphi}}$ and $\Delta = \Delta_{\boldsymbol{\varphi},\xi}$. By (2.2), we have

$$\boldsymbol{\varphi}^*_\xi(\nu) = \epsilon_i\big(\boldsymbol{\varphi}^*\partial_i + \Delta(\partial_i)\big)_{-1}(\boldsymbol{\varphi}^* db^i) = \epsilon_{ik}(h^k{}_i\partial_k)_{-1}(g^i{}_\ell db^\ell) + \epsilon_i\Delta(\partial_i)_{-1}(g^i{}_\ell db^\ell)$$

The first term above is computed as follows:

$$\begin{aligned}
&\epsilon_{ik}(h^k{}_i\partial_k)_{-1}(g^i{}_\ell db^\ell) \\
&= \epsilon_{ik}\Big(h^k{}_{i,-2}\,\partial_{k,1} + h^k{}_{i,-1}\,\partial_{k,0} + h^k{}_{i,0}\,\partial_{k,-1} - (h^k{}_i * \partial_k)_{-1}\Big) g^i{}_{\ell,0}\,db^\ell \\
&= \epsilon_k h^k{}_{i,-2}\,g^i{}_k + \epsilon_k h^k{}_{i,-1}\,dg^i{}_k + \epsilon_k\partial_{k,-1}db^k - \epsilon_k(\partial_k h^k{}_i)_{-1}\,g^i{}_{\ell,0}\,db^\ell + \epsilon_k(\partial_k h^k{}_i)_{-1}\,g^i{}_{\ell,0}\,db^\ell \\
&= \nu + \frac{1}{2}\,\mathrm{Str}\,\big((dh)_{-2}\,g\big) + \mathrm{Str}\,\big((dh)_{-1}\,dg\big) = \nu - \frac{1}{2}\,\mathrm{Str}\,(\theta_{-2}\mathbf{1}) - \frac{1}{2}\,\mathrm{Str}\,(\theta_{-1}\theta)
\end{aligned}$$

Then we compute the second term above:

$$\begin{aligned}
&\epsilon_i\Delta(\partial_i)_{-1}(g^i{}_\ell db^\ell) \\
&= \Big(-\epsilon_k\epsilon_r\epsilon_{ik}\epsilon_{ir}\epsilon_{kr}\partial_r h^k{}_i \cdot \theta^r{}_k - \frac{1}{2}\epsilon_{ik}\mathrm{Str}\,(\theta\otimes\theta)(h^k{}_i\partial_k \otimes \text{-}\,) - \frac{1}{2}\epsilon_{ik}\,\xi(h^k{}_i\partial_k,\,\text{-}\,)\Big)_{-1} g^i{}_{\ell,0}\,db^\ell \\
&= \epsilon_r\theta^r{}_{k,-1}\,h^k{}_{i,0}\,dg^i{}_r - \frac{1}{2}\epsilon_k\epsilon_r\mathrm{Str}\,(\theta\otimes\theta)(\partial_k\otimes\partial_r)_0(db^r)_{-1}db^k - \frac{1}{2}\epsilon_k\epsilon_r\xi(\partial_k,\partial_r)_0(db^r)_{-1}db^k \\
&= \mathrm{Str}\,(\theta_{-1}\theta) - \frac{1}{2}\mathrm{Str}\,(\theta_{-1}\theta) = \frac{1}{2}\mathrm{Str}\,(\theta_{-1}\theta)
\end{aligned}$$

where we have used the graded symmetry of $(db^r)_{-1}db^k = b^r_{-1}b^k_{-1}\mathbf{1}$. This yields

$$\boldsymbol{\varphi}^*_\xi(\nu) = \nu - \frac{1}{2}(\mathrm{Str}\,\theta)_{-2}\mathbf{1} = \nu^{-\mathrm{Str}\,\theta}. \qquad \square$$

Preparation. Given topological spaces X, X', a presheaf $\mathcal{S}$ on X and a presheaf $\mathcal{S}'$ on X' valued in some category, let $(\varphi, \Phi) : (X, \mathcal{S}) \to (X', \mathcal{S}')$ denote the data consisting of a continuous map $\varphi : X \to X'$ and a morphism of presheaves $\Phi : \mathcal{S}' \to \varphi_*\mathcal{S}$ on X'. Composition reads $(\varphi', \Phi') \circ (\varphi, \Phi) = (\varphi'\varphi,\ \varphi'_*\Phi \circ \Phi')$. Recall the sheaf of vertex superalgebras $\mathcal{D}^{\mathrm{ch}}_{p|q}$ described in §2.1.

Definition 2.5. A *sheaf of CDOs on a smooth $(p|q)$-dimensional cs-manifold* $\mathbf{M} = (M, C^\infty_{\mathbf{M}})$ is a sheaf of vertex superalgebras $\mathcal{V}$ on M with the following properties:

- The weight-zero component is $\mathcal{V}_0 = C^\infty_{\mathbf{M}}$.
- Given $x \in M$, there exist open sets $U \subset M$, $W \subset \mathbb{R}^p$ with $x \in U$, and an isomorphism between $(U, \mathcal{V}|_U)$ and $(W, \mathcal{D}^{\mathrm{ch}}_{p|q}|_W)$ as topological spaces equipped with sheaves of vertex superalgebras.

A *conformal structure* on $\mathcal{V}$ is an element $\nu \in \mathcal{V}(M)_2$ such that, under each isomorphism postulated above, $\nu|_U \in \mathcal{V}(U)$ corresponds to one of the conformal elements $\nu^\omega \in \mathcal{D}^{\mathrm{ch}}_{p|q}(W)$ described in (2.1).

Remark. For example, $\mathcal{D}^{\mathrm{ch}}_{p|q}$ is a sheaf of CDOs on $\mathbb{R}^{p|q}$ with a family of conformal structures ν^ω. While a general sheaf of CDOs is locally isomorphic to $\mathcal{D}^{\mathrm{ch}}_{p|q}$, the latter has up to this point been defined using coordinates in a manifest way (see §2.1 and appendix §A). The geometric data required to globalize the construction is the main content of Theorem 2.8.

Preparation. The sheaves of smooth functions, 1-forms and vector fields on a smooth cs-manifold $\mathbf{M}$ form a sheaf of extended Lie superalgebroids $(C^\infty_{\mathbf{M}}, \Omega^1_{\mathbf{M}}, \mathcal{T}_{\mathbf{M}})$ using the usual differential on functions, Lie brackets on vector fields, Lie derivations on functions and 1-forms by vector fields, and pairing between 1-forms and vector fields.

Lemma 2.6. *Let* $(\varphi, \Phi) : (U, \mathcal{V}|_U) \to (W, \mathcal{D}^{\mathrm{ch}}_{p|q}|_W)$ *be an isomorphism as postulated in Definition 2.5. Also let* $\mathbf{U} = \mathbf{M}|_U$, $\mathbf{W} = (\mathbb{R}^{p|q})|_W$.

(a) The data determine a diffeomorphism of cs-manifolds $\boldsymbol{\varphi} : \mathbf{U} \to \mathbf{W}$. *The presheaf (in fact, sheaf) of extended Lie superalgebroids associated to* $\mathcal{V}|_U$ *can be identified with* $(C^\infty_{\mathbf{U}}, \Omega^1_{\mathbf{U}}, \mathcal{T}_{\mathbf{U}})$ *in a canonical way. Under this identification, the isomorphism of sheaves of extended Lie superalgebroids induced by* Φ *is given by* $\boldsymbol{\varphi}^* : (C^\infty_{\mathbf{W}}, \Omega^1_{\mathbf{W}}, \mathcal{T}_{\mathbf{W}}) \to \varphi_*(C^\infty_{\mathbf{U}}, \Omega^1_{\mathbf{U}}, \mathcal{T}_{\mathbf{U}})$.

(b) The quotient map $\mathcal{V}_1|_U \to \mathcal{T}_{\mathbf{U}}$ *is split as a morphism of sheaves of* $\mathbb{C}$*-vector spaces, and* $\mathcal{V}|_U$ *is freely generated by any associated sheaf of vertex superalgebroids. Moreover,* Φ *is induced by an isomorphism of sheaves of vertex superalgebroids.*

Proof. (a) At weight zero, (φ, Φ) defines an isomorphism of ringed spaces $(U, C^\infty_{\mathbf{U}}) \to (W, C^\infty_{\mathbf{W}})$, which is the same as a diffeomorphism $\boldsymbol{\varphi} : \mathbf{U} \to \mathbf{W}$. Let $(C^\infty_{\mathbf{M}}, \Omega, \mathcal{T})$ be the presheaf of extended Lie superalgebroids associated to $\mathcal{V}$. The following isomorphisms, induced respectively by Φ and $\boldsymbol{\varphi}$

$$\varphi_*(C^\infty_{\mathbf{U}}, \Omega|_U, \mathcal{T}|_U) \xleftarrow{\cong} (C^\infty_{\mathbf{W}}, \Omega^1_{\mathbf{W}}, \mathcal{T}_{\mathbf{W}}) \xrightarrow{\cong} \varphi_*(C^\infty_{\mathbf{U}}, \Omega^1_{\mathbf{U}}, \mathcal{T}_{\mathbf{U}})$$

allow us to identify $(C^\infty_{\mathbf{U}}, \Omega|_U, \mathcal{T}|_U)$ with $(C^\infty_{\mathbf{U}}, \Omega^1_{\mathbf{U}}, \mathcal{T}_{\mathbf{U}})$ via identity on $C^\infty_{\mathbf{U}}$. Since any isomorphism with $(C^\infty_{\mathbf{U}}, \Omega^1_{\mathbf{U}}, \mathcal{T}_{\mathbf{U}})$ is determined by its first component, the above identification is independent of the choice of W and (φ, Φ).

(b) The statements about $\mathcal{V}|_U$ are true because their analogues for $\mathcal{D}^{\mathrm{ch}}_{p|q}|_W$ are true. The statement about Φ is then clear. □

Lemma 2.7. *Let* $\mathcal{V}$ *be a sheaf of CDOs on a smooth cs-manifold* $\mathbf{M} = (M, C^\infty_{\mathbf{M}})$.

(a) The presheaf (in fact, sheaf) of extended Lie superalgebroids associated to $\mathcal{V}$ *can be identified with* $(C^\infty_{\mathbf{M}}, \Omega^1_{\mathbf{M}}, \mathcal{T}_{\mathbf{M}})$ *in a canonical way.*

(b) The quotient map $\mathcal{V}_1 \to \mathcal{T}_{\mathbf{M}}$ *is split as a morphism of sheaves of* $\mathbb{C}$*-vector spaces, and* $\mathcal{V}$ *is freely generated by any associated sheaf of vertex superalgebroids.*

Proof. Let $\mathfrak{U} = \{U_a\}_{a \in I}$ be an open cover of M such that $(U_a, \mathcal{V}|_{U_a})$ admit isomorphisms as postulated in Definition 2.5. For $A \subset M$, let "$A \cap \mathfrak{U}$" denote the open cover $\{A \cap U_a\}_{a \in I}$ of A.

(a) Let $(C^\infty_{\mathbf{M}}, \Omega, \mathcal{T})$ be the *presheaf* of extended Lie superalgebroids associated to $\mathcal{V}$ and $U \subset M$ an arbitrary open set. Consider the diagram (natural in U)

$$\begin{array}{ccccccc} & & \Omega(U) & \xrightarrow{\varepsilon} & \check{C}^0(U \cap \mathfrak{U}, \Omega) & \xrightarrow{\delta} & \check{C}^1(U \cap \mathfrak{U}, \Omega) \\ & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ 0 & \longrightarrow & \Omega^1_{\mathbf{M}}(U) & \longrightarrow & \check{C}^0(U \cap \mathfrak{U}, \Omega^1_{\mathbf{M}}) & \xrightarrow{\delta} & \check{C}^1(U \cap \mathfrak{U}, \Omega^1_{\mathbf{M}}) \end{array}$$

where $(\check{C}^*(\,\cdot\,), \delta)$ denote Cech complexes and the isomorphisms are given by Lemma 2.6a. By the exactness of the bottom row, the dotted arrow ι can be filled in in a unique way. By construction, ι is compatible with the derivations $C^\infty_{\mathbf{M}} \to \Omega$, $C^\infty_{\mathbf{M}} \to \Omega^1_{\mathbf{M}}$, and this implies ι is surjective. On the other hand, since Ω is a subpresheaf of a sheaf, ε is injective, and so is ι. Hence we have an isomorphism $\Omega \cong \Omega^1_{\mathbf{M}}$. Now $\mathcal{T}$ must also be a sheaf. This is a formal consequence of: (i) $\mathcal{T}(U) := \mathcal{V}_1(U)/\Omega(U)$ for open sets $U \subset M$, (ii) $\mathcal{V}_1$ is a sheaf, and (iii) $\Omega \cong \Omega^1_{\mathbf{M}}$ is a fine sheaf. Then a diagram similar to the one above produces an isomorphism $\mathcal{T} \cong \mathcal{T}_{\mathbf{M}}$. By construction, the isomorphisms $\Omega \cong \Omega^1_{\mathbf{M}}$ and $\mathcal{T} \cong \mathcal{T}_{\mathbf{M}}$ respect the extended Lie superalgebroid structures.

(b) Let π denote the quotient map $\mathcal{V}_1 \to \mathcal{T}_{\mathbf{M}}$. By Lemma 2.6b, the restriction of π to each U_a has a splitting $s_a : \mathcal{T}_{\mathbf{M}}|_{U_a} \to \mathcal{V}_1|_{U_a}$. Let $\{f_a\}_{a \in I}$ be a smooth partition of unity on M subordinate to $\mathfrak{U}$. Use the operation ${}_{(-1)} : C^\infty_M \times \mathcal{V}_1 \to \mathcal{V}_1$ to define such morphisms of sheaves $(f_a)_{(-1)} s_a$ that extend from U_a to M. Since the said operation induces via π the usual $C^\infty_{\mathbf{M}}$-multiplication on $\mathcal{T}_{\mathbf{M}}$, the sum

$$s := \sum_{a \in I} (f_a)_{(-1)} s_a : \mathcal{T}_{\mathbf{M}} \to \mathcal{V}_1$$

splits π. Such a splitting yields a sheaf of vertex superalgebroids.

Given a sheaf of vertex superalgebroids associated to $\mathcal{V}$, its sections freely generate a *presheaf* of vertex superalgebras $\mathcal{V}'$. Moreover, there is a canonical morphism of presheaves of vertex superalgebras $\kappa : \mathcal{V}' \to \mathcal{V}$. Since $\kappa|_{U_a}$ are isomorphisms, so is κ if and only if $\mathcal{V}'$ is a sheaf. Now each weight component $\mathcal{V}'_k$, $k \geq 1$, admits a filtration whose associated graded presheaf is a sheaf (see §A.9). It follows formally from this fact that $\mathcal{V}'$ is indeed a sheaf as desired. □

Preparation. Suppose $\mathbf{M}$ is a smooth cs-manifold and ∇ a connection on $T\mathbf{M}$. Given $X \in \mathcal{T}(\mathbf{M})$, let $\nabla^t X$ denote the section of $\operatorname{End} T\mathbf{M}$ defined by $(\nabla^t X)(Y) = \nabla_X Y - [X, Y]$ for $Y \in \mathcal{T}(\mathbf{M})$. Notice that if ∇ is torsion-free, then $\nabla^t X = \nabla X$.

Theorem 2.8. *Let $\mathbf{M} = (M, C^\infty_{\mathbf{M}})$ be a smooth cs-manifold.*

(a) Suppose ∇ is a connection on $T\mathbf{M}$ with curvature operator R, and H is an even 3-form on $\mathbf{M}$ with $dH = \operatorname{Str}(R \wedge R)$. Given such data, a sheaf of vertex superalgebroids

$$\big(C^\infty_{\mathbf{M}}, \Omega^1_{\mathbf{M}}, \mathcal{T}_{\mathbf{M}}, *, \{\ \}, \{\ \}_\Omega\big)$$

can be defined on M using the following formulae

$$\begin{aligned} f * X \quad &= -(\nabla df)(X) \\ \{X, Y\} \quad &= -\operatorname{Str}(\nabla^t X \cdot \nabla^t Y) \\ \{X, Y\}_\Omega &= \operatorname{Str}\Big(- \nabla(\nabla^t X) \cdot \nabla^t Y + \nabla^t X \cdot \iota_Y R - \iota_X R \cdot \nabla^t Y \Big) + \frac{1}{2} \iota_X \iota_Y H \end{aligned}$$

and it freely generats a sheaf of CDOs on $\mathbf{M}$, denoted by $\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H}$. Up to isomorphism, this construction yields all sheaves of CDOs on $\mathbf{M}$.

(b) Conformal structures on $\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H}$ are in one-to-one correspondence with even 1-forms ω on $\mathbf{M}$ satisfying $d\omega = \operatorname{Str} R$. This correspondence is independent of the choice of H. Given ω as described, the corresponding conformal structure, denoted by ν^ω, is characterized by

$$L^\omega_1 X := \nu^\omega_{(2)} X = \operatorname{Str} \nabla^t X - \omega(X)$$

for vector fields X on $\mathbf{M}$.

Proof. (a) Suppose $\mathcal{V}$ is a sheaf of CDOs on $\mathbf{M}$. By Lemma 2.7, $\mathcal{V}$ is freely generated by a sheaf of vertex superalgebroids $\big(C^\infty_{\mathbf{M}}, \Omega^1_{\mathbf{M}}, \mathcal{T}_{\mathbf{M}}, *, \{\ \}, \{\ \}_\Omega\big)$. Let $\mathfrak{U} = \{U_a\}_{a \in I}$ be an open cover of M and

$$(\varphi_a, \Phi_a) : \big(U_a, \mathcal{V}|_{U_a}\big) \to \big(W_a, \mathcal{D}^{\mathrm{ch}}_{p|q}|_{W_a}\big), \quad W_a \subset \mathbb{R}^p \text{ open}, \quad a \in I$$

be isomorphisms as postulated in Definition 2.5. Also let $\mathbf{U}_a = \mathbf{M}|_{U_a}$ and $\mathbf{W}_a = (\mathbb{R}^{p|q})|_{W_a}$. By Lemma 2.6, there are diffeomorphisms $\boldsymbol{\varphi}_a : \mathbf{U}_a \to \mathbf{W}_a$ such that Φ_a are induced by isomorphisms of sheaves of vertex superalgebroids of the form

$$(\boldsymbol{\varphi}_a^*, \boldsymbol{\varphi}_a^* \Delta_a) : \big(C^\infty_{\mathbf{W}_a}, \Omega^1_{\mathbf{W}_a}, \mathcal{T}_{\mathbf{W}_a}, *^c, \{\ \}^c, \{\ \}^c_\Omega\big) \to \varphi_{a*}\big(C^\infty_{\mathbf{U}_a}, \Omega^1_{\mathbf{U}_a}, \mathcal{T}_{\mathbf{U}_a}, *, \{\ \}, \{\ \}_\Omega\big) \tag{2.3}$$

where $\Delta_a : \mathcal{T}_{\mathbf{W}_a} \to \Omega^1_{\mathbf{W}_a}$ are some even morphisms of sheaves on W_a.

Somewhat abusing notations, we will write φ_a, $\boldsymbol{\varphi}_a$, Φ_a, etc. also for their restrictions to various open subsets. For $a, a' \in I$, let $W_{aa'} = \varphi_a(U_a \cap U_{a'})$, $\mathbf{W}_{aa'} = (\mathbb{R}^{p|q})|_{W_{aa'}}$ and

$$\begin{gathered} \varphi_{a'a} = \varphi_{a'} \circ \varphi_a^{-1} : W_{aa'} \to W_{a'a}, \qquad \boldsymbol{\varphi}_{a'a} = \boldsymbol{\varphi}_{a'} \circ \boldsymbol{\varphi}_a^{-1} : \mathbf{W}_{aa'} \to \mathbf{W}_{a'a} \\ (\varphi_{a'a}, \Phi_{a'a}) = (\varphi_{a'}, \Phi_{a'}) \circ (\varphi_a, \Phi_a)^{-1} : \big(W_{aa'}, \mathcal{D}^{\mathrm{ch}}_{p|q}|_{W_{aa'}}\big) \to \big(W_{a'a}, \mathcal{D}^{\mathrm{ch}}_{p|q}|_{W_{a'a}}\big) \end{gathered}$$

Recall the notations in §A.16 and write $g_{\boldsymbol{\varphi}_{a'a}}$, $\theta_{\boldsymbol{\varphi}_{a'a}}$, $WZ_{\boldsymbol{\varphi}_{a'a}}$ more simply as $g_{a'a}$, $\theta_{a'a}$, $WZ_{a'a}$. According to §2.3, $\Phi_{a'a} = (\boldsymbol{\varphi}_{a'a})^*_{\xi_{a'a}}$ for some unique even 2-forms $\xi_{a'a}$ on $\mathbf{W}_{aa'}$ with $d\xi_{a'a} = WZ_{a'a}$, and it is induced by an isomorphism of sheaves of vertex superalgebroids $(\boldsymbol{\varphi}^*_{a'a}, \Delta_{a'a})$, where

$$\Delta_{a'a} = \Delta_{\boldsymbol{\varphi}_{a'a},\xi_{a'a}} : \mathcal{T}_{\mathbf{W}_{a'a}} \to (\varphi_{a'a})_*\Omega^1_{\mathbf{W}_{aa'}}$$

is defined as in Theorem A.17. The definition of $\Phi_{a'a}$ given above is equivalent to

$$\begin{aligned} & (\boldsymbol{\varphi}^*_{a'a}, \Delta_{a'a}) = (\varphi_{a'a})_*(\boldsymbol{\varphi}^*_a, \boldsymbol{\varphi}^*_a\Delta_a)^{-1} \circ (\boldsymbol{\varphi}^*_{a'}, \boldsymbol{\varphi}^*_{a'}\Delta_{a'}) \\ \Leftrightarrow \quad & \Delta_{a'a} = \boldsymbol{\varphi}^*_{a'a} \circ \Delta_{a'} - (\varphi_{a'a})_*\Delta_a \circ \boldsymbol{\varphi}^*_{a'a} \end{aligned} \tag{2.4}$$

For $a, a', a'' \in I$, let $W_{aa'a''} = \varphi_a(U_a \cap U_{a'} \cap U_{a''})$ and $\mathbf{W}_{aa'a''} = (\mathbb{R}^{p|q})|_{W_{aa'a''}}$. In $W_{aa'a''}$ we have

$$\boldsymbol{\varphi}_{a''a} = \boldsymbol{\varphi}_{a''a'} \circ \boldsymbol{\varphi}_{a'a}, \qquad (\boldsymbol{\varphi}_{a''a})^*_{\xi_{a''a}} = (\varphi_{a''a'})_*(\boldsymbol{\varphi}_{a'a})^*_{\xi_{a'a}} \circ (\boldsymbol{\varphi}_{a''a'})^*_{\xi_{a''a'}}$$

According to §2.3, the latter is equivalent to

$$\xi_{a''a'} = \xi_{a'a} + \boldsymbol{\varphi}^*_{a'a}\xi_{a''a'} + \sigma_{a''a'a} \tag{2.5}$$

where $\sigma_{a''a'a} = \sigma_{\boldsymbol{\varphi}_{a''a'},\boldsymbol{\varphi}_{a'a}} \in \Omega^2(\mathbf{W}_{aa'a''})$ is defined as in Theorem A.17.

Lemma 2.9. *Given $\boldsymbol{\varphi}_{a'a}$ and $\xi_{a'a}$ for $a, a' \in I$ as above (which determine $\Delta_{a'a}$), a collection of even morphisms of sheaves $\Delta_a : \mathcal{T}_{\mathbf{W}_a} \to \Omega^1_{\mathbf{W}_a}$ satisfy (2.4) if and only if they are of the form*

$$\Delta_a(X) = \epsilon_i\epsilon_{ij}\epsilon_j^{1+|X|}(\partial_j X^i)(\Gamma_a)^j{}_i + \frac{1}{2}\iota_X \mathrm{Str}\,(\Gamma_a \otimes \Gamma_a) + \frac{1}{2}\iota_X B_a + O_a(X)$$

for homogeneous X, where:

- *$\Gamma_a \in \Omega^1(\mathbf{W}_a) \otimes \mathfrak{gl}(p|q)$ are even, i.e. $|(\Gamma_a)^i{}_j| = |b^i| + |b^j|$, and*

$$g^{-1}_{a'a} \cdot \boldsymbol{\varphi}^*_{a'a}\Gamma_{a'} \cdot g_{a'a} - \Gamma_a = -\theta_{a'a} \tag{2.6}$$

- *$B_a \in \Omega^2(\mathbf{W}_a)$ are even and*

$$\boldsymbol{\varphi}^*_{a'a}B_{a'} - B_a = -\xi_{a'a} - \mathrm{Str}\,(\theta_{a'a} \wedge \Gamma_a) \tag{2.7}$$

and $O_a : \mathcal{T}_{\mathbf{W}_a} \to \Omega^1_{\mathbf{W}_a}$ are even and $\boldsymbol{\varphi}^*_{a'a} \circ O_{a'} = (\varphi_{a'a})_* O_a \circ \boldsymbol{\varphi}^*_{a'a}$.

Proof. If we assume Δ_a are first-order differential operators, we may write

$$\Delta_a(X) = \epsilon_i\epsilon_{ij}\epsilon_j^{1+|X|}(\partial_j X^i)(\Gamma_a)^j{}_i + \frac{1}{2}\iota_X(S_a + B_a)$$

for some $\mathfrak{gl}(p|q)$-valued 1-forms Γ_a, symmetric $(0,2)$-tensors S_a and 2-forms B_a on $\mathbf{W}_a$; their parities are dictated by that of Δ_a. Plugging this into (2.4), namely

$$\boldsymbol{\varphi}^*_{a'a}\Delta_{a'}(X) - \Delta_a(\boldsymbol{\varphi}^*_{a'a}X) = \Delta_{a'a}(X)$$

results in three sets of equations: (2.6), (2.7) and

$$\boldsymbol{\varphi}^*_{a'a}S_{a'} - S_a = -\mathrm{Str}\,(\Gamma_a \otimes \theta_{a'a}) - \mathrm{Str}\,(\theta_{a'a} \otimes \Gamma_a) + \mathrm{Str}\,(\theta_{a'a} \otimes \theta_{a'a}).$$

By (2.6), the last set of equations is satisfied by $S_a = \mathrm{Str}\,(\Gamma_a \otimes \Gamma_a)$. Observe that once we have a solution to (2.4), any other solution differs precisely by a term O_a with the properties stated in the lemma. ☐

Proof of Theorem 2.8 continued. Consider the formula of Δ_a obtained in Lemma 2.9. The condition on the term O_a lets us define a map $O : \mathcal{T}_{\mathbf{M}} \to \Omega^1_{\mathbf{M}}$ such that $O(\boldsymbol{\varphi}_a^* X) = \boldsymbol{\varphi}_a^* O_a(X)$ for $a \in I$. By Lemma A.10, O determines an isomorphism of sheaves of vertex superalgebroids

$$(\mathrm{id}, -O) : \big(C^\infty_{\mathbf{M}}, \Omega^1_{\mathbf{M}}, \mathcal{T}_{\mathbf{M}}, *, \{\ \}, \{\ \}_\Omega\big) \to \big(C^\infty_{\mathbf{M}}, \Omega^1_{\mathbf{M}}, \mathcal{T}_{\mathbf{M}}, *', \{\ \}', \{\ \}'_\Omega\big)$$

whose composition with (2.3) equals

$$\varphi_{a*}(\mathrm{id}, -O)|_{U_a} \circ (\boldsymbol{\varphi}_a^*, \boldsymbol{\varphi}_a^* \Delta_a) = (\boldsymbol{\varphi}_a^*, \boldsymbol{\varphi}_a^* \Delta_a - \boldsymbol{\varphi}_a^* O_a).$$

Therefore up to isomorphism of sheaves of CDOs, we may assume $O_a = 0$. The following lemma concerns the other ingredients in the formula of Δ_a.

Lemma 2.10. *Assume that U_a, $a \in I$, are contractible. Given $\boldsymbol{\varphi}_{a'a}$ for $a, a' \in I$ as above, the existence of the following are equivalent:*

(i) $\xi_{a'a} \in \Omega^2(\mathbf{W}_{a'a})$ that are even, and satisfy $d\xi_{a'a} = WZ_{a'a}$ and (2.5)

(ii) $\Gamma_a \in \Omega^1(\mathbf{W}_a) \otimes \mathfrak{gl}(p|q)$ and $B_a \in \Omega^2(\mathbf{W}_a)$ that are even, and satisfy (2.6) and

$$\boldsymbol{\varphi}_{a'a}^* dB_{a'} - dB_a = -\boldsymbol{\varphi}_{a'a}^* CS(\Gamma_{a'}) + CS(\Gamma_a) \tag{2.8}$$

where $CS(\Gamma_a) \in \Omega^3(\mathbf{W}_a)$ is defined below

(iii) a connection ∇ on $T\mathbf{M}$ and $H \in \Omega^3(\mathbf{M})$ that is even and satisfies

$$dH = \mathrm{Str}\,(R \wedge R) \tag{2.9}$$

where R is the curvature operator of ∇

Proof. First, a collection of $\Gamma_a \in \Omega^1(\mathbf{W}_a) \otimes \mathfrak{gl}(p|q)$ that are even and satisfy (2.6) is equivalent to a connection ∇ on $T\mathbf{M}$. Indeed, the two are related via

$$\nabla(\boldsymbol{\varphi}_a^* \partial_i) = \epsilon_i \epsilon_{ij}\, \boldsymbol{\varphi}_a^* \big((\Gamma_a)^j{}_i \otimes \partial_j\big) \tag{2.10}$$

for $i = 1, \dots, p+q$. The curvature operator $R \in \Omega^2(\mathbf{M}, \mathrm{End}\, T\mathbf{M})$ of ∇ is locally given by

$$R(\boldsymbol{\varphi}_a^* \partial_i) = \epsilon_i \epsilon_{ij}\, \boldsymbol{\varphi}_a^* \big((R_a)^j{}_i \otimes \partial_j\big), \qquad R_a = d\Gamma_a + \Gamma_a \wedge \Gamma_a$$

whose tensoriality means $g_{a'a}^{-1} \cdot \boldsymbol{\varphi}_{a'a}^* R_{a'} \cdot g_{a'a} = R_a$. Define the following even 3-forms on $\mathbf{W}_a$

$$CS(\Gamma_a) := \mathrm{Str}\,(\Gamma_a \wedge R_a) - \frac{1}{3} \mathrm{Str}\,(\Gamma_a \wedge \Gamma_a \wedge \Gamma_a).$$

Notice that $d\,CS(\Gamma_a) = \mathrm{Str}\,(R_a \wedge R_a)$ and (2.6) implies

$$\boldsymbol{\varphi}_{a'a}^* CS(\Gamma_{a'}) - CS(\Gamma_a) = WZ_{a'a} + d\,\mathrm{Str}\,(\theta_{a'a} \wedge \Gamma_a). \tag{2.11}$$

Now we prove the equivalences.

(i) ⇒ (ii): Choose a connection ∇ on $T\mathbf{M}$ and define Γ_a as in (2.10). Then Γ_a satisfy (2.6). The right hand side of (2.7), after being pulled back by $\boldsymbol{\varphi}_a^*$, defines a 1-cochain in the Čech complex $\check{C}^*(\mathfrak{U}, \Omega^2_{\mathbf{M}})$; it is a cocycle by (2.5) and (2.6). Since $\check{C}^*(\mathfrak{U}, \Omega^2_{\mathbf{M}})$ is acyclic, we may choose such even 2-forms B_a on $\mathbf{W}_a$ that satisfy (2.7). Then (2.8) follows from $d\xi_{a'a} = WZ_{a'a}$ and (2.11).

(ii) ⇒ (i): Define $\xi_{a'a}$ using (2.7). Then (2.6) implies (2.5). On the other hand, (2.8) and (2.11) together imply $d\xi_{a'a} = WZ_{a'a}$.

(ii) ⇒ (iii): Define ∇ as in (2.10). By (2.8), there is a global even 3-form H on $\mathbf{M}$ with

$$H|_{U_a} = \boldsymbol{\varphi}_a^* \big(dB_a + CS(\Gamma_a)\big). \tag{2.12}$$

Then (2.9) follows from $d\,CS(\Gamma_a) = \mathrm{Str}\,(R_a \wedge R_a)$.

(iii) ⇒ (ii): Define Γ_a as in (2.10). Then Γ_a satisfy (2.6). The 3-forms $H|_{U_a} - \boldsymbol{\varphi}_a^* CS(\Gamma_a)$ are closed because of (2.9) and the fact that $d\,CS(\Gamma_a) = \mathrm{Str}\,(R_a \wedge R_a)$. Since U_a are contractible, we may choose such even 2-forms B_a on $\mathbf{W}_a$ that satisfy (2.12), which implies (2.8). □

Proof of Theorem 2.8 continued. Now compute the maps $*^\dagger$, $\{\ \}_0^\dagger$, $\{\ \}_1^\dagger$. In view of (2.3), the restrictions of the three maps to U_a are given by

$$\begin{aligned} f * X \quad &= \boldsymbol{\varphi}_a^*\Big(f_a *^c X_a + \Delta_a(f_a X_a) - f_a\Delta_a(X_a)\Big) \\ \{X,Y\} \quad &= \boldsymbol{\varphi}_a^*\Big(\{X_a,Y_a\}^c - \Delta_a(X_a)(Y_a) - (-1)^{|X||Y|}\Delta_a(Y_a)(X_a)\Big) \\ \{X,Y\}_\Omega &= \boldsymbol{\varphi}_a^*\Big(\{X_a,Y_a\}^c_\Omega - L_{X_a}\Delta_a(Y_a) + (-1)^{|X||Y|}L_{Y_a}\Delta_a(X_a) - d\,\Delta_a(X_a)(Y_a) + \Delta_a([X_a,Y_a])\Big) \end{aligned} \tag{2.13}$$

where f_a, X_a, Y_a are such that $f = \boldsymbol{\varphi}_a^* f_a$, $X = \boldsymbol{\varphi}_a^* X_a$, $Y = \boldsymbol{\varphi}_a^* Y_a$. To evaluate (2.13), apply the formulae of $*^c$, $\{\ \}^c$, $\{\ \}^c_\Omega$ in (A.6), and that of Δ_a in Lemma 2.9 (with $O_a = 0$). Also use the data Γ_a, B_a in the formula of Δ_a to define a connection ∇ on $T\mathbf{M}$ and an even 3-form H on $\mathbf{M}$ as in (2.10) and (2.12); by the proof of Lemma 2.10 they satisfy (2.9). A lengthy but straightforward computation then yields the formulae of $*$, $\{\ \}$, $\{\ \}_\Omega$ stated in the theorem. This proves the last statement of part (a).

It remains to argue that the construction described in the theorem always produces a sheaf of CDOs on $\mathbf{M}$. Notations in this paragraph will have the same meaning as above. Choose a covering $\mathfrak{U} = \{U_a\}_{a\in I}$ of M by contractible open sets, and diffeomorphisms $\boldsymbol{\varphi}_a : \mathbf{U}_a \to \mathbf{W}_a$; let $\boldsymbol{\varphi}_{a'a} = \boldsymbol{\varphi}_{a'} \circ \boldsymbol{\varphi}_a^{-1}$. Starting with the given data ∇, H, define Γ_a, B_a as in the proof of Lemma 2.10, and then Δ_a as in Lemma 2.9 (with $O_a = 0$). By the same computation mentioned before, Δ_a and the given formulae of $*$, $\{\ \}$, $\{\ \}_\Omega$ satisfy (2.13). Then by Lemma A.10, $*$, $\{\ \}$, $\{\ \}_\Omega$ define a sheaf of vertex superalgebroids equipped with the isomorphisms (2.3). Its freely generated sheaf of vertex superalgebras is therefore a sheaf of CDOs.

(b) Use the notations in (a). Suppose ν is a conformal structure on $\mathcal{V}$. For $a \in I$

$$\nu|_{U_a} = \Phi_a\big(\nu^{\omega_a}\big)$$

for some even closed 1-forms ω_a on $\mathbf{W}_a$. For $a, a' \in I$, the isomorphism $\Phi_{a'a} = (\boldsymbol{\varphi}_{a'a})^*_{\xi_{a'a}}$ sends $\nu^{\omega_{a'}}$ to ν^{ω_a}. By Proposition 2.4, this is equivalent to the relation

$$\boldsymbol{\varphi}_{a'a}^*\omega_{a'} - \omega_a = \operatorname{Str}\theta_{a'a} = -\boldsymbol{\varphi}_{a'a}^*\operatorname{Str}\Gamma_{a'} + \operatorname{Str}\Gamma_a$$

where the second equality is given by (2.6). Hence there is an even 1-form ω on $\mathbf{M}$ with

$$\omega|_{U_a} = \boldsymbol{\varphi}_a^*(\omega_a + \operatorname{Str}\Gamma_a).$$

Since $d\omega_a = 0$ and $d\operatorname{Str}\Gamma_a = \operatorname{Str}R_a$, we have $d\omega = \operatorname{Str}R$. Observe that the construction of ω from ν is reversible. To relate ν and ω more explicitly, we compute $\Phi_a(\nu^{\omega_a})$ as follows:

$$\begin{aligned} \nu|_{U_a} &= \epsilon_i\big(\boldsymbol{\varphi}_a^*\partial_i + \boldsymbol{\varphi}_a^*\Delta_a(\partial_i)\big)_{-1}(\boldsymbol{\varphi}_a^* db^i) + \frac{1}{2}(\boldsymbol{\varphi}_a^*\omega_a)_{-2}\mathbf{1} \\ &= \epsilon_i\left(\frac{\partial}{\partial\boldsymbol{\varphi}_a^i}\right)_{-1} d\boldsymbol{\varphi}_a^i + \frac{1}{2}\operatorname{Str}\Big((\boldsymbol{\varphi}_a^*\Gamma_a)_{-1}(\boldsymbol{\varphi}_a^*\Gamma_a) - (\boldsymbol{\varphi}_a^*\Gamma_a)_{-2}\mathbf{1}\Big) + \frac{1}{2}\omega_{-2}\mathbf{1} \end{aligned} \tag{2.14}$$

where we first recall that Φ_a is induced by (2.3) and then use Lemma 2.9. The computation does not depend on B_a, hence not on H. Let $L_n^\omega = \nu_{(n+1)}$. Using (2.14) we have

$$\begin{aligned} L_1^\omega X\big|_{U_a} &= (d\boldsymbol{\varphi}_a^i)_1\left(\frac{\partial}{\partial\boldsymbol{\varphi}_a^i}\right)_0 X + \operatorname{Str}(\boldsymbol{\varphi}_a^*\Gamma_a)_1 X - \omega_1 X \\ &= d\boldsymbol{\varphi}_a^i\left(\left[\frac{\partial}{\partial\boldsymbol{\varphi}_a^i}, X\right]\right) + \operatorname{Str}(\boldsymbol{\varphi}_a^*\Gamma_a)(X) - \omega(X) \end{aligned}$$

for vector fields X. The sum of the first two terms is a local expression for $\operatorname{Str}\nabla^t X$. □

Remarks. (i) Given an open set $U \subset M$, let $\mathbf{U} = \mathbf{M}|_U$. The vertex superalgebra $\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H}(U)$ will also be written as $\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{U})$. A conformal structure ν on $\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H}$ restricts to a conformal structure $\nu|_U$ on $\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{U})$ of central charge $2(p-q)$. (ii) By definition, there are canonical identifications

$$(\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H})_0 = C^\infty_{\mathbf{M}}, \qquad (\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H})_1 = \Omega^1_{\mathbf{M}} \oplus \mathcal{T}_{\mathbf{M}}.$$

Consider the following $\mathbb{C}$-bilinear operation for each $k \geq 0$

$$C^\infty_{\mathbf{M}} \times (\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H})_k \to (\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H})_k, \qquad (f, v) \mapsto f_0 v = f_{(-1)} v.$$

For $k > 0$, this operation does not make $(\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H})_k$ a $C^\infty_{\mathbf{M}}$-module, [3] but it induces a $C^\infty_{\mathbf{M}}$-module structure on an associated graded sheaf $gr(\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H})_k$. Given a $C^\infty_{\mathbf{M}}$-module $\mathcal{E}$, we use the notation $\widehat{\mathrm{Sym}}_t\mathcal{E}$ for the formal sum $\sum_{n=0}^{\infty} t^n \widehat{\mathrm{Sym}}^n\mathcal{E}$, where t is a formal variable and $\widehat{\mathrm{Sym}}^n\mathcal{E}$ is the n-fold graded symmetric tensor power of $\mathcal{E}$ over $C^\infty_{\mathbf{M}}$. There is an isomorphism of $C^\infty_{\mathbf{M}}$-modules

$$C^\infty_{\mathbf{M}} \oplus \bigoplus_{k=1}^{\infty} q^k gr(\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H})_k \cong \bigotimes_{\ell=1}^{\infty} \widehat{\mathrm{Sym}}_{q^\ell}(\Omega^1_{\mathbf{M}} \oplus \mathcal{T}_{\mathbf{M}}).$$

For more details of the vertex superalgebra structure of $\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H}$, consult §A.7 and §A.9.

Theorem 2.11. *Let $\mathbf{M}$ be a smooth cs-manifold, ∇ a connection on $T\mathbf{M}$ with curvature operator R, and H, H' even 3-forms on $\mathbf{M}$ with $dH = dH' = \mathrm{Str}\,(R \wedge R)$. Define $*$, $\{\ \}$, $\{\ \}_\Omega$ (resp. $\{\ \}'_\Omega$) using ∇ and H (resp. H') as in Theorem 2.8a.*

(a) There is a one-to-one correspondence:

$$\left\{ \begin{array}{c} \text{isomorphisms of sheaves of CDOs } \mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H} \to \mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H'} \\ \text{whose weight-zero components are identity on } C^\infty_{\mathbf{M}} \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} B \in \Omega^2(\mathbf{M}), \text{even}, \\ dB = H' - H \end{array} \right\}$$

Given B as above, the corresponding isomorphism, denoted by id_B, is induced by an isomorphism between the associated sheaves of vertex superalgebroids

$$(\mathrm{id}, \Delta_B) : \big(C^\infty_{\mathbf{M}}, \Omega^1_{\mathbf{M}}, \mathcal{T}_{\mathbf{M}}, *, \{\ \}, \{\ \}_\Omega\big) \to \big(C^\infty_{\mathbf{M}}, \Omega^1_{\mathbf{M}}, \mathcal{T}_{\mathbf{M}}, *, \{\ \}, \{\ \}'_\Omega\big)$$

where the map $\Delta_B : \mathcal{T}_{\mathbf{M}} \to \Omega^1_{\mathbf{M}}$ is given by $\Delta_B(X) = \frac{1}{2}\iota_X B$.

(b) The isomorphism id_B preserves the correspondence in Theorem 2.8b, i.e. $\mathrm{id}_B(\nu^\omega) = \nu^\omega$.

Proof. (a) If an isomorphism between the two sheaves of CDOs equals the identity on $C^\infty_{\mathbf{M}}$, then it induces the identity on the sheaf of extended Lie superalgebroids $(C^\infty_{\mathbf{M}}, \Omega^1_{\mathbf{M}}, \mathcal{T}_{\mathbf{M}})$, and is therefore determined by an isomorphism of sheaves of vertex superalgebroids of the form

$$(\mathrm{id}, \Delta) : \big(C^\infty_{\mathbf{M}}, \Omega^1_{\mathbf{M}}, \mathcal{T}_{\mathbf{M}}, *, \{\ \}, \{\ \}_\Omega\big) \to \big(C^\infty_{\mathbf{M}}, \Omega^1_{\mathbf{M}}, \mathcal{T}_{\mathbf{M}}, *, \{\ \}, \{\ \}'_\Omega\big).$$

By definition, the even map $\Delta : \mathcal{T}_{\mathbf{M}} \to \Omega^1_{\mathbf{M}}$ has to satisfy precisely the following equations:

$$\Delta(fX) = f\Delta(X), \qquad \Delta(Y)(X) = -(-1)^{|X||Y|}\Delta(X)(Y)$$
$$L_X\Delta(Y) - (-1)^{|X||Y|} L_Y \Delta(X) + d\,\Delta(X)(Y) - \Delta([X,Y]) = \{X,Y\}_\Omega - \{X,Y\}'_\Omega$$

According to the first two equations, $B(X,Y) := 2\Delta(X)(Y)$ defines an even 2-form B on $\mathbf{M}$. Then the last equation can be rewritten as

$$\iota_X \iota_Y dB = \iota_X \iota_Y (H' - H).$$

(b) Since the said correspondence is independent of H, this is clear. This also follows from the local expression (2.14) of ν^ω and the graded symmetry of $(d\boldsymbol{\varphi}^i_a)_{-1} d\boldsymbol{\varphi}^j_a$. □

Example 2.12. Sheaves of CDOs on ΠE. Let M be a smooth manifold, $E \to M$ a smooth $\mathbb{C}$-vector bundle and $\mathbf{M} = \Pi E$ as a smooth cs-manifold. The canonical pullback embeds $\Omega^*(M)$ into $\Omega^*(\mathbf{M})$ quasi-isomorphically. [DM99] Choose connections ∇^M on TM and ∇^E on E, which determine a connection ∇

[3] For example, we have $f_0 X = fX + f * X = fX - (\nabla df)(X)$ for vector fields X.

on $T\mathbf{M}$ in the sense of §B.3; denote by R^M, R^E and R the corresponding curvature tensors. As stated in Lemma B.5, we have

$$\mathrm{Str}\,(R \wedge R) = \mathrm{Tr}\,(R^M \wedge R^M) - \mathrm{Tr}\,(R^E \wedge R^E), \qquad \mathrm{Str}\,R = \mathrm{Tr}\,R^M - \mathrm{Tr}\,R^E.$$

By Theorems 2.8a and 2.11a, $\mathbf{M}$ admits sheaves of CDOs if and only if $p_1(TM) - ch_2(E)$ vanishes in de Rham cohomology, and their isomorphism classes form an $H^3(M;\mathbb{C})$-torsor. By Theorem 2.8b, the sheaves of CDOs possess conformal structures if and only if $c_1(E)$ vanishes in de Rham cohomology as well.

Example 2.13. The smooth chiral de Rham complex. Consider the case $E = TM \otimes \mathbb{C}$ in the previous example. Both obstructions are now trivial, so that $\mathbf{M}$ always admits sheaves of CDOs equipped with conformal structures. In particular, we may define a sheaf of CDOs $\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla} = \mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,0}$ using the trivial 3-form and a conformal structure $\nu = \nu^0$ using the trivial 1-form.

Let J and Q be the vector fields on $\mathbf{M}$ defined in §B.2 and Example B.6. Regarded as elements of $\mathcal{D}^{\mathrm{ch}}_{\nabla}(\mathbf{M})$ of weight 1, they satisfy

$$2Q_0^2 = [Q_0, Q_0] = [Q, Q]_0 + (\{Q, Q\}_\Omega)_0 = 0, \qquad [J_0, Q_0] = [J, Q]_0 + (\{J, Q\}_\Omega)_0 = Q_0.$$

In view of the formulae in Theorem 2.8a and Lemma 2.14, the two equations follow from (B.7) and Lemma B.8, with the second also requiring Lemmas B.4b and B.5c. [4] Moreover, we have

$$Q_0\nu = -\frac{1}{2}T^2(L_1 Q) = 0, \qquad J_0\nu = -\frac{1}{2}T^2(L_1 J) = 0.$$

In view of Theorem 2.8b, the two equations follow from Lemmas B.8 and B.4a respectively. Therefore, with J_0 as the grading operator and Q_0 as the differential, $\mathcal{D}^{\mathrm{ch}}_{\nabla}(\mathbf{M})$ becomes a differential graded conformal vertex superalgebra. [5]

Lemma 2.14. *Consider a sheaf of CDOs $\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H}$ on a smooth cs-manifold $\mathbf{M}$ constructed as in Theorem 2.8. Given $\alpha \in \Omega^1(\mathbf{M})$, we have $\alpha_0 = 0$ on $\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M})$ if and only if $d\alpha = 0$.*

Proof. Since α_0 is a derivation, it acts trivially on the entire vertex superalgebra $\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M})$ if and only if it acts trivially on functions and vector fields. For $f \in C^\infty(\mathbf{M})$, we always have $\alpha_0 f = 0$. For $X \in \mathcal{T}(\mathbf{M})$, we compute

$$\alpha_0 X = \pm[X_{-1}, \alpha_0]\mathbf{1} = \pm(L_X\alpha - d\iota_X\alpha) = \pm\iota_X d\alpha$$

which proves the assertion. □

[4] In fact, we are assuming that ∇^M is Levi-Civita. The torsion-free condition is used to obtain various formulae in Example B.6 and subsequently Lemma B.8, while orthogonality ensures that the right hand side of Lemma B.5c vanishes.

[5] For a description of a richer structure on $\mathcal{D}^{\mathrm{ch}}_{\nabla}(\mathbf{M})$, see [BHS08].

§3. Chiral Dolbeault Algebras

Applying the description of CDOs obtained in Theorem 2.8, we study a vertex algebraic analogue of the Dolbeault complex of a complex manifold. This provides a new point of view on the relation between CDOs and elliptic genera.

§ 3.1. Dolbeault cs-manifolds. Let M be a complex manifold, TM its holomorphic tangent bundle, E a holomorphic vector bundle over M, and $\mathbf{M} = \Pi(\overline{TM} \oplus E)$ as a smooth cs-manifold. Let $d = \dim_{\mathbb{C}} M$ and $r = \operatorname{rank} E$. Under the identification

$$C^\infty(\mathbf{M}) \cong \Omega^{0,*}(M; \wedge^* E^\vee) \tag{3.1}$$

vector fields on $\mathbf{M}$ correspond to derivations of the $(0,*)$-forms on M valued in $\wedge^* E^\vee$. In particular, let

$$\left\{ \begin{array}{c} J^r \\ J^\ell \\ Q \end{array} \right\} = \begin{array}{l} \text{the vector field on } \mathbf{M} \\ \text{corresponding to the} \end{array} \left\{ \begin{array}{l} \text{Dolbeault degree} \\ \text{exterior degree in } \wedge^* E^\vee \\ \text{Dolbeault operator } \bar\partial \otimes 1 \end{array} \right\} \tag{3.2}$$

For more discussion of $\mathcal{T}(\mathbf{M})$, see Example B.9.

§ 3.2. Sheaves of CDOs on M. Choose connections ∇^M on TM and ∇^E on E such that both are of type $(1,0)$ and ∇^M is torsion-free (see footnote 15). Let ∇ be the induced connection on $T\mathbf{M}$ defined as in Example B.9. Denote by R^M, R^E and R the respective curvature tensors. Notice that the canonical pullback embeds $\Omega^*(M)$ into $\Omega^*(\mathbf{M})$ quasi-isomorphically [DM99] and recall Lemma B.12.

Assume that $ch_2(TM) - ch_2(E) = 0$ in de Rham cohomology and choose $H \in \Omega^3(M)$ such that

$$dH = \operatorname{Str}(R \wedge R) = \operatorname{Tr}(R^M \wedge R^M) - \operatorname{Tr}(R^E \wedge R^E). \tag{3.3}$$

By Theorems 2.8a and 2.11a, this determines a sheaf of CDOs $\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H}$ and every sheaf of CDOs on $\mathbf{M}$ is up to isomorphism of this form. Assume also that $c_1(TM) - c_1(E) = 0$ in de Rham cohomology and choose $\omega \in \Omega^1(M)$ such that

$$d\omega = \operatorname{Str} R = \operatorname{Tr} R^M - \operatorname{Tr} R^E. \tag{3.4}$$

By Theorem 2.8b, this determines a conformal structure ν^ω on $\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H}$ of central charge $2(d-r)$.

Theorem 3.3. *Regard Q as an element of $\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M})$ of weight 1. The odd derivation Q_0:*
(a) is a differential if and only if H has no $(1,2)$- or $(0,3)$-part, and
(b) respects the conformal structure ν^ω if and only if ω has no $(0,1)$-part.

Proof. (a) The supercommutator of Q_0 with itself is given by

$$2Q_0^2 = [Q_0, Q_0] = [Q, Q]_0 + (\{Q, Q\}_\Omega)_0 = \frac{1}{2}(\iota_Q \iota_Q H)_0$$

where the last step follows from (B.13), Theorem 2.8a, Lemma B.14 and Lemma 2.14. By Lemma 2.14 again, Q_0^2 vanishes if and only if $\iota_Q\iota_Q H$ is closed. In view of the identity

$$2\iota_Q\iota_Q H = L_{J^r}\iota_Q\iota_Q H = \iota_{J^r} d\iota_Q\iota_Q H$$

$\iota_Q\iota_Q H$ can only be closed when it is in fact trivial. When applied to a differential form on M, $\iota_Q\iota_Q$ picks out those components of type (i,j) with $j \geq 2$.

(b) Applying Q_0 to ν^ω yields

$$Q_0(\nu^\omega) = -[\nu^\omega_{(-1)}, Q_0]\mathbf{1} = -\frac{1}{2}T^2 L_1^\omega Q = \frac{1}{2}T^2\omega(Q)$$

where the last step follows from Theorem 2.8b and Lemma B.14. Hence Q_0 annihilates ν^ω if and only if $T\omega(Q) = d\omega(Q) = 0$. In view of the identity

$$\omega(Q) = J^r\omega(Q) = \iota_{J^r} d\omega(Q)$$

$\omega(Q)$ can only be constant when it is in fact trivial. When applied to a differential form on M, ι_Q picks out those components of type (i, j) with $j \geq 1$. ☐

Definition 3.4. For each $n \geq 0$, let $\Omega^{n,\mathrm{cl}}_{M,\mathrm{hol}}$ (resp. $\Omega^{n,\mathrm{cl}}_M$) denote the sheaf of holomorphic (resp. smooth) closed n-forms on M and define an element

$$c_n^{\mathrm{hol}}(E) \in H^n(\Omega^{n,\mathrm{cl}}_{M,\mathrm{hol}})$$

as follows. Since ∇^E is of type $(1, 0)$, its curvature R^E has only $(2, 0)$- and $(1, 1)$-parts. Thus the n-th Chern form $c_n(\nabla^E)$ lives in $\Omega^{n+*,*}(M)$. Consider the diagram of fine resolutions of sheaves:

$$\begin{array}{ccccc} 0 & \longrightarrow & \Omega^{n,\mathrm{cl}}_{M,\mathrm{hol}} & \longrightarrow & (\Omega^{n+*,*}_M, d) \\ & & \downarrow \cap & & \downarrow \cap \\ 0 & \longrightarrow & \Omega^{n,\mathrm{cl}}_M & \longrightarrow & (\Omega^{n+*}_M, d) \end{array}$$

In light of this diagram, $c_n(\nabla^E)$ represents an element "$c_n^{\mathrm{hol}}(E)$" of $H^n(\Omega^{n,\mathrm{cl}}_{M,\mathrm{hol}})$, whose image under

$$H^n(\Omega^{n,\mathrm{cl}}_{M,\mathrm{hol}}) \to H^n(\Omega^{n,\mathrm{cl}}_M) \cong H^{2n}(M; \mathbb{C}) \tag{3.5}$$

is the n-th Chern class $c_n(E)$. More generally, if $C(E)$ is a polynomial in the Chern classes $c_n(E)$, denote by $C^{\mathrm{hol}}(E)$ the corresponding polynomial in $c_n^{\mathrm{hol}}(E)$. The following result relates some of these cohomology classes to the conditions encountered in Theorem 3.3.

Proposition 3.5. *There exists:*
(a) $H \in \Omega^3(M)$ *satisfying (3.3) and* $H^{1,2} = H^{0,3} = 0$ *if and only if* $ch_2^{\mathrm{hol}}(TM) - ch_2^{\mathrm{hol}}(E) = 0$*;*
(b) $\omega \in \Omega^1(M)$ *satisfying (3.4) and* $\omega^{0,1} = 0$ *if and only if* $c_1^{\mathrm{hol}}(TM) - c_1^{\mathrm{hol}}(E) = 0$.

Proof. Recall Definition 3.4. Statement (a) holds because the said element of $H^2(\Omega^{2,\mathrm{cl}}_{M,\mathrm{hol}})$ is represented, via the fine resolution

$$0 \longrightarrow \Omega^{2,\mathrm{cl}}_{M,\mathrm{hol}} \longrightarrow \Omega^{2,0}_M \xrightarrow{d} \Omega^{3,0}_M \oplus \Omega^{2,1}_M \xrightarrow{d} \Omega^{4,0}_M \oplus \Omega^{3,1}_M \oplus \Omega^{2,2}_M \xrightarrow{d} \cdots$$

by the right hand side of (3.3) up to a constant factor. Statement (b) is similar. ☐

Remark. In the case M is Kähler, (3.5) is injective, as it can be identified with the inclusion

$$\bigoplus_{p \geq 0, p+q=n} \mathcal{H}^{n+p,q} \hookrightarrow \mathcal{H}^{2n}$$

where $\mathcal{H}^{n+p,q}$, $\mathcal{H}^{2n}$ are the spaces of harmonic $(n + p, q)$- and $2n$-forms respectively. Thus the conditions in Proposition 3.5 become equivalent to $ch_2(TM) - ch_2(E) = 0$ and $c_1(TM) - c_1(E) = 0$.

§ 3.6. Fermion numbers. The eigenvalues of J_0^r and J_0^ℓ on $\mathcal{D}^{\mathrm{ch}}_{\mathbf{M},\nabla,H}$ will be referred to respectively as right (i.e. antiholomorphic) and left (i.e. holomorphic) fermion numbers. Recall from (3.2) that in weight 0, these numbers correspond to the exterior degrees in $\wedge^*\overline{TM}^\vee$ and $\wedge^* E^\vee$.

Proposition 3.7. *The operator* Q_0 *always increases right fermion numbers by* 1 *if and only if the line bundle* $\det TM$ *is flat.*

Proof. Denote by $\bar{\nabla}^M$ the connection on $\overline{TM}$ induced by ∇^M and $\bar{R}^M$ its curvature tensor. The commutator between J_0^r and Q_0 is given by

$$[J_0^r, Q_0] = [J^r, Q]_0 + (\{J^r, Q\}_\Omega)_0 = Q_0 - (\iota_Q \mathrm{Tr}\, \bar{R}^M)_0$$

which follows from (B.13), Theorem 2.8a, Lemmas B.10b, B.12c and B.14. Hence by Lemma 2.14, Q_0 is compatible with right fermion numbers if and only if $\iota_Q \mathrm{Tr}\, \bar{R}^M$ is closed. By the same argument used in the proof of Theorem 3.3, $\iota_Q \mathrm{Tr}\, \bar{R}^M$ can only be closed when it is in fact trivial. Since ∇^M is of type $(1,0)$, $\bar{R}^M$ has only $(1,1)$- and $(0,2)$-parts, so that $\iota_Q \mathrm{Tr}\, \bar{R}^M = 0$ if and only if $\mathrm{Tr}\, \bar{R}^M = 0$. ☐

Proposition 3.8. *The operator Q_0 respects left fermion numbers if and only if $\mathrm{Tr}\, R^E$ has no $(1,1)$-part.*

Proof. The commutator between J_0^ℓ and Q_0 is given by

$$[J_0^\ell, Q_0] = [J^\ell, Q]_0 + (\{J^\ell, Q\}_\Omega)_0 = -(\iota_Q \mathrm{Tr}\, R^E)_0$$

which follows from (B.13), Theorem 2.8a, Lemmas B.11b, B.12c and B.14. Hence by Lemma 2.14, Q_0 commutes with J_0^ℓ if and only if $\iota_Q \mathrm{Tr}\, R^E$ is closed. By the same argument used in the proof of Theorem 3.3, $\iota_Q \mathrm{Tr}\, R^E$ can only be closed when it is in fact trivial. Since ∇^E is of type $(1,0)$, R^E has only $(2,0)$- and $(1,1)$-parts, so that ι_Q picks out the $(1,1)$-part. ☐

Remark. Given a hermitian metric on E, there exists a unique unitary connection ∇^E of type $(1,0)$, and its curvature R^E is of pure type $(1,1)$. [Wel80] If $\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M})$ has been defined using this ∇^E, then Q_0 respects left fermion numbers if and only if $\mathrm{Tr}\, R^E = 0$, i.e. the line bundle $\det E$ is flat.

Corollary 3.9. *Suppose $Q_0^2 = 0$ holds, so that $(\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M}), Q_0)$ is a differential vertex superalgebra.*

(a) If $\det TM \cong \det E$ as holomorphic line bundles, $(\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M}), Q_0)$ is a differential conformal vertex superalgebra.

(b) If $\det TM$ is flat, the grading by right fermion numbers makes $(\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M}), Q_0)$ a differential graded vertex superalgebra.

(c) If $\det E$ is flat, left fermion numbers are well-defined on the cohomology of $(\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M}), Q_0)$.

Proof. (a) Under the assumption we may compare the induced connections $\det \nabla^M$ and $\det \nabla^E$ via the isomorphism, and they differ by a $(1,0)$-form ω. This implies (3.4) and, by Theorem 3.3b, $Q_0(\nu^\omega) = 0$. (b)-(c) These are simply restatements of Propositions 3.7 and 3.8. ☐

For the rest of this section, M is always compact.

Theorem 3.10. *Suppose $Q_0^2 = 0$ holds and consider the vertex superalgebra*

$$V = H\big(\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M}), Q_0\big).$$

Let q be a formal variable. There is an identity of formal power series

$$\mathrm{Str}_V(q^{L_0}) = \int_M \mathrm{Td}(TM)\, ch \left(\bigotimes_{n=1}^\infty \mathrm{Sym}_{q^n}(TM \oplus TM^\vee) \otimes \bigotimes_{n=1}^\infty \wedge_{-q^n} E \otimes \bigotimes_{n=0}^\infty \wedge_{-q^n} E^\vee \right). \tag{3.6}$$

Let y be another formal variable. If $\det E$ is flat, there is a more refined identity

$$\mathrm{Str}_V(y^{J_0^\ell} q^{L_0}) = \int_M \mathrm{Td}(TM)\, ch \left(\bigotimes_{n=1}^\infty \mathrm{Sym}_{q^n}(TM \oplus TM^\vee) \otimes \bigotimes_{n=1}^\infty \wedge_{-y^{-1}q^n} E \otimes \bigotimes_{n=0}^\infty \wedge_{-yq^n} E^\vee \right).$$

Proof. By Proposition 3.8, if $\det R^E$ is flat, J_0^ℓ is well-defined on V. Otherwise, set $y = 1$ whenever it appears in the proof below.

Observe that Q_0 respects the filtration on $\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M})$ described in §A.9 and induces the operator L_Q on the associated graded space $gr(\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M}))$. Let

$$V' = H\big(gr(\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M})), L_Q\big).$$

The quantity we want to compute can be rephrased as follows:

$$\begin{aligned} &\text{supertrace of } y^{J_0^l} q^{L_0} \text{ on } V \\ =\ &\text{supertrace of } y^{J_0^l} q^{L_0} \text{ on } V' \\ =\ &\text{supertrace of } y^{J_0^l} \text{ on } H\left(\bigotimes_{n=1}^{\infty} \widehat{\mathrm{Sym}}_{q^n}\big(\Omega^1(\mathbf{M}) \oplus \mathcal{T}(\mathbf{M})\big), L_Q\right) \end{aligned} \tag{3.7}$$

where the graded symmetric tensor products are taken over $C^\infty(\mathbf{M})$. Recall the local coordinates

$$(\mathrm{Re}z^1, \mathrm{Im}z^1, \cdots, \mathrm{Re}z^d, \mathrm{Im}z^d, \bar\zeta^1, \cdots, \bar\zeta^d, \varepsilon^1, \cdots, \varepsilon^r)$$

defined in Example B.9. To compute (3.7), consider the following subspaces of $\Omega^1(\mathbf{M})$ and $\mathcal{T}(\mathbf{M})$:

$$\begin{aligned} \Omega^{1,0}(\mathbf{M}) &= \big\{\alpha \in \Omega^1(\mathbf{M}) \text{ s.t. } \alpha \text{ is locally a } C^\infty\text{-linear combination of } dz^i, d\varepsilon^k\big\} \\ \Omega^{1,0}(\mathbf{M})' &= \big\{\alpha \in \Omega^1(\mathbf{M}) \text{ s.t. } \alpha \text{ is locally a } C^\infty\text{-linear combination of } dz^i\big\} \\ \mathcal{T}^{1,0}(\mathbf{M}) &= \big\{X \in \mathcal{T}(\mathbf{M}) \text{ s.t. } X \text{ is locally a } C^\infty\text{-linear combination of } \partial/\partial z^i, \partial/\partial\varepsilon^k\big\} \\ \mathcal{T}^{1,0}(\mathbf{M})' &= \big\{X \in \mathcal{T}(\mathbf{M}) \text{ s.t. } X \text{ is locally a } C^\infty\text{-linear combination of } \partial/\partial\varepsilon^k\big\} \end{aligned}$$

Lemma 3.11. *The following inclusions*

$$\big(\Omega^{1,0}(\mathbf{M}), L_Q\big) \hookrightarrow \big(\Omega^1(\mathbf{M}), L_Q\big), \qquad \big(\mathcal{T}^{1,0}(\mathbf{M}), L_Q\big) \hookrightarrow \big(\mathcal{T}(\mathbf{M}), L_Q\big)$$

are quasi-isomorphisms.

Proof. Denote both of the projections $\Omega^1(\mathbf{M}) \to \Omega^{1,0}(\mathbf{M})$ and $\mathcal{T}(\mathbf{M}) \to \mathcal{T}^{1,0}(\mathbf{M})$ by $\pi^{1,0}$. It suffices to show that $\mathrm{id} - \pi^{1,0}$ are null homotopic. Define $G : \Omega^1(\mathbf{M}) \to \Omega^1(\mathbf{M})$ and $G : \mathcal{T}(\mathbf{M}) \to \mathcal{T}(\mathbf{M})$ locally by

$$G\alpha = (-1)^{|\alpha|}\alpha\Big(\frac{\partial}{\partial\bar\zeta^i}\Big)d\bar z^i, \qquad GX = (-1)^{|X|}d\bar z^i(X)\frac{\partial}{\partial\bar\zeta^i}$$

and notice that the expressions are independent of local coordinates. By a calculation we have

$$L_Q G + G L_Q = \mathrm{id} - \pi^{1,0}$$

on both $\Omega^1(\mathbf{M})$ and $\mathcal{T}(\mathbf{M})$, as desired. □

Lemma 3.12. *There are natural filtrations on* $(\Omega^{1,0}(\mathbf{M}), L_Q)$ *and* $(\mathcal{T}^{1,0}(\mathbf{M}), L_Q)$ *whose associated graded complexes are isomorphic respectively to*

$$\big(\Omega^{0,*}(M; E'), \bar\partial\big) \quad \textit{and} \quad \big(\Omega^{0,*}(M; E''), \bar\partial\big)$$

where $E' = \wedge^* E \otimes (TM^\vee \oplus E^\vee)$ *and* $E'' = \wedge^* E \otimes (TM \oplus E)$.

Proof. There are identifications defined by the following local expressions

$$\begin{aligned} &\Omega^{1,0}(\mathbf{M})' \cong C^\infty(\mathbf{M}) \otimes_{C^\infty(M)} \Omega^{1,0}(M), && dz^i \mapsto 1 \otimes dz^i \\ &\Omega^{1,0}(\mathbf{M})/\Omega^{1,0}(\mathbf{M})' \cong C^\infty(\mathbf{M}) \otimes_{C^\infty(M)} \Gamma(E^\vee), && d\varepsilon^k \bmod \Omega^{1,0}(\mathbf{M})' \mapsto 1 \otimes \varepsilon^k \\ &\mathcal{T}^{1,0}(\mathbf{M})' \cong C^\infty(\mathbf{M}) \otimes_{C^\infty(M)} \Gamma(E), && \partial/\partial\varepsilon^k \mapsto 1 \otimes \varepsilon_k \\ &\mathcal{T}^{1,0}(\mathbf{M})/\mathcal{T}^{1,0}(\mathbf{M})' \cong C^\infty(\mathbf{M}) \otimes_{C^\infty(M)} \mathcal{T}^{1,0}(M), && \partial/\partial z^i \bmod \mathcal{T}^{1,0}(\mathbf{M})' \mapsto 1 \otimes \partial/\partial z^i \end{aligned}$$

and $C^\infty(\mathbf{M})$-linearity. Then it remains to recall (3.1). □

Proof of Theorem 3.10 continued. Now we apply Lemmas 3.11 and 3.12 to compute (3.7). Since (graded symmetric) tensor products of quasi-isomorphic complexes are quasi-isomorphic and filtrations on complexes induce filtrations on their (graded symmetric) tensor products, we have

$$
\begin{aligned}
(3.7) &= \text{supertrace of } y^{J_0^l} \text{ on } H\left(\bigotimes_{n=1}^{\infty}\widehat{\mathrm{Sym}}_{q^n}\left(\Omega^{1,0}(\mathbf{M})\oplus\mathcal{T}^{1,0}(\mathbf{M})\right), L_Q\right)\\
&= \text{supertrace of } y^{J_0^l} \text{ on } H\left(\bigotimes_{n=1}^{\infty}\widehat{\mathrm{Sym}}_{q^n}\Omega^{0,*}(M;E'\oplus E''),\bar\partial\right)\\
&= \operatorname{sdim} H\left(\Omega^{0,*}\left(M;\wedge_{-y}E^\vee\otimes\bigotimes_{n=1}^{\infty}\widehat{\mathrm{Sym}}_{q^n}\left(TM\oplus TM^\vee\oplus(-y^{-1})E\oplus(-y)E^\vee\right)\right),\bar\partial\right)\\
&= \operatorname{sdim} H\left(\Omega^{0,*}\left(M;\bigotimes_{n=1}^{\infty}\mathrm{Sym}_{q^n}(TM\oplus TM^\vee)\otimes\bigotimes_{n=1}^{\infty}\wedge_{-y^{-1}q^n}E\otimes\bigotimes_{n=0}^{\infty}\wedge_{-yq^n}E^\vee\right),\bar\partial\right)
\end{aligned}
$$

Notice that the graded symmetric tensor products in the second expression are taken over $\Omega^{0,*}(M;\wedge^*E^\vee)$. To finish the proof, apply the Hirzebruch-Riemann-Roch Theorem. □

Remark. In terms of the Chern roots $x_1,\dots,x_d$ of TM and $x_1^E,\dots,x_r^E$ of E, we may write the integrand in (3.6) as follows

$$
\begin{aligned}
&\prod_{i=1}^{d}\left(\frac{x_i}{1-e^{-x_i}}\prod_{n=1}^{\infty}\frac{1}{(1-q^ne^{x_i})(1-q^ne^{-x_i})}\right)\cdot\prod_{j=1}^{r}\left((1-e^{-x_j^E})\prod_{n=1}^{\infty}(1-q^ne^{x_j^E})(1-q^ne^{-x_j^E})\right)\\
=&\prod_{i=1}^{d}\left(\frac{x_i/2}{\sinh(x_i/2)}\prod_{n=1}^{\infty}\frac{1}{(1-q^ne^{x_i})(1-q^ne^{-x_i})}\right)\cdot\prod_{j=1}^{r}\left(2\sinh\frac{x_j^E}{2}\prod_{n=1}^{\infty}(1-q^ne^{x_j^E})(1-q^ne^{-x_j^E})\right)\cdot\frac{e^{\frac12 c_1(TM)}}{e^{\frac12 c_1(E)}}
\end{aligned}
$$

If $c_1(TM)=c_1(E)$, this expression lives in $H^{4*}(M;\mathbb{C})$ if r is even, or in $H^{4*+2}(M;\mathbb{C})$ if r is odd, so that $\operatorname{Str}_V(q^{L_0})=0$ whenever $d+r$ is odd.

Example 3.13. The case $E=0$. By Theorem 3.3a and Proposition 3.5a, there exists a differential vertex superalgebra

$$\left(\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M}),Q_0\right),\qquad \mathbf{M}=\Pi\,\overline{TM}$$

if and only if $ch_2^{\mathrm{hol}}(TM)=0$; denote its cohomology by V. By Theorem 3.10

$$\operatorname{Str}_V(q^{L_0})=\int_M e^{\frac12 c_1(TM)}\cdot W(TM_{\mathbb{R}})\cdot\left(\prod_{n=1}^{\infty}\frac{1}{1-q^n}\right)^{2d}$$

where $W(TM_{\mathbb{R}})$ is the Witten class of the real tangent bundle of M. By Theorem 3.3b and Proposition 3.5b, if $c_1^{\mathrm{hol}}(TM)=0$ as well, V is conformal with central charge $2d$. Then, writing $q=e^{2\pi i\tau}$, we have

$$\operatorname{char}V=q^{-d/12}\operatorname{Str}_V(q^{L_0})=\frac{W(M)}{\Delta(\tau)^{d/12}}\tag{3.8}$$

where $W(M)$ is the Witten genus of M and

$$\Delta(\tau)=q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

The condition $c_1(TM)=c_2(TM)=0$ guarantees that $W(M)$ is a modular form of weight d, while $\Delta(\tau)$ is a modular form of weight 12, both over $SL(2,\mathbb{Z})$. The expression in (3.8) is the conjectured S^1-equivariant index of the Dirac operator on the free loop space LM. [Wit88]

Example 3.14. The case $E = TM$. By Theorem 3.3 and Proposition 3.5, there always exists a differential conformal vertex superalgebra

$$\big(\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M}), Q_0\big), \qquad \mathbf{M} = \Pi(\overline{TM} \oplus TM)$$

with no central charge; denote its cohomology by V. By Theorem 3.10, we have

$$\mathrm{Str}_V(q^{L_0}) = \chi(M)$$

and, if $\det TM$ is flat, also have

$$y^{-d}\,\mathrm{Str}_V(y^{J^l_0} q^{L_0}) = Ell_{y,q}(M)$$

namely the two-variable elliptic genus of M. [BL00] In particular, writing $q = e^{2\pi i\tau}$, we have the special value

$$\mathrm{Str}_V((-1)^{J^l_0} q^{L_0}) = \frac{\mathrm{Och}(M)}{\epsilon(\tau)^{d/4}} \tag{3.9}$$

where $\mathrm{Och}(M)$ is the Ochanine elliptic genus of M and

$$\epsilon(\tau) = \frac{1}{16}\prod_{n=1}^{\infty}\left(\frac{1-q^n}{1+q^n}\right)^8$$

respectively a modular form of weight d and a modular form of weight 4 over $\Gamma_0(2) \subset SL(2,\mathbb{Z})$. The expression in (3.9) is the S^1-equivariant signature of LM. [HBJ92]

Example 3.15. The case $E = \det TM$. Let $c = c_1(TM)$ and $c^{\mathrm{hol}} = c_1^{\mathrm{hol}}(TM)$. By Theorem 3.3a and Proposition 3.5a, there exists a differential vertex superalgebra

$$\big(\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M}), Q_0\big), \qquad \mathbf{M} = \Pi(\overline{TM} \oplus \det TM)$$

if and only if

$$ch_2^{\mathrm{hol}}(TM) - \frac{1}{2}(c^{\mathrm{hol}})^2 = 0; \tag{3.10}$$

denote its cohomology by V. By Theorem 3.3b and Proposition 3.5b, V is always conformal with central charge $2(d-1)$. By Theorem 3.10 and the remark below its proof, we have

$$\mathrm{Str}_V(q^{L_0}) = 2\int_M W(TM_{\mathbb{R}})\,\sinh\frac{c}{2}\prod_{n=1}^{\infty}\frac{(1-q^n e^c)(1-q^n e^{-c})}{(1-q^n)^2}\cdot\left(\prod_{n=1}^{\infty}\frac{1}{1-q^n}\right)^{2(d-1)} \tag{3.11}$$

which always vanishes if d is even. Now assume d is odd. This case provides a geometric interpretation of the notions introduced in [CHZ10] for certain spin^c manifolds of (real) dimension 2 mod 4. Firstly, condition (3.10) implies that M is *rationally string^c* in the sense of *loc. cit.*, namely

$$ch_2(TM) - \frac{1}{2}c^2 = 0 \quad \text{in } H^4(M;\mathbb{C}). \tag{3.12}$$

In the case M is Kähler, (3.10) and (3.12) are equivalent, as remarked after the proof of Proposition 3.5. Secondly, writing $q = e^{2\pi i\tau}$, we have

$$\mathrm{char}\,V = q^{-(d-1)/12}\mathrm{Str}_V(q^{L_0}) = \frac{2W_c(M)}{\Delta(\tau)^{(d-1)/12}}$$

where $W_c(M)$ is the *generalized Witten genus* of M defined in *loc. cit.* [6] The string^c condition (3.12) guarantees that $W_c(M)$ is a modular form of weight $d-1$ over $SL(2,\mathbb{Z})$.

[6] To recover the expression for $W_c(M)$ in [CHZ10], notice that they write $q = e^{\pi i\tau}$, and the factor $\sinh(c/2)$ in (3.11) may be replaced by $e^{c/2}$ since $d = \dim_{\mathbb{C}} M$ is odd.

Example 3.16. The case $E = (\det TM)^{\otimes 2} - \det TM$. [7] Let $c = c_1(TM)$ and $c^{\mathrm{hol}} = c_1^{\mathrm{hol}}(TM)$. By Theorem 3.3a and Proposition 3.5a, there exists a differential vertex superalgebra

$$\big(\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M}), Q_0\big), \qquad \mathbf{M} = \text{“}\Pi\big(\overline{TM} \oplus ((\det TM)^{\otimes 2} - \det TM)\big)\text{”}$$

if and only if

$$ch_2^{\mathrm{hol}}(TM) - \frac{3}{2}(c^{\mathrm{hol}})^2 = 0; \tag{3.13}$$

denote its cohomology by V. By Theorem 3.3b and Proposition 3.5b, V is always conformal with central charge $2d$. By Theorem 3.10 and the remark below its proof, we have

$$\begin{aligned}
\mathrm{Str}_V(q^{L_0}) &= 2\int_M W(TM_{\mathbb{R}})\,\cosh\frac{c}{2}\prod_{n=1}^{\infty}\frac{(1-q^ne^{2c})(1-q^ne^{-2c})}{(1-q^ne^{c})(1-q^ne^{-c})}\cdot\Big(\prod_{n=1}^{\infty}\frac{1}{1-q^n}\Big)^{2d} \\
&= 2\int_M W(TM_{\mathbb{R}})\,\cosh\frac{c}{2}\prod_{n=1}^{\infty}\Big[(1-q^{n-\frac12}e^{c})(1+q^{n-\frac12}e^{c})(1+q^ne^{c}) \\
&\qquad\cdot(1-q^{n-\frac12}e^{-c})(1+q^{n-\frac12}e^{-c})(1+q^ne^{-c})\Big]\cdot\Big(\prod_{n=1}^{\infty}\frac{1}{1-q^n}\Big)^{2d}
\end{aligned} \tag{3.14}$$

which always vanishes if d is odd. Now assume d is even. This case provides a geometric interpretation of the notions introduced in [CHZ10] for certain spin^c manifolds of (real) dimension divisible by 4. Firstly, condition (3.13) implies that M is *rationally string^c* in the sense of *loc. cit.*, namely

$$ch_2(TM) - \frac{3}{2}c^2 = 0 \quad \text{in } H^4(M;\mathbb{C}). \tag{3.15}$$

In the case M is Kähler, (3.13) and (3.15) are equivalent, as remarked after the proof of Proposition 3.5. Secondly, writing $q = e^{2\pi i\tau}$, we have

$$\mathrm{char}\, V = q^{-d/12}\mathrm{Str}_V(q^{L_0}) = \frac{2W_c(M)}{\Delta(\tau)^{d/12}}$$

where $W_c(M)$ is the *generalized Witten genus* of M defined in *loc. cit.* [8] The string^c condition (3.15) guarantees that $W_c(M)$ is a modular form of weight d over $SL(2,\mathbb{Z})$.

[7] The results obtained above may be *formally* applied to a virtual holomorphic vector bundle $E = E_1 - E_2$. This amounts to using “$C^\infty(\mathbf{M})$”$:= \Gamma(\wedge^* E_1^\vee \otimes \mathrm{Sym}^* E_2^\vee)$ in the construction of “$\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(\mathbf{M})$.”

[8] To recover the expression for $W_c(M)$ in [CHZ10], notice that they write $q = e^{\pi i\tau}$, and the factor $\cosh(c/2)$ in (3.14) may be replaced by $e^{c/2}$ since $d = \dim_{\mathbb{C}} M$ is even.

Appendix §A. Vertex Algebroids

The notion of a vertex algebroid, introduced in [GMS04], captures the part of structure of a vertex algebra involving only the two lowest weights. In this appendix, we review the category of vertex algebroids, the forgetful functor from vertex algebras to vertex algebroids, and its adjoint functor. Some examples are given, including the construction of local smooth CDOs.

Definition A.1. An *extended Lie algebroid* $(A, \Omega, \mathcal{T})$ consists of
- a commutative, associative $\mathbb{C}$-algebra with unit $(A, \mathbf{1})$
- two A-modules Ω and $\mathcal{T}$
- an A-derivation $d : A \to \Omega$ whose image generates Ω as an A-module
- a Lie bracket $[\ ]$ on $\mathcal{T}$
- an A-linear homomorphism of Lie algebras $\mathcal{T} \to \operatorname{End} A$, denoted $X \mapsto X$
- a $\mathbb{C}$-linear homomorphism of Lie algebras $\mathcal{T} \to \operatorname{End} \Omega$, denoted $X \mapsto L_X$
- an A-bilinear pairing $\Omega \times \mathcal{T} \to A$, denoted $(\alpha, X) \mapsto \alpha(X)$

Furthermore, we require that
- the $\mathcal{T}$-actions on A and Ω commute with d
- the $\mathcal{T}$-actions on A, Ω and $\mathcal{T}$ (via $[\ ]$) satisfy the Leibniz rule w.r.t. A-multiplication
- $df(X) = Xf$ for $f \in A$, $X \in \mathcal{T}$

Definition A.2. A *morphism of extended Lie algebroids* $\varphi : (A, \Omega, \mathcal{T}) \to (A', \Omega', \mathcal{T}')$ is a map of triples that respects the extended Lie algebroid structures. Composition of morphisms is defined in the obvious way.

Definition A.3. A *vertex algebroid* $(A, \Omega, \mathcal{T}, *, \{\ \}, \{\ \}_\Omega)$ consists of an extended Lie algebroid $(A, \Omega, \mathcal{T})$ and three $\mathbb{C}$-bilinear maps

$$* : A \times \mathcal{T} \to \Omega, \qquad \{\ \} : \mathcal{T} \times \mathcal{T} \to A, \qquad \{\ \}_\Omega : \mathcal{T} \times \mathcal{T} \to \Omega$$

that satisfy the following identities
- $\{X, Y\} = \{Y, X\}$
- $d\{X, Y\} = \{X, Y\}_\Omega + \{Y, X\}_\Omega$
- $(fg) * X - f * (gX) - f(g * X) = -(Xf)dg - (Xg)df$
- $\{X, fY\} - f\{X, Y\} = -(f * Y)(X) - YXf$
- $\{X, fY\}_\Omega - f\{X, Y\}_\Omega = -L_X(f * Y) + (Xf) * Y + f * [X, Y]$
- $X\{Y, Z\} - \{[X, Y], Z\} - \{Y, [X, Z]\} = \{X, Y\}_\Omega(Z) + \{X, Z\}_\Omega(Y)$
- $L_X\{Y, Z\}_\Omega - L_Y\{X, Z\}_\Omega + L_Z\{X, Y\}_\Omega + \{X, [Y, Z]\}_\Omega - \{Y, [X, Z]\}_\Omega - \{[X, Y], Z\}_\Omega$
  $= d\big(\{X, Y\}_\Omega(Z)\big)$

for $f, g \in A$ and $X, Y, Z \in \mathcal{T}$.

Remark. This definition is slightly different from but equivalent to the original one in [GMS04]. What we denote by $*, \{\ \}, \{\ \}_\Omega$ equal respectively $-\gamma, \langle\ \rangle, -c + \frac{1}{2} d \circ \langle\ \rangle$ in their notations.

Definition A.4. A *morphism of vertex algebroids*

$$(\varphi, \Delta) : (A, \Omega, \mathcal{T}, *, \{\ \}, \{\ \}_\Omega) \to (A', \Omega', \mathcal{T}', *', \{\ \}', \{\ \}'_\Omega)$$

consists of a morphism of extended Lie algebroids $\varphi : (A, \Omega, \mathcal{T}) \to (A', \Omega', \mathcal{T}')$ and a $\mathbb{C}$-linear map $\Delta : \mathcal{T} \to \Omega'$ such that
- $\varphi f *' \varphi X - \varphi(f * X) = \Delta(fX) - (\varphi f)\Delta(X)$
- $\{\varphi X, \varphi Y\}' - \varphi\{X, Y\} = -\Delta(X)(\varphi Y) - \Delta(Y)(\varphi X)$
- $\{\varphi X, \varphi Y\}'_\Omega - \varphi\{X, Y\}_\Omega = -L_{\varphi X}\Delta(Y) + L_{\varphi Y}\Delta(X) - d\big(\Delta(X)(\varphi Y)\big) + \Delta([X, Y])$

for $f \in A$ and $X, Y \in \mathcal{T}$. Composition of morphisms is given by

$$(\varphi', \Delta') \circ (\varphi, \Delta) = (\varphi'\varphi,\ \varphi'\Delta + \Delta'\varphi|_{\mathcal{T}}).$$

§ A.5. The vertex algebroid associated to a vertex algebra (and a "splitting"). Given a vertex algebra $(V, \mathbf{1}, T, Y)$, let

$$A := V_0, \qquad \Omega := A_{(-1)}(TA), \qquad \mathcal{T} := V_1/\Omega.$$

Choose a splitting $s : \mathcal{T} \to V_1$ of the quotient map to obtain an identification of vector spaces

$$\Omega \oplus \mathcal{T} \cong V_1, \qquad (\alpha, X) \mapsto \alpha + s(X). \tag{A.1}$$

The vertex algebra structure on V involving only the two lowest weights consists of an element $\mathbf{1} \in V_0$, a linear map $T : V_0 \to V_1$, and eight bilinear maps

$$_{(i+j-k-1)} : V_i \times V_j \to V_k, \quad i, j, k = 0, 1$$

satisfying a set of (Borcherds) identities. These data, when rephrased in terms of the identification (A.1), are equivalent to a vertex algebroid $(A, \Omega, \mathcal{T}, *, \{\ \}, \{\ \}_\Omega)$. The extended Lie algebroid $(A, \Omega, \mathcal{T})$ consists of precisely those data that are independent of the choice of s, namely

$$\begin{array}{lll} fg := f_{(-1)}g & f\alpha := f_{(-1)}\alpha & fX := f_{(-1)}s(X) \bmod \Omega \\ Xf := s(X)_{(0)}f & L_X\alpha := s(X)_{(0)}\alpha & [X, Y] := s(X)_{(0)}s(Y) \bmod \Omega \\ df := Tf & \alpha(X) := \alpha_{(1)}s(X) & \end{array} \tag{A.2}$$

[9] for $f, g \in A$, $\alpha \in \Omega$ and $X, Y \in \mathcal{T}$; on the other hand

$$\begin{array}{ll} f * X & := f_{(-1)}s(X) - s(fX) \\ \{X, Y\} & := s(X)_{(1)}s(Y) \\ \{X, Y\}_\Omega & := s(X)_{(0)}s(Y) - s([X, Y]) \end{array} \tag{A.3}$$

for $f \in A$ and $X, Y \in \mathcal{T}$.

§ A.6. The induced morphism of vertex algebroids. Consider a homomorphism of vertex algebras $\Phi : V \to V'$. Let $(A, \Omega, \mathcal{T}, *, \{\ \}, \{\ \}_\Omega)$, $(A', \Omega', \mathcal{T}', *', \{\ \}', \{\ \}'_\Omega)$ be the vertex algebroids associated to V, V' and some splittings $s : \mathcal{T} \to V_1$, $s' : \mathcal{T}' \to V'_1$. The part of data of Φ involving only the two lowest weights, when rephrased in terms of identifications like (A.1), are equivalent to a morphism (φ, Δ) between the two vertex algebroids. It consists of the obvious map of triples $\varphi : (A, \Omega, \mathcal{T}) \to (A', \Omega', \mathcal{T}')$ induced by Φ, and a map $\Delta : \mathcal{T} \to \Omega'$ given by

$$\Delta(X) = \Phi s(X) - s'(\varphi X), \quad X \in \mathcal{T}.$$

§ A.7. The vertex algebra freely generated by a vertex algebroid. Let $(A, \Omega, \mathcal{T}, *, \{\ \}, \{\ \}_\Omega)$ be a vertex algebroid. Throughout this discussion, we always have $f, g \in A$, $\alpha, \beta \in \Omega$, $X, Y \in \mathcal{T}$. Define an associative $\mathbb{C}$-algebra $\mathcal{W}$ with generators of the form f_n, α_n, X_n, $n \in \mathbb{Z}$ and the following relations

$$\begin{array}{l} (cf)_n = cf_n \qquad (c\alpha)_n = c\alpha_n \qquad (cX)_n = cX_n \\ \mathbf{1}_n = \delta_{n,0} \qquad (df)_n = -nf_n \qquad [f_n, g_m] = [f_n, \alpha_m] = [\alpha_n, \beta_m] = 0 \\ [X_n, f_m] = (Xf)_{n+m} \qquad [X_n, \alpha_m] = (L_X\alpha)_{n+m} + n\alpha(X)_{n+m} \\ [X_n, Y_m] = [X, Y]_{n+m} + (\{X, Y\}_\Omega)_{n+m} + n(\{X, Y\})_{n+m} \end{array} \tag{A.4}$$

where $c \in \mathbb{C}$, $n, m \in \mathbb{Z}$. The subalgebra $\mathcal{W}_+ \subset \mathcal{W}$ generated by f_n, $n > 0$ and α_n, X_n, $n \geq 0$ admits a trivial action on $\mathbb{C}$. Let $\widetilde{V} := \mathcal{W} \otimes_{\mathcal{W}_+} \mathbb{C}$ be the induced $\mathcal{W}$-module and $V := \widetilde{V}/\sim$ the quotient module obtained by imposing the following relations for $v \in \widetilde{V}$:

$$\begin{array}{lll} (fg)_n v & \sim & \sum_{k\in\mathbb{Z}} f_k g_{n-k} v \\ (f\alpha)_n v & \sim & \sum_{k\in\mathbb{Z}} f_k \alpha_{n-k} v \\ (fX)_n v & \sim & \sum_{k\leq 0} f_k X_{n-k} v + \sum_{k>0} X_{n-k} f_k v - (f * X)_n v \end{array} \tag{A.5}$$

[9] For example, the definition of Xf is indeed independent of s because $\alpha_{(0)}f = 0$ for $f \in A$ and $\alpha \in \Omega$.

Notice that the summations are always finite. It is a consequence of the axioms of a vertex algebroid that (A.4)–(A.5) are consistent. [10] Define a vertex algebra structure on V as follows. The vacuum $\mathbf{1} \in V$ is given by the coset of $1 \otimes 1 \in \widetilde{V}$. The infinitesimal translation T and weight operator L_0 are determined by the requirements

$$\begin{array}{llll} T\mathbf{1} = 0 & [T, f_n] = (1-n) f_{n-1} & [T, \alpha_n] = -n\alpha_{n-1} & [T, X_n] = -nX_{n-1} \\ L_0\mathbf{1} = 0 & [L_0, f_n] = -nf_n & [L_0, \alpha_n] = -n\alpha_n & [L_0, X_n] = -nX_n \end{array}$$

which are consistent with (A.4)–(A.5); notice that actions of f_n, α_n, X_n change weights by $-n$. Identify f, α, X with $f_0\mathbf{1}$, $\alpha_{-1}\mathbf{1}$, $X_{-1}\mathbf{1}$ and associate to them the following fields

$$\textstyle\sum_n f_n z^{-n}, \qquad \sum_n \alpha_n z^{-n-1}, \qquad \sum_n X_n z^{-n-1}$$

which are mutually local by (A.4); notice that $f_{(n)} = f_{n+1}$, $\alpha_{(n)} = \alpha_n$, $X_{(n)} = X_n$. Now apply the Strong Reconstruction Theorem [FB04].

Suppose $(A, \Omega, \mathcal{T}, *, \{\ \}, \{\ \}_\Omega)$ is the vertex algebroid associated with a vertex algebra V' and a splitting $s : \mathcal{T} \to V'_1$. There is a canonical homomorphism of vertex algebras $\Phi : V \to V'$, determined by $\Phi f = f$, $\Phi\alpha = \alpha$ and $\Phi X = s(X)$. If Φ is an isomorphism, V' is said to be *freely generated by a vertex algebroid*.

§ A.8. The induced homomorphism of vertex algebras. A morphism of vertex algebroids

$$(\varphi, \Delta) : (A, \Omega, \mathcal{T}, *, \{\ \}, \{\ \}_\Omega) \to (A', \Omega', \mathcal{T}', *', \{\ \}', \{\ \}'_\Omega)$$

induces a homomorphism $\Phi : V \to V'$ between the freely generated vertex algebras by the equations

$$\begin{array}{lll} \Phi f = \varphi f & \Phi\alpha = \varphi\alpha & \Phi X = \varphi X + \Delta(X) \\ \Phi \circ f_n = (\Phi f)_n \circ \Phi & \Phi \circ \alpha_n = (\Phi\alpha)_n \circ \Phi & \Phi \circ X_n = (\Phi X)_n \circ \Phi \end{array}$$

for $f \in A$, $\alpha \in \Omega$, $X \in \mathcal{T}, n \in \mathbb{Z}$. Indeed, these equations are consistent with (A.4)–(A.5).

§ A.9. More details on the constructions in §A.7 and §A.8. Given a possibly empty sequence of negative integers $\mathbf{n} = \{n_1 \le \cdots \le n_s < 0\}$, we write

$$|\mathbf{n}| = n_1 + \cdots + n_s \quad (0 \text{ if } \mathbf{n} = \{\}), \qquad \mathbf{n}(i) = \text{number of times } i \text{ appears in } \mathbf{n}$$

and regard $\mathbf{n}$ as a partition of $|\mathbf{n}|$. For $k \ge 0$, let I_k be the set of pairs $(\mathbf{n}, \mathbf{m})$ of such sequences with $-|\mathbf{n}| - |\mathbf{m}| = k$. Define a partial ordering on I_k such that $(\mathbf{n}, \mathbf{m}) \prec (\mathbf{n}', \mathbf{m}')$ if and only if

$$\begin{array}{lll} -|\mathbf{n}| < -|\mathbf{n}'| & \text{or} & |\mathbf{n}| = |\mathbf{n}'| \text{ and } \mathbf{n}' \text{ is a proper subpartition of } \mathbf{n} \\ & \text{or} & \mathbf{n} = \mathbf{n}' \text{ and } \mathbf{m} \text{ is a proper subpartition of } \mathbf{m}' \end{array}$$

For example, $(\{\}, \{-2,-2,-1\}) \prec (\{\}, \{-3,-2\}) \prec (\{-4\}, \{-1\}) \prec (\{-3,-1\}, \{-1\})$ in I_5.

Consider the vertex algebra V constructed in §A.7. Associate to each $\mathbf{n} = \{n_1 \le \cdots \le n_s < 0\}$ and s-tuples $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in \Omega^s$, $\boldsymbol{X} = (X_1, \ldots, X_s) \in \mathcal{T}^s$ the following operators on V

$$\boldsymbol{\alpha}_{\mathbf{n}} := \alpha_{1,n_1} \cdots \alpha_{s,n_s}, \qquad \boldsymbol{X}_{\mathbf{n}} := X_{1,n_1} \cdots X_{s,n_s} \qquad (\text{both } 1 \text{ if } \mathbf{n} = \{\})$$

For $k > 0$, we have $V_k = \operatorname{span}\{\boldsymbol{X}_{\mathbf{n}}\boldsymbol{\alpha}_{\mathbf{m}}\mathbf{1} \,|\, (\mathbf{n}, \mathbf{m}) \in I_k\}$; for $(\mathbf{n}, \mathbf{m}) \in I_k$, define the subspaces

$$\begin{aligned} \mathcal{F}_{\preceq(\mathbf{n},\mathbf{m})} &:= \operatorname{span}\{\boldsymbol{X}_{\mathbf{n}'}\boldsymbol{\alpha}_{\mathbf{m}'}\mathbf{1} \,|\, (\mathbf{n}', \mathbf{m}') \in I_k,\ (\mathbf{n}', \mathbf{m}') \preceq (\mathbf{n}, \mathbf{m})\} \subset V_k \\ \mathcal{F}_{\prec(\mathbf{n},\mathbf{m})} &:= \operatorname{span}\{\boldsymbol{X}_{\mathbf{n}'}\boldsymbol{\alpha}_{\mathbf{m}'}\mathbf{1} \,|\, (\mathbf{n}', \mathbf{m}') \in I_k,\ (\mathbf{n}', \mathbf{m}') \prec (\mathbf{n}, \mathbf{m})\} \subset V_k \end{aligned}$$

[10] For example, $[X_n, (fY)_m]$ can be computed by either taking the commutator first or expanding $(fY)_m$ first. The resulting identity is already implied by the vertex algebroid axioms and does not lead to a new relation.

The bilinear operation $A \times V_k \to V_k$ given by $(f, v) \mapsto f_0 v$ does not make V_k an A-module, but it preserves $\mathcal{F}_{\preceq(\mathbf{n},\mathbf{m})}$, $\mathcal{F}_{\prec(\mathbf{n},\mathbf{m})}$ and induces an A-module structure on their quotient. In fact,

$$\mathcal{F}_{\preceq(\mathbf{n},\mathbf{m})}/\mathcal{F}_{\prec(\mathbf{n},\mathbf{m})} \cong \left(\bigotimes_{i=-1}^{-\infty} \mathrm{Sym}_A^{\mathbf{n}(i)} \mathcal{T} \right) \otimes \left(\bigotimes_{j=-1}^{-\infty} \mathrm{Sym}_A^{\mathbf{m}(j)} \Omega \right)$$

as A-modules. This allows us to compute the "associated graded space" [11]

$$A \oplus \bigoplus_{k=1}^{\infty} \left(q^k \bigoplus_{(\mathbf{n},\mathbf{m}) \in I_k} \mathcal{F}_{\preceq(\mathbf{n},\mathbf{m})}/\mathcal{F}_{\prec(\mathbf{n},\mathbf{m})} \right) \cong \bigotimes_{\ell=1}^{\infty} \mathrm{Sym}_{q^\ell}(\mathcal{T} \oplus \Omega)$$

where q is a formal variable and $\mathrm{Sym}_t(\cdot) = \sum_{t=0}^{\infty} t^n \mathrm{Sym}_A^n(\cdot)$. The subspaces $\mathcal{F}_{\preceq(\mathbf{n},\mathbf{m})}$, $\mathcal{F}_{\prec(\mathbf{n},\mathbf{m})}$ and the isomorphisms stated here are natural, i.e. respected by the homomorphism Φ constructed in §A.8.

The omitted proofs of the following lemmas are straightforward (though somewhat tedious).

Lemma A.10. *Given the following data:*
- *a vertex algebroid* $(A, \Omega, \mathcal{T}, *, \{\ \}, \{\ \}_\Omega)$
- *an isomorphism of extended Lie algebroids* $\varphi : (A, \Omega, \mathcal{T}) \to (A', \Omega', \mathcal{T}')$
- *a* $\mathbb{C}$*-linear map* $\Delta : \mathcal{T} \to \Omega'$

if we define maps

$$*' : A' \times \mathcal{T}' \to \Omega', \qquad \{\ \}' : \mathcal{T}' \times \mathcal{T}' \to A', \qquad \{\ \}'_\Omega : \mathcal{T}' \times \mathcal{T}' \to \Omega'$$

by the equations in Definition A.4, then $(A', \Omega', \mathcal{T}', *', \{\ \}', \{\ \}'_\Omega)$ *is a vertex algebroid and* (φ, Δ) *is by construction an isomorphism between the two vertex algebroids.* ☐

Lemma A.11. *Given the following data:*
- *two vertex algebroids* $(A, \Omega, \mathcal{T}, *, \{\ \}, \{\ \}_\Omega)$ *and* $(A', \Omega', \mathcal{T}', *', \{\ \}', \{\ \}'_\Omega)$
- *a morphism of extended Lie algebroids* $\varphi : (A, \Omega, \mathcal{T}) \to (A', \Omega', \mathcal{T}')$
- *a* $\mathbb{C}$*-linear map* $\Delta : \mathcal{T} \to \Omega'$
- *a subset* $S \subset \mathcal{T}$ *that is closed under* $[\ ]$ *and spans* $\mathcal{T}$ *as an* A*-module*

if (φ, Δ) *satisfies the equations in Definition A.4 for* $(f, X, Y) \in A \times S^2$*, then it also does for* $(f, X, Y) \in A \times \mathcal{T}^2$ *and hence is a morphism between the two given vertex algebroids.* ☐

§ A.12. Super version. There is no difficulty in generalizing the discussions in this appendix to define *extended Lie superalgebroids*, *vertex superalgebroids*, and relate them to vertex superalgebras.

Example A.13. The vertex algebroids associated to a Lie algebra. Consider a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and a vertex algebroid of the form $(\mathbb{C}, 0, \mathfrak{g}, 0, \lambda, 0)$ with $\mathfrak{g}$ acting trivially on $\mathbb{C}$. The second, fourth and last components are trivial by necessity. The conditions on $\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ are

$$\lambda(X, Y) = \lambda(Y, X), \qquad \lambda([X, Y], Z) + \lambda(Y, [X, Z]) = 0$$

i.e. it is a symmetric invariant bilinear form on $\mathfrak{g}$. Let

$$V_\lambda(\mathfrak{g}) = \text{ the vertex algebra freely generated by } (\mathbb{C}, 0, \mathfrak{g}, 0, \lambda, 0).$$

In the case $\mathfrak{g}$ is simple, finite-dimensional and λ equals k times the normalized Killing form, this is the vertex algebra defined on the level-k vacuum representation of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$. [FB04]

[11] More precisely, the coefficient of q^k is the associated graded space of a certain filtration on V_k.

Example A.14. Polynomial CDOs. Given nonnegative integers p and q, let $\mathcal{W}$ be the associative $\mathbb{C}$-superalgebra generated by elements of the form

$$b^i_n,\, a_{i,n}, \qquad n\in\mathbb{Z},\ i=1,\dots,p+q, \qquad |b^i_n|=|a_{i,n}|=\begin{cases}\bar{0}, & i=1,\dots,p\\ \bar{1}, & i=p+1,\dots,p+q\end{cases}$$

($|\cdot|$ = parity) satisfying the following relations

$$[b^i_n,b^j_m]=0=[a_{i,n},a_{j,m}], \qquad [a_{i,n},b^j_m]=\delta^j_i\delta_{n,-m}$$

where [ ] is the supercommutator. The subalgebra $\mathcal{W}_+\subset\mathcal{W}$ generated by b^i_n, $n>0$ and $a_{i,n}$, $n\geq 0$ is supercommutative and admits a (purely even) trivial representation $\mathbb{C}$. The induced $\mathcal{W}$-module

$$\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^{p|q}):=\mathcal{W}\otimes_{\mathcal{W}_+}\mathbb{C}$$

has the structure of a vertex superalgebra. The vacuum is given by $\mathbf{1}=1\otimes 1$. The infinitesimal translation T and weight operator L_0 are determined by

$$\begin{array}{lll} T\mathbf{1}=0 & [T,b^i_n]=(1-n)b^i_{n-1} & [T,a_{i,n}]=-na_{i,n-1}\\ L_0\mathbf{1}=0 & [L_0,b^i_n]=-nb^i_n & [L_0,a_{i,n}]=-na_{i,n}\end{array}$$

The vertex operators of $b^i_0\mathbf{1}$ and $a_{i,-1}\mathbf{1}$ are given respectively by the fields

$$\textstyle\sum_n b^i_n z^{-n}, \qquad \sum_n a_{i,n}z^{-n-1}$$

while the other vertex operators follow from the Reconstruction Theorem [FB04]. [12]

The vertex superalgebra $\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^{p|q})$ is freely generated by the associated vertex superalgebroid. To describe the latter, consider the algebraic supermanifold

$$\mathbb{A}^{p|q}:=\mathrm{Spec}\left(\mathbb{C}[b^1,\cdots,b^d]\otimes\textstyle\bigwedge(b^{p+1},\cdots,b^{p+q})\right)$$

and identify its functions, 1-forms and vector fields with the following subquotients of $\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^{p|q})$:

- $\mathcal{O}(\mathbb{A}^{p|q})=\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^{p|q})_0$ via $b^i=b^i_0\mathbf{1}$, $b^ib^j=b^i_0b^j_0\mathbf{1}$, etc.
- $\Omega^1(\mathbb{A}^{p|q})\subset\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^{p|q})_1$ via $db^i=b^i_{-1}\mathbf{1}$
- $\mathcal{T}(\mathbb{A}^{p|q})=\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^{p|q})_1/\Omega^1(\mathbb{A}^{p|q})$ via $\partial_i=\partial/\partial b^i$ = coset of $a_{i,-1}\mathbf{1}$

[13] Then "the" vertex superalgebroid associated to $\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^{p|q})$ is of the form

$$\left(\mathcal{O}(\mathbb{A}^{p|q}),\Omega^1(\mathbb{A}^{p|q}),\mathcal{T}(\mathbb{A}^{p|q}),*^c,\{\ \}^c,\{\ \}^c_\Omega\right).$$

The extended Lie superalgebroid structure consists of the usual differential on functions, Lie bracket on vector fields, Lie derivations by vector fields on functions and 1-forms, and pairing between 1-forms and vector fields. Let $\epsilon_i:=(-1)^{|b^i|}$. If we use the splitting

$$s:\mathcal{T}(\mathbb{A}^{p|q})\to\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^{p|q})_1, \qquad X=X^i\partial_i\ \mapsto\ \epsilon_i^{1+|X|}a_{i,-1}X^i$$

the rest of the vertex superalgebroid structure, as given by (A.3), reads

$$\begin{aligned} f*^cX &= -(\epsilon_i\epsilon_j)^{1+|f|+|X|}(\partial_j\partial_i f)X^idb^j\\ \{X,Y\}^c &= -\epsilon_j^{1+|X|+|Y|}(\partial_jX^i)(\partial_iY^j)\\ \{X,Y\}^c_\Omega &= -(\epsilon_j\epsilon_k)^{1+|X|+|Y|}(\partial_k\partial_jX^i)(\partial_iY^j)db^k\end{aligned} \tag{A.6}$$

The superscript c refers to the dependence on coordinates.

[12] This vertex superalgebra is the tensor product of p copies of the $\beta\gamma$-system and q copies of the bc-system.

[13] From another point of view, making these identifications dictates our (sign) conventions for calculus on $\mathbb{A}^{p|q}$ (or $\mathbb{R}^{p|q}$). For example, it follows from $\alpha(X):=\alpha_{(1)}s(X)$ in (A.2) that $db^i(\partial_j)=\epsilon_j\delta^i_j$.

Example A.15. Local smooth CDOs. Let $b^1, \ldots, b^p$ and $b^{p+1}, \ldots, b^{p+q}$ be respectively the real, even and odd coordinates of $\mathbb{R}^{p|q}$, regarded as a smooth cs-manifold, namely

$$C^\infty(\mathbb{R}^{p|q}) = C^\infty(\mathbb{R}^p) \otimes \bigwedge(b^{p+1}, \ldots, b^{p+q}) \otimes \mathbb{C}.$$

Let $\mathbf{W}$ be the restriction of $\mathbb{R}^{p|q}$ to an open set in $\mathbb{R}^p$. Motivated by Example A.14, we define a vertex superalgebra $\mathcal{D}^{\mathrm{ch}}(\mathbf{W})$ as follows. The functions, 1-forms and vector fields on $\mathbf{W}$ form an extended Lie superalgebroid as in Example A.14, and formulae (A.6) again yield a vertex superalgebroid

$$\big(C^\infty(\mathbf{W}), \Omega^1(\mathbf{W}), \mathcal{T}(\mathbf{W}), *^c, \{\ \}^c, \{\ \}^c_\Omega\big);$$

then take the freely generated vertex superalgebra.

§ A.16. The Wess-Zumino form of a diffeomorphism. Suppose $\boldsymbol{\varphi} : \mathbf{W} \to \mathbf{W}'$ is a diffeomorphism between restrictions of $\mathbb{R}^{p|q}$ (as a cs-manifold) to open sets. The following notations will be used:

$$|\cdot| = \text{parity}, \quad \epsilon_i = (-1)^{|b^i|}, \quad \epsilon_{ij} = (-1)^{|b^i||b^j|}, \quad i,j = 1, \ldots, p+q$$

Let $g_{\boldsymbol{\varphi}} : \mathbf{W} \to GL(p|q)$ be the map of cs-manifolds whose components $(g_{\boldsymbol{\varphi}})^i{}_j$ are given by

$$\boldsymbol{\varphi}^* db^i = (g_{\boldsymbol{\varphi}})^i{}_j db^j \quad \Leftrightarrow \quad (g_{\boldsymbol{\varphi}})^i{}_j = \epsilon_j \epsilon_{ij} \partial_j \boldsymbol{\varphi}^i$$

where $\boldsymbol{\varphi}^i = \boldsymbol{\varphi}^* b^i$. [14] Define the following differential forms

$$\theta_{\boldsymbol{\varphi}} := g_{\boldsymbol{\varphi}}^{-1} \cdot dg_{\boldsymbol{\varphi}} \in \Omega^1(\mathbf{W}) \otimes \mathfrak{gl}(p|q), \qquad WZ_{\boldsymbol{\varphi}} := \frac{1}{3}\mathrm{Str}\,(\theta_{\boldsymbol{\varphi}} \wedge \theta_{\boldsymbol{\varphi}} \wedge \theta_{\boldsymbol{\varphi}}) \in \Omega^3(\mathbf{W}).$$

It follows from $d\theta_{\boldsymbol{\varphi}} = -\theta_{\boldsymbol{\varphi}} \wedge \theta_{\boldsymbol{\varphi}}$ that $WZ_{\boldsymbol{\varphi}}$ is closed.

Theorem A.17. *Let $\mathbf{W}, \mathbf{W}', \mathbf{W}''$ be restrictions of $\mathbb{R}^{p|q}$ (as a cs-manifold) to open sets in $\mathbb{R}^p$.*
(a) Suppose $\boldsymbol{\varphi} : \mathbf{W} \to \mathbf{W}'$ is a diffeomorphism. There is a one-to-one correspondence:

$$\left\{ \begin{array}{c} \textit{isomorphisms of vertex superalgebras } \mathcal{D}^{\mathrm{ch}}(\mathbf{W}') \to \mathcal{D}^{\mathrm{ch}}(\mathbf{W}) \\ \textit{whose weight-zero components are } \boldsymbol{\varphi}^* : C^\infty(\mathbf{W}') \to C^\infty(\mathbf{W}) \end{array} \right\} \stackrel{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \xi \in \Omega^2(\mathbf{W}),\ \textit{even} \\ \textit{and } d\xi = WZ_{\boldsymbol{\varphi}} \end{array} \right\}$$

Given ξ as above, the corresponding isomorphism, denoted by $\boldsymbol{\varphi}^*_\xi$, is induced by an isomorphism between the associated vertex superalgebroids

$$(\boldsymbol{\varphi}^*, \Delta_{\boldsymbol{\varphi},\xi}) : \big(C^\infty(\mathbf{W}'), \Omega^1(\mathbf{W}'), \mathcal{T}(\mathbf{W}'), *^c, \{\ \}^c, \{\ \}^c_\Omega\big) \to \big(C^\infty(\mathbf{W}), \Omega^1(\mathbf{W}), \mathcal{T}(\mathbf{W}), *^c, \{\ \}^c, \{\ \}^c_\Omega\big)$$

where $\Delta_{\boldsymbol{\varphi},\xi} : \mathcal{T}(\mathbf{W}') \to \Omega^1(\mathbf{W})$ is given by

$$\Delta_{\boldsymbol{\varphi},\xi}(X) = -\epsilon_i \epsilon_{ij} \epsilon_j^{1+|X|} \partial_j (\boldsymbol{\varphi}^* X)^i (\theta_{\boldsymbol{\varphi}})^j{}_i - \frac{1}{2}\iota_{\boldsymbol{\varphi}^* X} \mathrm{Str}\,(\theta_{\boldsymbol{\varphi}} \otimes \theta_{\boldsymbol{\varphi}}) - \frac{1}{2}\iota_{\boldsymbol{\varphi}^* X}\xi$$

for homogeneous elements.
(b) Suppose $\boldsymbol{\varphi}' : \mathbf{W}' \to \mathbf{W}''$ is another diffeomorphism, $\xi' \in \Omega^2(\mathbf{W}')$ is even, and $d\xi' = WZ_{\boldsymbol{\varphi}'}$. Then we have the composition

$$\boldsymbol{\varphi}^*_\xi \circ \boldsymbol{\varphi}'^*_{\xi'} = (\boldsymbol{\varphi}'\boldsymbol{\varphi})^*_\eta, \qquad \eta = \xi + \boldsymbol{\varphi}^* \xi' + \sigma_{\boldsymbol{\varphi}',\boldsymbol{\varphi}}$$

where $\sigma_{\boldsymbol{\varphi}',\boldsymbol{\varphi}} := \mathrm{Str}\,(\theta_{\boldsymbol{\varphi}} \wedge g_{\boldsymbol{\varphi}}^{-1} \cdot \boldsymbol{\varphi}^* \theta_{\boldsymbol{\varphi}'} \cdot g_{\boldsymbol{\varphi}})$.

Remarks. (i) This is a reformulation of a result in [GMS00] in the smooth case. (ii) As a consistency check, it follows from $\theta_{\boldsymbol{\varphi}'\boldsymbol{\varphi}} = \theta_{\boldsymbol{\varphi}} + g_{\boldsymbol{\varphi}}^{-1} \cdot \boldsymbol{\varphi}^* \theta_{\boldsymbol{\varphi}'} \cdot g_{\boldsymbol{\varphi}}$ that $WZ_{\boldsymbol{\varphi}'\boldsymbol{\varphi}} = WZ_{\boldsymbol{\varphi}} + \boldsymbol{\varphi}^* WZ_{\boldsymbol{\varphi}'} + d\sigma_{\boldsymbol{\varphi}',\boldsymbol{\varphi}}$.

[14] In this notation, the chain rule reads $g_{\boldsymbol{\varphi}'\boldsymbol{\varphi}} = (\boldsymbol{\varphi}^* g_{\boldsymbol{\varphi}'}) \cdot g_{\boldsymbol{\varphi}}$.

Proof of Theorem A.17. (a) Any morphism between the extended Lie superalgebroids associated to $\mathbf{W}$ and $\mathbf{W}'$ is induced by a map of cs-manifolds, which is $\boldsymbol{\varphi}$ in this case. Consider a morphism of vertex superalgebroids of the form

$$(\boldsymbol{\varphi}^*, \Delta) : \big(C^\infty(\mathbf{W}'), \Omega^1(\mathbf{W}'), \mathcal{T}(\mathbf{W}'), *^c, \{\ \}^c, \{\ \}^c_\Omega\big) \to \big(C^\infty(\mathbf{W}), \Omega^1(\mathbf{W}), \mathcal{T}(\mathbf{W}), *^c, \{\ \}^c, \{\ \}^c_\Omega\big)$$

and write $\Delta = \Delta_0 \circ \boldsymbol{\varphi}^*|_{\mathcal{T}(\mathbf{W}')}$ in terms of an even map $\Delta_0 : \mathcal{T}(\mathbf{W}) \to \Omega^1(\mathbf{W})$. Applying Lemma A.11 with $S = \{\boldsymbol{\varphi}_*\partial_i\}_{i=1}^{p+q}$ and using (A.6), we obtain a complete set of equations for Δ_0, namely

$$\begin{aligned}
&\Delta_0(f\partial_i) - f\Delta_0(\partial_i) && = -\epsilon_i\epsilon_j^{1+|f|}(\partial_j f)(\theta_{\boldsymbol{\varphi}})^j{}_i \\
&\Delta_0(\partial_i)(\partial_j) + \epsilon_{ij}\Delta_0(\partial_j)(\partial_i) && = -\mathrm{Str}\,(\theta_{\boldsymbol{\varphi}} \otimes \theta_{\boldsymbol{\varphi}})(\partial_i \otimes \partial_j) \\
&\partial_i\big(\Delta_0(\partial_j)(\partial_k)\big) - \epsilon_{ij}\partial_j\big(\Delta_0(\partial_i)(\partial_k)\big) + \epsilon_{ik}\epsilon_{jk}\partial_k\big(\Delta_0(\partial_i)(\partial_j)\big) && \\
& && = -\epsilon_r\epsilon_{jk}\epsilon_{kr}\epsilon_{ks}\,\partial_i\big((\theta_{\boldsymbol{\varphi}})^r{}_s(\partial_k)\big)\cdot(\theta_{\boldsymbol{\varphi}})^s{}_r(\partial_j)
\end{aligned}$$

The first equation implies that for any $X \in \mathcal{T}(\mathbf{W})$

$$\Delta_0(X) = -\epsilon_i\epsilon_{ij}\epsilon_j^{1+|X|}(\partial_j X^i)(\theta_{\boldsymbol{\varphi}})^j{}_i + X^i\Delta_0(\partial_i)\,;$$

the second equation allows us to write

$$\Delta_0(\partial_i)(\partial_j) = -\frac{1}{2}\mathrm{Str}\,(\theta_{\boldsymbol{\varphi}} \otimes \theta_{\boldsymbol{\varphi}})(\partial_i \otimes \partial_j) - \frac{1}{2}\xi_{ij}, \qquad \xi_{ji} = -\epsilon_{ij}\xi_{ij}\,;$$

then it follows from $d\theta_{\boldsymbol{\varphi}} = -\theta_{\boldsymbol{\varphi}} \wedge \theta_{\boldsymbol{\varphi}}$ that the third equation is equivalent to

$$d\xi = WZ_{\boldsymbol{\varphi}} = \frac{1}{3}\mathrm{Str}\,(\theta_{\boldsymbol{\varphi}} \wedge \theta_{\boldsymbol{\varphi}} \wedge \theta_{\boldsymbol{\varphi}})$$

where ξ is the even 2-form with $\xi(\partial_i, \partial_j) = \xi_{ij}$. Since $\mathcal{D}^{\mathrm{ch}}(\mathbf{W})$ and $\mathcal{D}^{\mathrm{ch}}(\mathbf{W}')$ are freely generated by vertex superalgebroids, an isomorphism between them is equivalent to an isomorphism between the associated vertex superalgebroids. This completes the proof of (a).

(b) By part (a), the composition in question must be of the form $(\boldsymbol{\varphi}'\boldsymbol{\varphi})^*_\eta$ for some $\eta \in \Omega^2(\mathbf{W})$. At the level of vertex superalgebroids, we have

$$\begin{aligned}
&(\boldsymbol{\varphi}^*\boldsymbol{\varphi}'^*, \Delta_{\boldsymbol{\varphi}'\boldsymbol{\varphi},\eta}) = (\boldsymbol{\varphi}^*, \Delta_{\boldsymbol{\varphi},\xi}) \circ (\boldsymbol{\varphi}'^*, \Delta_{\boldsymbol{\varphi}',\xi'}) \\
\Leftrightarrow \qquad &\Delta_{\boldsymbol{\varphi}'\boldsymbol{\varphi},\eta} = \boldsymbol{\varphi}^*\Delta_{\boldsymbol{\varphi}',\xi'} + \Delta_{\boldsymbol{\varphi},\xi}\boldsymbol{\varphi}'^*.
\end{aligned}$$

Evaluation at e.g. $\boldsymbol{\varphi}'_*\boldsymbol{\varphi}_*\partial_k$ then yields the desired formula for η. □

Appendix §B. Affine Connections on CS-Manifolds

Consider a smooth manifold M and a smooth $\mathbb{C}$-vector bundle $E \to M$. In this appendix, we construct an affine connection on the smooth cs-manifold $\mathbf{M} = \Pi E$ and obtain a number of formulae used in the computations of CDOs on $\mathbf{M}$.

§ B.1. Functions on M. Let $d = \dim M$ and $r = \operatorname{rank} E$. There is a canonical identification

$$C^\infty(\mathbf{M}) \cong \Gamma(\wedge^* E^\vee). \tag{B.1}$$

In particular, a set of local coordinates $(x^1, \cdots, x^d)$ on M and a local frame $(\varepsilon^1, \cdots, \varepsilon^r)$ of $E^\vee$ together determine a set of local coordinates $(x^1, \cdots, x^d, \varepsilon^1, \cdots, \varepsilon^r)$ on $\mathbf{M}$.

§ B.2. Vector fields on M. Choose a connection ∇^E on E and use the same notation for the induced connection on $\wedge^* E^\vee$. Let $(\varepsilon_1, \cdots, \varepsilon_r)$ be the local frame of E dual to $(\varepsilon^1, \cdots, \varepsilon^r)$. Let $X, Y \in \mathcal{T}(M)$ and $\sigma, \tau \in \Gamma(E)$. Under the identification (B.1), vector fields on $\mathbf{M}$ correspond to derivations on sections of $\wedge^* E^\vee$. In particular, denote by

$$\left\{\begin{array}{c} \mathcal{D}_X \\ \mathcal{I}_\sigma \\ J \end{array}\right\} \begin{array}{c}\text{the vector fields on } \mathbf{M} \\ \text{corresponding to the}\end{array} \left\{\begin{array}{l} \text{covariant differentiation } \nabla^E_X \\ \text{contraction with } \sigma \\ \text{exterior degree} \end{array}\right\} \tag{B.2}$$

The vector fields $\mathcal{D}_X$ and $\mathcal{I}_\sigma$ span $\mathcal{T}(\mathbf{M})$ over $C^\infty(\mathbf{M})$. The super Lie brackets of (B.2) are given by

$$[\mathcal{D}_X, \mathcal{D}_Y] = \mathcal{D}_{[X,Y]} - \varepsilon^k \mathcal{I}_{R^E_{X,Y}\varepsilon_k}, \quad [\mathcal{D}_X, \mathcal{I}_\sigma] = \mathcal{I}_{\nabla^E_X \sigma}, \quad [\mathcal{I}_\sigma, \mathcal{I}_\tau] = 0 = [J, \mathcal{D}_X], \quad [J, \mathcal{I}_\sigma] = -\mathcal{I}_\sigma \tag{B.3}$$

where R^E is the curvature of ∇^E.

§ B.3. An affine connection on M. Choose also a connection ∇^M on TM. Let $X, Y, Z \in \mathcal{T}(M)$ and $\sigma, \tau \in \Gamma(E)$. Define a connection ∇ on $T\mathbf{M}$ by

$$\nabla_{\mathcal{D}_X} \mathcal{D}_Y = \mathcal{D}_{\nabla^M_X Y}, \quad \nabla_{\mathcal{D}_X} \mathcal{I}_\sigma = \mathcal{I}_{\nabla^E_X \sigma}, \quad \nabla_{\mathcal{I}_\sigma} \mathcal{D}_X = \nabla_{\mathcal{I}_\sigma} \mathcal{I}_\tau = 0 \tag{B.4}$$

and the Leibniz rule. Using (B.3), we compute the curvature of ∇ as follows

$$R_{\mathcal{D}_X, \mathcal{D}_Y} \mathcal{D}_Z = \mathcal{D}_{R^M_{X,Y} Z}, \quad R_{\mathcal{D}_X, \mathcal{D}_Y} \mathcal{I}_\sigma = \mathcal{I}_{R^E_{X,Y}\sigma}, \quad R_{\mathcal{D}_X, \mathcal{I}_\sigma} = R_{\mathcal{I}_\sigma, \mathcal{I}_\tau} = 0 \tag{B.5}$$

where R^M is the curvature of ∇^M.

Lemma B.4. *(a) The operator $\nabla^t J$ sends $\mathcal{D}_X$ to 0, and $\mathcal{I}_\sigma$ to itself. (b) $\nabla(\nabla^t J) = 0$.*

Proof. Recall that $\nabla^t J := \nabla_J - [J, \text{-}]$. Using the fact that $J = \varepsilon^k \mathcal{I}_{\varepsilon_k}$, (a) follows readily from (B.3) and (B.4). Then (b) is clear. □

Lemma B.5. *Regarding $\Omega^*(M)$ as a subalgebra of $\Omega^*(\mathbf{M})$, we have*
(a) $\operatorname{Str} R = \operatorname{Tr} R^M - \operatorname{Tr} R^E$,
(b) $\operatorname{Str}(R \wedge R) = \operatorname{Tr}(R^M \wedge R^M) - \operatorname{Tr}(R^E \wedge R^E)$,
(c) $\operatorname{Str}(R \cdot \nabla^t J) = -\operatorname{Tr} R^E$.

Proof. All these statements follow easily from (B.5) and Lemma B.4a. □

Example B.6. The de Rham cs-manifold. In the case $E = TM \otimes \mathbb{C}$, (B.1) can be rewritten as

$$C^\infty(\mathbf{M}) \cong \Omega^*(M).$$

Let $\varepsilon^i = dx^i$ and $\varepsilon_i = \partial_i = \partial/\partial x^i$. Besides (B.2), consider also the odd vector field $Q = \varepsilon^i \partial_i$ on $\mathbf{M}$ corresponding to the de Rham operator d. Assume that ∇^M is torsion-free. This implies the identity $d = dx^i \wedge \nabla^M_{\partial_i}$, or equivalently

$$Q = \varepsilon^i \mathcal{D}_{\partial_i} = \mathcal{D}_Q \tag{B.6}$$

where the second equality should be understood as the definition of a new notation. Similar abuse of notation will appear below without further comment. The super Lie brackets with Q are given by

$$[Q, \mathcal{D}_X] = \mathcal{D}_{\nabla^M_Q X} - \mathcal{I}_{R^M_{Q,X} Q}, \quad [Q, \mathcal{I}_X] = \mathcal{I}_{\nabla^M_Q X} + \mathcal{D}_X, \quad [J, Q] = Q, \quad [Q, Q] = 0. \tag{B.7}$$

Indeed, the first two equations follow from the following calculations

$$\begin{aligned}
[Q, \mathcal{D}_X] &= [\varepsilon^i \mathcal{D}_{\partial_i}, \mathcal{D}_X] = \varepsilon^i [\mathcal{D}_{\partial_i}, \mathcal{D}_X] - (\mathcal{D}_X \varepsilon^i) \mathcal{D}_{\partial_i} = \varepsilon^i \mathcal{D}_{[\partial_i, X]} - \varepsilon^i \varepsilon^j \mathcal{I}_{R^M_{\partial_i, X} \partial_j} + \varepsilon^i \mathcal{D}_{\nabla^M_X \partial_i} \\
&= \varepsilon^i \mathcal{D}_{\nabla^M_{\partial_i} X} - \mathcal{I}_{R^M_{Q,X} Q} = \mathcal{D}_{\nabla^M_Q X} - \mathcal{I}_{R^M_{Q,X} Q} \\
[Q, \mathcal{I}_X] &= [\varepsilon^i \mathcal{D}_{\partial_i}, \mathcal{I}_X] = \varepsilon^i [\mathcal{D}_{\partial_i}, \mathcal{I}_X] + (\mathcal{I}_X \varepsilon^i) \mathcal{D}_{\partial_i} = \varepsilon^i \mathcal{I}_{\nabla^M_{\partial_i} X} + \mathcal{D}_X = \mathcal{I}_{\nabla^M_Q X} + \mathcal{D}_X
\end{aligned}$$

where we have used (B.6), (B.3) and the torsion-free condition. By (B.4) and (B.6), covariant differentiation with respect to Q is given by

$$\nabla_Q \mathcal{D}_X = \mathcal{D}_{\nabla^M_Q X}, \quad \nabla_Q \mathcal{I}_X = \mathcal{I}_{\nabla^M_Q X}. \tag{B.8}$$

Lemma B.7. *The operator $\nabla^t Q$ and its covariant derivatives are computed as follows:*
(a) $\nabla^t Q$ sends $\mathcal{D}_X$ to $\mathcal{I}_{R^M_{Q,X} Q}$, and $\mathcal{I}_X$ to $-\mathcal{D}_X$.
(b) $\nabla_{\mathcal{D}_X}(\nabla^t Q)$ sends $\mathcal{D}_Y$ to $\mathcal{I}_{(\nabla^M_X R^M)_{Q,Y} Q}$, and $\mathcal{I}_Y$ to 0.
(c) $\nabla_{\mathcal{I}_X}(\nabla^t Q)$ sends $\mathcal{D}_Y$ to $\mathcal{I}_{R^M_{X,Q} Y}$, and $\mathcal{I}_Y$ to 0.

Proof. Recall that $\nabla^t Q := \nabla_Q - [Q, \text{-}]$. (a) follows from (B.7) and (B.8). For (b) and (c) we compute

$$\begin{aligned}
\Big(\nabla_{\mathcal{D}_X}(\nabla^t Q)\Big)\mathcal{D}_Y &= \nabla_{\mathcal{D}_X} \mathcal{I}_{R^M_{Q,Y} Q} - (\nabla^t Q)\mathcal{D}_{\nabla^M_X Y} = \mathcal{I}_{\nabla^M_X R^M_{Q,Y} Q} + (\mathcal{D}_X \varepsilon^i \varepsilon^j) \mathcal{I}_{R^M_{\partial_i, Y} \partial_j} - \mathcal{I}_{R^M_{Q, \nabla^M_X Y} Q} \\
&= \mathcal{I}_{\nabla^M_X R^M_{Q,Y} Q} - \varepsilon^i \mathcal{I}_{R^M_{\nabla^M_X \partial_i, Y} Q} - \varepsilon^i \mathcal{I}_{R^M_{Q,Y} \nabla^M_X \partial_i} - \mathcal{I}_{R^M_{Q, \nabla^M_X Y} Q} = \mathcal{I}_{(\nabla^M_X R^M)_{Q,Y} Q} \\
\Big(\nabla_{\mathcal{D}_X}(\nabla^t Q)\Big)\mathcal{I}_Y &= -\nabla_{\mathcal{D}_X} \mathcal{D}_Y - (\nabla^t Q)\mathcal{I}_{\nabla^M_X Y} = -\mathcal{D}_{\nabla^M_X Y} + \mathcal{D}_{\nabla^M_X Y} = 0 \\
\Big(\nabla_{\mathcal{I}_X}(\nabla^t Q)\Big)\mathcal{D}_Y &= \nabla_{\mathcal{I}_X} \mathcal{I}_{R^M_{Q,Y} Q} = (\mathcal{I}_X \varepsilon^i \varepsilon^j) \mathcal{I}_{R^M_{\partial_i, Y} \partial_j} = \mathcal{I}_{R^M_{X,Y} Q} - \mathcal{I}_{R^M_{Q,Y} X} = \mathcal{I}_{R^M_{X,Q} Y} \\
\Big(\nabla_{\mathcal{I}_X}(\nabla^t Q)\Big)\mathcal{I}_Y &= -\nabla_{\mathcal{I}_X} \mathcal{D}_Y = 0
\end{aligned}$$

where we have used (B.4) and the first Bianchi identity. □

Lemma B.8. *The operators $\nabla^t Q$, $R \cdot \nabla^t Q$, $R \cdot \nabla^t Q \cdot \nabla^t Q$ and $\nabla(\nabla^t Q) \wedge \nabla(\nabla^t Q)$ all have supertrace zero. It follows that the supertrace of $\nabla(\nabla^t Q) \cdot \nabla^t Q$ is closed.*

Proof. The first three operators have no supertrace by Lemma B.7a and (B.5). For the third, notice that

$$(\nabla^t Q)^2 \mathcal{D}_X = -\mathcal{D}_{R^M_{Q,X} Q}, \quad (\nabla^t Q)^2 \mathcal{I}_X = -\mathcal{I}_{R^M_{Q,X} Q}.$$

The fourth operator has no supertrace by Lemma B.7b and c. The remaining assertion follows from

$$d\mathrm{Str}\Big(\nabla(\nabla^t Q) \cdot \nabla^t Q\Big) = 2\mathrm{Str}\left(R \cdot \nabla^t Q \cdot \nabla^t Q\right) - \mathrm{Str}\Big(\nabla(\nabla^t Q) \wedge \nabla(\nabla^t Q)\Big). \quad \square$$

Example B.9. Dolbeault cs-manifolds. Now we change our notations as follows: M is a complex manifold, TM its holomorphic tangent bundle, E a holomorphic vector bundle over M, and $\mathbf{M} = \Pi(\overline{TM} \oplus E)$ as a smooth cs-manifold. There is a canonical identification

$$C^\infty(\mathbf{M}) \cong \Omega^{0,*}(M; \wedge^* E^\vee). \tag{B.9}$$

Given a set of local holomorphic coordinates $(z^1, \cdots, z^d)$ on M and a local holomorphic frame $(\varepsilon^1, \cdots, \varepsilon^r)$ of $E^\vee$, there is an associated set of local coordinates on $\mathbf{M}$, namely

$$(\mathrm{Re}z^1, \mathrm{Im}z^1, \cdots, \mathrm{Re}z^d, \mathrm{Im}z^d, \bar\zeta^1, \cdots, \bar\zeta^d, \varepsilon^1, \cdots, \varepsilon^r)$$

where $\bar\zeta^i$ correspond to $d\bar z^i$ under (B.9). Let $\bar\partial_i = \partial/\partial\bar z^i$ and $(\varepsilon_1, \cdots, \varepsilon_k)$ be the dual local frame of E.

Choose connections ∇^M on TM and ∇^E on E of type $(1,0)$; denote by $\bar\nabla^M$ the induced connection on $\overline{TM}$. Let $X, Y, Z \in \mathcal{T}(M)$, $U, V \in \mathcal{T}^{0,1}(M)$ and $\sigma, \tau \in \Gamma(E)$. Under the identification (B.9), vector fields on $\mathbf{M}$ correspond to derivations of $(0,*)$-forms on M valued in $\wedge^* E^\vee$. In particular, denote by

$$\left\{\begin{array}{l} \mathcal{D}_X \\ \mathcal{I}_U, \mathcal{I}_\sigma \\ J^r, J^\ell \\ Q \end{array}\right\} \begin{array}{c}\text{the vector fields on } \mathbf{M} \\ \text{corresponding to the}\end{array} \left\{\begin{array}{l} \text{covariant differentiation } \bar\nabla^M_X \otimes 1 + 1 \otimes \nabla^E_X \\ \text{contractions with } U, \sigma \\ \text{exterior degrees in } \wedge^* \overline{TM}^\vee, \wedge^* E^\vee \\ \text{Dolbeault operator } \bar\partial \otimes 1 \end{array}\right\} \tag{B.10}$$

The vector fields $\mathcal{D}_X$, $\mathcal{I}_U$ and $\mathcal{I}_\sigma$ span $\mathcal{T}(\mathbf{M})$ over $C^\infty(\mathbf{M})$. Adopt an abuse of notation similar to that in Example B.6. The super Lie brackets among the first three types of vector fields in (B.10) are

$$\begin{gathered}
[\mathcal{D}_X, \mathcal{D}_Y] = \mathcal{D}_{[X,Y]} - \mathcal{I}_{\bar R^M_{X,Y} Q} - \varepsilon^k \mathcal{I}_{R^E_{X,Y}\varepsilon_k}, \quad [\mathcal{D}_X, \mathcal{I}_U] = \mathcal{I}_{\bar\nabla^M_X U}, \quad [\mathcal{D}_X, \mathcal{I}_\sigma] = \mathcal{I}_{\nabla^E_X \sigma} \\
[\mathcal{I}_U, \mathcal{I}_V] = [\mathcal{I}_U, \mathcal{I}_\sigma] = [\mathcal{I}_\sigma, \mathcal{I}_\tau] = 0 = [J^r, \mathcal{D}_X] = [J^r, \mathcal{I}_\sigma] = [J^\ell, \mathcal{D}_X] = [J^\ell, \mathcal{I}_U] = [J^r, J^\ell] \\
[J^r, \mathcal{I}_U] = -\mathcal{I}_U, \quad [J^\ell, \mathcal{I}_\sigma] = -\mathcal{I}_\sigma
\end{gathered} \tag{B.11}$$

Assume that ∇^M is torsion-free. [15] This implies the identity $\bar\partial = d\bar z^i \wedge \bar\nabla^M_{\bar\partial_i}$, or equivalently

$$Q = \bar\zeta^i \mathcal{D}_{\bar\partial_i} = \mathcal{D}_Q. \tag{B.12}$$

Then the various super Lie brackets with Q are given by

$$\begin{gathered}
[Q, \mathcal{D}_X] = \mathcal{D}_{\nabla^M_Q X^{1,0} + \bar\nabla^M_Q X^{0,1}} - \mathcal{I}_{\bar R^M_{Q,X} Q} + \varepsilon^k \mathcal{I}_{R^E_{Q,X}\varepsilon_k}, \quad [Q, \mathcal{I}_U] = \mathcal{I}_{\bar\nabla^M_Q U} + \mathcal{D}_U \\
[Q, \mathcal{I}_\sigma] = \mathcal{I}_{\nabla^E_Q \sigma}, \quad [J^r, Q] = Q, \quad [J^\ell, Q] = 0 = [Q, Q]
\end{gathered} \tag{B.13}$$

Indeed, the first two follow from (B.12) and calculations similar to those below (B.7).

Define a connection ∇ on $T\mathbf{M}$ as in §B.3. More explicitly, we define

$$\begin{gathered}
\nabla_{\mathcal{D}_X} \mathcal{D}_Y = \mathcal{D}_{\nabla^M_X Y^{1,0} + \bar\nabla^M_X Y^{0,1}}, \quad \nabla_{\mathcal{D}_X} \mathcal{I}_U = \mathcal{I}_{\bar\nabla^M_X U}, \quad \nabla_{\mathcal{D}_X} \mathcal{I}_\sigma = \mathcal{I}_{\nabla^E_X \sigma} \\
\nabla_{\mathcal{I}_U} \mathcal{D}_X = \nabla_{\mathcal{I}_U} \mathcal{I}_V = \nabla_{\mathcal{I}_U} \mathcal{I}_\sigma = \nabla_{\mathcal{I}_\sigma} \mathcal{D}_X = \nabla_{\mathcal{I}_\sigma} \mathcal{I}_U = \nabla_{\mathcal{I}_\sigma} \mathcal{I}_\tau = 0
\end{gathered} \tag{B.14}$$

By (B.12), covariant differentiation with respect to Q is given by

$$\nabla_Q \mathcal{D}_X = \mathcal{D}_{\nabla^M_Q X^{1,0} + \bar\nabla^M_Q X^{0,1}}, \quad \nabla_Q \mathcal{I}_U = \mathcal{I}_{\bar\nabla^M_Q U}, \quad \nabla_Q \mathcal{I}_\sigma = \mathcal{I}_{\nabla^E_Q \sigma}. \tag{B.15}$$

Using (B.11), we compute the curvature of ∇ as follows

$$\begin{gathered}
R_{\mathcal{D}_X, \mathcal{D}_Y} \mathcal{D}_Z = \mathcal{D}_{R^M_{X,Y} Z^{1,0} + \bar R^M_{X,Y} Z^{0,1}}, \quad R_{\mathcal{D}_X, \mathcal{D}_Y} \mathcal{I}_U = \mathcal{I}_{\bar R^M_{X,Y} U}, \quad R_{\mathcal{D}_X, \mathcal{D}_Y} \mathcal{I}_\sigma = \mathcal{I}_{R^E_{X,Y} \sigma} \\
R_{\mathcal{D}_X, \mathcal{I}_U} = R_{\mathcal{D}_X, \mathcal{I}_\sigma} = R_{\mathcal{I}_U, \mathcal{I}_V} = R_{\mathcal{I}_U, \mathcal{I}_\sigma} = R_{\mathcal{I}_\sigma, \mathcal{I}_\tau} = 0
\end{gathered} \tag{B.16}$$

[15] If ∇^M has a nontrivial torsion T, we can replace it with a new connection ∇'^M defined by $\nabla'^M_X = \nabla^M_X - \frac{1}{2} T_{X,-}$, which is also of type $(1,0)$ and is torsion-free.

The following statements and their proofs are similar to Lemmas B.4 and B.5.

Lemma B.10. *(a) The operator $\nabla^t J^r$ sends $\mathcal{D}_X$, $\mathcal{I}_\sigma$ to 0, and $\mathcal{I}_U$ to itself. (b) $\nabla(\nabla^t J^r) = 0$.*

Proof. Use (B.11), (B.14) and the fact that $J^r = \bar{\zeta}^i \mathcal{I}_{\bar{\partial}_i}$. □

Lemma B.11. *(a) The operator $\nabla^t J^\ell$ sends $\mathcal{D}_X$, $\mathcal{I}_U$ to 0, and $\mathcal{I}_\sigma$ to itself. (b) $\nabla(\nabla^t J^\ell) = 0$.*

Proof. Use (B.11), (B.14) and the fact that $J^\ell = \varepsilon^k \mathcal{I}_{\varepsilon_k}$. □

Lemma B.12. *Regarding $\Omega^*(M)$ as a subalgebra of $\Omega^*(\mathbf{M})$, we have*
(a) $\operatorname{Str} R = \operatorname{Tr} R^M - \operatorname{Tr} R^E$,
(b) $\operatorname{Str}(R \wedge R) = \operatorname{Tr}(R^M \wedge R^M) - \operatorname{Tr}(R^E \wedge R^E)$,
(c) $\operatorname{Str}(R \cdot \nabla^t J^r) = -\operatorname{Tr} \bar{R}^M$ *and* $\operatorname{Str}(R \cdot \nabla^t J^\ell) = -\operatorname{Tr} R^E$.

Proof. Use (B.16) and the previous two lemmas. □

The following statements and their proofs are similar to Lemmas B.7 and B.8.

Lemma B.13. *The operator $\nabla^t Q$ and its covariant derivatives are computed as follows:*
(a) $\nabla^t Q$ sends $\mathcal{D}_X$ to $\mathcal{I}_{\bar{R}^M_{Q,X} Q} - \varepsilon^k \mathcal{I}_{R^E_{Q,X}\varepsilon_k}$, $\mathcal{I}_U$ to $-\mathcal{D}_U$, and $\mathcal{I}_\sigma$ to 0.
(b) $\nabla_{\mathcal{D}_X}(\nabla^t Q)$ sends $\mathcal{D}_Y$ to $\mathcal{I}_{(\bar{\nabla}^M_X \bar{R}^M)_{Q,Y} Q} - \varepsilon^k \mathcal{I}_{(\nabla^E_X R^E)_{Q,Y}\varepsilon_k}$, and $\mathcal{I}_U$, $\mathcal{I}_\sigma$ to 0.
(c) $\nabla_{\mathcal{I}_U}(\nabla^t Q)$ sends $\mathcal{D}_X$ to $\mathcal{I}_{\bar{R}^M_{U,Q} X} + \varepsilon^k \mathcal{I}_{R^E_{U,X}\varepsilon_k}$, and $\mathcal{I}_V$, $\mathcal{I}_\sigma$ to 0.
(d) $\nabla_{\mathcal{I}_\sigma}(\nabla^t Q)$ sends $\mathcal{D}_X$ to $-\mathcal{I}_{R^E_{Q,X}\sigma}$, and $\mathcal{I}_U$, $\mathcal{I}_\sigma$ to 0.

Proof. Use (B.13), (B.14), (B.15) and the first Bianchi identity. □

Lemma B.14. *The operators $\nabla^t Q$, $R \cdot \nabla^t Q$, $R \cdot \nabla^t Q \cdot \nabla^t Q$ and $\nabla(\nabla^t Q) \wedge \nabla(\nabla^t Q)$ all have supertrace zero. It follows that the supertrace of $\nabla(\nabla^t Q) \cdot \nabla^t Q$ is closed.*

Proof. Use (B.16) and the previous lemma. For the third operator, also notice that $\bar{R}^M_{Q,X} Q = \frac{1}{2} \bar{R}^M_{Q,Q} X^{0,1}$ by the first Bianchi identity, and $R^E_{Q,U} = 0$ by our assumption on ∇^E. □

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