PRINCIPAL REALIZATION OF TWISTED YANGIAN $Y(\mathfrak{g}_N)$

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Abstract. We give the principal realization of the twisted Yangians of orthogonal and symplectic types. The new bases are interpreted in terms of discrete Fourier transform over the cyclic group $\mathbb{Z}_N$.

1. Introduction

Let $\mathfrak{g}$ be a simple complex Lie algebra. As one of the two important classes of quantum groups associated to $\mathfrak{g}$, the Yangian $Y(\mathfrak{g})$ was introduced by Drinfeld in the study of the Yang-Baxter equation [4, 5, 6, 3]. Other versions of Yangians were given by Olshanski [12] in connection with classical groups. All these Yangian algebras play important roles in conformal field theory, combinatorics and representation theory (see [11] for a beautiful survey and the monograph [9] for recent developments). For other presentations of Yangian algebras, see also [2].

The Yangian $Y(\mathfrak{gl}_N)$ of the general liner algebra captures and unifies, on a higher theoretic ground, many far reaching aspects of invariant theory and combinatorial theory [10, 8]. Motivated by the principal realization of the affine Kac-Moody Lie algebras, the authors in [1] introduced the corresponding set of generators for the Yangian $Y(\mathfrak{gl}_N)$ and show that the new basis is useful in studying representations of Yangians. Roughly speaking, the idea is based on replacing the Cartan-Weyl basis by the the Toeplitz basis in the general linear Lie algebra $\mathfrak{gl}_N$, this enables one to get new presentations of the Yangian.

We will study the principal generators for the Olshanski Yangian algebras $Y(\mathfrak{so}_N)$ and $Y(\mathfrak{sp}_N)$ in this letter and give their main properties. A new feature is that all principal generators are actually discrete Fourier transform of certain sequences defined by the Yangians over the cyclic group $\mathbb{Z}_N$. It seems that the twisted cases are also related to the Fourier transform over the abelian group $\mathbb{Z}_N \times \mathbb{Z}_N$. Under discrete Fourier transform, several interesting properties are formulated in the same pattern for the principal generators in the cases of the orthogonal and symplectic Yangians.

The paper is organized as follows. First in section two we recall the principal generators for type A and formulate the principal generators as Fourier transform. Several new results are proved for later usage. Section three discusses Olshanski

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twisted Yangian algebras. Section four gives the principal realizations for the twisted Yangian algebras of orthogonal and symplectic types.

2. Principal realization of Yangian algebra $Y(\mathfrak{gl}_N)$

In this section, we first recall the principal realization for $Y(\mathfrak{gl}_N)$ and then derive new relations using discrete Fourier transform for our later purpose.

2.1. **Principal basis of $\mathfrak{gl}_N$.** Let $\mathfrak{g} = \mathfrak{gl}_N$ be the Lie algebra of $N \times N$ complex matrices. The standard Cartan-Weyl basis consists of matrices E_{ij}, where $i, j \in \mathbb{Z}_N = \{0, 1...N-1\}$. For our purpose we will make full use of the additive structure of the index set $\mathbb{Z}_N$.

The cyclic element in $\mathfrak{gl}_N$ is

$$E = \sum_{i \in \mathbb{Z}_N} E_{i,i+1}$$

and its centralizer $C(E) = \bigoplus_{k \in \mathbb{Z}_N} \mathbb{C}E^k$ is a Cartan subalgebra of $\mathfrak{gl}_N$ called the principal Cartan subalgebra. With respect to this Cartan subalgebra, the principal root space decomposition is given as follows:

$$\mathfrak{gl}_N = \bigoplus_{i,j \in \mathbb{Z}_N} A_{ij}, A_{ij} = \sum_{k \in \mathbb{Z}_N} \omega^{ki} E_{k,k+j}, \tag{2.1}$$

where $\omega = e^{\frac{\mathrm{i}2\pi}{N}}$.

Let G be a finite abelian group with irreducible characters χ_i, $(i = 1, \ldots, |G|)$, the discrete Fourier transform of the function $f(g)$ on G is another function on $G^* = \{\chi_i | i = 1, \ldots, |G|\}$ defined by

$$\mathfrak{F}(f)(\chi) = \sum_{h \in G} \chi(h) f(h). \tag{2.2}$$

In the case of cyclic group $G = \mathbb{Z}_N$, $\mathbb{Z}_N^* = \{\chi^i | i = 0, \ldots, N-1\} \simeq Z_N$, and $\chi_i(j) = \omega^{ij}$. Therefore the discrete Fourier transform of the function f on $\mathbb{Z}_N$ is

$$\mathfrak{F}(f)(\chi_i) = \sum_{j=0}^{N-1} \omega^{ij} f(j). \tag{2.3}$$

The inverse Fourier transform is given by

$$\mathfrak{F}^{-1}(g)(i) = \frac{1}{N} \sum_{j=0}^{N-1} \omega^{-ij} g(j). \tag{2.4}$$

Fix j, denote the finite sequence $\{E_{k,k+j}\}$ by $\{\epsilon_j\}$, where k runs over $\mathbb{Z}_N$, i.e., $\epsilon_j(k) = E_{k,k+j}$. Then the principal basis elements A_{ij} are actually the ith term of the discrete Fourier transform of the sequence $\{\epsilon_j\}$, and formula (2.1) takes the following new form:

$$A_{ij} = \mathfrak{F}(\{\epsilon_j(k)\})(i).$$

The algebraic structure of principal basis A_{ij} is given by:

$$A_{ij}A_{kl} = \omega^{jk}A_{i+k,j+l}.$$

Under the standard inner product $(x|y) = tr(xy)$, we have

$$(A_{ij}|A_{kl}) = tr(A_{ij}A_{kl}) = N\omega^{-ij}\delta_{i,-k}\delta_{j,-l},$$

and the dual principal basis is $\{\frac{\omega^{ij}}{N}A_{-i,-j}\}$.

2.2. **Yangian** $Y(\mathfrak{gl}_N)$.

Definition 2.1. The Yangian algebra $Y(\mathfrak{gl}_N)$ is an unital associative algebra with generators $t_{ij}^{(r)}$ $(i, j \in \{1, 2...N\}, r \in \mathbb{Z}_+)$ subject to the relations:

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)}t_{il}^{(s)} - t_{kj}^{(s)}t_{il}^{(r)}, \tag{2.5}$$

where $t_{ij}^{(0)} = \delta_{ij}$.

Its matrix presentation is given in terms of the rational Yang-Baxter R-matrix. Let u be a formal variable and let

$$R(u) = 1 - \frac{P}{u} \in End(\mathbb{C}^N) \otimes End(\mathbb{C}^N)[[u^{-1}]], \tag{2.6}$$

where P is the permutation matrix: $P(u \otimes v) = v \otimes u$ for any $u, v \in \mathbb{C}^N$. The matrix $R(u)$ satisfies the quantum Yang-Baxter equation:

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u).$$

Set

$$T(u) = \sum_{i,j} t_{ij}(u) \otimes E_{ij} \in Y(\mathfrak{gl}_N)[[u^{-1}]] \otimes End(\mathbb{C}^N)$$

where

$$t_{ij}(u) = \delta_{ij} + \sum_{k=1}^{\infty} t_{ij}^{(k)}u^{-k} \in Y(\mathfrak{gl}_N)[[u^{-1}]].$$

The following well-known result (see [9]) gives the FRT formulation [7] of the Yangian algebra.

Proposition 2.2. *The defining relations of Yangian can be written compactly as*

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v).$$

2.3. **The principal realization of $Y(\mathfrak{gl}_N)$.** In [1] a new set of generators, the principal generators, $x_{ij}^{(k)}$ are introduced. Here the indices $i, j \in \mathbb{Z}_N$, $k \in \mathbb{N}$. Let $x_{ij}(u)$ be the generating series:

$$x_{ij}(u) = \sum_{n=0}^{\infty} x_{ij}^{(n)} u^{-n},$$

where $x_{ij}^{(0)} = \delta_{i,0}\delta_{j,0}$. Then

$$x_{ij}(u) = \sum_{k \in \mathbb{Z}_N} \frac{\omega^{-ki}}{N} t_{k,j+k}(u). \tag{2.7}$$

This can be viewed as the inverse of the Fourier transform on the finite sequence $\{t_{k,k+j}(u)\}$ where the variable $k \in \mathbb{Z}_N$. Then the formula (2.7) can be rewritten as

$$x_{ij}(u) = \mathfrak{F}^{-1}(\{t_{k,k+j}(u)\})(i)$$

Rewriting the T-matrix $T(u)$ by using the principal basis of $\mathfrak{gl}_N$ and $x_{ij}(u)$ as follows

$$T(u) = \sum_{k,l \in \mathbb{Z}_N} x_{kl}(u) \otimes A_{kl},$$

we obtain the principal realization of $Y(\mathfrak{gl}_N)$ as follows.

Proposition 2.3. [1] *The principal generators $x_{ij}^{(k)}$ of Yangian $Y(\mathfrak{gl}_N)$ satisfy the following relations:*

$$[x_{ij}(u), x_{kl}(v)] = \frac{1}{u-v}\Big(\sum_{a,b} \frac{\omega^{ib-bk-ab}}{N} x_{i-a,j-b}(u) x_{k+a,l+b}(v)$$
$$- \sum_{a,b} \frac{\omega^{ja-al-ab}}{N} x_{k+a,l+b}(v) x_{i-a,j-b}(u)\Big)$$

We can simplify the commutation relations and get a new compact formula as follows:

Theorem 2.4. *The principal generators $x_{ij}^{(k)}$ of Yangian $Y(\mathfrak{gl}_N)$ satisfy the following relations:*

$$(u-v)[x_{ij}(u), x_{kl}(v)] = \frac{1}{N} \sum_{a,b} \omega^{-ab} (x_{k+a,j+b}(u) x_{i-a,j-b}(v) - x_{k+a,j+b}(v) x_{i-a,j-b}(u)),$$

where a, b run through the group $\mathbb{Z}_N$.

It is easy to see that Proposition 2.3 is a consequence of Theorem 2.4.

3. Twisted Yangian $Y(\mathfrak{so}_N)$ and $Y(\mathfrak{sp}_N)$

We first describe the structure of the Oshanski twisted Yangian algebras associated to $\mathfrak{so}_N$ and $\mathfrak{sp}_N$ in this section.

3.1. **The Lie algebra $\mathfrak{so}_N$ and $\mathfrak{sp}_N$.** We will consider simultaneously both $\mathfrak{so}_N$ and $\mathfrak{sp}_N$. In the following let the index set $\mathbb{Z}_N = \{0, \ldots, N-1\}$ for matrices in $Mat(N)$. Let $A \mapsto A^t$ denote the transposition of $Mat(N)$ defined by

$$E^t_{ij} = \theta_i \theta_j E_{N-1-j,N-1-i},$$

where $i, j \in \mathbb{Z}_N$. The scalar θ_i is defined according to two cases as follows. For the symmetric case,

$$\theta_i = 1, \; i = 0, 1, \ldots, N-1,$$

and for the alternating or antisymmetric case with $N = 2n$,

$$\theta_i = \begin{cases} -1, & i = 0, 1, \ldots, n-1; \\ 1, & i = n, n+1, \ldots, 2n-1. \end{cases}$$

Introduce the following elements of the Lie algebra $\mathfrak{gl}_N$:

$$F_{ij} = E_{ij} - E^t_{ij} = E_{ij} - \theta_i \theta_j E_{N-1-j,N-1-i},$$

then the Lie subalgebra spanned by F_{ij} is isomorphic to $\mathfrak{so}_N$ in the symmetric case and to $\mathfrak{sp}_N$ in the alternating case. The resulting Lie algebra will be denoted by $\mathfrak{g}_N$. Thus,

$$\mathfrak{g}_N = \mathfrak{so}_N \text{ or } \mathfrak{sp}_N,$$

where the latter case N is supposed to be even. Corresponding to the principal basis of $\mathfrak{gl}_N$, we can derive the following simple result.

Proposition 3.1. *The subalgebra of $\mathfrak{gl}_N$ spanned by the elements $B_{ij} = A_{ij} - A^t_{ij}$ $i, j \in \mathbb{Z}_N$ is isomorphic to $\mathfrak{g}_N$, where*

$$A^t_{ij} = \frac{\omega^{-i(1+j)}}{N} \sum_{k,l \in \mathbb{Z}_N} \theta_k \theta_{k+j} \omega^{-k(i+l)} A_{lj}.$$

In particular, $A^t_{ij} = \omega^{-i(1+j)} A_{-i,j}$ in the symmetric case.

Proof. It follows from Equation (2.1) that

$$\begin{aligned} A_{ij}^t &= \sum_{k\in\mathbb{Z}_N} \omega^{ki} E_{k,k+j}^t \\ &= \sum_{k\in\mathbb{Z}_N} \omega^{ki}\theta_k\theta_{k+j} E_{N-1-k-j,N-1-k} \\ &= \sum_{k\in\mathbb{Z}_N} \omega^{(N-1-j-k)i}\theta_{N-1-k-j}\theta_{N-1-k} E_{k,k+j} \\ &= \omega^{-i(1+j)} \sum_{k\in\mathbb{Z}_N} \omega^{-ki}\theta_{k+j}\theta_k E_{k,k+j}, \end{aligned}$$

where we used $\theta_{N-1-k} = -\theta_k$ and $\omega^N = 1$ in the last equation. It follows from equation (2.1) and inverting the Fourier transform that

$$E_{k,k+j} = \frac{1}{N} \sum_{l\in\mathbb{Z}_N} \omega^{-kl} A_{lj}.$$

Therefore we have

$$A_{ij}^t = \frac{\omega^{-i(1+j)}}{N} \sum_{k,l\in\mathbb{Z}_N} \theta_{k+j}\theta_k \omega^{-k(i+l)} A_{lj}.$$

In particular, $\theta_k = \theta_{k+j} = 1$ in the symmetric case and

$$\begin{aligned} \sum_{l\in\mathbb{Z}_N} \omega^{-kl} A_{lj} &= \sum_{m,l\in\mathbb{Z}_N} \omega^{-kl}\omega^{ml} E_{m,m+j} \\ &= \sum_{m\in\mathbb{Z}_N} (\sum_{l\in\mathbb{Z}_N} \omega^{(-k+m)l}) E_{m,m+j} \\ &= \sum_{m\in\mathbb{Z}_N} N\delta_{m,k} E_{m,m+j} = N E_{k,k+j}. \end{aligned}$$

So in the case $\mathfrak{g}_N = \mathfrak{so}_N$

$$A_{ij}^t = \omega^{-i(1+j)} A_{-i,j}.$$

$\square$

Just as in the case of $\mathfrak{gl}_N$, we can also interpret the generators $B_{ij}'s$ using Fourier transform.

$$B_{ij} = \mathfrak{F}(\{F_{k,k+j}\})(i).$$

3.2. **Twisted Yangians** $Y(\mathfrak{g}_N)$.

Definition 3.2. The twisted Yangian corresponding to $\mathfrak{g}_N$ is a unital associative algebra with generators $s_{ij}^{(1)}, s_{ij}^{(2)}, \ldots$, where $i, j \in \mathbb{Z}_N$, and the defining relations are given in terms of generating series

$$s_{ij}(u) = \delta_{ij} + s_{ij}^{(1)} u^{-1} + s_{ij}^{(2)} u^{-2} + \ldots$$

as follows.

$$\begin{aligned}(u^2 - v^2)[s_{ij}(u), s_{kl}(v)] &= (u+v)(s_{kj}(u)s_{il}(v) - s_{kj}(v)s_{il}(u)) \\ &- (u-v)(\theta_k\theta_{j'} s_{ik'}(u)s_{j'l}(v) - \theta_i\theta_{l'} s_{ki'}(v)s_{l'j}(u)) \\ &+ \theta_i\theta_{j'}(s_{ki'}(u)s_{j'l}(v) - s_{ki'}(v)s_{j'l}(u)),\end{aligned}$$

and

$$\theta_i\theta_j s_{j'i'}(-u) = s_{ij}(u) \pm \frac{s_{ij}(u) - s_{ij}(-u)}{2}, \tag{3.1}$$

where $i' = N - 1 - i$. Whenever the double sign $\pm$ or $\mp$ occurs, the upper sign corresponds to the $\mathfrak{so}_N$ case and the lower sign to the $\mathfrak{sp}_N$ case.

These relations can also be given in an equivalent matrix form. For this purpose we define the partial transpose $R^t(u)$ for the Yang's R-matrix (2.6) by

$$R^t(u) = R(u)^{t_1} = 1 - Qu^{-1}, \quad Q = \sum_{i,j \in \mathbb{Z}_N} E_{ij}^t \otimes E_{ji}. \tag{3.2}$$

Furthermore, we regard $S(u)$ as an element of the algebra $Y(\mathfrak{g}_N) \otimes End\mathbb{C}^N$ given by

$$S(u) = \sum_{i,j \in \mathbb{Z}_N} s_{ij}(u) \otimes E_{ij}.$$

Then the twisted Yangian can be characterized by the following relations [9]:

$$R(u-v)S_1(u)R^t(-u-v)S_2(v) = S_2(v)R^t(-u-v)S_1(u)R(u-v), \tag{3.3}$$

and the symmetric relation:

$$S^t(-u) = S(u) \pm \frac{S(u) - S(-u)}{2u}. \tag{3.4}$$

The following relations between twisted Yangian and classical Lie algebras will be useful.

Proposition 3.3. [9] *The assignment*

$$s_{ij}(u) \mapsto \delta_{ij} + (u \pm \frac{1}{2})^{-1} F_{ij}$$

defines a homomorphism between $Y(\mathfrak{g}_N)$ *and* $U(\mathfrak{g}_N)$*. Moreover, the assignment*

$$F_{ij} \mapsto s_{ij}^{(1)}$$

defines an embedding $U(\mathfrak{g}_N) \hookrightarrow Y(\mathfrak{g}_N)$.

4. Principal realization of $Y(\mathfrak{g}_N)$

In this section we give the principal realization for twisted Yangian $Y(\mathfrak{g}_N)$ analogous to the $Y(\mathfrak{gl}_N)$ case. As before the generators are certain Fourier coefficients.

We start by recalling a well-known result in linear algebra. Let $\{e_i\}$ and $\{e^i\}$ be a pair of dual bases of simple Lie algebra $\mathfrak{g}$, then the rational r-matrix can be expressed as follows (cf. [1]):

$$r = \sum e_i \otimes e^i. \tag{4.1}$$

Moreover, this expression is independent of the choice of the dual bases.

Using the principal basis of $\mathfrak{gl}_N$, we get the following result.

Lemma 4.1. *The permutation matrix P can be written as*

$$P = \sum_{i,j} E_{ij} \otimes E_{ji} = \sum_{k,l \in \mathbb{Z}_N} \frac{\omega^{kl}}{N} A_{kl} \otimes A_{-k,-l}. \tag{4.2}$$

The partial transposition of P can be written as

$$Q = P^{t_1} = \sum_{k,l \in \mathbb{Z}_N} \frac{\omega^{kl}}{N} A^t_{kl} \otimes A_{-k,-l} = \sum_{k,l,a,b} \frac{\omega^{-a(k+b)-k}}{N^2} \theta_{a,a+l} A_{bl} \otimes A_{-k,-l}$$

In particular, when $\mathfrak{g}_N = \mathfrak{so}_N$

$$Q = P^{t_1} = \sum_{k,l \in \mathbb{Z}_N} \frac{\omega^{kl}}{N} A^t_{kl} \otimes A_{-k,-l} = \sum_{k,l \in \mathbb{Z}_N} \frac{\omega^{-k}}{N} A_{-k,l} \otimes A_{-k,-l}. \tag{4.3}$$

Proof. Note that $\{A_{kl}\}$ and $\{\frac{\omega^{kl}}{N} A_{-k,-l}\}$ are dual principal basis of $\mathfrak{gl}_N$. Invoking (4.1) we have

$$P = \sum_{k,l \in \mathbb{Z}_N} \frac{\omega^{kl}}{N} A_{kl} \otimes A_{-k,-l}.$$

Using Proposition 3.1, we get

$$\begin{aligned} Q &= \sum_{k,l \in \mathbb{Z}_N} \frac{\omega^{kl}}{N} A^t_{kl} \otimes A_{-k,-l} \\ &= \sum_{k,l,a,b} \frac{\omega^{-a(k+b)-k}}{N^2} \theta_{a,a+l} A_{bl} \otimes A_{-k,-l}. \end{aligned}$$

And in the case of $\mathfrak{g}_N = \mathfrak{so}_N$,

$$Q = \sum_{k,l\in\mathbb{Z}_N} \frac{\omega^{kl}}{N} A_{kl}^t \otimes A_{-k,-l}$$
$$= \sum_{k,l\in\mathbb{Z}_N} \frac{\omega^{-k}}{N} A_{-k,l} \otimes A_{-k,-l}.$$

□

We now introduce a new set of generators $y_{ij}^{(r)}$ of twisted Yangian $Y(\mathfrak{g}_N)$, where $i,j \in \mathbb{Z}_N$, $r \in \mathbb{Z}_+$. Rewrite the matrix of generators $S(u)$ as follows:

$$S(u) = \sum_{i,j} s_{ij}(u) \otimes E_{ij} = \sum_{k,l\in\mathbb{Z}_N} y_{kl}(u) \otimes A_{kl}, \tag{4.4}$$

where $y_{kl}(u)'$s are the generating series defined by:

$$y_{kl}(u) = \sum_{r=0}^{\infty} y_{kl}^{(r)} u^{-r},$$

and $y_{kl}^{(0)} = \delta_{k,0}\delta_{l,0}$.

Theorem 4.2. *The principal generators $y_{ij}^{(r)}$ of the Yangian $Y(\mathfrak{g}_N)$ satisfy the following relations:*

$$(u^2 - v^2)[y_{ij}(u), y_{kl}(v)] =$$
$$\frac{u+v}{N} \sum_{a,b\in\mathbb{Z}_N} (\omega^{b(i-k-a)} y_{i-a,j-b}(u) y_{k+a,l+b}(v) - \omega^{a(j-l-b)} y_{k+a,l+b}(v) y_{i-a,j-b}(u))$$
$$+ \frac{u-v}{N^2} \sum_{a,b,a',b'\in\mathbb{Z}_N} \theta_{a'}\theta_{a'+b} \omega^{-a-a'(a+b')+b'(j-b)-b(k+a))} y_{i-b',j-b}(u) y_{k+a,l+b}(v)$$
$$- \frac{u-v}{N^2} \sum_{a,b,a',b'\in\mathbb{Z}_N} \theta_{a'}\theta_{a'+b} \omega^{-a-a'(a+b')+b(i-b')-a(l+b))} y_{k+a,l+b}(v) y_{i-b',j-b}(v)$$
$$+ \frac{1}{N^3} \sum_{a,b,a',b',c,d\in\mathbb{Z}_N} \theta_{a'}\theta_{a'+b} \omega^{-a-a'(a+b')+b'(j-b-d)+b(k+a+c)+d(i-k-c)} y_{i-c-b',j-d-b}(u) y_{k+a+c,l+b+d}(v)$$
$$- \frac{1}{N^3} \sum_{a,b,a',b',c,d\in\mathbb{Z}_N} \theta_{a'}\theta_{a'+b} \omega^{-a-a'(a+b')+b(i-a-b')+a(l+b+d)+c(j-l-d)} y_{k+a+c,l+b+d}(v) y_{i-c-b',j-d-b}(u)$$

and the symmetric relation:

$$\frac{1}{N} \sum_{k,l} \theta_k \theta_{k+j} \omega^{-l(1+j)-k(i+l)} y_{l,j}(-u) = y_{ij}(u) \pm \frac{y_{ij}(u) - y_{ij}(-u)}{2u}.$$

In particular, when $\mathfrak{g}_N = \mathfrak{so}_N$ the defining relations can be reduced to the following relations:

$$
\begin{aligned}
&(u^2-v^2)[y_{ij}(u), y_{kl}(v)] = \\
&\frac{(u+v)}{N}\sum_{a,b\in\mathbb{Z}_N}(\omega^{b(i-k-a)}y_{i-a,j-b}(u)y_{k+a,l+b}(v)-\omega^{a(j-l-b)}y_{k+a,l+b}(v)y_{i-a,j-b}(u)) \\
&+\frac{(u-v)}{N}\sum_{a,b\in\mathbb{Z}_N}(\omega^{-a+bi-la}y_{k+a,l+b}(u)y_{i+a,j-b}(v)-\omega^{-a-aj-bk}y_{i+a,j-b}(u)y_{k+a,l+b}(v)) \\
&+\frac{1}{N^2}\sum_{a,b,a',b'\in\mathbb{Z}_N}\omega^{-a-aj+b'(a-k+i-a')+b(k+a')}y_{i+a-a',j-b-b'}(u)y_{k+a+a',l+b+b'}(v) \\
&-\frac{1}{N^2}\sum_{a,b,a',b'\in\mathbb{Z}_N}\omega^{-a+bi+a'(j-b-b'-l)-a(l+b')}y_{k+a+a',l+b+b'}(v)y_{i+a-a',j-b-b'}(u)
\end{aligned}
$$

and the symmetric relation:

$$\omega^{i(1+j)}y_{-i,j}(-u) = y_{ij}(u) + \frac{1}{2u}(y_{ij}(u) - y_{ij}(-u)).$$

Here we only prove the case $\mathfrak{g}_N = \mathfrak{so}_N$ as the other case is similar. In order to prove the theorem, we need the following lemma:

Lemma 4.3. *We have the following equations for $\mathfrak{g}_N = \mathfrak{so}_N$:*

$$
\begin{aligned}
PS_1(u)S_2(v) - S_2(v)S_1(u)P = \frac{1}{N}\sum_{a,b,i,j,k,l\in\mathbb{Z}_N}&(\omega^{b(i-k-a)}y_{i-a,j-b}(u)y_{k+a,l+b}(v) \\
&-\omega^{a(j-l-b)}y_{k+a,l+b}(v)y_{i-a,j-b}(u))A_{ij}\otimes A_{kl}.
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
S_2(v)QS_1(u) - S_1(u)QS_2(v) = \frac{1}{N}\sum_{a,b,i,j,k,l\in\mathbb{Z}_N}&(\omega^{-a+bi-la}y_{k+a,l+b}(u)y_{i+a,j-b}(v) \\
&-\omega^{-a-aj-bk}y_{i+a,j-b}(u)y_{k+a,l+b}(v))A_{ij}\otimes A_{kl}.
\end{aligned} \tag{4.6}
$$

(4.7)

$$
\begin{aligned}
&PS_1(u)QS_2(v) - S_2(v)QS_1(u)P = \\
&\frac{1}{N^2}\sum_{a,b,a',b',i,j,k,l\in\mathbb{Z}_N}\omega^{-a-aj+b'(a-k+i-a')+b(k+a')}y_{i+a-a',j-b-b'}(u)y_{k+a+a',l+b+b'}(v)A_{ij}\otimes A_{kl} - \\
&\frac{1}{N^2}\sum_{a,b,a',b',i,j,k,l\in\mathbb{Z}_N}\omega^{-a+bi+a'(j-b-b'-l)-a(l+b')}y_{k+a+a',l+b+b'}(v)y_{i+a-a',j-b-b'}(u)A_{ij}\otimes A_{kl}.
\end{aligned}
$$

Proof. Here we just check the equation (4.6), and the other two equations can be proved by the same method. Using the principal decomposition of the operator Q

(see formula (4.3)), we have

$$S_2(v)QS_1(u) = (\sum_{k,l\in\mathbb{Z}_N} y_{kl}(v)1\otimes A_{kl})(\frac{1}{N}\sum_{a,b\in\mathbb{Z}_N}\omega^{-a}A_{-a,b}\otimes A_{-a,-b})(\sum_{i,j\in\mathbb{Z}_N} y_{ij}(u)A_{ij}\otimes 1)$$
$$= \frac{1}{N}\sum_{a,b,i,j,k,l\in\mathbb{Z}_N}\omega^{-a}y_{kl}(v)y_{ij}(u)A_{-a,b}A_{ij}\otimes A_{kl}A_{-a,-b}$$
$$= \frac{1}{N}\sum_{a,b,i,j,k,l\in\mathbb{Z}_N}\omega^{-a+bi-la}y_{kl}(v)y_{ij}(u)A_{-a+i,b+j}\otimes A_{k-a,l-b}$$
$$(Let\ i' = -a+i, j' = b+j, k' = k-a, l' = l-b)$$
$$= \frac{1}{N}\sum_{a,b,i',j',k',l'\in\mathbb{Z}_N}\omega^{-a+bi'-l'a}y_{k'+a,l'+b}(v)y_{i'+a,j'-b}(u)A_{i'j'}\otimes A_{k'l'}$$
$$= \frac{1}{N}\sum_{a,b,i,j,k,l\in\mathbb{Z}_N}\omega^{-a+bi-la}y_{k+a,l+b}(v)y_{i+a,j-b}(u)A_{ij}\otimes A_{kl}.$$

Similarly, we can get

$$S_1(u)QS_2(v) = \frac{1}{N}\sum_{a,b,i,j,k,l\in\mathbb{Z}_N}\omega^{-a-aj-bk}y_{i+a,j-b}(u)y_{k+a,l+b}(v))A_{ij}\otimes A_{kl},$$

from which one gets Equation (4.6). □

Now we prove Theorem 4.2 (the case $\mathfrak{g}_N = \mathfrak{so}_N$) using the above lemma.

Proof. From the equation (3.3), it follows that

$$(u^2-v^2)(S_1(u)S_2(v) - S_2(v)S_1(u)) = (u+v)(PS_1(u)S_2(v) - S_2(v)S_1(u)P)$$
$$+ (u-v)(S_2(v)QS_1(u) - S_1(u)QS_2(v))$$
$$+ (PS_1(u)QS_2(v) - S_2(v)QS_1(u)P).$$

Then one derives the reflection relation in the Theorem 4.2 by using the equations in lemma 4.3.

Next we check the symmetric relation. It follows from definition that the symmetric relation is

$$S^t(-u) = S(u) + \frac{S(u)-S(-u)}{2u}.$$

Using the principal presentation of $S(u)$ we have:

$$S(u) = \sum_{i,j\in\mathbb{Z}_N} y_{ij}(u) \otimes A_{ij},$$

$$S^t(-u) = \sum_{i,j\in\mathbb{Z}_N} y_{ij}(-u) \otimes A^t_{ij}$$

$$= \sum_{i,j\in\mathbb{Z}_N} \omega^{-i(1+j)} y_{ij}(-u) \otimes A_{-i,j}$$

Then we can rewrite the symmetric relation as following:

$$\sum_{i,j\in\mathbb{Z}_N} \omega^{-i(1+j)} y_{ij}(-u) \otimes A_{-i,j} = \sum_{i,j\in\mathbb{Z}_N} y_{ij}(u) \otimes A_{ij} + \frac{1}{2u} \sum_{i,j\in\mathbb{Z}_N} (y_{ij}(u) - y_{ij}(-u)) \otimes A_{ij}.$$

Therefore

$$\omega^{i(1+j)} y_{-i,j}(-u) = y_{ij}(u) + \frac{1}{2u}(y_{ij}(u) - y_{ij}(-u)),$$

which is just the symmetric relation in the theorem. □

Theorem 4.4. *The mapping* $s_{ij}(u) \mapsto \sum_{k\in\mathbb{Z}_N} \omega^{ik} y_{k,j-i}(u)$ *defines an isomorphism of the two presentations of* $Y(\mathfrak{g}_N)$*. The inverse mapping is given by*

$$y_{kl}(u) \mapsto \sum_{i\in\mathbb{Z}_N} \frac{\omega^{-ki}}{N} s_{i,i+l}(u)$$

Proof. This can be quickly shown by the inverse Fourier transform. For completeness we give another proof. From the principal realizations of twisted Yangian $Y(\mathfrak{g}_N)$, we have

$$S(u) = \sum_{k,l\in\mathbb{Z}_N} y_{kl}(u) \otimes A_{kl},$$

where

$$A_{kl} = \sum_{a\in\mathbb{Z}_N} \omega^{ak} E_{a,a+l}.$$

Plugging back into the equation, one has

$$S(u) = \sum_{k,l,a\in\mathbb{Z}_N} \omega^{ak} y_{kl}(u) \otimes E_{a,a+l}$$

$$= \sum_{i,j,k} \omega^{ik} y_{k,j-i}(u) \otimes E_{ij},$$

which shows that

$$s_{ij}(u) = \sum_{k\in\mathbb{Z}_N} \omega^{ik} y_{k,j-i}.$$

Since $A_{k,l}$ and $\frac{\omega^{kl}}{N}A_{-k,-l}$ are dual bases of $\mathfrak{gl}_N$, it follows that:

$$\begin{aligned}
y_{kl}(u) &= (S(u)|A_{-k,-l})\frac{\omega^{kl}}{N} \\
&= \sum_{i,j}\frac{\omega^{kl}}{N}s_{ij}(u)(E_{ij}|A_{-k,-l}) \\
&= \sum_{i,j}\frac{\omega^{k(l-j)}}{N}\delta_{i,j-l}s_{ij}(u) \\
&= \sum_{i}\frac{\omega^{-ki}}{N}s_{i,i+l}(u)
\end{aligned}$$

□

From the Theorem 4.4, we know that as in the case $Y(\mathfrak{gl}_N)$, the principal generators $y_{ij}(u)$ of $Y(\mathfrak{g}_N)$ are actually obtained from the Fourier transform of the sequence $\{s_{k,k+j}(u)\}$, where $k \in \mathbb{Z}_N$.

$$y_{ij}(u) = \mathfrak{F}^{-1}(\{s_{k,k+j}(u)\})(i).$$

By Theorem 4.4 we can get the following algebra homomorphism between the twisted Yangian $Y(\mathfrak{g}_N)$ and the universal enveloping algebra $U(\mathfrak{g}_N)$.

Theorem 4.5. *The assignment*

$$y_{kl}(u) \mapsto \frac{1}{N}\delta_{k,0}\delta_{l,0} + \frac{(u \pm \frac{1}{2})^{-1}}{N}B_{-k,l}$$

gives a homomorphism between $Y(\mathfrak{g}_N)$ and $U(\mathfrak{g}_N)$. Moreover, the assignment

$$B_{ij} \mapsto Ny^{(1)}_{-i,j}$$

defines an embedding $U(\mathfrak{g}_N) \hookrightarrow Y(\mathfrak{g}_N)$.

Proof. From Theorem 4.4, it follows that the mapping

$$y_{kl}(u) \mapsto \sum_{i\in\mathbb{Z}_N}\frac{\omega^{-ki}}{N}s_{i,i+l}(u)$$

is an isomorphism of the two presentations of $Y(\mathfrak{g}_N)$. And from the proposition 3.3, we have that the assignment

$$s_{ij}(u) \mapsto \delta_{ij} + (u \pm \frac{1}{2})^{-1}F_{ij}$$

is an algebra homomorphism between $Y(\mathfrak{g}_N)$ and $U(\mathfrak{g}_N)$. Then we get an algebra homomorphism between $Y(\mathfrak{g}_N)$ and $U(\mathfrak{g}_N)$:

$$
\begin{aligned}
y_{kl}(u) &\mapsto \sum_{i\in\mathbb{Z}_N} \frac{\omega^{-ki}}{N}(\delta_{l,0} + (u \pm \frac{1}{2})^{-1} F_{i,i+l}) \\
&= \frac{1}{N}\delta_{l,0} \sum_{i\in\mathbb{Z}_N} \omega^{-ki} + \frac{(u \pm \frac{1}{2})^{-1}}{N} \sum_{i\in\mathbb{Z}_N} \omega^{-ki} F_{i,i+l} \\
&= \frac{1}{N}\delta_{l,0}\delta_{k,0} + \frac{(u \pm \frac{1}{2})^{-1}}{N} \sum_{i\in\mathbb{Z}_N} \omega^{-ki} F_{i,i+l}.
\end{aligned}
$$

Subsequently one has

$$
\begin{aligned}
&\sum_{i\in\mathbb{Z}_N} \omega^{-ki} F_{i,i+l} = \sum_{i\in\mathbb{Z}_N} \omega^{-ki} E_{i,i+l} - \sum_{i\in\mathbb{Z}_N} \omega^{-ki} E^t_{i,i+l} \\
&= A_{-k,l} - A^t_{-k,l}
\end{aligned}
$$

which is just $B_{-k,l}$. So the assignment

$$
y_{kl}(u) \mapsto \frac{1}{N}\delta_{k,0}\delta_{l,0} + \frac{(u + \frac{1}{2})^{-1}}{N} B_{-k,l}
$$

does define an algebra homomorphism between $Y(\mathfrak{g}_N)$ and $U(\mathfrak{g}_N)$.

Similarly, we can show that the assignment $B_{ij} \mapsto N y^{(1)}_{-i,j}$ defines an embedding $U(\mathfrak{g}_N) \hookrightarrow Y(\mathfrak{g}_N)$.

$\square$

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