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New Integral Representations of Whittaker Functions for Classical Lie Groups

A. Gerasimov

Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia,
School of Mathematics, Trinity College, Dublin 2, Ireland,
Hamilton Mathematics Institute, TCD, Dublin 2, Ireland,

D. Lebedev[1]

Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia,
Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany,

S. Oblezin [2]

Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia,
Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany.

Abstract

We propose integral representations of the Whittaker functions for the classical Lie algebras $\mathfrak{sp}_{2\ell}$, $\mathfrak{so}_{2\ell}$ and $\mathfrak{so}_{2\ell+1}$. These integral representations generalize the integral representation of $\mathfrak{gl}_{\ell+1}$-Whittaker functions first introduced by Givental. One of the salient features of the Givental representation is its recursive structure with respect to the rank ℓ of the Lie algebra $\mathfrak{gl}_{\ell+1}$. The proposed generalization of the Givental representation to the classical Lie algebras retains this property. It was shown elsewhere that the integral recursion operator for $\mathfrak{gl}_{\ell+1}$-Whittaker function in the Givental representation coincides with a degeneration of the Baxter $\mathcal{Q}$-operator for $\widehat{\mathfrak{gl}}_{\ell+1}$-Toda chains. We construct $\mathcal{Q}$-operator for affine Lie algebras $\widehat{\mathfrak{so}}_{2\ell}$, $\widehat{\mathfrak{so}}_{2\ell+1}$ and a twisted form of $\widehat{\mathfrak{gl}}_{2\ell}$. We demonstrate that the relation between recursion integral operators of the generalized Givental representation and degenerate $\mathcal{Q}$-operators remains valid for all classical Lie algebras.

[1]E-mail: lebedev@itep.ru
[2]E-mail: Sergey.Oblezin@itep.ru

1 Introduction

A remarkable integral representation for common eigenfunctions of $\mathfrak{gl}_{\ell+1}$-Toda chain Hamiltonian operators was proposed by A. Givental [Gi] (see also [JK]). The integral representation appears naturally in a construction of a mirror dual of the theory of Type A topological closed strings on $\mathfrak{gl}_{\ell+1}$-flag manifolds. The Givental integral representation has many interesting properties. It has an explicit recursive structure over the rank ℓ of the corresponding Lie algebra $\mathfrak{gl}_{\ell+1}$. The integrand in the integral representation allows for purely combinatorial description in terms of a simple graph. This graph captures a flat degenerations of flag manifolds to Gorenstein toric Fano varieties (torification) [Ba], [BCFKS].

In [GKLO], the Givental integral representation was reconsidered in the framework of the representation theory approach to quantum integrable systems. According to B. Kostant [Ko1],[Ko2] the common eigenfunctions of $\mathfrak{g}$-Toda chain Hamiltonian operators are given by generalizations of classical Whittaker functions and can be expressed in terms of the matrix elements of infinite-dimensional representations of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. It was demonstrated in [GKLO] that the Givental representation of $\mathfrak{gl}_{\ell+1}$-Toda eigenfunctions coincides with an integral representation of the relevant matrix elements obtained by using a particular parametrization of an open part of the $\mathfrak{gl}_{\ell+1}$-flag manifold. A conceptual explanation for the particular choice of coordinates on flag manifolds was proposed using a relation with the Baxter $\mathcal{Q}$-operator formalism. It was noticed that the Givental integral representation has a recursive structure connecting the $\mathfrak{gl}_{\ell}$ - and $\mathfrak{gl}_{\ell+1}$-Whittaker functions by simple integral transformations. The corresponding integral operator coincides with a particular degeneration of the Baxter $\mathcal{Q}$-operator for $\widehat{\mathfrak{gl}}_{\ell+1}$-Toda chain [PG]. It is well-known that $\mathcal{Q}$-operators realise the quantum Bäcklund transformations in quantum integrable systems. On the other hand, in the classical limit, the formalism of $\mathcal{Q}$-operators allows us to define special coordinate system on the phase space. Thus degenerate $\mathcal{Q}$-operators define particular coordinates on an open part of flag manifolds and therefore lead to the Givental integral representation of $\mathfrak{gl}_{\ell+1}$-Whittaker function.

Until now a generalization of the Givental integral representation of $\mathfrak{gl}_{\ell+1}$-Whittaker functions to Lie algebras other then $\mathfrak{gl}_{\ell+1}$ was not known. The only known generalization [BCFKS], [Ri] of the Givental construction is an integral representation for common eigenfunctions of certain degenerations $\mathfrak{gl}_{\ell+1}$-Toda chains [STS]. It is based on flat degenerations of partial flag manifolds G/P for $G = GL(\ell+1,\mathbb{C})$, P being a parabolic subgroup of G [BCFKS]. In this paper we propose a generalization of the Givental construction for classical Lie algebras $\mathfrak{sp}_{2\ell}$, $\mathfrak{so}_{2\ell}$ and $\mathfrak{so}_{2\ell+1}$. Our construction possesses all characteristic properties of the original Givental integral representation. The integral representations for the classical Lie algebras have recursive structure. The integrands of the integral representations have combinatorial descriptions in terms of graphs. The proposed generalization to the classical Lie algebras is based on a modification of a well-known factorized representation of generic elements of maximal unipotent subgroups of the corresponding Lie groups [Lu] (see also [FZ], [BZ]). The construction of the modified factorized representation essentially uses the realization of maximal unipotent subgroups of classical Lie groups as explicitly described subgroups of upper-triangular matrices (see e.g. [DS]). We define Baxter $\mathcal{Q}$-operators associated with the classical affine Lie algebras $\widehat{\mathfrak{so}}_{2\ell}$, $\widehat{\mathfrak{so}}_{2\ell+1}$ and a twisted form of $\widehat{\mathfrak{gl}}_{2\ell}$. We demonstrate that the relation between recursion integral operators of the generalized Givental representation and degenerate $\mathcal{Q}$-operators remains valid for all classical Lie algebras.

The novel feature of the constructed integral representation is that, in contrast with the $\mathfrak{gl}_{\ell+1}$ case (where the kernel of the recursion operator is a simple function), the integral kernels of the recursion operators for all other classical Lie groups are given by non-trivial integrals. This suggests that the recursion operators can be obtained as a composition of elementary operators. Indeed, for zero eigenvalues, recursion operators relating the Toda chain eigenfunctions of the Lie algebras with adjacent ranks can be represented as compositions of elementary recursion operators relating the Toda chain eigenfunctions of *different* classical series. This might not be so surprising due to the fact that we essentially use a realisation of all classical Lie groups as subgroups of general Lie groups of large enough rank.

Let us stress that the construction of integral representations of $\mathfrak{g}$-Whittaker functions presented in this paper also has a natural interpretation in terms of torification of flag manifolds associated with classical Lie groups. The graph encoding the integrand of the Givental representation for a classical Lie group allows us to describe toric degeneration of the corresponding flag manifold explicitly (thus generalizing the results of [BCFKS] to all classical Lie groups).

One of the interesting applications of the Givental integral representation of $\mathfrak{g}$-Whittaker functions for classical Lie algebras might be a construction of mirror duals for closed strings on flag spaces associated with classical Lie groups G, $\mathfrak{g} = \mathrm{Lie}(G)$. According to Givental [Gi] the mirror dual to Type A topological string theories on flag manifolds associated with Lie groups G should be Landau-Ginzburg models associated with Langlands dual Lie groups $G^{\vee}$ such that the generating function of the genus zero correlators is a $\mathfrak{g}^{\vee}$-Whittaker function, $\mathfrak{g}^{\vee} = \mathrm{Lie}(G^{\vee})$. In the case of $\mathfrak{g} = \mathfrak{gl}_{\ell+1}$ Givental provides a description of the dual Landau-Ginzburg model in terms of the integrand of the integral representation of the corresponding Whittaker function. Moreover the interpretation of the integral representation of a $\mathfrak{g}$-Whittaker function in terms of a torification of flag manifolds [Ba], [BCFKS] allows us to construct the mirror map explicitly. Thus using the same reasoning, the generalization of the Givental integral representation for classical Lie groups allows us to infer the superpotential of the corresponding Landau-Ginzburg model from the integrand. Moreover, similar to the case of $\mathfrak{g} = \mathfrak{gl}_{\ell+1}$, the interpretation of the integrand in terms of a toric degeneration of the flag manifold provides an explicit construction of the mirror map. We will discuss this construction elsewhere.

Finally, note that some of the results presented in this paper was announced previously in [GLO], [GLO1].

The plan of this paper is as follows. In Part I we formulate the results for the classical Lie algebras $\mathfrak{sp}_{2\ell}$, $\mathfrak{so}_{2\ell}$ and $\mathfrak{so}_{2\ell+1}$. The main results are formulated in the Theorems 2.3, 2.6, 2.10, 2.14 respectively. In Part II we collect the proofs of the results presented in Part I.

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2 Part I: Results

2.1 Toda chain eigenfunctions as matrix elements

Eigenfunctions of $\mathfrak{g}$-Toda chain are given by particular matrix elements of infinite-dimensional representations of Lie algebra $\mathfrak{g}$ [Ko1], [Ko2] (for detailed exposition see e.g. [Et]). In this section we provide integral representations of these matrix elements with integrands being expressed in terms of matrix elements of finite-dimensional representations of $\mathfrak{g}$. In the following sections we derive explicit expressions for the relevant matrix elements of finite-dimensional representations and thus obtain integral representations of $\mathfrak{g}$-Toda chain eigenfunctions generalizing the results of Givental for $\mathfrak{g} = \mathfrak{gl}_{\ell+1}$. The construction will be given for all classical Lie algebras. We start with standard definitions in the theory of Lie algebras mostly following [K] (for a discussion of root data of reductive groups see e.g. [S]).

2.1.1 Root data for reductive groups

Root datum is a quadruple $(X, \Phi, X^\vee, \Phi^\vee)$ where X is a lattice of a finite rank, $X^\vee$ is a dual lattice, Φ and $\Phi^\vee$ are subsets of X and $X^\vee$ supplied with a bijection $\alpha \mapsto \alpha^\vee$ of Φ onto $\Phi^\vee$ and the following conditions hold. One has $\langle \alpha, \alpha^\vee \rangle = 2$ for any $\alpha \in \Phi$. Subsets Φ and $\Phi^\vee$ should be stable with respect to any automorphisms s_α, $s_{\alpha^\vee}$:

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \qquad s_{\alpha^\vee}(y) = y - \langle y, \alpha \rangle \alpha^\vee, \qquad x \in X, \;\; y \in X^\vee, \;\; \alpha \in \Phi.$$

Let $Q \subset X$ be a sublattice generated by elements of Φ, and P be a lattice defined as

$$P = \{x \in X \otimes \mathbb{Q} | \, \langle x, \, \alpha^\vee \rangle \in \mathbb{Z}, \, \alpha \in \Phi\}.$$

One has $Q \subset P$ and P/Q is a finite group. Let $X_0 \subset X$ be sublattice defined as

$$X_0 = \{x \in X | \, \langle x, y \rangle = 0, \, y \in \Phi^\vee\}.$$

With any reductive Lie group one can associate root datum. Let G be a connected reductive complex Lie group and $H \subset G$ be a maximal torus. We associate to a pair (G, H) a root datum $(X, \Phi, X^\vee, \Phi^\vee)$ as follows. Here X is a free abelian finite rank group of $\mathbb{Q}$-characters of H, $X^\vee = \mathrm{Hom}(\mathbb{C}^*, H)$ is a dual group of one-parameter multiplicative subgroups of H. The pairing $\langle \, , \, \rangle : X \times X^\vee \to \mathbb{Z}$ is defined as

$$\lambda(u(t)) = t^{\langle \lambda, \, u \rangle}, \qquad \lambda \in X, \quad u \in X^\vee, \quad t \in \mathbb{C}^*.$$

Then Φ and $\Phi^\vee$ are finite subsets of X and $X^\vee$ respectively, and there is a bijection $\alpha \mapsto \alpha^\vee$ of Φ onto $\Phi^\vee$.

Adjoint action of H on a Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ defines a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathbb{C} e_\alpha, \qquad \mathfrak{h} = \mathrm{Lie}(H),$$

and thus defines a subset $\Phi \subset X$. Let B be Borel subgroup containing H. There is a unique ordering $>$ of Φ such that $\mathfrak{b} = \mathrm{Lie}(B)$ is generated by $\mathfrak{h} = \mathrm{Lie}(H)$ and e_α with $\alpha > 0$. One fixes a basis $\Pi = \{\alpha_i\}$ of Φ compatible with the ordering of Φ associated to B.

There is a decomposition $G = Z_0 \cdot G'$ where Z_0 is the identity component of the center Z of G and G' is a semisimple group (derived group of G). We have $H = Z_0 \cdot H'$ where H' is a maximal torus of G'. The root datum associated with (G', H') is $(X/X_0, \Phi, Q^\vee, \Phi^\vee)$, with $Q \subset X/X_0$. Given a basis $\{\alpha_i^\vee\}$, $i \in I$ in $Q^\vee$ and a basis $\{\omega_j\}$ $j \in J$ in X, one can choose a basis of representatives of the form $\{\omega_i' = \omega_i + X_0\}$, $i \in I \subset J$ in X/X_0 such that $\{\omega_i'\}$, $i \in I$ form a basis dual to $\{\alpha_i^\vee\}$, $i \in I$.

From now on if not explicitly mentioned $\mathfrak{g}$ be a semisimple Lie algebra. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and $\mathfrak{b}_\pm$ be a pair of opposite Borel subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$. We have a decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ where $\mathfrak{n}_\pm \subset \mathfrak{b}_\pm$ is a nilpotent radical. Denote by Γ the set of vertexes of Dynkin graph associated with the root system of $\mathfrak{g}$. Let $\Pi = \{\alpha_i \in \mathfrak{h}^*,\ i \in \Gamma\}$ be the set of simple roots, $\{\omega_i \in \mathfrak{h}^*,\ i \in \Gamma\}$ be the set of fundamental weights and $\Pi^\vee\{\alpha_i^\vee \in \mathfrak{h},\ i \in \Gamma\}$ with be the set of the co-roots defined by $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$. Let $A = \|a_{ij}\|,\ i, j = 1, \ldots, \ell$ be the Cartan matrix of $\mathfrak{g}$ defined by $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$. Denote R_+ the set of positive roots of $\mathfrak{g}$ and let ρ be a half of the sum of the positive roots $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$. There exist co-prime positive rational numbers $d_1, \ldots, d_\ell$ such that the matrix $\|b_{ij}\| = \|d_i a_{ij}\|$ is symmetric. Define a symmetric bilinear form on $\mathfrak{h}^*$ by $(\alpha_i, \alpha_j) = b_{ij}$. This form defines a non-degenerate pairing $\nu : \mathfrak{h} \widetilde{\rightarrow} \mathfrak{h}^*$ given by $\nu(\alpha_i^\vee) = d_i^{-1} \alpha_i$.

Let W be a Weyl group of root system associated with Lie algebra $\mathfrak{g}$. It is generated by simple reflections $s_1, \ldots, s_\ell$ acting by linear transformations in $\mathfrak{h}^*$:

$$s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \ \ \lambda \in \mathfrak{h}^*. \tag{2.1}$$

Defining relations can be represented as:

$$s_i^2 = 1, \quad (s_i s_j)^{m_{ij}} = 1, \ \ i, j = 1, \ldots, \ell, \tag{2.2}$$

where m_{ij} are equal to

$$m_{ij} = 2, , 3, \ 4, \ 6, \ \infty,$$

for

$$a_{ij} a_{ji} = 0, \ 1, \ 2, \ 3, \ \geq 4,$$

respectively. For any $w \in W$ a reduced word is a sequence of indexes $I_w = (i_1, \ldots, i_{l(w)})$, $i_k \in \Gamma$, of shortest possible length such that $w = s_{i_1} s_{i_2} \cdots s_{i_{l(w)}}$. The integer $l(w)$ is called the length of w. Denote by w_0 the unique element of maximal length in Weyl group and let $m = l(w_0)$. In the following we fix a lift $\dot{w} \in G$, $\mathfrak{g} = \mathrm{Lie}(G)$ of an element $w \in W$ such that $w(u) = \mathrm{Ad}_{\dot{w}} u$, $u \in \mathfrak{g}$. For simple reflections s_i we define

$$\dot{s}_i = e^{e_i} e^{-f_i} e^{e_i},$$

and for $w = s_{i_1} s_{i_2} \cdots s_{i_{l(w)}}$ we take $\dot{w} = \dot{s}_{i_1} \dot{s}_{i_2} \cdots \dot{s}_{i_{l(w)}}$. Thus defined $\dot{w}$ does not depend on the decomposition into the product of simple reflections (see e.g. [K], Lemma 3.8).

Denote by $e_i, f_i, h_i,\ i = 1, \ldots, \ell$ the set of standard generators of a semisimple Lie algebra $\mathfrak{g}$ satisfying the following relations:

$$[h_i, h_j] = 0, \tag{2.3}$$

$$[h_i, e_j] = a_{ij} e_j,\ [h_i, f_j] = -a_{ij} f_j,\ [e_i, f_j] = \delta_{ij} h_i, \tag{2.4}$$

$$(\text{ad } e_i)^{1-a_{ij}} e_j = 0,\ (\text{ad } f_i)^{1-a_{ij}} f_j = 0,\ for\ i \neq j. \tag{2.5}$$

The invariant symmetric bilinear form on $\mathfrak{g}$ is given by

$$(h_i, h_j) = b_{ij} d_i^{-1} d_j^{-1},\ (e_i, f_j) = \delta_{ij} d_i^{-1},\ (e_i, h_j) = (f_i, h_j) = 0.$$

The only example of a non-semisimple reductive Lie algebra that will be considered in this paper is the reductive Lie algebra $\mathfrak{gl}_{\ell+1}$. In this case we explicitly define Lie algebra by generators and relations as follows. Introduce the set of generators

$$\{e_{i,i\pm 1},\ i = 1, \ldots, \ell; \quad e_{k,k},\ k = 1, \ldots, \ell + 1\},$$

of $\mathfrak{gl}_{\ell+1}$. They satisfy the following relations

$$\begin{aligned}
&[e_{i,i}, e_{j,j}] = 0, && [e_{i,i+1}, e_{i+1,i}] = e_{i,i} - e_{i+1,i+1},\\
&[e_{i,i}, e_{i,i+1}] = e_{i,i+1}, && [e_{i+1,i+1}, e_{i,i+1}] = -e_{i,i+1},\\
&[e_{i,i}, e_{i+1,i}] = -e_{i+1,i}, && [e_{i+1,i+1}, e_{i+1,i}] = e_{i+1,i},
\end{aligned} \tag{2.6}$$

$$(\text{ad}_{e_{i,i+1}})^2 e_{j,j+1} = 0, \qquad (\text{ad}_{e_{i+1,i}})^2 e_{j+1,j} = 0, \qquad |i - j| = 1.$$

2.1.2 Whittaker model of principal series representations

Let $\mathcal{U}(\mathfrak{g})$ be a universal enveloping of $\mathfrak{g}$ and V, V' be $\mathcal{U}(\mathfrak{g})$-modules. Modules V and V' are called dual if there exists a non-degenerate pairing $\langle .,. \rangle : V' \times V \to \mathbb{C}$ such that $\langle v', Xv \rangle = -\langle Xv', v \rangle$ for all $v \in V$, $v' \in V'$ and $X \in \mathfrak{g}$. We will assume that the action of the Cartan subalgebra on V, V' is integrated to the action of the Cartan torus.

Let $B_- = N_- H$ and $B_+ = HN_+$ be a pair of opposed Borel subgroups where H is a maximal torus, and $N_\pm$ are opposite maximal unipotent subgroups of G. Characters of $\mathfrak{n}_\pm = \text{Lie}(N_\pm)$ are defined by their values on simple root generators. Let $\chi_\pm : \mathfrak{n}_\pm \to \mathbb{C}$ be the characters of $\mathfrak{n}_\pm$ defined by $\chi_+(e_i) := -1$ and $\chi_-(f_i) := -1$ for all $i = 1, \ldots, \ell$. A vector $\psi_R \in V$ is called a Whittaker vector with respect to χ_+ if

$$e_i \psi_R = -\psi_R\,, \qquad (i = 1, \ldots, \ell), \tag{2.7}$$

and a vector $\psi_L \in V'$ is called a Whittaker vector with respect to χ_- if

$$f_i \psi_L = -\psi_L\,, \qquad (i = 1, \ldots, \ell). \tag{2.8}$$

A Whittaker vector ψ is called cyclic in V if $\mathcal{U}(\mathfrak{g})\psi = V$, and $\mathcal{U}(\mathfrak{g})$-module V is a Whittaker module if it contains a cyclic Whittaker vector. The Whittaker $\mathcal{U}(\mathfrak{g})$-module V admits an infinitesimal character ξ i.e. there exists a homomorphism of the center $\mathcal{Z}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$ $\xi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ such that $zv = \xi(z)v$ for all $z \in \mathcal{Z}(\mathfrak{g})$ and $v \in V$.

Consider the principal series representation $\mathrm{Ind}_{B_-}^{G} \chi_\mu$ of G, induced from the character χ_μ of $B_- = HN_-$ trivial on N_-. It is realized in the space of functions $f \in L^2(G)$ satisfying

$$f(bg) = \chi_\mu(b)f(g). \tag{2.9}$$

The action of G is given by the right action $(g_1 \cdot f)(g_2) = f(g_2 g_1)$. We will be interested in the infinitesimal form $\mathrm{Ind}_{U(\mathfrak{b}_-)}^{U(\mathfrak{g})} \chi_\mu$ of this representation. The action of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is given by the infinitesimal form of the right action

$$(Xf)(g) = \frac{d}{d\epsilon} f(ge^{\epsilon X})|_{\epsilon\to 0} \ . \tag{2.10}$$

Denote V_μ the corresponding $U(\mathfrak{g})$-module.

Let $G(\mathbb{R})$ be a totally split real form of a reductive Lie group G, $\mathfrak{g}_\mathbb{R} = \mathrm{Lie}(G(\mathbb{R}))$ be a corresponding Lie algebra and $N(\mathbb{R})_+ \subset N_+ \cap G(\mathbb{R})$ be a nilpotent subalgebra of $G(\mathbb{R})$. Let $d\mu_{G(\mathbb{R})}$ be a bi-invariant (Haar) measure on $G(\mathbb{R})$. We have the Bruhat decomposition $G(\mathbb{R}) = \coprod_{w\in W} B_-(\mathbb{R})wB_+(\mathbb{R})$. Let $G_0(\mathbb{R}) = B_-(\mathbb{R})N_+(\mathbb{R})$ be a $w = 1$ component in this decomposition. Restriction of the measure $d\mu_{G(\mathbb{R})}$ on $G_0(\mathbb{R})$ up to normalization has the following form [He]:

$$d\mu_{G(\mathbb{R})}(g) = \delta_{B_+(\mathbb{R})}(b)\, d\mu_{B_+(\mathbb{R})}(b) \wedge d\mu_{N_+(\mathbb{R})}(x). \tag{2.11}$$

Here $\delta_{B_+(\mathbb{R})}$ is the modular function on $B_+(\mathbb{R})$. For any $b = ng_0 \in N_+(\mathbb{R})\, H$ it is equal to $\delta_{B_+(\mathbb{R})}(b) = \exp 2\langle \rho, \ln g_0\rangle$.

Let $\mu = i\lambda - \rho$. Consider the following non-degenerate pairing $V_\mu \times V_\mu \to \mathbb{C}$:

$$\langle f_1, f_2\rangle = \int_{N_+(\mathbb{R})} d\mu_{N_+(\mathbb{R})}(x)\, f_1(x)\, \overline{f_2(x)},$$

where $d\mu_{N_+(\mathbb{R})}$ is a restriction of (2.11) on $N_+(\mathbb{R})$. It defines on V_μ a structure of a unitary representation π_λ of $U(\mathfrak{g}_\mathbb{R})$ and we have $\langle f_1, Xf_2\rangle = -\langle Xf_1, f_2\rangle$ for any $X \in \mathfrak{g}_\mathbb{R}$.

We shall consider a slightly more general pairing defined as follows. Note that $N_+(\mathbb{R}) \subset N_+$ is a real non-compact middle dimension subspace. One has a natural holomorphic structure on a Lie algebra $\mathfrak{g} = \mathfrak{g}_\mathbb{R} \otimes \mathbb{C}$ which induces the holomorphic structure on N_+. Consider the space of holomorphic functions on N_+. It is a module with respect to the holomorphic action of a corresponding holomorphic subalgebra of $\mathfrak{g}$. The right-invariant measure $d\mu_{N_+(\mathbb{R})}$ can be extended to a holomorphic top-dimensional form $d\mu_{N_+}^{hol}$ on N_+. Let $C \subset N_+$ be an arbitrary non-compact middle-dimensional submanifold. Consider the following pairing

$$\langle f_1, f_2\rangle_C = \int_C d\mu_{N_+}^{hol}(x)\, f_1(x)\, \overline{f_2(\overline{x})},$$

on the space $\mathcal{S}_C^{hol}(N_+)$ of holomorphic functions on N_+ exponentially decreasing with all its derivatives when restricted to C. This pairing satisfies $\langle f_1, Xf_2\rangle_C = -\langle Xf_1, f_2\rangle_C$ for any holomorphic $X \in \mathfrak{g}$.

2.1.3 Whittaker function as Toda wave function

According to B. Kostant [Ko1],[Ko2] eigenfunctions of $\mathfrak{g}$-Toda chain can be written in terms of the invariant pairing on Whittaker modules as follows

$$\Psi^{\mathfrak{g}}_{\lambda}(x)=e^{-\langle\rho,x\rangle}\langle\psi_L\,,\pi_\lambda(e^{-h_x})\,\psi_R\rangle,\qquad\qquad x\in\mathfrak{h},\tag{2.12}$$

where $h_x:=\sum\limits_{i=1}^{\ell}\langle\omega_i,x\rangle\,h_i$. In the special case of $\mathfrak{g}=\mathfrak{sl}_2$ the function $\Psi^{\mathfrak{g}}_{\lambda}(x)$, $x\in\mathbb{R}$ coincides with the classical Whittaker function. In the following we will use the term $\mathfrak{g}$-Whittaker function for (2.12) (see e.g. [Et]). A slightly different notion of the Whittaker functions was used in [Ja], [Ha].

One can introduce a set of commuting differential operators $\mathcal{H}_k\in\mathrm{Diff}(\mathfrak{h})$, $k=1,\cdots,\ell$ corresponding to a set $\{c_k\}$ of generators of the center $\mathcal{Z}\subset\mathcal{U}(\mathfrak{g})$ as follows:

$$\mathcal{H}_k\Psi^{\mathfrak{g}}_{\lambda}(x)=e^{-\langle\rho,x\rangle}\langle\psi_L\,,\pi_\lambda(e^{-h_x})\,c_k\,\psi_R\rangle.\tag{2.13}$$

Operators $\mathcal{H}_k$ provide a complete set of commuting Hamiltonians of $\mathfrak{g}$-Toda chain [Ko1]. Connection with Toda chains can be seen as follows. Quadratic generator of the center $\mathcal{Z}(\mathfrak{g})$ (Casimir element) is given by

$$c_2=\frac{1}{2}\sum_{i,j=1}^{\ell}c_{ij}h_ih_j+\frac{1}{2}\sum_{\alpha\in R_+}(e_\alpha f_\alpha+f_\alpha e_\alpha),\tag{2.14}$$

where the matrix $\|c_{ij}\|=\|d_id_j(b^{-1})_{ij}\|$ is inverse to the matrix $\|(\alpha_i^\vee,\alpha_j^\vee)\|$. Let $\{\epsilon_i\}$ be an orthogonal bases $(\epsilon_i,\epsilon_j)=\delta_{ij}$ in $\mathfrak{h}$ and $x=\sum_{i=1}^{\ell}x_i\epsilon_i$ be a decomposition of $x\in\mathfrak{h}$ in this bases. Then the projection (2.13) of (2.14) gives the well-known Hamiltonian operator of $\mathfrak{g}$-Toda chain [STS]

$$\mathcal{H}^{\mathfrak{g}}_2=-\frac{1}{2}\sum_{i=1}^{\ell}\frac{\partial^2}{\partial {x_i}^2}+\sum_{i=1}^{\ell}d_ie^{\langle\alpha_i,x\rangle}.\tag{2.15}$$

The eigenfunctions (2.12) of $\mathfrak{g}$-Toda chain are written in an abstract form. To get explicit integral representations we start with representations of matrix elements (2.12) of infinite-dimensional representations in terms of matrix elements of finite-dimensional representations of $\mathcal{U}(\mathfrak{g})$. Let π_i be a set of fundamental representations corresponding to all fundamental weights ω_i of $\mathfrak{g}$ and $\xi^{+/-}_{\omega_i}$ be highest/lowest vectors in these representations such that $\langle\xi^-_{\omega_i}|\xi^+_{\omega_i}\rangle=1$. For highest weight vector $\xi^+_{\omega_i}$ in a fundamental representation V_{ω_i} we have $\dot{s}_i^{-1}\xi^+_{\omega_i}=f_i\xi^+_{\omega_i}$. Consider following matrix elements in fundamental finite-dimensional representations

$$\Delta_{\omega_i,\dot{w}}(g)=\langle\xi^-_{\omega_i}|\,\pi_i(g)\pi_i(\dot{w})|\,\xi^+_{\omega_i}\rangle,\qquad w\in W,\;\;g\in G.\tag{2.16}$$

Lemma 2.1 *The left/right Whittaker vectors defined by (2.7) and (2.8) are given by:*

$$\psi_R(v)=\exp\Big\{-\sum_{i=1}^{\ell}\frac{\Delta_{\omega_i,\dot{s}_i^{-1}}(v)}{\Delta_{\omega_i,1}(v)}\Big\},\tag{2.17}$$

$$\psi_L(v) = \prod_{i=1}^{\ell} (\Delta_{\omega_i, \dot{w}_0^{-1}}(v))^{\imath\langle\lambda,\alpha_i^\vee\rangle - 1} \times \exp\Big\{ \sum_{i=1}^{\ell} \frac{\Delta_{\omega_i, \dot{w}_0^{-1}\dot{s}_i^{-1}}(v)}{\Delta_{\omega_i, \dot{w}_0^{-1}}(v)} \Big\}, \tag{2.18}$$

The proof in given in Part II, Section 3.2.

Proposition 2.1 *Common eigenfunctions (2.12) of $\mathfrak{g}$-Toda chain can be represented in the following integral form:*

$$\Psi^{\mathfrak{g}}_\lambda(x) = e^{\imath\langle\lambda,x\rangle} \int_C d\mu^{hol}_{N_+}(v) \prod_{i=1}^{\ell} (\Delta_{\omega_i, \dot{w}_0^{-1}}(v))^{\imath\langle\lambda,\alpha_i^\vee\rangle - 1} \times \tag{2.19}$$
$$\times \exp\Big\{ \sum_{i=1}^{\ell} \left(\frac{\Delta_{\omega_i, \dot{w}_0^{-1}\dot{s}_i^{-1}}(v)}{\Delta_{\omega_i, \dot{w}_0^{-1}}(v)} - e^{\langle\alpha_i,x\rangle} \frac{\Delta_{\omega_i, \dot{s}_i^{-1}}(v)}{\Delta_{\omega_i, 1}(v)} \right) \Big\}.$$

Here $C \subset N_+$ is a middle-dimensional non-compact cycle such that the integrand decreases exponentially at the boundaries and infinities. The measure of the integration is the restriction on C of the holomorphic continuation $d\mu^{hol}_{N_+}$ of the right-invariant measure $d\mu_{N_+(\mathbb{R})}$ on $N_+(\mathbb{R})$.

The first example of this type of integral representation for $\mathfrak{gl}_n$-Whittaker function was considered in [GKMMMO]. Its generalization given above is straightforward. The proof of the Proposition is given in Part II, Section 3.2.

The expression (2.19) for a Whittaker function is much more detailed then (2.12) but does not yet provide explicit integral representation. To obtain explicit integral representations of Whittaker functions one should choose a parameterization of N_+ (or an open part of it) and express the measure $d\mu^{hol}_{N_+}$ and various matrix elements entering (2.19) in terms of the coordinates on N_+. Natural choice would be a factorized representation of the elements of an open part of a maximal unipotent subgroup of an arbitrary Lie group [Lu] (see also [BZ], [FZ]). For each $i = 1, \ldots, \ell$ let $X_i(t) = \exp\{te_i\}$ be a one-parameter subgroup in N_+. Pick a decomposition of the longest element w_0 in the Weyl group W corresponding to a reduced word $I_{w_0} = (i_1, \ldots, i_m)$, $l(w_0) = m = \dim N_+$. Then the following map

$$\mathbb{C}^m \longrightarrow N_+^{(0)}, \qquad (t_1, \ldots, t_m) \longmapsto v(t_1, \cdots, t_m) = X_{i_1}(t_1) \cdots X_{i_m}(t_m), \tag{2.20}$$

is a birational isomorphism. This provides a parametrization of an open part $N_+^{(0)}$ of N_+. Parametrizations corresponding to different choices of the reduced word I_{w_0} are related by birational transformations described explicitly by G. Lustzig [Lu]. The right-invariant measure has the following description in the factorized representation.

Lemma 2.2 *The right-invariant measure $d\mu^{hol}_{N_+}$ in the factorized parametrization is given by:*

$$d\mu^{hol}_{N_+}(v) = \prod_{i=1}^{\ell} \Delta_{\omega_i, \dot{w}_0^{-1}}(v) \bigwedge_{k=1}^{m} \frac{dt_k}{t_k}. \tag{2.21}$$

The proof is given in Part II, Section 3.1.

Thus the problem of finding explicit integral representations of Whittaker functions in the factorized parametrization (2.20) is reduced to a calculation of the matrix elements of finite-dimensional representations of $\mathfrak{g}$ in this parametrization. In the following we provide explicit expressions for finite-dimensional matrix elements for classical Lie groups and give corresponding integral representations of Whittaker functions. Let us stress however that thus obtained integral representation for $\mathfrak{g} = \mathfrak{gl}_{\ell+1}$ does not coincide with Givental representation [Gi]. Note that for classical series of Lie algebras the factorized parametrization (2.20) has a recursive structure over the rank ℓ reflecting the recursive structure of the reduced decomposition of $w_0 \in W$. This recursive structure is not translated, however, into a simple recursive structure of the infinite-dimensional matrix element in the factorized parametrization and does not reproduce the recursive structure of the Givental integral representation.

In [GKLO] a modification of the factorized parametrization (2.20) for $\mathfrak{g} = \mathfrak{gl}_{\ell+1}$ was proposed and it was shown that the integral representation (2.19) in this parametrization exactly reproduces the Givental integral representation of $\mathfrak{gl}_{\ell+1}$-Whittaker functions. In particular for this parametrization the recursive structure of the reduced decomposition of $w_0 \in W$ directly translates into the recursive structure of the integral representation of the corresponding Whittaker function.

Below we generalize the results of [GKLO] to all classical series of Lie algebras. We propose a modification of factorized parametrization (2.20) based on a particular realization of maximal unipotent subgroups $N_+ \subset G$ of classical Lie groups as explicitly defined subgroups of the maximal unipotent subgroups of general linear groups. For any classical simple Lie group, the maximal unipotent subgroup can be realized as a subgroup of a group of upper-triangular matrices of appropriate size with units on diagonal (see e.g. [DS]). The corresponding subset of upper-triangular matrices for classical Lie group can be describe explicitly. We define a parametrization of maximal unipotent subgroups of classical Lie groups by constructing a particular form of the parametrization of the corresponding subset of upper-triangular matrices. Using this parametrization we derive explicit integral representations of Whittaker functions associated with all classical groups and demonstrate that these integral representations have all characteristic properties of the Givental integral representation for $\mathfrak{gl}_{\ell+1}$-Whittaker functions. In particular the recursive structure of Whittaker functions is explicit in this new parametrization.

2.2 Integral representations of $\mathfrak{gl}_{\ell+1}$- and $\mathfrak{sl}_{\ell+1}$-Toda chain eigenfunctions

In this section we recall the construction of integral representations of $\mathfrak{gl}_{\ell+1}$- and $\mathfrak{sl}_{\ell+1}$-Toda eigenfuctions using factorized parametrization (2.20) of a maximal unipotent subgroup $N_+ \subset GL(\ell+1)$ and its modification introduced in [GKLO]. The second parametrization leads to an integral representation obtained earlier by Givental [Gi] using different approach. In the following these constructions will be generalized to $\mathfrak{g}$-Toda theory for arbitrary classical Lie algebras $\mathfrak{g}$.

We start with the case of the reductive Lie algebra $\mathfrak{gl}_{\ell+1}$. Let $(\epsilon_1, \ldots, \epsilon_{\ell+1})$ be an orthogonal basis in $\mathbb{R}^{\ell+1}$, $(\epsilon_i, \epsilon_j) = \delta_{ij}$. Roots and fundamental wights of $\mathfrak{gl}_{\ell+1}$ considered as vectors in $\mathbb{R}^{\ell+1}$ are given by:

$$\alpha_i = \epsilon_{i+1} - \epsilon_i, \ \ i = 1, \ldots, \ell, \qquad \omega_j = \epsilon_j, \ \ j = 1, \ldots, (\ell+1). \tag{2.22}$$

Coroots $\alpha_i^\vee$ can be identified with the corresponding roots α_i with respect to the pairing in $\mathbb{R}^{\ell+1}$. To this root/weight system one associates $\mathfrak{gl}_{\ell+1}$-Toda quantum integrable system having a set of $(\ell+1)$ mutually commuting functionally independent quantum Hamiltonians $H_k^{\mathfrak{gl}_{\ell+1}}$, $k = 1, \cdots, (\ell+1)$. We are interested in the explicit integral representations for common eigenfunctions of the full set of quantum Hamiltonian operators for $\mathfrak{gl}_{\ell+1}$. For instance linear and quadratic quantum Hamiltonians of $\mathfrak{gl}_{\ell+1}$-Toda chain are given by

$$\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}} = -\imath \sum_{i=1}^{\ell+1} \frac{\partial}{\partial x_i}, \tag{2.23}$$

$$\mathcal{H}_2^{\mathfrak{gl}_{\ell+1}} = -\frac{1}{2} \sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial {x_i}^2} + \sum_{i=1}^{\ell} e^{x_{i+1}-x_i}, \tag{2.24}$$

and the eigenfunction should satisfy the following equation

$$\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}}(x) \ \Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\cdots,\lambda_{\ell+1}}(x_1, \ldots, x_{\ell+1}) = \sum_{i=1}^{\ell+1} \lambda_i \ \Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\cdots,\lambda_{\ell+1}}(x_1, \ldots, x_{\ell+1}), \tag{2.25}$$

$$\mathcal{H}_2^{\mathfrak{gl}_{\ell+1}}(x) \ \Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\cdots,\lambda_{\ell+1}}(x_1, \ldots, x_{\ell+1}) = \frac{1}{2} \sum_{i=1}^{\ell+1} \lambda_i^2 \ \Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\cdots,\lambda_{\ell+1}}(x_1, \ldots, x_{\ell+1}). \tag{2.26}$$

Common eigenfunction of the quantum Hamiltonians has the following representation as a matrix element

$$\Psi_\lambda^{\mathfrak{gl}_{\ell+1}}(x) = e^{-\sum x_i\rho_i} \langle \psi_L \,, \pi_\lambda(e^{-\sum x_i E_{i,i}}) \, \psi_R \rangle, \tag{2.27}$$

where $\rho_i = \frac{1}{2}(\ell - 2i + 2)$ are the components of ρ in the standard basis $\{\epsilon_i\}$ in $\mathbb{R}^{\ell+1}$.

The construction for the semisimple Lie algebra $\mathfrak{sl}_{\ell+1}$ is quite similar to that for reductive Lie algebra $\mathfrak{gl}_{\ell+1}$. The roots and fundamental wights for semisimple Lie algebra $\mathfrak{sl}_{\ell+1}$ can be written in the following form (see [Bou]):

$$\alpha_i = \epsilon_{i+1} - \epsilon_i, \qquad \omega_i = -(\epsilon_1 + \ldots + \epsilon_i) + \frac{i}{\ell+1}(\epsilon_1 + \ldots + \epsilon_{\ell+1}), \tag{2.28}$$

for $i = 1, \ldots, \ell$. This representation of the A_ℓ root/weight system can be obtained from the root/weight system of the reductive Lie algebra $\mathfrak{gl}_{\ell+1}$ as follows. Let us pick an orthogonal basis of fundamental weights of $\mathfrak{gl}_{\ell+1}$:

$$\omega_i' = -\epsilon_1 - \ldots - \epsilon_i,$$

such that $\langle \omega_i', \, \alpha_j^\vee \rangle = \delta_{ij}$ for $i, j = 1, \ldots \ell$, and $\langle \omega_{\ell+1}', \, \alpha_j^\vee \rangle = 0$ for $j = 1, \ldots \ell$. Then $\omega_{\ell+1}'$ can be identified as a generator of X_0. Introducing

$$\omega_i = \omega_i' - \frac{i}{\ell+1} \omega_{\ell+1}',$$

one readily obtains the set (2.28) of fundamental weights for $\mathfrak{sl}_{\ell+1}$.

To this root/weight system one associates $\mathfrak{sl}_{\ell+1}$-Toda quantum integrable system possessing a set of ℓ mutually commuting functionally independent Hamiltonians $\mathcal{H}_k^{\mathfrak{sl}_{\ell+1}}$, $k = 1, \cdots, \ell$. It is convenient however to consider $\mathfrak{sl}_{\ell+1}$-Toda chain Hamiltonians as a subset $\mathcal{H}_k^{\mathfrak{gl}_{\ell+1}}$, $k = 2, \cdots, (\ell+1)$ of $\mathfrak{gl}_{\ell+1}$-Toda chain Hamiltonians acting on the kernel of the linear Hamiltonian $\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}}$. For instance the eigenfunction of a quadratic quantum Hamiltonian of $\mathfrak{sl}_{\ell+1}$-Toda chain should satisfy the equation

$$\mathcal{H}_2^{\mathfrak{sl}_{\ell+1}} \Psi^{\mathfrak{sl}_{\ell+1}}_{\lambda_1,\cdots,\lambda_{\ell+1}}(x_1, \dots, x_{\ell+1}) =$$

$$= (-\frac{1}{2}\sum_{i=1}^{\ell+1}\frac{\partial^2}{\partial {x_i}^2} + \sum_{i=1}^{\ell} e^{x_{i+1}-x_i}))\Psi^{\mathfrak{sl}_{\ell+1}}_{\lambda_1,\cdots,\lambda_{\ell+1}}(x_1, \dots, x_{\ell+1}) = \tag{2.29}$$

$$= \frac{1}{2}\sum_{i=1}^{\ell+1}\lambda_i^2 \; \Psi^{\mathfrak{sl}_{\ell+1}}_{\lambda_1,\cdots,\lambda_{\ell+1}}(x_1, \dots, x_{\ell+1}),$$

with an additional constraint $\lambda_1 + \ldots + \lambda_{\ell+1} = 0$. The eigenfunctions for $\mathfrak{sl}_{\ell+1}$-Toda chain can be also written using a reduced set of variable

$$\Psi^{\mathfrak{sl}_{\ell+1}}_{\nu_1,\dots,\nu_\ell}(y_1, \dots, y_\ell) \equiv \Psi^{\mathfrak{sl}_{\ell+1}}_{\lambda_1,\dots,\lambda_{\ell+1}}(x_1, \dots, x_{\ell+1}) \qquad \nu_i = \lambda_{i+1} - \lambda_i, \quad y_i = x_{i+1} - x_i \tag{2.30}$$

Note that without imposing the constraint $\lambda_1 + \ldots + \lambda_{\ell+1} = 0$, the eigenfunctions of $\mathfrak{sl}_{\ell+1}$-Toda chain can be expressed through eigenfunctions of $\mathfrak{gl}_{\ell+1}$-Toda theory in the following simple way

$$\Psi^{\mathfrak{sl}_{\ell+1}}_{\nu_1,\dots,\nu_\ell}(y_1, \dots, y_\ell) = \exp\Big\{ -\frac{\imath}{\ell+1}\sum_{i=1}^{\ell+1}\lambda_i \cdot \sum_{i=1}^{\ell+1} x_i \Big\} \cdot \Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\dots,\lambda_{\ell+1}}(x_1, \dots, x_{\ell+1}), \tag{2.31}$$

where we use notations (2.30). In the following we will consider mostly $\mathfrak{gl}_{\ell+1}$-Toda chain eigenfunctions making comments on the corresponding modifications for $\mathfrak{sl}_{\ell+1}$ case (we will mostly use the non-reduced form $\Psi^{\mathfrak{sl}_{\ell+1}}_\lambda(x)$).

2.2.1 $\mathfrak{gl}_{\ell+1}$-Whittaker function: factorized parametrization

To make the integral representation (2.19) for $\mathfrak{gl}_{\ell+1}$-Whitaker functions explicit one should pick a particular parametrization of $N_+ \subset GL(\ell+1)$. Let w_0 be an element of maximal length of the Weyl group $W = S_{\ell+1}$ of $\mathfrak{gl}_{\ell+1}$. Consider the reduced decomposition of w_0 corresponding to the following reduced word I_ℓ

$$I_\ell = (i_1, i_2, \dots, i_m) := (1, 21, 321, \dots, (\ell \dots 321)).$$

The reduced word I_ℓ has an obvious recursive structure: $I_{\ell+1} = I_\ell \sqcup (\ell+1 \dots 321)$. Thus the corresponding parametrization of unipotent elements $v^{(\ell)}$ in an open part $N_+^{(0)}$ of N_+ can be written in a recursive form:

$$v^{(\ell)} = v^{(\ell-1)} \cdot \mathfrak{X}^{\ell}_{\ell-1}, \tag{2.32}$$

where

$$\mathfrak{X}^{A_\ell}_{A_{\ell-1}} = X_\ell(y_{\ell,1})\cdots X_2(y_{2,\ell-1})X_1(y_{1,\ell}), \tag{2.33}$$

and $X_i(y) = \exp(ye_i)$. Parameters y_{ik} of one-parametric subgroups will be called factorization parameters. The action of Lie algebra $\mathfrak{gl}_{\ell+1}$ on G/B_- considered at the beginning of the previous section defines an action of the Lie algebra on the space of functions V_μ restricted to $N_+^{(0)}$.

Proposition 2.2 *The following differential operators define a realization of the representation π_λ of $\mathcal{U}(\mathfrak{gl}_{\ell+1})$ in V_μ in terms of factorized parametrization (2.32), (2.33):*

$$\begin{aligned}
E_{i,i} &= \mu_i - \sum_{l=1}^{\ell+1-i} y_{i,l}\frac{\partial}{\partial y_{i,l}} + \sum_{l=1}^{\ell+2-i} y_{i-1,l}\frac{\partial}{\partial y_{i-1,l}},\\
E_{i,i+1} &= \sum_{k=0}^{i-1}\prod_{s=0}^{k}\frac{y_{i-s,\ell+2-i}}{y_{i+1-s,\ell+1-i}}\frac{\partial}{\partial y_{i-k,\ell+1-i}} - \prod_{s=0}^{k}\frac{y_{i-(s+1),\ell+2-i}}{y_{i-s,\ell+1-i}}\frac{\partial}{\partial y_{i-(k+1),\ell+2-i}},
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
E_{i+1,i} &= \sum_{k=1}^{\ell}\Big[(\mu_{i+1}-\mu_i)y_{i,k+1-i} - y_{i,k+1-i}\Big(y_{i,k+1-i}\frac{\partial}{\partial y_{i,k+1-i}} - y_{i+1,k-i}\frac{\partial}{\partial y_{i+1,k-i}}\Big) +\\
&+ y_{i,k+1-i}\sum_{s=1}^{k-1}\Big(y_{i-1,s+2-i}\frac{\partial}{\partial y_{i-1,s+2-i}} - 2y_{i,s+1-i}\frac{\partial}{\partial y_{i,s+1-i}} + y_{i+1,s-i}\frac{\partial}{\partial y_{i+1,s-i}}\Big)\Big],
\end{aligned}$$

where $E_{i,j} = \pi_\lambda(e_{i,j})$, $\mu_k = \imath\lambda_k - \rho_k$ and $\rho_k = \frac{1}{2}(\ell - 2k + 2)$.

Proof. The proof is given in Part II, Section 3.4.1.

The calculation of matrix elements entering the integral (2.19) in the factorized parametrization (2.32), (2.33) can be done following [BZ] and [FZ] (see Section 3.3 for details). Another, more straightforward approach is to find left and right Whittaker vectors solving the equations (2.7)-(2.8) directly. In the following we will use the convention: $\sum_{i=k}^{j} = 0$, when $k > j$ and $\prod_{i=k}^{j} = 1$, when $k > j$.

Lemma 2.3 *The following expressions for the left/right Whittaker vectors in terms of factorization parameters hold:*

$$\psi_R(y) = \exp\Big\{-\sum_{i=1}^{\ell}\sum_{n=1}^{\ell+1-i} y_{i,n}\Big\}, \tag{2.35}$$

$$\begin{aligned}
\psi_L(y) = \prod_{i=1}^{\ell}\Big(\prod_{k=1}^{i}\prod_{n=i+1-k}^{\ell} y_{k,n}\Big)^{(\mu_{i+1}-\mu_i)} \times\\
\times \exp\Big\{-\sum_{k=1}^{\ell}\frac{1}{y_{\ell+1-k,k}}\Big(1 + \sum_{n=1}^{\ell-k}\prod_{i=1}^{n}\frac{y_{\ell+1-k-i,k+1}}{y_{\ell+1-k-i,k}}\Big)\Big\}.
\end{aligned} \tag{2.36}$$

Using (2.21) we have the following expression for $\mathfrak{gl}_{\ell+1}$-Whittaker function in the factorized parametrization.

Theorem 2.1 *Eigenfunctions of the $\mathfrak{gl}_{\ell+1}$-Toda chain (2.27) admit the integral representation:*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\ldots,\lambda_{\ell+1}}(x_1,\ldots,x_{\ell+1})=e^{\imath\sum_{k=1}^{\ell+1}\lambda_k x_k}\int\limits_C\bigwedge_{i=1}^{\ell}\bigwedge_{n=1}^{\ell+1-i}\frac{dy_{i,n}}{y_{i,n}}\prod_{i=1}^{\ell}\Big(\prod_{k=1}^{i}\prod_{n=i+1-k}^{\ell}y_{k,n}\Big)^{\imath(\lambda_{i+1}-\lambda_i)}$$

$$\exp\Big\{-\Big(\sum_{k=1}^{\ell}\frac{1}{y_{\ell+1-k,k}}\Big(1+\sum_{n=1}^{\ell-k}\prod_{i=1}^{n}\frac{y_{\ell+1-k-i,k+1}}{y_{\ell+1-k-i,k}}\Big)\;+\;\sum_{i=1}^{\ell}e^{x_{i+1}-x_i}\sum_{n=1}^{\ell+1-i}y_{i,n}\Big)\Big\}.\qquad(2.37)$$

Here $C\subset N_+$ is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at the boundaries and infinities. In particular one can take $C=\mathbb{R}_+^{\ell(\ell+1)/2}$.

The proof is given in Part II, Section 3.3.1.

2.2.2 $\mathfrak{gl}_{\ell+1}$-Whittaker function: modified factorized parametrization

Now we consider a modification of the factorized parametrization (2.32), (2.33) leading to the Givental integral representation of $\mathfrak{gl}_{\ell+1}$-Whittaker function. This modified factorized parametrization was first introduced in [GKLO]. There is an important difference between factorized and modified factorized parametrizations. Note that the parametrization (2.32), (2.33) is defined in terms of group elements of N_+. To define a modified factorized parametrization of N_+ we shall consider the image of a group element in a faithful finite-dimensional representation of G. In the case of $\mathfrak{gl}_{\ell+1}$ and $\mathfrak{sl}_{\ell+1}$ we use a tautological representation $\pi_{\ell+1}:\mathfrak{gl}_{\ell+1}\to End(\mathbb{C}^{\ell+1})$. Let $\epsilon_{i,j}$ be a set of elementary $(\ell+1)\times(\ell+1)$-matrices with units at (i,j)-places and zeros, otherwise. Consider the following set of diagonal matrices

$$U_k=\sum_{i=1}^{k}e^{-x_{k,i}}\epsilon_{i,i}+\sum_{i=k+1}^{N}\epsilon_{i,i}.$$

Define the following upper-triangular deformation of U_k

$$\tilde U_k=\sum_{i=1}^{k}e^{-x_{k,i}}\epsilon_{i,i}+\sum_{i=k+1}^{N}\epsilon_{i,i}+\sum_{i=1}^{k-1}e^{-x_{k-1,i}}\epsilon_{i,i+1}.\qquad(2.38)$$

The modified factorized parametrization of N_+ is then defined as follows.

Theorem 2.2 *i) The image of any generic unipotent element $v\in N_+$ in the tautological representation $\pi_{\ell+1}:\mathfrak{gl}_{\ell+1}\to End(\mathbb{C}_{\ell+1})$ can be represented in the form*

$$\pi_{\ell+1}(v)=\tilde U_2U_2^{-1}\tilde U_3U_3^{-1}\cdots\tilde U_{N-1}U_{N-1}^{-1}\tilde U_N,\qquad(2.39)$$

where we assume that $x_{\ell+1,i}=0,\quad i=1,\ldots,\ell+1$.

ii) This defines a parametrization of an open part $N_+^{(0)}$ of N_+.

Proof. Let $v(y)$ be elements of N_+ parametrized according to (2.32), (2.33). Let us now change the variables in the following way:

$$y_{i,n} = e^{x_{n+i,i+1}-x_{n+i-1,i}}, \tag{2.40}$$

where $x_{\ell+1,n}=0,\ \ n=1,\dots,\ell+1$ are assumed. By elementary operations it is easy to check that after the change of variables, the image $\pi_{\ell+1}(v)$ of v defined by (2.32), (2.33) transforms into (2.39). Taking into account that the change of variables (2.40) is invertible we get a parametrization of $N_+^{(0)}\subset N_+$ □

Considering the image of the factorized group element (2.32), (2.33) in the tautological representation $\pi_{\ell+1}$ we obtain the following relations between factorization and modified factorization parameters:

$$y_{i,n} = e^{x_{n+i,i+1}-x_{n+i-1,i}}, \tag{2.41}$$

where $x_{\ell+1,n}=0,\ \ n=1,\dots,\ell+1$ are assumed. Applying the change of variables (2.41) to the expressions in Proposition 2.2 one obtains the realization in the modified factorized parametrization.

Proposition 2.3 *The following differential operators define a realization of representation π_λ of $\mathfrak{gl}_{\ell+1}$ in V_μ in terms of modified factorized parametrization (2.39), (2.41) of N_+:*

$$\begin{aligned}
E_{i,i} &= \mu_i-\sum_{k=1}^{i-1}\frac{\partial}{\partial x_{\ell+1+k-i,k}}+\sum_{k=i}^{\ell}\frac{\partial}{\partial x_{k,i}},\\
E_{i,i+1} &= -\sum_{k=1}^{i}\left(\sum_{s=k}^{i}e^{x_{\ell+1+s-i,s}-x_{\ell+s-i,s}}\right)\left(\frac{\partial}{\partial x_{\ell+k-i,k}}-\frac{\partial}{\partial x_{\ell+k-i,k-1}}\right),\\
E_{i+1,i} &= -\sum_{k=1}^{\ell}e^{(x_{k,i}-x_{k+1,i+1})}\left(\mu_i-\mu_{i+1}+\sum_{s=1}^{k}\left(\frac{\partial}{\partial x_{s,i+1}}-\frac{\partial}{\partial x_{s,i}}\right)\right),
\end{aligned} \tag{2.42}$$

where $E_{i,j}=\pi_\lambda(e_{i,j})$, $\mu_k=\imath\lambda_k-\rho_k$, and $\rho_k=\frac{1}{2}(\ell+2k-2)$. We let $x_{\ell+1,k}=0$, $(k=1,\dots,\ell+1)$.

This realization of the principal series representation of $\mathfrak{gl}_{\ell+1}$ by differential operators is based on a particular parametrization of the maximal unipotent subgroup N_+ entering the Gauss decomposition of the group G and was inspired by the Givental integral formula. In [GKLO] we coined the term Gauss-Givental representation for this realization of the principal series representation. Applying the change of variables (2.41) to the expressions in Lemma 2.3 one obtains Whitaker vectors in the modified factorized parametrization.

Lemma 2.4 *The following expressions for the left/right Whittaker vectors hold:*

$$\begin{aligned}
\psi_R(x) &= \exp\Big\{-\sum_{i=1}^{\ell}\sum_{n=1}^{\ell+1-i}e^{x_{n+i,i+1}-x_{n+i-1,i}}\Big\}, \qquad (2.43)\\
\psi_L(x) &= \exp\Big\{\sum_{k=1}^{\ell}\sum_{i=1}^{k}(\mu_{k+1}-\mu_k)x_{k,i}\Big\}\exp\Big\{-\sum_{i=1}^{\ell}\sum_{k=1}^{\ell+1-i}e^{x_{k+i-1,k}-x_{k+i,k}}\Big\},
\end{aligned}$$

where we set $x_{\ell+1,i}=0, \quad i=1,\dots,\ell+1$.

Now we are ready to write down the integral representation of the pairing (2.12) using the modified factorized representation. Going from (2.12) to (2.19), (2.37) we chose to act by an element of the Cartan torus to the right in (2.12). Different choice (for example the action to the left) leads to the integrand that differs by total derivative. The choice made in (2.19), (2.37) is not the most symmetric one. One of the special features of Gauss-Givental representation is that up to a simple exponential term in $\psi_L(x)$ the left and right Whittaker vectors are very similar (compare in this respect with the case of factorized parameterization (2.17), (2.18)). We would like to maintain this symmetry in the integrand of the integral representation. Let us represent the Cartan group element in the following way:

$$e^H=e^{H_L}e^{H_R},$$

where

$$e^H = e^{-\sum x_iE_{i,i}}=\exp\Big\{\sum_{i=1}^{\ell}x_{\ell+1,i}\Big(\mu_i-\sum_{k=1}^{i-1}\frac{\partial}{\partial x_{\ell+1+k-i,k}}+\sum_{k=i}^{\ell}\frac{\partial}{\partial x_{k,i}}\Big)\Big\}, \tag{2.44}$$

$$e^{H_L} = \exp\Big\{\sum_{i=1}^{\ell+1}x_{\ell+1,i}\sum_{k=i}^{\ell}\frac{\partial}{\partial x_{k,i}}\Big\}, \tag{2.45}$$

$$e^{H_R} = \exp\{\sum_{i=1}^{\ell+1}x_{\ell+1,i}\mu_i-\sum_{i=1}^{\ell+1}x_{\ell+1,i}\sum_{k=1}^{i-1}\frac{\partial}{\partial x_{\ell+1+k-i,k}}\}. \tag{2.46}$$

In the calculation of the matrix element we will chose the differential operator H_L acting on the left vector and H_R acting on the right vector in (2.12). This way we obtain the following integral formula for eigenfunctions of the $\mathfrak{gl}_{\ell+1}$-Toda chain.

Theorem 2.3 *Eigenfunctions of the* $\mathfrak{gl}_{\ell+1}$*-Toda chain (2.27) admit the integral representation:*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\dots,\lambda_{\ell+1}}(x_1,\dots,x_{\ell+1})=\int_C\bigwedge_{k=1}^{\ell}\bigwedge_{i=1}^{k}dx_{k,i}\;e^{\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(x)}, \tag{2.47}$$

where the function $\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(x)$ *is given by*

$$\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(x)=\imath\sum_{k=1}^{\ell+1}\lambda_k\Big(\sum_{i=1}^{k}x_{k,i}-\sum_{i=1}^{k-1}x_{k-1,i}\Big)-\sum_{k=1}^{\ell}\sum_{i=1}^{k-1}\Big(e^{x_{k-1,i}-x_{k,i}}+e^{x_{k,i+1}-x_{k-1,i}}\Big). \tag{2.48}$$

Here $x_i=-x_{\ell+1,i},\;\; i=1,\dots,\ell+1$ *and* $C\subset N_+$ *is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at the boundaries and at infinities. In particular* C *can be chosen to be* $C=\mathbb{R}^{\frac{(\ell+1)\ell}{2}}$.

Corollary 2.1 *The $\mathfrak{sl}_{\ell+1}$-Whittaker function has the following integral representation:*

$$\Psi^{\mathfrak{sl}_{\ell+1}}_{\lambda_2-\lambda_1,\ldots,\lambda_{\ell+1}-\lambda_\ell}(x_2-x_1,\ldots,x_{\ell+1}-x_\ell)= \tag{2.49}$$

$$=\exp\Big\{\frac{\imath}{\ell+1}\sum_{i=1}^{\ell+1}x_i\Big(\ell\lambda_i-\sum_{j\neq i}\lambda_j\Big)\Big\}\ \int_C\bigwedge_{k=1}^{\ell}\bigwedge_{i=1}^{k}dx_{k,i}\ e^{\mathcal{F}^{\mathfrak{sl}_{\ell+1}}(x)},$$

where

$$\mathcal{F}^{\mathfrak{sl}_{\ell+1}}(x)=\imath\sum_{k=1}^{\ell}(\lambda_{k+1}-\lambda_k)\sum_{i=1}^{k}(x_k-x_{k,i})- \tag{2.50}$$

$$-\sum_{k=1}^{\ell}\sum_{i=1}^{k-1}\Big(e^{x_{k-1,i}-x_{k,i}}+e^{x_{k,i+1}-x_{k-1,i}}\Big).$$

This integral representation of the $\mathfrak{gl}_{\ell+1}$-Toda chain eigenfunctions was first obtained by A. Givental in his study of quantum cohomology of the $\mathfrak{gl}_{\ell+1}$ flag manifold [Gi] (see also [JK]). The description of the Givental integral formula in terms of the matrix element (2.27) was first obtained in [GKLO].

The function $\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(x)$ allows simple description in terms of the following diagram introduced by A. Givental

$$\begin{array}{ccccccccc}
x_{\ell,1} & & & & & & & & \\
\downarrow & & & & & & & & \\
x_{\ell-1,1} & \longrightarrow & x_{\ell,2} & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\vdots & & \ddots & & \ddots & & & & \\
\downarrow & & & & & & & & \\
x_{21} & \longrightarrow & \ddots & & \ddots & \longrightarrow & x_{\ell,\ell-1} & & \\
\downarrow & & \downarrow & & & & \downarrow & & \\
x_{11} & \longrightarrow & x_{22} & \longrightarrow & \cdots & \longrightarrow & x_{\ell-1,\ell-1} & \longrightarrow & x_{\ell,\ell}
\end{array} \tag{2.51}$$

We assign variables $x_{k,i}$ to vertexes (k,i) and functions e^{y-x} to arrows $(x\longrightarrow y)$ of the diagram (2.51). Then the potential function $\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(x)$ (2.48) at zero spectrum $\lambda_i=0$ is given by the sum of the functions assigned to all arrows.

As it was demonstrated in the Theorem 2.2 variables $\{x_{k,i}\}$ provide a parametrization of an open part $N_+^{(0)}$ of the flag manifold $X = SL(\ell+1,\mathbb{C})/B$. The non-compact manifold $N_+^{(0)}$ has a natural action of the torus $T^{l(w_0)}$ and can be compactified to a (singular) toric variety. The set of the monomial relations defining this compactification can be described as follows. Introduce new variables

$$a_{k,i} = e^{x_{k,i}-x_{k+1,i}}, \quad b_{k,i} = e^{x_{k+1,i+1}-x_{k,i}}, \qquad 1 \le k \le \ell,\ 1 \le i \le k,$$

assigned to arrows of the diagram (2.51). Then the following defining relations hold

$$\begin{gathered} a_{k,i}\cdot b_{k,i} \;=\; b_{k+1,i}\cdot a_{k+1,i+1}, \qquad 1 \le k < \ell,\ 1 \le i \le k, \\ a_{\ell,i}\cdot b_{\ell,i} = e^{x_{\ell,i+1}-x_{\ell,i}}. \end{gathered} \tag{2.52}$$

They can be interpreted as relations between various compositions of elementary paths having the same initial and final vertexes. The set of relations between more general paths (following from (2.52)) provides a toric embedding of the degeneration of flag manifold (see [BCFKS] for details).

2.2.3 Relation with $\widehat{\mathfrak{gl}}_{\ell+1}$-Toda chain Baxter $\mathcal{Q}$-operator

Integral representation (2.47), (2.48) of $\mathfrak{gl}_{\ell+1}$-Whittaker function has a recursive structure over the rank ℓ of the Lie algebra. Indeed the integral representation can be rewritten in the following form

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\ldots,\lambda_{\ell+1}}(\underline{x}_{\ell+1}) = \int_C \bigwedge_{k=1}^{\ell}\bigwedge_{i=1}^{k} dx_{k,i}\ \prod_{k=1}^{\ell+1} Q^{\mathfrak{gl}_k}_{\mathfrak{gl}_{k-1}}(\underline{x}_k;\underline{x}_{k-1};\lambda_k), \tag{2.53}$$

where

$$Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_k}(\underline{x}_{k+1};\underline{x}_k;\lambda_{k+1}) = \tag{2.54}$$

$$= \exp\left\{\imath\lambda_{k+1}\Big(\sum_{i=1}^{k+1} x_{k+1,i} - \sum_{i=1}^{k} x_{k,i}\Big) - \sum_{i=1}^{k}\Big(e^{x_{k,i}-x_{k+1,i}} + e^{x_{k+1,i+1}-x_{k,i}}\Big)\right\}.$$

Here we denote $\underline{x}_k = (x_{k,1},\ldots,x_{k,k})$ and assume that $Q^{\mathfrak{gl}_1}_{\mathfrak{gl}_0} = e^{\imath\lambda_1 x_{1,1}}$.

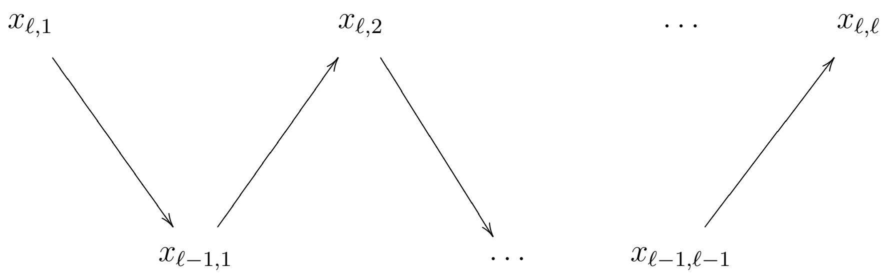
Let us chose linear coordinates $\underline{x}_k = (x_{k,1},\ldots,x_{k,k})$ in $\mathbb{C}^k$. Let C_k be a non-compact middle-dimensional submanifold in $\mathbb{C}^k$ such that (2.54) as a function of $\underline{x}_k$ decreases exponentially at possible boundaries and infinities of C_k. Consider the following integral operator

$$(Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_k} f)(\underline{x}_k) = \int_{C_k} Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_k}(\underline{x}_{k+1};\underline{x}_k;\lambda_{k+1}) f(\underline{x}_k) d\underline{x}_k.$$

acting on functions not growing too fast at possible boundaries and infinities of C_k. Integral operators $Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_k}$ provide a recursive construction of $\mathfrak{gl}_{\ell+1}$-Whittaker functions:

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\ldots,\lambda_{\ell+1}}(\underline{x}_{\ell+1}) = \int_C \bigwedge_{i=1}^{\ell} dx_{\ell,i}\ Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{x}_{\ell+1};\underline{x}_\ell;\lambda_{\ell+1})\Psi^{\mathfrak{gl}_\ell}_{\lambda_1,\ldots,\lambda_\ell}(\underline{x}_\ell). \tag{2.55}$$

There is a natural oriented path in the diagram (2.51), which can be associated with the recursive operator $Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_{\ell}}$:



(2.56)

Diagram (2.51) can be considered as a collection of the oriented pathes (2.56) and thus the recursive construction of the integral representation is encoded in the diagram (2.51) in an obvious way.

As a consequence of (2.55), integral operators $Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_{k}}$ with the kernels $Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_{k}}(\underline{x}_{k+1},\, \underline{x}_k; \lambda_{k+1})$ satisfy braiding relations with the Quantum Toda chain Hamiltonians. For example the following relation between quadratic Hamiltonians $\mathcal{H}^{\mathfrak{gl}_{k+1}}_2(\underline{x}_{k+1})$ and $\mathcal{H}^{\mathfrak{gl}_{k}}_2(\underline{x}_{k})$, holds

$$\mathcal{H}^{\mathfrak{gl}_{k+1}}_2(\underline{x}_{k+1})Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_{k}}(\underline{x}_{k+1},\, \underline{x}_k; \lambda_{k+1}) = Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_{k}}(\underline{x}_{k+1},\, \underline{x}_k; \lambda_{k+1})\mathcal{H}^{\mathfrak{gl}_{k}}_2(\underline{x}_{k}) + \frac{1}{2}\lambda^2_{k+1}. \tag{2.57}$$

We shall assume that in the relation above and similar ones, Hamiltonian operators on l.h.s. act to the right and Hamiltonians on r.h.s. act to the left. Similar braiding relations hold for higher quantum Hamiltonian operators (see [GKLO] for details).

The recursion operators $Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_{k}}$ appear to be related with an important object in the theory of Quantum Integrable Systems, $\mathcal{Q}$-operator. $\mathcal{Q}$-operator was introduced by R. Baxter [B] for certain statistical models as a tool to solve quantum integrable models explicitly. In the case of $\widehat{\mathfrak{gl}}_{\ell+1}$-Toda chain, with the quadratic Hamiltonian

$$\mathcal{H}^{\widehat{\mathfrak{gl}}_{\ell+1}}_2 = -\frac{1}{2}\sum_{i=1}^{\ell+1}\frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{\ell} e^{x_{i+1}-x_i} + g e^{x_1 - x_{\ell+1}}\,, \tag{2.58}$$

where g is an arbitrary coupling constant, the $\mathcal{Q}$-operator has the following integral kernel

$$\mathcal{Q}^{\widehat{\mathfrak{gl}}_{\ell+1}}(\underline{x}^{(\ell+1)}, \underline{y}^{(\ell+1)}; \lambda) = \exp\Big\{\, \imath\lambda\sum_{i=1}^{\ell+1}(x_i - y_i) - \tag{2.59}$$

$$-\Big(\sum_{i=1}^{\ell}\left(e^{x_i-y_i} + e^{y_{i+1}-x_i}\right) + e^{x_{\ell+1}-y_{\ell+1}} + g e^{y_1 - x_{\ell+1}}\Big)\Big\}.$$

Here we use notations $\underline{x}^{(\ell+1)} = (x_1, \dots, x_{\ell+1})$ and $\underline{y}^{(\ell+1)} = (y_1, \dots, y_{\ell+1})$. This $\mathcal{Q}$-operator was first constructed in [PG]. It commutes with all Hamiltonians of $\widehat{\mathfrak{gl}}_{\ell+1}$-Toda chain and generates quantum Bäcklund transformations [PG]. For instance, for the quadratic Hamiltonians we have:

$$\mathcal{H}_2^{\widehat{\mathfrak{gl}}_{\ell+1}}(\underline{x}^{(\ell+1)})\mathcal{Q}^{\widehat{\mathfrak{gl}}_{\ell+1}}(\underline{x}^{(\ell+1)},\, \underline{y}^{(\ell+1)},\lambda) = \mathcal{Q}^{\widehat{\mathfrak{gl}}_{\ell+1}}(\underline{x}^{(\ell+1)},\, \underline{y}^{(\ell+1)},\lambda)\mathcal{H}_2^{\widehat{\mathfrak{gl}}_{\ell+1}}(\underline{y}^{(\ell+1)}). \tag{2.60}$$

To establish a relation between Baxter $\mathcal{Q}$-operator for $\widehat{\mathfrak{gl}}_{k+1}$-Toda theory and a recursion operator for $\mathfrak{gl}_{k+1}$-Toda theory it is useful to introduce a slightly modified recursion operator $Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_k\oplus\mathfrak{gl}_1}$ with the kernel:

$$Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_k\oplus\mathfrak{gl}_1}(\underline{x}^{(k+1)},\underline{y}^{(k+1)},\lambda) = \exp\{\imath\lambda\, y_{k+1}\}\; Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_k}(\underline{x}^{(k+1)},\underline{y}^{(k)},\lambda) = \tag{2.61}$$

$$\exp\left\{\imath\lambda\Big(\sum_{i=1}^{k+1}x_i-\sum_{i=1}^{k}y_i\Big)-\sum_{i=1}^{k}\Big(e^{y_i-x_i}+e^{x_{i+1}-y_i}\Big)\right\},$$

where $\underline{x}^{(k+1)}=(x_1,\ldots,x_{k+1})$, $\underline{y}^{(k)}=(y_1,\ldots,y_k)$ and $\underline{y}^{(k+1)}=(y_1,\ldots,y_k,y_{k+1})$.

This modified operator intertwines Hamiltonian operators of $\mathfrak{gl}_{k+1}$- and $\mathfrak{gl}_k\oplus\mathfrak{gl}_1$-Toda chains (the new variable y_{k+1} enters only $\mathfrak{gl}_1$-Toda chain). Thus for quadratic Hamiltonian operators we have

$$\mathcal{H}_2^{\mathfrak{gl}_{k+1}}(\underline{x}^{(k+1)})Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_k\oplus\mathfrak{gl}_1}(\underline{x}^{(k+1)},\, \underline{y}^{(k+1)},\lambda) = Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_k\oplus\mathfrak{gl}_1}(\underline{x}^{(k+1)},\, \underline{y}^{(k+1)},\lambda)(\mathcal{H}_2^{\mathfrak{gl}_k}(\underline{y}^{(k)})+\mathcal{H}_2^{\mathfrak{gl}_1}(y_{k+1})),$$

where $\mathcal{H}_2^{\mathfrak{gl}_1}(y_{k+1})=-\frac{1}{2}\frac{\partial^2}{\partial y_{k+1}^2}$. Obviously the projection of the above relation on the subspace of functions $F(\underline{y},y_{k+1})=\exp(\imath\lambda\, y_{k+1})\,f(\underline{x})$ leads to (2.57).

Now consider a one-parameter family of integral operators

$$\mathcal{Q}^{\widehat{\mathfrak{gl}}_{\ell+1}}(\underline{x}^{(k+1)},\underline{y}^{(k+1)};\lambda;\varepsilon) = \varepsilon^{\imath\lambda}\exp\Big\{\,\imath\lambda\sum_{i=1}^{\ell+1}(x_i-y_i)\,- \tag{2.62}$$

$$-\Big(\sum_{i=1}^{\ell}\big(e^{x_i-y_i}+e^{y_{i+1}-x_i}\big)\;+\varepsilon\, e^{x_{\ell+1}-y_{\ell+1}}+\varepsilon^{-1}g e^{y_1-x_{\ell+1}}\Big)\Big\}.$$

obtained from (2.59) by a shift of the variable $x_{\ell+1}\to x_{\ell+1}+\ln\varepsilon$. The limiting behavior of (2.62) when $\varepsilon\to 0$, $g\epsilon^{-1}\to 0$ can be described as follows

$$Q^{\mathfrak{gl}_k\oplus\mathfrak{gl}_1}_{\mathfrak{gl}_{k+1}}(\underline{x}^{(k+1)},\, \underline{y}^{(k+1)},\lambda) = \lim_{\varepsilon\to 0,\, g\varepsilon^{-1}\to 0}\;\varepsilon^{-\imath\lambda}\,\mathcal{Q}^{\widehat{\mathfrak{gl}}_{k+1}}(\underline{x}^{(k+1)},\underline{y}^{(k+1)},\lambda,\varepsilon). \tag{2.63}$$

This provides a relation between the Baxter $\mathcal{Q}$-operator and the (modified) recursion operator.

2.3 Integral representations of $\mathfrak{so}_{2\ell+1}$-Toda chain eigenfunctions

In this subsection we provide a generalization of the Givental integral representation of $\mathfrak{gl}_{\ell+1}$-Whittaker functions to the case of $\mathfrak{so}_{2\ell+1}$. We start with a derivation of the integral representation of $\mathfrak{so}_{2\ell+1}$-Whittaker functions using the factorized representation. Then we consider a modification of the factorized representation that directly leads to a Givental type integral representation.

Consider B_ℓ type root system corresponding to Lie algebra $\mathfrak{so}_{2\ell+1}$. Let $(\epsilon_1, \ldots, \epsilon_\ell)$ be an orthogonal basis in $\mathbb{R}^\ell$. We realize B_ℓ roots, coroots and fundamental weights as vectors in $\mathbb{R}^\ell$ in the following way:

$$\begin{array}{lll} \alpha_1 = \epsilon_1, & \alpha_1^\vee = 2\epsilon_1, & \omega_1 = \frac{1}{2}(\epsilon_1 + \ldots + \epsilon_\ell), \\ \alpha_2 = \epsilon_2 - \epsilon_1, & \alpha_2^\vee = \epsilon_2 - \epsilon_1, & \omega_2 = \epsilon_2 + \ldots + \epsilon_\ell, \\ \ldots & \ldots & \ldots \\ \alpha_\ell = \epsilon_\ell - \epsilon_{\ell-1}, & \alpha_\ell^\vee = \epsilon_\ell - \epsilon_{\ell-1}, & \omega_\ell = \epsilon_\ell. \end{array} \tag{2.64}$$

The Cartan matrix is then given by $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ and positive rational numbers $d_1 = \frac{1}{2}, d_2 = 1, \ldots, d_\ell = 1$ are such that the matrix $\|b_{ij}\| = \|d_i a_{ij}\|$ is symmetric. One associates with these data a Quantum Toda chain with a quadratic Hamiltonian

$$\mathcal{H}_2^{B_\ell} \;=\; -\frac{1}{2}\sum_{i=1}^{\ell} \frac{\partial^2}{\partial {x_i}^2} + \frac{1}{2} e^{x_1} + \sum_{i=1}^{\ell-1} e^{x_{i+1}-x_i}. \tag{2.65}$$

One can complete (2.65) to a full set of ℓ mutually commuting functionally independent Hamiltonians $H_k^{B_\ell}$ of the $\mathfrak{so}_{2\ell+1}$-Toda chain. We are looking for common eigenfunction integral representations of the commuting set of the Hamiltonians. Corresponding eigenfunction problem for the quadratic Hamiltonian can be written in the following form

$$\mathcal{H}_2^{B_\ell}(x)\;\; \Psi^{B_\ell}_{\lambda_1,\cdots,\lambda_\ell}(x_1, \ldots, x_\ell) = \frac{1}{2}\sum_{i=1}^{\ell} \lambda_i^2 \;\; \Psi^{B_\ell}_{\lambda_1,\cdots,\lambda_\ell}(x_1, \ldots, x_\ell). \tag{2.66}$$

2.3.1 $\mathfrak{so}_{2\ell+1}$-Whittaker function: factorized parametrization

The reduced word for the element w_0 of maximal length in the Weyl group of B_ℓ-type can be represented in the recursive form:

$$I = (i_1, i_2, \ldots, i_m) := (1, 212, 32123, \ldots, (\ell \ldots 212 \ldots \ell)),$$

where indexes i_k correspond to elementary reflections with respect to roots α_k. Let $N_+ \subset G$ be a maximal unipotent subgroup of $G = SO(2\ell+1)$. One associates with the reduced word I the following recursive parametrization of a generic unipotent element $v^{B_\ell} \in N_+$:

$$v^{B_\ell} = v^{B_{\ell-1}}\, \mathfrak{X}^{B_\ell}_{B_{\ell-1}}, \tag{2.67}$$

where

$$\mathfrak{X}^{B_\ell}_{B_{\ell-1}} = X_\ell(y_{\ell,1}) \cdots X_k(y_{k,2(\ell+1-k)-1}) \cdots X_2(y_{2,2\ell-3}) \times \\ \times X_1(y_{1,\ell}) X_2(y_{2,2\ell-2}) \cdots X_k(y_{k,2(\ell+1-k)}) \cdot X_\ell(y_{\ell,2}). \tag{2.68}$$

Here $X_i(y) = e^{ye_i}$ and $e_i \equiv e_{\alpha_i}$ are simple root generators. The subset $N_+^{(0)}$ of elements allowing the representation is an open part of N_+. The action of the Lie algebra $\mathfrak{so}_{2\ell+1}$ on N_+ (2.10) considered at the beginning of the previous section defines an action of the Lie algebra on $N_+^{(0)}$. The following proposition explicitly describes this action on the space V_μ considered as a space of functions on $N_+^{(0)}$.

Proposition 2.4 *The following differential operators define a realization of a principal series representation π_λ of $\mathcal{U}(\mathfrak{so}_{2\ell+1})$ in terms of factorized parametrization of $N_+^{(0)}$:*

$$\begin{aligned}
E_1 &= \frac{\partial}{\partial y_{1,\ell}} + \sum_{n=1}^{\ell-1} \Big\{ \Big(\frac{\partial}{\partial y_{1,n}} - \frac{\partial}{\partial y_{1,n+1}} \Big) \prod_{j=n}^{\ell-1} \frac{y_{2,2i}}{y_{2,2i-1}} + \\
&+ \quad 2 \Big(\frac{\partial}{\partial y_{2,2n-1}} - \frac{\partial}{\partial y_{2,2n}} \Big) \frac{y_{2,2(n-1)}}{y_{1,n}} \prod_{i=n+1}^{\ell-1} \frac{y_{2,2i}}{y_{2,2i-1}} \Big\}, \\
E_k &= \frac{\partial}{\partial y_{k,2(\ell+1-k)}} + \sum_{n=1}^{n-k} \Big\{ \Big(\frac{\partial}{\partial y_{k,2n}} - \frac{\partial}{\partial y_{k,2n+1}} \Big) \prod_{i=n}^{\ell-k} \frac{y_{k+1,2i}}{y_{k+1,2i-1}} \frac{y_{k,2(i+1)-1}}{y_{k,2(i+1)}} + \\
&+ \quad \Big(\frac{\partial}{\partial y_{k+1,2n-1}} - \frac{\partial}{\partial y_{k+1,2n}} \Big) \frac{y_{k+1,2n}}{y_{k,2(n+1)}} \prod_{i=n+1}^{\ell-k} \frac{y_{k+1,2i}}{y_{k+1,2i-1}} \frac{y_{k,2(i+1)-1}}{y_{k,2(i+1)}} \Big\}, \qquad 1 < k < \ell, \\
E_\ell &= \frac{\partial}{\partial y_{\ell,2}},
\end{aligned} \tag{2.69}$$

$$H_k = \langle \mu \,, \alpha_k^\vee \rangle + \sum_{i=1}^{\ell} a_{k,i} \sum_{j=1}^{n_i} y_{i,j} \frac{\partial}{\partial y_{i,j}}, \qquad 1 \le k \le \ell, \tag{2.70}$$

$$
\begin{aligned}
F_1 &= \sum_{n=1}^{\ell} y_{1,n}\Big(\langle\mu,\alpha_1^\vee\rangle + \sum_{j=1}^{2(n-1)-1} 2y_{2,j}\frac{\partial}{\partial y_{2,j}} - 2\sum_{j=1}^{n-1} y_{1,j}\frac{\partial}{\partial y_{1,j}} - y_{1,n}\frac{\partial}{\partial y_{1,n}}\Big),\\
F_2 &= \sum_{n=1}^{2(\ell-1)} y_{2,n}\Big(\langle\mu,\alpha_2^\vee\rangle + 2\sum_{j=1}^{[n/2]+1} y_{1,j}\frac{\partial}{\partial y_{1,j}} - 2\sum_{j=1}^{n-1} y_{2,j}\frac{\partial}{\partial y_{2,j}} +\\
&+ \sum_{j=1}^{2[(n+1)/2]-3} y_{3,j}\frac{\partial}{\partial y_{3,j}} - y_{2,n}\frac{\partial}{\partial y_{2,n}}\Big),\\
F_k &= \sum_{n=1}^{2(\ell+1-k)} y_{k,n}\Big(\langle\mu,\alpha_k^\vee\rangle + \sum_{j=1}^{2[n/2]+1} y_{k-1,j}\frac{\partial}{\partial y_{k-1,j}} - 2\sum_{j=1}^{n-1} y_{k,j}\frac{\partial}{\partial y_{k,j}} +\\
&+ \sum_{j=1}^{2[(n+1)/2]-3} y_{k+1,j}\frac{\partial}{\partial y_{k+1,j}} - y_{k,n}\frac{\partial}{\partial y_{k,n}}\Big), \qquad 2<k<\ell,\\
F_\ell &= (y_{\ell,1}+y_{\ell,2})\left(\langle\mu,\alpha_\ell^\vee\rangle + y_{\ell-1,1}\frac{\partial}{\partial y_{\ell-1,1}} + y_{\ell-1,2}\frac{\partial}{\partial y_{\ell-1,2}}\right) +\\
&+ y_{\ell,2}\left(y_{\ell-1,3}\frac{\partial}{\partial y_{\ell-1,3}} + y_{\ell-1,4}\frac{\partial}{\partial y_{\ell-1,4}}\right) - \left(y_{\ell,1}^2\frac{\partial}{\partial y_{\ell,1}} + 2y_{\ell,1}y_{\ell,2}\frac{\partial}{\partial y_{\ell,1}} + y_{\ell,2}^2\frac{\partial}{\partial y_{\ell,2}}\right),
\end{aligned}
\tag{2.71}
$$

where $\pi_\lambda(e_i)=E_i$, $\pi_\lambda(f_i)=F_i$, $\pi_\lambda(h_i)=H_i$ $i=1,\dots,\ell$, $n_1=\ell$, $n_k=2(\ell+1-k)$ for $1<k\le\ell$, a_{ij} is a Cartan matrix and we assume that the terms containing $y_{i,j}$ with the indexes not in the set $\{1\le i,j\le\ell\}$ should be omitted.

For the proof see Part II, Section 3.4.2.

Left/right Whittaker vectors in the factorized parametrization have the following expressions.

Lemma 2.5 *The following expressions for the left/right Whittaker vectors hold:*

$$\psi_R(y) = \exp\Big\{ -\Big(\sum_{n=1}^{\ell} y_{1,n} + \sum_{k=2}^{\ell}\sum_{n=1}^{n_k} y_{k,n}\Big)\Big\},$$

$$\psi_L(y) = \Big(\prod_{n=1}^{\ell} y_{1,n} \prod_{i=2}^{\ell}\prod_{n=1}^{n_i/2} y_{i,2n-1}\Big)^{\langle\mu,\alpha_1^\vee\rangle} \times$$

$$\times \prod_{k=2}^{\ell}\Big(\prod_{n=2}^{\ell} y_{1,n}^2 \prod_{i=k+1}^{\ell}\prod_{n=1}^{n_i/2} y_{i,2n-1}^2 \prod_{i=2}^{k}\prod_{n=1}^{n_i/2} y_{i,2n-1}y_{i,2n}\Big)^{\langle\mu,\alpha_k^\vee\rangle} \times \tag{2.72}$$

$$\times \exp\Big\{ -\Big(\sum_{n=1}^{\ell}\frac{1}{y_{1,n}}\Big(1+\frac{y_{2,2(n-1)}}{y_{2,2(n-1)-1}}\Big)\prod_{i=n+1}^{\ell}\frac{y_{2,2(i-1)}}{y_{2,2(i-1)-1}} +$$

$$+\sum_{k=2}^{\ell}\sum_{n=1}^{n_k/2}\frac{1}{y_{k,2n}}\Big(1+\frac{y_{k+1,2(n-1)}}{y_{k+1,2(n-1)-1}}\Big)\prod_{i=n+1}^{n_k/2}\frac{y_{k+1,2(i-1)}}{y_{k+1,2(-1)-1}}\frac{y_{k,2i-1}}{y_{k,2i}}\Big)\Big\},$$

where $n_1=\ell$ *and* $n_k = 2(\ell+1-k)$, $k=2,\ldots,\ell$.

For the proof see Part II, Section 3.3.2.

Using (2.12) and (2.21) we obtain an integral representation of $\mathfrak{so}_{2\ell+1}$-Whittaker functions in the factorized parametrization.

Theorem 2.4 *The eigenfunctions of the* $\mathfrak{so}_{2\ell+1}$*-Toda chain (2.12) admit the following integral representation:*

$$\Psi^{B_\ell}_{\lambda_1,\ldots,\lambda_\ell}(x_1,\ldots,x_\ell) = e^{\imath\lambda_1x_1+\ldots+\imath\lambda_\ell x_\ell}\int_C \bigwedge_{i=1}^{\ell}\bigwedge_{k=1}^{n_i}\frac{dy_{i,k}}{y_{i,k}}\Big(\prod_{n=1}^{\ell} y_{1,n}\prod_{i=2}^{\ell}\prod_{n=1}^{n_i/2} y_{i,2n-1}\Big)^{2\imath\lambda_1} \times$$

$$\times\prod_{k=2}^{\ell}\Big(\prod_{n=2}^{\ell} y_{1,n}^2 \prod_{i=k+1}^{\ell}\prod_{n=1}^{n_i/2} y_{i,2n-1}^2 \prod_{i=2}^{k}\prod_{n=1}^{n_i/2} y_{i,2n-1}y_{i,2n}\Big)^{\imath(\lambda_k-\lambda_{k-1})} \times$$

$$\times \exp\Big\{ -\Big(\sum_{n=1}^{\ell}\frac{1}{y_{1,n}}\Big(1+\frac{y_{2,2(n-1)}}{y_{2,2(n-1)-1}}\Big)\prod_{i=n+1}^{\ell}\frac{y_{2,2(i-1)}}{y_{2,2(i-1)-1}} + \tag{2.73}$$

$$+\sum_{k=2}^{\ell}\sum_{n=1}^{n_k/2}\frac{1}{y_{k,2n}}\Big(1+\frac{y_{k+1,2(n-1)}}{y_{k+1,2(n-1)-1}}\Big)\prod_{i=n+1}^{n_k/2}\frac{y_{k+1,2(i-1)}}{y_{k+1,2(-1)-1}}\frac{y_{k,2i-1}}{y_{k,2i}} +$$

$$+e^{x_1}\sum_{n=1}^{\ell} y_{1,n} + \sum_{k=2}^{\ell} e^{x_k-x_{k-1}}\sum_{n=1}^{n_k} y_{k,n}\Big)\Big\},$$

where $n_1=\ell$, $n_k = 2(\ell+1-k)$, $k=2,\ldots,\ell$ *and* $C\subset N_+$ *is a middle-dimensional non-compact submanifold such the integrand decays exponentially at the boundaries and infinities. In particular one can chose* $C=\mathbb{R}_+^{\ell^2}$.

The proof is given in Part II, Section 3.3.2.

Example 2.1 *Let $\ell = 2$. In this case, the general formula (2.73) acquirs the form*

$$\Psi^{B_2}_{\lambda_1,\lambda_2}(x_1,x_2) = e^{\imath\lambda_1 x_1+\imath\lambda_2 x_2}\int\limits_C \bigwedge_{i=1}^{2}\bigwedge_{k=1}^{2}\frac{dy_{i,k}}{y_{i,k}}(y_{11}y_{21}y_{12})^{2\imath\lambda_1}(y_{21}y_{12}^2y_{22})^{\imath\lambda_2-\imath\lambda_1}\cdot \tag{2.74}$$

$$\exp\Big\{-\Big(\Big\{\frac{1}{y_{12}}+\frac{y_{22}}{y_{21}}\Big(\frac{1}{y_{11}}+\frac{1}{y_{12}}\Big)\Big\}+\frac{1}{y_{22}}+e^{x_1}(y_{11}+y_{12})+e^{x_2-x_1}(y_{21}+y_{22})\Big)\Big\},$$

where one can chose $C=\mathbb{R}^4_+$.

2.3.2 $\mathfrak{so}_{2\ell+1}$-Whittaker function: modified factorized parametrization

In this part we introduce a modified factorized parametrization of N_+. We use this parametrization to construct the integral representations for $\mathfrak{so}_{2\ell+1}$-Whittaker functions. In contrast with the integral representations described above these integral representations have a simple recursive structure over the rank ℓ and can be described in purely combinatorial terms using suitable graphs. Thus these representations can be considered as a generalization of Givental integral representations to $\mathfrak{so}_{2\ell+1}$.

There exists a realization of a tautological representation $\pi_{2\ell+1}:\mathfrak{so}_{2\ell+1}\to End(\mathbb{C}_{2\ell+1})$ such that Weyl generators corresponding to Borel (Cartan) subalgebra of $\mathfrak{so}_{2\ell+1}$ are realized by upper triangular (diagonal) matrices. This defines an embedding $\mathfrak{so}_{2\ell+1}\subset\mathfrak{gl}_{2\ell+1}$ such that Borel (Cartan) subalgebra maps into Borel (Cartan) subalgebra (see e.g. [DS]). To define the corresponding embedding of the groups consider the following involution on $GL(2\ell+1)$:

$$g\longmapsto g^*:=\dot{w}_0\cdot(g^{-1})^t\cdot\dot{w}_0^{-1}, \tag{2.75}$$

where a^t is induced by the standard transposition of the matrix a and $\dot{w}_0$ is a lift of the maximal length element w_0 of the Weyl group of $\mathfrak{gl}_{2\ell+1}$. In a matrix form it can be written as

$$\dot{w}_0 = S\cdot J,$$

where $S=\mathrm{diag}(1,-1,\ldots,-1,1)$ and $J=\|J_{i,j}\|=\|\delta_{i+j,2\ell+2}\|$. The orthogonal group $G=SO(2\ell+1)$ then can be defined as a following subgroup of $GL(2\ell+1)$

$$SO(2\ell+1)=\{g\in GL(2\ell+1):g^*=g\}.$$

Let $\epsilon_{i,j}$ be elementary $(2\ell+1)\times(2\ell+1)$ matrices with units at the (i,j) place and zeros, otherwise. For any $n=2,\ldots,\ell$ introduce matrices U_n, $\widetilde{U}_n$ and V_n, $\widetilde{V}_n$:

$$U_n=\sum_{i=1}^{\ell-n}(\epsilon_{i,i}+\epsilon_{2\ell+2-i,2\ell+2-i})+\sum_{i=1}^{n}\epsilon_{\ell-n+i,\ell-n+i}+e^{-z_{n,1}}\epsilon_{\ell+1,\ell+1}+ \tag{2.76}$$

$$+e^{z_{n,1}}\epsilon_{\ell+2,\ell+2}+\sum_{i=1}^{n-1}e^{-z_{\ell,\ell+1-i}}\epsilon_{\ell+n+2-i,\ell+n+2-i},$$

$$\widetilde{U}_n = \sum_{i=1}^{\ell-n}(\epsilon_{i,i} + \epsilon_{2\ell+2-i,2\ell+2-i}) + \sum_{i=1}^{n} \epsilon_{\ell-n+i,\ell-n+i} + e^{-z_{n,1}}\epsilon_{\ell+1,\ell+1} + e^{z_{n,1}}\epsilon_{\ell+2,\ell+2} + \tag{2.77}$$

$$+\sum_{i=1}^{n-1} e^{-z_{\ell,\ell+1-i}}\epsilon_{\ell+n+2-i,\ell+n+2-i} + e^{x_{n-1,1}}\epsilon_{\ell+1,\ell+2} + \sum_{i=2}^{n-2} e^{-x_{n-1,i}}\epsilon_{\ell+i,\ell+i+1},$$

and

$$V_n = \sum_{i=1}^{\ell-n}(\epsilon_{i,i} + \epsilon_{2\ell+2-i,2\ell+2-i}) + \tag{2.78}$$

$$+\sum_{i=1}^{n} e^{x_{n,n+1-i}}\epsilon_{\ell-n+i,\ell-n+i} + e^{-x_{n,1}}\epsilon_{\ell+1,\ell+1} + \sum_{i=1}^{n} \epsilon_{\ell+i+1,\ell+i+1},$$

$$\widetilde{V}_n = \sum_{i=1}^{\ell-n}(\epsilon_{i,i} + \epsilon_{2\ell+2-i,2\ell+2-i}) + \sum_{i=1}^{n} e^{x_{n,n+1-i}}\epsilon_{\ell-n+i,\ell-n+i} + e^{-x_{n,1}}\epsilon_{\ell+1,\ell+1} + \tag{2.79}$$

$$+\sum_{i=1}^{n} \epsilon_{\ell+i+1,\ell+i+1} + \sum_{i=1}^{n-1} e^{z_{n,n+1-i}}\epsilon_{\ell-n+i,\ell-n+i+1} + e^{-z_{n,1}}\epsilon_{\ell,\ell+1},$$

$$U_1 \;=\; \sum_{i=1}^{\ell-1} \epsilon_{i,i} + e^{-z_{11}}\epsilon_{\ell,\ell} + e^{+z_{11}}\epsilon_{\ell+1,\ell+1} + \sum_{i=\ell+2}^{2\ell+1} \epsilon_{i,i}, \tag{2.80}$$

$$\widetilde{U}_1 \;=\; \sum_{i=1}^{\ell-1} \epsilon_{i,i} + e^{-z_{11}}\epsilon_{\ell,\ell} + e^{z_{11}}\epsilon_{\ell+1,\ell+1} + \sum_{i=\ell+2}^{2\ell+1} \epsilon_{i,i} + e^{x_{11}}\epsilon_{\ell,\ell+1}, \tag{2.81}$$

$$V_1 \;=\; \sum_{i=1}^{\ell} \epsilon_{i,i} + e^{-z_{11}}\epsilon_{\ell+1,\ell+1} + e^{z_{11}}\epsilon_{\ell+2,\ell+2} + \sum_{i=\ell+3}^{2\ell+1} \epsilon_{i,i}, \tag{2.82}$$

$$\widetilde{V}_1 \;=\; U_1^* = \sum_{i=1}^{\ell} \epsilon_{i,i} + e^{-z_{11}}\epsilon_{\ell+1,\ell+1} + e^{z_{11}}\epsilon_{\ell+2,\ell+2} + \sum_{i=\ell+3}^{2\ell+1} \epsilon_{i,i} + e^{x_{11}}\epsilon_{\ell+1,\ell+2}, \tag{2.83}$$

where $x_{\ell,k} = 0,\ k = 1,\dots,\ell$ are assumed. Note that $\widetilde{V}_i$, $\widetilde{U}_i$ can be considered as off-diagonal deformations of V_i, U_i. Now we can define a modified factorized representation for $N_+ \subset SO(2\ell+1)$.

Theorem 2.5 *i) The image of any generic unipotent element $v^{B_\ell} \in N_+$ in the tautological representation $\pi_{2\ell+1} : \mathfrak{so}_{2\ell+1} \to End(\mathbb{C}_{2\ell+1})$ can be represented in the form*

$$\pi_{2\ell+1}(v^{B_\ell}) = \mathfrak{X}_1 \mathfrak{X}_2 \cdots \mathfrak{X}_\ell, \tag{2.84}$$

where

$$\mathfrak{X}_1 = \widetilde{U}_1 U_1^{-1} \widetilde{V}_1 V_1^{-1}, \tag{2.85}$$

$$\mathfrak{X}_n = \widetilde{U}_n U_n^{-1} [\widetilde{U}_n U_n^{-1}]^* \widetilde{V}_n V_n^{-1} [\widetilde{V}_n V_n^{-1}]^*, \qquad n = 2, \ldots, \ell$$

and $x_{\ell,k} = 0$ for $k = 1, \ldots \ell$ are assumed.

ii) This defines a parametrization of an open part $N_+^{(0)}$ of N_+.

Proof. Let $v^{B_\ell}(y)$ be a parametrization of an open part of N_+ according to (2.67)-(2.68). Let $\tilde{X}_i(y) = e^{y e_{i,i+1}}$ be a one-parametric unipotent subgroup in $GL(2\ell+1)$, then $\tilde{X}_i(y)^* = \tilde{X}_{2\ell+1-i}(y)$. Embed an elementary unipotent element $X_i(y)$ of $SO(2\ell+1)$ into $GL(2\ell+1)$ as follows:

$$X_i(y) = \tilde{X}_i(y)^* \cdot \tilde{X}_i(y).$$

This maps an arbitrary regular unipotent element v^{B_ℓ} into unipotent subgroup of $GL(2\ell+1)$. Let us now change the variables in the following way:

$$y_{11} = e^{x_{11} - z_{11}}, \qquad y_{1,k} = \Big(e^{x_{k-1,1} - z_{k,1}} + e^{x_{k,1} - z_{k,1}} \Big), \tag{2.86}$$

$$y_{k,2r-1} = e^{z_{k+r-1,k} - x_{k+r-2,k-1}}, \qquad k = 2, \ldots, \ell,$$

$$y_{k,2r} = e^{z_{k+r-1,k} - x_{k+r-1,k-1}}, \qquad r = 1, \ldots, \ell + 1 - k,$$

where the conditions $x_{\ell,k} = 0, \ k = 1, \ldots \ell$ are assumed. By elementary operations it is easy to check that after the change of variables, the image $\pi_{2\ell+1}(v^{B_\ell})$ of v^{B_ℓ} defined by (2.67)-(2.68) transforms into the (2.84). Taking into account that the change of variables (2.86) is invertible we get a parametrization of $N_+^{(0)} \subset N_+$ $\Box$

The modified factorized parameterization of a unipotent group $N_+ \subset SO(2\ell+1)$ defines a particular realization of a principal series representation of $U(\mathfrak{so}_{2\ell+1})$ by differential operators. It can be obtained using the change of variables (2.86) applied to the representation given in Proposition 2.4. We shall use the term Gauss-Givental representation for this realization.

Proposition 2.5 *The following differential operators define a representation π_λ of $\mathfrak{so}_{2\ell+1}$ in V_μ in terms of the modified factorized parametrization:*

$$E_1 = -2 \sum_{n=1}^{\ell} e^{z_{n,1}} \left(\frac{1}{2} \frac{\partial}{\partial z_{n,1}} + \frac{e^{x_{n,1}}}{e^{x_{n-1,1}} + e^{x_{n,1}}} \frac{\partial}{\partial z_{n,2}} + \right. \tag{2.87}$$

$$\left. + \sum_{n=1}^{\ell-1} \Big(\frac{\partial}{\partial z_{\ell,1}} + \frac{\partial}{\partial z_{\ell,2}} + \frac{\partial}{\partial x_{\ell,1}} \Big) \right),$$

$$E_2=\left(\frac{\partial}{\partial z_{11}}+\frac{\partial}{\partial x_{11}}\right)\left(e^{x_{22}-z_{22}}+\sum_{k=3}^{\ell}e^{x_{k-1,2}-z_{k,2}}+e^{x_{k,2}-z_{k,2}}\right)+ \tag{2.88}$$

$$+\sum_{n=2}^{\ell}\frac{\partial}{\partial\, z_{n,1}}\left(e^{x_{n-1,1}-z_{n,1}}\frac{e^{x_{n-1,2}-z_{n,2}}+e^{x_{n,2}-z_{n,2}}}{e^{x_{n-1,1}-z_{n,1}}+e^{x_{n,1}-z_{n,1}}}+\sum_{k=n+1}^{\ell}e^{x_{k-1,2}-z_{k,2}}+e^{x_{k,2}-z_{k,2}}\right)+$$

$$+\sum_{n=2}^{\ell}\left(\frac{\partial}{\partial z_{n,2}}-\frac{\partial}{\partial z_{n,3}}\right)\left(e^{x_{n,2}-z_{n,2}}+\sum_{k=n+1}^{\ell}e^{x_{k-1,2}-z_{k,2}}+e^{x_{k,2}-z_{k,2}}\right)+$$

$$+\sum_{n=2}^{\ell}\left(\frac{\partial}{\partial x_{n,1}}-\frac{\partial}{\partial x_{n,2}}\right)\sum_{k=n+1}^{\ell}\left(e^{x_{k-1,2}-z_{k,2}}+e^{x_{k,2}-z_{k,2}}\right),$$

$$E_k=\sum_{n=k-1}^{\ell-1}\left(\frac{\partial}{\partial x_{n,k-1}}-\frac{\partial}{\partial x_{n,k}}\right)\sum_{i=n+1}^{\ell}\left(e^{x_{i-1,k}-z_{i,k}}+e^{x_{i,k}-z_{i,k}}\right)+ \tag{2.89}$$

$$+\sum_{n=k}^{\ell}\left(\frac{\partial}{\partial z_{n,k}}-\frac{\partial}{\partial z_{n,k+1}}\right)\left(e^{x_{n,k}-z_{n,k}}+\sum_{i=n+1}^{\ell}e^{x_{i-1,k}-z_{i,k}}+e^{x_{i,k}-z_{i,k}}\right),\quad 3\le k\le\ell,$$

$$H_k=\langle\mu,\alpha_k^{\vee}\rangle+\sum_{n=1}^{\ell}a_{k,n}\sum_{i=n}^{\ell}\frac{\partial}{\partial z_{i,n}},\quad 1\le k\le\ell, \tag{2.90}$$

$$F_1=\langle\mu,\alpha_1^{\vee}\rangle\left(e^{x_{11}-z_{11}}+\sum_{k=2}^{\ell}e^{x_{k-1,1}-z_{k,1}}+e^{x_{k,1}-z_{k,1}}\right)+ \tag{2.91}$$

$$+\sum_{n=1}^{\ell}\left(e^{x_{n,1}-z_{n,1}}-e^{x_{n-1,1}-z_{n,1}}\right)\frac{\partial}{\partial z_{n,1}}-$$

$$+2\sum_{n=1}^{\ell}\frac{\partial}{\partial\, x_{n,1}}\sum_{k=n+1}^{\ell}\left(e^{x_{k-1,1}-z_{k,1}}+e^{x_{k,1}-z_{k,1}}\right),$$

$$F_2=\left(\langle\mu,\alpha_2^{\vee}\rangle+\frac{\partial}{\partial x_{11}}\right)\sum_{k=2}^{\ell}\left(e^{z_{k,2}-x_{k-1,1}}+e^{z_{k,2}-x_{k,1}}\right)- \tag{2.92}$$

$$-\sum_{n=2}^{\ell}\frac{\partial}{\partial\, z_{n,2}}\left(e^{z_{n,2}-x_{n,1}}+\sum_{k=n+1}^{\ell}e^{z_{k,2}-x_{k-1,1}}+e^{z_{k,2}-x_{k,1}}\right)+$$

$$+\sum_{n=2}^{\ell}\left(\frac{\partial}{\partial\, x_{n,1}}-\frac{\partial}{\partial\, x_{n,2}}\right)\sum_{k=n+1}^{\ell}\left(e^{z_{k,2}-x_{k-1,1}}+e^{z_{k,2}-x_{k,1}}\right),$$

$$F_k = \Big(\langle\mu,\alpha_k^\vee\rangle + \frac{\partial}{\partial x_{k-1,k-1}} + \frac{\partial}{\partial z_{k-1,k-1}}\Big)\sum_{n=k}^{\ell}\Big(e^{z_{n,k}-x_{n-1,k-1}} + e^{z_{n,k}-x_{n,k-1}}\Big) -$$

$$-\sum_{n=k}^{\ell}\Big(\frac{\partial}{\partial z_{n,k}} - \frac{\partial}{\partial z_{n,k-1}}\Big)\Big(e^{z_{n,k}-x_{n,k-1}} + \sum_{i=n+1}^{\ell} e^{z_{i,k}-x_{i-1,k-1}} + e^{z_{i,k}-x_{i,k-1}}\Big) +$$

$$+\sum_{n=k}^{\ell}\Big(\frac{\partial}{\partial x_{n,k-1}} - \frac{\partial}{\partial x_{n,k}}\Big)\sum_{i=n+1}^{\ell}\Big(e^{z_{i,k}-x_{i-1,k-1}} + e^{z_{i,k}-x_{i,k-1}}\Big), \qquad 3\le k\le \ell,$$

where $E_i = \pi_\lambda(e_i)$, $F_i = \pi_\lambda(f_i)$, $H_i = \pi_\lambda(h_i)$, $x_{\ell,k} = 0, k = 1,\dots,\ell$ are assumed and the derivatives over $x_{i,k}$, $z_{i,k}$, $i < k$, $x_{\ell,n}$, $n = 1,\dots,\ell$ are omitted.

We are going to write down the matrix element (2.12) explicitly in Gauss-Givental representation. Whittaker vectors ψ_R and ψ_L in this representation satisfy the system of differential

$$E_i\psi_R(x) = -\psi_R(x), \qquad F_i\psi_L(x) = -\psi_L(x), \qquad 1\le i\le \ell. \tag{2.93}$$

Its solution has the following form.

Lemma 2.6 *The functions*

$$\psi_L(x,z) = e^{2\mu_1 x_{1,1}}\prod_{n=2}^{\ell}\Big(e^{x_{n,1}} + e^{x_{n-1,1}}\Big)^{2\mu_n}\times \tag{2.94}$$

$$\times\prod_{n=1}^{\ell}\exp\Big\{-\mu_n\Big(\sum_{i=1}^{n} x_{n,i} + 2z_{n,1} - 2\sum_{i=2}^{n} z_{n,i} + \sum_{i=1}^{n-1} x_{n-1,i}\Big)\Big\}\times$$

$$\times\exp\Big\{-\Big(\sum_{k=1}^{\ell} e^{z_{k,1}} + \sum_{k=2}^{\ell} e^{x_{k,k}-z_{k,k}} + \sum_{k=2}^{\ell}\sum_{n=k+1}^{\ell}\Big(e^{x_{n-1,k}-z_{n,k}} + e^{x_{n,k}-z_{n,k}}\Big)\Big)\Big\},$$

$$\psi_R(x,z) = \exp\Big\{-\Big(e^{x_{11}-z_{11}} + \sum_{n=2}^{\ell}\Big(e^{x_{n-1,1}-z_{n,1}} + e^{x_{n,1}-z_{n,1}}\Big)\Big) + \tag{2.95}$$

$$+\sum_{k=2}^{\ell}\sum_{n=k}^{\ell}\Big(e^{z_{n,k}-x_{n-1,k-1}} + e^{z_{n,k}-x_{n,k-1}}\Big)\Big\},$$

are solutions of the linear differential equations (2.93). We let $x_{\ell,k} = 0$ for $k = 1,\dots,\ell$, and $\mu_k = \imath\lambda_k - \rho_k$, where $\rho_k = \frac{2k-1}{2}$.

Now we are ready to find the integral representation of the pairing (2.12) in terms of modified factorization parameters. To get explicit expression for the integrand, one uses the same type of a Cartan element decomposition as in the case of $\mathfrak{gl}_{\ell+1}$:

$$e^{-H_x} = \pi_\lambda(\exp(-\sum_{i=1}^{\ell}\langle\omega_i, x\rangle h_i)) = e^{H_L}e^{H_R},$$

where

$$- H_x = H_L + H_R = -\sum_{i=1}^{\ell}\langle\omega_i, x\rangle\langle\mu, \alpha_i^\vee\rangle + \tag{2.96}$$

$$+x_{\ell,1}\sum_{n=1}^{\ell}\frac{\partial}{\partial z_{n,1}} + \sum_{k=1}^{\ell-1}(x_{\ell,i} - x_{\ell,i+1})\sum_{n=k}^{\ell}\frac{\partial}{\partial z_{n,k}},$$

with

$$H_L = \sum_{k=1}^{\ell-1}\sum_{n=1}^{k} x_{\ell,n}\frac{\partial}{\partial x_{k,n}} + \sum_{k=2}^{\ell}\sum_{n=2}^{k} x_{\ell,n}\frac{\partial}{\partial z_{k,n}}, \tag{2.97}$$

$$H_R = -H_x - H_L. \tag{2.98}$$

We imply that the differential operator H_L acts on the left vector, and H_R acts on the right vector in (2.12). Taking into account the results of the Proposition 2.6 one obtains the following theorem.

Theorem 2.6 *The eigenfunctions of $\mathfrak{so}_{2\ell+1}$-Toda chain (2.12) admit the integral representation:*

$$\Psi^{B_\ell}_{\lambda_1,\ldots,\lambda_\ell}(x_{\ell,1},\ldots,x_{\ell,\ell}) \;=\; \int\limits_{C}\bigwedge_{k=1}^{\ell-1}\bigwedge_{i=1}^{k} dx_{k,i}\bigwedge_{k=1}^{\ell}\bigwedge_{i=1}^{k} dz_{k,i}\; e^{\mathcal{F}^{B_\ell}},$$

where

$$\begin{gathered} \mathcal{F}^{B_\ell} = -\imath\lambda_1(-x_{1,1} + 2z_{1,1}) - \\ -\imath\sum_{n=2}^{\ell}\lambda_n\Big(\sum_{i=1}^{n} x_{n,i} + 2z_{n,1} - 2\sum_{i=2}^{n} z_{n,i} + \sum_{i=1}^{n-1} x_{n-1,i} - 2\ln(e^{x_{n,1}} + e^{x_{n-1,1}})\Big) - \\ -\Big\{\sum_{n=1}^{\ell} e^{z_{n,1}} + \sum_{k=2}^{\ell}\sum_{n=k+1}^{\ell}\left(e^{x_{n-1,k}-z_{n,k}} + e^{x_{n,k}-z_{n,k}}\right) + \\ +\sum_{n=k}^{\ell}\left(e^{z_{n,k}-x_{n-1,k-1}} + e^{z_{n,k}-x_{n,k-1}}\right) + \sum_{n=1}^{\ell} e^{x_{n,n}-z_{n,n}}\Big\}, \end{gathered} \tag{2.99}$$

where we set $x_i := x_{\ell,i}$, $1 \le i \le \ell$. Here $C \subset N_+$ is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at the boundaries and at infinities. In particular the domain of integration can be chosen to be $C = \mathbb{R}^m$, where $m = l(w_0)$.

Example 2.2 *For $\ell = 2$ the general formula (2.99) has the following form:*

$$\Psi^{B_2}_{\lambda_1,\lambda_2}(x_{2,1}, x_{2,2}) = \int_C dz_{1,1} \wedge dx_{1,1} \wedge dz_{2,1} \wedge dz_{2,2} \times \tag{2.100}$$

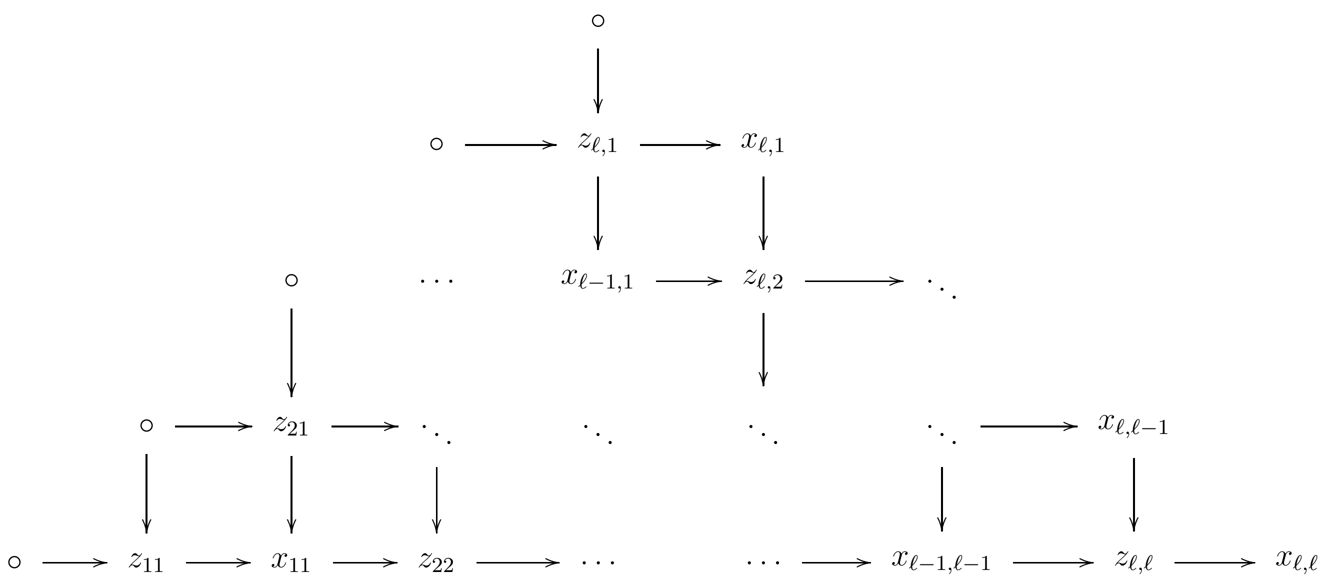
$$\times e^{-\imath\lambda_1(-x_{1,1}+2z_{1,1})-\imath\lambda_2(2z_{2,1}-2z_{2,2}+x_{1,1}+x_{2,1}+x_{22})}\Big(e^{x_{2,1}} + e^{x_{1,1}}\Big)^{2\imath\lambda_2} \times$$

$$\times \exp\Big\{ - \Big(e^{z_{1,1}} + e^{z_{2,1}} + e^{x_{2,2}-z_{2,2}} + e^{x_{1,1}-z_{1,1}} + e^{x_{1,1}-z_{2,1}} +$$

$$+ e^{x_{2,1}-z_{2,1}} + e^{z_{2,2}-x_{1,1}} + e^{z_{2,2}-x_{2,1}}\Big)\Big\},$$

where we set $x_1 = x_{21}, x_2 = x_{22}$ and the contour of integration $C = \mathbb{R}^4_+$.

There is a simple combinatorial description of the potential $\mathcal{F}^{B_\ell}$ for zero spectrum $\{\lambda_i = 0\}$. Namely, it can be presented as the sum over all the arrows in the following diagram. The diagram for B_ℓ reads



Here we use the same rules for assigning variables to the arrows of the diagram as in A_ℓ case. In addition we assign functions e^x to the arrows $\circ \longrightarrow x$.

Note that the diagram for B_ℓ can be obtained by a factorization of the diagram (2.51) for $A_{2\ell}$. Consider the following involution

$$\iota : \quad X \longmapsto \dot{w}_0^{-1} X^t \dot{w}_0, \tag{2.101}$$

where $\dot{w}_0$ is the longest element of $A_{2\ell}$ Weyl group and X^t denotes the standard transposition. Corresponding action on the modified factorization parameters is given by

$$w_0 : \qquad x_{k,i} \longleftrightarrow -x_{k,k+1-i}. \tag{2.102}$$

This defines a factorization of $A_{2\ell}$-diagram that gives the diagram for B_ℓ.

An analog of $\mathfrak{gl}_{\ell+1}$-monomial relations (2.52) can be described as follows. Associate variables $a_{k,i}$, $b_{k,i}$, $c_{k,i}$, $d_{k,i}$ to the arrows of the Givental diagram as

$$a_{k,i} = e^{z_{k,i}-x_{k-1,i-1}},\ \ b_{k,i} = e^{z_{k,i}-x_{k,i-1}},\ \ c_{k,i} = e^{z_{k,i}-x_{k,i}},\ \ d_{l,j} = e^{x_{l,j}-z_{l+1,j}},$$
$$1 \le k \le \ell,\ 1 \le i \le k, \quad 1 \le l \le \ell-1,\ 1 \le j \le l. \tag{2.103}$$

Then the following relations hold:

$$\begin{aligned}
a_{k,1} &= b_{k,1}, && 1 \le k \le \ell,\\
d_{k,i}\cdot a_{k+1,i+1} &= c_{k+1,i}\cdot b_{k+1,i+1}, && 1 \le k < \ell-1,\ 1 \le i \le k,\\
b_{k,i}\cdot c_{k,i} &= a_{k+1,i}\cdot d_{k,i}, && 1 \le k < \ell-1,\ 1 \le i \le k,\\
b_{\ell,i}\cdot c_{\ell,i} &= e^{x_{\ell,i}-x_{\ell,i-1}}.
\end{aligned} \tag{2.104}$$

The above relations can be considered as relations between elementary paths on the Givental diagram. Using a set of relations for more general paths that follows from (2.104) one can define a toric degeneration of the $\mathfrak{so}_{2\ell+1}$ flag manifolds thus generalizing the results in [BCFKS].

2.3.3 Recursion for $\mathfrak{so}_{2\ell+1}$-Whittaker functions and $\mathcal{Q}$-operator for $B_\ell^{(1)}$-Toda chain

The integral representation (2.99) of $\mathfrak{so}_{2\ell+1}$-Whittaker functions possesses a remarkable recursive structure over the rank ℓ. Let us introduce integral operators $Q^{B_n}_{B_{n-1}}$, $n = 2,\dots,\ell$ with the kernels $Q^{B_n}_{B_{n-1}}(\underline{x}_n;\,\underline{x}_{n-1};\lambda_n)$ defined as follows

$$Q^{B_n}_{B_{n-1}}(\underline{x}_n;\,\underline{x}_{n-1};\lambda_n) = \int\bigwedge_{i=1}^{n} dz_{n,i}\ \Big(e^{x_{n,1}}+e^{x_{n-1,1}}\Big)^{2\imath\lambda_n}\times \tag{2.105}$$
$$\times\exp\Big\{-\imath\lambda_n\Big(\sum_{i=1}^{n}x_{n,i}+2z_{n,1}-2\sum_{i=2}^{n}z_{n,i}+\sum_{i=1}^{n-1}x_{n-1,i}\Big)\Big\}\ \times$$
$$\times Q^{B_n}_{BC_n}(\underline{x}_n;\,\underline{z}_n)\ \ Q^{BC_n}_{B_{n-1}}(\underline{z}_n;\,\underline{x}_{n-1}),$$

where

$$Q^{BC_n}_{B_{n-1}}(\underline{z}_n;\,\underline{x}_{n-1}) = \exp\Big\{-\Big(\frac{1}{2}e^{z_{n,1}}+\sum_{i=1}^{n-1}\Big(e^{x_{n-1,i}-z_{n,i}}+e^{z_{n,i+1}-x_{n-1,i}}\Big)\Big)\Big\}, \tag{2.106}$$

$$Q^{B_n}_{BC_n}(\underline{x}_n;\,\underline{z}_n) =$$
$$= \exp\Big\{-\Big(\frac{1}{2}e^{z_{n,1}}+\sum_{i=1}^{n-1}\Big(e^{x_{n,i}-z_{n,i}}+e^{z_{n,i+1}-x_{n,i}}\Big)+e^{x_{n,n}-z_{n,n}}\Big)\Big\}. \tag{2.107}$$

We set for $n=1$

$$Q^{B_1}_{B_0}(x_{1,1};\lambda_1)=\int dz_{1,1}e^{\imath\lambda_1 x_{1,1}-2\imath\lambda_1 z_{1,1}}\exp\Big\{-\Big(e^{z_{1,1}}+e^{x_{1,1}-z_{1,1}}\Big)\Big\}.$$

Using integral operators $Q^{B_n}_{B_{n-1}}$ the integral representation (2.99) can be written in a recursive form.

Theorem 2.7 *The eigenfunction of B_ℓ-Toda chain can be written as*

$$\Psi^{B_\ell}_{\lambda_1,\ldots,\lambda_\ell}(x_1,\ldots,x_\ell)\;=\;\int\limits_C\bigwedge_{k=1}^{\ell-1}\bigwedge_{i=1}^{k}dx_{k,i}\,\prod_{k=1}^{\ell}Q^{B_k}_{B_{k-1}}(\underline{x}_k;\,\underline{x}_{k-1};\lambda_k),$$

or equivalently

$$\Psi^{B_\ell}_{\lambda_1,\ldots,\lambda_\ell}(x_1,\ldots,x_\ell)\;=\;\int\limits_C\bigwedge_{i=1}^{\ell-1}dx_{\ell-1,i}\,Q^{B_\ell}_{B_{\ell-1}}(\underline{x}_\ell;\,\underline{x}_{\ell-1};\lambda_\ell)\Psi^{B_{\ell-1}}_{\lambda_1,\ldots,\lambda_{\ell-1}}(\underline{x}_{\ell-1}),\qquad(2.108)$$

where we assume $x_n:=x_{\ell,n},\;\;1\leq n\leq\ell$ and $C\subset N_+$ is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at possible boundaries and at infinities. In particular as the domain of integration one can chose $C=\mathbb{R}^m$, where $m=l(w_0)$.

Let us note that in contrast with the case of $\mathfrak{gl}_{\ell+1}$ integral representations, kernels of $Q^{B_n}_{B_{n-1}}$, $n=1,\ldots,\ell$ have more complicated form. Curious new structure appeares if we consider the Whittaker functions for zero spectrum[3] $\{\lambda_i=0\}$. As it is clear from (2.105) the kernel of $Q^{B_n}_{B_{n-1}}$ is given by a convolution of two kernels $Q^{B_n}_{BC_n}(\underline{x}_n;\,\underline{z}_n)$ and $Q^{BC_n}_{B_{n-1}}(\underline{z}_n;\,\underline{x}_{n-1})$. Corresponding integral operators $Q^{B_n}_{BC_n}$, $Q^{BC_n}_{B_{n-1}}$ can be regarded as elementary intertwiners relating Toda chains for B_n, BC_n and BC_n, B_{n-1} root systems. BC_ℓ-Toda chain[4] is defined in terms of the non-reduced root system BC_ℓ in a standrad fashion. Let us recall the construction of the non-reduced root system BC_ℓ. Root system of BC_ℓ type can be realized in terms of an orthogonal bases $\{\epsilon_i\}$ in $\mathbb{R}^\ell$ as

$$\alpha_0=2\epsilon_1,\qquad\alpha_1=\epsilon_1,\qquad\alpha_{i+1}=\epsilon_{i+1}-\epsilon_i,\qquad1\leq i\leq\ell-1,\qquad(2.109)$$

and the corresponding Dynkin diagram is

$$\begin{matrix}\alpha_0\\ \alpha_1\end{matrix}\;\Longleftrightarrow\;\alpha_2\;\text{——}\;\cdots\;\text{——}\;\alpha_{\ell-1}\;\text{——}\;\alpha_\ell$$

where the first vertex from the left is a doubled vertex corresponding to a reduced $\alpha_1=\epsilon_1$ and non-reduced $\alpha_0=2\epsilon_1$ roots. Then for example the quadratic Hamiltonian operator of BC_ℓ-Toda chain is given by

$$\mathcal{H}^{BC_\ell}_2(\underline{x}^{(\ell)})=-\frac{1}{2}\sum_{i=1}^{\ell}\frac{\partial^2}{\partial x_i^2}+\frac{1}{4}\Big(e^{x_1}+\frac{1}{2}e^{2x_1}\Big)+\sum_{i=1}^{\ell-1}e^{x_{i+1}-x_i}.\qquad(2.110)$$

[3] Note that the zero spectrum Whittaker functions are directly related to the quntum cohomology of flag manifolds in Givental description.

[4] BC_ℓ-Toda chain can be also considered as a most general form of C_ℓ-Toda chain (see e.g. [RSTS], Remark p.61). In the following we will use the term BC_ℓ-Toda chain to distinguish it from a more standard C_ℓ-Toda chain that will be consider below.

Integral operators $Q^{B_n}_{BC_n}$ and $Q^{BC_n}_{B_{n-1}}$ intertwine Hamitonian operators of different Toda chains. Thus for quadratic Hamiltonians one can directly check the following relations.

Proposition 2.6 *1. The operators $Q^{B_n}_{BC_n}$ and $Q^{BC_n}_{B_{n-1}}$ defined by the kernels (2.106), (2.107) intertwine quadratic Hamiltonians of B and BC Toda chains:*

$$\mathcal{H}_2^{BC_n}(\underline{z}_n)Q^{BC_n}_{B_{n-1}}(\underline{z}_n,\,\underline{x}_{n-1}) = Q^{BC_n}_{B_{n-1}}(\underline{z}_n,\,\underline{x}_{n-1})\mathcal{H}_2^{B_{n-1}}(\underline{x}_{n-1}), \tag{2.111}$$

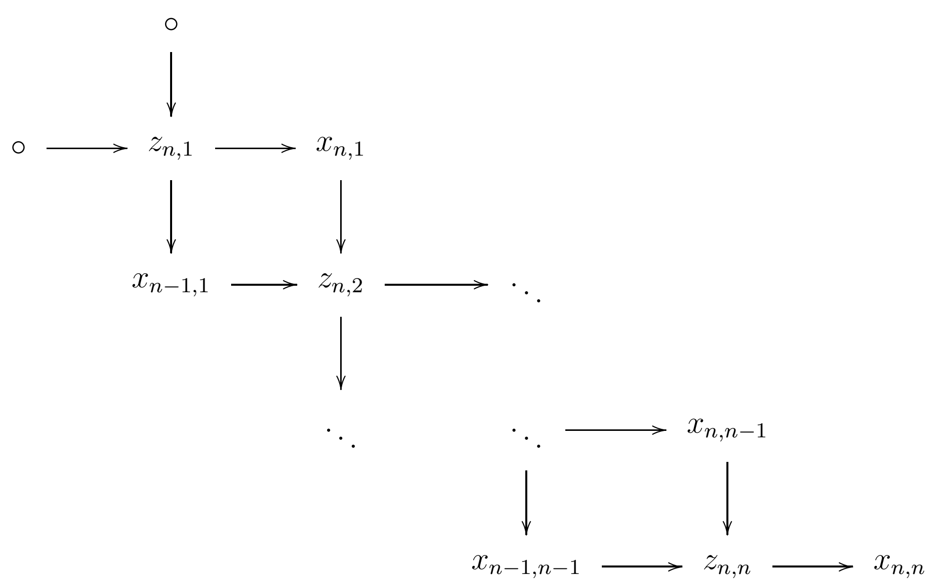
$$\mathcal{H}_2^{B_n}(\underline{x}_n)Q^{B_n}_{BC_n}(\underline{x}_n,\,\underline{z}_n) = Q^{B_n}_{BC_n}(\underline{x}_n,\,\underline{z}_n)\mathcal{H}_2^{BC_n}(\underline{z}_n). \tag{2.112}$$

2. Integral operator $Q^{B_n}_{B_{n-1}}$ at $\lambda_n = 0$ intertwines Hamiltonians $\mathcal{H}_2^{B_n}$ and $\mathcal{H}_2^{B_{n-1}}$:

$$\mathcal{H}_2^{B_n}(\underline{x}_n)Q^{B_n}_{B_{n-1}}(\underline{x}_n,\,\underline{x}_{n-1};\lambda_n = 0) = Q^{B_n}_{B_{n-1}}(\underline{x}_n,\,\underline{x}_{n-1};\lambda_n = 0)\mathcal{H}_2^{B_{n-1}}(\underline{x}_{n-1}). \tag{2.113}$$

The kernel $Q^{B_n}_{B_{n-1}}(\underline{x}_n,\,\underline{x}_{n-1};\lambda_n = 0)$: can be succinctly encoded into the following subdiagramm of $\mathfrak{so}_{2\ell+1}$ Givental diagram

(2.114)



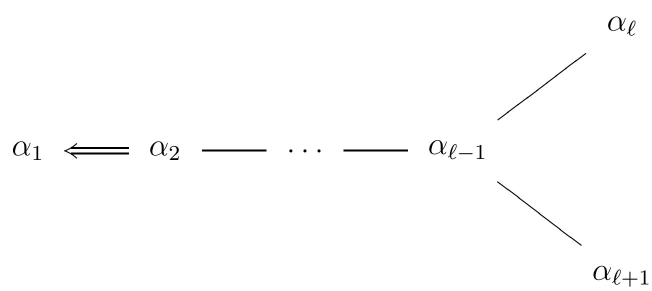
Here the upper and lower descending paths of the oriented diagram correspond to the kernels of elementary intertwiners $Q^{B_n}_{BC_n}$ and $Q^{BC_n}_{B_{n-1}}$ respectively. The convolution of the kernels $Q^{B_n}_{BC_n}$ and $Q^{BC_n}_{B_{n-1}}$ in (2.105) at $\lambda_n = 0$ corresponds to the integration over the variables $z_{n,i}$ associated with the inner vertexes of the subdiagram (2.114).

Similarly to the case of $\mathfrak{gl}_{\ell+1}$, recursion operators $Q^{B_n}_{B_{n-1}}$ can be considered as particular degenerations of Baxter $\mathcal{Q}$-operators for affine $B^{(1)}_\ell$-Toda chains. Below we provide the integral representations for these $\mathcal{Q}$-operator. Let us stress that up to now $\mathcal{Q}$-operators were known only for $\widehat{\mathfrak{gl}}_{\ell+1}$-case. We will not present here the complete set of properties characterizing the introduced $\mathcal{Q}$-operators and only consider the commutation relations with quadratic affine Toda chain Hamiltonians. The detailed account will be given elsewhere.

We start with a description of $B_\ell^{(1)}$-Toda chain. The set of simple roots of the affine root system $B_\ell^{(1)}$ can be represented in the following form:

$$\alpha_1 = \epsilon_1, \qquad \alpha_{i+1} = \epsilon_{i+1} - \epsilon_i, \qquad 1 \le i \le \ell - 1 \qquad \alpha_{\ell+1} = -\epsilon_\ell - \epsilon_{\ell-1}. \tag{2.115}$$

The corresponding Dynkin diagram is



These root data allows to define affine $B_\ell^{(1)}$-Toda chain with a quadratic Hamiltonian given by

$$\mathcal{H}_2^{B_\ell^{(1)}} = -\frac{1}{2}\sum_{i=1}^{\ell+1}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}e^{x_1} + \sum_{i=1}^{\ell-1} e^{x_{i+1}-x_i} + g e^{-x_\ell - x_{\ell-1}}\,. \tag{2.116}$$

Here g is an arbitrary coupling constant.

Define the Baxter $\mathcal{Q}$-operator of $B_\ell^{(1)}$-Toda chain as an integral operator with the following kernel

$$\mathcal{Q}^{B_\ell^{(1)}}(\underline{x}^{(\ell)}, \underline{y}^{(\ell)}\,,\,\lambda) = \int \bigwedge_{i=1}^{\ell} dz_i\ \Big(e^{x_1} + e^{y_1}\Big)^{2\imath\lambda}\Big(e^{-x_\ell} + e^{-y_\ell}\Big)^{-2\imath\lambda} \times \tag{2.117}$$

$$\times \exp\Big\{ -\imath\lambda\Big(\sum_{i=1}^{\ell} x_i + 2z_1 - 2\sum_{i=2}^{\ell} z_i + \sum_{i=1}^{\ell} y_i\Big)\Big\}\ Q^{B_\ell^{(1)}}_{BC_\ell^{(1)}}(x_i;\, z_i)\ \ Q^{BC_\ell^{(1)}}_{B_\ell^{(1)}}(z_i;\, y_i),$$

where

$$Q_{\ B_\ell^{(1)}}^{\ BC_\ell^{(1)}}(z_i,\, y_i) =$$

$$= \exp\Big\{ -\Big(\frac{1}{2}e^{z_1} + \sum_{i=1}^{\ell-1}\Big(e^{y_i - z_i} + e^{z_{i+1}-y_i}\Big) + e^{y_\ell - z_\ell} + g e^{-y_\ell - z_\ell}\Big)\Big\}, \tag{2.118}$$

and

$$Q_{\ BC_n^{(1)}}^{\ B_n^{(1)}}(x_i,\, z_i) = Q_{\ B_n^{(1)}}^{\ BC_n^{(1)}}(z_i,\, x_i). \tag{2.119}$$

Here we denote $\underline{x}^{(\ell)} = (x_1, \dots, x_\ell)$, and $\underline{y}^{(\ell)} = (y_1, \dots, y_\ell)$.

The following Proposition can be proved by a direct check.

Proposition 2.7 *The $\mathcal{Q}$-operator (2.117) commutes with quadratic Hamiltonian of the $B_\ell^{(1)}$ Toda chain, that is the kernel intertwines the Hamiltonians $\mathcal{H}_2^{B_\ell^{(1)}}$*

$$\mathcal{H}_2^{B_\ell^{(1)}}(\underline{x}^{(\ell)})\mathcal{Q}^{B_\ell^{(1)}}(\underline{x}^{(\ell)}\,;\,\underline{y}^{(\ell)}\,,\,\lambda)=\mathcal{Q}^{B_\ell^{(1)}}(\underline{x}^{(\ell)}\,,\,\underline{y}^{(\ell)}\,;\,\lambda)\mathcal{H}_2^{B_\ell^{(1)}}(\underline{y}^{(\ell)}). \tag{2.120}$$

Now we will demonstrate that recursion operator $Q^{B_\ell}_{B_{\ell-1}}$ can be considered as a degeneration of Baxter $\mathcal{Q}$-operators for $B_\ell^{(1)}$. Let us introduce a slightly modified recursion operator $Q^{B_\ell}_{B_{\ell-1}\oplus B_1}$ with the kernel:

$$Q^{B_\ell}_{B_{\ell-1}\oplus B_1}(\underline{x}^{(\ell)}\,,\,\underline{y}^{(\ell)}\,,\,\lambda):=e^{\imath\lambda y_\ell}\;Q^{B_\ell}_{B_{\ell-1}}(\underline{x}^{(\ell)}\,,\,\underline{y}^{(\ell-1)}\,,\,\lambda), \tag{2.121}$$

where $\underline{y}^{(\ell-1)}=(y_1,\ldots,y_{\ell-1})$. Operator (2.121) intertwines Hamiltonians of $\mathfrak{so}_{2\ell+1}$- and $\mathfrak{so}_{2\ell-1}\oplus\mathfrak{so}_2$-Toda chains. Thus for quadratic Hamiltonians we have

$$\mathcal{H}_2^{B_\ell}(\underline{x}^{(\ell)})Q^{B_\ell}_{B_{\ell-1}\oplus B_1}(\underline{x}^{(\ell)}\,,\,\underline{y}^{(\ell)},\lambda)=Q^{B_\ell}_{B_{\ell-1}\oplus B_1}(\underline{x}^{(\ell)}\,,\,\underline{y}^{(\ell)},\lambda)\Big(\mathcal{H}_2^{B_{\ell-1}}(\underline{y}^{(\ell-1)})+\mathcal{H}_2^{B_1}(y_\ell)\Big),$$

where $\mathcal{H}_2^{B_1}(y_\ell)=-\frac{1}{2}\Big(\partial^2/\partial y_\ell^2\Big)$. Obviously the projection of the above relation on the subspace of functions $F(\underline{y}^{(\ell)},y_\ell)=\exp(\imath\lambda y_\ell)f(\underline{y}^{(\ell-1)})$ recovers the genuine recursion operator satisfying:

$$\mathcal{H}_2^{B_\ell}(\underline{x}^{(\ell)})Q^{B_\ell}_{B_{\ell-1}}(\underline{x}^{(\ell)}\,,\,\underline{y}^{(\ell)}\,;\,\lambda)=Q^{B_\ell}_{B_{\ell-1}}(\underline{x}^{(\ell)}\,,\,\underline{y}^{(\ell-1)}\,,\,\lambda)\Big(\mathcal{H}_2^{B_{\ell-1}}(\underline{y}^{(\ell-1)})+\frac{1}{2}\lambda^2\Big). \tag{2.122}$$

Let us introduce a one-parameter family of the kernels

$$\mathcal{Q}^{B_\ell^{(1)}}(\underline{x}^{(\ell)},\,\underline{y}^{(\ell)};\lambda;\varepsilon)=\varepsilon^{\imath\lambda}e^{\imath\lambda y_\ell}\int\bigwedge_{i=1}^{\ell}dz_i\;\Big(e^{x_1}+e^{y_1}\Big)^{2\imath\lambda}\Big(\varepsilon e^{y_\ell-x_\ell}+1\Big)^{-2\imath\lambda}\times$$

$$\times\exp\Big\{-\imath\lambda\Big(\sum_{i=1}^{\ell}x_i+2z_1-2\sum_{i=2}^{\ell}z_i+\sum_{i=1}^{\ell-1}y_i\Big)\Big\}\;Q^{B_\ell^{(1)}}_{BC_\ell^{(1)}}(x_i;\,z_i)\;\;Q^{BC_\ell^{(1)}}_{B_\ell^{(1)}}(z_i;\,y_i;\varepsilon), \tag{2.123}$$

where

$$Q^{BC_\ell^{(1)}}_{B_\ell^{(1)}}(\underline{z}_\ell,\,\underline{y}_\ell;\varepsilon)=\exp\Big\{-\Big(\frac{1}{2}e^{z_1}+\sum_{i=1}^{\ell-1}\Big(e^{y_i-z_i}+e^{z_{i+1}-y_i}\Big)+$$

$$+\varepsilon e^{y_\ell-z_\ell}+\varepsilon^{-1}ge^{-y_\ell-z_\ell}\Big)\Big\},$$

obtained from the kernel of the operator $\mathcal{Q}^{B_\ell^{(1)}}$ by the change of the variable $y_\ell=y_\ell+\ln\varepsilon$. Consider limiting behavior of (2.124) when $\varepsilon\to 0,\; g\varepsilon^{-1}\to 0$. Then the following relation between $\mathcal{Q}$-operator for $B_\ell^{(1)}$-Toda chain and (modified) recursion operator for $\mathfrak{so}_{2\ell+1}$-Whittaker function holds

$$Q^{B_\ell}_{B_{\ell-1}\oplus B_1}(\underline{x}^{(\ell)},\underline{y}^{(\ell)};\lambda)=\lim_{\varepsilon\to 0,\;g\epsilon^{-1}\to 0}\varepsilon^{-\imath\lambda}\mathcal{Q}^{B_\ell^{(1)}}(\underline{x}^{(\ell)},\,\underline{y}^{(\ell)};\lambda;\varepsilon). \tag{2.124}$$

2.4 Integral representations of $\mathfrak{sp}_{2\ell}$-Toda chain eigenfunctions

In this subsection we provide an analog of the Givental integral representation of Whittaker functions for $\mathfrak{sp}_{2\ell}$ Lie algebras. As in the case of $\mathfrak{so}_{2\ell+1}$, we start with a derivation of the integral representation of $\mathfrak{sp}_{2\ell}$-Whittaker functions using the factorized parametrization. Then we consider a modification of the factorized parametrization leading to a Givental type integral representation of $\mathfrak{sp}_{2\ell}$-Whittaker functions.

Consider C_ℓ type root system corresponding to a Lie algebra $\mathfrak{sp}_{2\ell}$. Let $(\epsilon_1,\dots,\epsilon_\ell)$ be an orthogonal basis in $\mathbb{R}^\ell$. We use the following realization of simple roots, coroots and fundamental weights as vectors in $\mathbb{R}^\ell$:

$$\begin{array}{lll}\alpha_1=2\epsilon_1, & \alpha_1^\vee=\epsilon_1, & \omega_1=\epsilon_1+\ldots+\epsilon_\ell\\ \alpha_2=\epsilon_2-\epsilon_1, & \alpha_2^\vee=\epsilon_2-\epsilon_1, & \omega_2=\epsilon_2+\ldots+\epsilon_\ell,\\ \ldots & \ldots & \ldots\\ \alpha_\ell=\epsilon_\ell-\epsilon_{\ell-1}, & \alpha_\ell^\vee=\epsilon_\ell-\epsilon_{\ell-1}, & \omega_\ell=\epsilon_\ell.\end{array} \tag{2.125}$$

Cartan matrix $\|a_{ij}\|=\|\langle\alpha_i^\vee,\alpha_j\rangle\|$ can be made symmetric $\|b_{ij}\|=\|d_ia_{ij}\|$ with $d_1=2$, $d_i=1$, $i=2,\dots,\ell$. One associates with these root data a $\mathfrak{sp}_{2\ell}$-Toda chain with a quadratic Hamiltonian given by

$$\mathcal{H}_2^{C_\ell} \;=\; -\frac{1}{2}\sum_{i=1}^{\ell}\frac{\partial^2}{\partial {z_i}^2}+2e^{2z_1}+\sum_{i=1}^{\ell-1}e^{z_{i+1}-z_i}. \tag{2.126}$$

One can complete (2.126) to a full set of ℓ mutually commuting operators $H_k^{C_\ell}$ of C_ℓ-Toda chain. We are looking for integral representations of common eigenfunctions of the full commuting set of Hamiltonians. The corresponding eigenfunction problem for quadratic Hamiltonian can be written in the following form

$$\mathcal{H}_2^{C_\ell}\;\;\Psi^{C_\ell}_{\lambda_1,\cdots,\lambda_\ell}(z_1,\dots,z_\ell)=\frac{1}{2}\sum_{i=1}^{\ell}\lambda_i^2\;\;\Psi^{C_\ell}_{\lambda_1,\cdots,\lambda_\ell}(z_1,\dots,z_\ell). \tag{2.127}$$

2.4.1 $\mathfrak{sp}_{2\ell}$-Whittaker function: factorized parametrization

The reduced word for the maximal length element w_0 in the Weyl group of $\mathfrak{sp}_{2\ell}$ can be represented in the recursive form:

$$I=(i_1,i_2,\dots,i_m):=(1,212,32123,\dots,(\ell\ldots212\ldots\ell)),$$

where indexes i_k correspond to elementary reflections with respect to the roots α_k. Let $N_+\subset G$ be a maximal unipotent subgroup of $G=Sp(2\ell)$. One associates with the reduced word I the following recursive parametrization of a generic element $v^{C_\ell}\in N_+$:

$$v^{C_\ell}=v^{C_{\ell-1}}\cdot\mathfrak{X}^{C_\ell}_{C_{\ell-1}}, \tag{2.128}$$

where

$$\begin{aligned}\mathfrak{X}^{C_\ell}_{C_{\ell-1}}&=X_\ell(y_{\ell,1})\,X_k(y_{k,2(\ell+1-k)-1})\,X_2(y_{2,2\ell-3})\times\\ &\times X_1(y_{1,\ell})X_2(y_{2,2\ell-2})\,X_k(y_{k,2(\ell+1-k)})\cdot X_\ell(y_{\ell,2}).\end{aligned} \tag{2.129}$$

Here $X_i(y) = e^{ye_i}$ and $e_i \equiv e_{\alpha_i}$ are simple root generators. The subset $N_+^{(0)}$ allowing representation (2.128), (2.129) is an open part of N_+. The action of the Lie algebra $\mathfrak{sp}_{2\ell}$ on N_+ given by (2.10) defines an action on the space of functions on $N_+^{(0)}$. The following proposition explicitly describes the action on the space V_μ of (twisted) functions on $N_+^{(0)}$.

Proposition 2.8 *The following differential operators define a realization of the representation π_λ of $\mathcal{U}(\mathfrak{sp}_{2\ell})$ in V_μ in terms of factorized parametrization of $N_+^{(0)}$:*

$$\begin{aligned}
E_1 &= \sum_{n=1}^{\ell}\Big(\frac{\partial}{\partial y_{1,n}}-\frac{\partial}{\partial y_{1,n+1}}\Big)\prod_{j=n}^{\ell-1}\Big(\frac{y_{2,2j}}{y_{2,2j-1}}\Big)^2 + \\
&+ \sum_{n=1}^{\ell-1}\Big(\frac{\partial}{\partial y_{2,2n-1}}-\frac{\partial}{\partial y_{2,2n}}\Big)\frac{y_{2,2n}}{y_{1,n}}\Big(1+\frac{y_{2,2n}}{y_{2,2n-1}}\Big)\prod_{j=n+1}^{\ell-1}\Big(\frac{y_{2,2j}}{y_{2,2j-1}}\Big)^2, \qquad (2.130)\\
E_k &= \sum_{n=1}^{\ell+1-k}\Big(\frac{\partial}{\partial y_{k,2n}}-\frac{\partial}{\partial y_{k,2n+1}}\Big)\prod_{i=n}^{\ell-k}\frac{y_{k+1,2j}}{y_{k+1,2j-1}}\frac{y_{k,2(j+1)-1}}{y_{k,2(j+1)}} + \\
&+ \sum_{n=1}^{\ell-k}\Big(\frac{\partial}{\partial y_{k+1,2n-1}}-\frac{\partial}{\partial y_{k+1,2n}}\Big)\frac{y_{k+1,2n}}{y_{k,2(n-1)}}\prod_{i=n}^{\ell-k}\frac{y_{k+1,2j}}{y_{k+1,2j-1}}\frac{y_{k,2(j+1)-1}}{y_{k,2(j+1)}}, \quad 1<k<\ell,\\
E_\ell &= \frac{\partial}{\partial y_{\ell,2}},
\end{aligned}$$

$$H_k = \langle\mu\,,\alpha_k^\vee\rangle + \sum_{i=1}^{\ell}a_{k,i}\sum_{j=1}^{n_i}y_{i,j}\frac{\partial}{\partial y_{i,j}}, \qquad (2.131)$$

$$\begin{aligned}
F_1 &= \sum_{n=1}^{\ell}y_{1,n}\Big(-\langle\mu,\alpha_1^\vee\rangle+\sum_{j=1}^{2(n-1)-1}y_{2,j}\frac{\partial}{\partial y_{2,j}}-2\sum_{j=1}^{n-1}y_{1,j}\frac{\partial}{\partial y_{1,j}}-y_{1,n}\frac{\partial}{\partial y_{1,n}}\Big),\\
F_2 &= \sum_{n=1}^{2(\ell-1)}y_{2,n}\Big(\langle\mu,\alpha_2^\vee\rangle+2\sum_{j=1}^{[n/2]+1}y_{1,j}\frac{\partial}{\partial y_{1,j}}-2\sum_{j=1}^{n-1}y_{2,j}\frac{\partial}{\partial y_{2,j}}+\\
&+\sum_{j=1}^{2[(n+1)/2]-3}y_{3,j}\frac{\partial}{\partial y_{3,j}}-y_{2,n}\frac{\partial}{\partial y_{2,n}}\Big),
\end{aligned}$$

$$\begin{aligned}
F_k = \sum_{n=1}^{2(\ell+1-k)}y_{k,n}\Big(\langle\mu,\alpha_k^\vee\rangle+2\sum_{j=1}^{2[n/2]+1}y_{k-1,j}\frac{\partial}{\partial y_{k-1,j}}-2\sum_{j=1}^{n-1}y_{k,j}\frac{\partial}{\partial y_{k,j}}+ \qquad (2.132)\\
+\sum_{j=1}^{2[(n+1)/2]-3}y_{k+1,j}\frac{\partial}{\partial y_{k+1,j}}-y_{k,n}\frac{\partial}{\partial y_{k,n}}\Big),
\end{aligned}$$

for $2 < k < \ell$

$$F_\ell \;=\; (y_{\ell,1}+y_{\ell,2})\left(-\langle\mu,\alpha_\ell^\vee\rangle + y_{\ell-1,1}\frac{\partial}{\partial y_{\ell-1,1}} + y_{\ell-1,2}\frac{\partial}{\partial y_{\ell-1,2}}\right) + \qquad (2.133)$$

$$+y_{\ell,2}\left(y_{\ell-1,3}\frac{\partial}{\partial y_{\ell-1,3}} + y_{\ell-1,4}\frac{\partial}{\partial y_{\ell-1,4}}\right) - \left(y_{\ell,1}^2\frac{\partial}{\partial y_{\ell,1}} + 2y_{\ell,1}y_{\ell,2}\frac{\partial}{\partial y_{\ell,1}} + y_{\ell,2}^2\frac{\partial}{\partial y_{\ell,2}}\right),$$

where $E_i = \pi_\lambda(e_i), H_i = \pi_\lambda(h_i), F_i = \pi_\lambda(f_i),\;\; i = 1,\ldots,\ell.$ *and* $n_1 = \ell$, $n_k = 2(\ell+1-k)$ *for* $1 < k \le \ell$.

The proof is given in Part II, Section 3.4.3.

For left/right Whittaker vectors in the factorized parametrization we have the following expressions.

Lemma 2.7 *Left/right Whittaker vectors in the factorized parametrization are given by:*

$$\psi_R(y) = \exp\Big\{-\Big(\sum_{n=1}^{\ell} y_{1,n} \;+\; \sum_{k=2}^{\ell}\sum_{n=1}^{n_k} y_{k,n}\Big)\Big\},$$

$$\psi_L(y) = \prod_{i=1}^{\ell}\Big(\prod_{n=1}^{\ell} y_{1,n} \times \prod_{k=2}^{i}\prod_{n=1}^{2(\ell+1-k)} y_{k,n} \times \prod_{k=i+1}^{\ell}\prod_{n=1}^{\ell+1-k} y_{k,2n-1}^2\Big)^{\langle\mu,\alpha_i^\vee\rangle} \times \qquad (2.134)$$

$$\times \exp\Big\{-\Big(\sum_{n=1}^{\ell}\frac{1}{y_{1,n}}\Big(1+\frac{y_{2,2(n-1)}}{y_{2,2(n-1)-1}}\Big)^2 \prod_{i=n+1}^{\ell}\Big(\frac{y_{2,2(i-1)}}{y_{2,2(i-1)-1}}\Big)^2 +$$

$$+\sum_{k=2}^{\ell}\sum_{n=1}^{n_k/2}\frac{1}{y_{k,2n}}\Big(1+\frac{y_{k+1,2(n-1)}}{y_{k+1,2(n-1)-1}}\Big)\prod_{i=n+1}^{n_k/2}\frac{y_{k+1,2(i-1)}}{y_{k+1,2(-1)-1}}\frac{y_{k,2i-1}}{y_{k,2i}}\Big)\Big\},$$

where $n_1 = \ell$ *and* $n_k = 2(\ell+1-k)$, *for* $k = 2,\ldots,\ell$.

Proof is given in Part II, Section 3.3.3.

Using the expressions (2.134) for the left/right Whittaker vectors we obtain the integral representation of $\mathfrak{sp}_{2\ell}$-Whittaker function in terms of factorized parametrization.

Theorem 2.8 *The eigenfunctions of the $\mathfrak{sp}_{2\ell}$-Toda chain admit the integral representation:*

$$\begin{aligned}\Psi^{C_\ell}_{\lambda_1,\ldots,\lambda_\ell}(z_1,\ldots,z_\ell) = e^{\imath\lambda_1 z_1+\ldots+\imath\lambda_\ell z_\ell}\int_C \bigwedge_{i=1}^{\ell}\bigwedge_{k=1}^{n_i}\frac{dy_{i,k}}{y_{i,k}}\Big(\prod_{n=1}^{\ell} y_{1,n}\prod_{k=2}^{\ell}\prod_{n=1}^{n_k/2} y^2_{k,2n-1}\Big)^{\imath\lambda_1}\times \\ \times\prod_{i=2}^{\ell}\Big(\prod_{n=1}^{\ell}y_{1,n}\prod_{k=2}^{i}\prod_{n=1}^{n_k}y_{k,n}\prod_{k=i+1}^{\ell}\prod_{n=1}^{n_k/2}y^2_{k,2n-1}\Big)^{\imath(\lambda_i-\lambda_{i-1})}\times \\ \times\exp\Big\{-\Big(\sum_{n=1}^{\ell}\frac{1}{y_{1,n}}\Big(1+\frac{y_{2,2(n-1)}}{y_{2,2(n-1)-1}}\Big)^2\prod_{i=n+1}^{\ell}\Big(\frac{y_{2,2(i-1)}}{y_{2,2(i-1)-1}}\Big)^2+ \\ +\sum_{k=2}^{\ell}\sum_{n=1}^{n_k/2}\frac{1}{y_{k,2n}}\Big(1+\frac{y_{k+1,2(n-1)}}{y_{k+1,2(n-1)-1}}\Big)\prod_{i=n+1}^{n_k/2}\frac{y_{k+1,2(i-1)}}{y_{k+1,2(-1)-1}}\frac{y_{k,2i-1}}{y_{k,2i}}+ \\ +e^{2z_1}\sum_{n=1}^{\ell}y_{1,n}+\sum_{k=2}^{\ell}e^{z_k-z_{k-1}}\sum_{n=1}^{n_k}y_{k,n}\Big)\Big\},\end{aligned} \tag{2.135}$$

where $n_1=\ell$ and $n_k=2(\ell+1-k)$, for $k=2,\ldots,\ell$. The domain of integration $C\subset N_+$ is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at the boundaries and infinities. In particular one can chose $C=\mathbb{R}^{\ell^2}_+$.

The proof is given in Part II, Section 3.3.3.

Example 2.3 *For $\ell=2$ the general formula (2.135) acquires the form*

$$\begin{aligned}\Psi^{C_2}_{\lambda_1,\lambda_2}(z_1,z_2) = e^{\imath\lambda_1 z_1+\imath\lambda_2 z_2}\int\limits_C\bigwedge_{i,k=1}^{2}\frac{dy_{i,k}}{y_{i,k}}(y_{1,1}y^2_{2,1}y_{1,2})^{\imath\lambda_1}(y_{2,1}y_{1,2}y_{2,2})^{\imath\lambda_2-\imath\lambda_1}\times \\ \times\exp\Big\{-\Big(\frac{1}{y_{1,1}}\Big(\frac{y_{2,2}}{y_{2,1}}\Big)^2+\frac{1}{y_{1,2}}\Big(\frac{y_{2,2}}{y_{2,1}}+1\Big)^2+\frac{1}{y_{2,2}}+ \\ +e^{2z_1}(y_{1,1}+y_{1,2})+e^{z_2-z_1}(y_{2,1}+y_{2,2})\Big\},\end{aligned} \tag{2.136}$$

with one can take $C=\mathbb{R}^4$.

2.4.2 $\mathfrak{sp}_{2\ell}$-Whittaker function: modified factorized parametrization

In this part we introduce a modified factorized parametrization of an open part of $N_+\subset Sp(2\ell)$. We use this parametrization to construct integral representations for $\mathfrak{sp}_{2\ell}$-Whittaker functions. Similar to integral representation of $\mathfrak{so}_{2\ell+1}$-Whittaker functions considered above these integral representations have a simple recursive structure over the rank ℓ and can be describe in purely combinatorial terms using suitable graphs. These representations can be considered as a generalization of Givental integral representations to the case of $\mathfrak{g}=\mathfrak{sp}_{2\ell}$.

We follow the same approach that was used in the description of modified factorized representation for $\mathfrak{so}_{2\ell+1}$. There exists a realization of a tautological representation $\pi_{2\ell}$: $\mathfrak{sp}_{2\ell} \to End(\mathbb{C}_{2\ell})$ such that Weyl generators corresponding to Borel (Cartan) subalgebra of $\mathfrak{sp}_{2\ell}$ are realized by upper triangular (diagonal) matrices. This defines an embedding $\mathfrak{sp}_{2\ell} \subset \mathfrak{gl}_{2\ell}$ such that Borel (Cartan) subalgebra maps into Borel (Cartan) subalgebra (see e.g. [DS]). To define the corresponding embedding of the groups consider the following involution on $GL(2\ell)$:

$$g \longmapsto g^* := \dot{w}_0 \cdot (g^{-1})^t \cdot \dot{w}_0^{-1}, \tag{2.137}$$

where $a \to a^t$ is induced by the standard transposition matrices and $\dot{w}_0$ is a lift of the longest element of the Weyl group of $\mathfrak{gl}_{2\ell}$. In the matrix form it can be written as

$$\pi_{2\ell}(\dot{w}_0) = S \cdot J,$$

where $S = \mathrm{diag}(1, -1, \ldots, -1, 1)$ and $J = \|J_{i,j}\| = \|\delta_{i+j,2\ell+2}\|$. The symplectic group $G = Sp(2\ell)$ then can be defined as a following subgroup of $GL(2\ell)$ (see i.e. [DS]):

$$Sp(2\ell) = \{g \in GL(2\ell) : g^* = g\}.$$

Let $\epsilon_{i,j}$ stands for an elementary $(2\ell \times 2\ell)$ matrix with a unit at (i, j) place and zeros otherwise. Introduce the following $(2\ell \times 2\ell)$ matrices:

$$U_n = \sum_{i=1}^{\ell} \epsilon_{i,i} + e^{-z_{n-1,1}} \epsilon_{\ell+1,\ell+1} + \sum_{i=1}^{n-1} e^{z_{n-1,i}} \epsilon_{\ell+1-i,\ell+1+i} \quad , \tag{2.138}$$

$$\begin{aligned} \widetilde{U}_n = \sum_{i=1}^{\ell} \epsilon_{i,i} + e^{-z_{n-1,1}} \epsilon_{\ell+1,\ell+1} + \sum_{i=1}^{n-1} e^{z_{n-1,i}} \epsilon_{\ell+1-i,\ell+1+i} + \\ + \sum_{i=2}^{n} e^{x_{n,i}} \epsilon_{\ell+i-1,\ell+i} + \sum_{i=1}^{\ell-n} \epsilon_{\ell+n+i,\ell+n+i} \quad , \end{aligned} \tag{2.139}$$

$$\begin{aligned} \widetilde{U}'_n = \sum_{i=1}^{\ell} \epsilon_{i,i} + e^{-z_{n-1,1}} \epsilon_{\ell+1,\ell+1} + \sum_{i=1}^{n-1} e^{z_{n-1,i}} \epsilon_{\ell+1-i,\ell+1+i} + \\ + \sum_{i=1}^{n} e^{x_{n,i}} \epsilon_{\ell+i-1,\ell+i} + \sum_{i=1}^{\ell-n} \epsilon_{\ell+n+i,\ell+n+i} \quad , \end{aligned} \tag{2.140}$$

$$V_n = \sum_{i=1}^{\ell-1} \epsilon_{i,i} + e^{-z_{n,1}} \epsilon_{\ell,\ell} + \sum_{i=1}^{n} e^{z_{n,i}} \epsilon_{\ell+i,\ell+i} \quad . \tag{2.141}$$

$$\widetilde{V}_n = \sum_{i=1}^{\ell-1} \epsilon_{i,i} + e^{-z_{n,1}} \epsilon_{\ell,\ell} + \sum_{i=1}^{n} e^{z_{n,i}} \epsilon_{\ell+i,\ell+i} + \tag{2.142}$$

$$+ \sum_{i=2}^{n} e^{x_{n,i}} \epsilon_{\ell+i-1,\ell+i} + \sum_{i=1}^{\ell-n} \epsilon_{\ell+n+i,\ell+n+i} \quad ,$$

$$\widetilde{V}'_n = \sum_{i=1}^{\ell-1} \epsilon_{i,i} + e^{-z_{n,1}} \epsilon_{\ell,\ell} + \sum_{i=1}^{n} e^{z_{n,i}} \epsilon_{\ell+i,\ell+i} + \tag{2.143}$$

$$+ \sum_{i=1}^{n} e^{x_{n,i}} \epsilon_{\ell+i-1,\ell+i} + \sum_{i=1}^{\ell-n} \epsilon_{\ell+n+i,\ell+n+i} \quad ,$$

We can define a modified factorized parametrization as follows.

Theorem 2.9 *i) The image of any generic unipotent element $v^{C_\ell} \in N_+$ in the tautological representation $\pi_{2\ell} : \mathfrak{sp}_{2\ell} \to End(\mathbb{C}^{2\ell})$ can be presented in the form*

$$\pi_{2\ell}(v^{C_\ell}) = \mathfrak{X}_1 \mathfrak{X}_2 \cdots \mathfrak{X}_\ell, \tag{2.144}$$

where

$$\begin{aligned} \mathfrak{X}_1 &= 1 + e^{x_{11}+z_{11}} \epsilon_{\ell-1,\ell} \ , \\ \mathfrak{X}_n &= [\widetilde{U}_n U_n^{-1}]^* \widetilde{U}'_n (U'_n)^{-1} [(V'_n)^{-1} \widetilde{V}'_n]^* V_n^{-1} \widetilde{V}_n, \qquad n = 2, \ldots, \ell, \end{aligned} \tag{2.145}$$

and $z_{\ell,k} = 0, \ k = 1, \ldots \ell$ are assumed.

ii) This defines a parametrization of an open part $N_+^{(0)}$ in N_+.

Proof. Let $v^{C_\ell}(y)$ be parametrization of an open part of N_+ according to (2.128)-(2.129). Let $\tilde{X}_i(y) = e^{y e_{i,i+1}}$ be a one-parametric unipotent subgroup in $GL(2\ell)$. Then we have $\tilde{X}_i(y)^* = \tilde{X}_{2\ell+1-i}(y)$. Embed elementary unipotent subgroups $X_i(y)$ of $Sp(2\ell)$ into $GL(2\ell)$ as follows:

$$X_i(y) = \tilde{X}_i(y)^* \, \tilde{X}_i(y).$$

This maps an arbitrary regular unipotent element v^{C_ℓ} into unipotent subgroup of $GL(2\ell)$. Let us now change the variables in the following way:

$$\begin{aligned} y_{11} = e^{x_{11}+z_{11}}, \qquad y_{1,k} &= \left(e^{z_{k-1,1}+x_{k,1}} + e^{z_{k,1}+x_{k,1}} \right), \\ y_{k,2r-1} = e^{x_{k+r-1,k}-z_{k+r-2,k-1}}, & \qquad k = 2, \ldots, \ell, \\ y_{k,2r} = e^{x_{k+r-1,k}-z_{k+r-1,k-1}}, & \qquad r = 1, \ldots, \ell+1-k. \end{aligned} \tag{2.146}$$

Here $z_{\ell,k} = 0$ for $k = 1, \ldots \ell$ are assumed.

By elementary manipulations it is easy to check that after the change of variables (2.146), the image $\pi_{2\ell}(v^{C_\ell})$ of v^{C_ℓ} defined by (2.128)-(2.129) transforms into the (2.144) -(**??**). Taking into account that the change of variables (2.146) is invertible we get a parametrization of $N_+^{(0)} \subset N_+$ $\Box$

The modified factorized parametrization of a unipotent group N_+ defines a particular realization of a principal series representation of $\mathcal{U}(\mathfrak{sp}_{2\ell})$ by differential operators. It can be obtained using the change of variables (2.146) applied to the realization given in Proposition 2.8. We shall use the term Gauss-Givental representation for this realization of representation of $\mathcal{U}(\mathfrak{sp}_{2\ell})$.

Proposition 2.9 *The following differential operators define a representation* π_λ *of* $\mathcal{U}(\mathfrak{sp}_{2\ell})$ *in* V_μ *in terms of the modified factorized parametrization:*

$$E_1 = \sum_{n=1}^{\ell}\left(\frac{\partial}{\partial x_{n,1}} - \frac{\partial}{\partial x_{n+1,1}}\right)\left(e^{z_{n,1}-x_{n,1}} + \sum_{i=n+1}^{\ell}\left(e^{z_{i-1,1}-x_{i,1}} + e^{z_{i,1}-x_{i,1}}\right)\right) - \tag{2.147}$$

$$-\sum_{n=1}^{\ell-1}\frac{\partial}{\partial z_{n,1}}\sum_{i=n+1}^{\ell}\left(e^{z_{i-1,1}-x_{i,1}} + e^{z_{i,1}-x_{i,1}}\right) ,$$

$$E_2 = \left(\frac{\partial}{\partial z_{11}} - \frac{\partial}{\partial x_{11}}\right)\left(e^{z_{22}-x_{22}} + \sum_{i=3}^{\ell} e^{z_{i-1,2}-x_{i,2}} + e^{z_{i,2}-x_{i,2}}\right) + \tag{2.148}$$

$$+\sum_{n=2}^{\ell}\left(\frac{\partial}{\partial x_{n,2}} - \frac{\partial}{\partial x_{n,3}}\right)\left(e^{z_{n,2}-x_{n,2}} + \sum_{i=n+1}^{\ell} e^{z_{i-1,2}-x_{i,2}} + e^{z_{i,2}-x_{i,2}}\right) +$$

$$+\sum_{n=2}^{\ell}\left(\frac{\partial}{\partial z_{n,1}} - \frac{\partial}{\partial z_{n,2}}\right)\sum_{i=n+1}^{\ell}\left(e^{z_{i-1,2}-x_{i,2}} + e^{z_{i,2}-x_{i,2}}\right) -$$

$$-\sum_{n=2}^{\ell}\frac{\partial}{\partial x_{n,1}}\left(e^{z_{n-1,1}+x_{n,1}}\frac{e^{z_{n-1,2}-x_{n,2}} + e^{z_{n,2}-x_{n,2}}}{e^{z_{n-1,1}+x_{n,1}} + e^{z_{n,1}+x_{n,1}}} + \sum_{i=n+1}^{\ell} e^{z_{i-1,2}-x_{i,2}} + e^{z_{i,2}-x_{i,2}}\right) ,$$

$$E_k = \left(\frac{\partial}{\partial z_{k-1,k-1}} + \frac{\partial}{\partial x_{k,k}}\right)\left(e^{z_{k,k}-x_{k,k}} + \sum_{i=k+1}^{\ell} e^{z_{i-1,k}-x_{i,k}} + e^{z_{i,k}-x_{i,k}}\right) + \tag{2.149}$$

$$+\sum_{n=k}^{\ell}\left(\frac{\partial}{\partial z_{n,k-1}} - \frac{\partial}{\partial z_{n,k}}\right)\sum_{i=n+1}^{\ell}\left(e^{z_{i-1,k}-x_{i,k}} + e^{z_{i,k}-x_{i,k}}\right) +$$

$$+\sum_{n=k+1}^{\ell}\left(\frac{\partial}{\partial x_{n,k}} - \frac{\partial}{\partial x_{n,k+1}}\right)\left(e^{z_{n,k}-x_{n,k}} + \sum_{i=n+1}^{\ell} e^{z_{i-1,k}-x_{i,k}} + e^{z_{i,k}-x_{i,k}}\right), \qquad 2 < k < \ell,$$

$$E_\ell = e^{-x_{\ell,\ell}} \left(\frac{\partial}{\partial z_{k-1,k-1}} + \frac{\partial}{\partial x_{k,k}} \right) \quad , \tag{2.150}$$

$$H_k = \langle \mu, \alpha_k^\vee \rangle + \sum_{n=1}^{\ell} a_{k,n} \sum_{i=n}^{\ell} \frac{\partial}{\partial x_{i,n}} \quad , \tag{2.151}$$

$$\begin{aligned} F_1 = -\Big(e^{z_{11}+x_{11}} + \sum_{i=2}^{\ell} e^{z_{i-1,1}+x_{i,1}} + e^{z_{i,1}+x_{i,1}} \Big) \left(\langle \mu, \alpha_1^\vee \rangle + \frac{\partial}{\partial x_{11}} \right) - \\ - \sum_{n=2}^{\ell} \frac{\partial}{\partial x_{n,1}} \Big(e^{z_{n,1}+x_{n,1}} + \sum_{i=n+1}^{\ell} e^{z_{i-1,1}+x_{i,1}} + e^{z_{i,1}+x_{i,1}} \Big) - \\ - \sum_{n=2}^{\ell} \frac{\partial}{\partial z_{n,1}} \sum_{i=n+1}^{\ell} \Big(e^{z_{i-1,1}+x_{i,1}} + e^{z_{i,1}+x_{i,1}} \Big), \end{aligned} \tag{2.152}$$

$$\begin{aligned} F_2 = -\left(\langle \mu, \alpha_2^\vee \rangle - \frac{\partial}{\partial x_{11}} - \frac{\partial}{\partial z_{11}} \right) \sum_{i=2}^{\ell} \Big(e^{x_{i,2}-z_{i-1,1}} + e^{x_{i,2}-z_{i,1}} \Big) + \\ + \sum_{n=2}^{\ell} \left(\frac{\partial}{\partial x_{n,1}} - \frac{\partial}{\partial x_{n,2}} \right) \Big(e^{x_{n,2}-z_{n,1}} + \sum_{i=n+1}^{\ell} e^{x_{i,2}-z_{i-1,1}} + e^{x_{i,2}-z_{i,1}} \Big) + \\ + \sum_{n=2}^{\ell} \left(\frac{\partial}{\partial z_{n,1}} - \frac{\partial}{\partial z_{n,2}} \right) \sum_{i=n+1}^{\ell} \Big(e^{x_{i,2}-z_{i-1,1}} + e^{x_{i,2}-z_{i,1}} \Big), \end{aligned} \tag{2.153}$$

$$\begin{aligned} F_k = -\left(\langle \mu, \alpha_k^\vee \rangle - \frac{\partial}{\partial x_{k-1,k-1}} - \frac{\partial}{\partial z_{k-1,k-1}} \right) \sum_{i=k}^{\ell} \Big(e^{x_{i,k}-z_{i-1,k-1}} + e^{x_{i,k}-z_{i,k-1}} \Big) + \\ + \sum_{n=k}^{\ell} \left(\frac{\partial}{\partial x_{n,k-1}} - \frac{\partial}{\partial x_{n,k}} \right) \Big(e^{x_{n,k}-z_{n,k-1}} + \sum_{i=n+1}^{\ell} e^{x_{i,k}-z_{i-1,k-1}} + e^{x_{i,k}-z_{i,k-1}} \Big) + \\ + \sum_{n=k}^{\ell} \left(\frac{\partial}{\partial z_{n,k-1}} - \frac{\partial}{\partial z_{n,k}} \right) \sum_{i=n+1}^{\ell} \Big(e^{x_{i,k}-z_{i-1,k-1}} + e^{x_{i,k}-z_{i,k-1}} \Big), \qquad 2 < k < \ell, \end{aligned} \tag{2.154}$$

$$\begin{aligned} F_\ell \, = \, \Big(e^{x_{\ell,\ell}-z_{\ell-1,\ell-1}} + e^{x_{\ell,\ell}} \Big) \left(-\langle \mu, \alpha_\ell^\vee \rangle + \frac{\partial}{\partial x_{k-1,k-1}} + \frac{\partial}{\partial z_{k-1,k-1}} \right) + \\ + e^{x_{\ell,\ell}} \left(\frac{\partial}{\partial x_{\ell,\ell-1}} - \frac{\partial}{\partial x_{\ell,\ell}} \right), \end{aligned} \tag{2.155}$$

where $z_{\ell,k} = 0, k = 1, \ldots, \ell$ are assumed and the derivatives over $x_{i,k}$, $z_{i,k}$, $i < k$ are omitted. Here we denote $E_i = \pi_\lambda(e_i)$, $F_i = \pi_\lambda(f_i)$, $H_i = \pi_\lambda(h_i)$ $i = 1, \ldots, \ell$.

We are going to write down matrix element (2.12) explicitly using Gauss-Givental representation defined above. Whittaker vectors ψ_R and ψ_L in this representation satisfiy the system of differential equations

$$E_i\psi_R(x) = -\psi_R(x), \qquad F_i\psi_L(x) = -\psi_L(x), \qquad 1 \le i \le \ell. \tag{2.156}$$

Its solution has the following form. Using the explicit change of the variables (2.146) we obtain the expressions for Whittaker vectors in modified factorized parametrization.

Lemma 2.8 *The following expressions for left/right Whittaker vectors hold:*

$$\psi_R = \exp\Big\{ -\Big(e^{x_{11}+z_{11}} + \sum_{n=2}^{\ell}\Big(e^{z_{n-1,1}-x_{n,1}} + e^{z_{n,1}-x_{n,1}}\Big)\Big) - \tag{2.157}$$

$$-\sum_{k=2}^{\ell}\sum_{n=k}^{\ell}\Big(e^{x_{n,k}-z_{n-1,k-1}} + e^{x_{n,k}-z_{n,k-1}}\Big)\Big\},$$

$$\psi_L = e^{\mu_1 z_{1,1}}\prod_{n=2}^{\ell}\Big(e^{z_{n,1}} + e^{z_{n-1,1}}\Big)^{\mu_n} \times \tag{2.158}$$

$$\times \prod_{n=1}^{\ell}\exp\Big\{ -\mu_n\Big(\sum_{i=1}^{n} z_{n,i} - x_{n,1} - 2\sum_{i=2}^{n} x_{n,i} + \sum_{i=1}^{n-1} z_{n-1,i}\Big)\Big\} \times$$

$$\times \exp\Big\{ -\sum_{k=1}^{\ell}\Big(e^{z_{k,k}-x_{k,k}} + \sum_{n=k+1}^{\ell} e^{z_{n-1,k}-x_{n,k}} + e^{z_{n,k}-x_{n,k}}\Big)\Big\},$$

where $z_{\ell,k} = 0$ and $\mu_k = \imath\lambda_k - \rho_k$, $\rho_k = k$ for $k = 1, \ldots, \ell$.

Now we are ready to find an integral representation of the pairing (2.12) for $\mathfrak{g} = \mathfrak{sp}_{2\ell}$. To get an explicit expression for the integrand, one uses the same type of decomposition of the Cartan element as for $\mathfrak{gl}_{\ell+1}$ and $\mathfrak{sp}_{2\ell}$ before:

$$e^{-H_z} = \pi_\lambda(\exp(-\sum_{i=1}^{\ell}\langle\omega_i, z\rangle h_i)) = e^{H_L}e^{H_R},$$

where

$$-H_z = H_L + H_R = -\langle\mu, z_\ell\rangle - 2z_{\ell,1}\sum_{n=1}^{\ell}\frac{\partial}{\partial x_{n,1}} + \sum_{k=1}^{\ell-1}(z_{\ell,i} - z_{\ell,i+1})\sum_{n=k}^{\ell}\frac{\partial}{\partial x_{n,k}}, \tag{2.159}$$

with

$$H_L=\sum_{k=1}^{\ell} z_{\ell,k}\Big(\sum_{n=k}^{\ell-1}\frac{\partial}{\partial z_{n,k}}+\sum_{n=k}^{\ell}\frac{\partial}{\partial x_{n,k}}\Big), \tag{2.160}$$

$$H_R=-\langle\mu,z_\ell\rangle-\sum_{k=1}^{\ell-1} z_{\ell,k}\Big(\sum_{n=k}^{\ell}\frac{\partial}{\partial x_{n,k}}-\sum_{n=k}^{\ell-1}\frac{\partial}{\partial z_{n,k}}-\sum_{n=k+1}^{\ell}\frac{\partial}{\partial x_{n,k+1}}\Big). \tag{2.161}$$

We imply that H_L acts on the left vector and H_R acts on the right vector in (2.12). Taking into account Lemma 2.8 one obtains the following theorem.

Theorem 2.10 *The eigenfunctions of $\mathfrak{sp}_{2\ell}$-Toda chain (2.12) admit the integral representation*

$$\Psi^{C_\ell}_{\lambda_1,\dots,\lambda_\ell}(z_{\ell,1},\dots,z_{\ell,\ell})\;=\;\int\limits_C\bigwedge_{k=1}^{\ell-1}\bigwedge_{i=1}^{k}dz_{k,i}\bigwedge_{k=1}^{\ell}\bigwedge_{i=1}^{k}dx_{k,i}\,e^{\mathcal{F}^{C_\ell}}, \tag{2.162}$$

where

$$\mathcal{F}^{C_\ell}=\imath\lambda_1z_{1,1}-\sum_{n=2}^{\ell}\imath\lambda_n\Big(\sum_{i=1}^{n}z_{n,i}-x_{n,1}-2\sum_{i=2}^{n}x_{n,i}+\sum_{i=1}^{n-1}z_{n-1,i}-\ln(e^{z_{n,1}}+e^{z_{n-1,1}})\Big)-$$

$$-\Big\{\sum_{k=1}^{\ell}\Big(e^{z_{k,k}-x_{k,k}}+\sum_{n=k+1}^{\ell}e^{z_{n-1,k}-x_{n,k}}+e^{z_{n,k}-x_{n,k}}\Big)+e^{x_{11}+z_{11}}+ \tag{2.163}$$

$$+\sum_{n=2}^{\ell}\Big(e^{z_{n-1,1}-x_{n,1}}+e^{z_{n,1}-x_{n,1}}\Big)+\sum_{k=2}^{\ell}\sum_{n=k}^{\ell}\Big(e^{x_{n,k}-z_{n-1,k-1}}+e^{x_{n,k}-z_{n,k-1}}\Big)\Big\}\;,$$

where we set $z_i:=z_{\ell,i}$, $1\le i\le\ell$. Here $C\subset N_+$ is a middle-dimensional non-compact submanifold such that the integrand decays exponentially at the boundaries and at infinities. In particular the domain of integration can be chosen to be $C=\mathbb{R}^m$, where $m=l(w_0)$.

Example 2.4 *For $\ell=2$ the general expression (2.202) acquires the form*

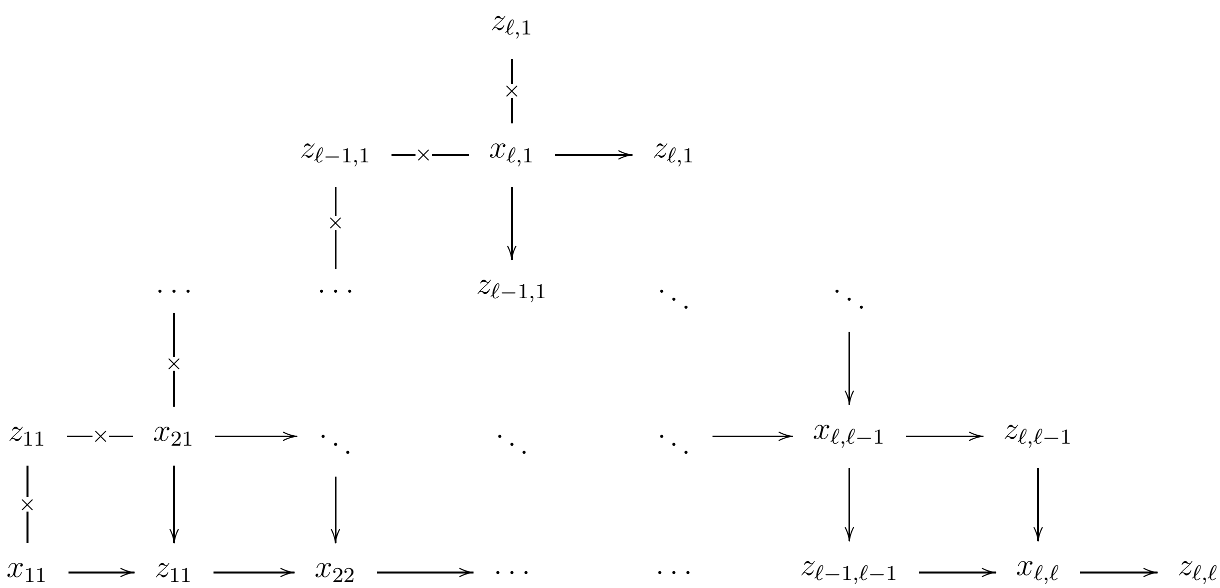
$$\Psi^{C_2}_{\lambda_1,\lambda_2}(z_1,z_\ell)\;=\;\int\limits_C dx_{11}\wedge x_{11}\wedge dx_{21}\wedge dx_{22}\wedge dz_{11}\times$$

$$\times\exp\Big\{\imath\lambda_1x_{11}-\imath\lambda_2\Big(z_{21}+z_{22}-x_{21}-2x_{22}+z_{11}-\log(e^{z_{21}}+e^{z_{11}})\Big)- \tag{2.164}$$

$$-\Big(e^{z_{11}-x_{11}}+e^{z_{22}-x_{22}}+e^{x_{11}+z_{11}}+e^{z_{11}-x_{21}}+e^{z_{21}-x_{21}}+e^{x_{22}-z_{11}}+e^{x_{22}-z_{21}}\Big)\Big\},$$

where $z_1=z_{2,1},z_2=z_{2,2}$. In particular one can chose $C=\mathbb{R}^4$.

There is a simple combinatorial description of the potential $\mathcal{F}^{C_\ell}$ for zero spectrum $\{\lambda_i = 0\}$. Namely, it can be presented as the sum over all arrows in the following diagram.



We use the same rule to assign variables to the arrows of the diagram as for A_ℓ. In addition we assign to the symbol z —×— x the exponent e^{-z-x}.

Note that the diagram for C_ℓ can be obtained by a factorization of the diagram for $A_{2\ell-1}$. Consider the following involution

$$\iota : \quad X \longmapsto \dot{w}_0^{-1} X^t \dot{w}_0, \tag{2.165}$$

where $\dot{w}_0$ is a lift the longest element of $A_{2\ell-1}$ Weyl group and X^t denotes the standard transposition. Corresponding action on the modified factorization parameters is given by

$$\dot{w}_0 : \qquad x_{k,i} \longleftrightarrow -x_{k,k+1-i}. \tag{2.166}$$

This defines a factorization of $A_{2\ell-1}$-diagram that produces the diagram for C_ℓ.

One can easily write down C_ℓ-analog of A_ℓ-monomial relations (2.52). Let us introduce the variables

$$\begin{aligned} a_{k,1} &= e^{x_{k,1}+z_{k-1,1}}, \qquad & a_{k,i} &= e^{x_{k,i}-z_{k-1,i-1}}, \\ b_{k,1} &= e^{x_{k,1}+z_{k,1}}, \qquad & b_{k,i} &= e^{z_{k,i}-x_{k,i-1}}, \\ c_{k,i} &= e^{z_{k,i}-x_{k,i}}, \qquad & d_{k,i} &= e^{z_{k,i}-x_{k+1,i}}. \end{aligned} \tag{2.167}$$

Then the following relations hold

$$\begin{aligned} c_{k,i} \cdot b_{k,i} \; &= \; d_{k,i} \cdot a_{k+1,i}, \qquad & a_{k,i} \cdot d_{k-1,i-1} \; &= \; b_{k,i} \cdot c_{k,i-1}, \\ b_{\ell,1} c_{\ell,1} &= e^{2z_{\ell,1}}, \qquad & c_{\ell,i} \cdot b_{\ell,i} &= e^{z_{\ell,i}-z_{\ell,i-1}}. \end{aligned} \tag{2.168}$$

The above relations can be considered as relations between elementary paths on the Givental diagram. Using relations for more general paths that follows from (2.168) one can define a toric degeneration of the C_ℓ-flag manifolds thus generalizing results of [BCFKS].

2.4.3 Recursion for $\mathfrak{sp}_{2\ell}$-Whittaker functions and $\mathcal{Q}$-operator for $A^{(2)}_{2\ell-1}$-Toda chain

The integral representation (2.162), (2.163) of $\mathfrak{sp}_{2\ell}$-Whittaker functions possesses a recursive structure over the rank ℓ. For any $n = 2, \ldots, \ell$ let us introduce integral operators $Q^{C_n}_{C_{n-1}}$ with the integral kernels

$$Q^{C_n}_{C_{n-1}}(\underline{z}_n;\,\underline{z}_{n-1};\lambda_n) = \int \bigwedge_{i=1}^{n} dx_{n,i}\;\Big(e^{z_{n,1}}+e^{z_{n-1,1}}\Big)^{\imath\lambda_n}\times$$
$$\times\exp\Big\{-\imath\lambda_n\Big(\sum_{i=1}^{n} z_{n,i}-x_{n,1}-2\sum_{i=2}^{n}x_{n,i}+\sum_{i=1}^{n-1}z_{n-1,i}\Big)\Big\}\;\times \tag{2.169}$$
$$\times Q^{C_n}_{D_n}(\underline{z}_n;\,\underline{x}_n)\;Q^{D_n}_{C_{n-1}}(\underline{x}_n;\,\underline{z}_{n-1}),$$

where

$$Q^{D_n}_{C_{n-1}}(\underline{x}_n;\,\underline{z}_{n-1}) =$$
$$=\exp\Big\{-\Big(e^{x_{n,1}+z_{n-1,1}}+\sum_{i=1}^{n-1}\Big(e^{z_{n-1,i}-x_{n,i}}+e^{x_{n,i+1}-z_{n-1,i}}\Big)\Big)\Big\}, \tag{2.170}$$

$$Q^{C_n}_{D_n}(\underline{z}_n;\,\underline{x}_n) =$$
$$=\exp\Big\{-\Big(e^{x_{n,1}+z_{n,1}}+\sum_{i=1}^{n-1}\Big(e^{z_{n,i}-x_{n,i}}+e^{x_{n,i+1}-z_{n,i}}\Big)+e^{z_{n,n}-x_{n,n}}\Big)\Big\}. \tag{2.171}$$

For $n=1$ we define

$$Q^{C_1}_{C_0} = \int dx_{11}e^{\imath\lambda_1 x_{11}}\exp\Big\{-\Big(e^{x_{11}+z_{11}}+e^{z_{11}-x_{11}}\Big)\Big\}$$

Using integral operators $Q^{C_n}_{C_{n-1}}$, the integral representation for $\mathfrak{sp}_{2\ell}$-Whittaker function can be written in the recursive form.

Theorem 2.11 *The integral representations of $\mathfrak{sp}_{2\ell}$-Toda chain eigenfunctions (2.162) can be written as*

$$\Psi^{C_\ell}_{\lambda_1,\ldots,\lambda_\ell}(z_1,\ldots,z_\ell)\;=\;\int\limits_{\mathcal{C}}\bigwedge_{k=1}^{\ell-1}\bigwedge_{i=1}^{k}dz_{k,i}\;\prod_{n=1}^{\ell}Q^{C_n}_{C_{n-1}}(\underline{z}_n;\,\underline{z}_n;\lambda_n), \tag{2.172}$$

or equivalently

$$\Psi^{C_\ell}_{\lambda_1,\ldots,\lambda_\ell}(z_{\ell,1},\ldots,z_{\ell,\ell})\;=\;\int\limits_{\mathcal{C}_\ell}\bigwedge_{i=1}^{\ell-1}dz_{\ell-1,i}\;\times \tag{2.173}$$
$$\times Q^{C_\ell}_{C_{\ell-1}}(\underline{z}_\ell;\,\underline{z}_{\ell-1};\lambda_\ell)\Psi^{C_{\ell-1}}_{\lambda_1,\ldots,\lambda_{\ell-1}}(z_{\ell-1,1},\ldots,z_{\ell-1,\ell-1}).$$

Here $z_n := z_{\ell,n},\ 1 \le n \le \ell$ and $\mathcal{C} \subset N_+$ is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at the boundaries and at infinities. In particular as a domain of integration one can chose $\mathcal{C} = \mathbb{R}^{\ell^2}$.

This recursive form of the integral representation is similar to the case of $\mathfrak{so}_{2\ell+1}$. Its recursive kernel $Q^{C_n}_{C_{n-1}}$ is given by a nontrivial integral in contrast with $\mathfrak{gl}_{\ell+1}$-case (2.54). Similar to $\mathfrak{so}_{2\ell+1}$-Whittaker function new structure appears if we consider the Whittaker function for zero spectrum $\{\lambda_i = 0\}$. As it is clear from (2.169) the kernels $Q^{C_n}_{C_{n-1}}$ at $\lambda_n = 0$ are given by convolutions of the kernels $Q^{C_n}_{D_n}(\underline{z}_n, \underline{x}_n)$ and $Q^{D_n}_{C_{n-1}}(\underline{x}_n, \underline{z}_{n-1})$. The corresponding integral operators $Q^{C_n}_{D_n}$, $Q^{D_n}_{C_{n-1}}$ can be regarded as elementary intertwiners relating Hamiltonians of Toda chains for C_n, D_n and D_n, C_{n-1} root systems correspondingly. For example it is easy to check directly intertwinig realtions with quadratic Hamiltonians. Indeed, D_ℓ-Toda chain (for more detailed discussion see Subsection 2.5.3) has the following quadratic Hamiltonians

$$\mathcal{H}^{D_\ell}_2(\underline{x}^{(\ell)}) = -\frac{1}{2}\sum_{i=1}^{\ell}\frac{\partial^2}{\partial x_i^2} + e^{x_1+x_2} + \sum_{i=1}^{\ell-1} e^{x_{i+1}-x_i}. \qquad (2.174)$$

Proposition 2.10 *The integral operators $Q^{C_n}_{C_{n-1}}$, $Q^{C_n}_{D_n}$ and $Q^{D_n}_{C_{n-1}}$ satisfy the following relations.*

1. *Operators $Q^{C_n}_{D_n}$ and $Q^{D_n}_{C_{n-1}}$ intertwine quadratic Hamiltonians of C- and D-Toda chains:*

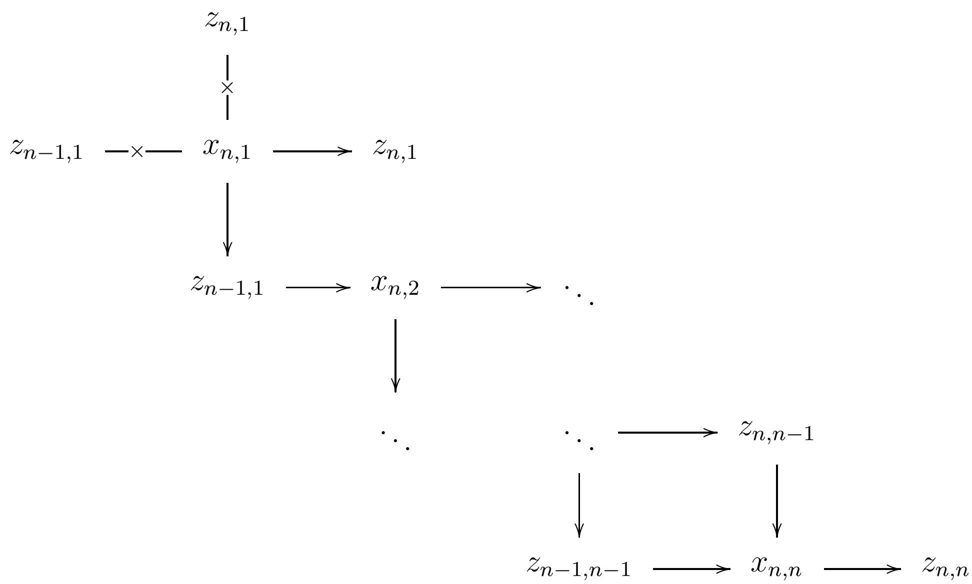
$$\mathcal{H}^{D_n}_2(\underline{x}_n)Q^{D_n}_{C_{n-1}}(\underline{x}_n,\, \underline{z}_{n-1}) = Q^{D_n}_{C_{n-1}}(\underline{x}_n,\, \underline{z}_{n-1})\mathcal{H}^{C_{n-1}}_2(\underline{x}_{n-1}), \qquad (2.175)$$

$$\mathcal{H}^{C_n}_2(\underline{z}_n)Q^{C_n}_{D_n}(\underline{z}_n,\, \underline{x}_n) = Q^{C_n}_{D_n}(\underline{z}_n,\, \underline{x}_n)\mathcal{H}^{D_n}_2(\underline{x}_n). \qquad (2.176)$$

2. *The operator $Q^{C_n}_{C_{n-1}}$ at $\lambda_n = 0$ intertwines the Hamiltonians $\mathcal{H}^{C_n}_2$ and $\mathcal{H}^{C_{n-1}}_2$:*

$$\mathcal{H}^{C_n}_2(\underline{z}_n)Q^{C_n}_{C_{n-1}}(\underline{z}_n,\, \underline{z}_{n-1}) = Q^{C_n}_{C_{n-1}}(\underline{z}_n,\, \underline{z}_{n-1})\mathcal{H}^{C_{n-1}}_2(\underline{z}_{n-1}). \qquad (2.177)$$

The integral kernel of $Q^{C_n}_{C_{n-1}}$ can be succinctly encoded into the following diagram



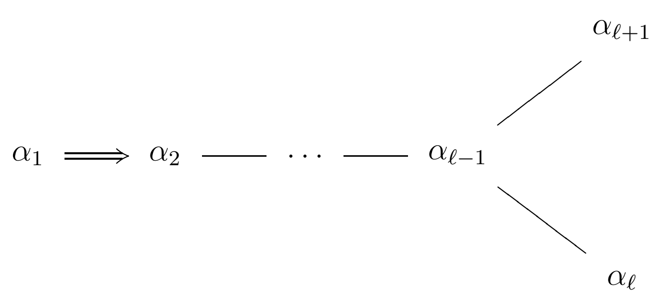
Here the upper (lower) boundary of the oriented diagram corresponds to the kernels of elementary intertwiner $Q^{C_n}_{D_n}$ ($Q^{D_n}_{C_{n-1}}$) and the convolution of the two kernel is given by the integration over variables $x_{n,1},\dots,x_{n,n}$ on the diagonal of the diagram.

Similarly to the cases of $\mathfrak{gl}_{\ell+1}$ and $\mathfrak{sp}_{2\ell}$ recursion operators $Q^{C_n}_{C_{n-1}}$ can be considered as degenerations of a Baxter $\mathcal{Q}$-operators for twisted affine $A^{(2)}_{2\ell-1}$-Toda chain introduced below. Let us stress that up to now $\mathcal{Q}$-operators for $A^{(2)}_{2\ell-1}$ were not known. We will not present here a complete set of the characteristic properties of the introduced $\mathcal{Q}$-operators and only consider commutation relations with quadratic affine Toda chain Hamiltonians. The detailed account will be given elsewhere.

We start with a description of $A^{(2)}_{2\ell-1}$-Toda chains. The set of simple roots of the affine root system $A^{(2)}_{2\ell-1}$ can be represented in terms of the orthogonal bases $\{\epsilon_i\}$, $i=1,\dots,\ell$ in $\mathbb{R}^\ell$ as follows:

$$\alpha_1=2\epsilon_1,\qquad \alpha_{i+1}=\epsilon_{i+1}-\epsilon_i,\qquad 1\leq i\leq\ell-1,\qquad \alpha_{\ell+1}=-\epsilon_\ell-\epsilon_{\ell-1},\qquad (2.178)$$

and corresponding Dynkin diagram is given by



These root data allows to define affine $A^{(2)}_{2\ell-1}$-Toda chain with the quadratic Hamiltonian given by

$$\mathcal{H}_2^{A^{(2)}_{2\ell-1}}(\underline{z}^{(\ell)}) \;=\; -\frac{1}{2}\sum_{i=1}^{\ell}\frac{\partial^2}{\partial z_i^2}+2e^{2z_1}+\sum_{i=1}^{\ell-1}e^{z_{i+1}-z_i}+ge^{-z_{\ell-1}-z_\ell}, \tag{2.179}$$

where g is an arbitrary parameter.

Define the Baxter $\mathcal{Q}$-operator of $A^{(2)}_{2\ell-1}$-Toda chain as an integral operator with the following integral kernel:

$$\mathcal{Q}^{A^{(2)}_{2\ell-1}}(\underline{z}^{(\ell)}\,,\underline{y}^{(\ell)},\lambda)=\int\bigwedge_{i=1}^{\ell+1}dx_i\,\Big(e^{z_1}+e^{y_1}\Big)^{\imath\lambda}\Big(e^{-z_\ell}+e^{-y_\ell}\Big)^{-2\imath\lambda}\times \tag{2.180}$$

$$\times\exp\Big\{-\imath\lambda\Big(\sum_{i=1}^{\ell}z_i-x_1-2\sum_{i=2}^{\ell}x_i+\sum_{i=1}^{\ell}y_i\Big)\Big\}\times$$

$$\times Q^{A^{(2)}_{2\ell-1}}_{A^{(2)}_{2\ell-1}}(z_1,\ldots,z_\ell\,;x_1,\ldots,x_{\ell+1})\;\,Q^{A^{(2)}_{2\ell-1}}_{A^{(2)}_{2\ell-1}}(x_1,\ldots,x_{\ell+1}\,;y_1,\ldots,y_\ell),$$

where

$$Q^{A^{(2)}_{2\ell-1}}_{A^{(2)}_{2\ell-1}}(z_1,\ldots,z_\ell\,;\,x_1,\ldots,x_{\ell+1})= \tag{2.181}$$

$$=\exp\Big\{-\Big(e^{z_1+x_1}+\sum_{i=1}^{\ell-1}\Big(e^{z_i-x_i}+e^{x_{i+1}-z_i}\Big)+e^{z_\ell-x_\ell}+ge^{-z_\ell-x_\ell}\Big)\Big\},$$

and

$$Q^{A^{(2)}_{2\ell-1}}_{A^{(2)}_{2\ell-1}}(x_1,\ldots,x_{\ell+1}\,;\,y_1,\ldots,y_\ell)= \tag{2.182}$$

$$=\exp\Big\{-\Big(e^{y_1+x_1}+\sum_{i=1}^{\ell-1}\Big(e^{y_i-x_i}+e^{x_{i+1}-y_i}\Big)+e^{y_\ell-x_\ell}+ge^{-y_\ell-x_\ell}\Big)\Big\}.$$

Here we use the following notations $\underline{z}^{(\ell)}=(z_1,\ldots,z_\ell)$, $\underline{y}^{(\ell)}=(y_1,\ldots,y_\ell)$.

The following statement can be verified straightforwardly.

Proposition 2.11 *The $\mathcal{Q}$-operator (2.180) commutes with the quadratic Hamiltonian of $A^{(2)}_{2\ell-1}$-Toda chain:*

$$\mathcal{H}^{A^{(2)}_{2\ell-1}}(\underline{z}^{(\ell)})Q^{A^{(2)}_{2\ell-1}}(\underline{z}^{(\ell)},\,\underline{y}^{(\ell)})=Q^{A^{(2)}_{2\ell-1}}(\underline{z}^{(\ell)},\,\underline{y}^{(\ell)})\mathcal{H}^{A^{(2)}_{2\ell-1}}(\underline{y}^{(\ell)}). \tag{2.183}$$

Now we will demonstrate that the recursion operator $Q^{C_\ell}_{C_{\ell-1}}$ can be obtained by a degeneration of the Baxter $\mathcal{Q}$-operator for $A^{(2)}_{2\ell-1}$. Consider a slightly modified recursion operator $Q^{C_\ell}_{C_{\ell-1}\oplus C_1}$ with the kernel given by

$$Q^{C_\ell}_{C_{\ell-1}\oplus C_1}(\underline{z}^{(\ell)}\,,\,\underline{y}^{(\ell)}\,,\lambda):=e^{\imath\lambda y_\ell}\;Q^{C_\ell}_{C_{\ell-1}}(\underline{z}^{(\ell)}\,,\,\underline{y}^{(\ell-1)}\,,\,\lambda),$$

where $\underline{y}^{(\ell-1)}=(y_1,\ldots,y_{\ell-1})$. Thus defined operator intertwines Hamiltonians of $\mathfrak{sp}_{2\ell}$- and $\mathfrak{sp}_{2\ell-2}\oplus\mathfrak{sp}_2$-Toda chains. Thus for quadratic Hamiltonians we have

$$\mathcal{H}_2^{C_\ell}(\underline{z}^{(\ell)})Q^{C_\ell}_{C_{\ell-1}\oplus C_1}(\underline{z}^{(\ell)}\,,\,\underline{y}^{(\ell)},\lambda)=Q^{C_\ell}_{C_{\ell-1}\oplus C_1}(\underline{z}^{(\ell)}\,,\underline{y}^{(\ell)}\,,\lambda)\Big(\mathcal{H}_2^{C_\ell}(\underline{y}^{(\ell-1)})+\mathcal{H}_2^{C_1}(y_\ell)\Big),$$

where $\mathcal{H}_2^{C_1}(y_\ell)=-\frac{1}{2}\Big(\partial^2/\partial y_\ell^2\Big)$. Obviously the projection of above equation on the subspace of functions $F(\underline{y}^{(\ell)})=\exp(\imath\lambda y_\ell)f(\underline{y}^{(\ell-1)})$ recovers the genuine recursion operator satisfying:

$$\mathcal{H}_2^{C_\ell}(\underline{z}^{(\ell)})Q^{C_\ell}_{C_{\ell-1}}(\underline{z}^{(\ell)}\,,\,\underline{y}^{(\ell-1)}\,,\,\lambda)=Q^{C_\ell}_{C_{\ell-1}}(\underline{z}^{(\ell)}\,,\,\underline{y}^{(\ell-1)}\,,\,\lambda)\Big(\mathcal{H}_2^{C_\ell}(\underline{y}^{(\ell-1)})+\frac{1}{2}\lambda^2)\Big). \tag{2.184}$$

Consider a one-parameter family of the kernels:

$$\begin{aligned}\mathcal{Q}^{A^{(2)}_{2\ell-1}}(\underline{z}^{(\ell)},\,\underline{y}^{(\ell)},\lambda,\,\varepsilon)=\varepsilon^{\imath\lambda}e^{\imath\lambda y_\ell}\int\bigwedge_{i=1}^{\ell+1}dx_i\,\Big(e^{z_1}+e^{y_1}\Big)^{\imath\lambda}\Big(\varepsilon e^{y_\ell-z_\ell}+1\Big)^{-2\imath\lambda}\times\\ \times\exp\Big\{-\imath\lambda\Big(\sum_{i=1}^{\ell}z_i-x_1-2\sum_{i=2}^{\ell}x_i+\sum_{i=1}^{\ell-1}y_i\Big)\Big\}\times\\ \times Q^{A^{(2)}_{2\ell-1}}_{A^{(2)}_{2\ell-1}}(z_1,\ldots,z_\ell\,;x_1,\ldots,x_{\ell+1})\;Q^{A^{(2)}_{2\ell-1}}_{A^{(2)}_{2\ell-1}}(x_1,\ldots,x_{\ell+1}\,;y_1,\ldots,y_\ell;\,\varepsilon),\end{aligned} \tag{2.185}$$

where

$$\begin{aligned}Q^{A^{(2)}_{2\ell-1}}_{A^{(2)}_{2\ell-1}}(x_1,\ldots,x_{\ell+1}\,;y_1,\ldots,y_\ell;\varepsilon)=\exp\Big\{-\Big(e^{y_1+x_1}+\sum_{i=1}^{\ell-1}\Big(e^{y_i-x_i}+e^{x_{i+1}-y_i}\Big)+\\ +\varepsilon e^{y_\ell-x_\ell}+\varepsilon^{-1}ge^{-y_\ell-x_\ell}\Big)\Big\},\end{aligned} \tag{2.186}$$

is obtained by shifting the variable $y_\ell=y_\ell+\ln\varepsilon$ in (2.180). Then the following relation between $\mathcal{Q}$-operator for $A^{(2)}_{2\ell-1}$-Toda chain and recursive operator for the $\mathfrak{sp}_{2\ell}$-Whittaker function holds

$$Q^{C_\ell}_{C_{\ell-1}\oplus C_1}(\underline{z}^{(\ell)}\,,\,\underline{y}^{(\ell)},\lambda)=\lim_{\varepsilon\to0,\varepsilon^{-1}g\to0}\varepsilon^{-\imath\lambda}\mathcal{Q}^{A^{(2)}_{2\ell-1}}(\underline{z}^{(\ell)}\,,\,\underline{y}^{(\ell)},\lambda;\,\varepsilon). \tag{2.187}$$

2.5 Integral representations of $\mathfrak{so}_{2\ell}$-Toda chain eigenfunctions

In this subsection we provide an analog of the Givental integral representation of Whittaker functions for $\mathfrak{so}_{2\ell}$ Lie algebras. As in the previously considered cases, we start with a derivation of an integral representation of $\mathfrak{so}_{2\ell}$-Whittaker functions using the factorized representation. Then we consider a modification of the factorized representation leading to a Givental type integral representation of $\mathfrak{so}_{2\ell}$-Whittaker functions.

Consider D_ℓ root system corresponding to Lie algebra $\mathfrak{so}_{2\ell}$. Let $(\epsilon_1, \dots, \epsilon_\ell)$ be an orthogonal basis in $\mathbb{R}^\ell$. We realize D_ℓ simple root and fundamental weights as the following vectors in $\mathbb{R}^\ell$:

$$\begin{aligned} &\alpha_1 = \epsilon_2 - \epsilon_1, && \omega_1 = (-\epsilon_1 + \epsilon_2 + \ldots + \epsilon_\ell)/2, \\ &\alpha_2 = \epsilon_2 + \epsilon_1, && \omega_2 = (\epsilon_1 + \epsilon_2 + \ldots + \epsilon_\ell)/2, \\ &\alpha_3 = \epsilon_3 - \epsilon_2, && \omega_3 = \epsilon_3 + \ldots + \epsilon_\ell, \\ &\ldots && \ldots \\ &\alpha_n = \epsilon_\ell - \epsilon_{\ell-1}, && \omega_\ell = \epsilon_\ell. \end{aligned} \tag{2.188}$$

Coroots $\alpha_i^\vee$ can be identified with the corresponding roots α_i using the scalar product in $\mathbb{R}^\ell$. One associates with these root data $\mathfrak{so}_{2\ell}$-Toda chain with a quadratic Hamiltonian given by

$$\mathcal{H}_2^{D_\ell} \;=\; -\frac{1}{2}\sum_{i=1}^{\ell}\frac{\partial^2}{\partial {x_i}^2} + e^{x_1+x_2} + \sum_{i=1}^{\ell-1} e^{x_{i+1}-x_i}. \tag{2.189}$$

One can complete (2.189) to a full set of ℓ mutually commuting functionally independent Hamiltonians $H_k^{D_\ell}$ of the $\mathfrak{so}_{2\ell}$-Toda chain. We are looking for integral representations of common eigenfunctions of the full set of the Hamiltonians. Corresponding eigenfunction problem for the quadratic Hamiltonian can be written in the following form

$$\mathcal{H}_2^{D_\ell}\;\Psi^{D_\ell}_{\lambda_1,\cdots,\lambda_\ell}(x_1,\dots,x_\ell) = \frac{1}{2}\sum_{i=1}^{\ell}\lambda_i^2\;\Psi^{D_\ell}_{\lambda_1,\cdots,\lambda_\ell}(x_1,\dots,x_\ell). \tag{2.190}$$

2.5.1 $\mathfrak{so}_{2\ell}$-Whittaker function: factorized parametrization

The reduced word for the maximal length element w_0 in the Weyl group of $\mathfrak{so}_{2\ell}$ can be represented in the following recursive way:

$$I = (i_1, i_2, \dots, i_m) := (12, 3123, \dots, (\ell\ldots 3123\ldots\ell)),$$

where index i_k corresponds to an elementary reflection with respect to the root α_k. Let $N_+ \subset G$ be a maximal unipotent subgroup of $G = SO(2\ell)$. One associates with the reduced word I the following recursive parametrization of a generic element $v^{D_\ell} \in N_+$:

$$v^{D_\ell} = v^{D_{\ell-1}}\cdot \mathfrak{X}^{D_\ell}_{D_{\ell-1}}, \tag{2.191}$$

where

$$\begin{aligned}\mathfrak{X}^{D_\ell}_{D_{\ell-1}} = X_\ell(y_{\ell,1})\cdots X_k(y_{k,2(\ell+1-k)-1})\cdots X_3(y_{3,2\ell-5})X_1(y_{1,\ell-1})\,\cdot \\ X_2(y_{2,\ell-1})X_3(y_{3,2\ell-4})\cdots X_k(y_{k,2(\ell+1-k)})\cdots X_\ell(y_{\ell,2}).\end{aligned} \tag{2.192}$$

Here $X_i(y) = e^{ye_i}$ and $e_i \equiv e_{\alpha_i}$ are simple root generators. The subset $N_+^{(0)}$ of the elements allowing representation (2.191), (2.192) is an open part of N_+. The action of the Lie algebra $\mathfrak{so}_{2\ell}$ on N_+ (2.10) defines an action on the space of functions on $N_+^{(0)}$. The explicit description of the action on the space V_μ of (twisted) functions on $N_+^{(0)}$ is given below.

Proposition 2.12 *The following differential operators define a realization of a principal series representation π_λ of $\mathcal{U}(\mathfrak{so}_{2\ell})$ in V_μ in terms of factorized parametrization of $N_+^{(0)}$:*

$$E_1 = \frac{\partial}{\partial y_{1,\ell-1}} + \sum_{n=1}^{[\ell/2]} \Big(\frac{\partial}{\partial y_{2,\ell-n-1}} - \frac{\partial}{\partial y_{2,\ell-n}}\Big) \prod_{k=1}^{2n-1} \Big(\frac{y_{1,\ell-k}}{y_{2,\ell-k}}\Big)^{(-1)^k} \frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}} + \tag{2.193}$$
$$+ \sum_{n=2}^{[\ell/2]} \Big(\frac{\partial}{\partial y_{1,\ell-n-1}} - \frac{\partial}{\partial y_{1,\ell-n}}\Big) \prod_{k=1}^{2(n-1)} \Big(\frac{y_{1,\ell-k}}{y_{2,\ell-k}}\Big)^{(-1)^k} \frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}} +$$
$$+ \sum_{n=1}^{[\frac{\ell-1}{2}]} \Big(\frac{\partial}{\partial y_{3,2(2n-1)-1}} - \frac{\partial}{\partial y_{3,2(2n-1)}}\Big) \frac{y_{3,2(2n-1)}}{y_{1,2n}} \prod_{k=1}^{\ell-2n-1} \Big(\frac{y_{1,\ell-k}}{y_{2,\ell-k}}\Big)^{(-1)^k} \frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}} +$$
$$+ \sum_{n=1}^{[\frac{\ell-2}{2}]} \Big(\frac{\partial}{\partial y_{3,4n-1}} - \frac{\partial}{\partial y_{3,4n}}\Big) \frac{y_{3,4n}}{y_{2,2n+1}} \prod_{k=1}^{\ell-2(n+1)} \Big(\frac{y_{1,\ell-k}}{y_{2,\ell-k}}\Big)^{(-1)^k} \frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}},$$

$$E_2 = \frac{\partial}{\partial y_{2,\ell-1}} + \sum_{n=2}^{[\ell/2]} \Big(\frac{\partial}{\partial y_{2,\ell-n-1}} - \frac{\partial}{\partial y_{2,\ell-n}}\Big) \prod_{k=1}^{2(n-1)} \Big(\frac{y_{2,\ell-k}}{y_{1,\ell-k}}\Big)^{(-1)^k} \frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}} + \tag{2.194}$$
$$+ \sum_{n=1}^{[\ell/2]} \Big(\frac{\partial}{\partial y_{1,\ell-n-1}} - \frac{\partial}{\partial y_{1,\ell-n}}\Big) \prod_{k=1}^{2n-1} \Big(\frac{y_{2,\ell-k}}{y_{1,\ell-k}}\Big)^{(-1)^k} \frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}} +$$
$$+ \sum_{n=1}^{[\frac{\ell-2}{2}]} \Big(\frac{\partial}{\partial y_{3,4n-1}} - \frac{\partial}{\partial y_{3,4n}}\Big) \frac{y_{3,4n}}{y_{1,2n+1}} \prod_{k=1}^{\ell-2(n+1)} \Big(\frac{y_{2,\ell-k}}{y_{1,\ell-k}}\Big)^{(-1)^k} \frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}} +$$
$$+ \sum_{n=1}^{[\frac{\ell-1}{2}]} \Big(\frac{\partial}{\partial y_{3,2(2n-1)-1}} - \frac{\partial}{\partial y_{3,2(2n-1)}}\Big) \frac{y_{3,2(2n-1)}}{y_{1,2n}} \prod_{k=1}^{\ell-2n-1} \Big(\frac{y_{2,\ell-k}}{y_{1,\ell-k}}\Big)^{(-1)^k} \frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}},$$

$$E_k = \frac{\partial}{\partial y_{k,2(\ell+1-k)}} + \sum_{n=1}^{\ell-k} \Big(\frac{\partial}{\partial y_{k,2n}} - \frac{\partial}{\partial y_{k,2n+1}}\Big) \prod_{i=1}^{\ell+1-n-k} \frac{y_{k,2(i+1)-1}}{y_{k,2(k+1)}} \frac{y_{k+1,2i}}{y_{k+1,2i-1}} + \tag{2.195}$$

$$+ \sum_{n=1}^{\ell-k} \Big(\frac{\partial}{\partial y_{k+1,2n-1}} - \frac{\partial}{\partial y_{k+1,2n}}\Big) \frac{y_{k+1,2n}}{y_{k,2(n+1)}} \prod_{i=2}^{\ell+1-n-k} \frac{y_{k,2(i+1)-1}}{y_{k,2(k+1)}} \frac{y_{k+1,2i}}{y_{k+1,2i-1}}, \qquad 2 < k < \ell,$$

$$E_\ell = \frac{\partial}{\partial y_{\ell,2}}, \tag{2.196}$$

$$H_i = \langle \mu,\, \alpha_i^\vee \rangle + \sum_{k=1}^{\ell} a_{i,k} \sum_{j=1}^{n_k} y_{k,j} \frac{\partial}{\partial y_{k,j}}, \tag{2.197}$$

$$F_i = -\langle \mu, \alpha_i^\vee \rangle \sum_{n=1}^{\ell-1} y_{i,n} - \sum_{n=1}^{\ell-1} \Big(y_{i,n}^2 \frac{\partial}{\partial y_{i,n}} + 2 \sum_{k=n+1}^{\ell-1} y_{i,k} y_{i,n} \frac{\partial}{\partial\, y_{i,n}} \Big) + \tag{2.198}$$

$$+ \sum_{n=1}^{2(\ell-2)-1} \sum_{k=[n/2]+2} y_{i,k} y_{3,n} \frac{\partial}{\partial\, y_{3,n}}, \qquad i = 1, 2,$$

$$F_k = -\langle \mu, \alpha_k^\vee \rangle \sum_{n=1}^{2(\ell+1-k)} y_{k,n} - \sum_{n=1}^{2(\ell+1-k)} \Big(y_{k,n}^2 \frac{\partial}{\partial\, y_{k,n}} + 2 \sum_{i=n+1}^{2(\ell+1-k)} y_{k,i} y_{k,n} \frac{\partial}{\partial\, y_{k,n}} \Big) + \tag{2.199}$$

$$+ \sum_{n=1}^{\ell+2-k} \sum_{i=2(n-1)}^{2(\ell+1-k)} y_{k,i} \Big(y_{k-1,2n-1} \frac{\partial}{\partial\, y_{k-1,2n-1}} + y_{k-1,2n} \frac{\partial}{\partial\, y_{k-1,2n}} \Big) +$$

$$+ \sum_{n=1}^{2(\ell-k)-1} \sum_{i=2[n/2]+3} y_{k,i} y_{k+1,n} \frac{\partial}{\partial\, y_{k+1,n}}, \qquad 3 \le k \le \ell,$$

where $n_1 = n_2 = \ell - 1,\ n_k = 2(\ell + 1 - k)$, *for* $2 < k \le \ell$.

The proof is given in Part II, Section 3.4.4.

The left/right Whittaker vectors in the factorized parametrization can be found explicitly.

Lemma 2.9 *The following expressions for the left/right Whittaker vectors hold:*

$$\psi_R(y) = \exp\Big\{ - \Big(\sum_{n=1}^{\ell-1} y_{1,n} + \sum_{n=1}^{\ell-1} y_{2,n} + \sum_{k=3}^{\ell} \sum_{n=1}^{2(\ell+1-k)} y_{k,n} \Big) \Big\}, \tag{2.200}$$

$$\psi_L(y) = \Big(\prod_{n=1}^{\ell/2} y_{1,2n-1}\prod_{n=1}^{\frac{\ell-1}{2}} y_{2,2n}\prod_{n=3}^{\ell}\prod_{i=3}^{n} y_{i,2(n+1-i)-1}\Big)^{\langle\mu,\alpha_1^\vee\rangle}\times \tag{2.201}$$

$$\times\Big(\prod_{n=1}^{\ell/2} y_{2,2n-1}\prod_{n=1}^{\frac{\ell-1}{2}} y_{1,2n}\prod_{n=3}^{\ell}\prod_{i=3}^{n} y_{i,2(n+1-i)-1}\Big)^{\langle\mu,\alpha_2^\vee\rangle}\times$$

$$\times\prod_{k=3}^{\ell}\Big(\prod_{i=1}^{k}\prod_{n=1}^{n_i} y_{i,n}\prod_{i=k+1}^{\ell}\prod_{n=1}^{n_i/2} y_{i,2n-1}^2\Big)^{\langle\mu,\alpha_k^\vee\rangle}\times$$

$$\times\exp\Big\{-\Big(\sum_{n=1}^{\ell-1}\Big[\frac{1}{y_{1,\ell-1}}\prod_{k=1}^{n-1}\Big(\frac{y_{1,\ell-k}}{y_{1,\ell-k-1}}\Big)^{p_{k-1}}\Big(\frac{y_{2,\ell-k}}{y_{2,\ell-k-1}}\Big)^{p_k}+$$

$$+\frac{1}{y_{2,\ell-1}}\prod_{k=1}^{n-1}\Big(\frac{y_{1,\ell-k}}{y_{1,\ell-k-1}}\Big)^{p_k}\Big(\frac{y_{2,\ell-k}}{y_{2,\ell-k-1}}\Big)^{p_{k+1}}\Big]\times$$

$$\times\Big(1+\frac{y_{3,2(\ell-n-1)}}{y_{3,2(\ell-n-1)-1}}\Big)\prod_{k=1}^{n-1}\frac{y_{3,2(\ell-k-1)}}{y_{3,2(\ell-k-1)-1}}+\sum_{k=3}^{\ell}\frac{1}{y_{k,2(\ell+1-k)}}\Big)\Big\},$$

where $n_1 = n_2 = \ell - 1$ and $n_k = 2(\ell+1-k)$, $k > 2$, and $p_k = (1-(-1)^k)$ is the parity of k.

The proof is given in Part II, Section 3.3.4.

Using (2.17) and (2.18) it is easy to obtain the integral representations of $\mathfrak{so}_{2\ell}$-Whittaker function in the factorized parametrization.

Theorem 2.12 *The eigenfunctions of $\mathfrak{so}_{2\ell}$-Toda chain admit the following integral representation:*

$$\Psi^{D_\ell}_{\lambda_1,\dots,\lambda_\ell}(x_1,\dots,x_\ell) = e^{\imath\lambda_1x_1+\ldots+\imath\lambda_\ell x_\ell}\int_C\bigwedge_{i=1}^{\ell}\bigwedge_{k=1}^{n_i}\frac{dy_{i,k}}{y_{i,k}}\times \tag{2.202}$$

$$\times\Big(\prod_{n=1}^{\ell/2} y_{1,2n-1}\prod_{n=1}^{\frac{\ell-1}{2}} y_{2,2n}\prod_{n=3}^{\ell}\prod_{i=3}^{n} y_{i,2(n+1-i)-1}\Big)^{\imath(\lambda_2-\lambda_1)}\times$$

$$\times\Big(\prod_{n=1}^{\ell/2} y_{2,2n-1}\prod_{n=1}^{\frac{\ell-1}{2}} y_{1,2n}\prod_{n=3}^{\ell}\prod_{i=3}^{n} y_{i,2(n+1-i)-1}\Big)^{\imath(\lambda_1+\lambda_2)}\times$$

$$\times\prod_{k=3}^{\ell}\Big(\prod_{i=1}^{k}\prod_{n=1}^{n_i} y_{i,n}\prod_{i=k+1}^{\ell}\prod_{n=1}^{n_i/2} y_{i,2n-1}^2\Big)^{\imath(\lambda_k-\lambda_{k-1})}\times$$

$$\times \exp\Big\{ -\Big(\sum_{n=1}^{\ell-1} \Big[\frac{1}{y_{1,\ell-1}} \prod_{k=1}^{n-1} \Big(\frac{y_{1,\ell-k}}{y_{1,\ell-k-1}}\Big)^{p_{k-1}} \Big(\frac{y_{2,\ell-k}}{y_{2,\ell-k-1}}\Big)^{p_k} +$$

$$+ \frac{1}{y_{2,\ell-1}} \prod_{k=1}^{n-1} \Big(\frac{y_{1,\ell-k}}{y_{1,\ell-k-1}}\Big)^{p_k} \Big(\frac{y_{2,\ell-k}}{y_{2,\ell-k-1}}\Big)^{p_{k+1}} \Big] \times$$

$$\times \Big(1 + \frac{y_{3,2(\ell-n-1)}}{y_{3,2(\ell-n-1)-1}}\Big) \prod_{k=1}^{n-1} \frac{y_{3,2(\ell-k-1)}}{y_{3,2(\ell-k-1)-1}} + \sum_{k=3}^{\ell} \frac{1}{y_{k,2(\ell+1-k)}} +$$

$$+ e^{x_2-x_1} \sum_{n=1}^{\ell-1} y_{1,n} + e^{x_1+x_2} \sum_{n=1}^{\ell-1} y_{2,n} + \sum_{k=3}^{\ell} e^{x_k-x_{k-1}} \sum_{n=1}^{2(\ell+1-k)} y_{k,n} \Big) \Big\}.$$

Here we assume $x_1 = x_{\ell,1}, \ldots, x_{\ell,\ell}$, $p_k = (1-(-1)^k)/2$, $n_1 = n_2 = \ell - 1$ and $n_k = 2(\ell+1-k)$, $k > 2$. Domain of integration $C \subset N_+$ is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at the possible boundaries and infinities. In particular one can chose $C = \mathbb{R}_+^m$, where $m = l(w_0)$.

The proof is given in Part II, Section 3.3.4.

Example 2.5 *For $\ell = 3$ the general formula (2.202) acquires the following form*

$$\Psi^{D_3}_{\lambda_1,\lambda_2,\lambda_3}(x_{31}, x_{32}, x_{33}) = \int_C \bigwedge_{i=1}^{3} \bigwedge_{k=1}^{2} \frac{dy_{i,k}}{y_{i,k}} \times \tag{2.203}$$

$$\times (y_{11}y_{31}y_{22})^{\imath(\lambda_2-\lambda_1)} (y_{21}y_{31}y_{12})^{\imath(\lambda_2+\lambda_1)} (y_{31}y_{12}y_{22}y_{32})^{\imath(\lambda_3-\lambda_2)} \times$$

$$\times \exp\Big\{ \frac{1}{y_{12}}\Big(1 + \frac{y_{32}}{y_{31}}\Big) + \frac{1}{y_{12}}\frac{y_{22}}{y_{21}}\frac{y_{32}}{y_{31}} + \frac{1}{y_{22}}\Big(1 + \frac{y_{32}}{y_{31}}\Big) + \frac{1}{y_{22}}\frac{y_{12}}{y_{11}}\frac{y_{32}}{y_{31}} + \frac{1}{y_{32}} +$$

$$+ e^{x_{32}-x_{31}}(y_{11}+y_{12}) + e^{x_{32}+x_{31}}(y_{21}+y_{22}) + e^{x_{33}-x_{32}}(y_{31}+y_{32}) \Big\}.$$

For the domain of integration one can chose $C = \mathbb{R}^6$.

2.5.2 $\mathfrak{so}_{2\ell}$-Whittaker function: modified factorized parametrization

In this part we introduce a modified factorized parametrization of an open part $N_+^{(0)}$ of maximal unipotent subgroup $N_+ \subset SO(2\ell)$. We use this parametrization to construct integral representations for $\mathfrak{so}_{2\ell}$-Whittaker functions. Similar to other series of classical Lie algebras these integral representations for $\mathfrak{so}_{2\ell}$-Whittaker functions have a simple recursive structure over the rank ℓ and can be describe in purely combinatorial terms using suitable graphs. These representations can be considered as a generalization of Givental integral representations to $\mathfrak{g} = \mathfrak{so}_{2\ell}$.

We follow the same approach that was used in the description of modified factorized representation for other classical groups. There exists a realization of a tautological representation $\pi_{2\ell} : \mathfrak{so}_{2\ell} \to End(\mathbb{C}^{2\ell})$ such that Weyl generators corresponding to Borel (Cartan) subalgebra of $\mathfrak{so}_{2\ell}$ are realized by upper triangular (diagonal) matrices. This defines an embedding $\mathfrak{so}_{2\ell} \subset \mathfrak{gl}_{2\ell}$ such that Borel (Cartan) subalgebra maps into Borel (Cartan) subalgebra (see e.g. [DS]). To define the corresponding embedding of the groups consider the following involution on $GL(2\ell)$:

$$g \longmapsto g^* := \dot{w}_0 \cdot (g^{-1})^t \cdot \dot{w}_0^{-1}, \tag{2.204}$$

where a^t is induced by the standard transposition of the matrix a and $\dot{w}_0$ is a lift of the longest element of the Weyl group of $\mathfrak{gl}_{2\ell}$. In the matrix form it can be written as

$$\pi_{2\ell}(\dot{w}_0) = S \cdot J,$$

where $S = \mathrm{diag}(1, -1, \ldots, -1, 1)$ and $J = \|J_{i,j}\| = \|\delta_{i+j,2\ell+2}\|$. The orthogonal group $G = SO(2\ell)$ then can be defined as a following subgroup of $GL(2\ell)$ (see i.e. [DS]):

$$SO(2\ell) = \{g \in GL(2\ell) : g^* = g\}.$$

Let $\epsilon_{i,j}$ be elementary $(2\ell \times 2\ell)$ matrices with unites at (i,j) place and zeroes otherwise. Introduce the following matrices

$$U_n = \sum_{i=1}^{n} \epsilon_{\ell-n+i,\ell-n+i} + e^{-x_{n-1,1}} \epsilon_{\ell+1,\ell+1} + \sum_{i=1}^{n-1} e^{x_{n-1,i}} \epsilon_{\ell+i+1,\ell+i+1} + \tag{2.205}$$

$$+ e^{z_{n-1,1}} \epsilon_{\ell,\ell+1} + \sum_{i=1}^{n-1} e^{z_{n-1,i}} \epsilon_{\ell+i,\ell+i+1} + \sum_{i=1}^{\ell-n} (\epsilon_{i,i} + \epsilon_{2\ell+1-i,2\ell+1-i}),$$

$$\widetilde{U}'_n = \sum_{i=1}^{n+2} \epsilon_{\ell-n+i,\ell-n+i} + \sum_{i=2}^{n-1} e^{x_{n-1,i}} \epsilon_{\ell+i+1,\ell+i+1} + \sum_{i=2}^{n-1} e^{z_{n-1,i}} \epsilon_{\ell+i,\ell+i+1} + \tag{2.206}$$

$$+ \sum_{i=1}^{\ell-n} (\epsilon_{i,i} + \epsilon_{2\ell+1-i,2\ell+1-i}),$$

$$\widetilde{U}''_n = \sum_{i=1}^{n} \epsilon_{\ell-n+i,\ell-n+i} + e^{-x_{n-1,1}} \epsilon_{\ell+1,\ell+1} + e^{x_{n-1,1}} \epsilon_{\ell+2,\ell+2} + \tag{2.207}$$

$$+ e^{z_{n-1,1}} (\epsilon_{\ell,\ell+1} + \epsilon_{\ell+1,\ell+2}) + \sum_{i=3}^{n} \epsilon_{\ell+i,\ell+i} + \sum_{i=1}^{\ell-n} (\epsilon_{i,i} + \epsilon_{2\ell+1-i,2\ell+1-i}),$$

$$\widetilde{U}_n = \widetilde{U}'_n \widetilde{U}''_n = \sum_{i=1}^{n} \epsilon_{\ell-n+i,\ell-n+i} + e^{-x_{n-1,1}} \epsilon_{\ell+1,\ell+1} + \sum_{i=1}^{n-1} e^{x_{n-1,i}} \epsilon_{\ell+i+1,\ell+i+1} + \tag{2.208}$$

$$+ e^{z_{n-1,1}} \epsilon_{\ell,\ell+1} + \sum_{i=1}^{n-1} e^{z_{n-1,i}} \epsilon_{\ell+i,\ell+i+1},$$

$$V_n = \sum_{i=1}^{n} e^{x_{n,n+1-i}} \epsilon_{\ell-n+i,\ell-n+i} + e^{-x_{n,1}} \epsilon_{\ell+1,\ell+1} + \sum_{i=2}^{n} \epsilon_{\ell+i,\ell+i} + \tag{2.209}$$

$$+ \sum_{i=1}^{\ell-n} (\epsilon_{i,i} + \epsilon_{2\ell+1-i,2\ell+1-i}),$$

$$\widetilde{V}'_n = \sum_{i=1}^{n-1} e^{x_{n,n+1-i}} \epsilon_{\ell-n+i,\ell-n+i} + \sum_{i=-1}^{n} \epsilon_{\ell+i,\ell+i} + + \sum_{i=1}^{n-2} e^{z_{n-1,n-i}} \epsilon_{\ell-n+i,\ell-n+i+1} + \tag{2.210}$$

$$+ \sum_{i=1}^{\ell-n} (\epsilon_{i,i} + \epsilon_{2\ell+1-i,2\ell+1-i}),$$

$$\widetilde{V}''_n = e^{x_{n,1}} \epsilon_{\ell,\ell} + e^{-x_{n,1}} \epsilon_{\ell+1,\ell+1} + e^{z_{n-1,1}} (\epsilon_{\ell-1,\ell} + \epsilon_{\ell,\ell+1}) + \tag{2.211}$$

$$+ \sum_{i=1}^{\ell-n} (\epsilon_{i,i} + \epsilon_{2\ell+1-i,2\ell+1-i}),$$

$$\widetilde{V}_n = \widetilde{V}''_n \widetilde{V}'_n = \sum_{i=1}^{n} e^{x_{n,n+1-i}} \epsilon_{\ell-n+i,\ell-n+i} + e^{-x_{n,1}} \epsilon_{\ell+1,\ell+1} + \sum_{i=2}^{n} \epsilon_{\ell+i,\ell+i} + \tag{2.212}$$

$$+ \sum_{i=1}^{n-1} e^{z_{n-1,n-i}} \epsilon_{\ell-n+i,\ell-n+i+1} + e^{z_{n-1,1}} \epsilon_{\ell,\ell+1} + \sum_{i=1}^{\ell-n} (\epsilon_{i,i} + \epsilon_{2\ell+1-i,2\ell+1-i}).$$

Theorem 2.13 *i) The image of a generic unipotent element $v^{D_\ell} \in N_+$ in the tautological representation $\pi_{2\ell} : \mathfrak{so}_{2\ell} \to End(\mathbb{C}^{2\ell})$ can be presented in the form*

$$v^{D_\ell} = \mathfrak{X}_2 \mathfrak{X}_3 \cdots \mathfrak{X}_\ell, \tag{2.213}$$

with

$$\begin{aligned} \mathfrak{X}_2 &= S_1 \widetilde{U}_2 U_2^{-1} S_1 \cdot S_3 (\widetilde{U}_2 U_2^{-1})^* S_3 \cdot S_1 (\tilde{V}_2 V_2^{-1})^* S_1 \cdot S_3 \tilde{V}_2 V_2^{-1} S_3 \\ \mathfrak{X}_n &= (\widetilde{U}'_n (U'_n)^{-1})^* \cdot S_{n-1} \widetilde{U}_n U_n^{-1} S_{n-1} \cdot S_{n+1} (\widetilde{U}''_n (U''_n)^{-1})^* S_{n+1} \cdot \\ &\cdot S_{n-1} (\widetilde{V}''_n (V''_n)^{-1})^* S_{n-1} \cdot S_{n+1} \widetilde{V}_n V_n^{-1} S_{n+1} \cdot (\widetilde{V}'_n (V'_n)^{-1})^*, \end{aligned} \tag{2.214}$$

where $x_{\ell,k} = 0,\ k = 1, \dots \ell$ is assumed and S_i is defined as follows:

$$S_i = \sum_{k=1}^{i-1} \epsilon_{k,k} + \epsilon_{i,i+1} + \epsilon_{i+1,i} + \sum_{k=i+2}^{2\ell} \epsilon_{k,k}. \tag{2.215}$$

ii) This defines a parametrization of an open part $N_+^{(0)}$ of N_+.

Proof. Let $v^{D_\ell}(y)$ be a parametrization of N_+ according to (2.191)-(2.192). Let $\tilde{X}_i(y) = e^{ye_{i,i+1}}$ be a one-parametric unipotent subgroup in $GL(2\ell)$, then $\tilde{X}_i(y)^* = \tilde{X}_{2\ell+1-i}(y)$. Embed elementary unipotent subgroups $X_i(y)$ of $SO(2\ell)$ into $GL(2\ell)$ as follows:

$$X_i(y) = \tilde{X}_i(y)^* \cdot \tilde{X}_i(y).$$

This defines a map of an arbitrary regular unipotent element v^{D_ℓ} into unipotent subgroup of $GL(2\ell)$. Let us change the variables in the following way:

$$\begin{aligned}
y_{1,n} &= \Big(e^{z_{n,1}-x_{n,1}} + e^{z_{n,1}-x_{n+1,1}}\Big), && n = 1,\ldots,\ell-1,\\
y_{2,n} &= \Big(e^{z_{n,1}+x_{n,1}} + e^{z_{n,1}+x_{n+1,1}}\Big), && n = 1,\ldots,\ell-1,\\
y_{k,2r-1} &= e^{z_{k+r-2,k-1}-x_{k+r-2,k-1}}, && k = 3,\ldots,\ell,\\
y_{k,2r} &= e^{z_{k+r-2,k-1}-x_{k+r-1,k-1}}, && r = 1,\ldots,\ell+1-k.
\end{aligned} \tag{2.216}$$

Here the conditions $x_{\ell,k} = 0,\ \ k = 1,\ldots,\ell$ are implied. By elementary manipulations it is easy to check that after the change of variables (2.216), the image $\pi_{2\ell}(v^{D_\ell})$ of v^{D_ℓ} defined by (2.191)-(2.192) transforms into the (2.213) -(2.214). Taking into account that the change of variables (3.63) is invertible we obtain a parametrization of $N_+^{(0)} \subset N_+$ $\Box$

The modified factorized parametrization of a unipotent group N_+ defines a particular realization of a principal series representation of $\mathcal{U}(\mathfrak{so}_{2\ell})$ by differential operators. It can be obtained using the change of variables (3.63) applied to a realization given in Proposition 2.12. We shall use the term Gauss-Givental representation for this realization of representation of $\mathcal{U}(\mathfrak{so}_{2\ell})$.

Proposition 2.13 *The following differential operators define a representation π_λ of $\mathcal{U}(\mathfrak{so}_{2\ell})$ in V_μ in terms of modified factorized parametrization of $N_+^{(0)}$:*

$$\begin{aligned}
E_1 = e^{x_{22}-z_{11}}\frac{e^{x_{11}}}{e^{x_{11}}+e^{x_{21}}}\Big(-\frac{\partial}{\partial x_{11}} - \frac{e^{x_{11}}}{e^{x_{11}}+e^{x_{21}}}\frac{\partial}{\partial z_{11}}\Big) + \\
+\sum_{k=2}^{\ell-1}\Big\{\frac{p_k e^{x_{k,1}} + p_{k+1}e^{x_{k+1,1}}}{e^{x_{k,1}}+e^{x_{k+1,1}}}\Big(e^{x_{k,2}-z_{k,1}} + e^{x_{k+1,2}-z_{k,1}}\Big)\times \\
\times\Big(-\frac{\partial}{\partial x_{11}} - \frac{e^{x_{21}}-e^{x_{11}}}{e^{x_{11}}+e^{x_{21}}}\frac{\partial}{\partial z_{11}} + \frac{\partial}{\partial x_{21}} - \frac{\partial}{\partial x_{22}} + \\
+\sum_{i=3}^{k}(-1)^i\frac{\partial}{\partial x_{i,1}} - \frac{\partial}{\partial x_{i,2}} + (-1)^{i-1}\frac{e^{x_{i,1}}-e^{x_{i-1,1}}}{e^{x_{i,1}}+e^{x_{i-1,1}}}\frac{\partial}{\partial z_{i-1,1}} - \frac{\partial}{\partial z_{i-1,2}}\Big) + \\
+\Big(e^{x_{k+1,2}+(-1)^k x_{k+1,1}}\frac{1}{e^{z_{k,1}+x_{k,1}}+e^{z_{k,1}+x_{k+1,1}}}\frac{p_k e^{x_{k,1}}+p_{k+1}e^{x_{k+1,1}}}{e^{x_{k,1}}+e^{x_{k+1,1}}} - \\
-e^{x_{k,2}+(-1)^{k-1}x_{k,1}}\frac{1}{e^{z_{k,1}-x_{k,1}}+e^{z_{k,1}-x_{k+1,1}}}\frac{p_{k-1}e^{x_{k,1}}+p_k e^{x_{k+1,1}}}{e^{x_{k,1}}+e^{x_{k+1,1}}}\Big)\frac{\partial}{\partial z_{k,1}} - \\
-e^{x_{k+1,2}-z_{k,1}} - \frac{p_k e^{x_{k,1}}+p_{k+1}e^{x_{k+1,1}}}{e^{x_{k,1}}+e^{x_{k+1,1}}}\frac{\partial}{\partial z_{k,2}}\Big\},
\end{aligned} \tag{2.217}$$

$$E_2 = e^{x_{22}-z_{11}} \frac{e^{x_{21}}}{e^{x_{11}}+e^{x_{21}}}\Big(\frac{\partial}{\partial x_{11}} + \frac{e^{x_{21}}}{e^{x_{11}}+e^{x_{21}}}\frac{\partial}{\partial z_{11}}\Big) + \tag{2.218}$$

$$+\sum_{k=2}^{\ell-1}\Big\{ \frac{p_{k-1}e^{x_{k,1}}+p_k e^{x_{k+1,1}}}{e^{x_{k,1}}+e^{x_{k+1,1}}}\Big(e^{x_{k,2}-z_{k,1}}+e^{x_{k+1,2}-z_{k,1}}\Big)\times$$

$$\times\Bigg(\frac{\partial}{\partial x_{11}} + \frac{e^{x_{21}}-e^{x_{11}}}{e^{x_{11}}+e^{x_{21}}}\frac{\partial}{\partial z_{11}} - \frac{\partial}{\partial x_{21}} - \frac{\partial}{\partial x_{22}} +$$

$$+\sum_{i=3}^{k}(-1)^{i-1}\frac{\partial}{\partial x_{i,1}} - \frac{\partial}{\partial x_{i,2}} + (-1)^i \frac{e^{x_{i,1}}-e^{x_{i-1,1}}}{e^{x_{i,1}}+e^{x_{i-1,1}}}\frac{\partial}{\partial z_{i-1,1}} - \frac{\partial}{\partial z_{i-1,2}}\Bigg) +$$

$$+\Bigg(e^{x_{k+1,2}+(-1)^{k-1}x_{k+1,1}}\frac{1}{e^{z_{k,1}+x_{k,1}}+e^{z_{k,1}+x_{k+1,1}}}\frac{p_{k-1}e^{x_{k,1}}+p_k e^{x_{k+1,1}}}{e^{x_{k,1}}+e^{x_{k+1,1}}} -$$

$$-e^{x_{k,2}+(-1)^k x_{k,1}}\frac{1}{e^{z_{k,1}-x_{k,1}}+e^{z_{k,1}-x_{k+1,1}}}\frac{p_k e^{x_{k,1}}+p_{k+1}e^{x_{k+1,1}}}{e^{x_{k,1}}+e^{x_{k+1,1}}}\Bigg)\frac{\partial}{\partial z_{k,1}} -$$

$$-e^{x_{k+1,2}-z_{k,1}} - \frac{p_{k-1}e^{x_{k,1}}+p_k e^{x_{k+1,1}}}{e^{x_{k,1}}+e^{x_{k+1,1}}}\frac{\partial}{\partial z_{k,2}}\Big\}.$$

Here $p_k = (1-(-1)^k)/2$ is the parity of k.

$$E_k = \Big(\frac{\partial}{\partial z_{k-1,k-1}} - \frac{\partial}{\partial x_{k-1,k-1}}\Big)\Big(e^{x_{k,k}-z_{k-1,k-1}} + \sum_{n=k}^{\ell-1} e^{x_{n,k}-z_{n,k-1}} + e^{x_{n+1,k}-z_{n,k}}\Big) + \tag{2.219}$$

$$+\sum_{i=k}^{\ell-1}\Big(\frac{\partial}{\partial x_{i,k-1}} - \frac{\partial}{\partial\, x_{i,k}}\Big)\sum_{n=i}^{\ell-1}\Big(e^{x_{n,k}-z_{n,k-1}} + e^{x_{n+1,k}-z_{n,k-1}}\Big) +$$

$$+\sum_{i=k}^{\ell-1}\Big(\frac{\partial}{\partial z_{i,k-1}} - \frac{\partial}{\partial\, z_{i,k}}\Big)\Big(e^{x_{i+1,k}-z_{i,k-1}} + \sum_{n=i+1}^{\ell-1} e^{x_{n,k}-z_{n,k-1}} + e^{x_{n+1,k}-z_{n,k-1}}\Big),$$

where $2 < k < \ell$ and

$$E_\ell = e^{-z_{\ell-1,\ell-1}}\Big(\frac{\partial}{\partial\, z_{\ell-1,\ell-1}} + \frac{\partial}{\partial\, x_{\ell-1,\ell-1}}\Big), \tag{2.220}$$

$$H_1 = \langle\mu, \alpha_1^\vee\rangle) + 2\Big(- \frac{\partial}{\partial x_{\ell-1,1}} - \frac{\partial}{\partial x_{11}} + \tag{2.221}$$

$$+\sum_{n=1}^{\ell-1}\frac{p_n e^{x_{n,1}}+p_{n+1}e^{x_{n+1,1}}}{e^{x_{n,1}}+e^{x_{n+1,1}}}\frac{\partial}{\partial z_{n,1}}\Big) - \sum_{k=2}^{\ell-1}\frac{\partial}{\partial z_{k,2}},$$

$$H_2 = \langle \mu, \alpha_2^\vee \rangle + 2\Big(\frac{\partial}{\partial x_{\ell-1,1}} + \frac{\partial}{\partial x_{11}} + \tag{2.222}$$

$$+ \sum_{n=1}^{\ell-1} \frac{p_{n-1} e^{x_{n,1}} + p_n e^{x_{n+1,1}}}{e^{x_{n,1}} + e^{x_{n+1,1}}} \frac{\partial}{\partial z_{n,1}} \Big) - \sum_{k=2}^{\ell-1} \frac{\partial}{\partial z_{k,2}},$$

where $p_k = (1 - (-1)^k)/2$.

$$H_i = \langle \mu, \alpha_i^\vee \rangle + \sum_{k=2}^{\ell} a_{i,k} \sum_{j=k-1}^{\ell-1} \frac{\partial}{\partial z_{j,k-1}}, \qquad 2 < i \leq \ell, \tag{2.223}$$

$$F_1 = -\sum_{n=1}^{\ell-1} \Big(e^{z_{n,1} - x_{n+1,1}} + e^{z_{n,1} - x_{n+1,1}} \Big) \Big[\langle \mu, \alpha_1^\vee \rangle - \frac{\partial}{\partial x_{11}} + \tag{2.224}$$

$$+ \frac{e^{x_{11}}}{e^{x_{11}} + e^{x_{21}}} \frac{\partial}{\partial z_{11}} \Big] - e^{x_{\ell,1} - x_{\ell,2}} \sum_{k=2}^{\ell-1} \Big[- \frac{\partial}{\partial x_{k,1}} + \frac{\partial}{\partial x_{k,2}} +$$

$$+ \frac{e^{x_{k,1}}}{e^{x_{k,1}} + e^{x_{k+1,1}}} \frac{\partial}{\partial z_{k,1}} + \frac{e^{x_{k,1}}}{e^{x_{k-1,1}} + e^{x_{k,1}}} \frac{\partial}{\partial z_{k-1,1}} \Big] \sum_{n=k}^{\ell-1} \Big(e^{z_{n,1} - x_{n+1,1}} + e^{z_{n,1} - x_{n+1,1}} \Big),$$

$$F_2 = -\sum_{n=1}^{\ell-1} \Big(e^{z_{n,1} + x_{n+1,1}} + e^{z_{n,1} + x_{n+1,1}} \Big) \Big[\langle \mu, \alpha_2^\vee \rangle + \frac{\partial}{\partial x_{11}} + \tag{2.225}$$

$$+ \frac{e^{x_{21}}}{e^{x_{11}} + e^{x_{21}}} \frac{\partial}{\partial z_{11}} \Big] - \sum_{k=2}^{\ell-1} \Big[\frac{\partial}{\partial x_{k,1}} + \frac{\partial}{\partial x_{k,2}} +$$

$$+ \frac{e^{x_{k+1,1}}}{e^{x_{k,1}} + e^{x_{k+1,1}}} \frac{\partial}{\partial z_{k,1}} + \frac{e^{x_{k-1,1}}}{e^{x_{k-1,1}} + e^{x_{k,1}}} \frac{\partial}{\partial z_{k-1,1}} \Big] \sum_{n=k}^{\ell-1} \Big(e^{z_{n,1} + x_{n+1,1}} + e^{z_{n,1} + x_{n+1,1}} \Big),$$

$$
F_k = \Big(-\langle \mu, \alpha_k \rangle + \tag{2.226}
$$

$$
+\frac{\partial}{\partial x_{k-1,k-1}} + \frac{\partial}{\partial z_{k-2,k-2}}\Big) \sum_{n=k-1}^{\ell-1} \Big(e^{z_{n,k-1}-x_{n,k-1}} + e^{z_{n,k-1}-x_{n+1,k-1}} \Big) -
$$

$$
- \sum_{i=k-1}^{\ell-1} \Big(\frac{\partial}{\partial z_{i,k-1}} - \frac{\partial}{\partial z_{i,k-2}} \Big) \Big(e^{z_{i,k-1}-x_{i+1,k-1}} +
$$

$$
+ \sum_{j=i+1}^{\ell-1} e^{z_{j,k-1}-x_{j,k-1}} + e^{z_{j,k-1}-x_{j+1,k-1}} \Big) -
$$

$$
- \sum_{i=k}^{\ell-1} \Big(\frac{\partial}{\partial x_{i,k-1}} - \frac{\partial}{\partial x_{i,k-2}} \Big) \sum_{j=i+1}^{\ell-1} \Big(e^{z_{j,k-1}-x_{j,k-1}} + e^{z_{j,k-1}-x_{j+1,k-1}} \Big),
$$

where $3 \leq k \leq \ell$ *and* $x_{\ell,k} = 0$ *is assumed.*

We are going to write down the matrix element (2.12) for $\mathfrak{g} = \mathfrak{so}_{2\ell}$ explicitly using Gauss-Givental representation defined above. Whittaker vectors ψ_R and ψ_L in this representation should satisfiy the system of differential equations

$$
E_i \psi_R(x) = -\psi_R(x), \qquad F_i \psi_L(x) = -\psi_L(x), \qquad 1 \leq i \leq \ell. \tag{2.227}
$$

Its solution has the following form.

Lemma 2.10 *The following expressions for the left/right Whittaker vectors hold:*

$$
\psi_R = \exp\Big\{ -\sum_{n=1}^{\ell-1} \Big(e^{z_{n,1}-x_{n,1}} + e^{z_{n,1}-x_{n+1,1}} + e^{z_{n,1}+x_{n,1}} + e^{z_{n,1}+x_{n+1,1}} \Big) - \tag{2.228}
$$

$$
- \sum_{k=3}^{\ell} \sum_{n=1}^{\ell+1-k} \Big(e^{z_{k+n-2,k-1}-x_{k+n-2,k-1}} + e^{z_{k+n-2,k-1}-x_{k+n-1,k-1}} \Big) \Big\},
$$

$$
\psi_L = e^{2\mu_1 x_{1,1}} \prod_{n=2}^{\ell} \Big(e^{x_{n,1}} + e^{x_{n-1,1}} \Big)^{2\mu_n} \times \tag{2.229}
$$

$$
\times \prod_{n=1}^{\ell} \exp\Big\{ -\mu_n \Big(\sum_{i=1}^{n} x_{n,i} - 2\sum_{i=1}^{n-1} z_{n-1,i} + \sum_{i=1}^{n-1} x_{n-1,i} \Big) \Big\} \times
$$

$$
\times \exp\Big\{ -\sum_{k=1}^{\ell-1} \Big(e^{x_{k+1,k+1}-z_{k,k}} + \sum_{i=k+1}^{\ell-1} e^{x_{i,k+1}-z_{i,k}} + e^{x_{i+1,k+1}-z_{i,k}} \Big) \Big\},
$$

where we set $x_{\ell,k} = 0, \quad k = 1, \ldots, \ell$ *and* $\mu_n = \imath\lambda_n - \rho_n, \quad \rho_1 = 0$ *and* $\rho_n = n-1$ *for* $1 < n \leq \ell$. ($\sum_i^j = 0$ *when* $j < i$).

Now we are ready to find the integral representation of the pairing (2.12) for $\mathfrak{g} = \mathfrak{so}_{2\ell}$. To get an explicit expression for the integrand, one uses the same type of decomposition of the Cartan element as for other classical groups in the previous subsections:

$$e^{-H_x} = \pi_\lambda(\exp(-\sum_{i=1}^{\ell}\langle\omega_i, x\rangle h_i)) = e^{H_L}e^{H_R},$$

where

$$-H_x = H_L + H_R = \sum_{i=1}^{\ell}\mu_i x_{\ell,i} + \sum_{k=3}^{\ell}(x_{\ell,k} - x_{\ell,k-1})\sum_{i=k-1}^{\ell-1}\frac{\partial}{\partial z_{i,k-1}} + x_{\ell,2}\sum_{i=1}^{\ell-1}\frac{\partial}{\partial z_{i,1}} + \tag{2.230}$$

$$+x_{\ell,1}\Big(\frac{\partial}{\partial x_{\ell-1,1}} + \frac{\partial}{\partial x_{1,1}} - \sum_{k=1}^{\ell-1}(-1)^k\frac{e^{x_{k+1,1}} - e^{x_{k,1}}}{e^{x_{k,1}} + e^{x_{k+1,1}}}\frac{\partial}{\partial z_{k,1}}\Big),$$

with

$$H_L = \sum_{k=1}^{\ell} x_{\ell,k}\Big(\sum_{i=k}^{\ell-1}\frac{\partial}{\partial x_{i,k}} + \sum_{i=k-1}^{\ell-1}\frac{\partial}{\partial z_{i,k-1}}\Big), \tag{2.231}$$

$$H_R = -H_x - H_L. \tag{2.232}$$

We imply that H_L acts on the left vector and H_R acts on the right vector in (2.12). Taking into account Proposition 2.10 one obtains the following theorem.

Theorem 2.14 *The eigenfunctions of $\mathfrak{so}_{2\ell}$-Toda chain (2.12) admit the integral representation:*

$$\Psi^{D_\ell}_{\lambda_1,\ldots,\lambda_\ell}(x_{\ell,1},\ldots,x_{\ell,\ell}) = \int\limits_C \bigwedge_{k=1}^{\ell-1}\bigwedge_{i=1}^{k} dx_{k,i}\wedge dz_{k,i}\quad e^{\mathcal{F}^{D_\ell}},$$

where

$$\mathcal{F}^{D_\ell} = \imath\lambda_1 x_{1,1} - \sum_{n=2}^{\ell}\imath\lambda_n\Big(\sum_{i=1}^{n}x_{n,i} - \tag{2.233}$$

$$-2\sum_{i=1}^{n-1}z_{n-1,i} + \sum_{i=1}^{n-1}x_{n-1,i} - 2\ln\left(e^{x_{n,1}} + e^{x_{n-1,1}}\right)\Big) -$$

$$-\sum_{k=1}^{\ell-1}\left(e^{x_{k+1,k+1}-z_{k,k}} + \sum_{i=k+1}^{\ell-1}e^{x_{i,k+1}-z_{i,k}} + e^{x_{i+1,k+1}-z_{i,k}}\right) -$$

$$-\sum_{n=1}^{\ell-1}\left(e^{z_{n,1}-x_{n,1}} + e^{z_{n,1}-x_{n+1,1}} + e^{z_{n,1}+x_{n,1}} + e^{z_{n,1}+x_{n+1,1}}\right) -$$

$$-\sum_{k=3}^{\ell}\sum_{n=1}^{\ell+1-k}\left(e^{z_{k+n-2,k-1}-x_{k+n-2,k-1}} + e^{z_{k+n-2,k-1}-x_{k+n-1,k-1}}\right),$$

where $x_i := x_{\ell,i}$, $1 \le i \le \ell$ *and* $C \subset N_+$ *is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at the boundaries and at infinities. In particular one can take* $C = \mathbb{R}^m$, $m = l(w_0)$ *as a domain of integration.*

Example 2.6 *For* $\ell = 2$ *the general expression (2.233) acquires the following form*

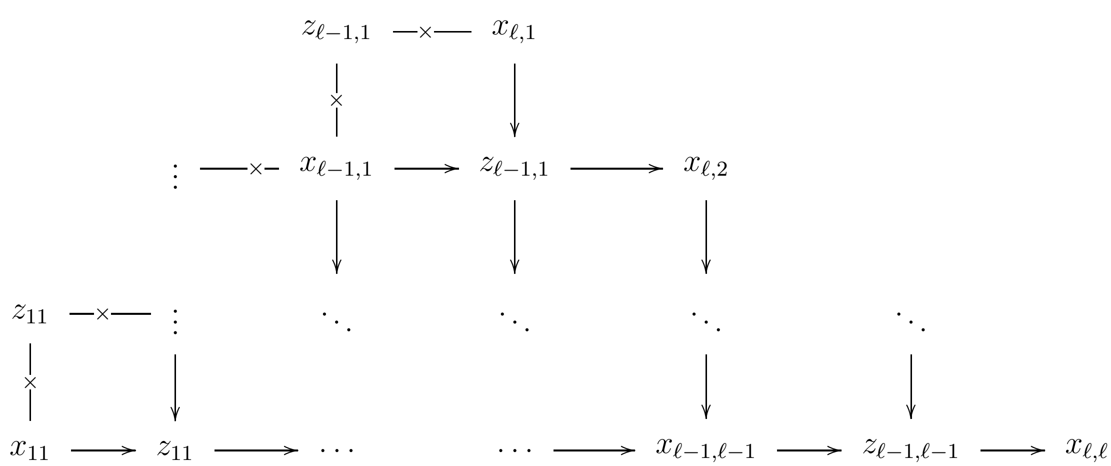
$$\Psi^{D_2}_{\lambda_1,\lambda_2}(x_{21}, x_{22}) = \int_C dx_{11} dz_{11} \left(e^{-x_{22}} + e^{-x_{11}} \right)^{2\imath\lambda_2} \times \tag{2.234}$$

$$\times \exp\left\{ \imath\lambda_2(x_{21} + x_{22} - 2z_{11} + x_{11}) - \imath\lambda_1 x_{11} \right\} \times$$

$$\times \exp\left\{ -\left(e^{z_{11}-x_{21}} + e^{x_{22}-z_{11}} + e^{-x_{22}-z_{11}} + e^{x_{11}-z_{11}} + e^{-x_{11}-z_{11}} \right) \right\}.$$

One can chose $C = \mathbb{R}^2$ *as an integration domain.*

There is a simple combinatorial description of the potential $\mathcal{F}^{D_\ell}$ for zero spectrum $\{\lambda_i = 0\}$. Namely, it can be presented as a sum over arrows in the following diagram.



Note that the diagram for D_ℓ can be obtained by a factorization of the diagram for $A_{2\ell-1}$. Consider the following involution:

$$\iota : \quad X \longmapsto \dot{w}_0^{-1} X^t \dot{w}_0, \tag{2.235}$$

where $\dot{w}_0$ is a lift the longest element of $A_{2\ell-1}$ Weyl group and X^t denotes the standard transposition. Corresponding action on the modified factorization parameters is given by

$$\dot{w}_0 : \qquad x_{k,i} \longleftrightarrow -x_{k,k+1-i} \tag{2.236}$$

This defines a factorization of $A_{2\ell-1}$-diagram that produce the diagram for D_ℓ. Note that diagram for D_ℓ can be also obtained by erasing the last row of vertexes and arrows on the right slope from the diagram for C_ℓ

An analog of the monomial relations (2.52) is as follows. Introduce variables $a_{i,k}$, $b_{i,k}$, $c_{i,k}$, $d_{i,k}$ associated with the arrows of the diagram

$$\begin{aligned} a_{k,1} &= e^{x_{k,1}+z_{k-1,1}}, & a_{k,i} &= e^{z_{k-1,i-1}-x_{k,i}}, \\ b_{k,1} &= e^{x_{k,1}+z_{k,1}}, & b_{k,i} &= e^{x_{k,i}-z_{k,i-1}}, \\ c_{k,i} &= e^{z_{k,i}-x_{k,i}}, & d_{k,i} &= e^{z_{k,i}-x_{k+1,i}}. \end{aligned} \tag{2.237}$$

Then the following relations hold

$$b_{k,i}c_{k,i} = a_{k+1,i}d_{k,i}, \qquad a_{k+1,i+1}d_{k,i} = b_{k+1,i+1}c_{k+1,i}, \tag{2.238}$$

$$a_{\ell,1}a_{\ell,2} = e^{x_{\ell,1}+x_{\ell,2}}, \qquad a_{\ell,i}d_{\ell-1,i-1} = e^{x_{\ell,i}-x_{\ell,i-1}}. \tag{2.239}$$

2.5.3 Recursion for $\mathfrak{so}_{2\ell}$-Whittaker functions and $\mathcal{Q}$-operator for $D^{(1)}_\ell$-Toda chain

The integral representation (2.233) of $\mathfrak{so}_{2\ell}$-Whittaker functions possesses a recursive structure over the rank ℓ. For any $n = 2, \ldots, \ell$ let us introduce integral operators $Q^{D_n}_{D_{n-1}}$ with the kernels $Q^{D_n}_{D_{n-1}}(\underline{x}_n;\, \underline{x}_{n-1}; \lambda_n)$ defined as follows

$$\begin{aligned} Q^{D_n}_{D_{n-1}}(\underline{x}_n;\, \underline{x}_{n-1}; \lambda_n) = \int \bigwedge_{i=1}^{n-1} dz_{n,i}\, \Big(e^{x_{n-1,1}} + e^{x_{n,1}}\Big)^{2\imath\lambda_n} \times \\ \times \exp\Big\{ - \imath\lambda_n \Big(\sum_{i=1}^{n} x_{n,i} - 2\sum_{i=1}^{n-1} z_{n-1,i} + \sum_{i=1}^{n-1} x_{n-1,i}\Big)\Big\} \times \\ \times Q^{D_n}_{C_{n-1}}(\underline{x}_n;\, \underline{z}_{n-1})\;\; Q^{C_{n-1}}_{D_{n-1}}(\underline{z}_{n-1};\, \underline{x}_{n-1}), \end{aligned}$$

where

$$Q^{D_n}_{C_{n-1}}(\underline{x}_n;\, \underline{z}_{n-1}) = \exp\Big\{ - \Big(e^{x_{n,1}+z_{n-1,1}} + \sum_{i=1}^{n-1} \Big(e^{z_{n-1,i}-x_{n,i}} + e^{x_{n,i+1}-z_{n-1,i}}\Big)\Big)\Big\}, \tag{2.240}$$

$$\begin{aligned} Q^{C_{n-1}}_{D_{n-1}}(\underline{z}_{n-1};\, \underline{x}_{n-1}) = \exp\Big\{ - \Big(e^{x_{n-1,1}+z_{n-1,1}} + \\ + \sum_{i=1}^{n-2} \Big(e^{z_{n-1,i}-x_{n-1,i}} + e^{x_{n-1,i+1}-z_{n-1,i}}\Big) + e^{z_{n-1,n-1}-x_{n-1,n-1}}\Big)\Big\}, \end{aligned}$$

and for $n = 1$ we define

$$Q^{D_1}_{D_0}(x_{1,1}; \lambda_1) = e^{\imath\lambda_1 x_{1,1}}.$$

Using $Q^{D_n}_{D_{n-1}}$, $n = 1, \ldots, \ell$ the integral representation (2.233) can be written in the recursive form.

Theorem 2.15 *The eigenfunction for $\mathfrak{so}_{2\ell}$-Toda chain can be written in the following recursive form:*

$$\Psi^{D_\ell}_{\lambda_1,\dots,\lambda_\ell}(x_1,\dots,x_\ell) \;=\; \int\limits_{\mathcal{C}} \bigwedge_{k=1}^{\ell-1}\bigwedge_{i=1}^{k} dx_{k,i} \prod_{k=1}^{\ell} Q^{D_k}_{D_k-1}(\underline{x}_k;\,\underline{x}_{k-1};\lambda_k), \tag{2.241}$$

or equivalently

$$\Psi^{D_\ell}_{\lambda_1,\dots,\lambda_\ell}(x_{\ell,1},\dots,x_{\ell,\ell}) \;=\; \int\limits_{\mathcal{C}_\ell} \bigwedge_{i=1}^{\ell-1} dx_{\ell-1,i}\, Q^{D_\ell}_{D_{\ell-1}}(\underline{x}_\ell;\,\underline{x}_{\ell-1};\lambda_\ell)\Psi^{D_{\ell-1}}_{\lambda_1,\dots,\lambda_{\ell-1}}(x_{\ell-1,1},\dots,x_{\ell-1,\ell-1}),$$

where we assume $x_n := x_{\ell,n},\ 1\le n\le \ell$. Here $\mathcal{C}\subset N_+$ is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at the boundaries and at infinities. In particular the domain of integration can be chosen to be $\mathcal{C}=\mathbb{R}^m$, where $m=l(w_0)$.

As for other classical Lie algebras, different from $\mathfrak{gl}_{\ell+1}$, the specailization to zero spectrum $\{\lambda_n=0\}$ revieals a more refined recursive structure. In this case the kernel of the operator $Q^{D_n}_{D_{n-1}}$ is reduced to a convolution of two kernels $Q^{D_n}_{C_{n-1}}(\underline{x}_n;\,\underline{z}_{n-1})$ and $Q^{C_{n-1}}_{D_{n-1}}(\underline{z}_{n-1};\,\underline{x}_{n-1})$. The corresponding integral operators $Q^{D_n}_{C_{n-1}}$, $Q^{C_{n-1}}_{D_{n-1}}$ can be regarded as elementary intertwiners relating Toda chains for D_n, C_{n-1} and C_{n-1}, D_{n-1} root systems. Thus for quadratic Hamiltonians one can directly check the following relations

Lemma 2.11 *The operators $Q^{D_n}_{D_{n-1}}$, $Q^{D_n}_{C_{n-1}}$ and $Q^{C_{n-1}}_{D_{n-1}}$ satisfy the following intertwining relations with quadratic Toda Hamiltonians.*

1. *Operators $Q^{D_n}_{C_{n-1}}$ and $Q^{C_{n-1}}_{D_{n-1}}$ intertwine quadratic Hamiltonians of C- and D-Toda chains:*

$$\mathcal{H}^{D_n}_2(\underline{x}_n)Q^{D_n}_{C_{n-1}}(\underline{x}_n,\,\underline{z}_{n-1}) = Q^{D_n}_{C_{n-1}}(\underline{x}_n,\,\underline{z}_{n-1})\mathcal{H}^{C_{n-1}}_2(\underline{x}_{n-1}), \tag{2.242}$$

$$\mathcal{H}^{C_n}_2(\underline{z}_n)Q^{C_n}_{D_n}(\underline{z}_n,\,\underline{x}_n) = Q^{C_n}_{D_n}(\underline{z}_n,\,\underline{x}_n)\mathcal{H}^{D_n}_2(\underline{x}_n). \tag{2.243}$$

2. *Operator $Q^{D_n}_{D_{n-1}}$ for $\lambda_n=0$ intertwines Hamiltonians $\mathcal{H}^{D_n}_2$ and $\mathcal{H}^{D_{n-1}}_2$:*

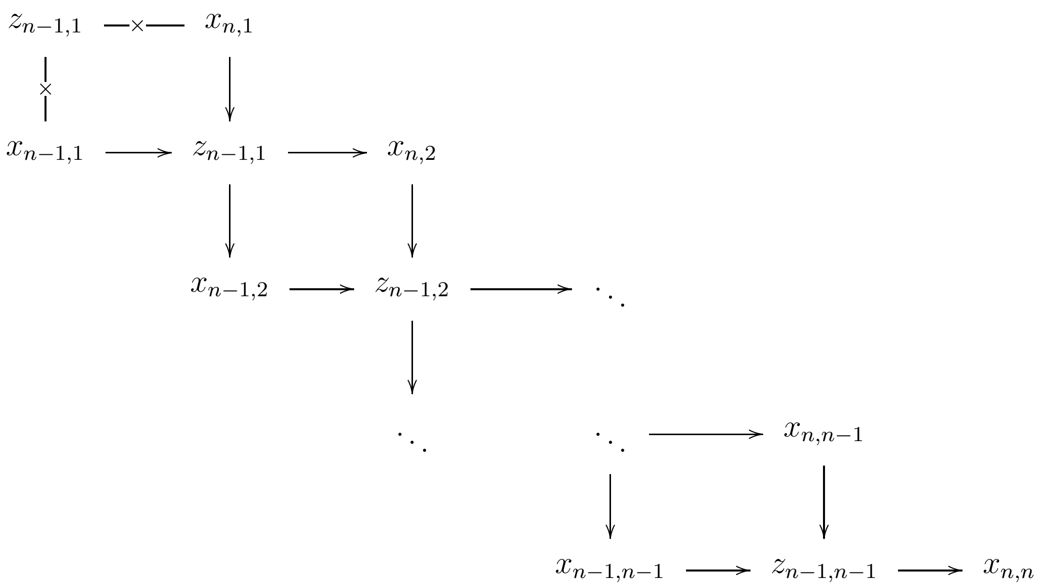
$$\mathcal{H}^{D_n}_2(\underline{x}_n)Q^{D_n}_{D_{n-1}}(\underline{x}_n;\,\underline{x}_{n-1};\lambda_n=0) = Q^{D_n}_{D_{n-1}}(\underline{x}_n;\,\underline{x}_{n-1};\lambda_n=0)\mathcal{H}^{D_{n-1}}_2(\underline{x}_{n-1}), \tag{2.244}$$

where

$$\mathcal{H}^{C_n}_2 = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial z_i^2} + 2e^{2z_1} + \sum_{i=1}^{n-2} e^{z_{i+1}-z_i}, \tag{2.245}$$

$$\mathcal{H}^{D_n}_2 = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + e^{x_1+x_2} + \sum_{i=1}^{n-1} e^{x_{i+1}-x_i}. \tag{2.246}$$

The integral kernel of the operator $Q^{D_n}_{D_{n-1}}(\underline{x}_n; \underline{x}_{n-1})$ at $\lambda_n = 0$ can be succinctly encoded into the following diagram



(2.247)

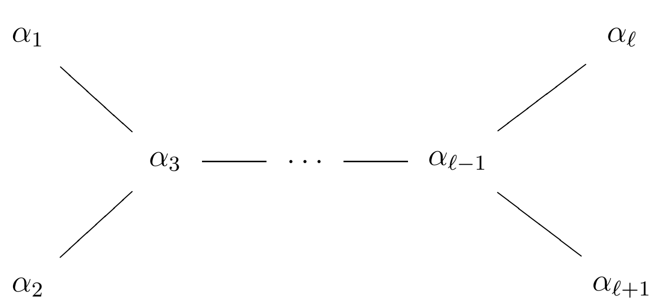
Here the upper and lower descending paths of the oriented diagram correspond to the kernels of elementary intertwiners $Q^{D_n}_{C_{n-1}}$ and $Q^{C_n}_{D_{n-1}}$ respectively. The convolution of the kernels $Q^{D_n}_{C_n}$ and $Q^{C_{n-1}}_{D_{n-1}}$ corresponds to the integration over the variables $z_{n-1,i}$ associated with the inner vertexes of the sub-diagram (2.247.)

Similarly to the cases of other classical series of Lie algebras recursion operators $Q^{D_n}_{D_{n-1}}$ can be considered as degenerations of Baxter $\mathcal{Q}$-operators for affine $D^{(1)}_\ell$-Toda chains. Let us recall the root data for $D^{(1)}_\ell$. Simple roots of the affine root system $D^{(1)}_\ell$ can be represented as vectors in $\mathbb{R}^\ell$ in the following way

$$\alpha_1 = \epsilon_1 + \epsilon_2, \qquad \alpha_{i+1} = \epsilon_{i+1} - \epsilon_i, \quad 1 \leq i \leq \ell - 1,$$

$$\alpha_{\ell+1} = -\epsilon_\ell - \epsilon_{\ell-1},$$

and Dynkin diagram is given by



The corresponding $D_\ell^{(1)}$-Toda chain quadratic Hamiltonian is defined by

$$\mathcal{H}_2^{D_\ell^{(1)}} \;=\; -\frac{1}{2}\sum_{i=1}^{\ell}\frac{\partial^2}{\partial x_i^2}+e^{x_1+x_2}+\sum_{i=1}^{\ell-2}e^{x_{i+1}-x_i}+ge^{x_\ell-x_{\ell-1}}+ge^{-x_\ell-x_{\ell-1}}. \qquad (2.248)$$

Define the Baxter $\mathcal{Q}$-operator of $D_\ell^{(1)}$-Toda chain as an integral operator with the following integral kernel

$$\mathcal{Q}^{D_\ell^{(1)}}(\underline{x}^{(\ell)},\underline{y}^{(\ell)},\lambda)=\int\bigwedge_{i=1}^{\ell-1}dz_i\,\Big(e^{x_1}+e^{y_1}\Big)^{2\imath\lambda}\Big(e^{-x_\ell}+e^{-y_\ell}\Big)^{-2\imath\lambda}\times$$
$$\times\exp\Big\{-\imath\lambda\Big(\sum_{i=1}^{\ell}x_i-2\sum_{i=1}^{\ell-1}z_i+\sum_{i=1}^{\ell}y_i\Big)\Big\}\times \qquad (2.249)$$
$$\times Q^{D_\ell^{(1)}}_{C_{\ell-1}^{(1)}}(x_1,\ldots,x_\ell\,;z_1,\ldots,z_{\ell-1})\quad Q^{C_{\ell-1}^{(1)}}_{D_\ell^{(1)}}(z_1,\ldots,z_{\ell-1}\,;y_1,\ldots,y_\ell),$$

where

$$Q^{D_\ell^{(1)}}_{C_{\ell-1}^{(1)}}(x_1,\ldots,x_\ell\,;\,z_1,\ldots,z_{\ell-1})= \qquad (2.250)$$
$$=\exp\Big\{e^{z_1+x_1}+\sum_{i=1}^{\ell-1}\Big(e^{z_i-x_i}+e^{x_{i+1}-z_i}\Big)+ge^{-x_\ell-z_{\ell-1}}\Big\},$$

and

$$Q^{D_\ell^{(1)}}_{C_{\ell-1}^{(1)}}(x_1,\ldots,x_\ell;\,z_1,\ldots,z_\ell)=Q^{C_{\ell-1}^{(1)}}_{D_\ell^{(1)}}(z_1,\ldots,z_\ell;x_1,\ldots,x_\ell). \qquad (2.251)$$

Here we use the following notations $\underline{x}^{(\ell)}=(x_1,\ldots,x_\ell)$, $\underline{y}^{(\ell)}=(y_1,\ldots,y_\ell)$.

Proposition 2.14 *The $\mathcal{Q}$-operator (2.249) commutes with quadratic Hamiltonian of the $D_\ell^{(1)}$-Toda chain:*

$$\mathcal{H}^{D_\ell^{(1)}}(\underline{x}^{(\ell)})\mathcal{Q}^{D_\ell^{(1)}}(\underline{x}^{(\ell)},\underline{y}^{(\ell)})=\mathcal{Q}^{D_\ell^{(1)}}(\underline{x}^{(\ell)},\,\underline{y}^{(\ell)})\mathcal{H}^{D_\ell^{(1)}}(\underline{y}^{(\ell)}). \qquad (2.252)$$

Now we will demonstrate that recursion operator $Q^{D_\ell}_{D_{\ell-1}}$ can be considered as a degeneration of Baxter $\mathcal{Q}$-operators for $D_\ell^{(1)}$. Let us introduce a slightly modified recursion operator with the kernel: $Q^{D_\ell}_{D_{\ell-1}\oplus D_1}$:

$$Q^{D_\ell}_{D_{\ell-1}\oplus D_1}(\underline{x}^{(\ell)}\,,\,\underline{y}^{(\ell)},\,\lambda)=e^{\imath\lambda y_\ell}Q^{D_\ell}_{D_{\ell-1}}(\underline{x}^{(\ell)}\,,\,\underline{y}^{(\ell-1)}\,,\,\lambda), \qquad (2.253)$$

where we use the notations $\underline{y}^{(\ell-1)}=(y_1,\ldots,y_{\ell-1})$. This operator intertwines Hamiltonians of $\mathfrak{so}_{2\ell}$- and $\mathfrak{so}_{2\ell-2}\oplus\mathfrak{so}_2$-Toda chains. For instance we have for quadratic Hamiltonians

$$\mathcal{H}_2^{D_\ell}(\underline{x}^{(\ell)})Q^{D_\ell}_{D_{\ell-1}\oplus D_1}(\underline{x}^{(\ell)}\,,\,\underline{y}^{(\ell)},\lambda)=Q^{D_\ell}_{D_{\ell-1}\oplus D_1}(\underline{x}^{(\ell)}\,,\,\underline{y}^{(\ell)},\lambda)\Big(\mathcal{H}_2^{D_{\ell-1}}(\underline{y}^{(\ell-1)})+\mathcal{H}_2^{D_1}(y_\ell)\Big),$$

where $\mathcal{H}_2^{D_1}(y_\ell)=-\frac{1}{2}\Big(\partial^2/\partial y_\ell^2\Big)$. Obviously the projection of the above relation on the subspace of functions $F(\underline{y}^{(\ell)})=\exp(\imath\lambda y_\ell)f(\underline{y}^{(\ell-1)})$ recovers the genuine recursion operator satisfying:

$$\mathcal{H}_2^{D_\ell}(\underline{x}^{(\ell)})Q^{D_\ell}_{D_{\ell-1}}(\underline{x}^{(\ell)}\,,\,\underline{y}^{(\ell-1)}\,,\,\lambda)=Q^{D_\ell}_{D_{\ell-1}}(\underline{x}^{(\ell)}\,,\,\underline{y}^{(\ell-1)}\,,\,\lambda)\Big(\mathcal{H}_2^{D_{\ell-1}}(\underline{y}^{(\ell-1)})+\frac{1}{2}\lambda^2\Big). \tag{2.254}$$

Let us introduce a one-parameter family of the operators with the kernels

$$\mathcal{Q}^{D_\ell^{(1)}}(\underline{x}^{(\ell)},\,\underline{y}^{(\ell)},\lambda;\varepsilon):=\varepsilon^{\imath\lambda}e^{\imath\lambda y_\ell}\int\bigwedge_{i=1}^{\ell-1}dz_i\,\Big(e^{x_1}+e^{y_1}\Big)^{2\imath\lambda}\Big(\varepsilon e^{-x_\ell+y_\ell}+1\Big)^{-2\imath\lambda}\times \tag{2.255}$$
$$\times\exp\Big\{-\imath\lambda\Big(\sum_{i=1}^{\ell}x_i-2\sum_{i=1}^{\ell-1}z_i+\sum_{i=1}^{\ell-1}y_i\Big)\Big\}\times$$
$$\times Q^{D_\ell^{(1)}}_{C_{\ell-1}^{(1)}}(x_1,\ldots,x_\ell\,;z_1,\ldots,z_{\ell-1})\quad Q^{C_{\ell-1}^{(1)}}_{D_\ell^{(1)}}(z_1,\ldots,z_{\ell-1}\,;y_1,\ldots,y_\ell;\varepsilon),$$

where

$$Q^{D_\ell^{(1)}}_{C_{\ell-1}^{(1)}}(x_1,\ldots,x_\ell\,;\,z_1,\ldots,z_{\ell-1})= \tag{2.256}$$
$$=\exp\Big\{e^{z_1+x_1}+\sum_{i=1}^{\ell-1}\Big(e^{z_i-x_i}+e^{x_{i+1}-z_i}\Big)+ge^{-x_\ell-z_{\ell-1}}\Big\},$$

and

$$Q^{C_{\ell-1}^{(1)}}_{D_\ell^{(1)}}(z_1,\ldots,z_{\ell-1}\,,\,y_1,\ldots,y_\ell;\varepsilon)=$$
$$=\exp\Big\{-\Big(e^{z_1+y_1}+\sum_{i=1}^{\ell-2}\Big(e^{z_i-y_i}+e^{y_{i+1}-z_i}+e^{z_{\ell-1}-y_{\ell-1}}\Big)+ \tag{2.257}$$
$$+\varepsilon e^{y_\ell-z_{\ell-1}}+\varepsilon^{-1}ge^{-x_\ell-z_{\ell-1}}\Big)\Big\}.$$

These operators are obtained by a shift of the variable $y_\ell=y_\ell+\ln\varepsilon$ in (2.240). Then the following relation between $\mathcal{Q}$-operator for $D_\ell^{(1)}$-Toda chain and (modified) recursion operator for $\mathfrak{so}_{2\ell}$-Whittaker function holds

$$Q^{D_\ell}_{D_{\ell-1}\oplus D_1}(\underline{x}^{(\ell)}\,,\;\underline{y}^{(\ell)},\,\lambda)=\lim_{\varepsilon\to 0}\lim_{\varepsilon^{-1}g\to 0}\varepsilon^{-\imath\lambda}Q^{D_\ell^{(1)}}(\underline{x}^{(\ell)}\,,\,\underline{y}^{(\ell)}\,,\,\lambda\,;\,\varepsilon). \tag{2.258}$$

3 Part II. Proofs

Let G be a complex connected simply-connected semisimple Lie group of finite rank ℓ, $\mathfrak{g} = \mathrm{Lie}(G)$ be the corresponding semisimple Lie algebra with the Chevalley generators f_i, h_i, e_i. Let us fix a Borel subgroup B_+ and let T be the maximal torus $T \subset B_+$. This defines a pair N_+, N_- of opposite unipotent subgroups in G, $N_+ \subset B_+$. Let Γ be the set of vertices of Dynkin graph of $\mathfrak{g}$, $\{\alpha_i,\, i \in \Gamma\}$ be the set of simple roots, $\{\gamma_k, k = 1, \ldots, \frac{1}{2}(\dim \mathfrak{g}/\mathfrak{h})\}$ be the set of all positive roots and $\{\alpha_i^\vee,\, i \in \Gamma\}$ be the set of simple coroots. For every $i \in \Gamma$ there is a group homomorphism

$$\varphi_i : \quad SL_2 \longrightarrow G, \tag{3.1}$$

defined as follows. Introduce a set of one-parameter subgroups $e^{te_i} = X_i(t) \subset N_+$, $e^{tf_i} = Y_i(t) \subset N_-$ and $e^{th_i} = \alpha_i^\vee(t) \subset T$. Homomorphisms (3.1) are defined as

$$\varphi_i(e^{te}) = e^{te_i}, \qquad \varphi_i(e^{tf}) = e^{tf_i}, \qquad \varphi_i(e^{th}) = \alpha_i^\vee(t), \tag{3.2}$$

where e, f, h are standard generators of $\mathfrak{sl}_2$. Let us fix the lifts $\dot{s}_i \subset G$, $\dot{s} \subset SL(2)$ of the generators s_i of the Weyl group of G and the generator of the Weyl group of $SL(2)$

$$\dot{s} = e^e e^{-f} e^e, \qquad \dot{s}_i = e^{e_i} e^{-f_i} e^{e_i}. \tag{3.3}$$

Thus defined lifts of Weyl group generators are obviously compatible $\varphi_i(\dot{s}) = \dot{s}_i$ with homomorphisms (3.1). We have the following relations

$$\dot{s}^{-1} f \, \dot{s} \;=\; -e, \qquad \dot{s}_i^{-1} f_i \, \dot{s}_i \;=\; -e_i. \tag{3.4}$$

The action $w_0(\alpha_i) = -\alpha_{i^*}$ of the maximal length element w_0 of the Weyl group on simple roots defines an involution $i \mapsto i^*$. The corresponding action of $\dot{w}_0$ is given by

$$\dot{w}_0^{-1} f_i \, \dot{w}_0 \;=\; -e_{i^*}. \tag{3.5}$$

Remark 3.1 *For classical Lie groups one has $i^* = \ell + 1 - i$ for $G = SL(\ell+1)$, $i^* = i$ for $G = SO(2\ell+1)$ and for $G = Sp(2\ell)$. In the case $G = SO(2\ell)$ (for $\ell \geq 2$) the action of the involution $*$ is as follows:*

$$* : \qquad 1 \longmapsto \begin{cases} 1\,, & \ell \text{ even} \\ 2\,, & \ell \text{ odd} \end{cases} \qquad\qquad 2 \longmapsto \begin{cases} 2\,, & \ell \text{ even} \\ 1\,, & \ell \text{ odd} \end{cases} \tag{3.6}$$

$$k^* = k\,, \qquad 2 < k \leq \ell,$$

where the enumeration of roots of $SO(2\ell)$ is given by (2.188).

In the following we will be considering matrix elements of finite-dimensional representations V_{ω_i} of $\mathfrak{g}$ corresponding to the fundamental weights ω_i, $i \in \Gamma$. Let $\xi^+_{\omega_i}$and $\xi^-_{\omega_i}$ be highest and lowest vectors in V_{ω_i} such that $\langle\xi^-_{\omega_i}|\xi^+_{\omega_i}\rangle = 1$. For the lift (3.3) of the elements of the Weyl group we have (see e.g. [K] Lemma 3.8, [FZ] eq. (2.29))

$$\dot{w}_0^{-1}\xi^+_{\omega_i} = \xi^-_{\omega_i}, \qquad \dot{s}_i^{-1}\xi^+_{\omega_i} = f_i\xi^+_{\omega_i}. \tag{3.7}$$

Consider the following parametrization of a generic group element $g \in G$

$$g = g^{(-)}\, g^{(0)}\, g^{(+)} = \exp(\sum_{\alpha\in\Delta_+} u_{-\alpha}f_\alpha)\ \exp(\sum_{i=1}^{\ell} u_i h_i)\ \exp(\sum_{\alpha\in\Delta_+} u_\alpha e_\alpha). \tag{3.8}$$

For coordinates u_i corresponding to Cartan generators h_i and for coordinates $u_{\pm\alpha_i}$ corresponding to simple root generators e_{α_i}, f_{α_i} there exits simple expressions in terms of matrix elements of fundamental representations V_{ω_i}:

$$u_{\alpha_i}(g) = \frac{\langle\xi^-_{\omega_i}|\pi_i(g)\pi_i(f_i)|\xi^+_{\omega_i}\rangle}{\langle\xi^-_{\omega_i}|\pi_i(g)|\xi^+_{\omega_i}\rangle}, \qquad u_{-\alpha_i}(g) = \frac{\langle\xi^-_{\omega_i}|\pi_i(g)\pi_i(e_i)|\xi^+_{\omega_i}\rangle}{\langle\xi^-_{\omega_i}|\pi_i(g)|\xi^+_{\omega_i}\rangle}, \tag{3.9}$$

$$u_i(g) = \langle\xi^-_{\omega_i}|\pi_i(g)|\xi^+_{\omega_i}\rangle,$$

where $\pi_i \equiv \pi_{\omega_i}$ is a fundamental representation in V_{ω_i}. Define generalized twisted minors as

$$\Delta_{\omega_i,\dot{w}}(g) = \langle\xi^-_{\omega_i}|\pi_{\omega_i}(g)\pi_{\omega_i}(\dot{w})|\xi^+_{\omega_i}\rangle, \qquad g \in G. \tag{3.10}$$

Then coordinate u_i and u_{α_i} of a twisted unipotent element $v\dot{w}_0^{-1} \in G$ (where $v \in N_+$) can be expressed in terms of twisted minors (3.10) as follows

$$e^{u_i(v\dot{w}_0^{-1})} = \Delta_{\omega_i,\dot{w}_0^{-1}}(v),$$

$$u_{\alpha_i}(v\dot{w}_0^{-1}) = \frac{\langle\xi_i^-|\pi_i(v\dot{w}_0^{-1})\pi_i(f_i)|\xi_i^+\rangle}{\langle\xi_i^-|\pi_i(v\dot{w}_0^{-1})|\xi_i^+\rangle} = \frac{\langle\xi_i^-|\pi_i(v)\pi_i(\dot{w}_0^{-1})\pi_i(\dot{s}_i^{-1})|\xi^+_{\omega_i}\rangle}{\langle\xi_i^-|\pi_i(v\dot{w}_0^{-1})|\xi^+_{\omega_i}\rangle} = \tag{3.11}$$

$$= -\frac{\langle\xi^-_{\omega_i}|\pi_i(v)\pi_i(e_{i^*})\pi_i(\dot{w}_0^{-1})|\xi^+_{\omega_i}\rangle}{\langle\xi_i^-|\pi_i(v\dot{w}_0^{-1})|\xi_i^+\rangle} = \frac{\Delta_{\omega_i,\dot{w}_0^{-1}\dot{s}_i^{-1}}(v)}{\Delta_{\omega_i,\dot{w}_0^{-1}}(v)}.$$

In the following we will use the shorthand notations

$$\begin{aligned}\Delta_i'(v) :\ &=\ \langle\xi^-_{\omega_i}|\pi_i(ve_{i*}\dot{w}_0^{-1})\xi^+_{\omega_i}\rangle = -\Delta_{\omega_i,\dot{w}_0^{-1}\dot{s}_i^{-1}}(v)\ , \\ \Delta_i(v) :\ &=\ \Delta_{\omega_i,\dot{w}_0^{-1}}(v).\end{aligned} \tag{3.12}$$

3.1 Measure on N_+: Proof of Lemma 2.2

In this part we derive an explicit expression (2.21) for a measure $d\mu_{N_+}(x)$ on a unipotent subgroup $N_+ \subset G$ of any classical Lie group using a factorized parametrization (2.20) of N_+. Recall that for a reduced word $I_\ell = (i_1, \ldots, i_{m_\ell})$ of w_0 there is a birational isomorphism $\mathbb{C}^{m_\ell} \to N_+$. Particularly, given an unipotent element $v \in N_+$ the following factorized representation holds.

$$v(t) = X_{i_1}(t_1)X_{i_2}(t_2)\cdot\ldots\cdot X_{i_m}(t_{m_\ell}), \tag{3.13}$$

where $X_i(t) = e^{te_i}$. The variables t_i are called factorization parameters of v.

Proposition 3.1 *Let $v(t) \in N_+^{(0)}$ be a factorized parametrization (3.13) corresponding to a reduced word $I = (i_1, \ldots, i_{m_\ell})$. Then*

$$d\mu_{N_+}(v(t)) = \prod_{k=1}^{\ell}\prod_{i=1}^{m_\ell}(t_i)^{\langle\omega_k,\gamma_i\rangle} \cdot \bigwedge_{i=1}^{m_\ell}\frac{dt_i}{t_i}, \tag{3.14}$$

is a restriction of the right-invariant measure $d\mu_{N_+}$ to $N_+^{(0)}$, that is

$$d\mu_{N_+}(v(t)) = d\mu_{N_+}(v(t)\cdot X_j(\tau)), \quad j = 1, \ldots, \ell. \tag{3.15}$$

Proof. To prove the Proposition consider a dependence on a choice of a reduced word $I = (i_1, \ldots, i_m)$ explicitly. Let $t^I = (t_1^I, \ldots, t_{m_\ell}^I)$ be factorization parameters corresponding to a reduced word I. According to [BZ] (Theorem 4.3) one has the following expressions for matrix elements

$$\Delta_k(t^I) := \Delta_k(x(t^I)w_0^{-1}) = \prod_{i=1}^{m_\ell}(t_i^I)^{\langle\omega_k,\gamma_i\rangle} \tag{3.16}$$

Two parameterizations $x(t^I)$ and $x(t^{I'})$ of $N_+^{(0)}$ corresponding to reduced words I and I' are related by a birational transformation.

Lemma 3.1 *For any reduced decompositions of w_0 corresponding to reduced words I and I' the following relations hold*

1.

$$\Delta_k(t^I) = \Delta_k(t^{I'}), \qquad 1 \leqq k \leqq \ell. \tag{3.17}$$

2.

$$\bigwedge_{j=1}^{m_\ell}\frac{dt_j^I}{t_j^I} = \bigwedge_{j=1}^{m_\ell}\frac{dt_j^{I'}}{t_j^{I'}}. \tag{3.18}$$

Proof of Lemma. It is shown in [Lu] that birational transformations $R_I^{I'}$ of N_+ corresponding to any two reduced words I and I' can be represented as a composition of elementary transformations (so-called 3- and 4-moves). Therefore to prove (3.17), (3.18) one should check these identities for the elementary moves only in the following In the case of classical Lie groups it is enough to consider the following two birational transformations $R_I^{I'} : t^I \to t^{I'}$

1. $X_i(t_1)X_j(t_2)X_i(t_3) = X_j(t_1')X_i(t_2')X_j(t_3')$ for $a_{ij} = a_{ji} - 1$,
2. $X_j(t_1)X_i(t_2)X_j(t_3)X_i(t_4) = X_i(t_1')X_j(t_2')X_i(t_3')X_j(t_4')$ for $a_{ij} = -1$ and $a_{ji} = -2$,

where we denote $t = t^I$ and $t' = t^{I'}$.

The proof of the identity (3.17) for elementary 3- and 4-moves follows straightforwardly from the results in [BZ]. Thus we consider only the proof of (3.18) below.

1) In the case $a_{ij} = a_{ji} = -1$ we should consider the birational transformation between the parametrizations associated with reduced words $I = (\ldots iji \ldots)$ and $I' = (\ldots jij \ldots)$. We have the following relation between parameters

$$v = X_i(t_1)X_j(t_2)X_i(t_3) \ = \ X_j(t'_1)X_i(t'_2)X_j(t'_3),$$

where

$$t'_1 = \frac{t_2 t_3}{t_1 + t_3}, \qquad t'_2 = t_1 + t_3, \qquad t'_3 = \frac{t_1 t_2}{t_1 + t_3}\,.$$

Direct check gives

$$d\log t'_1 \wedge d\log t'_2 \wedge d\log t'_3 \ = \ d\log t_1 \wedge d\log t_2 \wedge d\log t_3\,. \tag{3.19}$$

2) In the case $a_{ij} = -1$, $a_{ji} = -2$ we should consider the birational transformation between the parameterizations associated with reduced words $I = (\ldots jiji \ldots)$ and $I' = (\ldots ijij \ldots)$. Thus we have the following relation between parameters

$$X = X_j(t_1)X_i(t_2)X_j(t_3)X_i(t_4) \ = \ X_i(t'_1)X_j(t'_2)X_i(t'_3)X_j(t'_4),$$

with

$$t'_1 = \frac{t_2 t_3^2 t_4}{t_1^2 t_2 + (t_1+t_3)^2 t_4}\,, \qquad t'_2 = \frac{t_1^2 t_2 + (t_1+t_3)^2 t_4}{t_1 t_2 + (t_1+t_3) t_4}\,, \tag{3.20}$$

$$t'_3 = \frac{\Big(t_1 t_2 + (t_1+t_3)t_4\Big)^2}{t_1^2 t_2 + (t_1+t_3)^2 t_4}\,, \qquad t'_4 = \frac{t_1 t_2 t_3}{t_1 t_2 + (t_1+t_3) t_4}.$$

One can readily verify the following identity:

$$d\log t'_1 \wedge d\log t'_2 \wedge d\log t'_3 \wedge d\log t'_4 \ = d\log t_1 \wedge d\log t_2 \wedge d\log t_3 \wedge d\log t_4.$$

This completes the proof of the Lemma.

Now we can complete the proof of the Proposition 3.1. To establish the right-invariance of measure $d\mu_{N_+}(v)$ we use (3.17), (3.18). For any simple root α_i one can find a reduced word $I(\alpha_i) = (j_1, \ldots, j_m)$ with $m = m_\ell$ such that $j_m = i$. Then identities (3.17), (3.18) imply that

$$d\mu_{N_+}(v(t^{I(\alpha_i)})) = d\mu_{N_+}(v(t^{I_\ell})).$$

In this way we obtain

$$v(t^{I(\alpha_i)}) \cdot X_i(\tau) = X_{j_1}(t_1) \cdot \ldots \cdot X_{j_{m-1}}(t_{m-1}) X_i(t_m + \tau) \tag{3.21}$$

By construction the factorization parameter t_m enters only in the (monomial) expression for $\Delta_j(v(t))$ as a homogeneous factor of degree one. In this way, the factorization parameter t_m appears in the measure $d\mu_{N_+}$ only in the α_j-component $\Delta_j(v(t)) d\ln t_m$, and hence, the measure $d\mu_{N_+}$ is invariant under the shift $t_m \to t_m + \tau$. Thus the measure is right-invariant with respect to the action of $X_j(\tau)$ for any $j = 1, \ldots, \ell$, and eventually it is right-invariant with respect to the whole N_+. This completes the proof of Lemma 2.2.

3.2 Whittaker vectors for classical Lie groups: Proof of Lemma 2.1 and Proposition 2.1

In this subsection we derive expressions for left and right $\mathfrak{g}$-Whittaker vectors in terms of the matrix elements of finite-dimensional representations $\mathfrak{g}$. The Whittaker vectors satisfy the following equations

$$e_i\psi_R = -\psi_R\,, \qquad f_i\psi_L = -\psi_L\,, \qquad i = 1, \ldots, \ell. \tag{3.22}$$

Integrating actions of the nilpotent Lie subalgebras $\mathfrak{n}_\pm \subset \mathfrak{g}$ to actions of the nilpotent Lie subgroups $N_\pm \subset G$, equations on $\mathfrak{g}$-Whittaker can be written in terms of one-parameter subgroups $X_i(t) \subset N_+$, $Y_i(t) \subset N_-$ as follows

$$\pi_\lambda(X_i(t))\psi_R(v) = e^{-t}\psi_R(v), \quad \pi_\lambda(Y_i(t))\psi_L(v) = e^{-t}\psi_L(v), \quad i = 1, \ldots, \ell, \quad v \in N_+.$$

Equivalently one has for any $z_\pm \in N_\pm$

$$\pi_\lambda(z_+)\psi_R(v) = \exp\Big\{-\sum_{i=1}^{\ell}(z_+)_i\Big\}\psi_R(v), \quad \pi_\lambda(z_-)\psi_L(v) = \exp\Big\{-\sum_{i=1}^{\ell}(z_-)_i\Big\}\psi_L(v), \tag{3.23}$$

where $(z_\pm)_i := u_{\pm\alpha_i}(z_\pm)$. Construction of the right Whittaker vector is pretty straightforward. Note that we have a simple identity

$$u_{\alpha_i}(v_1\, v_2) = u_{\alpha_i}(v_1) + u_{\alpha_i}(v_2), \qquad v_1, v_2 \in N_+.$$

Then from (3.23) we infer that the right Whittaker vector is given by a multiplicative character of the maximal unipotent subgroup N_+

$$\psi_R(v) = \exp\Big\{-\sum_{i=1}^{\ell} v_i\Big\} = \exp\Big\{-\frac{\Delta_{\omega_i, \dot{s}_i^{-1}}(v)}{\Delta_{\omega_i,1}(v)}\Big\}, \qquad v \in N_+. \tag{3.24}$$

where $v_i := u_{\alpha_i}(v)$ and we use (3.9) to express v_i in terms of matrix elements.

To construct the left Whittaker vector in terms of matrix elements we use an inner automorphism of G, acting on $z \in G$ as $z^\tau = \dot{w}_0^{-1} z \dot{w}_0$. Taking into account that $\dot{w}_0^{-1} X_{i^*}(-t)\dot{w}_0 = Y_i(t)$ we have $\dot{w}_0^{-1} N_+ \dot{w}_0 = N_-$. Now the equation for the left Whittaker

$$\pi_\lambda(Y_i(t))\psi_L(v) = e^{-t}\psi_L(v), \quad i = 1, \ldots, \ell$$

can be written in the following form

$$\pi_\lambda(z^\tau)\psi_L(v) = \exp\Big\{-\sum_{i=1}^{\ell} z_i\Big\}\psi_L(v), \qquad z \in N_+, \tag{3.25}$$

The left Whittaker vector can be obtained by the twist of the right vector

$$\psi_L(v) = \psi_R(v\dot{w}_0^{-1})$$

where the function ψ_R is considered as a B_--equivariant function on G (see (2.9) for the precise definition). Using Gauss decomposition and the parametrization (3.8), (3.9) we get for the left Whittaker vector

$$\psi_L(v) = e^{\langle \imath\lambda-\rho, \sum_{i=1}^{\ell} u_i(v\dot{w}_0^{-1})h_i\rangle} \;\; e^{\sum_i u_{\alpha_i}(v\dot{w}_0^{-1})}.$$

In terms of the matrix elements of finite-dimensional representations we have the following representation

$$\begin{aligned}\psi_L(v) &= \prod_{i=1}^{\ell} \Delta_{\omega_i,\dot{w}_0^{-1}}(v)^{\langle \imath\lambda-\rho,\alpha_i^\vee\rangle} \cdot \exp\Big\{\frac{\Delta_{\omega_i,\dot{w}_0^{-1}\dot{s}_i^{-1}}(v)}{\Delta_{\omega_i,\dot{w}_0^{-1}}(v)}\Big\} = \\ &= \prod_{i=1}^{\ell} \Delta_i(v)^{\langle \imath\lambda-\rho,\alpha_i^\vee\rangle} \;\exp\Big\{-\frac{\Delta'_i(v)}{\Delta_i(v)}\Big\}.\end{aligned} \tag{3.26}$$

This completes the proof of Lemma 2.1. The proof of the Proposition 2.1 is then obtained by combining the expressions for right Whittaker vector and left Whittaker vector twisted by the action of Cartan generator $\exp h_x = \exp -(\sum_{i=1}^{\ell}\langle\omega_i, x\rangle h_i)$.

3.3 Explicit evaluation of matrix elements

To construct integral representations of Whittaker functions one should express various matrix elements entering the integral formulas (2.19) using factorized and modified factorized parametrizations of group elements. This can be done rather straightforwardly using results of [BZ], [**?**]. Below we shall use a recursive structure of reduced word I corresponding to a maximal length element w_0 of Weyl group of classical Lie algebras. This recursive structure translates into recursive formulas for the relevant ratios of matrix elements. Resolving recursive equations we find explicit expressions of ψ_L and ψ_R in a (modified) factorized parametrization. This provides corresponding integral representations for Whittaker functions of classical Lie groups. In the case of the modified factorized parametrization we obtain a generalization of Givental integral representation for $\mathfrak{g} = \mathfrak{gl}_{\ell+1}$.

3.3.1 Expressions for $\mathfrak{gl}_{\ell+1}$-matrix elements: Proofs of Theorem 2.1 and Theorem 2.3

In this subsection we introduce expressions for matrix elements relevant for the construction of integral representations of $\mathfrak{gl}_{\ell+1}$-Whittaker functions using factorized parametrization of an open part of $N_+ \subset GL(\ell+1)$. This provides a proof of the integral representations of $\mathfrak{gl}_{\ell+1}$-Whittaker functions presented in Part I.

The eigenfunctions of $\mathfrak{gl}_{\ell+1}$ and $\mathfrak{sl}_{\ell+1}$ Toda chains differ by a simple factor (2.31), and the Whittaker vectors ψ_L, ψ_R are the same for both Lie algebras. Thus we use the $\mathfrak{sl}_{\ell+1}$ root data for calculations of the matrix elements $\Delta_{\omega_i,\dot{w}_0^{-1}}(v)$, $\Delta_{\omega_i,\dot{w}_0^{-1}\dot{s}_i^{-1}}(v)$, $i = 1,\dots,\ell$ in the fundamental representations of $\mathfrak{sl}_{\ell+1}$ and set in addition $\Delta_{\omega_{\ell+1},\dot{w}_0^{-1}}(v) = 1$. The $\mathfrak{sl}_{\ell+1}$ root data given by (2.28). Reduced decomposition $w_0 = s_{i_1}s_{i_2}\cdots s_{i_m}$ of the maximal length element $w_0 \in W$ corresponding to a reduced word $I_\ell = (i_1, \ldots, i_m)$ with $m = m_\ell = \ell(\ell+1)/2$, provides a total ordering of positive co-roots by $R_+^\vee = \{\gamma_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_k^\vee\}$ of $\mathfrak{sl}_{\ell+1}$. We consider a decomposition of w_0 described by the following reduced word

$$I_\ell = (1, 21, \ldots, (\ell \ldots 21)).$$

Corresponding ordering of positive co-roots is given by:

$$\gamma_1^\vee = \alpha_1^\vee, \qquad \begin{array}{l} \gamma_2^\vee = \alpha_1^\vee + \alpha_2^\vee, \\ \gamma_3^\vee = \alpha_2^\vee, \end{array} \qquad \ldots \qquad \begin{array}{l} \gamma_{m_{\ell-1}+1}^\vee = \alpha_1^\vee + \ldots + \alpha_\ell^\vee, \\ \vdots \\ \gamma_{m_\ell}^\vee = \alpha_\ell^\vee. \end{array} \tag{3.27}$$

Recursive parametrization of an open part $N_+^{(0)}$ of N_+ corresponding to a reduced word I_ℓ is as follows. Given $v^{A_\ell} \in N_+^{(0)}$ we have

$$v^{A_\ell}(y) = \mathfrak{X}_1(y)\mathfrak{X}_2(y) \cdot \ldots \cdot \mathfrak{X}_\ell(y)\,, \tag{3.28}$$

where

$$\mathfrak{X}_k(y) = X_k(y_{k,n_{k,k}}) \cdot \ldots \cdot X_2(y_{2,n_{k,2}})X_1(y_{1,n_{k,1}}),$$

and $\mathfrak{X}_1 = X_1(y_{11})$. Here we adopt the following notations. Let $|I_\ell| = m_\ell$ be the length of w_0. For the root system of type A_ℓ one has $m_\ell = \ell(\ell+1)/2$. Then for any $k \in \{1, \ldots, \ell\}$ consider a subword

$$I_k = (i_1, \ldots, i_k) \subset I_\ell = (i_1, \ldots, i_k,\, i_{k+1}, \ldots, i_\ell),$$

with $|I_k| = m_k = k(k+1)/2$. Let A_k be a corresponding root subsystem in R_+ and $v^{C_k} = \mathfrak{X}_1 \cdots \mathfrak{X}_k$ be a factorized parametrization of the corresponding subgroup. Factorization parameters for $v^{A_k}(y)$ can be naturally enumerated as $\{y_{i,n}\}$ with $1 \leq i \leq k$, $1 \leq n \leq n_{k,i}$ and

$$n_{k,i} = k + 1 - i\,, \qquad\qquad 1 \leq i \leq \ell. \tag{3.29}$$

We are interested in explicit expressions for the following matrix elements in terms of factorization parameters $\{y_{i,n}\}$

$$\Delta_i(v) := \langle \xi_{\omega_i}^- | \pi_i(v\dot{w}_0^{-1})\, | \xi_{\omega_i}^+ \rangle, \qquad \Delta_i'(v) := \langle \xi_{\omega_i}^- | \pi_i(v e_{i*} \dot{w}_0^{-1})\, | \xi_{\omega_i}^+ \rangle,$$

where $\pi_i = \pi_{\omega_i}$ is a fundamental representation with the highest weight ω_i, $\xi_{\omega_i}^+$ and $\xi_{\omega_i}^-$ are the highest and lowest weight vectors in the representation π_i such that $\langle \xi_{\omega_i}^+ |\, \xi_{\omega_i}^- \rangle = 1$. Note that for Lie algebra $\mathfrak{sl}_{\ell+1}$ according to (3.5) we have $i^* = \ell + 1 - i$. The proof of the following statement is obtained by an iterative evaluation of the matrix elements taking into account Serre relations and defining ideals of the fundamental representations and using the technique of [BZ].

Lemma 3.2

$$\Delta_i(v)^{A_\ell} = \Big(\prod_{k=1}^{i} y_{\ell+1-k,i} \Big) \Delta_i(v)^{A_\ell}, \qquad\qquad i = 1, \ldots, \ell,$$

$$\left(\frac{\Delta'_i(v)}{\Delta_i(v)}\right)^{A_\ell} = e^{x_{\ell,\ell-i+1}-x_{\ell+1,\ell-i+1}} + \frac{e^{x_{\ell,\ell-i+1}-x_{\ell,\ell-i}}}{e^{x_{\ell+1,\ell-i+1}-x_{\ell+1,\ell-i}}}\left(\frac{\Delta'_i(v)}{\Delta_i(v)}\right)^{A_{\ell-1}}, \quad i=1,\dots \ell-1, \tag{3.30}$$

$$\left(\frac{\Delta'_\ell(v)}{\Delta_\ell(v)}\right)^{A_\ell} \;=\; \frac{1}{y_{1,\ell}},$$

for $k=2,\dots,\ell-1$.

The matrix elements then can be found by resolving the recursive relations (3.30).

Lemma 3.3 *Let* v *be defined by (2.32) and (2.33). The following relations for matrix elements of* v *in terms of the variables* $y_{i,k}$ *hold:*

$$\begin{aligned}
\Big(\Delta_{\omega_i,\dot{s}_i}(v)\Big)^{A_\ell} &= \sum_{n=i}^{\ell} y_{i,n}, \qquad\qquad i=1,\dots,\ell,\\
\Big(\Delta_{\omega_i,\dot{w}_0^{-1}}(v)\Big)^{A_\ell} &= \prod_{k=i}^{\ell}\prod_{n=1}^{i} y_{k+1-n,n},\\
\left(\frac{\Delta'_k(v)}{\Delta_k(v)}\right)^{A_\ell} &= \frac{1}{y_{\ell+1-k,k}}\Big(1+\sum_{n=1}^{\ell-k}\prod_{i=1}^{n}\frac{y_{\ell+1-k-i,k+1}}{y_{\ell+1-k-i,k}}\Big),\\
\left(\frac{\Delta'_\ell(v)}{\Delta_\ell(v)}\right)^{A_\ell} &= \frac{1}{y_{1,\ell}}, \qquad k=2,\dots,\ell-1.
\end{aligned} \tag{3.31}$$

Combining these expressions with the expression (2.21) for the invariant measure on N_+ and substituting into (2.17), (2.18) and (2.19) one completes the proof of Theorem 2.1.

Now consider an integral representation for $\mathfrak{gl}_{\ell+1}$-Whittaker function in a modified factorized parametrization (2.40). We start with an analog of the recursive relations (3.30) for matrix elements in the modified factorized parametrization. To simplify the formulation of the recursive relations it turns out to be useful to consider a twisted version

$$y_{i,n} = e^{x_{\ell+1,i}-x_{\ell+1,i+1}}e^{x_{n+i,i+1}-x_{n+i-1,i}}, \tag{3.32}$$

of the modified parametrization (2.40) by taking into account the action of the part H_R of the Cartan generators (2.46). The simple change of variables (3.32) applied to (3.30) gives the following.

Lemma 3.4 *1. In the modified factorized parametrization (3.32) recursive relations (3.30) are given by*

$$\left(\frac{\Delta'_i(v)}{\Delta_i(v)}\right)^{A_\ell} = e^{x_{\ell,\ell-i+1}-x_{\ell+1,\ell-i+1}} + \frac{e^{x_{\ell,\ell-i+1}-x_{\ell,\ell-i}}}{e^{x_{\ell+1,\ell-i+1}-x_{\ell+1,\ell-i}}}\left(\frac{\Delta'_i(v)}{\Delta_i(v)}\right)^{A_{\ell-1}}. \tag{3.33}$$

2. Solution of the recursive equations read as

$$\Big(\frac{\Delta'_i(v)}{\Delta_i(v)}\Big)^{A_\ell}=\sum_{n=1}^{\ell+1-i}e^{x_{n+i-1,n}-x_{n+i,n}},\qquad i=1,\ldots,\ell, \tag{3.34}$$

$$\Big(\Delta_{\omega_i,\dot{w}_0^{-1}}(v)\Big)^{A_\ell}\;=\;\exp\Big\{\sum_{n=1}^{i}(x_{\ell+1,n}-x_{i,n})\Big\},\qquad 1\leq i\leq\ell, \tag{3.35}$$

$$\Big(\frac{\Delta_i(v)}{\Delta_{i+1}(v)}\Big)^{A_\ell}\;=\;e^{-x_{\ell+1,i+1}}\exp\Big\{\sum_{k=1}^{i+1}x_{i+1,k}-\sum_{k=1}^{i}x_{i,k}\Big\},$$

where $\Delta_{\ell+1}(v)=1$ is assumed.

Now substitute (3.34), (3.35) into (2.17), (2.18) we obtain Whittaker vectors in the parametrization (3.32). Taking $\{x_{\ell,k}=0\}$ we recover the expressions for Whittaker vectors given in Lemma 2.4. To prove the Theorem 2.3 one remains to take into account the measure $d\mu_{N_+}$ in the modified factorized parametrization. This completes the proofs of the Theorem 2.3.

3.3.2 Expressions for $\mathfrak{so}_{2\ell+1}$-matrix elements: Proofs of Theorem 2.4 and Theorem 2.6

In this subsection we introduce expressions for matrix elements relevant for the construction of integral representations of $\mathfrak{so}_{2\ell+1}$-Whittaker functions using factorized parametrization of an open part of $N_+\subset SO(2\ell+1)$. This provides a proof of the integral representations of $\mathfrak{so}_{2\ell+1}$-Whittaker functions presented in Part I.

We are using the root data given by (2.64). Reduced decomposition $w_0=s_{i_1}s_{i_2}\cdots s_{i_{m_\ell}}$ of the maximal length element $w_0\in W$ corresponding to a reduced word $I_\ell=(i_1,\ldots,i_{m_\ell})$ with $m_\ell=\ell^2$ provides a total ordering of positive coroots by $R_+^\vee=\{\gamma_k=s_{i_1}\cdots s_{i_{k-1}}\alpha_k^\vee\}$ of $\mathfrak{so}_{2\ell+1}$. We consider a decomposition of w_0 described by the following reduced word

$$I_\ell=(1,212,\ldots,(\ell\ldots212\ldots\ell)).$$

Corresponding ordering of positive co-roots is given by:

$$\gamma_1^\vee=\alpha_1^\vee,\qquad \begin{array}{l}\gamma_2^\vee=\alpha_1^\vee+\alpha_2^\vee,\\ \gamma_3^\vee=\alpha_1^\vee+2\alpha_2^\vee,\\ \gamma_4^\vee=\alpha_2^\vee,\end{array}\qquad\ldots \tag{3.36}$$

$$\begin{array}{l}\gamma^\vee_{(\ell-1)^2+1}=\alpha_1^\vee+2(\alpha_2^\vee+\ldots+\alpha_{\ell-1}^\vee)+\alpha_\ell^\vee,\\ \gamma^\vee_{(\ell-1)^2+2}=\alpha_1^\vee+2(\alpha_2^\vee+\ldots+\alpha_{\ell-2}^\vee)+\alpha_{\ell-1}^\vee+\alpha_\ell^\vee,\\ \vdots\\ \gamma^\vee_{\ell(\ell-1)}=\alpha_1^\vee+\alpha_2^\vee+\ldots+\alpha_\ell^\vee,\\ \gamma^\vee_{\ell(\ell-1)+1}=\alpha_1^\vee+2(\alpha_2^\vee+\ldots+\alpha_\ell^\vee),\\ \gamma^\vee_{\ell(\ell-1)+2}=\alpha_2^\vee+\ldots+\alpha_\ell^\vee,\\ \vdots\\ \gamma^\vee_{\ell^2}=\alpha_\ell^\vee.\end{array}$$

Recursive parametrization of an open part $N_+^{(0)}$ of N_+ corresponding to a reduced word I_ℓ is as follows. Given $v^{B_\ell} \in N_+^{(0)}$ we have

$$v^{B_\ell}(y) = \mathfrak{X}_1(y)\mathfrak{X}_2(y) \cdot \ldots \cdot \mathfrak{X}_\ell(y)\,, \tag{3.37}$$

where

$$\mathfrak{X}_k(y) = X_k(y_{k,n_{k,k}-1}) \cdot \ldots \cdot X_2(y_{2,n_{k,2}-1})X_1(y_{1,n_{k,1}})X_2(y_{2,n_{k,2}}) \cdot \ldots \cdot X_k(y_{k,n_{k,k}}),$$

and $\mathfrak{X}_1 = X_1(y_{11})$. Here we adopt the following notations. Let $|I_\ell| = m_\ell$ be the length of w_0. For the root system of type B_ℓ one has $m_\ell = \ell^2$. Then for any $k \in \{1, \ldots, \ell\}$ consider a sub-word

$$I_k = (i_1, \ldots, i_k) \subset I_\ell = (i_1, \ldots, i_k,\, i_{k+1}, \ldots, i_\ell),$$

with $|I_k| = m_k = k^2$. Let B_k be a corresponding root subsystem in R_+ and $v^{B_k} = \mathfrak{X}_1 \cdots \mathfrak{X}_k$ be a factorized parametrization of the corresponding subgroup. Factorization parameters for $v^{B_k}(y)$ can be naturally enumerated as $\{y_{i,n}\}$ with $1 \leq i \leq k$, $1 \leq n \leq n_{k,i}$ and

$$n_{k,1} = k\,, \qquad n_{k,i} = 2(k+1-i), \quad 1 < i \leq \ell. \tag{3.38}$$

We also use the notation $n_i := n_{\ell,i}$.

We are interested in explicit expressions for the following matrix elements in terms of factorization parameters $\{y_{i,n}\}$

$$\Delta_i(v) := \langle \xi^-_{\omega_i} | \pi_i(v\dot{w}_0^{-1})\, | \xi^+_{\omega_i} \rangle, \qquad \Delta'_i(v) := \langle \xi^-_{\omega_i} | \pi_i(ve_{i*}\dot{w}_0^{-1})\, | \xi^+_{\omega_i} \rangle,$$

where $\pi_i = \pi_{\omega_i}$ is a fundamental representation with the highest weight ω_i, $\xi^+_{\omega_i}$ and $\xi^-_{\omega_i}$ are the highest and lowest weight vectors in the representation π_i such that $\langle \xi^+_{\omega_i} |\, \xi^-_{\omega_i} \rangle = 1$. Note that for Lie algebra $\mathfrak{so}_{2\ell+1}$ we have $i \to i^*$ for the involution defined by (3.5). The proof of the following statement is obtained by an iterative evaluation of the matrix elements taking into account Serre relations and defining ideals of the fundamental representations and using the technique of [BZ].

Lemma 3.5 *Let $v := v^{B_\ell}$ be defined by (3.37). The following recursive equations hold:*

$$\Delta_1(v)^{B_\ell} \;=\; \Big(\prod_{k=1}^{\ell} y_{k,n_k-1} \Big) \cdot \Delta_1(v^{B_{\ell-1}}),$$

$$\Delta_i(v)^{B_\ell} \;=\; \Big(y_{1,\ell}^2 \prod_{k=2}^{i} y_{k,n_k-1} y_{k,n_k} \prod_{k=i+1}^{\ell} y_{k,n_k-1}^2 \Big) \cdot \Delta_i(v^{B_{\ell-1}}), \qquad 1 < i < \ell,$$

$$\begin{aligned}
\Big(\frac{\Delta_1'(v)}{\Delta_1(v)}\Big)^{B_\ell} &= \frac{1}{y_{1,\ell}}\Big(1+\frac{y_{2,2(\ell-1)}}{y_{2,2(\ell-1)-1}}\Big)+\frac{y_{2,2(\ell-1)}}{y_{2,2(\ell-1)-1}}\Big(\frac{\Delta_1'(v)}{\Delta_1(v)}\Big)^{B_{\ell-1}},\\
\Big(\frac{\Delta_k'(v)}{\Delta_k(v)}\Big)^{B_\ell} &= \frac{1}{y_{k,2(\ell+1-k)}}\Big(1+\frac{y_{k+1,2(\ell-k)}}{y_{k+1,2(\ell-k)-1}}\Big)+\\
&\quad+\frac{y_{k,2(\ell+1-k)-1}}{y_{k,2(\ell+1-k)}}\frac{y_{k+1,2(\ell-k)}}{y_{k+1,2(\ell-k)-1}}\Big(\frac{\Delta_k'(v)}{\Delta_k(v)}\Big)^{B_{\ell-1}},\\
\Big(\frac{\Delta_\ell'(v)}{\Delta_\ell(v)}\Big)^{B_\ell} &= \frac{1}{y_{\ell,2}},\qquad k=2,\dots,\ell-1.
\end{aligned}\tag{3.39}$$

Now matrix elements can be found by resolving recursive relations given above.

Lemma 3.6 *Let v is defined by (2.67) and (2.68.) The following expressions of matrix elements of v in terms of the variables $y_{i,k}$ hold:*

$$\Big(\Delta_{\omega_i,s_i}(v)\Big)^{B_\ell}=\sum_{n=i}^{\ell}y_{i,n},\tag{3.40}$$

$$\Big(\Delta_1(v)\Big)^{B_\ell}=\prod_{n=1}^{\ell}y_{1,n}\times\prod_{k=2}^{\ell}\prod_{n=1}^{\ell+1-k}y_{k,2n-1},\tag{3.41}$$

$$\Big(\Delta_k(v)\Big)^{B_\ell}=\prod_{n=2}^{\ell}y_{1,n}^2\times\prod_{i=k+1}^{\ell}\prod_{n=1}^{\ell+1-i}y_{i,2n-1}^2\times\prod_{i=2}^{k}\prod_{n=1}^{\ell+1-i}y_{i,2n-1}y_{i,2n},\tag{3.42}$$

$$i=1,\dots,\ell,\quad k=2,\dots,\ell.$$

$$\Big(\frac{\Delta_1'(v)}{\Delta_1(v)}\Big)^{B_\ell}=\sum_{n=1}^{\ell}\frac{1}{y_{1,n}}\Big(1+\frac{y_{2,2(n-1)}}{y_{2,2(n-1)-1}}\Big)\prod_{i=n+1}^{\ell}\frac{y_{2,2(i-1)}}{y_{2,2(i-1)-1}},\tag{3.43}$$

$$\Big(\frac{\Delta_k'(v)}{\Delta_k(v)}\Big)^{B_\ell}=\sum_{n=1}^{n_k/2}\frac{1}{y_{k,2n}}\Big(1+\frac{y_{k+1,2(n-1)}}{y_{k+1,2(n-1)-1}}\Big)\prod_{i=n+1}^{n_k/2}\frac{y_{k+1,2(i-1)}}{y_{k+1,2(-1)-1}}\frac{y_{k,2i-1}}{y_{k,2i}},\tag{3.44}$$

$$k=2,\dots,\ell,\tag{3.45}$$

where $n_1=\ell$ and $n_k=2(\ell+1-k)$.

Now consider an integral representation for $\mathfrak{so}_{2\ell+1}$-Whittaker function in a modified factorized parametrization (2.86), (2.87). We start with an analog of the recursive relations

(3.39) for the matrix elements in the modified factorized parametrization. To simplify the formulation of the recursive relations it turns out to be useful to consider a twisted version

$$y_{1,1} = e^{-x_{\ell,1}}e^{x_{11}-z_{11}}, \qquad y_{1,k} = e^{-x_{\ell,1}}\Big(e^{x_{k-1,1}-z_{k,1}} + e^{x_{k,1}-z_{k,1}}\Big),$$
$$y_{k,2r-1} = e^{x_{\ell,k-1}-x_{\ell,k}}e^{z_{k+r-1,k}-x_{k+r-2,k-1}}, \tag{3.46}$$
$$y_{k,2r} = e^{x_{\ell,k-1}-x_{\ell,k}}e^{z_{k+r-1,k}-x_{k+r-1,k-1}}$$

for $k = 2,\dots,\ell$ and $r = 1,\dots,\ell+1-k$. of the modified parametrization (2.86) by taking into account the action of the part H_R of the Cartan generators (2.96).

Lemma 3.7 *Choose an unipotent element $v \in N_+$. The following expressions for the matrix elements of v in variables $x_{k,i}, z_{k,i}$ defined by (3.46) hold:*

1.

$$\begin{aligned}
\frac{\Delta_k(v)}{\Delta_{k+1}(v)} &= \exp\Big\{-\sum_{i=1}^{k} x_{k,i} - 2z_{k,1} + 2\sum_{i=2}^{k} z_{k,i} - \sum_{i=1}^{k-1} x_{k-1,i}\Big\}\Big(e^{x_{k-1,1}} + e^{x_{k,1}}\Big)^2, \\
\frac{\Delta_1^2(v)}{\Delta_2(v)} &= e^{x_{11}-2z_{11}}, \qquad k = 2,\dots,\ell,
\end{aligned} \tag{3.47}$$

and $\Delta_{\ell+1}(v) = 1$ is assumed.

2.

$$\frac{\Delta_1'(v)}{\Delta_1(v)} = \sum_{k=1}^{\ell} e^{z_{k,1}} \tag{3.48}$$

$$\frac{\Delta_k'(v)}{\Delta_k(v)} = e^{x_{k,k}-z_{k,k}} + \sum_{n=k+1}^{\ell}\Big(e^{x_{n-1,k}-z_{n,k}} + e^{x_{n,k}-z_{n,k}}\Big), \qquad k = 2,\dots,\ell.$$

Here we let $x_{\ell,k} = 0$, $k = 1,\dots,\ell$. We assume, that the terms like $e^{z_{\ell+1,i}}$ in (3.48) are deleted and as usual we suppose that $\sum_{n=i}^{j} = 0$ whenever $i > j$.

Now substitute (3.34), (3.35) into (2.17), (2.18) we obtain Whittaker vectors in the parametrization (3.46). Taking $\{x_{\ell,k} = 0\}$ we recover the expressions for Whittaker vectors given in Lemma 2.6. To prove the Theorem 2.6 one remains to take into account the measure $d\mu_{N_+}$ in the modified factorized parametrization. This completes the proofs of the Theorem 2.6.

3.3.3 Expressions for $\mathfrak{sp}_{2\ell}$-matrix elements: Proofs of Theorem 2.8 and Theorem 2.10

In this subsection we introduce expressions for matrix elements relevant for the construction of integral representations of $\mathfrak{sp}_{2\ell}$-Whittaker functions using the factorized parametrization of an open part of $N_+ \subset Sp(2\ell)$. This provides proof of the integral representations of $\mathfrak{sp}_{2\ell}$-Whittaker functions presented in Part I.

We are using the root data for $\mathfrak{g} = \mathfrak{sp}_{2\ell}$ given by (2.125). Reduced decomposition $w_0 = s_{i_1}s_{i_2}\cdots s_{i_m}$ of the maximal length element $w_0 \in W$ corresponding to a reduced word $I_\ell = (i_1,\ldots,i_m)$ provides a total ordering of positive coroots by $R_+^\vee = \{\gamma_k = s_{i_1}\cdots s_{i_{k-1}}\alpha_k^\vee\}$. We consider a decomposition of w_0 described by the following reduced word

$$I_\ell = (1, 212, \ldots, (\ell \ldots 212 \ldots \ell)).$$

Corresponding ordering of positive coroots is given by:

$$\gamma_1^\vee = \alpha_1^\vee \qquad \begin{array}{l} \gamma_2^\vee = 2\alpha_1^\vee + \alpha_2^\vee, \\ \gamma_3^\vee = \alpha_1^\vee + \alpha_2^\vee, \\ \gamma_4^\vee = \alpha_2^\vee, \end{array} \qquad \ldots \qquad \begin{array}{l} \gamma_{(\ell-1)^2+1}^\vee = 2\alpha_1^\vee + \ldots + 2\alpha_{\ell-1}^\vee + \alpha_\ell^\vee, \\ \vdots \\ \gamma_{\ell(\ell-1)}^\vee = 2\alpha_1^\vee + \alpha_2^\vee \ldots + \alpha_\ell^\vee, \\ \gamma_{(\ell-1)^2+\ell}^\vee = \alpha_1^\vee + \ldots + \alpha_\ell^\vee, \\ \vdots \\ \gamma_{\ell^2}^\vee = \alpha_\ell^\vee. \end{array} \tag{3.49}$$

Recursive parametrization of an open part $N_+^{(0)}$ of N_+ defined by the reduced word I_ℓ is as follows. Given $v^{C_\ell} \in N_+^{(0)}$ we have

$$v^{C_\ell}(y) = \mathfrak{X}_1(y)\mathfrak{X}_2(y)\cdot\ldots\cdot\mathfrak{X}_\ell(y)\,, \tag{3.50}$$

where

$$\mathfrak{X}_k(y) = X_k(y_{k,n_{k,k}-1})\cdot\ldots\cdot X_2(y_{2,n_{k,2}-1})X_1(y_{1,n_{k,1}})X_2(y_{2,n_{k,2}})\cdot\ldots\cdot X_k(y_{k,n_{k,k}}),$$

and $\mathfrak{X}_1 = X_1(y_{11})$. Here we adopt the following notations. Let $|I_\ell| = m_\ell$ be the length of w_0. For the root system of type C_ℓ one has $m_\ell = \ell^2$. Then for any $k \in \{1,\ldots,\ell\}$ consider a subword

$$I_k = (i_1,\ldots,i_k) \subset I_\ell = (i_1,\ldots,i_k,\, i_{k+1},\ldots,i_\ell)$$

with $|I_k| = m_k = k^2$. Let C_k be a corresponding root subsystem in R_+ and $v^{C_k} = \mathfrak{X}_1\cdots\mathfrak{X}_k$ be a factorized parametrization of the corresponding subgroup. Factorization parameters for $v^{C_k}(y)$ can be naturally enumerated as $\{y_{i,n}\}$ with $1 \le i \le k$, $1 \le n \le n_{k,i}$ and

$$n_{k,1} = k\,, \qquad n_{k,i} = 2(k+1-i), \quad 1 < i \le \ell. \tag{3.51}$$

Denote also $n_i := n_{\ell,i}$.

We are interested in explicit expressions for the following matrix elements in terms of factorization parameters $\{y_{i,n}\}$

$$\Delta_i(v) := \langle\xi_{\omega_i}^-|\pi_i(v\dot{w}_0^{-1})\,|\xi_{\omega_i}^+\rangle, \qquad \Delta_i'(v) := \langle\xi_{\omega_i}^-|\pi_i(ve_{i*}\dot{w}_0^{-1})\,|\xi_{\omega_i}^+\rangle,$$

where $\pi_i = \pi_{\omega_i}$ is a fundamental representation with the highest weight ω_i, $\xi_{\omega_i}^+$ and $\xi_{\omega_i}^-$ are the highest and lowest weight vectors in the representation π_i such that $\langle\xi_{\omega_i}^+|\,\xi_{\omega_i}^-\rangle = 1$. Note that for Lie algebra $\mathfrak{sp}_{2\ell}$ the involution $i \to i^*$ defined by (3.5) is trivial $i^* = i$. The proof of the following statement is obtained by an iterative evaluation of the matrix elements taking into account Serre relations and defining ideals of the fundamental representations and using the technique of [BZ].

Lemma 3.8

$$\Delta_i(v)^{C_\ell} = \Big(y_{1,n_1}\prod_{k=2}^{i} y_{k,n_k-1}\, y_{k,n_k}\prod_{k=i+1}^{\ell} y^2_{k,n_k-1}\Big)\,\Delta_i(v)^{C_{\ell-1}}, \qquad i=1,\dots \ell,$$

$$\begin{aligned}
\Big(\frac{\Delta_1'(v)}{\Delta_1(v)}\Big)^{C_\ell} &= \frac{1}{y_{1,\ell}}\Big(1+\frac{y_{2,2(\ell-1)}}{y_{2,2(\ell-1)-1}}\Big)^2+\Big(\frac{y_{2,2(\ell-1)}}{y_{2,2(\ell-1)-1}}\Big)^2\Big(\frac{\Delta_1'(v)}{\Delta_1(v)}\Big)^{C_{\ell-1}},\\
\Big(\frac{\Delta_k'(v)}{\Delta_k(v)}\Big)^{C_\ell} &= \frac{1}{y_{k,2(\ell+1-k)}}\Big(1+\frac{y_{k+1,2(\ell-k)}}{y_{k+1,2(\ell-k)-1}}\Big)+ \\
&\quad +\frac{y_{k,2(\ell+1-k)-1}}{y_{k,2(\ell+1-k)}}\frac{y_{k+1,2(\ell-k)}}{y_{k+1,2(\ell-k)-1}}\Big(\frac{\Delta_k'(v)}{\Delta_k(v)}\Big)^{C_{\ell-1}},\\
\Big(\frac{\Delta_\ell'(v)}{\Delta_\ell(v)}\Big)^{C_\ell} &= \frac{1}{y_{\ell,2}},
\end{aligned}\tag{3.52}$$

for $k=2,\dots,\ell-1$.

Now matrix elements can be found by resolving recursive relations (3.52).

Lemma 3.9 *Let* v *be defined by (2.128) and (2.129). The following relations for matrix elements of* v *in terms of the variables* $y_{i,k}$ *hold:*

$$\begin{aligned}
\Big(\Delta_{\omega_i,\dot s_i}(v)\Big)^{C_\ell} &= \sum_{n=i}^{\ell} y_{i,n}, \quad i=1,\dots,\ell,\\
\Big(\Delta_{\omega_i,\dot w_0^{-1}}(v)\Big)^{C_\ell} &= \prod_{n=1}^{\ell} y_{1,n}\times\prod_{k=2}^{i}\prod_{n=1}^{2(\ell+1-k)} y_{k,n}\times\prod_{k=i+1}^{\ell}\prod_{n=1}^{\ell+1-k} y^2_{k,2n-1},
\end{aligned}\tag{3.53}$$

$$\begin{aligned}
\Big(\frac{\Delta_1'(v)}{\Delta_1(v)}\Big)^{C_\ell} &= \sum_{n=1}^{\ell}\frac{1}{y_{1,n}}\Big(1+\frac{y_{2,2(n-1)}}{y_{2,2(n-1)-1}}\Big)^2\prod_{i=n+1}^{\ell}\Big(\frac{y_{2,2(i-1)}}{y_{2,2(i-1)-1}}\Big)^2,\\
\Big(\frac{\Delta_k'(v)}{\Delta_k(v)}\Big)^{C_\ell} &= \sum_{n=1}^{\ell+1-k}\frac{1}{y_{k,2n}}\Big(1+\frac{y_{k+1,2(n-1)}}{y_{k+1,2(n-1)-1}}\Big)\prod_{i=n+1}^{\ell+1-k}\frac{y_{k+1,2(i-1)}}{y_{k+1,2(-1)-1}}\frac{y_{k,2i-1}}{y_{k,2i}},\\
\Big(\frac{\Delta_\ell'(v)}{\Delta_\ell(v)}\Big)^{C_\ell} &= \frac{1}{y_{\ell,2}},
\end{aligned}$$

for $k=2,\dots,\ell-1$.

Combining these expressions with the expression (2.21) for the invariant measure on N_+ and substituting into (2.17), (2.18), (2.19) one completes the proof of Theorem 2.8.

Now consider an integral representation for $\mathfrak{sp}_{2\ell}$-Whittaker function in a modified factorized parametrization (2.144)-(2.145). We start with an analog of the recursive relations (3.52) for the matrix elements in the modified factorized parametrization. To simplify the formulation of the recursive relations it turns out to be useful to consider a twisted version of the modified parametrization (2.146) by taking into account the action of the part H_R of the Cartan generators (2.159). Thus we consider the following change of the variables:

$$y_{11}=e^{-2z_{\ell,1}}e^{x_{11}+z_{11}},\qquad y_{1,k}=e^{-2z_{\ell,1}}\Big(e^{z_{k-1,1}+x_{k,1}}+e^{z_{k,1}+x_{k,1}}\Big),$$

$$y_{k,2r-1}=e^{z_{\ell,k-1}-z_{\ell,k}}e^{x_{k+r-1,k}-z_{k+r-2,k-1}},$$

$$y_{k,2r}=e^{z_{\ell,k-1}-z_{\ell,k}}e^{x_{k+r-1,k}-z_{k+r-1,k-1}},\qquad r=1,\ldots,\ell+1-k.$$

Here $k=2,\ldots,\ell$.

Lemma 3.10 *1. In a modified factorized parametrization recursive relations (3.52) are given by*

$$\Big(\frac{\Delta'_k}{\Delta_k}\Big)^{C_n}=e^{z_{n-1,k}-x_{n,k}}+e^{z_{n,k}-x_{n,k}}+\frac{e^{\langle\alpha_k\,,\underline{z}_{n-1}\rangle}}{e^{\langle\alpha_k\,,\underline{z}_n\rangle}}\Big(\frac{\Delta'_k}{\Delta_k}\Big)^{C_{n-1}},\qquad 1\le k<n<\ell,$$

with the solution

$$\Big(\frac{\Delta'_k(v)}{\Delta_k(v)}\Big)^{C_n}=e^{z_{k,k}-x_{k,k}}+\sum_{n=k+1}^{\ell}\Big(e^{z_{n-1,k}-x_{n,k}}+e^{z_{n,k}-x_{n,k}}\Big),\qquad k=1,\ldots,\ell,\tag{3.54}$$

where $\underline{z}_n=(z_{n,1},\ldots,z_{n,n})$ *and we define:* $\langle\alpha_k\,,\underline{z}_n\rangle=z_{n,k+1}-z_{n,k}$, $\langle\alpha_k\,,\underline{z}_{n-1}\rangle=z_{n-1,k+1}-z_{n-1,k}$.

2. The following expressions for $\Delta_k(v)$ *in terms of variables* $x_{k,i},z_{k,i}$ *hold:*

$$\Big(\frac{\Delta_1(v)}{\Delta_2(v)}\Big)^{C_n}=e^{-z_{\ell,1}}e^{x_{11}},\tag{3.55}$$

$$\Big(\frac{\Delta_k(v)}{\Delta_{k+1}(v)}\Big)^{C_n}=e^{-z_{\ell,k}}\Big(e^{z_{k,1}}+e^{z_{k-1,1}}\Big)\exp\Big\{-\sum_{i=1}^{k}z_{k,i}+x_{k,1}+2\sum_{i=2}^{k}x_{k,i}-\sum_{i=1}^{k-1}z_{k-1,i}\Big\},$$

where $k=2,\ldots,\ell$ *and* $\Delta_{\ell+1}=1$ *is assumed.*

Now substitute (3.54),(3.55) into (2.17), (2.18) we obtain the left/right Whittaker vectors in a twisted parametrization (3.54). Taking $\{z_{\ell,k}=0\}$ we recover the formulas for Whittaker vectors given in Lemma 2.8. To prove the Theorem 2.10 one remains to take into account the measure $d\mu_{N_+}$ in the modified factorized parametrization. This completes the proofs of the Theorem 2.10.

3.3.4 Expressions for $\mathfrak{so}_{2\ell}$-matrix elements: Proofs of Theorem 2.12 and Theorem 2.14

In this subsection we introduce expressions for matrix elements relevant for the construction of integral representations of $\mathfrak{so}_{2\ell}$-Whittaker functions using the factorized parametrization of an open part of $N_+ \subset SO(2\ell)$. This provides a proof of the integral representations of $\mathfrak{so}_{2\ell}$-Whittaker functions presented in Part I.

We are using the root data given by (2.188). Reduced decomposition $w_0 = s_{i_1}s_{i_2}\cdots s_{i_{m_\ell}}$ of the maximal length element $w_0 \in W$ corresponding to a reduced word $I_\ell = (i_1, \dots, i_{m_\ell})$ with $m_\ell = \ell(\ell-1)$ provides a total ordering of positive co-roots by $R_+^\vee = \{\gamma_k = s_{i_1}\cdots s_{i_{k-1}}\alpha_k^\vee\}$ of $\mathfrak{sp}_{2\ell}$. We consider a decomposition of w_0 described by the following reduced word

$$I_\ell = (12, 3123, \dots, (\ell \dots 3123 \dots \ell)).$$

Corresponding ordering of positive coroots is given by:

$$\begin{array}{lll} & \gamma_3^\vee = \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee, & \\ \gamma_1^\vee = \alpha_1^\vee, & \gamma_4^\vee = \alpha_2^\vee + \alpha_3^\vee, & \\ \gamma_2^\vee = \alpha_2^\vee, & \gamma_5^\vee = \alpha_2^\vee + \alpha_3^\vee, & \dots \\ & \gamma_6^\vee = \alpha_3^\vee, & \end{array} \tag{3.56}$$

$$\begin{aligned} &\gamma_{m_{\ell-1}+1}^\vee = \alpha_1^\vee + \alpha_2^\vee + 2(\alpha_3^\vee + \ldots + \alpha_{\ell-1}^\vee) + \alpha_\ell^\vee, \\ &\gamma_{m_{\ell-1}+2}^\vee = \alpha_1^\vee + \alpha_2^\vee + 2(\alpha_3^\vee + \ldots + \alpha_{\ell-2}^\vee) + \alpha_{\ell-1}^\vee + \alpha_\ell^\vee, \\ &\vdots \\ &\gamma_{m_{\ell-1}+\ell-2}^\vee = \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee + \ldots + \alpha_\ell^\vee, \\ &\gamma_{m_{\ell-1}+\ell-1}^\vee = p_{\ell-1}\alpha_1^\vee + p_\ell\alpha_2^\vee + \alpha_3^\vee + \ldots + \alpha_\ell^\vee, \\ &\gamma_{m_{\ell-1}+\ell}^\vee \ = p_\ell\alpha_1^\vee + p_{\ell+1}\alpha_2^\vee + \alpha_3^\vee + \ldots + \alpha_\ell^\vee, \\ &\gamma_{m_{\ell-1}+\ell+1}^\vee = \alpha_3^\vee + \ldots + \alpha_\ell^\vee, \\ &\vdots \\ &\gamma_{m_\ell}^\vee = \alpha_\ell^\vee. \end{aligned}$$

Recursive parametrization of an open part $N_+^{(0)}$ of N_+ corresponding to a reduced word I_ℓ is as follows. Given $v^{D_\ell} \in N_+^{(0)}$ we have

$$v^{D_\ell}(y) = \mathfrak{X}_2(y)\mathfrak{X}_2(y)\cdot \ldots \cdot \mathfrak{X}_\ell(y)\,, \tag{3.57}$$

where

$$\mathfrak{X}_k(y) = X_k(y_{k,n_{k,k}-1})\cdot\ldots\cdot X_3(y_{2,n_{k,3}-1})X_1(y_{1,n_{k,1}})X_2(y_{1,n_{k,2}})X_3(y_{2,n_{k,3}})\cdot\ldots\cdot X_k(y_{k,n_{k,k}}),$$

and $\mathfrak{X}_2 = X_1(y_{11})X_2(y_{21})$. Here we adopt the following notations. Let $|I_\ell| = m_\ell$ be the length of w_0. For the root system of type D_ℓ one has $m_\ell = \ell(\ell-1)$. Then for any $k \in \{1, \dots, \ell\}$ consider a subword

$$I_k = (i_1, \dots, i_k) \subset I_\ell = (i_1, \dots, i_k,\, i_{k+1}, \dots, i_\ell),$$

with $|I_k| = m_k = k(k-1)$. Let D_k be a corresponding root subsystem in R_+ and $v^{D_k} = \mathfrak{X}_2 \cdots \mathfrak{X}_k$ be a factorized parametrization of the corresponding subgroup. Factorization parameters for $v^{D_k}(y)$ can be naturally enumerated as $\{y_{i,n}\}$ with $1 \le i \le k$, $1 \le n \le n_{k,i}$ and

$$n_{k,1} = n_{k,2} = k-1\,, \qquad n_{k,i} = 2(k+1-i), \quad 2 < i \le \ell \tag{3.58}$$

We also denote $n_i = n_{\ell,i}$.

We are interested in explicit expressions for the following matrix elements in terms of factorization parameters $\{y_{i,n}\}$

$$\Delta_i(v) := \langle \xi^-_{\omega_i} | \pi_i(v\dot{w}_0^{-1})\, | \xi^+_{\omega_i} \rangle, \qquad \Delta'_i(v) := \langle \xi^-_{\omega_i} | \pi_i(ve_{i*}\dot{w}_0^{-1})\, | \xi^+_{\omega_i} \rangle,$$

where $\pi_i = \pi_{\omega_i}$ is a fundamental representation with the highest weight ω_i, $\xi^+_{\omega_i}$ and $\xi^-_{\omega_i}$ are the highest and lowest weight vectors in the representation π_i such that $\langle \xi^+_{\omega_i} | \, \xi^-_{\omega_i} \rangle = 1$. Note that for Lie algebra $\mathfrak{so}_{2\ell}$ the we have $i \to i^*$ for the involution defined by (3.5). The proof of the following statement is obtained by an iterative evaluation of the matrix elements taking into account Serre relations and defining ideals of the fundamental representations and using the technique of [BZ].

Lemma 3.11 *The following recursive relations hold.*

$$\begin{aligned}
\Delta_1(v)^{D_\ell} &= \Big((y_{1,n_1})^{p_{\ell-1}} (y_{2,n_2})^{p_\ell} \prod_{k=3}^{\ell} y_{k,n_k-1} \Big) \Delta_1(v)^{D_\ell - 1}, \\
\Delta_2(v)^{D_\ell} &= \Big((y_{1,n_1})^{p_\ell} (y_{2,n_2})^{p_{\ell+1}} \prod_{k=3}^{\ell} y_{k,n_k-1} \Big) \Delta_2(v)^{D_\ell - 1}, \\
\Delta_i(v)^{D_\ell} &= \Big(y_{1,n_1} y_{2,n_2} \prod_{k=3}^{i} y_{k,n_k-1} y_{k,n_k} \prod_{k=i+1}^{\ell} y^2_{k,n_k-1} \Big) \Delta_i(v)^{D_{\ell-1}}, \quad 2 < i < \ell,
\end{aligned}$$

$$\begin{aligned}
\Big(\frac{\Delta'_1}{\Delta_1} \Big)^{D_{2r}} &= \frac{1}{y_{1,2r-1}} \Big(1 + \frac{y_{3,2(2r-2)}}{y_{3,2(2r-2)-1}} \Big) + \frac{y_{2,2r-1}}{y_{1,2r-1}} \frac{y_{3,2(2r-2)}}{y_{3,2(2r-2)-1}} \Big(\frac{\Delta'_1}{\Delta_1} \Big)^{D_{2r-1}}, \\
\Big(\frac{\Delta'_1}{\Delta_1} \Big)^{D_{2r+1}} &= \frac{1}{y_{1,2r}} \Big(1 + \frac{y_{3,2(2r-1)}}{y_{3,2(2r-1)-1}} \Big) + \frac{y_{1,2r}}{y_{2,2r}} \frac{y_{3,2(2r-1)}}{y_{3,2(2r-1)-1}} \Big(\frac{\Delta'_1}{\Delta_1} \Big)^{D_{2r}},
\end{aligned} \tag{3.59}$$

$$\begin{aligned}
\Big(\frac{\Delta'_2}{\Delta_2} \Big)^{D_{2r}} &= \frac{1}{y_{1,2r-1}} \Big(1 + \frac{y_{3,4(r-1)}}{y_{3,4(r-1)-1}} \Big) + \frac{y_{1,2r-1}}{y_{1,2r-1}} \frac{y_{3,4(r-1)}}{y_{3,4(r-1)-1}} \Big(\frac{\Delta'_2}{\Delta_2} \Big)^{D_{2r-1}}, \\
\Big(\frac{\Delta'_2}{\Delta_2} \Big)^{D_{2r+1}} &= \frac{1}{y_{1,2r-1}} \Big(1 + \frac{y_{3,2(2r-1)}}{y_{3,2(2r-1)-1}} \Big) + \frac{y_{2,2r}}{y_{1,2r}} \frac{y_{3,2(2r-1)}}{y_{3,2(2r-1)-1}} \Big(\frac{\Delta'_2}{\Delta_2} \Big)^{D_{2r}}.
\end{aligned}$$

The matrix elements can be evaluated by resolving recursion equations and the results of calculation are presented in the following lemma.

Lemma 3.12 *Let v be defined by (3.57). The following expressions of matrix elements of v in terms of the variables $y_{i,k}$ hold:*

$$\Delta_1(v)=\prod_{n=1}^{\ell/2} y_{1,2n-1}\prod_{n=1}^{\frac{\ell-1}{2}} y_{2,2n}\prod_{n=3}^{\ell}\prod_{i=3}^{n} y_{i,2(n+1-i)-1},$$

$$\Delta_2(v)=\prod_{n=1}^{\ell/2} y_{2,2n-1}\prod_{n=1}^{\frac{\ell-1}{2}} y_{1,2n}\prod_{n=3}^{\ell}\prod_{i=3}^{n} y_{i,2(n+1-i)-1}, \tag{3.60}$$

$$\Delta_k(v)=\prod_{i=1}^{k}\prod_{n=1}^{n_i} y_{i,n}\prod_{i=k+1}^{\ell}\prod_{n=1}^{n_i/2} y^2_{i,2n-1},$$

where $n_1=n_2=\ell-1$ and $n_k=2(\ell+1-k)$, $2<k\leq\ell$.

$$\Big(\frac{\Delta_1'}{\Delta_1}+\frac{\Delta_2'}{\Delta_2}\Big)^{D_\ell}=\sum_{n=1}^{\ell-1}\Big\{\frac{1}{y_{1,\ell-1}}\prod_{k=1}^{n-1}\Big(\frac{y_{1,\ell-k}}{y_{1,\ell-k-1}}\Big)^{p_{k-1}}\Big(\frac{y_{2,\ell-k}}{y_{2,\ell-k-1}}\Big)^{p_k}+ \tag{3.61}$$

$$\frac{1}{y_{2,\ell-1}}\prod_{k=1}^{n-1}\Big(\frac{y_{1,\ell-k}}{y_{1,\ell-k-1}}\Big)^{p_k}\Big(\frac{y_{2,\ell-k}}{y_{2,\ell-k-1}}\Big)^{p_{k+1}}\Big\}\Big(1+\frac{y_{3,2(\ell-n-1)}}{y_{3,2(\ell-n-1)-1}}\Big)\prod_{k=1}^{n-1}\frac{y_{3,2(\ell-k-1)}}{y_{3,2(\ell-k-1)-1}}$$

where $p_s=(1+(-1)^s)/2$.

$$\Big(\frac{\Delta_k'(v)}{\Delta_k(v)}\Big)^{D_\ell}=\frac{1}{y_{k,2(\ell+1-k)}}, \tag{3.62}$$

for $k=3,\dots,\ell$.

Now consider an integral representation for $\mathfrak{so}_{2\ell}$-Whittaker function in a modified factorized parametrization (2.216). We start with an analog of the recursive relations (3.59) for the matrix elements in the modified factorized parametrization. To simplify the formulation of the recursive relations it turns out to be useful to consider a twisted version of the modified parametrization (2.216) by taking into account the action of the part H_R of the Cartan generators (2.230). Thus we consider the following change of the variables:

$$\begin{aligned}
y_{1,n} &= e^{x_{\ell,1}-x_{\ell,2}}\Big(e^{z_{n,1}-x_{n,1}}+e^{z_{n,1}-x_{n+1,1}}\Big), \quad n=1,\dots\ell-1,\\
y_{2,n} &= e^{-x_{\ell,1}-x_{\ell,2}}\Big(e^{z_{n,1}+x_{n,1}}+e^{z_{n,1}+x_{n+1,1}}\Big), \quad n=1,\dots\ell-1,\\
y_{k,2r-1} &= e^{x_{\ell,k-1}-x_{\ell,k}}e^{z_{k+r-2,k-1}-x_{k+r-2,k-1}},\\
y_{k,2r} &= e^{x_{\ell,k-1}-x_{\ell,k}}e^{z_{k+r-2,k-1}-x_{k+r-1,k-1}},
\end{aligned} \tag{3.63}$$

for $k=3,\dots,\ell$ and $r=1,\dots,\ell+1-k$.

Lemma 3.13 *The following recursive relations in the variables defined by (3.63) hold.*

1.

$$\Big(\frac{\Delta_1'}{\Delta_1}\Big)^{D_n}+\Big(\frac{\Delta_2'}{\Delta_2}\Big)^{D_n}=e^{x_{n-1,2}-z_{n-1,1}}+e^{x_{n,2}-z_{n-1,1}}+ \tag{3.64}$$

$$\frac{e^{x_{n-1}(\alpha_1)}}{e^{x_n(\alpha_1)}}\Big(\frac{\Delta_1'}{\Delta_1}\Big)^{D_{n-1}}+\frac{e^{x_{n-1}(\alpha_2)}}{e^{x_n(\alpha_2)}}\Big(\frac{\Delta_2'}{\Delta_2}\Big)^{D_{n-1}}\qquad n=2r-1$$

2.

$$\Big(\frac{\Delta_1'}{\Delta_1}\Big)^{D_n}+\Big(\frac{\Delta_2'}{\Delta_2}\Big)^{D_n}=e^{x_{n-1,2}-z_{n-1,1}}+e^{x_{n,2}-z_{n-1,1}}+ \tag{3.65}$$

$$\frac{e^{x_{n-1}(\alpha_2)}}{e^{x_n(\alpha_2)}}\Big(\frac{\Delta_1'}{\Delta_1}\Big)^{D_{n-1}}+\frac{e^{x_{n-1}(\alpha_1)}}{e^{x_n(\alpha_1)}}\Big(\frac{\Delta_2'}{\Delta_2}\Big)^{D_{n-1}}\qquad n=2r.$$

Resolving recursive equations one can easily obtains the following result.

Lemma 3.14 *Given an unipotent element $v\in N_+$, the following expressions for the matrix elements of v in a modified parametrization hold:*

$$\begin{aligned}\frac{\Delta_1'}{\Delta_1}+\frac{\Delta_2'}{\Delta_2}&=\sum_{k=1}^{n-1}(e^{z_{k,k}-x_{k,k-1}}+e^{z_{k,k}-x_{k+1,k}}),\\ \frac{\Delta_k'}{\Delta_k}&=e^{z_{n-1,n+1-k}-x_{n,n+1-k}},\qquad k=3,\dots,n,\end{aligned}$$

$$\begin{aligned}\frac{\Delta_2}{\Delta_1}&=e^{-x_{\ell,1}}e^{x_{11}},\\ \frac{\Delta_1\Delta_2}{\Delta_3}&=e^{-x_{\ell,2}}\exp\Big\{-(x_{21}+x_{22})+2z_{11}-x_{11}\Big\}\Big(e^{x_{11}}+e^{x_{21}}\Big)^2,\\ \frac{\Delta_k}{\Delta_{k+1}}&=e^{-x_{\ell,k}}\exp\Big\{-\sum_{i=1}^{k}x_{k,i}+\\ &+2\sum_{i=1}^{k-1}z_{k-1,i}-\sum_{i=1}^{k-1}x_{k-1,i}\Big\}\Big(e^{x_{k-1,1}}+e^{x_{k,1}}\Big)^2,\qquad k=3,\dots,n,\end{aligned} \tag{3.66}$$

and $\Delta_{n+1}=1$ is assumed.

Now substitute (3.66) into (2.17), (2.18) we obtain Whittaker vectors in a parametrization (3.63. Taking $\{x_{\ell,k}=0\}$ we recover the expressions for Whittaker vectors given in Lemma 2.10. To prove the Theorem 2.14 one remains to take into account the measure $d\mu_{N_+}$ in the modified factorized parametrization. This completes the proofs of the Theorem 2.14.

3.4 Realization of $\mathcal{U}(\mathfrak{g})$ by differential operators

In this part we prove formulae for realization of classical Lie algebra generators by differential operators acting in the space of (twisted) functions on N_+ supplied with the factorize parametrization. The analogous formulae for realization of Lie algebra generators in the modified factorized parametrization (Gauss-Givental representation) can be straightforwardly obtained by a simple change of the variables discussed in Part I and will not be considered in this section.

Let us outline the general strategy for the derivation of the realization of Lie algebra by differential operators used below. Let V_μ be a space of equivariant functions on $B_-\backslash G$

$$f(bg) = \chi_\mu(b)\ f(g)\,, \qquad\qquad b \in B_-, \tag{3.67}$$

where χ_μ is a character of the Borel subgroup $B_- \subset G$. Principle series representation of $\mathcal{U}(\mathfrak{g})$ in V_μ is defined as

$$(Xf)(v) = \frac{\partial}{\partial\varepsilon} f(ve^{\varepsilon X})|_{\varepsilon\to 0}\,, \qquad\qquad X \in \mathfrak{g}. \tag{3.68}$$

Let $I = (i_1, \ldots, i_m)$ be a reduced word corresponding to the reduced decomposition $w_0 = s_{i_1}\cdots s_{i_m}$ of the longest Weyl group element w_0. For classical Lie algebras one can chose I having recursive structure with respect to the rank ℓ of the Lie algebra. Consider corresponding recursive factorized parametrization of unipotent elements of a classical Lie group G, $\mathfrak{g} = \mathrm{Lie}(G)$:

$$v^{(\ell)} = \mathfrak{X}_1^2 \cdots \mathfrak{X}_{\ell-1}^\ell = v^{(\ell-1)} \cdot \mathfrak{X}_{\ell-1}^\ell. \tag{3.69}$$

We will derive explicit formulae defining representations of $\mathcal{U}(\mathfrak{g})$ in V_μ in two steps. At the first step we use recursive structure (3.69) to construct recursive relations between classical Lie algebra generators for Lie algebras of adjacent ranks. At the second step we resolve recursion relations to get explicit formulae for Lie generators of all classical Lie algebra.

We start with a list of relevant relations between one-parameter subgroups in G (see e.g. [Lu], [BZ]). Let e_i, h_i, f_i be a Chevalley basis of $\mathfrak{g}$, and let $A = \|a_{ij}\|$ be the Cartan matrix. Let us introduce one-parameter subgroups:

$$X_i(y) = e^{ye_i}\,, \qquad \alpha_i^\vee(y) = e^{yh_i}\,, \qquad Y_i(y) = e^{yf_i}. \tag{3.70}$$

Then the following relations hold:

$$X_i(y)\alpha_j^\vee(1+\varepsilon) \;=\; \alpha_j^\vee(1+\varepsilon)X_i(y - a_{ji}\varepsilon y) \quad mod(\varepsilon^2), \tag{3.71}$$

$$X_i(y)Y_i(\varepsilon) \;=\; Y_i(\varepsilon)\alpha_i^\vee(1+\varepsilon y)X_i(y-\varepsilon y^2) \quad mod(\varepsilon^2). \tag{3.72}$$

For $a_{ij} = a_{ji} = -1$ we have

$$X_i(y_1)X_j(y_2)X_i(\varepsilon) \;=\; X_j\Big(\varepsilon\frac{y_2}{y_1}\Big)\, X_i\Big(y_1+\varepsilon\Big)\, X_j\Big(y_2 - \varepsilon\frac{y_2}{y_1}\Big) \quad mod(\varepsilon^2). \tag{3.73}$$

For $a_{ij} = -2$ and $a_{ji} = -1$ we have

$$X_j(y_1)X_i(y_2)X_j(y_3)X_i(\varepsilon) = \tag{3.74}$$

$$= X_i\Big(\varepsilon\frac{y_3}{y_1}\Big)\, X_j\Big(y_1 + 2\varepsilon\frac{y_3}{y_2}\Big)\, X_i\Big(y_2 + \varepsilon - \varepsilon\frac{y_3}{y_1}\Big)\, X_j\Big(y_3 - 2\varepsilon\frac{y_3}{y_2}\Big) \quad mod(\varepsilon^2).$$

For $a_{ji} = -2$ and $a_{ij} = -1$ we have

$$X_j(y_1)X_i(y_2)X_j(y_3)X_i(\varepsilon) = \tag{3.75}$$

$$= X_i\Big(\varepsilon\frac{y_3^2}{y_1^2}\Big)\, X_j\Big(y_1 + \varepsilon\frac{y_1y_3 + y_3^2}{y_1y_2}\Big)\, X_i\Big(y_2 + \varepsilon\frac{y_1^2 - y_3^2}{y_1^2}\Big)\, X_j\Big(y_3 - \varepsilon\frac{y_1y_3 + y_3^2}{y_1y_2}\Big)\; mod(\varepsilon^2).$$

The derivation of the recursive relation for the generators of Lie algebra is as follows. Consider the right action of one-parameter subgroups (3.70) on the recursive factorized representation (3.69) of an element $v^\ell \subset N_+$. One uses the relations (3.71)-(3.75) to move the generators one step to the left. For example, in the case of the one-parameter subgroup generated by e_i we have:

$$v^{(\ell)}\, X_i(\varepsilon) = v^{(\ell-1)}\, \mathfrak{X}^\ell_{\ell-1}(y)\, X_i(\varepsilon) = v^{(\ell-1)}\; X_i(c_i(y)\varepsilon)\; \mathfrak{X}^\ell_{\ell-1}(y'(y)) \quad mod(\varepsilon^2). \tag{3.76}$$

This leads to recursive relations expressing generators of rank ℓ classical Lie algebra in terms of the generators of rank $(\ell-1)$ classical Lie algebra and the differential operators over $y_{i,n}$ parametrizing $\mathfrak{X}^\ell_{\ell-1}$. At the final step of the reduction we use (3.67). In the following subsections we provide recursive relations and resolved formulae for generators of all classical Lie algebras without further comments.

3.4.1 Generators of $\mathfrak{gl}_{\ell+1}$: Proof of Proposition 2.2

Let $E^{(\ell+1)}_{i,i+1}, E^{(\ell+1)}_{i,i}, E^{(\ell+1)}_{i+1,i}$ be Chevalley generators of $\mathfrak{gl}_{\ell+1}$. Below we present recursive relations and resolved expressions for these generators.

Recursive relations are given by:

$$E^{(\ell+1)}_{i,i+1} = \left(\frac{\partial}{\partial y_{i,\ell+1-i}} + \frac{y_{i-1,\ell+2-i}}{y_{i,\ell+1-i}}\left(E^{(\ell)}_{i-1,i} - \frac{\partial}{\partial y_{i-1,\ell+2-i}}\right)\right),$$

$$E^{(\ell+1)}_{i,i} = \left(\mu^{(\ell+1)}_i - \mu^{(\ell)}_i + E^{(\ell)}_{i,i} + y_{i-1,\ell+2-i}\frac{\partial}{\partial y_{i-1,\ell+2-i}} - y_{i,\ell+1-i}\frac{\partial}{\partial y_{i,\ell+1-i}}\right), \quad i \neq \ell+1,$$

$$E^{(\ell+1)}_{\ell+1,\ell+1} = (\mu^{(\ell+1)}_{\ell+1} + y_{\ell,1}\frac{\partial}{\partial y_{\ell,1}}), \tag{3.77}$$

$$E_{i+1,i}^{(\ell+1)} = \Big(E_{i+1,i}^{(\ell)} + y_{i,\ell+1-i}\Big(E_{i,i}^{(\ell)} - E_{i+1,i+1}^{(\ell)}\Big) -$$
$$-y_{i,\ell+1-i}\left(y_{i,\ell+1-i}\frac{\partial}{\partial y_{i,\ell+1-i}} + y_{i+1,\ell-i}\frac{\partial}{\partial y_{i+1,\ell-i}}\right)\Big).$$

Resolving the recursion we obtain:

$$E_{i,i+1}^{(\ell+1)} = \sum_{k=0}^{i-1}\prod_{s=0}^{k}\frac{y_{i-s,\ell+2-i}}{y_{i+1-s,\ell+1-i}}\frac{\partial}{\partial y_{i-k,\ell+1-i}} - \prod_{s=0}^{k}\frac{y_{i-(s+1),\ell+2-i}}{y_{i-s,\ell+1-i}}\frac{\partial}{\partial y_{i-(k+1),\ell+2-i}},$$

$$E_{i,i}^{(\ell+1)} = \mu_i^{(\ell+1)} - \sum_{l=1}^{\ell+1-i} y_{i,l}\frac{\partial}{\partial y_{i,l}} + \sum_{l=1}^{\ell+2-i} y_{i-1,l}\frac{\partial}{\partial y_{i-1,l}}, \tag{3.78}$$

$$E_{i+1,i}^{(\ell+1)} = \sum_{k=1}^{\ell}\Bigg[(\mu_{i+1}^{(\ell+1)} - \mu_i^{(\ell+1)})y_{i,k+1-i} - y_{i,k+1-i}\Big(y_{i,k+1-i}\frac{\partial}{\partial y_{i,k+1-i}} - y_{i+1,k-i}\frac{\partial}{\partial y_{i+1,k-i}}\Big)+$$
$$+ y_{i,k+1-i}\sum_{s=1}^{k-1}\Big(y_{i-1,s+2-i}\frac{\partial}{\partial y_{i-1,s+2-i}} - 2y_{i,s+1-i}\frac{\partial}{\partial y_{i,s+1-i}} + y_{i+1,s-i}\frac{\partial}{\partial y_{i+1,s-i}}\Big)\Bigg].$$

This completes the proof of Proposition 2.2.

3.4.2 Generators of $\mathfrak{so}_{2\ell+1}$: Proof of Proposition 2.4

Let $e_i^{(\ell)}, h_i^{(\ell)}, f_i^{(\ell)}$ be Chevalley generators of $\mathfrak{so}_{2\ell+1}$. Below we present recursive relations and resolved expressions for these generators.

Recursive relations are given by:

$$e_1^{(\ell)} = \frac{\partial}{\partial y_{1,\ell}} + \frac{y_{2,2(\ell-1)}}{y_{2,2(\ell-1)-1}}\left[e_1^{(\ell-1)} - \frac{\partial}{\partial y_{1,\ell}}\right] + 2\,\frac{y_{2,2(\ell-1)}}{y_{1,\ell}}\left(\frac{\partial}{\partial y_{2,2(\ell-1)-1}} - \frac{\partial}{\partial y_{2,2(\ell-1)}}\right),$$

$$e_k^{(\ell)} = \frac{\partial}{\partial y_{k,2(\ell+1-k)}} + \frac{y_{k,2(\ell+1-k)-1}}{y_{k,2(\ell+1-k)}}\frac{y_{k+1,2(\ell-k)}}{y_{k+1,2(\ell-k)-1}}\left[e_k^{(\ell-1)} - \frac{\partial}{\partial y_{k,2(\ell+1-k)-1}}\right] +$$
$$+\frac{y_{k+1,2(\ell-k)}}{y_{k,2(\ell+1-k)}}\left(\frac{\partial}{\partial y_{k+1,2(\ell-k)-1}} - \frac{\partial}{\partial y_{k+1,2(\ell-k)}}\right), \qquad 1 < k < \ell,$$

$$f_1^{(\ell)} = f_1^{(\ell-1)} - y_{1,\ell}h_1^{(\ell-1)} + 2y_{1,\ell}y_{2,2(\ell-1)-1}\frac{\partial}{\partial y_{2,2(\ell-1)-1}} - y_{1,\ell}^2\frac{\partial}{\partial y_{1,\ell}}, \tag{3.79}$$

$$f_2^{(\ell)} = f_2^{(\ell-1)} - \Big(y_{2,2(\ell-1)-1} + y_{2,2(\ell-1)}\Big)h_2^{(\ell-1)} +$$

$$+\Big(y_{2,2(\ell-1)-1}+y_{2,2(\ell-1)}\Big)y_{3,2(\ell-2)-1}\frac{\partial}{\partial y_{3,2(\ell-2)-1}}+2y_{2,2(\ell-1)}y_{1,\ell}\frac{\partial}{\partial y_{1,\ell}}-$$

$$-\left(y^2_{2,2(\ell-1)-1}\frac{\partial}{\partial y_{2,2(\ell-1)-1}}+2y_{2,2(\ell-1)}y_{2,2(\ell-1)-1}\frac{\partial}{\partial y_{2,2(\ell-1)-1}}+y^2_{2,2(\ell-1)}\frac{\partial}{\partial y_{2,2(\ell-1)}}\right),$$

$$f_k^{(\ell)} \;=\; f_k^{(\ell-1)} \;-\; \Big(y_{k,2(\ell+1-k)-1}+y_{k,2(\ell+1-k)}\Big)h_k^{(\ell-1)}+$$

$$+\Big(y_{k,2(\ell+1-k)-1}+y_{k,2(\ell+1-k)}\Big)y_{k+1,2(\ell-k)-1}\frac{\partial}{\partial y_{k+1,2(\ell-k)-1}}+$$

$$+y_{k,2(\ell+1-k)}\left(y_{k-1,2(\ell+2-k)-1}\frac{\partial}{\partial y_{k-1,2(\ell+2-k)-1}}+y_{k-1,2(\ell+2-k)}\frac{\partial}{\partial y_{k-1,2(\ell+2-k)}}\right)-$$

$$-\left(y^2_{k,2(\ell+1-k)-1}\frac{\partial}{\partial y_{k,2(\ell+1-k)-1}}+2y_{k,2(\ell+1-k)-1}y_{k,2(\ell+1-k)}\frac{\partial}{\partial y_{k,2(\ell+1-k)-1}}+\right.$$

$$\left.+y^2_{k,2(\ell+1-k)}\frac{\partial}{\partial y_{k,2(\ell+1-k)}}\right).$$

We have for h_i:

$$h_k^{(\ell)}=\langle\mu^{(\ell)}\,,\alpha_k^\vee\rangle\;+\;\sum_{i=1}^{\ell}a_{k,i}\sum_{j=1}^{n_i}y_{i,j}\frac{\partial}{\partial y_{i,j}}, \tag{3.80}$$

where $n_1=\ell$, $n_k=2(\ell+1-k)$ for $1<k\le\ell$.

Resolving the recursion one obtains:

$$e_1^{(\ell)}=\frac{\partial}{\partial y_{1,\ell}}\;+\sum_{n=1}^{\ell-1}\left(\frac{\partial}{\partial y_{1,n}}-\frac{\partial}{\partial y_{1,n+1}}\right)\prod_{i=n}^{\ell-1}\frac{y_{2,2i}}{y_{2,2i-1}}\;+$$

$$+2\left(\frac{\partial}{\partial y_{2,2n-1}}-\frac{\partial}{\partial y_{2,2n}}\right)\frac{y_{2,2(n-1)}}{y_{1,n}}\prod_{i=n+1}^{\ell-1}\frac{y_{2,2i}}{y_{2,2i-1}},$$

$$e_k^{(\ell)}=\frac{\partial}{\partial y_{k,2(\ell+1-k)}}\;+\sum_{n=1}^{n-k}\left(\frac{\partial}{\partial y_{k,2n}}-\frac{\partial}{\partial y_{k,2n+1}}\right)\prod_{i=n}^{\ell-k}\frac{y_{k+1,2i}}{y_{k+1,2i-1}}\frac{y_{k,2(i+1)-1}}{y_{k,2(i+1)}}\;+ \tag{3.81}$$

$$+\left(\frac{\partial}{\partial y_{k+1,2n-1}}-\frac{\partial}{\partial y_{k+1,2n}}\right)\frac{y_{k+1,2n}}{y_{k,2(n+1)}}\prod_{i=n+1}^{\ell-k}\frac{y_{k+1,2i}}{y_{k+1,2i-1}}\frac{y_{k,2(i+1)-1}}{y_{k,2(i+1)}},$$

$$e_\ell^{(\ell)}=\frac{\partial}{\partial y_{\ell,2}}, \tag{3.82}$$

$$f_1^{(\ell)} = \sum_{n=1}^{\ell} y_{1,n}\Big(\langle\mu,\alpha_1^\vee\rangle + \sum_{j=1}^{2(n-1)-1} 2y_{2,j}\frac{\partial}{\partial y_{2,j}} - 2\sum_{j=1}^{n-1} y_{1,j}\frac{\partial}{\partial y_{1,j}} - y_{1,n}\frac{\partial}{\partial y_{1,n}}\Big), \tag{3.83}$$

$$f_2^{(\ell)} = \sum_{n=1}^{2(\ell-1)} y_{2,n}\Big(\langle\mu,\alpha_2^\vee\rangle + 2\sum_{j=1}^{[n/2]+1} y_{1,j}\frac{\partial}{\partial y_{1,j}} - 2\sum_{j=1}^{n-1} y_{2,j}\frac{\partial}{\partial y_{2,j}} + $$

$$+ \sum_{j=1}^{2[(n+1)/2]-3} y_{3,j}\frac{\partial}{\partial y_{3,j}} - y_{2,n}\frac{\partial}{\partial y_{2,n}}\Big),$$

$$f_k^{(\ell)} = \sum_{n=1}^{2(\ell+1-k)} y_{k,n}\Big(\langle\mu,\alpha_k^\vee\rangle + \sum_{j=1}^{2[n/2]+1} y_{k-1,j}\frac{\partial}{\partial y_{k-1,j}} - 2\sum_{j=1}^{n-1} y_{k,j}\frac{\partial}{\partial y_{k,j}} +$$

$$+ \sum_{j=1}^{2[(n+1)/2]-3} y_{k+1,j}\frac{\partial}{\partial y_{k+1,j}} - y_{k,n}\frac{\partial}{\partial y_{k,n}}\Big), \qquad 2<k<\ell,$$

$$f_\ell^{(\ell)} = (y_{\ell,1}+y_{\ell,2})\left(\langle\mu,\alpha_\ell^\vee\rangle + y_{\ell-1,1}\frac{\partial}{\partial y_{\ell-1,1}} + y_{\ell-1,2}\frac{\partial}{\partial y_{\ell-1,2}}\right) +$$

$$+y_{\ell,2}\left(y_{\ell-1,3}\frac{\partial}{\partial y_{\ell-1,3}} + y_{\ell-1,4}\frac{\partial}{\partial y_{\ell-1,4}}\right) - \left(y_{\ell,1}^2\frac{\partial}{\partial y_{\ell,1}} + 2y_{\ell,1}y_{\ell,2}\frac{\partial}{\partial y_{\ell,1}} + y_{\ell,2}^2\frac{\partial}{\partial y_{\ell,2}}\right).$$

This completes the proof of Proposition 2.4.

3.4.3 Generators of $\mathfrak{sp}_{2\ell}$: Proof of Proposition 2.8

Let $e_i^{(\ell)}, h_i^{(\ell)}, f_i^{(\ell)}$ be Chevalley generators of $\mathfrak{sp}_{2\ell}$. Below we present recursive relations and resolved expressions for these generators.

Recursive relations are given by:

$$e_1^{(\ell)} = \frac{\partial}{\partial y_{1,\ell}} + \left(\frac{y_{2,2(\ell-1)}}{y_{2,2(\ell-1)-1}}\right)^2 \left[e_1^{(\ell-1)} - \frac{\partial}{\partial y_{1,\ell}}\right] +$$

$$+\frac{y_{2,2(\ell-1)-1}y_{2,2(\ell-1)} + y_{2,2(\ell-1)}^2}{y_{1,\ell}y_{2,2(\ell-1)}}\left(\frac{\partial}{\partial y_{2,2(\ell-1)-1}} - \frac{\partial}{\partial y_{2,2(\ell-1)}}\right),$$

$$e_k^{(\ell)} = \frac{\partial}{\partial y_{k,2(\ell+1-k)}} + \frac{y_{k,2(\ell+1-k)-1}}{y_{k,2(\ell+1-k)}}\frac{y_{k+1,2(\ell-k)}}{y_{k+1,2(\ell-k)-1}}\left[e_k^{(\ell-1)} - \frac{\partial}{\partial y_{k,2(\ell+1-k)-1}}\right] +$$

$$+\frac{y_{k+1,2(\ell-k)}}{y_{k,2(\ell+1-k)}}\left(\frac{\partial}{\partial y_{k+1,2(\ell-k)-1}} - \frac{\partial}{\partial y_{k+1,2(\ell-k)}}\right), \qquad 1<k<\ell,$$

$$e_\ell^{(\ell)} = \frac{\partial}{\partial y_{\ell,2}}, \tag{3.84}$$

$$f_1^{(\ell)} \;=\; f_1^{(\ell-1)} \;-\; y_{1,\ell} h_1^{(\ell-1)} \;+\; y_{1,\ell} y_{2,2(\ell-1)-1} \frac{\partial}{\partial y_{2,2(\ell-1)-1}} - y_{1,\ell}^2 \frac{\partial}{\partial y_{1,\ell}},$$

$$f_2^{(\ell)} \;=\; f_2^{(\ell-1)} \;-\; \Big(y_{2,2(\ell-1)-1} + y_{2,2(\ell-1)}\Big) h_2^{(\ell-1)} + 2 y_{2,2(\ell-1)-1} y_{1,\ell} \frac{\partial}{\partial y_{1,\ell}} +$$

$$+\Big(y_{2,2(\ell-1)-1} + y_{2,2(\ell-1)}\Big) y_{3,2(\ell-2)-1} \frac{\partial}{\partial y_{3,2(\ell-2)-1}} - \left(y_{2,2(\ell-1)-1}^2 \frac{\partial}{\partial y_{2,2(\ell-1)-1}} + \right.$$

$$\left. + \, 2 y_{2,2(\ell-1)-1} y_{2,2(\ell-1)-1} \frac{\partial}{\partial y_{2,2(\ell-1)-1}} + y_{2,2(\ell-1)}^2 \frac{\partial}{\partial y_{2,2(\ell-1)}} \right),$$

$$f_k^{(\ell)} \;=\; f_k^{(\ell-1)} \;-\; \Big(y_{k,2(\ell+1-k)-1} + y_{k,2(\ell+1-k)}\Big) h_k^{(\ell-1)} \;+$$

$$+\Big(y_{k,2(\ell+1-k)-1} + y_{k,2(\ell+1-k)}\Big) y_{k+1,2(\ell-k)-1} \frac{\partial}{\partial y_{k+1,2(\ell-k)-1}} +$$

$$+ y_{k,2(\ell+1-k)} \left(y_{k-1,2(\ell+2-k)-1} \frac{\partial}{\partial y_{k-1,2(\ell+2-k)-1}} + y_{k-1,2(\ell+2-k)} \frac{\partial}{\partial y_{k-1,2(\ell+2-k)}} \right) -$$

$$- \left(y_{k,2(\ell+1-k)-1}^2 \frac{\partial}{\partial y_{k,2(\ell+1-k)-1}} + 2 y_{k,2(\ell+1-k)-1} y_{k,2(\ell+1-k)} \frac{\partial}{\partial y_{k,2(\ell+1-k)-1}} + \right.$$

$$\left. + \, y_{k,2(\ell+1-k)}^2 \frac{\partial}{\partial y_{k,2(\ell+1-k)}} \right), \qquad 2 < k < \ell,$$

$$f_\ell^{(\ell)} \;=\; (y_{\ell,1} + y_{\ell,2}) \left(-\langle \mu, \alpha_\ell^\vee \rangle + y_{\ell-1,1} \frac{\partial}{\partial y_{\ell-1,1}} + y_{\ell-1,2} \frac{\partial}{\partial y_{\ell-1,2}} \right) +$$

$$+ y_{\ell,2} \left(y_{\ell-1,3} \frac{\partial}{\partial y_{\ell-1,3}} + y_{\ell-1,4} \frac{\partial}{\partial y_{\ell-1,4}} \right) - \left(y_{\ell,1}^2 \frac{\partial}{\partial y_{\ell,1}} + 2 y_{\ell,1} y_{\ell,2} \frac{\partial}{\partial y_{\ell,1}} + y_{\ell,2}^2 \frac{\partial}{\partial y_{\ell,2}} \right).$$

For the Cartan generators h_i we have:

$$h_k^{(\ell)} = \langle \mu \,, \alpha_k^\vee \rangle \;+\; \sum_{i=1}^{\ell} a_{k,i} \sum_{j=1}^{n_i} y_{i,j} \frac{\partial}{\partial y_{i,j}}, \tag{3.85}$$

where $n_1 = \ell$ and $n_k = 2(\ell + 1 - k)$ for $1 < k \le \ell$.

Resolving the recursion we obtain:

$$e_1 = \sum_{n=1}^{\ell} \left(\frac{\partial}{\partial y_{1,n}} - \frac{\partial}{\partial y_{1,n+1}} \right) \prod_{j=n}^{\ell-1} \Big(\frac{y_{2,2j}}{y_{2,2j-1}} \Big)^2 \;+$$

$$+ \sum_{n=1}^{\ell-1} \left(\frac{\partial}{\partial y_{2,2n-1}} - \frac{\partial}{\partial y_{2,2n}} \right) \frac{y_{2,2n}}{y_{1,n}} \Big(1 + \frac{y_{2,2n}}{y_{2,2n-1}} \Big) \prod_{j=n+1}^{\ell-1} \Big(\frac{y_{2,2j}}{y_{2,2j-1}} \Big)^2,$$

$$e_k = \sum_{n=1}^{\ell+1-k} \left(\frac{\partial}{\partial y_{k,2n}} - \frac{\partial}{\partial y_{k,2n+1}}\right) \prod_{i=n}^{\ell-k} \frac{y_{k+1,2j}}{y_{k+1,2j-1}} \frac{y_{k,2(j+1)-1}}{y_{k,2(j+1)}} +$$

$$+ \sum_{n=1}^{\ell-k} \left(\frac{\partial}{\partial y_{k+1,2n-1}} - \frac{\partial}{\partial y_{k+1,2n}}\right) \frac{y_{k+1,2n}}{y_{k,2(n-1)}} \prod_{i=n}^{\ell-k} \frac{y_{k+1,2j}}{y_{k+1,2j-1}} \frac{y_{k,2(j+1)-1}}{y_{k,2(j+1)}}, \qquad 1 < k < \ell,$$

$$f_1 = \sum_{n=1}^{\ell} y_{1,n}\Big(- \langle \mu, \alpha_1^\vee \rangle + \sum_{j=1}^{2(n-1)-1} y_{2,j} \frac{\partial}{\partial y_{2,j}} - 2 \sum_{j=1}^{n-1} y_{1,j} \frac{\partial}{\partial y_{1,j}} - y_{1,n} \frac{\partial}{\partial y_{1,n}}\Big),$$

$$f_2 = \sum_{n=1}^{2(\ell-1)} y_{2,n}\Big(\langle \mu, \alpha_2^\vee \rangle + 2 \sum_{j=1}^{[n/2]+1} y_{1,j} \frac{\partial}{\partial y_{1,j}} - 2 \sum_{j=1}^{n-1} y_{2,j} \frac{\partial}{\partial y_{2,j}} + \tag{3.86}$$

$$+ \sum_{j=1}^{2[(n+1)/2]-3} y_{3,j} \frac{\partial}{\partial y_{3,j}} - y_{2,n} \frac{\partial}{\partial y_{2,n}}\Big),$$

$$f_k = \sum_{n=1}^{2(\ell+1-k)} y_{k,n}\Big(\langle \mu, \alpha_k^\vee \rangle + 2 \sum_{j=1}^{2[n/2]+1} y_{k-1,j} \frac{\partial}{\partial y_{k-1,j}} - 2 \sum_{j=1}^{n-1} y_{k,j} \frac{\partial}{\partial y_{k,j}} +$$

$$+ \sum_{j=1}^{2[(n+1)/2]-3} y_{k+1,j} \frac{\partial}{\partial y_{k+1,j}} - y_{k,n} \frac{\partial}{\partial y_{k,n}}\Big), \qquad 2 < k < \ell.$$

This completes the proof of Proposition 2.8.

3.4.4 Generators of $\mathfrak{so}_{2\ell}$: Proof of Proposition 2.12

Let $e_i^{(\ell)}, h_i^{(\ell)}, f_i^{(\ell)}$ be Chevalley generators of $\mathfrak{so}_{2\ell}$. Below we present recursive relations and resolved expressions for these generators.

Recursive relations are given by:

$$e_1^{(\ell)} = \frac{\partial}{\partial y_{1,\ell-1}} + \frac{y_{2,\ell-1}}{y_{1,\ell-1}} \frac{y_{3,2(\ell-2)}}{y_{3,2(\ell-2)-1}} \Big(e_2^{(\ell-1)} - \frac{\partial}{\partial y_{2,\ell-1}}\Big) + \frac{y_{3,2(\ell-2)}}{y_{1,\ell-1}} \Big(\frac{\partial}{\partial\, y_{y_{3,2(\ell-2)-1}}} - \frac{\partial}{\partial\, y_{y_{3,2(\ell-2)}}}\Big),$$

$$e_2^{(\ell)} = \frac{\partial}{\partial y_{2,\ell-1}} + \frac{y_{1,\ell-1}}{y_{2,\ell-1}} \frac{y_{3,2(\ell-2)}}{y_{3,2(\ell-2)-1}} \Big(e_1^{(\ell-1)} - \frac{\partial}{\partial y_{1,\ell-1}}\Big) + \frac{y_{3,2(\ell-2)}}{y_{2,\ell-1}} \Big(\frac{\partial}{\partial\, y_{y_{3,2(\ell-2)-1}}} - \frac{\partial}{\partial\, y_{y_{3,2(\ell-2)}}}\Big),$$

$$e_k^{(\ell)} = \frac{\partial}{\partial y_{k,2(\ell+1-k)}} + \frac{y_{k,2(\ell+1-k)-1}}{y_{k,2(\ell+1-k)}} \frac{y_{k+1,2(\ell-k)}}{y_{k+1,2(\ell-k)-1}} \Big(e_k^{(\ell-1)} - \frac{\partial}{\partial y_{k,2(\ell+1-k)-1}}\Big) + \tag{3.87}$$

$$+\frac{y_{k+1,2(\ell-k)}}{y_{k,2(\ell+1-k)}}\Big(\frac{\partial}{\partial\, y_{y_{k,2(\ell+1-k)-1}}}-\frac{\partial}{\partial\, y_{y_{k,2(\ell+1-k)}}}\Big), \qquad 2<k<\ell,$$

$$e_\ell=\frac{\partial}{\partial y_{\ell,2}}, \tag{3.88}$$

$$f_i^{(\ell)}=f_i^{(\ell-1)}-y_{i,\ell-1}^2\frac{\partial}{\partial y_{i,\ell-1}}+y_{i,\ell-1}\Big(h_i^{(\ell-1)}+y_{3,2(\ell-2)-1}\frac{\partial}{\partial y_{3,2(\ell-2)-1}}\Big), \qquad i=1,2,$$

$$f_3^{(\ell)}=f_3^{(\ell-1)}-\Big(y_{3,2(\ell-2)-1}^2\frac{\partial}{\partial y_{3,2(\ell-2)-1}}+y_{3,2(\ell-2)}^2\frac{\partial}{\partial y_{3,2(\ell-2)}}\Big)+$$
$$+(y_{3,2(\ell-2)-1}+y_{3,2(\ell-2)})\Big(h_3^{(\ell-1)}+y_{4,2(\ell-3)-1}\frac{\partial}{\partial y_{4,2(\ell-3)-1}}\Big)+$$
$$+y_{3,2(\ell-2)}\Big(y_{1,\ell-1}\frac{\partial}{\partial y_{1,\ell-1}}+y_{2,\ell-1}\frac{\partial}{\partial y_{2,\ell-1}}\Big),$$

$$f_k^{(\ell)}=f_k^{(\ell-1)}-\Big(y_{k,2(\ell+1-k)-1}^2\frac{\partial}{\partial y_{k,2(\ell+1-k)-1}}+y_{k,2(\ell+1-k)}^2\frac{\partial}{\partial y_{k,2(\ell+1-k)}}\Big)+$$
$$+(y_{k,2(\ell+1-k)-1}+y_{k,2(\ell+1-k)})\Big(h_k^{(\ell-1)}+y_{k+1,2(\ell-k)-1}\frac{\partial}{\partial y_{k+1,2(\ell-k)-1}}\Big)+$$
$$+y_{k,2(\ell+1-k)}\Big(y_{k-1,2(\ell+2-k)-1}\frac{\partial}{\partial y_{k-1,2(\ell+2-k)-1}}+y_{k-1,2(\ell+2-k)}\frac{\partial}{\partial y_{k-1,2(\ell+2-k)}}\Big), \qquad 3<k\le\ell.$$

For Cartan generators we have:

$$h_i^{(\ell)}=\langle\mu,\,\alpha_i^\vee\rangle+\sum_{k=1}^{\ell}a_{i,k}\sum_{j=1}^{n_k}y_{k,j}\frac{\partial}{\partial y_{k,j}}, \tag{3.89}$$

where $n_1=n_2=\ell-1$, $n_k=2(\ell+1-k)$ for $2<k\le\ell$.

Resolving the recursion we obtain:

$$e_1^{(\ell)}=\frac{\partial}{\partial y_{1,\ell-1}}+\sum_{n=1}^{[\ell/2]}\Big(\frac{\partial}{\partial y_{2,\ell-n-1}}-\frac{\partial}{\partial y_{2,\ell-n}}\Big)\prod_{k=1}^{2n-1}\Big(\frac{y_{1,\ell-k}}{y_{2,\ell-k}}\Big)^{(-1)^k}\frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}}+$$
$$+\sum_{n=2}^{[\ell/2]}\Big(\frac{\partial}{\partial y_{1,\ell-n-1}}-\frac{\partial}{\partial y_{1,\ell-n}}\Big)\prod_{k=1}^{2(n-1)}\Big(\frac{y_{1,\ell-k}}{y_{2,\ell-k}}\Big)^{(-1)^k}\frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}}+$$
$$+\sum_{n=1}^{[\frac{\ell-1}{2}]}\Big(\frac{\partial}{\partial y_{3,2(2n-1)-1}}-\frac{\partial}{\partial y_{3,2(2n-1)}}\Big)\frac{y_{3,2(2n-1)}}{y_{1,2n}}\prod_{k=1}^{\ell-2n-1}\Big(\frac{y_{1,\ell-k}}{y_{2,\ell-k}}\Big)^{(-1)^k}\frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}}+$$
$$+\sum_{n=1}^{[\frac{\ell-2}{2}]}\Big(\frac{\partial}{\partial y_{3,4n-1}}-\frac{\partial}{\partial y_{3,4n}}\Big)\frac{y_{3,4n}}{y_{2,2n+1}}\prod_{k=1}^{\ell-2(n+1)}\Big(\frac{y_{1,\ell-k}}{y_{2,\ell-k}}\Big)^{(-1)^k}\frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}},$$

$$e_2^{(\ell)} = \frac{\partial}{\partial y_{2,\ell-1}} + \sum_{n=2}^{[\ell/2]} \Big(\frac{\partial}{\partial y_{2,\ell-n-1}} - \frac{\partial}{\partial y_{2,\ell-n}} \Big) \prod_{k=1}^{2(n-1)} \Big(\frac{y_{2,\ell-k}}{y_{1,\ell-k}} \Big)^{(-1)^k} \frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}} + \tag{3.90}$$

$$+ \sum_{n=1}^{[\ell/2]} \Big(\frac{\partial}{\partial y_{1,\ell-n-1}} - \frac{\partial}{\partial y_{1,\ell-n}} \Big) \prod_{k=1}^{2n-1} \Big(\frac{y_{2,\ell-k}}{y_{1,\ell-k}} \Big)^{(-1)^k} \frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}} +$$

$$+ \sum_{n=1}^{[\frac{\ell-2}{2}]} \Big(\frac{\partial}{\partial y_{3,4n-1}} - \frac{\partial}{\partial y_{3,4n}} \Big) \frac{y_{3,4n}}{y_{1,2n+1}} \prod_{k=1}^{\ell-2(n+1)} \Big(\frac{y_{2,\ell-k}}{y_{1,\ell-k}} \Big)^{(-1)^k} \frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}} +$$

$$+ \sum_{n=1}^{[\frac{\ell-1}{2}]} \Big(\frac{\partial}{\partial y_{3,2(2n-1)-1}} - \frac{\partial}{\partial y_{3,2(2n-1)}} \Big) \frac{y_{3,2(2n-1)}}{y_{1,2n}} \prod_{k=1}^{\ell-2n-1} \Big(\frac{y_{2,\ell-k}}{y_{1,\ell-k}} \Big)^{(-1)^k} \frac{y_{3,2(\ell-1-k)}}{y_{3,2(\ell-1-k)-1}},$$

$$e_k^{(\ell)} = \frac{\partial}{\partial y_{k,2(\ell+1-k)}} + \sum_{n=1}^{\ell-k} \Big(\frac{\partial}{\partial y_{k,2n}} - \frac{\partial}{\partial y_{k,2n+1}} \Big) \prod_{i=1}^{\ell+1-n-k)} \frac{y_{k,2(i+1)-1}}{y_{k,2(k+1)}} + \frac{y_{k+1,2i}}{y_{k+1,2i-1}} +$$

$$+ \sum_{n=1}^{\ell-k} \Big(\frac{\partial}{\partial y_{k+1,2n-1}} - \frac{\partial}{\partial y_{k+1,2n}} \Big) \frac{y_{k+1,2n}}{y_{k,2(n+1)}} \prod_{i=2}^{\ell+1-n-k)} \frac{y_{k,2(i+1)-1}}{y_{k,2(k+1)}} \frac{y_{k+1,2i}}{y_{k+1,2i-1}}, \qquad 2 < k < \ell,$$

$$f_i^{(\ell)} = -\langle \mu, \alpha_i^\vee \rangle \sum_{n=1}^{\ell-1} y_{i,n} - \sum_{n=1}^{\ell-1} \Big(y_{i,n}^2 \frac{\partial}{\partial y_{i,n}} + 2 \sum_{k=n+1}^{\ell-1} y_{i,k} y_{i,n} \frac{\partial}{\partial\, y_{i,n}} \Big) +$$

$$+ \sum_{n=1}^{2(\ell-2)-1} \sum_{k=[n/2]+2} y_{i,k} y_{3,n} \frac{\partial}{\partial\, y_{3,n}}, \qquad i = 1, 2,$$

$$f_3^{(\ell)} = -\langle \mu, \alpha_3^\vee \rangle \sum_{n=1}^{2(\ell-2)} y_{3,n} - \sum_{n=1}^{2(\ell-2)} \Big(y_{3,n}^2 \frac{\partial}{\partial\, y_{3,n}} + 2 \sum_{k=n+1}^{2(\ell-2)} y_{3,k} y_{3,n} \frac{\partial}{\partial\, y_{3,n}} \Big) +$$

$$+ \sum_{n=1}^{\ell-1} \sum_{k=2(n-1)}^{2(\ell-2)} y_{3,k} \Big(y_{i,n} \frac{\partial}{\partial\, y_{1,n}} + y_{2,n} \frac{\partial}{\partial\, y_{i,n}} \Big) + \sum_{n=1}^{2(\ell-3)-1} \sum_{k=2[n/2]+3} y_{3,k} y_{4,n} \frac{\partial}{\partial\, y_{4,n}},$$

$$f_k^{(\ell)} = -\langle \mu, \alpha_k^\vee \rangle \sum_{n=1}^{2(\ell+1-k)} y_{k,n} - \sum_{n=1}^{2(\ell+1-k)} \Big(y_{k,n}^2 \frac{\partial}{\partial\, y_{k,n}} + 2 \sum_{i=n+1}^{2(\ell+1-k)} y_{k,i} y_{k,n} \frac{\partial}{\partial\, y_{k,n}} \Big) +$$

$$+ \sum_{n=1}^{\ell+2-k} \sum_{i=2(n-1)}^{2(\ell+1-k)} y_{k,i} \Big(y_{k-1,2n-1} \frac{\partial}{\partial\, y_{k-1,2n-1}} + y_{k-1,2n} \frac{\partial}{\partial\, y_{k-1,2n}} \Big) +$$

$$+\sum_{n=1}^{2(\ell-k)-1}\sum_{i=2[n/2]+3} y_{k,i}y_{k+1,n}\frac{\partial}{\partial\, y_{k+1,n}}, \qquad 3<k\leq \ell.$$

This completes the proof of Proposition 2.12.

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