SIMPLE FINITE SUBGROUPS OF THE CREMONA GROUP OF RANK 3

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Abstract. We classify all finite simple subgroups of the Cremona group $\mathrm{Cr}_3(\mathbb{C})$.

1. Introduction

Let $\Bbbk$ be a field. The Cremona group $\mathrm{Cr}_d(\Bbbk)$ is the group of birational automorphisms of the projective space $\mathbb{P}^d_{\Bbbk}$, or, equivalently, the group of $\Bbbk$-automorphisms of the rational function field $\Bbbk(t_1, \ldots, t_d)$. It is well-known that $\mathrm{Cr}_1(\Bbbk) = \mathrm{PGL}_2(\Bbbk)$. For $d \geq 2$, the structure of $\mathrm{Cr}_d(\Bbbk)$ and its subgroups is very complicated. For example, the classification of finite subgroups in $\mathrm{Cr}_2(\mathbb{C})$ is an old classical problem. Recently this classification almost has been completed by Dolgachev and Iskovskikh [DI06]. The following is a consequence of the list in [DI06].

Theorem 1.1 ([DI06]). *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a non-abelian simple finite subgroup. Then G is isomorphic to one of the following groups:*

$$\mathfrak{A}_5, \quad \mathfrak{A}_6, \quad \mathrm{PSL}_2(7). \tag{1.2}$$

However, the methods and results of [DI06] show that one cannot expect a reasonable classification of all finite subgroups of Cremona groups of higher rank. In this paper we restrict ourselves with the case of simple finite subgroups of $\mathrm{Cr}_3(\mathbb{C})$. Our main result is the following:

Theorem 1.3. *Let $G \subset \mathrm{Cr}_3(\mathbb{C})$ be a non-abelian simple finite subgroup. Then G is isomorphic to one of the following groups:*

$$\mathfrak{A}_5, \quad \mathfrak{A}_6, \quad \mathfrak{A}_7, \quad \mathrm{PSL}_2(7), \quad \mathrm{SL}_2(8), \quad \mathrm{PSp}_4(3). \tag{1.4}$$

All the possibilities occur.

In particular, we give the affirmative answer to a question of J.-P. Serre [Ser09, Question 6.0]: there are a lot of finite groups which do not admit any embeddings into $\mathrm{Cr}_3(\mathbb{C})$. More generally we classify simple finite subgroups

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in the group of birational automorphisms of an arbitrary three-dimensional rationally connected variety and in many cases we determine all birational models of the action:

Theorem 1.5. *Let X be a rationally connected threefold and let $G \subset \mathrm{Bir}(X)$ be a non-abelian simple finite subgroup. Then G is isomorphic either to* $\mathrm{PSL}_2(11)$ *or to one of the groups in the list* (1.4). *All the possibilities occur. Furthermore, if G does not admit any embeddings into* $\mathrm{Cr}_2(\mathbb{C})$ (*see Theorem* 1.1), *then G is conjugate to one of the following:*

(i) $\mathfrak{A}_7$ *acting on some special smooth intersection of a quadric and a cubic* $X'_6 \subset \mathbb{P}^5$ (*see Example* 2.5),
(ii) $\mathfrak{A}_7$ *acting on* $\mathbb{P}^3$ (*see Theorem* 3.3),
(iii) $\mathrm{PSp}_4(3)$ *acting on* $\mathbb{P}^3$ (*see Theorem* 3.3),
(iv) $\mathrm{PSp}_4(3)$ *acting on the Burkhardt quartic* $X_4^{\mathrm{b}} \subset \mathbb{P}^4$ (*see Example* 2.8),
(v) $\mathrm{SL}_2(8)$ *acting on some smooth Fano threefold* $X_{12}^{\mathrm{m}} \subset \mathbb{P}^8$ *of genus* 7 (*see Example* 2.11),
(vi) $\mathrm{PSL}_2(11)$ *acting on the Klein cubic* $X_3^{\mathrm{k}} \subset \mathbb{P}^4$ (*see Example* 2.6),
(vii) $\mathrm{PSL}_2(11)$ *acting on some smooth Fano threefold* $X_{14}^{\mathrm{a}} \subset \mathbb{P}^9$ *of genus* 8 (*see Example* 2.9).

Moreover, any equivariant action of G on a Fano-Mori fiber space is isomorphic to one of the above cases.

However we should mention that in contrast with [DI06] we do not describe *actions* of groups $\mathfrak{A}_5$, $\mathfrak{A}_6$ and $\mathrm{PSL}_2(7)$. We also do not answer to the question about conjugacy groups (iii)-(iv), (vi)-(vii), and (i)-(ii). [1]

Remark 1.6. The cooresponding varieties in (ii)-(v) of the above theorem are rational. Hence these actions define embeddings of G into $\mathrm{Cr}_3(\mathbb{C})$. Varieties X_3^{k} and X'_{14} are birationally equivalent and non-rational (see Remark 2.10 and [CG72]). It is known that a general intersection of a quadric and a cubic is non-rational. As far as I know the non-rationality of any smooth threefold in this family is still not proved.

Remark 1.7. (i) The orders of the above groups are as follows:

G	$\mathfrak{A}_5$	$\mathfrak{A}_6$	$\mathfrak{A}_7$	$\mathrm{PSL}_2(7)$	$\mathrm{SL}_2(8)$	$\mathrm{PSp}_4(3)$	$\mathrm{PSL}_2(11)$
$\lvert G\rvert$	60	360	2520	168	504	25920	660

[1] It was recently proved that X_3^{k} and X_{14}^{a} are not birationally G-isomorphic (see [Cheltsov I. and Shramov C. arXiv:0909.0918]). I. Cheltsov also pointed out to me that non-conjugacy of the actions of $\mathrm{PSp}_4(3)$ on the Burkhardt quartic X_4^{b} and $\mathbb{P}^3$ follows from results of M. Mella and C. Shramov (see [Mella M. Math. Ann. (2004) **330**, 107–126], [Shramov C. arXiv:0803.4348]).

(ii) There are well-known isomorphisms $\mathrm{PSp}_4(3) \simeq \mathrm{SU}_4(2) \simeq \mathrm{O}_5(3)'$, $\mathfrak{A}_5 \simeq \mathrm{SL}_2(4) \simeq \mathrm{PSL}_2(5)$, $\mathrm{PSL}_2(7) \simeq \mathrm{GL}_3(2)$, and $\mathfrak{A}_6 \simeq \mathrm{PSL}_2(9)$ (see, e.g., [CCN+85]).

The idea of the proof is quite standard. It follows the classical ideas (cf. [DI06]) but has much more technical difficulties. Here is an outline of our approach.

By running the equivariant Minimal Model Program we may assume that our group G acts on a Mori-Fano fiber space X/Z. Here Z is either a point, a rational curve or a rational surface (because a rationally connected surface over $\mathbb{C}$ must be rational). Since the group is simple and because G does not admit any embeddings into $\mathrm{Cr}_2(\mathbb{C})$, we may assume that Z is a point. The latter means that X is a $G\mathbb{Q}$-Fano threefold.

Definition 1.8. A *G-variety* is a variety X provided with a biregular action of a finite group G. We say that a normal G-variety X is *$G\mathbb{Q}$-factorial* if any G-invariant Weil divisor on X is $\mathbb{Q}$-Cartier. A projective normal G-variety X is called *$G\mathbb{Q}$-Fano* if it is $G\mathbb{Q}$-factorial, has at worst terminal singularities, $-K_X$ is ample, and $\operatorname{rk}\operatorname{Pic}(X)^G = 1$.

Thus the G-equivariant Minimal Model Program reduces our problem to the classification of finite simple subgroups in automorphism groups of $G\mathbb{Q}$-Fano threefolds. Smooth Fano threefolds are completely classified by Iskovskikh [Isk80] and Mori–Mukai [MM82]. To study the singular case we use estimates for the number of singular points and analyze the action of G on the singular set.

The structure of the paper is as follows. In §2 we collect some examples and show that all the cases in our list really occur. Reduction to the case of $G\mathbb{Q}$-Fano threefolds is explained in §4. In §5 and §6 we study the cases where X is Gorenstein and non-Gorenstein, respectively.

Conventions. All varieties are defined over the complex number field $\mathbb{C}$. $\mathfrak{S}_n$ and $\mathfrak{A}_n$ denote the symmetric and the alternating groups, respectively. For linear groups over a field $\Bbbk$ we use the standard notations $\mathrm{GL}_n(\Bbbk)$, $\mathrm{SO}_n(\Bbbk)$, $\mathrm{Sp}_n(\Bbbk)$ etc. If the field $\Bbbk$ is finite and contains q elements, then, for short, the above groups are denoted by $\mathrm{GL}_n(q)$, $\mathrm{SO}_n(q)$, $\mathrm{Sp}_n(q)$ etc. For a group G, we denote by $Z(G)$ and $[G,G]$ its center and derived subgroup, respectively. If the group G acts on a set Ω, then, for an element $P \in \Omega$, its stabilizer is denoted by G_P. All simple groups are supposed to be non-abelian. The Picard number of a variety X is denoted by $\rho(X)$. For a normal variety X, $\mathrm{Cl}(X)$ is the Weil divisor class group. Note that there is a difference between conjugate/non-conjugate *embeddings* $G \hookrightarrow \mathrm{Cr}_n(\Bbbk)$ and conjugate/non-conjugate *subgroups* $G \subset \mathrm{Cr}_n(\Bbbk)$. In this paper we discuss *subgroups* $G \subset \mathrm{Cr}_n(\Bbbk)$.

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2. Examples

In this section we collect examples.

First of all, the group $\mathfrak{A}_5$ acts on $\mathbb{P}^1$ and $\mathbb{P}^2$. This gives a lot of embeddings into $\mathrm{Cr}_3(\mathbb{C})$ (by different actions on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2 \times \mathbb{P}^1$). The groups $\mathfrak{A}_6$ and $\mathrm{PSL}_2(7)$ admit embeddings into $\mathrm{Cr}_2(\mathbb{C})$, so they are also can be embedded to $\mathrm{Cr}_3(\mathbb{C})$.

Example 2.1. Consider the embedding of $\mathfrak{A}_5 \subset \mathrm{PGL}_2(\mathbb{C})$ as a binary icosahedron group. Let $H \subset \mathrm{PGL}_2(\mathbb{C})$ be another finite subgroup. Then there is an action of $\mathfrak{A}_5$ on a rational homogeneous variety $\mathrm{PGL}_2(\mathbb{C})/H$. This gives a series of embeddings of $\mathfrak{A}_5$ into $\mathrm{Cr}_3(\mathbb{C})$.

Trivial examples also provide subgroups of $\mathrm{PGL}_4(\mathbb{C})$ (see Theorem 3.3): $\mathfrak{A}_5$, $\mathfrak{A}_6$, $\mathfrak{A}_7$, $\mathrm{PSL}_2(7)$, $\mathrm{PSp}_4(3)$. In the examples below we show that a finite simple group acts on a (possibly singular) Fano thereefold. According to [Zha06] Fano varieties with log terminal singularities are rationally connected, so our constructions give embeddings of a finite simple group into the automorphism group of some rationally connected variety.

Example 2.2. The group $\mathfrak{A}_5$ acts on the smooth cubic $\{\sum_{i=1}^4 x_i^3 = 0\} \subset \mathbb{P}^4$ and on the smooth quartic $\{\sum_{i=1}^4 x_i^4 = 0\} \subset \mathbb{P}^4$. These varieties are not rational [CG72], [IM71].

Example 2.3. The *Segre cubic* X_3^{s} is a subvariety in $\mathbb{P}^5$ given by the equations $\sum x_i = \sum x_i^3 = 0$. This cubic has 10 nodes, it is obviously rational, and $\mathrm{Aut}\, X_3^{\mathrm{s}} \simeq \mathfrak{S}_6$. In particular, alternating groups $\mathfrak{A}_5$ and $\mathfrak{A}_6$ act on X_3^{s}. Since $\mathfrak{A}_5$ can be embedded into $\mathfrak{A}_6$ in two ways, this construction gives two embeddings of $\mathfrak{A}_5$ into $\mathrm{Cr}_3(\mathbb{C})$. We do not know if they are conjugate or not.

Example 2.4. Assume that G acts on $\mathbb{C}^5$ so that there are (irreducible) invariants ϕ_2 and ϕ_3 of degree 2 and 3, respectively. Let $Y \subset \mathbb{P}^4$ be a (possibly singular) cubic hypersurface given by $\phi_3 = 0$ and let $R \subset Y$ be the surface given by $\phi_2 = \phi_3 = 0$. Then $R \in |-K_Y|$. Consider the double cover $X \to Y$ ramified along R. Then X is a Fano threefold. It can be realized as an intersection of a cubic and quadratic cone in $\mathbb{P}^5$. The action of G lifts to X. There are two interesting cases (cf. [Muk88b]):

(a) $Y = \{\sum_{i=0}^5 x_i = \sum_{i=0}^5 x_i^3 = 0\} \subset \mathbb{P}^5$ is the Segre cubic, and R is cut out by the equation $\sum_{i=0}^5 x_i^2 = 0$, $G = \mathfrak{A}_6$;
(b) $Y = \{\sum_{i=0}^4 x_i = \sum_{i=0}^4 x_i^3 = 0\} \subset \mathbb{P}^5$ is a cubic cone, and R is cut out by the equation $\sum_{i=0}^5 x_i^2 = 0$, $G = \mathfrak{A}_5$.

Example 2.5. Consider the subvariety in $X_6' \subset \mathbb{P}^6$ given by the equations $\sigma_1 = \sigma_2 = \sigma_3 = 0$, where σ_i are symmetric polynomials in $x_1, \ldots, x_7$. Then X_6' is Fano smooth threefold, an intersection of a quadric and a cubic in $\mathbb{P}^5$. The alternating group $\mathfrak{A}_7$ naturally acts on X_6'. A general variety X_6 in this family is not rational [IP96]. The (non-)rationality of X_6' is not known[2].

Example 2.6. The automorphism group of the cubic $X_3^{\mathrm{k}} \subset \mathbb{P}^4$ given by the equation

$$x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_1 = 0 \tag{2.7}$$

is isomorphic to $\mathrm{PSL}_2(11)$. This was discovered by F. Klein, see [Adl78] for a complete modern proof. [3]

Example 2.8. The *Burkhardt quartic* X_4^{b} is a subvariety in $\mathbb{P}^5$ given by $\sigma_1 = \sigma_4 = 0$, where σ_i is i-th symmetric function in $x_1, \ldots, x_6$. The automorphism group of X_4^{b} is isomorphic to $\mathrm{PSp}_4(3)$, see [ST54]. [4]

Example 2.9. Let W be a 5-dimensional irreducible representation of $\tilde{G} := \mathrm{SL}_2(11)$. Consider the following skew symmetric matrix whose entries are

[2]The non-rationality of X_6' was proved recently by A. Beauville [Non-rationality of the symmetric sextic Fano threefold, arXiv:1102.1255].

[3]As pointed out by J. Ellenberg, the existence of this action can also be seen from the fact that the cubic X_3^{k} is birational to $\mathcal{A}_{11}^{lev}$, the moduli space of abelian surfaces with $(1, 11)$-polarization and canonical level structure, see [M. Gross and S. Popescu. The moduli space of $(1, 11)$-polarized abelian surfaces is unirational. *Compositio Math.*, 126:1–23, 2001].

[4]Similar to the previous example the existence of this action follows also from an interpretation of X_4^{b} as a moduli space, see, e.g., [B. van Geemen. Projective models of Picard modular varieties. in *Classification of irregular varieties* (Trento, 1990), 68–99, Lecture Notes in Math., **1515**, Springer, Berlin, 1992]

linear forms on W:

$$A := \begin{pmatrix} 0 & x_4 & x_5 & x_1 & x_2 & x_3 \\ -x_4 & 0 & 0 & x_3 & -x_1 & 0 \\ -x_5 & 0 & 0 & 0 & x_4 & -x_2 \\ -x_1 & -x_3 & 0 & 0 & 0 & x_5 \\ -x_2 & x_1 & -x_4 & 0 & 0 & 0 \\ -x_3 & 0 & x_2 & -x_5 & 0 & 0 \end{pmatrix}$$

The matrix A can be regarded as a non-trivial G-equivariant linear map from W to $\wedge^2 V$, where V is a 6-dimensional irreducible representation of $\tilde{G}$, see [AR96, §47]. Thus the representation $\wedge^2 V$ is decomposed as $\wedge^2 V = W \oplus W^\perp$, where $\dim W^\perp = 10$. Let $X_{14}^{\mathrm{a}} := \mathbb{P}(W^\perp) \cap \mathrm{Gr}(2, V) \subset \mathbb{P}(\wedge^2 V)$.

It is easy to check that $\operatorname{rk} A(w) \geq 4$ for any $w \in W$, $w \neq 0$. Thus A is a regular net of skew forms in the sense of [Kuz04]. The Pfaffian of A defines a cubic hypersurface $X_3 \subset \mathbb{P}^4$. This hypersurface X_3 is given by the equation (2.7) because the action of $\mathrm{SL}_2(11)$ on $\mathbb{C}^5$ has only one invariant of degree 3 (see [Adl78]). So, $X_3 = X_3^{\mathrm{k}}$. Hence it is smooth and so is X_{14}^{a} by [Kuz04, Prop. A.4]. Then by the adjunction formula X_{14}^{a} is a Fano threefold of Picard number one and genus 8 [IP99]. By construction X_{14}^{a} admits a non-trivial action of G.

Remark 2.10. It turns out that X_{14}^{a} and X_3^{k} are birationally equivalent (and not rational [CG72]), so our construction does not give an embedding of G into $\mathrm{Cr}_3(\mathbb{C})$. The birational equivalence of X_{14}^{a} and X_3^{k} can be seen from the following construction [Put82]. Given a smooth section $X_{14} = \mathrm{Gr}(2, 6) \cap \mathbb{P}^9$, let $Y \subset \mathbb{P}^5$ be the variety swept out by lines representing points of $X_{14} \subset \mathrm{Gr}(2, 6)$. Then Y is a singular quartic fourfold. It is called the *Palatini quartic* of X_{14}. In our case $X_{14} = X_{14}^{\mathrm{a}}$ the equation of Y is as follows [AR96, Cor. 50.2]:

$$\begin{aligned} x_0^4 + x_0(x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_1) + \\ + x_1^2 x_3 x_5 + x_2^2 x_4 x_1 + x_3^2 x_5 x_2 + x_4^2 x_1 x_3 + x_5^2 x_2 x_4 = 0. \end{aligned}$$

Let H be a general hyperplane section of Y. Then H is a quartic threefold with 25 singular points which is birational to both X_{14}^{a} and X_3^{k} [Put82], [CG72, Theorem 13.11]. Note however that our birational construction is not G-equivariant. We do not know whether our two embeddings of G into $\mathrm{Bir}(X_{14}^{\mathrm{a}}) \simeq \mathrm{Bir}(X_3^{\mathrm{k}})$ are conjugate or not.

Example 2.11 ([Muk92])**.** There is a curve C^{m} of genus 7 for which the Hurwitz bound of the automorphism group is achieved [Mac65]. In fact, $\operatorname{Aut} C^{\mathrm{m}} \simeq \mathrm{SL}_2(8)$. The "dual" Fano threefold of genus 7 has the same automorphism group. The construction due to S. Mukai [Muk92, Muk95] is as follows. Let $Q \subset \mathbb{P}^8$ be a smooth quadric. All 3-dimensional projective subspaces of $\mathbb{P}^8$ contained in Q are parameterized by a smooth irreducible

$\mathrm{SO}_9(\mathbb{C})$-homogeneous variety $\mathrm{LGr}(4,9)$, so-called, Lagrangian Grassmannian. In fact, $\mathrm{LGr}(4,9)$ is a Fano manifold of dimension 10 and Fano index 8 with $\rho(\mathrm{LGr}(4,9)) = 1$. The positive generator of $\mathrm{Pic}(\mathrm{LGr}(4,9)) \simeq \mathbb{Z}$ determines an embedding $\mathrm{LGr}(4,9) \hookrightarrow \mathbb{P}^{15}$. In fact, this embedding is given by the spinor coordinates on $\mathrm{LGr}(4,9)$. It is known that any smooth Fano threefold X_{12}^{m} of genus 7 with $\rho(X_{12}^{\mathrm{m}}) = 1$ is isomorphic to a section of $\mathrm{LGr}(4,9) \subset \mathbb{P}^{15}$ by a subspace of dimension 8 [Muk88a]. Similarly, any canonical curve C of genus 7 is isomorphic to a section of $\mathrm{LGr}(4,9) \subset \mathbb{P}^{15}$ by a subspace of dimension 6 if and only if C has no g_4^1 [Muk95]. The group $\mathrm{SL}_2(8)$ has a 9-dimensional representation U and there is an invariant quadric $Q \subset \mathbb{P}(U)$. Hence $\mathrm{SL}_2(8)$ naturally acts on $\mathrm{LGr}(4,9)$. This action lifts to $\mathbb{P}^{15}$ so that there are two invariant subspaces Π_1 and Π_2 of dimension 6 and 8, respectively. The intersections $\mathrm{LGr}(4,9) \cap \Pi_1$ and $\mathrm{LGr}(4,9) \cap \Pi_2$ are our curve C^{m} [Muk95, Table 1] and a smooth Fano threefold of genus 7 with $\rho = 1$ (see [IM04, Lemma 3.2]). Recall that any smooth Fano threefold of genus 7 with $\rho = 1$ is rational (see, e.g., [IP99]). Therefore, the above construction provides an embedding of $\mathrm{SL}_2(8)$ into $\mathrm{Cr}_3(\mathbb{C})$.

3. Finite linear and permutation groups

3.1. Finite linear groups. Let V be a vector space. An irreducible subgroup $G \subset \mathrm{GL}(V)$ is said to be *imprimitive* if there exists a non-trivial decomposition $V = \oplus V_i$ such that G permutes subspaces V_i. In this case G contains a non-trivial reducible normal subgroup N such that $gV_i = V_i$ for all $g \in N$ and all i. A group G is said to be *primitive* if it is irreducible and not imprimitive. Clearly, a simple linear group has to be primitive if it is irreducible.

All finite primitive linear groups of small degree are classified, see [Bli17], [Bra67], and [Lin71]. Basically we need only the list of the simple ones.

Theorem 3.2 ([Bli17])**.** *Let $G \subset \mathrm{PGL}_3(\mathbb{C})$ be a finite irreducible simple subgroup and let $\tilde{G} \subset \mathrm{SL}_3(\mathbb{C})$ be its preimage under the natural map $\mathrm{SL}_3(\mathbb{C}) \to \mathrm{PGL}_3(\mathbb{C})$ such that $[\tilde{G}, \tilde{G}] \supset Z(\tilde{G})$. Then only one of the following cases is possible:*

(i) *the icosahedral group, $G \simeq \tilde{G} \simeq \mathfrak{A}_5$;*
(ii) *the Valentiner group, $G \simeq \mathfrak{A}_6$, $Z(\tilde{G}) \simeq \boldsymbol{\mu}_3$;*
(iii) *the Klein group, $G \simeq \tilde{G} \simeq \mathrm{PSL}_2(7)$;*
(iv) *the Hessian group, $G \simeq (\boldsymbol{\mu}_3)^2 \rtimes \mathrm{SL}_2(3)$, $|G| = 216$, $Z(\tilde{G}) \simeq \boldsymbol{\mu}_3$;*
(v) *subgroups of the Hessian group of index 3 and 6.*

Theorem 3.3 ([Bli17])**.** *Let $G \subset \mathrm{PGL}_4(\mathbb{C})$ be a finite irreducible simple subgroup and let $\tilde{G} \subset \mathrm{SL}_4(\mathbb{C})$ be its preimage under the natural map $\mathrm{SL}_4(\mathbb{C}) \to \mathrm{PGL}_4(\mathbb{C})$ such that $[\tilde{G}, \tilde{G}] \supset Z(\tilde{G})$. Then one of the following cases is possible:*

(i) $G \simeq \tilde{G} \simeq \mathfrak{A}_5$,
(ii) $G \simeq \mathfrak{A}_6$, $Z(\tilde{G}) \simeq \boldsymbol{\mu}_2$,
(iii) $G \simeq \mathfrak{A}_7$, $Z(\tilde{G}) \simeq \boldsymbol{\mu}_2$,
(iv) $G \simeq \mathfrak{A}_5$, $\tilde{G} \simeq \mathrm{SL}_2(5)$, $Z(\tilde{G}) \simeq \boldsymbol{\mu}_2$,
(v) $G \simeq \mathrm{PSL}_2(7)$, $\tilde{G} \simeq \mathrm{SL}_2(7)$, $Z(\tilde{G}) \simeq \boldsymbol{\mu}_2$,
(vi) $G \simeq \mathrm{PSp}_4(3)$, $\tilde{G} = \mathrm{Sp}_4(3)$, $Z(\tilde{G}) \simeq \boldsymbol{\mu}_2$.

Theorem 3.4 ([Bra67]). *Let $G \subset \mathrm{PGL}_5(\mathbb{C})$ be a finite irreducible simple subgroup and let $\tilde{G} \subset \mathrm{SL}_5(\mathbb{C})$ be its preimage under the natural map $\mathrm{SL}_5(\mathbb{C}) \to \mathrm{PGL}_5(\mathbb{C})$ such that $[\tilde{G}, \tilde{G}] \supset Z(\tilde{G})$. Then $G \simeq \tilde{G}$ and only one of the following cases is possible:*

$$\mathfrak{A}_5, \quad \mathfrak{A}_6, \quad \mathrm{PSL}_2(11), \quad \mathrm{PSp}_4(3).$$

Theorem 3.5 ([Lin71]). *Let $G \subset \mathrm{GL}_6(\mathbb{C})$ be a finite irreducible simple subgroup. Then G is isomorphic to one of the following groups:*

$$\mathfrak{A}_7, \quad \mathrm{PSL}_2(7), \quad \mathrm{PSp}_4(3), \quad \mathrm{SU}_3(3).$$

Lemma 3.6. *Let G be a finite simple group. Assume that G admits an embedding into $\mathrm{PSO}_n(\mathbb{C})$ with $n \le 6$ and does not admit any embeddings into $\mathrm{Cr}_2(\mathbb{C})$. Then $n = 6$ and G is isomorphic to $\mathfrak{A}_7$ or $\mathrm{PSp}_4(3)$.*

Proof. We may assume that $G \subset \mathrm{PSO}_6(\mathbb{C})$ (we can use embeddings $\mathrm{PSO}_r(\mathbb{C}) \subset \mathrm{PSO}_6(\mathbb{C})$) for $r < 6$). Thus G acts faithfully on a smooth quadric $Q \subset \mathbb{P}^5$. It is well known that Q contains two 3-dimensional families F_1, F_2 of planes [GH78, Ch. 6, §1]. Regarding Q as the Grassmann variety $\mathrm{Gr}(2,4)$ we see $F_1 \simeq F_2 \simeq \mathbb{P}^3$ [GH78, Ch. 6, §2]. We get a non-trivial action of G on $\mathbb{P}^3$. Now the assertion follows by Theorems 3.3. □

Transitive simple permutation groups. Let G be a group acting transitively on a finite set Ω. A nonempty subset $\Omega' \subset \Omega$ is called a *block* for G if for each $\delta \in G$ either $\delta(\Omega') = \Omega'$ or $\delta(\Omega') \cap \Omega' = \varnothing$. If $\Omega' \subset \Omega$ is a block for G, then for any $\delta \in G$ the image $\delta(\Omega')$ is also a block and the system of all such blocks forms a partition of Ω. Moreover, the setwise stabilizer $G_{\Omega'}$ acts on Ω' transitively. The action of G is said to be *imprimitive* if there is a block $\Omega' \subset \Omega$ containing more than one element. Otherwise the action is said to be *primitive*.

Below we list all finite simple transitive permutation groups acting on $n \le 26$ symbols [DM96].

Theorem 3.7. *Let G be a finite transitive permutation group acting on the set Ω with $|\Omega| \le 26$. Assume that G is simple and is not contained in the list* (1.2)*. Then the action is primitive and we have one of the following cases:*

$\|\Omega\|$	G	Ω	degrees of irreducible representations in the interval $[2, 14]$	G_P
primitive groups				
n	$\mathfrak{A}_n$	standard	6, 10, 14 if $n = 7$	$\mathfrak{A}_{n-1}$
			7, 14 if $n = 8$	
			$n - 1$ if $n \geq 9$	
9	$\mathrm{SL}_2(8)$	$\mathbb{P}^1(\mathbb{F}_8)$	7, 8, 9	$(\boldsymbol{\mu}_2)^3 \rtimes \boldsymbol{\mu}_7$
11	$\mathrm{PSL}_2(11)$		5, 10, 11, 12	$\mathfrak{A}_5$
11	M_{11}	standard	10, 11	M_{10}
12	M_{11}		10, 11	$\mathrm{PSL}_2(11)$
12	M_{12}	standard	11	M_{11}
12	$\mathrm{PSL}_2(11)$	$\mathbb{P}^1(\mathbb{F}_{11})$	5, 10, 11, 12	$\boldsymbol{\mu}_{11} \rtimes \boldsymbol{\mu}_5$
13	$\mathrm{SL}_3(3)$	$\mathbb{P}^2(\mathbb{F}_3)$	12, 13	$(\boldsymbol{\mu}_3)^2 \rtimes \mathrm{GL}_2(3)$
14	$\mathrm{PSL}_2(13)$	$\mathbb{P}^1(\mathbb{F}_{13})$	7, 12, 13, 14	$\boldsymbol{\mu}_{13} \rtimes \boldsymbol{\mu}_6$
15	$\mathfrak{A}_7$		6, 10, 14	$\mathrm{PSL}_2(7)$
15	$\mathfrak{A}_8 \simeq \mathrm{SL}_4(2)$	$\mathbb{P}^3(\mathbb{F}_2)$	7, 14	$(\boldsymbol{\mu}_2)^3 \rtimes \mathrm{SL}_3(2)$
17	$\mathrm{SL}_2(16)$	$\mathbb{P}^1(\mathbb{F}_{16})$	–	$(\boldsymbol{\mu}_2)^4 \rtimes \boldsymbol{\mu}_{15}$
18	$\mathrm{PSL}_2(17)$	$\mathbb{P}^1(\mathbb{F}_{17})$	9	$\boldsymbol{\mu}_{17} \rtimes \boldsymbol{\mu}_8$
20	$\mathrm{PSL}_2(19)$	$\mathbb{P}^1(\mathbb{F}_{19})$	9	$\boldsymbol{\mu}_{19} \rtimes \boldsymbol{\mu}_9$
21	$\mathfrak{A}_7$		6, 10, 14	$\mathfrak{S}_5$
21	$\mathrm{PSL}_3(4)$	$\mathbb{P}^2(\mathbb{F}_4)$	–	$(\boldsymbol{\mu}_2)^4 \rtimes \mathrm{SL}_2(4)$
22	M_{22}	standard	–	$\mathrm{PSL}_3(4)$
23	M_{23}	standard	–	M_{22}
24	M_{24}	standard	–	M_{23}
24	$\mathrm{PSL}_2(23)$	$\mathbb{P}^1(\mathbb{F}_{23})$	11	$\boldsymbol{\mu}_{23} \rtimes \boldsymbol{\mu}_{11}$
26	$\mathrm{PSL}_2(25)$	$\mathbb{P}^1(\mathbb{F}_{25})$	13	$(\boldsymbol{\mu}_5)^2 \rtimes \boldsymbol{\mu}_{12}$
imprimitive groups				
22	M_{11}		10, 11	$\mathfrak{A}_6$
26	$\mathrm{SL}_3(3)$	$\mathbb{A}^3(\mathbb{F}_3)\backslash\{0\}$	12, 13	$(\boldsymbol{\mu}_3)^2 \rtimes \mathrm{SL}_2(3)$

Here M_k denotes the Mathieu group, G_P is the stabilizer of $P \in \Omega$ and $\mathbb{P}^m(\mathbb{F}_q)$ (resp. $\mathbb{A}^m(\mathbb{F}_q)$) denotes the projective (resp. affine) space over the finite field $\mathbb{F}_q$.

All primitive permutation groups are taken from the book [DM96]. Their irreducible representations can be found in [CCN$^+$85]. So we need to consider only imprimitive case.

Proof of Theorem 3.7 *in the imprimitive case.* If the group G is imprimitive, then G acts on the system of blocks Λ, where $|\Lambda| = |\Omega|/m \leq 13$ and $m \geq 2$ is the number of elements in a block. Then $m \leq 3$, the action on Λ is primitive, and for a block Ω', the setwise stabilizer $G_{\Omega'}$ acts on Ω' transitively. This gives us two possibilities: M_{11} and $\mathrm{SL}_3(3)$. □

Remark 3.8. We will show that the group $\mathfrak{A}_8$ cannot act non-trivially on a rationally connected threefold. Hence the same holds for all $\mathfrak{A}_n$ with $n \geq 9$. Therefore, in order to prove Theorems 1.3 and 1.5 we should not worry about groups $\mathfrak{A}_n$ for $n \geq 9$.

Corollary 3.9. *In notation of Theorem* 3.7 *the stabilizer* G_P *has a faithful representation of degree* ≤ 4 *only in the following cases:*

(i) $G \simeq \mathrm{PSL}_2(11)$, $|\Omega| = 11$, $G_P \simeq \mathfrak{A}_5$;
(ii) $G \simeq \mathfrak{A}_7$, $|\Omega| = 15$, $G_P \simeq \mathrm{PSL}_2(7)$;
(iii) $G \simeq \mathfrak{A}_7$, $|\Omega| = 21$, $G_P \simeq \mathfrak{S}_5$.

Proof. Clearly, in the above cases (i)-(iii) the group G_P has a faithful representation of degree ≤ 4.

Cases $G_P \simeq \mathrm{PSL}_3(4)$, $\mathfrak{A}_{n-1}$ with $n \geq 7$, M_n with $n = 10, 11, 22, 23$. Then G_P has no faithful representations of degree ≤ 4 (see, e.g., [CCN$^+$85] or Theorems 3.2 and 3.3).

In the remaining cases of Theorem 3.7 the group G_P is a semi-direct product $A \rtimes B$, where A is abelian and the action of B on A is faithfull.

Claim 3.9.1. (i) *A is a maximal abelian normal subgroup of G_P.*
(ii) *No non-trivial subgroups of A are normal in G_P.*
(iii) *Any non-trivial normal subgroup of G_P contains A.*

Proof of the claim. The statement of (i) is obvious because the action of B on A is faithfull.

(ii) Let $\{1\} \neq N \subsetneq A$ be a subgroup that is normal in G_P. Then A cannot be a cyclic group of prime order. In cases $G_P \simeq (\boldsymbol{\mu}_2)^3 \rtimes \boldsymbol{\mu}_7$, $(\boldsymbol{\mu}_2)^3 \rtimes \mathrm{SL}_3(2)$, $(\boldsymbol{\mu}_2)^4 \rtimes \boldsymbol{\mu}_{15}$, $(\boldsymbol{\mu}_2)^4 \rtimes \mathrm{SL}_2(4)$ the group B transitively acts on $A \setminus \{1\}$. So N cannot be normal. In the remaining cases $G_P \simeq (\boldsymbol{\mu}_3)^2 \rtimes \mathrm{GL}_2(3)$, $(\boldsymbol{\mu}_5)^2 \rtimes \boldsymbol{\mu}_{12}$, $(\boldsymbol{\mu}_3)^2 \rtimes \mathrm{SL}_2(3)$ the group N must be a cyclic group of prime order and one can see immediately that N is not normal in G_P.

(iii) Let $N \subset G_P$ is a non-trivial normal subgroup. By (ii) we may assume that $N \cap A = \{1\}$. Thus $A \times N$ is a normal subgroup of G_P. Since the action of B on A is faithfull, this is impossible. □

Assume that G_P has a faithful representation V of degree ≤ 4. Take V so that its degree is minimal possible. If V is reducible, then $V = V_1 \oplus V_2$, where both V_i are non-trivial non-faithful representations. By Claim 3.9.1 kernels of these representations contain A. So V is not faithful, a contradiction.

Thus V is irreducible. Then the action of G on V is imprimitive (because A contains an abelian normal subgroup) and the induced action on eigenspaces $V_1, \dots, V_n$ of A induces a transitive embedding of B into $\mathfrak{S}_n$ with $n \leq 4$. But in our cases B is isomorphic to either $\mathrm{GL}_2(3)$, $\mathrm{SL}_3(2)$, $\mathrm{SL}_2(4)$, $\mathrm{SL}_2(3)$, or $B \simeq \boldsymbol{\mu}_l$, with $l \geq 5$. This group does not admit any embeddings into $\mathfrak{S}_4$, a contradiction. $\square$

4. Main reduction

4.1. Terminal singularities. Here we list only some of the necessary results on three-dimensional terminal singularities. For more complete information we refer to [Rei87]. Let (X, P) be a germ of a three-dimensional terminal singularity. Then (X, P) is isolated, i.e, $\mathrm{Sing}(X) = \{P\}$. The *index* of (X, P) is the minimal positive integer r such that rK_X is Cartier. If $r = 1$, then (X, P) is Gorenstein. In this case $\dim T_{P,X} = 4$, $\mathrm{mult}(X, P) = 2$, and (X, P) is analytically isomorphic to a hypersurface singularity in $\mathbb{C}^4$. If $r > 1$, then there is a cyclic, étale outside of P cover $\pi : (X^\sharp, P^\sharp) \to (X, P)$ of degree r such that $(X^\sharp, P^\sharp)$ is a Gorenstein terminal singularity (or a smooth point). This π is called the *index-one cover* of (X, P). If $(X^\sharp, P^\sharp)$ is smooth, then the point (X, P) is analytically isomorphic to a quotient $\mathbb{C}^3/\boldsymbol{\mu}_r$, where the weights (w_1, w_2, w_3) of the action of $\boldsymbol{\mu}_r$ up to permutations satisfy the relations $w_1 + w_2 \equiv 0 \mod r$ and $\gcd(w_i, r) = 1$. This point is called a *cyclic quotient* singularity.

For any three-dimensional terminal singularity (X, P) of index $r \geq 1$ there exists a one-parameter deformation $\mathfrak{X} \to \Delta \ni 0$ over a small disk $\Delta \subset \mathbb{C}$ such that the central fiber $\mathfrak{X}_0$ is isomorphic to X and the general fiber $\mathfrak{X}_\lambda$ has only cyclic quotient terminal singularities $P_{\lambda,k}$. Thus, one can associate with a fixed threefold X with terminal singularities a collection $\mathbf{B} = \{(\mathfrak{X}_\lambda, P_{\lambda,k})\}$ of cyclic quotient singularities. This collection is uniquely determined by the variety X and is called the *basket* of singularities of X.

If (X, P) is a singularity of index one, then it is an isolated hypersurface singularity. Hence $X \setminus \{P\}$ is simply-connected and the (local) Weil divisor class group $\mathrm{Cl}(X)$ is torsion free. If (X, P) is of index $r > 1$, then the index one cover induces the topological universal cover $X^\sharp \setminus \{P^\sharp\} \to X \setminus \{P\}$.

4.2. G-equivariant minimal model program. Let X be a rationally connected three-dimensional algebraic variety and let $G \subset \mathrm{Bir}(X)$ be a finite subgroup. By shrinking X we may assume that G acts on X biregularly. The quotient $Y = X/G$ is quasiprojective, so there exists a projective completion $\hat{Y} \supset Y$. Let $\hat{X}$ be the normalization of $\hat{Y}$ in the function field

$\mathbb{C}(X)$. Then $\hat{X}$ is a projective variety birational to X admitting a biregular action of G. There is an equivariant resolution of singularities $\tilde{X} \to \hat{X}$, see [AW97]. Run the G-equivariant minimal model program: $\tilde{X} \to \bar{X}$, see [Mor88, 0.3.14]. Running this program we stay in the category of projective normal varieties with at worst terminal $G\mathbb{Q}$-factorial singularities. Since X is rationally connected, on the final step we get a Fano-Mori fibration $f : \bar{X} \to Z$. Here $\dim Z < \dim X$, Z is normal, f has connected fibers, the anticanonical Weil divisor $-K_{\bar{X}}$ is ample over Z, and the relative G-invariant Picard number $\rho(\bar{X})^G$ is one. Obviously, we have the following possibilities:

(i) Z is a rational surface and a general fiber $F = f^{-1}(y)$ is a conic;
(ii) $Z \simeq \mathbb{P}^1$ and a general fiber $F = f^{-1}(y)$ is a smooth del Pezzo surface;
(iii) Z is a point and $\bar{X}$ is a $G\mathbb{Q}$-Fano threefold.

Now we assume that G is a simple group. If Z is not a point, then G non-trivially acts either on the base Z or on a general fiber. Both of them are rational varieties. Hence $G \subset \mathrm{Cr}_2(\mathbb{C})$ in this case. Thus we may assume that we are in the case (iii). Replacing X with $\bar{X}$ we may assume that our original X is a $G\mathbb{Q}$-Fano threefold.

In some statements below this assumption will be weakened. For example we will assume sometimes that $-K_X$ is just nef and big (not ample). We need this for some technical reasons (see §6).

The following is an easy consequence of the Kawamata-Viehweg vanishing theorem (see, e.g., [IP99, Prop. 2.1.2]).

Lemma 4.3. *Let X be a variety with at worst (log) terminal singularities such that $-K_X$ is nef and big. Then $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$ is torsion free. Moreover, the numerical equivalence of Cartier divisors on X coincides with the linear one.*

Corollary 4.4. *Let X be a threefold with at worst Gorenstein terminal singularities such that $-K_X$ is nef and big. Then the Weil divisor class group $\mathrm{Cl}(X)$ is torsion free.*

Lemma 4.5. *Let X be a threefold with at worst terminal singularities and let $G \subset \mathrm{Aut}(X)$ be a finite simple group. If there is a G-fixed point P on X, then G is isomorphic to a subgroup of $\mathrm{Cr}_2(\mathbb{C})$.*

Proof. If $P \in X$ is Gorenstein, we consider the natural representation of G in the Zariski tangent space $T_{P,X}$. First of all note that this representation is faithful. Recall also that $P \in X$ is an isolated hypersurface singularity so the dimension of its tangent space is at most 4. Therefore, $G \subset \mathrm{GL}(T_{P,X})$, where $\dim T_{P,X} = 3$ or 4. Then by Theorems 3.2 and 3.3 the group G

is isomorphic to either $\mathfrak{A}_5$, $\mathfrak{A}_6$ or $\mathrm{PSL}_2(7)$. In these cases G admit an embedding into $\mathrm{Cr}_2(\mathbb{C})$ (see Theorem 1.1).

Assume that $P \in X$ is non-Gorenstein of index $r > 1$. Take a small G-invariant neighborhood $P \ni U \subset X$ and consider the index-one cover $\pi\colon (U^\sharp, P^\sharp) \to (U, P)$ (see §4.1). Here $(U, P) = (U^\sharp, P^\sharp)/\boldsymbol{\mu}_r$, $(U^\sharp, P^\sharp)$ is a Gorenstein terminal point, and $U^\sharp \setminus \{P^\sharp\} \to U \setminus \{P\}$ is the topological universal cover. Let $\tilde{G} \subset \mathrm{Aut}(U^\sharp, P^\sharp)$ be the natural lifting of G. There is the following exact sequence

$$1 \longrightarrow \boldsymbol{\mu}_r \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.$$

Since G is a simple group, the above sequence is a central extension. If the representation of G in $T_{P^\sharp,U^\sharp}$ has a non-trivial irreducible subrepresentation $T \subset T_{P^\sharp,U^\sharp}$, then we can apply Theorem 3.2 to the action on T. Thus assume that the representation of $\tilde{G}$ in $T_{P^\sharp,U^\sharp}$ is irreducible. Then $\boldsymbol{\mu}_r$ must act on $T_{P^\sharp,U^\sharp}$ by scalar multiplications. On the other hand, if $P \in X$ is not a cyclic quotient singularity, then, according to the classification of terminal singularities [Rei87, Th. 6.1], the action of $\boldsymbol{\mu}_r$ on $T_{P^\sharp,U^\sharp}$ in not free along a line. Hence, $P \in X$ must be a cyclic quotient singularity. In this case again according to [Rei87, Th. 5.2] $\boldsymbol{\mu}_r$ acts on $T_{P^\sharp,U^\sharp}$ with weights (w_1, w_2, w_3), where $(w_i, r) = 1$ and $w_1 + w_2 \equiv 0 \mod r$ (up to permutation of coordinates). This is possible only if $r = 2$ and $\dim T_{P^\sharp,U^\sharp} = 3$. Then we can apply Theorem 3.2 again. $\square$

Corollary 4.6. *Let X be a threefold with at worst Gorenstein terminal singularities such that $-K_X$ is nef and big and let $G \subset \mathrm{Aut}(X)$ be a finite simple group which does not admit an embedding into $\mathrm{Cr}_2(\mathbb{C})$. Then any G-orbit on X contains at least 7 elements.*

Proof. Follows by Lemma 4.5 and Theorem 3.7. $\square$

Lemma 4.7. *Let X be a G-threefold with at worst terminal singularities where G is a finite simple group which does not admit an embedding into $\mathrm{Cr}_2(\mathbb{C})$. Assume that that $-K_X$ is nef and big. Let S be a G-invariant effective integral Weil $\mathbb{Q}$-Cartier divisor numerically proportional to $-K_X$. Then $K_X + S$ is nef. Furthermore, if $K_X + S \sim 0$, then the pair (X, S) is LC (log canonical, see e.g.* [Kol92, ch. 2]) *and the surface S is reducible. If moreover X is $G\mathbb{Q}$-factorial, then the group G transitively acts on components of S.*

Proof. Assume that the divisor $-(K_X + S)$ is nef. Clearly, S is nef and big. We apply quite standard connectedness arguments of Shokurov [Sho93]:

Claim 4.7.1 (cf. [MP09, Prop. 2.6])**.** *If either $-(K_X + S)$ is big or the pair (X, S) is not LC, then for a suitable G-invariant boundary D, the pair (X, D) is LC, the divisor $-(K_X + D)$ is nef and big, and the minimal locus V of log canonical singularities of (X, D) is non-empty and G-invariant.*

Proof of the claim. Take $c \in \mathbb{Q}$ so that (X, cS) is maximally LC. Then $c \leq 1$ and $-(K_X + cS)$ is nef and big. If the pair (C, cS) is PLT (purely log terminal, see e.g. [Kol92, ch. 2]), then we can take $D = cS$ and $V = \lfloor cS \rfloor$. Thus we may assume that there is a center of log canonical singularities W for (X, cS) of dimension ≤ 1. Let A' be an invariant very ample divisor on X. Now take an element $F_1 \in |-n(K_X + cS) - A'|$, $n \gg 0$. We may assume that F_1 contains W. Let $F_1, \dots, F_m$ be the G-orbit. Then $F := \sum F_i$ is a G-invariant divisor contained in $|-nm(K_X + cS) - mA'|$. Thus there is a G-invariant decomposition $-(K_X + cS) \equiv A + E$, where $A := \frac{1}{n}A'$ is ample, $E := \frac{1}{nm}F$ is effective, and $W \subset S \cap \operatorname{Supp}(E)$. Put $D_{\epsilon,\delta} := (c-\epsilon)S + \delta E$. The divisor $-(K_X + D_{\epsilon,\delta}) \equiv \epsilon S - (1-\delta)(K_X + cS) + \delta A$ is ample for all $0 < \delta \leq 1$, $\epsilon \geq 0$. Fix some $0 < \delta \ll 1$ and then take ϵ so that the pair $(X, D_{\epsilon,\delta})$ is maximally LC. Let V be a minimal center of log canonical singularities for $(X, D_{\epsilon,\delta})$. Take a general very ample divisor H_1 containing V. Let $H_1, \dots, H_r$ be the G-orbit. Fix some $0 < \lambda \ll 1$ and then take γ so that the pair $(X, \lambda \sum H_i + (1-\gamma)D_{\epsilon,\delta})$ is maximally LC. Put $D := \lambda \sum H_i + (1-\gamma)D_{\epsilon,\delta}$. It is easy to see that $-(K_X + D)$ is ample, V is a minimal center of log canonical singularities for (X, D), and V does not meet other centers of log canonical singularities. Finally, by Shokurov's connectedness principle [Sho93], [Kol92, ch. 17] the whole locus of log canonical singularities (X, D) is connected. Hence it coincides with V. Thus V is G-invariant. □

Proof of Lemma 4.7 (*continued*). Assume either $-(K_X + S)$ is big or the pair (X, S) is not LC. By Lemma 4.5 we may assume that G has no fixed points. Hence, in the above claim, $\dim V \geq 1$. Then $G \subset \operatorname{Aut}(V) = \operatorname{Aut}(\mathbb{P}^1)$. If $\dim V = 1$, then V is a smooth rational curve [Kaw97], so $G \subset \operatorname{Aut}(\mathbb{P}^1)$, a contradiction. Thus V is an irreducible surface. Then by the Inversion of Adjunction [Sho93], [Kol92, Th. 17.6] the surface V is normal, has only log terminal singularities and $(K_X + D)|_V = K_V + D_V$, where D_V is an effective Weil divisor on V such that the pair (V, D_V) is Kawamata log terminal (so-called *different*, see [Sho93, §3], [Kol92, ch. 16]). This implies that (V, D_V) is a weak log del Pezzo surface, so V is rational (see e.g. [IP99]). Therefore, $G \subset \operatorname{Aut}(V) \subset \operatorname{Cr}_2(\mathbb{C})$. Again we get a contradiction.

Thus we may assume that the pair (X, S) is LC and $K_X + S \sim 0$. If the pair (X, S) is PLT, then, as above, by the Inversion of Adjunction the surface S is normal and has only Du Val singularities. Moreover, $K_S \sim 0$ and $H^1(S, \mathscr{O}_S) = 0$. Let $\tilde{S} \to S$ be the minimal resolution. Then $\tilde{S}$ is a smooth K3 surface and G naturally acts on $\tilde{S}$. Recall that an automorphism φ of a K3 surface V is *symplectic* if φ acts trivially on $H^0(V, K_V) \simeq \mathbb{C}$. Since G is a simple group, the action of G on $\tilde{S}$ is symplectic. According to [Muk88b] the group G is isomorphic to one of the following: $\mathfrak{A}_5$, $\mathfrak{A}_6$, $\operatorname{PSL}_2(7)$, so G can be embedded to $\operatorname{Cr}_2(\mathbb{C})$.

Therefore, the pair (X, S) is LC but not PLT. Assume that S is irreducible and let $\nu\colon S' \to S$ be the normalization. Recall that G acts on S faithfully by Lemma 4.5. If S is rational, then we are in cases (1.2) because a faithfull action of a group on a rational surface gives an embedding of this group to the Cremona group of rank 2. So we assume that S is not rational. Write $0 \sim \nu^*(K_X + S)|_S = K_{S'} + D'$, where D' is the different, an effective integral Weil divisor on S' such that the pair (S', D') is LC (see [Sho93, §3], [Kol92, ch. 16], [Kaw07]). The group G acts naturally on S' and ν is G-equivariant. Now consider the minimal resolution $\mu\colon \tilde{S} \to S'$ and let $\tilde{D}$ be the (uniquely defined) $\mathbb{Q}$-divisor such that

$$K_{\tilde{S}} + \tilde{D} = \mu^*(K_{S'} + D') \sim 0, \qquad \mu_*\tilde{D} = D'.$$

Thus $\tilde{D}$ is usually called *log crepant pull-back* of D'. Here $\tilde{D}$ is again an effective reduced divisor. Hence $\tilde{S}$ is a ruled non-rational surface. Consider the Albanese map $\alpha : \tilde{S} \to C$. Clearly α is G-equivariant and the action of G on C is not trivial (otherwise G non-trivially acts on a general fiber which is a rational curve). The curve C cannot be elliptic because otherwise G is contained into $\operatorname{Aut}(C)$ which is a semi-direct product of the (abelian) group of translations and a group of order ≤ 6. Hence, $g(C) > 1$. Let $\tilde{D}_1 \subset \tilde{D}$ be a α-horizontal component. Since the surface is smooth, by the genus formula $p_a(\tilde{D}_1) \leq 1$. So, $\tilde{D}_1$ is either a rational or elliptic curve. This contradicts, $g(C) > 1$.

Therefore the surface S is reducible. If the action on components $S_i \subset S$ is not transitive and X is $G\mathbb{Q}$-factorial, we have an invariant divisor $S' < S$ which should be $\mathbb{Q}$-Cartier. This contradicts the above considered cases. $\square$

Corollary 4.8. *Let X be a $G\mathbb{Q}$-factorial G-threefold with at worst terminal singularities where G is a finite simple group which does not admit an embedding into $\mathrm{Cr}_2(\mathbb{C})$. Assume that $-K_X$ is nef and big. Let $\mathscr{H}$ be a G-invariant linear system such that $\dim \mathscr{H} > 0$ and $-(K_X + \mathscr{H})$ is nef. Then $\mathscr{H}$ has no fixed components.*

Proof. Assume the converse $\mathscr{H} = F + \mathscr{M}$, where F is the fixed part and $\mathscr{M}$ is a linear system without fixed components. Then F is an invariant divisor. This contradicts Lemma 4.7. $\square$

Lemma 4.9. *Let X be a $G\mathbb{Q}$-factorial G-threefold with at worst terminal singularities where G is a finite simple group which does not admit an embedding into $\mathrm{Cr}_2(\mathbb{C})$. Assume that $-K_X$ is nef and big. Then $\dim H^0(X, -K_X)^G \leq 1$.*

Proof. Assume that there is a pencil $\mathscr{H}$ of invariant anticanonical sections. By Corollary 4.8 $\mathscr{H}$ has no fixed components. We claim that a general member of $\mathscr{H}$ is irreducible. Indeed, otherwise $\mathscr{H} = m\mathscr{L}$, $m > 1$ and the pencil $\mathscr{L}$ determines a G-equivariant rational map $X \dashrightarrow \mathbb{P}^1$ so that the action on $\mathbb{P}^1$ is trivial. Hence, the fibers are $\mathbb{Q}$-Cartier divisors and $-K_X \sim m\mathscr{L}$. This contradicts Lemma 4.7 applied to $S \in \mathscr{L}$. So, a general member $H \in \mathscr{H}$ is irreducible and G-invariant. Again we get a contradiction by Lemma 4.7. □

5. Case: X is Gorenstein

Assumption 5.1. In this section X denotes a threefold with at worst terminal Gorenstein singularities such that the anticanonical divisor $-K_X$ is nef and big. Let $G \subset \mathrm{Aut}(X)$ be a finite simple group which does not admit any embeddings into $\mathrm{Cr}_2(\mathbb{C})$. Write $-K_X^3 = 2g - 2$ for some g. This g is called the *genus* of a Fano threefold. By Kawamata-Viehweg vanishing and Riemann-Roch we have $\dim |-K_X| = g + 1$. In particular, g is an integer.

Lemma 5.2. *The linear system $|-K_X|$ is base point free.*

Proof. Assume that $\mathrm{Bs}\,|-K_X| \neq \varnothing$. If $\dim \mathrm{Bs}\,|-K_X| > 0$, then by [Shi89] $\mathrm{Bs}\,|-K_X|$ a smooth rational curve contained into the smooth locus of X. By Lemma 4.5 the action of G on this curve is non-trivial. Hence $G \subset \mathrm{Aut}(\mathbb{P}^1)$ and so $G \simeq \mathfrak{A}_5$. This contradicts Assumption 5.1. Thus $\dim \mathrm{Bs}\,|-K_X| = 0$. Again by [Shi89] $\mathrm{Bs}\,|-K_X|$ is a single point. This is impossible by Lemma 4.5. □

Lemma 5.3. *The linear system $|-K_X|$ determines a birational morphism $X \to \mathbb{P}^{g+1}$ whose image is a Fano threefold $\bar{X}_{2g-2} \subset \mathbb{P}^{g+1}$ with at worst canonical Gorenstein singularities. In particular, $g \geq 3$.*

Proof. Assume that the linear system $|-K_X|$ determines a morphism $\varphi\colon X \to \mathbb{P}^{g+1}$ and φ is not an embedding. Let $Y = \varphi(X)$. Then φ is a generically double cover and $Y \subset \mathbb{P}^{g+1}$ is a subvariety of degree $g - 1$ [Isk80], [Isk77], [PCS05]. Note that the action of G on X induces a non-trivial (hence faithful) action of G on $\varphi(X)$ since the map $\varphi : X \to \varphi(X)$ is given by $|-K_X|$.

If Y is a projective cone, then its vertex is either a point or $\mathbb{P}^1$. Since G is not embeddable to $\mathrm{Cr}_2(\mathbb{C})$, we get a contradiction by Corollary 4.6.

Thus we assume that Y is not a cone. According to the Enriques theorem the variety $Y \subset \mathbb{P}^{g+1}$ is one of the following (see, e.g., [Isk77, Lemma 2.8], [Isk80, Th. 3.11]):

(i) $\mathbb{P}^3$;
(ii) a smooth quadric in $\mathbb{P}^4$;
(iii) a rational scroll $\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})$, where $\mathscr{E}$ is a rank 3 vector bundle on $\mathbb{P}^1$.

In the first case $\varphi : X \to \mathbb{P}^3$ is a generically double cover with branch divisor $B \subset \mathbb{P}^3$ of degree 6. By Theorem 3.3 $G \simeq \mathfrak{A}_7$ or $\mathrm{PSp}_4(3)$. However both these groups have no non-trivial representations of degree 4, a contradiction.

The second case does not occur by Lemma 3.6. In the last case $\rho(Y) = 2$. Hence G acts trivially on $\mathrm{Pic}(Y)$ and so the projection $Y \to \mathbb{P}^1$ is G-equivariant. We get an embedding of G into $\mathrm{Aut}(\mathbb{P}^1)$ or $\mathrm{Aut}(F)$, where $F \simeq \mathbb{P}^2$ is a fiber. $\square$

Lemma 5.4. *In notation of Lemma* 5.3 *one of the following holds:*

(i) *the variety* $\bar{X} = \bar{X}_{2g-2} \subset \mathbb{P}^{g+1}$ *is an intersection of quadrics* (*in particular,* $g \geq 5$)*;*

(ii) $g = 3$*,* $\bar{X} = \bar{X}_4 \subset \mathbb{P}^4$ *is quartic, and* $G \simeq \mathrm{PSp}_4(3)$ (*see Example* 2.5)*;*

(iii) $g = 4$*,* $\bar{X} = \bar{X}_6 \subset \mathbb{P}^5$ *is an intersection of a quadric and a cubic, and* $G \simeq \mathfrak{A}_7$ (*see Example* 2.5)*.*

Proof. Assume that the linear system $|-K_X|$ determines a birational morphism but its image $\bar{X} = \bar{X}_{2g-2}$ is not an intersection of quadrics. Let $Y \subset \mathbb{P}^{g+1}$ be the variety that cut out by quadrics through $\bar{X}$. Then Y is a four-dimensional irreducible subvariety in $\mathbb{P}^{g+1}$ of minimal degree [Isk80], [PCS05]. As in the proof of Lemma 5.3 we can use the Enriques theorem. Assume that Y is a cone with vertex L over S. Since G is not contained in the list (1.2), L is a point and S is a three-dimensional variety of minimal degree (and $S \not\simeq \mathbb{P}^3$). We get a contradiction as in the proof of Lemma 5.3. Hence Y is smooth and we have the following possibilities:

(i) $Y \simeq \mathbb{P}^4$;

(ii) $Y \subset \mathbb{P}^5$ is a smooth quadric;

(iii) a rational scroll $\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})$, where $\mathscr{E}$ is a rank 4 vector bundle on $\mathbb{P}^1$.

In the first case $g = 3$ and $\bar{X} = \bar{X}_4 \subset \mathbb{P}^4$ is a quartic. Consider the representation of G in $H^0(\bar{X}, -K_{\bar{X}}) \simeq \mathbb{C}^5$. If this representation is reducible, then by our assumptions $\bar{X}$ has an invariant hyperplane section $S \in |-K_{\bar{X}}|$. Since $\deg S = 4$, this S must be irreducible (otherwise S has a G-invariant rational component). By Lemma 4.7 this is impossible. Then by Theorem 3.4 and Assumption 5.1 we have the case (ii) of the lemma or the group G is isomorphic to $\mathrm{PSL}_2(11)$. On the other hand, the group $\mathrm{PSL}_2(11)$ has no invariant quartics (see [AR96, §29]), a contradiction.

In the second case $\bar{X} = \bar{X}_6 \subset \mathbb{P}^5$ is an intersection of a quadric and a cubic. By Lemma 3.6 we obtain either $G \simeq \mathfrak{A}_7$ or $\mathrm{PSp}_4(3)$. The second possibility is does not occur because the action of $\mathrm{PSp}_4(3)$ on $\mathbb{C}^6$ has no invariants of degree 3. (In fact, $\mathrm{PSp}_4(3)$ can be embedded into a group of order 51840 generated by reflections, see [ST54, Table VII, No. 35]). Thus $G \simeq \mathfrak{A}_7$. We get a situation of Example 2.5 because the group $\mathfrak{A}_7$ has only one irreducible representation of degree 6.

In the last case, as in Lemma 5.3, we have a G-equivariant contraction $Y \to \mathbb{P}^1$ whose fibers are isomorphic to $\mathbb{P}^3$. The restriction map $X \to \mathbb{P}^1$ is a fibration whose general fiber F is a surface with big and nef anticanonical divisor. Such a surface must be rational. Hence either $G \subset \mathrm{Aut}(\mathbb{P}^1)$ or $G \subset \mathrm{Aut}(F)$. $\square$

Corollary 5.5. *In case* (i) *of Lemma* 5.4 *the variety* $\bar{X} = \bar{X}_{2g-2} \subset \mathbb{P}^{g+1}$ *is an intersection of* $(g-2)(g-3)/2$ *quadrics.*

Proof. Let $S \subset \mathbb{P}^g$ be a general hyperplane section of $\bar{X}$ and let $C \subset \mathbb{P}^{g-1}$ be a general hyperplane section of S. Then S is a smooth K3 surface and C is a canonical curve of genus g. Let $\mathscr{I}_{\bar{X}}$ (resp. $\mathscr{I}_S$, $\mathscr{I}_C$) be the ideal sheaf of $\bar{X} \subset \mathbb{P}^{g+1}$ (resp. $S \subset \mathbb{P}^g$, $C \subset \mathbb{P}^{g-1}$). The space $H^0(\mathscr{I}_{\bar{X}}(2))$ is the space of quadrics in $H^0(\bar{X}, -K_{\bar{X}})$ passing through $\bar{X}$. The standard cohomological arguments (see, e.g., [Isk77], [Isk80, Lemma 3.4]) show that $H^0(\mathscr{I}_{\bar{X}}(2)) \simeq H^0(\mathscr{I}_S(2)) \simeq H^0(\mathscr{I}_C(2))$. This gives us

$$\dim H^0(\bar{X}, \mathscr{I}_{\bar{X}}(2)) = \frac{1}{2}(g-2)(g-3).$$

$\square$

Theorem 5.6 ([Nam97])**.** *Let X be a Fano threefold with terminal Gorenstein singularities. Then X is smoothable, that is, there is a flat family X_t such that $X_0 \simeq X$ and a general member X_t is a smooth Fano threefold. Further, the number of singular points is bounded as follows:*

$$|\operatorname{Sing}(X)| \le 21 - \frac{1}{2}\operatorname{Eu}(X_t) = 20 - \rho(X_t) + h^{1,2}(X_t). \tag{5.7}$$

where $\operatorname{Eu}(X)$ *is the topological Euler number and* $h^{1,2}(X)$ *is the Hodge number.*

Remark 5.8. (i) In the above notation the total family $\mathfrak{X}$ has at worst isolated terminal factorial singularities and there are natural identifications $\mathrm{Pic}(X) \simeq \mathrm{Pic}(\mathfrak{X}) \simeq \mathrm{Pic}(X_t)$ (see [JR06, §1]). In particular, $\rho(X_t) = \rho(X)$, $-K_{X_t}^3 = -K_X^3$, and varieties X and X_t have the same Fano index.

(ii) The estimate (5.7) is very far from being sharp. For example, for cubic hypersurface $X \subset \mathbb{P}^4$ (5.7) gives us $|\operatorname{Sing}(X)| \le 24$ but the sharp bound is $|\operatorname{Sing}(X)| \le 10$ and achevied for the Segre cubic. However, for our purposes, (5.7) is sufficient.

Theorem 5.9 (see, e.g., [Isk80], [IP99])**.** *Let X be a smooth Fano threefold with* $\mathrm{Pic}(X) = \mathbb{Z}\cdot(-K_X)$*. Then the possible values of its genus g and Hodge numbers $h^{1,2}(X)$ are given by the following table:*

g	2	3	4	5	6	7	8	9	10	12
$h^{1,2}(X)$	52	30	20	14	10	7	5	3	2	0

Assumption 5.10. From now on and till the end of this section additionally to 5.1 we assume that $-K_X$ is ample, X is $G\mathbb{Q}$-factorial, and $\rho(X)^G = 1$, i.e., X is a Gorenstein $G\mathbb{Q}$-Fano threefold. Moreover, the anticanonical linear system determines an embedding $X = X_{2g-2} \subset \mathbb{P}^{g+1}$ and its image is an intersection of $(g-2)(g-3)/2$ quadrics.

Lemma 5.11. *Under the assumptions of* 5.10 *we have* $\rho(X) = 1$.

Proof. Assume that $\rho(X) > 1$. We have a natural action of G on $\operatorname{Pic}(X) \simeq \mathbb{Z}^\rho$ such that $\operatorname{Pic}(X)^G \simeq \mathbb{Z}$. In particular, there is a non-trivial representation $V \subsetneq \operatorname{Pic}(X) \otimes \mathbb{R}$. Hence G admits an embedding into $\operatorname{PSO}_{\rho-1}(\mathbb{R})$. By Lemma 3.6 we have $\rho(X) \geq 7$. Consider a smoothing X_t of X. Here X_t is a smooth Fano threefold with $\rho(X_t) = \rho(X)$ and $-K_{X_t}^3 = -K_X^3$ (see Remark 5.8, (i)). From the classification of smooth Fano threefolds with $\rho > 1$ [MM82] one can see that $X_t \simeq S \times \mathbb{P}^1$, where S is a del Pezzo surface.

Again by Remark 5.8, (i) there is natural identification $\operatorname{Pic}(X) \simeq \operatorname{Pic}(X_t)$ that preserves the intersection form. So we assume that G acts on $\operatorname{Pic}(X_t)$ (but not on X_t). Let F be a fiber of the projection $X_t = S \times \mathbb{P}^1 \to \mathbb{P}^1$. Take an element $\tau \in G$ sending F to F' that is not proportional to F. Then $F' \sim \alpha F + f^*L$ for some $0 \neq L \in \operatorname{Pic}(S)$ and $\alpha \in \mathbb{Z}$. Since $F^2 \equiv 0$, we have

$$0 = F'^2 \cdot F = f^*L^2 \cdot F.$$

Hence, $L^2 = 0$ and $2\alpha F \cdot f^*L \equiv F'^2 \equiv 0$. So, $\alpha = 0$ and $F' = f^*L$. Further, by Riemann-Roch $K_S \cdot L$ is even and

$$K_S^2 = K_{X_t}^2 \cdot F = K_{X_t}^2 \cdot F' = K_{X_t}^2 \cdot f^*L = (2F - f^*K_S)^2 \cdot f^*L = -4K_S \cdot L.$$

Therefore, $K_S^2 = 8$ and $\rho(X) = \rho(X_t) = 3$, a contradiction. □

Recall that the Fano index of a Gorenstein Fano variety X is the maximal positive integer dividing the class of $-K_X$ in $\operatorname{Pic}(X)$.

Lemma 5.12. *Under the assumptions of* 5.10 *we have either*

(i) *the Fano index of* X *is one, or*

(ii) $G \simeq \operatorname{PSL}_2(11)$ *and* $X_3^{\mathrm{k}} \subset \mathbb{P}^4$ *is the Klein cubic* (*see Example* 2.6).

Proof. Let q be the Fano index of X. Write $-K_X = qH$, where H is an ample Cartier divisor. Clearly, the class of H is G-stable. Assume that $q > 1$. If $q > 2$, then X is either $\mathbb{P}^3$ or a quadric in $\mathbb{P}^4$. Thus we may assume that $q = 2$. Below we use some facts on Gorenstein Fano threefolds of Fano index 2 with at worst canonical singularities, see [Isk80], [Shi89]. Denote $d = H^3$.

As in the proof of Lemma 5.11 there is a flat family X_t such that $X_0 \simeq X$ and a general member X_t is a smooth Fano threefold with the same Picard number, anticanonical degree, and Fano index. Since $\rho(X) = 1$, by the classification of smooth Fano threefolds [Isk80], [IP99] $d \leq 5$.

If $d = 1$, then $\mathrm{Bs}\,|H|$ is a single point contained into the smooth part of X. This point must be G-invariant. This contradicts Lemma 4.5. If $d = 2$, then the linear system $|H|$ determines a G-equivariant double cover $X \to \mathbb{P}^3$ with branch divisor $B = B_4 \subset \mathbb{P}^3$ of degree 4. Clearly, B has only isolated singularities. If B has at worst Du Val singularities, then according to [Muk88b] the group G is isomorphic to one of the following: $\mathfrak{A}_5$, $\mathfrak{A}_6$, $\mathrm{PSL}_2(7)$, so G can be embedded to $\mathrm{Cr}_2(\mathbb{C})$, a contradiction. Hence B is not Du Val. The non-Du Val locus of B coincides with the locus of log canonical singularities $\mathrm{LCS}(\mathbb{P}^3, B)$ of the pair $(\mathbb{P}^3, B)$. By a generalization of Shokurov's connectedness principle [Sho93, Th. 6.9] the set $\mathrm{LCS}(\mathbb{P}^3, B)$ is either connected or has two connected components. Then G has a fixed point on B and on X. This contradicts Lemma 4.5.

For $d > 2$, the linear system $|H|$ is very ample and determines a G-equivariant embedding $X \hookrightarrow \mathbb{P}^{d+1}$. Therefore, $G \subset \mathrm{PGL}_{d+2}(\mathbb{C})$. Take a lifting $\tilde{G} \subset \mathrm{GL}_{d+2}(\mathbb{C})$ so that $\tilde{G}/Z(\tilde{G}) \simeq G$ and $Z(\tilde{G}) \subset [\tilde{G}, \tilde{G}]$. We have a natural non-trivial representation of $\tilde{G}$ in $H^0(X, H)$, where $\dim H^0(X, H) = d + 2 \le 7$. We claim that this representation is irreducible. Indeed, assume that $H^0(X, H)$ is reducible as a $\tilde{G}$-module. By Lemma 4.7 the variety X has no invariant hyperplane sections, i.e., the representation of $\tilde{G}$ in $H^0(X, H)$ has no one-dimensional subrepresentations. Hence $H^0(X, H)$ has an irreducible subrepresentation V of dimension 2 or 3. In this case, $G \simeq \tilde{G}/Z(\tilde{G})$ acts faithfully on $\mathbb{P}(V) \subset \mathbb{P}(H^0(X, H))$. So, G admits an embedding to $\mathrm{Cr}_2(\mathbb{C})$. This contradicts our assumption 5.1 and proves the claim.

Consider the case $d = 3$. Assuming that G is not contained in $\mathrm{Cr}_2(\mathbb{C})$ by Theorem 3.4 we have either $G \simeq \mathrm{PSL}_2(11)$ or $G \simeq \mathrm{PSp}_4(3)$. In the first case, the only cubic invariant of this group is the Klein cubic (2.7), see [AR96, §29]. We get Example 2.6. The second case is impossible because the group $\mathrm{PSp}_4(3)$ has no invariants of degree 3, see [ST54].

Consider the case $d = 4$. Then $X = X_4 \subset \mathbb{P}^5$ is an intersection of two quadrics, say Q_1 and Q_2. The action of G on the pencil generated by Q_1, Q_2 must be trivial. Hence G acts on a degenerate quadric $Q' \in \langle Q_1, Q_2 \rangle$. In particular, G acts on the singular locus of Q' which is a linear subspace, a contradiction.

Consider the case $d = 5$. Then $X \subset \mathbb{P}^6$ is an intersection of 5 quadrics [Shi89]. Let $V = H^0(X, \mathscr{I}_X(2))$, where $\mathscr{I}_X$ be the ideal sheaf of X in $\mathbb{P}^6$. Then V is a 5-dimensional G-invariant subspace of $H^0(X, \mathscr{O}_X(2)) = H^0(X, -K_X)$. If the action of G on V is trivial, then, as above, there is a G-stable singular quadric $Q \subset \mathbb{P}^6$. But then the singular locus of Q is a G-stable linear subspace in $\mathbb{P}^6$, a contradiction. Thus $G \subset \mathrm{SL}_5(\mathbb{C})$. Assuming that G is not contained in the list (1.2) by Theorems 3.3 and 3.4 the group G is isomorphic to either $\mathrm{PSp}_4(3)$ or $\mathrm{PSL}_2(11)$. In both cases, the Schur multiplier of G is a group of order 2 and the covering group $\tilde{G}$ is isomorphic to $\mathrm{Sp}_4(3)$ and $\mathrm{SL}_2(11)$, respectively, see [CCN+85]. Since the order of $\mathrm{Sp}_4(3)$ and $\mathrm{SL}_2(11)$ is not divisible by 7, these groups have no irreducible representations of degree 7, a contradiction. $\square$

Assumption 5.13. Thus in what follows additionally to 5.1 and 5.10 we assume that the Fano index of X is one.

Lemma 5.14. *Under the assumptions of* 5.13 *we have* $H^0(X, -K_X)^G = 0$.

Proof. Assume that G has an invariant hyperplane section S. By Lemma 4.7 the pair (X, S) is LC, $S = \sum S_i$ and G acts transitively on $\Omega := \{S_i\}$. Let $m := |\Omega|$. Recall that $4 \le g \le 12$ and $g \ne 11$ by Theorem 5.9 and Lemma 5.11. We have $m \deg S_i = 2g - 2 \le 22$. Since $m \ge 7$, $\deg S_i \le 3$. The action of G on Ω induces a transitive embedding $G \subset \mathfrak{S}_m$.

If $\deg S_i = 2$, then $m = g - 1 \le 11$, $m \ne 10$. Recall that the natural representation of G in $H^0(X, -K_X) = \mathbb{C}^{g+2} = \mathbb{C}^{m+3}$ has no two-dimensional trivial subrepresentations. Taking this into account and using table in Theorem 3.7 we get only one case: $m = 7$, $g = 8$, $G \simeq \mathfrak{A}_7$, and the action of $\mathfrak{A}_7$ on $\{S_1, \dots, S_7\}$ is the standard one. Moreover, S_i is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a quadratic cone $\mathbb{P}(1,1,2)$. Therefore the stabilizer $G_{S_i} \simeq \mathfrak{A}_6$ acts trivially on S_i. The ample divisor $\sum S_i$ is connected. Hence, $S_i \cap S_j \ne \varnothing$ for some $i \ne j$. Then the stabilizer G_P of the point $P \in S_i \cap S_j$ contains the subgroup generated by G_{S_i} and G_{S_j}. So, $G_P = G$. This contradicts Lemma 4.5.

Hence $\deg S_i \ne 2$. Then $\deg S_i$ is odd, m is even, and $m \ge 8$. This implies that $\deg S_i = 1$, i.e., S_i is a plane. Moreover, $m = 2g - 2 \le 22$, $m \ne 20$. As above, using the fact that the representation of G in $H^0(X, -K_X) = \mathbb{C}^{m/2+3}$ has no two-dimensional trivial subrepresentations and Theorem 3.7 we get only one case: $m = 8$, $g = 5$, and $G \simeq \mathfrak{A}_8$. Similar to the previous case we derive a contradiction. The lemma is proved. $\square$

Corollary 5.15. *If in the assumptions of* 5.13 $g \le 7$, *then the representation of* G *in* $H^0(X, -K_X)$ *is irreducible.*

Proof. Follows from Theorem 3.3 and Lemma 5.14. $\square$

Lemma 5.16. *Under the assumptions of* 5.13 *we have* $g \ge 7$.

Proof. Assume that $g = 5$. Then by Corollary 5.5 we have $\dim H^0(\mathscr{I}_X(2)) = 3$ and $X \subset \mathbb{P}^6$ is a complete intersection of three quadrics. The group G acts on $H^0(\mathscr{I}_X(2)) \simeq \mathbb{C}^3$ and we may assume that this action is trivial (otherwise G acts on $\mathbb{P}^2 = \mathbb{P}(H^0(\mathscr{I}_X(2))))$. Thus we have a net of invariant quadrics $\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3$. In particular, there is an invariant degenerate quadric $Q' \in \lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3$. By Lemma 3.6 Q' is a cone with zero-dimensional vertex P. Thus $P \in \mathbb{P}^7$ is an invariant point and there is an invariant hyperplane section, a contradiction.

Now assume that $g = 6$. Again by Corollary 5.5 we have $\dim H^0(\mathscr{I}_X(2)) = 6$. If the action of G on $\dim H^0(X, \mathscr{I}_X(2))^G > 1$, then G acts on a singular irreducible 6-dimensional quadric $Q \subset \mathbb{P}^7$. In particular, the singular locus of Q, a projective space L of dimension ≤ 4 must be G-invariant. This contradicts the irreducibility of $H^0(X, -K_X)$. Therefore, $\dim H^0(X, \mathscr{I}_X(2))^G \leq 1$. In particular, G acts on $H^0(X, \mathscr{I}_X(2)) \simeq \mathbb{C}^6$ non-trivially and so G has an irreducible representation of degree 5 or 6. Since G is simple and because we assume that G is not contained in the list (1.2) by the classification theorems 3.4 and 3.5 we have only four possibilities: $G \simeq \mathfrak{A}_7$, $\mathrm{PSp}_4(3)$, $\mathrm{PSL}_2(11)$, or $\mathrm{SU}_3(3)$. But in all cases G has no irreducible representations of degree 8 (see [CCN+85]), a contradiction. □

Lemma 5.17. *Under the assumptions of* 5.13 *the variety X is smooth.*

Proof. Assume that X is singular. Let $\Omega \subset \mathrm{Sing}(X)$ be a G-orbit and let $n := |\Omega|$. Let $x_1, \dots, x_n \in H^0(X, -K_X)^*$ be the vectors corresponding to the points of Ω. By (5.7) we have $n \leq 26$. Let $P \in \mathrm{Sing}(X)$ and let G_P be the stabilizer of P. Then the natural representation of G_P in $T_{P,X}$ is faithful. On the other hand, by Corollary 3.9 the group G_P has a faithful representation of degree ≤ 4 only in the following cases:

(i) $G \simeq \mathrm{PSL}_2(11)$, $|\Omega| = 11$, $G_P \simeq \mathfrak{A}_5$;
(ii) $G \simeq \mathfrak{A}_7$, $|\Omega| = 21$, $G_P \simeq \mathfrak{S}_5$;
(iii) $G \simeq \mathfrak{A}_7$, $|\Omega| = 15$, $G_P \simeq \mathrm{PSL}_2(7)$.

Locally near P the singularity $X \ni P$ is given by a G_P-semi-invariant equation $\phi(x, \dots, t) = 0$. Write $\phi = \phi_2 + \phi_3 + \dots$, where ϕ_d is the homogeneous part of degree d. By the classification of terminal singularities, $\phi_2 \neq 0$. The last case $G_P \simeq \mathrm{PSL}_2(7)$ is impossible because, then the representation of G_P in $T_{P,X}$ is reducible: $T_{P,X} = T_1 \oplus T_3$, where T_3 is an irreducible representation of degree 3. Since the action of G_P on T_3 has no invariants of degree 2 and 3 (see [ST54]), we have $\phi_2 = \ell^2$ and $\phi_3 = \ell^3$, where ℓ is a linear form. But this contradicts the classification of terminal singularities [Rei87, Th. 6.1]. Therefore, $G_P \simeq \mathfrak{A}_5$ or $\mathfrak{S}_5$ and we are in cases (i) or (ii).

Claim 5.17.1. *If X is singular, then $g = 8$.*

Proof. The natural representation of G in $H^0(X, -K_X) \simeq \mathbb{C}^{g+2}$ has no trivial subrepresentations. Recall that $g = 7$, 8, 9, 10, or 12.

Consider the case $G \simeq \mathfrak{A}_7$. Then the degrees of irreducible representations in the interval $[2, 14]$ are 6, 10, 14 (see Theorem 3.7). Hence, $g = 8$, 10, or 12. On the other hand, X has at least 21 singular points (because we are in the case (ii) above). By (5.7) we have $h^{1,2}(X') \geq 2$. So, $g \neq 12$. Let χ be the character of G on $H^0(X, -K_X)^*$. We need the character table for

$G = \mathfrak{A}_7$ (see, e.g., [CCN+85]):

(5.17.2)

G	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3'$	$\mathcal{C}_6$	$\mathcal{C}_3''$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_7'$	$\mathcal{C}_7''$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	6	2	3	-1	0	0	1	-1	-1
χ_3	10	-2	1	1	1	0	0	α	$\bar{\alpha}$
χ_4	10	-2	1	1	1	0	0	$\bar{\alpha}$	α
...	...								

Here $\alpha = (-1+\sqrt{-7})/2$. (We omit characters of degree ≥ 14). Assume that $g = 10$. Since the representation of G in $H^0(X, -K_X)$ has no trivial subrepresentations, the only possibility is $H^0(X, -K_X)^* = W \oplus W'$, as G-module, where $W \simeq W'$ is a 6-dimensional representation (i.e. $\chi = \chi_2 \oplus \chi_2$). Thus $H^0(X, -K_X)^*$ contains a one-dimensional family W_λ of subrepresentations isomorphic to W. Replacing W with W_λ we can take the decomposition above so that the first copy W contains the vector x_1 (corresponding to $P_1 \in \Omega$). Then $P_1 \in \mathbb{P}^5 = \mathbb{P}(W)$ and obviously $\Omega \subset \mathbb{P}(W)$. Consider the set $S := \mathbb{P}(W) \cap X$, the base locus of the linear system of hyperplane sectons passing through $\mathbb{P}(W)$. By Corollary 4.8 $\dim S \leq 1$. Assume that $\dim S = 0$. Take a general hyperplane section H passing through $\mathbb{P}(W)$. By Bertini's theorem H is a normal surface with isolated singularities. Moreover, H is singular at points of Ω, so $|\operatorname{Sing}(H)| \geq |\Omega| = 21$. By the adjunction $K_H \sim 0$. Hence, by a generalization of Shokurov's connectedness principle [Sho93, Th. 6.9], H has at most two non-Du Val singularities. Since G has no fixed points on X, the surface H has only Du Val singularities. Therefore, the minimal resolution $\tilde{H}$ of H is a K3 surface and so $\rho(\tilde{H}) \leq \dim H^{1,1}(\tilde{H}) = 20$. On the other hand, $\rho(\tilde{H}) > |\operatorname{Sing}(H)| \geq 21$, a contradiction. Thus $\dim S = 1$. Let S' be the union of an orbit of a one-dimensional component. Since the representation W is irreducible, S' spans $\mathbb{P}(W)$. By Lemma 5.4 $S \subset \mathbb{P}^5$ is an intersection of quadrics. Since $S' \subset S$ and $\dim S = 1$, $\deg S' \leq 16$. If S' is reducible, then G interchanges its components S_i. In this case, $\deg S_i \leq 2$. By Theorem 3.7 the number of components is either 7 or 15. The stabilizer G_{S_i} ($\simeq \mathfrak{A}_6$ or $\mathrm{PSL}_2(7)$) acts on S_i which is a rational curve, a contradiction. Therefore, $S' \subset \mathbb{P}^5$ is an irreducible curve contained in S, an intersection of quadrics. Let $S'' \to S'$ be its normalization. By the Castelnuovo bound $g(S'') \leq 21$ (see e.g. [ACGH85, ch. 3, §2]). On the other hand, by the Hurwitz bound [ACGH85, ch. 1, §6, F] we have $|G| \leq \operatorname{Aut}(S'') \leq 84(g(S'')-1)$, a contradiction.

Now consider the case $G \simeq \mathrm{PSL}_2(11)$. As above, since the natural representation of G in $H^0(X, -K_X) \simeq \mathbb{C}^{g+2}$ has no trivial subrepresentations, we have $g = 8$, 9, or 10 (see Theorem 3.7). Moreover, if $g = 10$, then the representation of G in $H^0(X, -K_X)$ is irreducible. On the other hand, 11 points of the set $\Omega \subset \mathbb{P}(H^0(X, -K_X)^*) = \mathbb{P}^{11}$ generate and invariant subspace, a contradiction. If $g = 9$, then 11 points of the set $\Omega \subset \mathbb{P}(H^0(X, -K_X)^*)$ are in general position. Then the corresponding vectors $x_i \in H^0(X, -K_X)^*$ are linearly independent and the representation of G in $H^0(C, -K_X)$ is induced from the trivial representation of G_P in $\langle x_1 \rangle$. But in this case the G-invariant vector $\sum_{\delta \in G} \delta(x_1)$ is not zero, a contradiction. Thus $g = 8$. $\square$

Claim 5.17.3. *If X is singular, then $G \not\simeq \mathfrak{A}_7$.*

Proof. Assume that $G \simeq \mathfrak{A}_7$. Then $G_P \simeq \mathfrak{S}_5$. We compare the character tables for $\mathfrak{A}_7$ (see (5.17.2)) and for $\mathfrak{S}_5$:

(5.17.4)

G_P	$\mathcal{C}_1$	$\mathcal{C}_2'$	$\mathcal{C}_2''$	$\mathcal{C}_3$	$\mathcal{C}_6$	$\mathcal{C}_4$	$\mathcal{C}_5$
χ_1'	1	-1	1	1	-1	-1	1
χ_2'	4	-2	0	1	1	0	-1
χ_3'	5	-1	1	-1	-1	1	0
χ_4'	6	0	-2	0	0	0	1
χ_5'	5	1	1	-1	1	-1	0
χ_6'	4	2	0	1	-1	0	-1
χ_7'	1	1	1	1	1	1	1

Let χ be the character of the representation of G in $H^0(X, -K_X)^*$. By Lemma 5.14 and (5.17.2) χ is irreducible and either $\chi = \chi_3$ or $\chi = \chi_4$ (recall that χ_3 and χ_4 are characters of $\mathfrak{A}_7$). Using (5.17.2), in notations of (5.17.4), for the restriction $\chi|_{\mathfrak{S}_5} = \chi_3|_{\mathfrak{S}_5} = \chi_4|_{\mathfrak{S}_5}$ we obtain

$$\chi|_{\mathfrak{S}_5}(\mathcal{C}_1, \mathcal{C}_2', \mathcal{C}_2'', \mathcal{C}_3, \mathcal{C}_6, \mathcal{C}_4, \mathcal{C}_5) = (10, -2, -2, 1, 1, 0, 0).$$

Hence, $\chi|_{\mathfrak{S}_5} = \chi_2' \oplus \chi_4'$. In particular, the representation of $G_P \simeq \mathfrak{S}_5$ in $H^0(X, -K_X)^*$ has no trivial subrepresentations, a contradiction. $\square$

Thus we may assume that $G \simeq \mathrm{PSL}_2(11)$ and $G_P \simeq \mathfrak{A}_5$.

Claim 5.17.5. *If X is singular, then the natural representation of G_P in $T_{P,X}$ is irreducible and $P \in X$ is an ordinary double point, that is,* $\operatorname{rk} \phi_2 = 4$.

Proof. Let $x \in H^0(X, -K_X)^*$ be a vector corresponding to P. There is a G_P-equivariant embedding $T_{P,X} \hookrightarrow H^0(X, -K_X)^*$ so that $x \notin T_{P,X}$. Thus $H^0(X, -K_X)^*$ has a trivial G_P-representation $\langle x \rangle$ which is not contained in $T_{P,X}$. Let χ be the character of G on $H^0(X, -K_X)^*$. We need character tables for $G = \mathrm{PSL}_2(11)$ and $G_P = \mathfrak{A}_5$ (see, e.g., [CCN+85]):

G	$\mathcal{C}_1$	$\mathcal{C}_5'$	$\mathcal{C}_5''$	$\mathcal{C}_{11}'$	$\mathcal{C}_{11}''$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_6$
χ_1	1	1	1	1	1	1	1	1
χ_2	5	0	0	β	$\bar\beta$	1	-1	1
χ_3	5	0	0	$\bar\beta$	β	1	-1	1
χ_4	10	0	0	-1	-1	-2	1	1
χ_5	10	0	0	-1	-1	2	1	-1
...	...							

G_P	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_5'$	$\mathcal{C}_5''$
χ_1'	1	1	1	1	1
χ_2'	3	-1	0	α	α^*
χ_3'	3	-1	0	α^*	α
χ_4'	4	0	1	-1	-1
χ_5'	5	1	-1	0	0

Here $\beta = (-1+\sqrt{-11})/2$, $\alpha = (1-\sqrt{5})/2$, and $\alpha^* = (1+\sqrt{5})/2$. (We omit characters of degree > 10). Assume that the representation of G_P in $T_{P,X}$ is reducible. Then the restriction $\chi|_{G_P}$ contains χ_1' with multiplicity ≥ 2 and either χ_2' or χ_3'. Comparing the above tables we see that the restrictions $\chi_2|_{G_P}$ and $\chi_3|_{G_P}$ are irreducible (and coincide with χ_5'). Hence, $\chi = \chi_4$ or χ_5. In particular, $\chi(\mathcal{C}_5') = \chi(\mathcal{C}_5'') = 0$ and $\chi|_{G_P}$ contains both χ_2' and χ_3'. Thus $\chi|_{G_P} = \chi_2' + \chi_3' + 4\chi_1'$ and so $\chi(\mathcal{C}_3) = 4$. This contradicts $\chi_4(\mathcal{C}_3) = \chi_4(\mathcal{C}_5) = 1$. Therefore the character of the representation of G_P in $T_{P,X}$ coincides with χ_4' (and irreducible).

Then the vertex of the tangent cone $TC_{P,X} \subset T_{P,X}$ to X at P must be zero-dimensional. Hence, $TC_{P,X}$ a cone over a smooth quadric in $\mathbb{P}^3$. This shows that $P \in X$ is an ordinary double point (node). $\square$

Now we claim that $\operatorname{rk}\operatorname{Cl}(X) = 1$. Indeed, assume that $\operatorname{rk}\operatorname{Cl}(X) > 1$. Then we have a non-trivial representation of G in $\operatorname{Cl}(X)\otimes\mathbb{Q}$ such that $\operatorname{rk}\operatorname{Cl}(X)^G = 1$. By [CCN+85] the group G has no non-trivial rational representations of degree < 10. Hence, $\operatorname{rk}\operatorname{Cl}(X) \geq 11$. Let $F \subset X$ be a prime divisor and let $d := F \cdot K_X^2$ be its degree. Consider the G-orbit $F_1 = F, \dots, F_m$. Then $\sum F_i$ is a Cartier divisor on X (because the local Weil divisor class group of every singular point is torsion free). Hence, $\sum F_i \sim -rK_X$ for some r and so $md = (2g-2)r = 14r$. Since m divides $|G| = 660$, d is divisible by 7. In particular, X contains no surfaces of degree ≤ 6. Then by [Kal11, Cor. 3.12] $\operatorname{rk}\operatorname{Cl}(X) \leq 7$, a contradiction. Therefore, $\operatorname{rk}\operatorname{Cl}(X) = 1$.Then by Claim 5.17.6 below the number of singular points of X is at most 5. The contradiction proves the lemma. $\square$

Claim 5.17.6. *Let X be a Gorenstein Fano threefold whose singularities are only (isolated) ordinary double points. Let N be the number of singular points. Then*

$$N \leq \operatorname{rk}\operatorname{Cl}(X) - \rho(X) + h^{1,2}(X') - h^{1,2}(\hat{X}) \leq \operatorname{rk}\operatorname{Cl}(X) - 1 + h^{1,2}(X'),$$

where X' is a smoothing of X and $\hat{X} \to X$ is the blowup of singular points.

Proof. Let $D \in |-K_X|$ be a general member, let $\tilde{X} \to X$ be a small (not necessarily projective) resolution, and let $\tilde{D} \subset \tilde{X}$ be the pull-back of D. By the proof of Theorem 13 in [Nam97] we can write

$$N \le \dim \operatorname{Def}(X, D) - \dim \operatorname{Def}(\tilde{X}, \tilde{D}) = h^1(X', T_{X'}(-\log D'))-$$
$$- h^1(\tilde{X}, T_{\tilde{X}}(-\log \tilde{D})) = \tfrac{1}{2}\operatorname{Eu}(\tilde{X}) - \tfrac{1}{2}\operatorname{Eu}(X') = \tfrac{1}{2}\operatorname{Eu}(\hat{X}) - N - \tfrac{1}{2}\operatorname{Eu}(X'),$$

where $\operatorname{Def}(X, D)$ (resp. $\operatorname{Def}(\tilde{X}, \tilde{D})$) denotes the deformation space of the pair (X, D) (resp. $(\tilde{X}, \tilde{D})$) and (X', D') is a general member of the deformation family $\operatorname{Def}(X, D)$. Hence, $4N \le \operatorname{Eu}(\hat{X}) - \operatorname{Eu}(X')$. Note that $\operatorname{rk}\operatorname{Cl}(X) = \rho(\hat{X}) - N$. Since both X' and $\hat{X}$ are projective varieties with $H^i(X', \mathscr{O}_{X'}) = H^i(\hat{X}, \mathscr{O}_{\hat{X}}) = 0$, we get the disired inequality. □

Lemma 5.18. *Under the assumptions of* 5.13 *we have* $g \le 8$.

Proof. First we consider the case $g = 12$. Then the family of conics on X is parameterized by the projective plane $\mathbb{P}^2$, see [KS04]. By our assumption the induced action of G on $\mathbb{P}^2$ is trivial. Hence G acts non-trivially on each conic, a contradiction.

Now assume that $g = 9$ or 10. We claim that in the case $g = 9$ the order of G is divisible by 5 or 11. This follows from Theorem 3.3 whenever G has an irreducible representation of degree 4. Otherwise the representation of G in $H^0(X, -K_X) \simeq \mathbb{C}^{11}$ is either irreducible or has 5-dimensional irreducible subrepresentation. By Theorem 5.9 and our assumptions the action of G on $H^{1,2}(X)$ is trivial, so is the action on $H^3(X, \mathbb{C})$. Let $\delta \in G$ be an element of prime order $p \ge 5$. If $g = 9$, then we take $p = 5$ or 11. Assume that δ has no fixed points. Then the quotient $X/\langle\delta\rangle$ is a smooth Fano threefold. On the other hand, Fano manifolds are simply-connected, a contradiction. Therefore, δ has at least one fixed point on X. By the Lefschetz fixed point formula we have $\operatorname{Lef}(X, \delta) = 4 - \dim H^3(X, \mathbb{C}) = 2g - 20$. If $g = 9$ or 10, then $\operatorname{Lef}(X, \delta) \le 0$. Therefore, the set $\operatorname{Fix}(\delta)$ of δ-fixed points has positive diminsion. Let $\Phi(X) \subset X$ be the surface swept out by lines. Then $\operatorname{Fix}(\delta) \cap \Phi(X) \ne \varnothing$. Take a point $P \in \operatorname{Fix}(\delta) \cap \Phi(X)$. Since X is an intersection of quadrics, there are at most four lines passing through P, see [IP99, Prop. 4.2.2]. The group $\langle\delta\rangle$ cannot interchange these lines. Hence, there is a $\langle\delta\rangle$-invariant line $\ell \subset X$. Now consider the double projection digram (see [Isk90], [IP99, Th. 4.3.3]):

$$\begin{tikzcd}
& \tilde{X} \arrow[r, dashed, "\chi"] \arrow[dl, "\sigma"'] & \tilde{X}^+ \arrow[dr, "\varphi"] & \\
X \arrow[rrr, dashed] & & & Y
\end{tikzcd}$$

where σ is the blowup of ℓ and χ is a flop. If $g \ge 9$, then Y is a smooth Fano threefold and φ is the blowup of a smooth curve $\Gamma \subset Y$. Moreover,

(i) if $g = 9$, then $Y \simeq \mathbb{P}^3$, $\Gamma \subset \mathbb{P}^3$ is a non-hyperelliptic curve of genus 3 and degree 7 contained in a unique irreducible cubic surface $F \subset \mathbb{P}^3$,

(ii) if $g = 10$, then $Y = Y_2 \subset \mathbb{P}^4$ is a smooth quadric, Γ is a (hyperelliptic) curve of genus 2 and degree 7 contained in a unique irreducible surface $F \subset Y$ of degree 4.

Clearly, the above diagram is $\langle\delta\rangle$-equivariant. Since the linear span of Γ coincides with $\mathbb{P}^3$ for $g = 9$ (resp. $\mathbb{P}^4$ for $g = 10$), the group $\langle\delta\rangle$ non-trivially acts on Γ. On the other hand, the action of $\langle\delta\rangle$ on $H^1(\Gamma, \mathbb{Z}) \simeq H^3(X, \mathbb{Z})$ is trivial. This contradicts the Lefschetz fixed point formula. $\square$

Now we are going to finish our treatment of the Gorenstein case. It remains to consider two cases: $g = 8$ and $g = 7$, where $X = X_{2g-2} \subset \mathbb{P}^{g+1}$ is a smooth Fano threefold with $\operatorname{Pic}(X) = -K_X \cdot \mathbb{Z}$. Here we need the following result of S. Mukai.

Theorem 5.19 ([Muk88a]). (i) (*see also* [Gus83]) *Let $X = X_{14} \subset \mathbb{P}^9$ be a smooth Fano threefold of genus* 8 *with $\rho(X) = 1$. Then X is isomorphic to a linear section of the Grassmannian* $\operatorname{Gr}(2, 6) \subset \mathbb{P}^{14}$ *by a subspace of codimension* 5*. Any isomorphism $X = X_{14} \xrightarrow{\sim} X' = X'_{14}$ of two such smooth sections is induced by an isomorphism of the Grassmannian* $\operatorname{Gr}(2, 6)$*.*

(ii) *Let $X = X_{12} \subset \mathbb{P}^8$ be a smooth Fano threefold of genus* 7 *with $\rho(X) = 1$. Then X is isomorphic to a linear section of the Lagrangian Grassmannian* $\operatorname{LGr}(4, 9) \subset \mathbb{P}^{15}$ *by a subspace of dimension* 8 *(see Example* 2.11*). Any isomorphism $X = X_{12} \xrightarrow{\sim} X' = X'_{12}$ of two such smooth sections is induced by an isomorphism of the Lagrangian Grassmannian* $\operatorname{LGr}(4, 9)$*.*

Consider the case $g = 8$. By the above theorem the group G acts on $\operatorname{Gr}(2, 6)$ and on $\mathbb{P}^{14} = \mathbb{P}(\wedge^2\mathbb{C}^5) = \mathbb{P}(H^0(\operatorname{Gr}(2, 6), \mathscr{T}^*))$, where $\mathscr{T}$ is tautological rank two vector bundle on $\operatorname{Gr}(2, 6)$. The linear span of $X = X_{12}$ in $\mathbb{P}^{14}$ is a G-invariant $\mathbb{P}^9$. Let $\mathbb{P}^4 \subset \mathbb{P}^{14*} = \mathbb{P}(\wedge^2\mathbb{C}^{5*})$ be the G-invariant orthogonal subspace. The locus of all degenerate skew-forms is the Pfaffian cubic hypersurface $Y_3 \subset \mathbb{P}(\wedge^2\mathbb{C}^{5*})$. Put $X_3 = Y_3 \cap \mathbb{P}^4$. Then $X_3 \subset \mathbb{P}^4$ is a G-invariant cubic. Since the variety $X = X_{14}$ is smooth (see Lemma 5.17), so is our cubic $X_3 \subset \mathbb{P}^4$, see [Kuz04, Prop. A.4]. Then by Lemma 5.12 we get $G \simeq \operatorname{PSL}_2(11)$, $X_3^{\mathrm{k}} \subset \mathbb{P}^4$ is the Klein cubic and we get Example 2.6.

Finally consider the case $g = 7$. The group G acts on the Lagrangian Grassmannian $\operatorname{LGr}(4, 9) \subset \mathbb{P}^{15}$. Let $C := \operatorname{LGr}(4, 9) \cap \mathbb{P}^6$, where $\mathbb{P}^6 \subset \mathbb{P}^{15}$ is the subspace orthogonal to $\mathbb{P}^8$ with respect to the G-invariant quadratic form on $\mathbb{P}^{14}$. Then $C \subset \mathbb{P}^6$ is a smooth canonical curve of genus 7 [IM04, Lemma 3.2]. Hence $G \subset \operatorname{Aut}(C)$. On the other hand, by the Hurwitz bound we have $|G| \le |\operatorname{Aut}(C)| \le 504$. Furthermore, the group has an irreducible representation in $H^0(X, -K_X) \simeq \mathbb{C}^9$. Hence, $|G|$ is divisible by 9. Now it is an easy exercise to show that either $G \simeq \mathrm{SL}_2(8)$ or G is contained in the list (1.2). For example, according to Theorem 3.7 we may assume that G has no subgroups of index ≤ 26, i.e., of order ≥ 19. Hence $234 = 26 \cdot 9 \le |G|$. Now we write the Hurwitz formula for the quotient $\pi : C \to C/G = C'$:

$$12 = 2g(C) - 2 = |G|(2g(C') - 2) + |G| \sum_{i=1}^{s} (1 - 1/a_i),$$

where $\sum (a_i - 1) Q_i$ is the ramification divisor on C'. By the above $a_i \le 18$ for all i. There are only two integer solutions: $|G| = 288$, $(a_1, \dots, a_s) = (2, 3, 8)$ and $|G| = 504$, $(a_1, \dots, a_s) = (2, 3, 7)$. In the first case the Sylow 17-subgroup has index 14 in G, a contradiction. In the second case the curve C is unique up to isomorphism and $G \simeq \mathrm{PSL}_2(8)$, see [Mac65]. By the construction in Example 2.11 the threefold X_{12} is uniquely determined by C, so $X_{12} = X_{12}^{\mathrm{m}}$. This finishes the treatment of the case of Gorenstein X.

6. Case: X is not Gorenstein

In this section, as in §5, we assume that G is a simple group which does not admit any embeddings into $\mathrm{Cr}_2(\mathbb{C})$. We assume X is a $G\mathbb{Q}$-Fano threefold such that K_X is not Cartier. Let $\Omega \subset \mathrm{Sing}(X)$ be the set of all non-Gorenstein points and let $n := |\Omega|$.

Lemma 6.1. *In the above assumptions the group G transitively acts on Ω, $n \ge 9$, and each point $P \in \Omega$ is a cyclic quotient singularity of index* 2.

Proof. Let $\Omega = \Omega_1 \cup \dots \cup \Omega_m$ be the orbit decomposition, and let $n_i := |\Omega_i|$. For a point $P_i \in \Omega_i$, let $Q_{ij} \in \mathbf{B}$, $j = 1, \dots, l_i$ be "virtual" points in the basket over P_i and let r_{ij} be the index of Q_{ij}. The orbifold Riemann-Roch and Myaoka-Bogomolov inequality give us (see [Kaw92], [KMMT00]) [5]

$$24 > \sum_{i=1}^{m} n_i \sum_{j=1}^{l_i} \left(r_{ij} - \frac{1}{r_{ij}} \right) \ge \frac{3}{2} \sum_{i=1}^{m} n_i. \tag{6.2}$$

By Theorem 3.7 and our assumptions we have $n_1, \dots, n_m \ge 7$.

Assume that $P_1 \in X$ is not a cyclic quotient singularity. Then over each $P_i \in \Omega_1$ there are at least two virtual points Q_{ij}, i.e, $l_1 > 1$. By (6.2) we have

$$24 > n_1 \sum_{j=1}^{l_1} \left(r_{1j} - \frac{1}{r_{1j}} \right) \ge 7 \sum_{j=1}^{l_1} \left(r_{1j} - \frac{1}{r_{1j}} \right).$$

There is only one possibility: $l_1 = 2$, $n_1 = 7$, and $r_{11} = r_{12} = 2$. In this case, by the classification [Rei87, Th. 6.1] the point $P_1 \in X$ is of

[5] From [KMMT00] we have the inequality $\sum (r - 1/r) \le 24$. The strict inequality follows from the proof in [Kaw92] because $\rho(X)^G = 1$. I would like to thank Professor Y. Kawamata for pointing me out this fact.

type $\{xy + \phi(z^2, t)\}/\boldsymbol{\mu}_2(1, 1, 1, 0)$, where $\operatorname{ord} \phi(0, t) = 2$, or $\{x^2 + y^2 + \phi(z, t)\}/\boldsymbol{\mu}_2(0, 1, 1, 1)$ (because the "axial multiplicity" is equal to 2).

By Theorem 3.7 we have $G \simeq \mathfrak{A}_7$ and $G_P \simeq \mathfrak{A}_6$. As in the proof of Lemma 4.5 we have an embedding $\tilde{G}_P \subset \mathrm{GL}(T_{P^\sharp, U^\sharp})$, where $\dim T_{P^\sharp, U^\sharp} = 4$ and $\tilde{G}_P$ is a central extension of G_P by $\boldsymbol{\mu}_2$. The action of $\tilde{G}_P$ preserves the tangent cone $TC_{P^\sharp, U^\sharp} \subset T_{P^\sharp, U^\sharp}$ which is given by a quadratic form of rank ≥ 2. Since $G_P \simeq \mathfrak{A}_6$ cannot act non-trivially on a smooth quadric in $\mathbb{P}^3$, $\operatorname{rk} q \neq 4$. Hence, $\operatorname{rk} q = 2$ or 3 and the representation of $\tilde{G}_P$ in $T_{P^\sharp, U^\sharp} \simeq \mathbb{C}^4$ is reducible: the singular locus of $TC_{P^\sharp, U^\sharp}$ is a $\tilde{G}_P$-invariant linear subspace. On the other hand, $\tilde{G}_P \simeq \mathfrak{A}_6$ has no faithful representations of degree ≤ 3 (see, e.g., Theorem 3.2 or [CCN+85]), a contradiction.

Therefore, all the points in Ω are cyclic quotient singularities. Then (6.2) can be rewritten as follows:

$$24 > \sum_{i=1}^{m} n_i \left(r_i - \frac{1}{r_i} \right) \geq \frac{3}{2} \sum_{i=1}^{m} n_i, \tag{6.3}$$

where r_i is the index of the point $P_i \in \Omega_i$. Assume that $n_1 \leq 8$, then by Theorem 3.7 $G \simeq \mathfrak{A}_n$ with $n = 7$ or 8, and $G_P \simeq \mathfrak{A}_{n-1}$. As above $\tilde{G}_P \subset \mathrm{GL}(T_{P^\sharp, U^\sharp})$, where $\dim T_{P^\sharp, U^\sharp} = 3$ (because $U^\sharp$ is smooth) and $\tilde{G}_P$ is a central extension of G_P by $\boldsymbol{\mu}_{r_1}$. Clearly, the representation $\tilde{G}_P$ in $\mathrm{GL}(T_{P^\sharp, U^\sharp})$ is irreducible. Hence $\boldsymbol{\mu}_{r_1}$ acts on $T_{P^\sharp, U^\sharp}$ by scalar multiplication. As in the proof of Lemma 4.5, by the classification of terminal singularities (Terminal Lemma) [Rei87], we have $r_1 = 2$. But then the group $\tilde{G}_P$ has no non-trivial representations in $\mathbb{C}^3$ by Theorem 3.2. The contradiction shows that $n_1 \geq 9$ and, by symmetry, $n_i \geq 9$ for all i. Then by (6.3) we have $24 > 9m \cdot 3/2$. Hence $m = 1$, i.e., Ω consists of one orbit. Further, $24 > 9(r_i - 1/r_i)$. Hence $r_i = 2$ for all i. □

Lemma 6.4. (i) $Z(G_P) = \{1\}$, $Z(\tilde{G}_P) = \boldsymbol{\mu}_2$.

(ii) *The representation of $\tilde{G}_P$ in $T_{P^\sharp, U^\sharp} \simeq \mathbb{C}^3$ is irreducible.*

(iii) *The action of $\tilde{G}_P$ on $T_{P^\sharp, U^\sharp} \simeq \mathbb{C}^3$ is primitive.*

(iv) *The only possible case is $G \simeq \mathrm{PSL}_2(11)$, $n = 11$, $G_P \simeq \mathfrak{A}_5$.*

Proof. (i) follows from the explicit description of groups G_P in Theorem 3.7.

(ii) Assume that $T_{P^\sharp, U^\sharp} = T_1 \oplus T_2$. Then the kernel of the homomorphism $\tilde{G}_P \to \mathrm{GL}(T_2)$ is contained into $Z(\tilde{G}_P)$. Hence, $\tilde{G}_P \to \mathrm{GL}(T_2)$ is injective and so G_P effectively acts on $\mathbb{P}^1$, a contradiction.

(iii) Assume the converse. Then there is an abelian subgroup $\tilde{A} \subset \tilde{G}_P$ such that $\tilde{G}_P/\tilde{A} \simeq \mathfrak{A}_3$ or $\mathfrak{S}_3$. Hence there is an abelian subgroup $A \subset G_P$ such that $G_P/A \simeq \tilde{G}_P/\tilde{A} \simeq \mathfrak{A}_3$ or $\mathfrak{S}_3$. In particular, G_P is not simple and its order is divisible by 3. Thus by Theorem 3.7 there are only three possibilities: $G \simeq \mathrm{SL}_3(3)$, $\mathrm{PSL}_2(13)$, and $\mathrm{SL}_4(2)$.

In the case $G \simeq \mathrm{PSL}_2(13)$ the group $G_P \simeq \boldsymbol{\mu}_{13} \rtimes \boldsymbol{\mu}_6$ has no surjective homomorphisms to $\mathfrak{S}_3$. So, $G_P/A \simeq \mathfrak{A}_3$ and $A \simeq \boldsymbol{\mu}_{26}$. On the other hand, G_P contains no elements of order 26, a contradiction. Consider the case $G \simeq \mathrm{SL}_3(3)$. Then $G_P \supset \mathrm{GL}_2(3)$. Since $\mathrm{GL}_2(3)/Z(\mathrm{GL}_2(3)) \simeq \mathfrak{S}_4$, for $A \cap \mathrm{GL}_2(3)$ we have only one possibility: it is a group of order 8. But the group $A \cap \mathrm{GL}_2(3)$ is not abelian, a contradiction. Finally, in the case $G \simeq \mathrm{SL}_4(2)$ the group $\mathrm{SL}_3(2) \subset G_P$ is simple of order 168, a contradiction.

(iv) Follows from Theorem 3.2. □

From now on we assume that $G \simeq \mathrm{PSL}_2(11)$ and $n = 11$.

Lemma 6.5. $\dim |-K_X| > 0$.

Proof. By [Kaw92] we have $-K_X \cdot c_2(X) = 24 - 3n/2$. Hence by the orbifold Riemann-Roch (see [Rei87])

$$\dim |-K_X| = \frac{1}{2}(-K_X)^3 - \frac{1}{12} K_X \cdot c_2(X) + \sum_{P \in \Omega} c_P(-K_X) =$$

$$= \frac{1}{2}(-K_X)^3 + 2 - \frac{n}{4} = \frac{1}{2}(-K_X)^3 - \frac{3}{4}.$$

Put $\dim |-K_X| = l$. Then $(-K_X)^3 = 2l + 3/2$. In particular, $l \geq 0$ and $|-K_X| \neq \varnothing$. Assume that $\dim |-K_X| = 0$. Then $(-K_X)^3 = 3/2$. Let $S \in |-K_X|$ be (a unique) member. By Lemma 4.7 the surface S is reducible and G transitively acts on its components. Write $S = \sum_{i=1}^m S_i$. Then $m(-K_X)^2 \cdot S_i = (-K_X)^3 = 3/2$. Since $2(-K_X)^2 \cdot S_i$ is an integer, we have $m \leq 3$, a contradiction. □

Lemma 6.6. *The pair $(X, |-K_X|)$ is canonical.*

Proof. Put $\mathscr{H} := |-K_X|$. By Corollary 4.8 the linear system $\mathscr{H}$ has no fixed components. We apply a G-equivariant version of a construction [Ale94, §4]. Take c so that the pair $(X, c\mathscr{H})$ is canonical but not terminal. By our assumption $0 < c < 1$. Let $f \colon (\tilde{X}, c\tilde{\mathscr{H}}) \to (X, c\mathscr{H})$ be a G-equivariant $G\mathbb{Q}$-factorial terminal modification (terminal model). We can write

$$\begin{aligned} K_{\tilde{X}} + c\tilde{\mathscr{H}} &= f^*(K_X + c\mathscr{H}), \\ K_{\tilde{X}} + \tilde{\mathscr{H}} + \sum a_i E_i &= f^*(K_X + \mathscr{H}) \sim 0, \end{aligned}$$

where E_i are f-exceptional divisors and $a_i > 0$. Run $(\tilde{X}, c\tilde{\mathscr{H}})$-MMP:

$$(\tilde{X}, c\tilde{\mathscr{H}}) \dashrightarrow (\bar{X}, c\bar{\mathscr{H}}).$$

As in 4.2 $\bar{X}$ is a Fano threefold with $G\mathbb{Q}$-factorial terminal singularities and $\rho(\bar{X})^G = 1$. We also have $0 \sim K_{\bar{X}} + \bar{\mathscr{H}} + \sum a_i \bar{E}_i$. Here $\sum a_i \bar{E}_i$ is a non-trivial effective invariant divisor such that $-(K_{\bar{X}} + \sum a_i \bar{E}_i) \sim \bar{\mathscr{H}}$ is ample. This contradicts Lemma 4.7. □

Lemma 6.7. *The image of the (G-equivariant) rational map $\phi : X \dashrightarrow \mathbb{P}^l$ given by the linear system $|-K_X|$ is three-dimensional.*

Proof. Let $Y := \phi(X)$. Since X is rationally connected, G acts trivially on Y. This contradicts Lemma 4.9. $\square$

Recall that non-Gorenstein points $P_1, \dots, P_{11}$ of X are of type $\frac{1}{2}(1,1,1)$. Let $f : \tilde{X} \to X$ be blow up of $P_1, \dots, P_{11}$ and let $E = \sum E_i$ be the exceptional divisor, where $E_i = f^{-1}(P_i)$. Then $\tilde{X}$ is smooth over P_i, it has at worst Gorenstein terminal singularities, $E_i \simeq \mathbb{P}^2$, and $\mathscr{O}_{E_i}(-K_{\tilde{X}}) = \mathscr{O}_{\mathbb{P}^2}(1)$. Put $\mathscr{H} := |-K_X|$ and let $\tilde{\mathscr{H}}$ be the birational transform. Since the pair $(X, \mathscr{H})$ is canonical, we have

$$K_{\tilde{X}} + \tilde{\mathscr{H}} \sim f^*(K_X + \mathscr{H}) \sim 0.$$

Hence, $|-K_{\tilde{X}}| = \tilde{\mathscr{H}}$.

Lemma 6.8. *The linear system $\tilde{\mathscr{H}}$ is base point free.*

Proof. Note that the restriction $\tilde{\mathscr{H}}|_{E_i} = |-K_{\tilde{X}}||_{E_i}$ is a (not necessarily complete) linear system of lines on $E_i \simeq \mathbb{P}^2$. Since this linear system is G_{P_i}-invariant, where $G_{P_i} \simeq \mathfrak{A}_5$, it is base point free. Hence $\operatorname{Bs} \tilde{\mathscr{H}} \cap E_i = \varnothing$. In particular, this implies that for any curve $C \subset \operatorname{Bs} \tilde{\mathscr{H}}$, we have $E_i \cdot C = 0$ and so $\tilde{\mathscr{H}} \cdot C = \mathscr{H} \cdot f(C) > 0$. Hence, $\tilde{\mathscr{H}}$ is nef. By Lemma 6.7 it is big. Then the assertion follows by Lemma 5.2. $\square$

Now we are in position to finish the proof of Theorems 1.3 and 1.5. Note that the divisors E_i are linear independent elements of $\operatorname{Pic}(\tilde{X})$. Hence, $\rho(\tilde{X}) > 11$. If $\tilde{X}$ is a Fano threefold, then by Theorem 5.6 and Remark 5.8 there is a smoothing $\tilde{X}_t$ with $\rho(\tilde{X}_t) > 11$. This contradicts the classification of smooth Fano threefolds with $\rho > 1$ [MM82]. By Lemma 5.3 the linear system $|-K_{\tilde{X}}|$ determines a birational contraction $\varphi : \tilde{X} \to \bar{X} = \bar{X}_{2g-2} \subset \mathbb{P}^{g+1}$ whose image is an anticanonically embedded Fano threefold with at worst canonical singularities. Here g is the genus of $\bar{X}$ (see 5.1). By Lemma 5.4 the variety $\bar{X}_{2g-2} \subset \mathbb{P}^{g+1}$ is an intersection of quadrics. In particular, $g \geq 5$. Since $\rho(\tilde{X})^G = 2$, $\rho(\bar{X})^G = 1$. Let $\bar{E}_i := \varphi(E_i)$ and $\bar{E} := \varphi(E)$.

Claim 6.8.1. *The group $\operatorname{Cl}(X)^G$ is generated by $-K_X$.*

Proof. Assume that $\operatorname{Cl}(X)^G$ contains a torsion element, say T. Then $2T$ is Cartier and so $2T \sim 0$ (because $\operatorname{Pic}(X)$ is torsion free). As above, let $\Omega \subset X$ be the set of points where K_X is not Cartier. Since G acts on Ω transitively, T is not Cartier at all points $P \in \Omega$. On the other hand, by the orbifold Riemann-Roch (see [Rei87]) and Kawamata-Viehweg vanishing theorem we have

$$0 = \chi(\mathscr{O}_X(T)) = 1 + \sum_{P \in \Omega} c_P(T) = 1 - \frac{|\Omega|}{8},$$

where $c_P(T) = -1/8$ for all $P \in \Omega$ (because this P is a cyclic quotient of type $\frac{1}{2}(1,1,1)$). This gives us $|\Omega| = 8$, a contradiction.

Therefore, $\mathrm{Cl}(X)^G \simeq \mathbb{Z}$. Let A be the ample generator of $\mathrm{Cl}(X)^G$ and let $-K_X = qA$ for some $q \in \mathbb{Z}_{>0}$. Since K_X is not Cartier, $q > 2$. Again by the orbifold Riemann-Roch

$$\chi(\mathscr{O}_X(-A)) = -\frac{A^3}{12}(q-1)(q-2) - \frac{A \cdot c_2}{12} + \sum_{P \in \Omega} c_P(-A) + 1 <$$

$$< \sum_{P \in \Omega} c_P(-A) + 1 = -\frac{11}{8} + 1 < 0.$$

On the other hand, by the Kawamata-Viehweg vanishing theorem $\chi(\mathscr{O}_X(-A)) = 0$, a contradiction. □

Take a general member $\bar{H} \in |-K_{\bar{X}}|$. By Bertini's theorem $\bar{H}$ is a K3 surface with at worst Du Val singularities. Put $C_i := \bar{E}_i \cap \bar{H}$.

Claim 6.8.2. *$C_1, \ldots, C_{11}$ are disjointed smooth rational curves contained into the smooth locus of $\bar{H}$.*

Proof. Since $\bar{H}$ is Cartier, the number $\bar{E}_i \cdot \bar{E}_j \cdot \bar{H}$, where $1 \le i, j \le 11$, is well-defined and coincides with the intersection number $C_i \cdot C_j$ of curves $C_i := \bar{E}_i \cap \bar{H}$ and $C_j := \bar{E}_j \cap \bar{H}$ on $\bar{H}$. Clearly, the numbers $C_i^2 = \bar{E}_i^2 \cdot \bar{H}$ for $1 \le i \le 11$ do not depend on i. Since the action of G on $\{\bar{E}_i\}$ is doubly transitive [CCN+85], the numbers $C_i \cdot C_j = \bar{E}_i \cdot \bar{E}_j \cdot \bar{H}$ for $1 \le i \ne j \le 11$ also do not depend on i, j.

Since $(-K_{\tilde{X}})^2 \cdot E_i = 1$, the surfaces $\bar{E}_i$ are planes in $\mathbb{P}^{g+1}$ and every C_i is a line on $\bar{E}_i$. If $C_i \cdot C_j > 0$ for some $i \ne j$, then $\bar{E}_i \cap \bar{E}_j$ is a line. Since G acts doubly transitive on $\{\bar{E}_i\}$, the intersection $\bar{E}_i \cap \bar{E}_j$ is a line for all $i \ne j$. Hence, the linear span of $\bar{E}_1 \cup \bar{E}_2 \cup \bar{E}_3$ is a three-dimensional projective subspace $\mathbb{P}^3 \subset \mathbb{P}^{g+1}$. In this case, $\bar{X} \cap \mathbb{P}^3$ cannot be an intersection of quadrics. This contradicts Lemma 5.4.

Thus we may assume that $C_i \cdot C_j = 0$ for all $i \ne j$. By the Hodge index theorem $C_k^2 \le 0$ for all k. If $C_1^2 = 0$, then C_1 is a nef $\mathbb{Q}$-Cartier divisor on a K3 surface with at worst Du Val singularities. By the cone theorem, for some m, the linear system $|mC_1|$ determines an elliptic fibration $\psi : \bar{H} \to \mathbb{P}^1$ and all the curves C_k are degenerate fibers of ψ. Let $\mu : \hat{H} \to \bar{H}$ be the minimal resolution, let $F_k := \mu^{-1}(C_k)$ be the degenerate fiber corresponding to C_k, and let $\hat{C}_k$ be the proper transform of C_k. Then $\hat{H}$ is a smooth K3 surface. Since C_k is smooth, $\hat{C}_k \cdot (F_k - \hat{C}_k) = 1$. Using Kodaira's classification of degenerate fibers of elliptic fibrations we see that F_k has at least three components. But then $\rho(\hat{H}) \ge 23$, a contradiction.

Therefore, $C_k^2 < 0$ for all k. In particular, $\mathrm{rk}\,\mathrm{Cl}(\bar{H}) \ge 12$. If $\bar{H}$ is singular at a point on C_k, then, as above, considering the minimal resolution

$\mu : \hat{H} \to \bar{H}$ one can show that $\rho(\hat{H}) \geq 23$, a contradiction. Hence $\bar{H}$ is smooth near C_k. So, all the C_k are (-2)-curves contained into the smooth part of $\bar{H}$. $\square$

Clearly, fibers of φ meet $\sum E_i$ (otherwise φ is an isomorphism near E_i and then $\rho(\bar{X})^G > 1$). Since $E_i \simeq \mathbb{P}^2$, φ cannot contract divisors to points. Assume that φ contracts divisors D_l to curves Γ_l. Then $\Gamma_l \subset E_i$ for some i. Since φ is K-trivial, $\bar{X}$ is singular along Γ_l and $\bar{H}$ is singular at point $\Gamma_l \cap \bar{H}$. Since $\Gamma_l \cap \bar{H} \subset C_i$, we get a contradiction with the above claim.

Therefore φ does not contract any divisors, i.e., it contracts only a finite number of curves. Then $\bar{X}$ is a Fano threefold with Gorenstein terminal (but not $G\mathbb{Q}$-factorial) singularities. Consider the following diagram (cf. [IP99, Ch. 4]):

$$\begin{tikzcd}
 & \tilde{X} \arrow[dl, "f"'] \arrow[dr, "\varphi"] \arrow[rr, dashed, "\chi"] & & X^+ \arrow[dl, "\varphi^+"'] \arrow[dr, "f^+"] & \\
X & & \bar{X} & & Y
\end{tikzcd}$$

Here χ is a G-equivariant flop, φ^+ is a small modification, and f^+ is a K-negative G-equivariant G-extremal contraction. As in 4.2 we may assume that Y is $G\mathbb{Q}$-Fano threefold with $\rho(Y)^G = 1$. Let $E^+ = \sum E_i^+ \subset X^+$ be the proper transform of $E = \sum E_i$. Recall that $G \simeq \mathrm{PSL}_2(11)$. We can write

$$\begin{aligned}
-K_{\tilde{X}}^3 &= -K_{X^+}^3 = -K_{\bar{X}}^3 = 2g - 2, \\
(-K_{\tilde{X}})^2 \cdot E &= (-K_{X^+})^2 \cdot E^+ = (-K_{\bar{X}})^2 \cdot \bar{E} = 11, \\
-K_{\tilde{X}} \cdot E^2 &= -K_{X^+} \cdot E^{+2} = -K_{\bar{X}} \cdot \bar{E}^2 = -22.
\end{aligned}$$

Let $D := \sum D_i$ be the f^+-exceptional divisor. By Claim 6.8.1 we have $D \sim -\alpha K_{X^+} - \beta E^+$ for some $\alpha, \beta \in \mathbb{Z}_{>0}$. Therefore,

$$\begin{aligned}
(-K_{\bar{X}})^2 \cdot D &= (2g-2)\alpha - 11\beta, \\
-K_{\bar{X}} \cdot D^2 &= (2g-2)\alpha^2 - 22\alpha\beta - 22\beta^2.
\end{aligned}$$

Assume that Y is not Gorenstein. Then Y is of the same type as X. In particular, Y has 11 cyclic quotient singular points of index 2. In this case D has exactly 11 components and

$$(-K_{\bar{X}})^2 \cdot D = 11, \qquad -K_{\bar{X}} \cdot D^2 = -22. \tag{6.9}$$

In particular, either $g - 1$ or α is divisible by 11. Assume that $g - 1 = 11k$, $k \in \mathbb{Z}_{>0}$. Then the above equalities can be rewritten as follows:

$$\begin{aligned}
\beta &= 2k\alpha - 1, \\
0 &= -1 - k\alpha^2 + \alpha\beta + \beta^2.
\end{aligned}$$

Eliminating β we get

$$0 = -1 - k\alpha^2 + \alpha(2k\alpha - 1) + (2k\alpha - 1)^2 = (\alpha + 4k)(k\alpha - 1).$$

Since α, $k > 0$ we get $k = 1$ and $g = 12$. Hence $\dim H^0(\tilde{X}, -K_{\tilde{X}}) = 14$ and so $\dim H^0(\tilde{X}, -K_{\tilde{X}})^G \geq 2$ (because the degrees of irreducible representations of $G = \mathrm{PSL}_2(11)$ are 1, 5, 5, 10, 10, 11, 12, 12). This contradicts Lemma 4.9. Therefore, $\alpha = 11k$, $k \in \mathbb{Z}_{>0}$. Then, as above,

$$\begin{aligned}\beta &= 2(g-1)k - 1,\\ 0 &= -1 - 11(g-1)k^2 + 11k\beta + \beta^2.\end{aligned}$$

Thus

$$\begin{aligned}0 = -1 - 11(g-1)k^2 + 11k(2(g-1)k - 1) + (2(g-1)k - 1)^2 =\\ = (11 + 4(g-1))((g-1)k - 1).\end{aligned}$$

Since $g > 2$ (see Lemma 5.3) we have a contradiction.

Finally assume that Y is Gorenstein. By the results of §5 either $Y \simeq X_3^{\mathrm{k}} \subset \mathbb{P}^4$ or $Y \simeq X_{14}^{\mathrm{a}} \subset \mathbb{P}^9$. In particular, Y is smooth. If $\dim f^+(D) = 0$, then f^+ is just blowup of points $Q_1, \dots, Q_l \in Y$ [Cut88]. Note that $\mathrm{rk}\,\mathrm{Cl}(X^+) = \mathrm{rk}\,\mathrm{Cl}(\tilde{X}) \geq 12$, so $l \geq 11$. But then $-K_{X^+}^3 = -K_Y^3 - 8l < 0$, a contradiction. Therefore $f^+(D)$ is a (reducible) curve $\Gamma = \sum_{i=1}^{l} \Gamma_i$. Here again $l \geq 11$. Write

$$\varphi^{+*}(-K_{\bar{X}}) = -K_{X^+} = f^{+*}(-K_Y) - D.$$

Since the linear system $|-K_{\bar{X}}|$ is very ample, the map $Y \dashrightarrow \bar{X} = \bar{X}_{2g-2} \subset \mathbb{P}^{g+1}$ is given by a subsystem $f_*^+|-K_{X^+}|$ of the linear system $|-K_Y|$ consisting of elements passing through Γ. On the other hand, in the case $Y \simeq X_{14}^{\mathrm{a}} \subset \mathbb{P}^9$, the representation of G in $H^0(Y, -K_Y)$ is irreducible (see Example 2.9). Therefore, $Y \simeq X_3^{\mathrm{k}} \subset \mathbb{P}^4$. Moreover, $\dim|-K_{\bar{X}}| < \dim|-K_Y| = 14$. By Lemma 4.7 the representation of G in $\dim H^0(Y, -K_Y)^G$ has no trivial subrepresentations. Since $g \geq 5$ we have only one possibility: $\dim|-K_{\bar{X}}| = 9$, $g = 8$. Further, by Claim 6.8.1 the group $\mathrm{Cl}(X^+)^G$ is a free $\mathbb{Z}$-module generated by $-K_{X^+}$ and E. On the other hand, $\mathrm{Cl}(X^+)^G$ is generated by $\frac{1}{2}f^{+*}(-K_Y)$ and D. Hence $\beta = 2$, i.e., $D \sim -\alpha K_{X^+} - 2E^+$. Similar to (6.9) we have (see e.g. [Kal11])

$$\begin{aligned}-K_Y \cdot \Gamma - 2p_a(\Gamma) + 2 &= (-K_{\bar{X}})^2 \cdot D &= 14\alpha - 22,\\ 2p_a(\Gamma) - 2 &= -K_{\bar{X}} \cdot D^2 &= 14\alpha^2 - 44\alpha - 88.\end{aligned}$$

Thus $\deg\Gamma = -\frac{1}{2}K_Y \cdot \Gamma = 7\alpha^2 - 15\alpha - 55 \geq 45$. On the other hand, Γ is a scheme intersection of members of the linear system $f_*^+|-K_{X^+}| \subset |-K_Y|$ (because $|-K_{X^+}|$ is base point free). Hence, $\deg\Gamma \leq 12$, a contradiction. This finishes our proof of Theorems 1.3 and 1.5.

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