Quantum sphere $\mathbb{S}^4$ as a non-Levi conjugacy class

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Abstract

We construct a $U_\hbar\big(\mathfrak{sp}(4)\big)$-equivariant quantization of the four-dimensional complex sphere $\mathbb{S}^4$ regarded as a conjugacy class, $Sp(4)/Sp(2)\times Sp(2)$, of a simple complex group with non-Levi isotropy subgroup, through an operator realization of the quantum polynomial algebra $\mathbb{C}_\hbar[\mathbb{S}^4]$ on a highest weight module of $U_\hbar\big(\mathfrak{sp}(4)\big)$.

Key words: quantum groups, quantization, Verma modules.

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1 Introduction

There are two types of closed conjugacy classes in a simple complex algebraic group G. One type consists of classes that are isomorphic to orbits in the adjoint representation on the Lie algebra $\mathfrak{g}$. They are homogeneous spaces of G whose stabilizer of the initial point is a Levi subgroup in G. Our concern is equivariant quantization of classes of second type, i. e. whose isotropy subgroup *is not* Levi. Regarding the classical series, such classes are present only in the orthogonal and symplectic groups.

The group G supports a (Drinfeld-Sklyanin) Poisson bivector field $\pi_0\in\Lambda^2(G)$ associated with a solution of the classical Yang-Baxter equation. This structure makes G a Poisson group, whose multiplication $G\times G\to G$ is a Poisson map (here $G\times G$ is equipped with the Poisson structure of Cartesian product). The Drinfeld-Sklyanin bracket gives rise to the quantum group $U_\hbar(\mathfrak{g})$, which is a deformation, along the parameter $\hbar$, of the universal enveloping algebra $U(\mathfrak{g})$ in the class of Hopf algebras, [D].

There is a Poisson structure $\pi_1 \in \Lambda^2(G)$ compatible with the conjugacy action of the Poisson group on itself, [S]. It means that action map from the Cartesian product of (G, π_0) and (G, π_1) to (G, π_1) is Poisson. Then G is said to be a Poisson space over the Poisson group G, under the conjugacy action

The Poisson bivector field π_1 restricts to every closed conjugacy class making it a Poisson G-variety, [AM]. In this sense, the group G is analogous to $\mathfrak{g} \simeq \mathfrak{g}^*$ equipped with the canonical G-invariant bracket.

Quantization of conjugacy classes with Levi isotropy subgroups has been constructed in various settings, namely, as a star product and in terms of generators and relations, [EEM, M2]. Both approaches rely upon the representation theory of the quantum group $U_\hbar(\mathfrak{g})$ and make use of the following facts: a) the universal enveloping algebra $U(\mathfrak{l})$ of the isotropy subgroup is quantized to a Hopf subalgebra $U_\hbar(\mathfrak{l}) \subset U_\hbar(\mathfrak{g})$, b) there is a triangular factorization of $U(\mathfrak{g})$ relative to $U(\mathfrak{l})$, which amounts to a factorization of quantum groups and facilitates parabolic induction. In particular, quantum conjugacy classes of the Levi type have been realized by operators on scalar parabolic Verma modules in [M2].

The above mentioned conditions are violated for non-Levi conjugacy classes, which makes the conventional methods of quantization inapplicable in this case. In this paper, we show how to overcome these obstructions for the simplest non-Levi conjugacy class $Sp(4)/Sp(2) \times Sp(2)$. This is the class of symplectic invertible 4×4-matrices with eigenvalues ± 1, each of multiplicity 2. As an affine variety, it is isomorphic to the four-dimensional complex sphere $\mathbb{S}^4$. Although the quantization of $\mathbb{S}^4$ can be obtained by other methods, e. g. as in [FRT], we are interested in $\mathbb{S}^4$ as an illustration of our approach to a general non-Levi class.

The idea is to find a suitable highest weight $U_\hbar(\mathfrak{g})$-module where the quantum sphere could be represented by linear operators. We consider an auxiliary parabolic Verma module $\hat{M}_\lambda$ as a starting point. For a special value of weight λ, the module $\hat{M}_\lambda$ has a singular vector generating a submodule in $\hat{M}_\lambda$. The quotient M_λ of $\hat{M}_\lambda$ over that submodule is irreducible. The deformation of the polynomial algebra $\mathbb{C}[\mathbb{S}^4]$ is realized by a $U_\hbar(\mathfrak{g})$-invariant subalgebra in $\mathrm{End}(M_\lambda)$. This also allows us to describe the quantized polynomial algebra $\mathbb{C}_\hbar[\mathbb{S}^4]$ in terms of generators and relations.

Irreducibility of M_λ implies non-degeneracy of the Shapovalov form on it. In the simple case of $Sp(4)/Sp(2) \times Sp(2)$ this form can be calculated explicitly. This provides a bi-differential operator relating the multiplication in $\mathbb{C}_\hbar[\mathbb{S}^4]$ to the multiplication in the dual

Hopf algebra $U_\hbar^*(\mathfrak{g})$, as explained in [KST].

We start from description of the classical conjugacy class $Sp(4)/Sp(2)\times Sp(2)$ and the Poisson structure on it. Next we collect the necessary facts about the quantum group $U_\hbar(\mathfrak{g})$. Further we describe the quantization of the polynomial algebra $\mathbb{C}_\hbar[G]$ and its properties. After that we construct the module M_λ and analyze the submodule structure of the tensor product $\mathbb{C}^4\otimes M_\lambda$. This allows us to realize $\mathbb{C}_\hbar[\mathbb{S}^4]$ by operators on M_λ and describe it in generators and relations. In conclusion, we calculate the invariant pairing between M_λ and its dual and discuss the star product on $\mathbb{C}_\hbar[\mathbb{S}^4]$.

2 The classical conjugacy class $Sp(4)/Sp(2)\times Sp(2)$

Let $Sp(4)$ denote the complex algebraic group of matrices preserving the antisymmetric skew-diagonal bilinear form $C_{ij}=\epsilon_i\delta_{ij'}$, where $i'=5-i$, $(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4)=(1,1,-1,-1)$, and δ_{ij} is the Kronecker symbol. We are interested in the conjugacy class of symplectic matrices with eigenvalues ±1 each of multiplicity 2. It is an $Sp(4)$-orbit with respect to the conjugation action on itself. The initial point A_o of the class and its isotropy subgroup can be taken as

$$A_o=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},\quad Sp(2)\times Sp(2)=\begin{pmatrix}*&0&0&*\\0&*&*&0\\0&*&*&0\\*&0&0&*\end{pmatrix}\subset Sp(4).$$

This conjugacy class is a subvariety in $Sp(4)$ defined by the system of equations

$$ACA^t-C=0,\quad \mathrm{Tr}(A)=0,\quad A^2-1=0, \tag{2.1}$$

where 1 in the third equality is the matrix unit. This is a system of polynomial equations on the matrix coefficients A_{ij}, which can be written in an alternative way:

$$A^t+CAC=0,\quad \mathrm{Tr}(A)=0,\quad A^2-1=0.$$

The first two equations are linear and allow for the following non-zero entries:

$$A=\begin{pmatrix}a&b&y&0\\c&-a&0&-y\\z&0&-a&b\\0&-z&c&a\end{pmatrix}.$$

The quadratic equation is then equivalent to

$$a^2 + bc + yz - 1 = 0. \tag{2.2}$$

Thus, the conjugacy class of A_o is isomorphic to the complex sphere $\mathbb{S}^4$. The ideal generated by the entries of the matrix equations (2.1) along with the zero trace condition is, in fact, generated by a single irreducible polynomial and is the defining ideal of the class.

Consider the r-matrix

$$r = \sum_{i=1}^{4}(e_{ii} \otimes e_{ii} - e_{ii} \otimes e_{i'i'}) + 2\sum_{\substack{i,j=1\\ i>j}}^{4}(e_{ij} \otimes e_{ji} - \epsilon_i\epsilon_j e_{ij} \otimes e_{i'j'}) \in \mathfrak{sp}(4) \otimes \mathfrak{sp}(4)$$

solving the classical Yang-Baxter equation, [D]. It induces a Drinfeld-Sklyanin bivector field π_0 on $Sp(4)$ making it a Poisson group, [D]. We are concerned with the following Poisson structure, π_1, on $Sp(4)/Sp(2) \times Sp(2) \simeq \mathbb{S}^4$:

$$\{A_1, A_2\} = \frac{1}{2}(A_2 r_{21} A_1 - A_1 r A_2 + A_2 A_1 r - r_{21} A_1 A_2). \tag{2.3}$$

This equation is understood in $\mathrm{End}(\mathbb{C}^4) \otimes \mathrm{End}(\mathbb{C}^4) \otimes \mathbb{C}[\mathbb{S}^4]$ and is a shorthand matrix form of the system of $n^2 \times n^2$ identities defining the Poisson brackets $\{A_{ij}, A_{kl}\}$ of the coordinate functions. The subscripts indicate the copy of $\mathrm{End}(\mathbb{C}^4)$ in the tensor square, as usual in the quantum group literature. Explicitly, the brackets of the generators $a, b, c, y, z \in \mathbb{C}[\mathbb{S}^4]$ read

$$\{a, b\} = ab, \quad \{a, c\} = -ac, \quad \{a, y\} = ay, \quad \{a, z\} = -az,$$

$$\{b, y\} = by, \quad \{b, z\} = -bz, \quad \{c, y\} = cy, \quad \{c, z\} = -cz,$$

$$\{y, z\} = 2a^2 + 2bc, \quad \{b, c\} = 2a^2.$$

This Poisson structure restricts from $Sp(4)$ and makes $\mathbb{S}^4$ a Poisson manifold under the conjugacy action of the Poisson group $Sp(4)$, [S]. In can be shown that such a Poisson structure on $\mathbb{S}^4$ is unique.

3 Quantum group $U_\hbar(\mathfrak{sp}(4))$

Throughout the paper, $\mathfrak{g}$ stands for the Lie algebra $\mathfrak{sp}(4)$. We are looking for quantization of the polynomial algebra $\mathbb{C}[\mathbb{S}^4]$ along the Poisson bracket (2.3) that is invariant under an action of the quantized universal enveloping algebra $U_\hbar(\mathfrak{g})$. In this section we recall the definition of $U_\hbar(\mathfrak{g})$, following [D].

The root system of $\mathfrak{g}$ is generated by the simple positive roots α, β, which are defined in the orthogonal basis ε_1, ε_2 as

$$\alpha = \varepsilon_1 - \varepsilon_2, \quad \beta = 2\varepsilon_2.$$

The other positive roots are $\gamma = \alpha + \beta$ and $\delta = 2\alpha + \beta$. Root vectors and Cartan elements are represented by the matrices

$$\begin{array}{c} e_\alpha = e_{12} - e_{34}, \quad e_\beta = e_{23}, \quad e_\gamma = e_{13} + e_{24}, \quad e_\delta = e_{14}, \\ f_\alpha = e_{21} - e_{43}, \quad f_\beta = e_{32}, \quad f_\gamma = e_{31} + f_{42}, \quad f_\delta = e_{41}, \\ h_\alpha = e_{11} - e_{22} + e_{33} - e_{44}, \quad h_\beta = 2e_{22} - 2e_{33}, \end{array} \tag{3.4}$$

where $\{e_{ij}\}$ is the standard matrix basis.

The quantized universal enveloping algebra (quantum group) $U_\hbar(\mathfrak{g})$ is a $\mathbb{C}[[\hbar]]$-algebra generated by the elements $e_\alpha, e_\beta, f_\alpha, f_\beta, h_\alpha, h_\beta$ subject to the commutator relations

$$[h_\alpha, e_\alpha] = 2e_\alpha, \quad [h_\alpha, f_\alpha] = -2f_\alpha, \quad [h_\beta, e_\beta] = 4e_\beta, \quad [h_\beta, f_\beta] = -4f_\beta,$$

$$[h_\alpha, e_\beta] = -2e_\beta, \quad [h_\alpha, f_\beta] = 2f_\beta, \quad [h_\beta, e_\alpha] = -2f_\alpha, \quad [h_\beta, f_\alpha] = 2f_\alpha,$$

$$[e_\alpha, f_\alpha] = \frac{q^{h_\alpha} - q^{-h_\alpha}}{q - q^{-1}}, \quad [e_\alpha, f_\beta] = 0 = [e_\beta, f_\alpha], \quad [e_\beta, f_\beta] = \frac{q^{h_\beta} - q^{-h_\beta}}{q^2 - q^{-2}},$$

plus the Serre relations

$$e_\alpha^3 e_\beta - (q^2 + 1 + q^{-2}) e_\alpha^2 e_\beta e_\alpha + (q^2 + 1 + q^{-2}) e_\alpha e_\beta e_\alpha^2 - e_\beta e_\alpha^3 = 0,$$

$$e_\beta^2 e_\alpha - (q^2 + q^{-2}) e_\beta e_\alpha e_\beta + e_\alpha e_\beta^2 = 0,$$

and similar relations for f_α, f_β. Here and further on $q = e^\hbar$.

The comultiplication Δ and antipode γ are defined on the generators by

$$\Delta(h) = h \otimes 1 + 1 \otimes h, \quad \gamma(h) = -h, \quad h \in \mathfrak{h},$$

$$\Delta(e_\mu) = e_\mu \otimes 1 + q^{h_\mu} \otimes e_\mu, \quad \gamma(e_\mu) = -q^{-h_\mu} e_\mu, \quad \mu = \alpha, \beta,$$

$$\Delta(f_\mu) = f_\mu \otimes q^{-h_\mu} + 1 \otimes f_\mu, \quad \gamma(f_\mu) = -f_\mu q^{h_\mu}, \quad \mu = \alpha, \beta.$$

The counit homomorphism $\varepsilon \colon U_\hbar(\mathfrak{g}) \to \mathbb{C}[[\hbar]]$ is nil on the generators.

Remark 3.1. The quantum group $U_\hbar(\mathfrak{g})$ is regarded as a $\mathbb{C}[[\hbar]]$-algebra, bearing in mind its application to deformation quantization. Accordingly, all its modules are understood as free $\mathbb{C}[[\hbar]]$-modules. However, we will suppress the reference to $\mathbb{C}[[\hbar]]$ in order to simplify the formulas. For instance, the vector representation of $U_\hbar(\mathfrak{g})$ will be denoted simply as $\mathbb{C}^4$. The tensor products and linear maps are also understood over $\mathbb{C}[[\hbar]]$.

Let us introduce higher root vectors $e_\gamma, f_\gamma, e_\delta, f_\delta \in U_\hbar(\mathfrak{g})$ (the coincidence in the notation for the weight and the antipode should not cause a confusion) by

$$f_\gamma = f_\beta f_\alpha - q^{-2} f_\alpha f_\beta, , \quad f_\delta = f_\gamma f_\alpha - q^2 f_\alpha f_\gamma,$$

$$e_\gamma = e_\alpha e_\beta - q^2 e_\beta e_\alpha, \quad e_\delta = e_\alpha e_\gamma - q^{-2} e_\gamma e_\alpha.$$

Our definition of e_δ, f_δ is different from the usual definition $e_\delta = [e_\alpha, e_\gamma]$, $f_\delta = [f_\gamma, f_\alpha]$, corresponding to $(\alpha, \gamma) = 0$, [ChP]. The reason for that will be clear later on. The elements h_α, h_β span the Cartan subalgebra $\mathfrak{h}$ and generate the Hopf subalgebra $U_\hbar(\mathfrak{h}) \subset U_\hbar(\mathfrak{g})$. The vectors e_α, e_β along with $\mathfrak{h}$ generate the positive Borel subalgebra $U_\hbar(\mathfrak{b}^+)$ in $U_\hbar(\mathfrak{g})$. Similarly, f_α, f_β, and $\mathfrak{h}$ generate the negative Borel subalgebra $U_\hbar(\mathfrak{b}^-)$. They are Hopf subalgebras of $U_\hbar(\mathfrak{g})$.

Lemma 3.2. *The root vectors satisfy the relations*

$$e_\gamma e_\beta - q^{-2} e_\beta e_\gamma = 0, \quad [e_\alpha, e_\delta] = 0, \quad [e_\beta, e_\delta] = 0, \quad [e_\gamma, e_\delta] = 0,$$

$$f_\beta f_\gamma - q^2 f_\gamma f_\beta = 0, \quad [f_\alpha, f_\delta] = 0, \quad [f_\beta, f_\delta] = 0, \quad [f_\gamma, f_\delta] = 0.$$

Proof. The first two equalities in both lines are simply a rephrase of the Serre relations in the new terms. The last equalities follow from the second and third. Let us check the third equality, say, in the first line:

$$\begin{aligned}
e_\beta e_\delta &= e_\beta(e_\alpha e_\gamma - q^{-2} e_\gamma e_\alpha) = e_\beta e_\alpha e_\gamma - e_\gamma e_\beta e_\alpha \\
&= q^{-2} e_\alpha e_\beta e_\gamma - q^{-2} e_\gamma^2 - q^{-2} e_\gamma e_\alpha e_\beta + q^{-2} e_\gamma^2 = q^{-2} e_\alpha e_\beta e_\gamma - q^{-2} e_\gamma e_\alpha e_\beta \\
&= e_\alpha e_\gamma e_\beta - q^{-2} e_\gamma e_\alpha e_\beta = e_\delta e_\beta,
\end{aligned}$$

as required. □

Denote by $U_\hbar(\mathfrak{n}_0^\pm)$ the subalgebras generated by, respectively, positive and negative Chevalley generators. The Borel subalgebras $U_\hbar(\mathfrak{b}^\pm)$ are freely generated by $U_\hbar(\mathfrak{n}_0^\pm)$ over $U_\hbar(\mathfrak{h})$ with respect to right or left multiplication. These equalities facilitate the following

Corollary 3.3. *The positive (respectively, negative) root vectors generate a Poincarè-Birkgoff-Witt basis in $U_\hbar(\mathfrak{n}_0^\pm)$.*

Proof. The presence of PBW basis in the quantum group is a well known fact. However, we use a non-standard definition of the root vectors e_δ, f_δ, therefore the lemma is substantial. To prove it, say, for $U_\hbar(\mathfrak{n}_0^-)$ one should check that the system of monomials $f_\alpha^a f_\gamma^c f_\delta^d f_\beta^b = f_\alpha^a f_\delta^d f_\gamma^c f_\beta^b$, where a, b, c, and d are non-negative integers, is linearly independent and complete in $U_\hbar(\mathfrak{n}^-)$. The ordered sequence of the elements $f_\alpha, f'_\delta = [f_\gamma, f_\alpha], f_\gamma, f_\beta$ does generate a PBW basis, [ChP]. Using this fact along with Lemma 3.2 relations, one can easily check the statement via the substitution $f'_\delta = f_\delta + (q^2-1)f_\alpha f_\gamma$. □

4 The algebra of quantized polynomials on $Sp(4)$

We adopt the convention throughout the paper that G stands for the complex algebraic group $Sp(4)$. The conjugacy class of our interest is a closed affine variety in G, and its polynomial ring is a quotient of the polynomial ring $\mathbb{C}[G]$ by a certain ideal. Our goal is to obtain an analogous description of the quantum conjugacy class. To that end, we need to describe the quantum analog of the algebra $\mathbb{C}[G]$ first.

Recall from [J, B] that the image of the universal R-matrix of the quantum group $U_\hbar(\mathfrak{g})$ in the vector representation is equal, up to a scalar factor, to

$$R = \sum_{i,j=1}^{4} q^{\delta_{ij}-\delta_{ij'}} e_{ii}\otimes e_{jj} + (q-q^{-1})\sum_{\substack{i,j=1\\ i>j}}^{4}(e_{ij}\otimes e_{ji} - q^{\rho_i-\rho_j}\epsilon_i\epsilon_j e_{ij}\otimes e_{i'j'}),$$

where $(\rho_1,\rho_2,\rho_3,\rho_4) = (2,1,-1,-2)$.

Denote by S the $U_\hbar(\mathfrak{g})$-invariant operator $PR \in \mathrm{End}(\mathbb{C}^4)\otimes\mathrm{End}(\mathbb{C}^4)$, where P is the ordinary flip of $\mathbb{C}^4\otimes\mathbb{C}^4$. This operator has three invariant projectors to its eigenspaces, among which there is a one-dimensional projector $\sim \sum_{i,j=1}^4 q^{\rho_i-\rho_j}\epsilon_i\epsilon_j e_{i'j}\otimes e_{ij'}$ to the trivial $U_\hbar(\mathfrak{g})$-submodule, call it κ.

Denote by $\mathbb{C}_\hbar[G]$ the associative algebra generated by the entries of the matrix $K = ||k_{ij}||_{i,j=1}^4 \in \mathrm{End}(\mathbb{C}^4)\otimes\mathbb{C}_\hbar[G]$ modulo the relations

$$S_{12}K_2S_{12}K_2 = K_2S_{12}K_2S_{12}, \quad K_2S_{12}K_2\kappa = -q^{-5}\kappa = \kappa K_2S_{12}K_2. \tag{4.5}$$

These relations are understood in $\mathrm{End}(\mathbb{C}^4)\otimes\mathrm{End}(\mathbb{C}^4)\otimes\mathbb{C}_\hbar[G]$, and the indices distinguish the two copies of $\mathrm{End}(\mathbb{C}^4)$, as usual.

The algebra $\mathbb{C}_\hbar[G]$ is an equivariant quantization of $\mathbb{C}[G]$, [RS, FRT], which is different from the RTT-quantization and is not a Hopf algebra. It carries a $U_\hbar(\mathfrak{g})$-action, which is a deformation of the conjugation $U(\mathfrak{g})$-action on $\mathbb{C}[G]$. It admits a $U_\hbar(\mathfrak{g})$-equivariant algebra monomorphism to $U_\hbar(\mathfrak{g})$, where the latter is regarded as the adjoint module. The monomorphism is implemented by the assignment

$$K \mapsto (\phi \otimes \mathrm{id})(\mathcal{R}_{21}\mathcal{R}) = \mathcal{Q} \in \mathrm{End}(\mathbb{C}^4) \otimes U_\hbar(\mathfrak{g}),$$

where $\phi\colon U_\hbar(\mathfrak{g}) \to \mathrm{End}(\mathbb{C}^4)$ is the vector representation and $\mathcal{R}$ is the universal R-matrix of $U_\hbar(\mathfrak{g})$. The matrix $\mathcal{Q}$ is important for our presentation, and the reader is referred to [M2] for detailed explanation of its role in quantization and for its basic characteristics.

5 The generalized Verma module M_λ

Denote by $\mathfrak{l}$ the Levi subalgebra in $\mathfrak{g} = \mathfrak{sp}(4)$ spanned by $e_\beta, f_\beta, h_\beta, h_\alpha$. It is a Lie subalgebra of maximal rank, and its semisimple part is isomorphic to $\mathfrak{sl}(2) \simeq \mathfrak{sp}(2)$. The universal enveloping algebra $U(\mathfrak{l})$ is quantized as a Hopf subalgebra in $U_\hbar(\mathfrak{g})$. Denote by $\mathfrak{n}^+$ and $\mathfrak{n}^-$ the nilpotent subalgebras in $\mathfrak{g}$ spanned, respectively, by $\{e_\alpha, e_\gamma, e_\delta\}$ and $\{f_\alpha, f_\gamma, f_\delta\}$. The sum $\mathfrak{l} + \mathfrak{n}^\pm$ is a parabolic subalgebra $\mathfrak{p}^\pm \subset \mathfrak{g}$ whose universal enveloping algebra is quantized to a Hopf subalgebra in $U_\hbar(\mathfrak{p}^\pm) \subset U_\hbar(\mathfrak{g})$.

Let $U_\hbar(\mathfrak{n}^\pm)$ be the subalgebras in $U_\hbar(\mathfrak{g})$ generated by the quantum root vectors $\{e_\alpha, e_\gamma, e_\delta\}$ and $\{f_\alpha, f_\gamma, f_\delta\}$, respectively. The quantum group $U_\hbar(\mathfrak{g})$ is a free $U_\hbar(\mathfrak{n}^-) - U_\hbar(\mathfrak{n}^+)$-bimodule generated by $U_\hbar(\mathfrak{l})$:

$$U_\hbar(\mathfrak{p}^-) = U_\hbar(\mathfrak{n}^-)U_\hbar(\mathfrak{l}), \quad U_\hbar(\mathfrak{g}) = U_\hbar(\mathfrak{n}^-)U_\hbar(\mathfrak{l})U_\hbar(\mathfrak{n}^+), \quad U_\hbar(\mathfrak{p}^+) = U_\hbar(\mathfrak{l})U_\hbar(\mathfrak{n}^+). \tag{5.6}$$

The factorizations of $U_\hbar(\mathfrak{p}^\pm)$ have the structure of smash product.

Fix a weight $\lambda \in \mathfrak{h}^*$ orthogonal to β. It can be regarded as a one-dimensional representation of $U_\hbar(\mathfrak{l})$,

$$\lambda\colon e_\beta, f_\beta, h_\beta \mapsto 0, \quad \lambda\colon h_\alpha \mapsto (\alpha, \lambda),$$

which can be extended to a representation of $U_\hbar(\mathfrak{p}^+)$ by $\lambda\colon e_\alpha \mapsto 0$. Let $\mathbb{C}_\lambda$ denote the one-dimensional vector space supporting this representation.

Consider the scalar parabolic Verma module $\hat{M}_\lambda$ induced from $\mathbb{C}_\lambda$,

$$\hat{M}_\lambda = U_\hbar(\mathfrak{g}) \otimes_{U_\hbar(\mathfrak{p}^+)} \mathbb{C}_\lambda.$$

As a module over $U_\hbar(\mathfrak{n}^-)$, it is freely generated by its highest weight vector v_λ. As a module over the Cartan subalgebra, it isomorphic to $U_\hbar(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda$, where $U_\hbar(\mathfrak{n}^-)$ is the natural module over $U_\hbar(\mathfrak{h})$.

The $U_\hbar(\mathfrak{g})$-module $\hat{M}_\lambda$ is irreducible except for special values of λ, when $\hat{M}_\lambda$ may contain singular vectors. Recall that a weight vector is called singular if it is annihilated by the positive Chevalley generators. Such vectors generate submodules in $\hat{M}_\lambda$, where they carry the highest weight. We are looking for such λ that $\hat{M}_\lambda$ admits a singular vector of weight $\lambda-\delta$. Quotienting out the corresponding submodule yields a module that supports quantization of $\mathbb{C}[\mathbb{S}^4]$.

Proposition 5.1. *The module $\hat{M}_\lambda$ admits a singular vector of weight $\lambda - \delta$ if and only if $q^{2(\alpha,\lambda)} = -q^{-2}$. Then $f_\delta v_\lambda$ is the singular vector.*

Proof. The general expression for the vector of weight $\lambda - \delta$ in M_λ is

$$\big(f_\alpha^2 f_\beta - (a+b) f_\alpha f_\beta f_\alpha + ab f_\beta f_\alpha^2\big) v_\lambda = \big(-(a+b) f_\alpha f_\beta f_\alpha + ab f_\beta f_\alpha^2\big) v_\lambda,$$

where a, b are some scalars. For this vector being singular, we have a system of two equations on a, b resulted from the action of e_β and e_α:

$$\left\{\begin{array}{rcl} \big(-(a+b) f_\alpha [e_\beta, f_\beta] f_\alpha + ab [e_\beta, f_\beta] f_\alpha^2\big) v_\lambda &=& 0, \\ \big(-(a+b)[e_\alpha, f_\alpha] f_\beta f_\alpha + ab f_\beta [e_\alpha, f_\alpha] f_\alpha + ab f_\beta f_\alpha [e_\alpha, f_\alpha]\big) v_\lambda &=& 0. \end{array}\right.$$

The non-zero solution of this system is unique (up to permutation $a \leftrightarrow b$) and equal to

$$q^{2(\alpha,\lambda)} = -q^{-2}, \quad a = q^2, \quad b = q^{-2},$$

as required. Finally, notice that $f_\delta = f_\alpha^2 f_\beta - (q^2 + q^{-2}) f_\alpha f_\beta f_\alpha + f_\beta f_\alpha^2$. This completes the proof. □

Denote by M_λ the quotient of $\hat{M}_\lambda$ by the submodule $U_\hbar(\mathfrak{g}) f_\delta v_\lambda$. By Corollary 3.3, the vectors $f_\alpha^k f_\gamma^l f_\delta^m v_\lambda$ for all non-negative integer k, l, m form a basis in $\hat{M}_\lambda$. Therefore, M_λ is spanned by $f_\alpha^k f_\gamma^l v_\lambda$, $k, l \geqslant 0$.

Proposition 5.2. *The module M_λ is irreducible.*

Proof. Irreducibility follows from non-degeneracy of the invariant bilinear pairing of M_λ with its dual, see Section 8. One can also verify that M_λ has no singular vector. Omitting the details, the action of the positive Chevalley generators on M_λ is given by

$$\begin{array}{rcl} e_\alpha f_\alpha^k f_\gamma^m v_\lambda &=& q^{(\alpha,\lambda)+1} \dfrac{q^{2k} - q^{-2k}}{(q - q^{-1})^2} f_\alpha^{k-1} f_\gamma^m v_\lambda, \\ e_\beta f_\alpha^k f_\gamma^m v_\lambda &=& \dfrac{q^{2m} - q^{-2m}}{q^2 - q^{-2}} f_\alpha^{k+1} f_\gamma^{m-1} v_\lambda. \end{array}$$

Here we assume that $k > 0$ in the first line and $m > 0$ in the second; otherwise the right hand side is nil. This immediately implies the absence of singular vectors in M_λ. □

6 The $U_\hbar(\mathfrak{g})$-module $\mathbb{C}^4 \otimes M_\lambda$

The tautological assignment (3.4) defines the four-dimensional irreducible representation of $U(\mathfrak{g})$. Similar assignment on the quantum Chevalley generators and Cartan elements defines a representation of $U_\hbar(\mathfrak{g})$. Our next object of interest is the $U_\hbar(\mathfrak{g})$-module $\mathbb{C}^4 \otimes M_\lambda$. In particular, we shall study the decomposition of $\mathbb{C}^4 \otimes M_\lambda$ into direct sum of irreducible submodules.

Choose the standard basis $\{w_i\}_{i=1}^4 \subset \mathbb{C}^4$ of columns with the only nonzero entry 1 in the i-TtH place from the top. Their weights are $\varepsilon_1, \varepsilon_2, -\varepsilon_2, -\varepsilon_1$, respectively. As a $U_\hbar(\mathfrak{l})$-module, $\mathbb{C}^4$ splits into the sum of two one-dimensional blocks of weights $\pm\varepsilon_1$ and one two-dimensional block of highest weights ε_2. The parabolic Verma module contains three blocks of highest weights $\varepsilon_1+\lambda, \varepsilon_2+\lambda, -\varepsilon_1+\lambda$, which we denote by $\hat{V}_{\varepsilon_1+\lambda}$, $\hat{V}_{\varepsilon_2+\lambda}$, $\hat{V}_{-\varepsilon_1+\lambda}$. For generic λ these submodules are irreducible, and

$$\mathbb{C}^4 \otimes \hat{M}_\lambda = \hat{V}_{\varepsilon_1+\lambda} \oplus \hat{V}_{\varepsilon_2+\lambda} \oplus \hat{V}_{-\varepsilon_1+\lambda}. \tag{6.7}$$

All these blocks are parabolic Verma modules corresponding to the $U_\hbar(\mathfrak{l})$-submodules of $\mathbb{C}^4$.

Clearly $\lambda+\varepsilon_1$ is the highest weight of $\mathbb{C}^4 \otimes \hat{M}_\lambda$ and $w_1 \otimes v_\lambda$ is the highest weight vector. The other singular vectors in $\mathbb{C}^4 \otimes \hat{M}_\lambda$ are given next.

Lemma 6.1. *The vectors*

$$\begin{aligned}
u_{\varepsilon_1} &= w_1 \otimes v_\lambda, \\
u_{\varepsilon_2} &= w_1 \otimes f_\alpha v_\lambda - q\frac{q^{(\alpha,\lambda)} - q^{-(\alpha,\lambda)}}{q-q^{-1}} w_2 \otimes v_\lambda, \\
u_{-\varepsilon_1} &= f_\delta w_1 \otimes v_\lambda + (q^{(\lambda,\alpha)+1} + q^{-(\lambda,\alpha)-1}) \times \\
&\times \left(q w_2 \otimes f_\beta f_\alpha v_\lambda - q^3 w_3 \otimes f_\alpha v_\lambda - q^4 \frac{q^{(\lambda,\alpha)} - q^{-(\lambda,\alpha)}}{q-q^{-1}} w_4 \otimes v_\lambda\right)
\end{aligned}$$

are singular and generate the submodules $\hat{V}_{\varepsilon_1+\lambda}$, $\hat{V}_{\varepsilon_2+\lambda}$, $\hat{V}_{-\varepsilon_1+\lambda}$, respectively.

Proof. One should check that u_{ε_1}, u_{ε_2}, $u_{-\varepsilon_1}$ are annihilated by e_α and e_β. That is obvious for u_{ε_1} and relatively easy for u_{ε_2}. The case of $u_{-\varepsilon_1}$ requires bulky but straightforward calculation, which is omitted here. □

We denote by $V_{\varepsilon_1+\lambda}$, $V_{\varepsilon_2+\lambda}$, $V_{-\varepsilon_1+\lambda}$ the images of $\hat{V}_{\varepsilon_1+\lambda}$, $\hat{V}_{\varepsilon_2+\lambda}$, $\hat{V}_{-\varepsilon_1+\lambda}$ under the projection $\mathbb{C}^4 \otimes \hat{M}_\lambda \to \mathbb{C}^4 \otimes M_\lambda$, assuming $q^{2(\alpha,\lambda)} = -q^{-2}$. An important fact is that for $q^{2(\alpha,\lambda)} = -q^{-2}$ the singular vector $u_{-\varepsilon_1}$ turns into $w_1 \otimes f_\delta v_\lambda$ and thus disappears from $\mathbb{C}^4 \otimes M_\lambda$. The submodule $\hat{V}_{-\varepsilon_1+\lambda}$ is killed by the projection $\mathbb{C}^4 \otimes \hat{M}_\lambda \to \mathbb{C}^4 \otimes M_\lambda$, so $V_{-\varepsilon_1+\lambda} = \{0\}$.

Proposition 6.2. *The module $\mathbb{C}^4 \otimes M_\lambda$ is a direct sum of the submodules $V_{\varepsilon_1+\lambda}$ and $V_{\varepsilon_2+\lambda}$.*

Proof. The modules $V_{\varepsilon_1+\lambda}$ and $V_{\varepsilon_2+\lambda}$ have zero intersection, as they carry different eigenvalues of the invariant matrix $\mathcal{Q}$, see below. We must show that the sum $V_{\varepsilon_1+\lambda} \oplus V_{\varepsilon_2+\lambda}$ exhausts all of $\mathbb{C}^4 \otimes M_\lambda$. To that end, it is sufficient to show that $\mathbb{C}^4 \otimes v_\lambda$ lies in $V = V_{\varepsilon_1+\lambda} \oplus V_{\varepsilon_2+\lambda}$. Indeed, then for all $u \in U_\hbar(\mathfrak{g})$ and $w \in \mathbb{C}^4$,

$$w \otimes u v_\lambda = \Delta(u^{(2)})\big(\gamma^{-1}(u^{(1)}) w \otimes v\big) \in V,$$

as required.

In what follows $\equiv$ will mean equality modulo V. Obviously, $w_1 \otimes v_\lambda \equiv 0$. Applying f_α to $w_1 \otimes v_\lambda$ gives $w_1 \otimes f_\alpha v_\lambda + q^{-(\alpha,\lambda)} w_2 \otimes v_\lambda \equiv 0$. Comparing this with $u_{\varepsilon_2+\lambda} \in V$ we conclude that $w_2 \otimes v_\lambda \equiv 0$. Applying f_β to $w_2 \otimes v_\lambda$ gives $w_3 \otimes v_\lambda \equiv 0$.

Thus, we are left to check that $w_4 \otimes v \in V$. We have

$$0 \equiv f_\alpha(w_1 \otimes v_\lambda) \equiv w_1 \otimes f_\alpha v_\lambda, \quad 0 \equiv f_\alpha(w_2 \otimes v_\lambda) = w_2 \otimes f_\alpha v_\lambda,$$

$$0 \equiv f_\alpha^2(w_1 \otimes v_\lambda) \equiv f_\alpha(w_1 \otimes f_\alpha v_\lambda) = w_1 \otimes f_\alpha^2 v_\lambda + q^{-2-(\alpha,\lambda)} w_2 \otimes f_\alpha v_\lambda \equiv w_1 \otimes f_\alpha^2 v_\lambda,$$

$$0 \equiv f_\beta f_\alpha^2(w_1 \otimes v_\lambda) \equiv f_\beta(w_1 \otimes f_\alpha^2 v_\lambda) = w_1 \otimes f_\beta f_\alpha^2 v_\lambda. \tag{6.8}$$

Further,

$$0 \equiv f_\beta f_\alpha(w_1 \otimes v_\lambda) \equiv f_\beta(w_1 \otimes f_\alpha v_\lambda) = w_1 \otimes f_\beta f_\alpha v_\lambda,$$

$$0 \equiv f_\alpha f_\beta f_\alpha(w_1 \otimes v_\lambda) \equiv f_\alpha(w_1 \otimes f_\beta f_\alpha v_\lambda) = w_1 \otimes f_\alpha f_\beta f_\alpha v_\lambda + q^{-(\alpha,\lambda)} w_2 \otimes f_\beta f_\alpha v_\lambda. \tag{6.9}$$

Combining (6.8) and (6.9), we calculate $f_\delta(w_1 \otimes v_\lambda) \in V$:

$$0 \equiv \big(-(q^2+q^{-2}) f_\alpha f_\beta f_\alpha + f_\beta f_\alpha^2\big)(w_1 \otimes v_\lambda) \equiv w_1 \otimes f_\delta v_\lambda - (q^2+q^{-2}) q^{-(\alpha,\lambda)} w_2 \otimes f_\beta f_\alpha v_\lambda.$$

The first equality takes place because $f_\beta(w_1 \otimes v_\lambda) = 0$. Therefore $w_2 \otimes f_\beta f_\alpha v_\lambda \equiv 0$, and

$$0 \equiv f_\beta(w_2 \otimes f_\alpha v_\lambda) = w_2 \otimes f_\beta f_\alpha v_\lambda + q^2 w_3 \otimes f_\alpha v_\lambda \equiv q^2 w_3 \otimes f_\alpha v_\lambda.$$

Finally,

$$0 \equiv f_\alpha(w_3 \otimes v_\lambda) = w_3 \otimes f_\alpha v_\lambda - q^{-(\alpha,\lambda)} w_4 \otimes v_\lambda \equiv -q^{-(\alpha,\lambda)} w_4 \otimes v_\lambda,$$

as required. □

Now consider the action of the matrix $\mathcal{Q}$ on $\mathbb{C}^4 \otimes \hat{M}_\lambda$. It satisfies a cubic polynomial equation, and its eigenvalues in $\mathbb{C}^4 \otimes \hat{M}_\lambda$ can be found in [M2]:

$$q^{2(\lambda,\varepsilon_1)} = -q^{-2},$$
$$q^{2(\lambda+\rho,\varepsilon_2)-2(\rho,\nu)} = q^{2(\rho,\varepsilon_2)-2(\rho,\nu)} = q^{-2},$$
$$q^{2(\lambda+\rho,-\varepsilon_1)-2(\rho,\nu)} = q^{-2(\lambda,\varepsilon_1)-4(\rho,\nu)} = -q^{-6}.$$

The operator $\mathcal{Q}$ is semisimple on $\mathbb{C}^4 \otimes \hat{M}_\lambda$ for generic λ. Due to Proposition 6.2, it is semisimple on $\mathbb{C}^4 \otimes M_\lambda$ and has eigenvalues $\pm q^{-2}$.

7 Quantization of $\mathbb{S}^4$

Let ϕ denote the representation homomorphism $U_\hbar(\mathfrak{g}) \to \mathrm{End}(\mathbb{C}^4)$. The q-trace of $\mathcal{Q}$ is a weighted trace $\mathrm{Tr}_q(\mathcal{Q}) = \mathrm{Tr}(D\mathcal{Q})$, where D is the diagonal matrix $\mathrm{diag}(q^4, q^2, q^{-2}, q^{-4})$. It belongs to the center of $U_\hbar(\mathfrak{g})$ and hence the center of $\mathbb{C}_\hbar[G] \subset U_\hbar(\mathfrak{g})$. A module of highest weight λ defines a central character χ_λ of the algebra $\mathbb{C}_\hbar[G]$, which returns zero on $\mathrm{Tr}_q(\mathcal{Q})$:

$$\begin{aligned} \chi_\lambda\big(\mathrm{Tr}_q(\mathcal{Q})\big) &= \mathrm{Tr}\big(\phi(q^{h_\lambda+h_\rho})\big) = q^{2(\lambda+\rho,\varepsilon_1)} + q^{2(\lambda+\rho,\varepsilon_2)} + q^{2(\lambda+\rho,-\varepsilon_2)} + q^{2(\lambda+\rho,-\varepsilon_1)} \\ &= q^{2(\lambda,\varepsilon_1)+4} + q^2 + q^{-2} + q^{-2(\lambda,\varepsilon_1)-4} = -q^2 + q^2 + q^{-2} - q^{-2} = 0, \end{aligned}$$

cf. [M2]. Thus, the q-trace of the matrix $\mathcal{Q}$ vanishes in M_λ. Also, the entries of the matrix $\mathcal{Q}^2 - q^{-4}$ are annihilated in $\mathrm{End}(M_\lambda)$, as discussed in the previous section.

Proposition 7.1. *The image of* $\mathbb{C}_\hbar[G]$ *in* $\mathrm{End}(M_\lambda)$ *is a quantization of* $\mathbb{C}_\hbar[\mathbb{S}^4]$. *It is isomorphic to the subalgebra in* $U_\hbar(\mathfrak{g})$ *generated by the entries of the matrix* $\mathcal{Q} = (\phi \otimes \mathrm{id})(\mathcal{R}_{21}\mathcal{R})$, *modulo the relations*

$$\mathcal{Q}^2 = q^{-4}, \quad \mathrm{Tr}_q(\mathcal{Q}) = 0. \tag{7.10}$$

Proof. The center of $\mathbb{C}_\hbar[G]$ is formed by $U_\hbar(\mathfrak{g})$-invariants, which are also central in $U_\hbar(\mathfrak{g})$. Therefore, $\ker \chi_\lambda$ lies in the kernel of the representation $\mathbb{C}_\hbar[G] \to \mathrm{End}(M_\lambda)$. The quotient of $\mathbb{C}_\hbar[G]$ by the ideal generated by $\ker \chi_\lambda$ is free over $\mathbb{C}[[\hbar]]$ and is a direct sum of isotypical $U_\hbar(\mathfrak{g})$-components of finite multiplicities, [M1]. Therefore, the image of $\mathbb{C}_\hbar[G]$ in $\mathrm{End}(M_\lambda)$ is a direct sum of isotypical $U_\hbar(\mathfrak{g})$-components which are free and finite over $\mathbb{C}[[\hbar]]$.

The ideal in $\mathbb{C}_\hbar[G]$ generated by (7.10) lies in the kernel of the homomorphism $\phi\colon \mathbb{C}_\hbar[G] \to \mathrm{End}(M_\lambda)$ and turns into the defining ideal of $\mathbb{S}^4$ modulo $\hbar$. Therefore this ideal coincides with $\ker \phi$, and the quotient of $\mathbb{C}_\hbar[G]$ by this ideal is a quantization of $\mathbb{C}[\mathbb{S}^4]$, see [M2] for details. ∎

We will give a more explicit description of $\mathbb{C}_\hbar[\mathbb{S}^4]$. The matrix $\mathcal{Q}$ is the image of the matrix K from Section 4 under the embedding $\mathbb{C}_\hbar[G] \to U_\hbar(\mathfrak{g})$. The algebra $\mathbb{C}_\hbar[\mathbb{S}^4]$ is generated by elements a, b, c, y, z arranged in the matrix

$$\begin{pmatrix} a & b & y & 0 \\ c & -q^2a & 0 & -y \\ z & 0 & -q^2a & q^2b \\ 0 & -z & q^2c & q^4a \end{pmatrix}.$$

This matrix is obtained from K by imposing the linear relations on its entries derived from (4.5) by the substitution $K^2 = q^{-4}$, $\mathrm{Tr}_q(K) = 0$. The generators of $\mathbb{C}_\hbar[\mathbb{S}^4]$ obey the relations

$$ab = q^2ba, \quad ac = q^{-2}ca, \quad ay = q^2ya, \quad az = q^{-2}za,$$

$$by = q^2yb, \quad bz = q^{-2}zb, \quad cy = q^2yc, \quad cz = q^{-2}zc,$$

$$[b, c] = (q^4 - 1)a^2, \quad [y, z] = (q^4 - 1)a^2 + (q^4 - 1)bc,$$

plus

$$a^2 + bc + yz = q^{-4},$$

which is a deformation of (2.2).

Remark that $\mathbb{C}_\hbar[\mathbb{S}^4]$ has a 1-dimensional representation $a, b, c \mapsto 0$, $y, z \mapsto q^{-2}$. Therefore it can be realized as a subalgebra in the Hopf algebra dual to $U_\hbar(\mathfrak{g})$, as explained in [DM].

8 On invariant star product on $\mathbb{S}^4$

It follows from [KST] that the star product on the conjugacy class $Sp(4)/Sp(2) \times Sp(2)$ can be calculated by means of the invariant pairing between the modules $M^-_{-\lambda}$ and M^+_λ, where $M^+_\lambda = M_\lambda$ and $M^-_{-\lambda}$ is its restricted dual. The module $M^-_{-\lambda}$ is the quotient of the lower parabolic Verma module $\hat{M}^-_{-\lambda} = U_\hbar(\mathfrak{g}) \otimes_{U_\hbar(\mathfrak{p}^-)} \mathbb{C}_{-\lambda}$ by the submodule $U_\hbar(\mathfrak{g})e_\delta v_{-\lambda}$. Explicitly, the pairing is given by the assignment

$$xv_{-\lambda} \otimes yv_\lambda \mapsto \langle xv_{-\lambda}, yv_\lambda \rangle = \lambda([\gamma(x)y]_\mathfrak{l}).$$

Here $x \mapsto [x]_\mathfrak{l}$ is the projection $U_\hbar(\mathfrak{g}) \to U_\hbar(\mathfrak{l})$ along $U'_\hbar(\mathfrak{n}^-)U_\hbar(\mathfrak{g}) + U_\hbar(\mathfrak{g})U'_\hbar(\mathfrak{n}^+)$, where the prime designates the kernel of the counit. This projection is facilitated by the triangular factorization (5.6) and it is a homomorphism of $U_\hbar(\mathfrak{l})$-bimodules.

The modules $M^{\pm}_{\pm\lambda}$ are irreducible if and only if the pairing is non-degenerate, [Ja]. Our next goal is to calculate it explicitly. Put

$$x_1 = e_\alpha, \quad x_2 = e_\gamma, \quad \tilde{x}_1 = e_\alpha, \quad \tilde{x}_2 = q^4 e_\beta e_\alpha - q^2 e_\alpha e_\beta, \quad y_1 = f_\alpha, \quad y_2 = f_\gamma.$$

The twiddled root vectors are related to non-twiddled via the antipode:

$$\gamma(\tilde{e}_\gamma) = q^4 q^{-h_\beta} e_\beta q^{-h_\alpha} e_\alpha - q^2 q^{-h_\alpha} e_\alpha q^{-h_\beta} e_\beta = -q^{-h_\gamma} e_\gamma,$$

with a similar relation for the root α.

The following system of monomials constitute bases in $M^-_{-\lambda}$ and M^+_λ:

$$\left(y_1^k y_2^m v_\lambda\right)_{k,m=0}^\infty \subset M^+_\lambda, \quad \left(\tilde{x}_1^k \tilde{x}_2^m v_\lambda\right)_{k,m=0}^\infty \subset M^-_{-\lambda}.$$

Further we need the identities

$$[e_\alpha, f_\gamma] = -(q+q^{-1}) q^{-2} f_\beta q^{-h_\alpha}, \quad [e_\beta, f_\gamma] = f_\alpha q^{h_\beta}, \tag{8.11}$$

$$[e_\gamma, f_\alpha] = -(q+q^{-1}) e_\beta q^{h_\alpha}, \quad [e_\gamma, f_\beta] = q^2 e_\beta q^{-h_\beta}, \tag{8.12}$$

which can be proved directly from the defining relations $U_\hbar(\mathfrak{g})$ and the definition of e_γ and f_γ. Also, one can check that

$$[e_\gamma, f_\gamma] = \frac{q^{h_\gamma} - q^{-h_\gamma}}{q - q^{-1}}. \tag{8.13}$$

Thus, for $\nu = \alpha, \gamma$ and any positive integer k we have

$$[e_\nu, f_\nu^k] = q^{h_\nu + 1} \frac{1 - q^{-2k}}{(q - q^{-1})^2} + q^{-h_\nu - 1} \frac{1 - q^{2k}}{(q - q^{-1})^2}. \tag{8.14}$$

Lemma 8.1. *The matrix coefficient $\langle \tilde{x}_1^i \tilde{x}_2^j v_{-\lambda}, y_1^k, y_2^m v_\lambda \rangle$ is zero unless $i = k$, $j = m$.*

Proof. It follows that $[e_\alpha, f_\gamma^k]$ belongs to the left ideal $U_\hbar(\mathfrak{g}) f_\beta$, hence $x_1 y_2^k v_\lambda = 0$. We have

$$x_1 y_1^k y_2^m v_\lambda = e_\alpha f_\alpha^k f_\gamma^m v_\lambda \sim f_\alpha^{k-1} f_\gamma^m v_\lambda + f_\alpha^{k-1} [e_\alpha, f_\gamma^m] v_\lambda = f_\alpha^{k-1} f_\gamma^m v_\lambda = y_1^{k-1} y_2^m v_\lambda.$$

Using this, we find

$$\langle \tilde{x}_1^i \tilde{x}_2^j v_{-\lambda}, y_1^k, y_2^m v_\lambda \rangle = \langle v_{-\lambda}, x_2^j x_1^i y_1^k y_2^m v_\lambda \rangle \sim \langle v_{-\lambda}, x_2^j x_1^{i-k} y_2^m v_\lambda \rangle = 0,$$

assuming $i > k$. If $i < k$, then y_1 is pulled to the left in a similar way, so the matrix coefficient can be non-zero only if $i = k$. Suppose $i = k$ but $j > m$. Then

$$\langle \tilde{x}_1^k \tilde{x}_2^j v_{-\lambda}, y_1^k, y_2^m v_\lambda \rangle \sim \langle v_{-\lambda}, x_2^k x_1^i y_1^k y_2^m v_\lambda \rangle \sim \langle v_{-\lambda}, x_2^j y_2^m v_\lambda \rangle \sim \langle v_{-\lambda}, x_2^{j-m} v_\lambda \rangle = 0.$$

The case $i = k$, $j < m$ is verified similarly. ∎

Define $[2l]_q!! = \prod_{i=1}^{l}[2l]_q$ for all positive integer l and put $[0]_q!! = 1$.

Proposition 8.2. *The matrix coefficients of the invariant pairing are given by the formula*

$$\langle \tilde{x}_1^k \tilde{x}_2^m v_{-\lambda}, y_1^k y_2^m v_\lambda \rangle = q^{m(m-2)+k(k-2)} \Big(\frac{1}{q-q^{-1}}\Big)^{k+m} [2k]_q!![2m]_q!!,$$

for all $k, m = 0, 1, \ldots$.

Proof. The matrix coefficient $\langle \tilde{x}_1^k \tilde{x}_2^m v_{-\lambda}, y_1^k y_2^m v_\lambda \rangle$ is equal to

$$\langle v_{-\lambda}, (-q^{-h_\gamma} e_\gamma)^m (-q^{-h_\alpha} e_\alpha)^k f_\alpha^k, f_\gamma^m v_\lambda \rangle = c \langle v_{-\lambda}, e_\gamma^m e_\alpha^k f_\alpha^k f_\gamma^m v_\lambda \rangle,$$

where $c = (-1)^{k+m} q^{(k+m)(\alpha,\lambda)+m(m-1)+k(k-1)}$. Further, $\langle v_{-\lambda}, e_\gamma^m e_\alpha^k f_\alpha^k f_\gamma^m v_\lambda \rangle$ is found to be

$$\langle v_{-\lambda}, e_\gamma^m e_\alpha^{k-1} f_\alpha^{k-1} \Big(q^{h_\alpha+1} \frac{1-q^{-2k}}{(q-q^{-1})^2} + q^{-h_\alpha-1} \frac{1-q^{2k}}{(q-q^{-1})^2} \Big) f_\gamma^m v_\lambda \rangle + \ldots$$

The omitted term is zero, as it involves $e_\alpha f_\gamma^m v_\lambda = 0$. We continue in this way and get

$$[k]_q! \prod_{i=1}^{k} [(\alpha,\lambda)+1-i]_q [m]_q! \prod_{i=1}^{m} [(\alpha,\lambda)+1-i]_q = [2k]_q!![2m]_q!! \Big(\frac{q^{(\alpha,\lambda)} q}{q-q^{-1}} \Big)^{k+m},$$

since $[(\alpha,\lambda)+1-i]_q = \frac{q^{(\alpha,\lambda)+1-i} - q^{-(\alpha,\lambda)-1+i}}{q-q^{-1}} = q^{(\alpha,\lambda)+1} \frac{q^{-i}+q^{i}}{q-q^{-1}} = \frac{q^{(\alpha,\lambda)+1}[2i]_q}{(q-q^{-1})[i]_q}$. Combining this result with the multiplier c calculated earlier and taking into account $q^{2(\alpha,\lambda)} = -q^{-2}$ we prove the statement. ☐

Let $\mathbb{C}_\hbar[G_{DS}]$ denote the affine coordinate ring on the quantum group $Sp_q(4)$, i.e. the quantization of the algebra of polynomial functions on G along the Drinfeld-Sklyanin bracket. It is the Hopf dual to the quantized universal enveloping algebra $U_\hbar(\mathfrak{g})$, and the reader should not confuse it with $\mathbb{C}_\hbar[G]$ defined in Section 4. It is known that the multiplication in $\mathbb{C}_\hbar[G]$, call it $\cdot_\hbar$, is a star product, [T]. Denote by $\mathbb{C}_\hbar[G_{DS}]^{\mathfrak{l}}$ the subalgebra of $U_\hbar(\mathfrak{l})$-invariants in $\mathbb{C}_\hbar[G]$ under the left co-regular action. It is a natural right $U_\hbar(\mathfrak{g})$-module algebra under the right co-regular action and is also a quantization a closed conjugacy class (quotient by the Levi subgroup).

The Shapovalov form on $\hat{M}_\lambda$ is invertible for non-special λ and its inverse provides an associative multiplication on $\mathbb{C}_\hbar[G_{DS}]^{\mathfrak{l}}$. For the special value of λ, it has a pole, while its regular part still defines an associative multiplication on a certain subspace of $\mathbb{C}_\hbar[G_{DS}]$, as argued in [KST]. Description of this subspace boils down to description of the "stabilizer" of the quantum conjugacy class. It should be an appropriate deformation of $U(\mathfrak{k}) \supset U(\mathfrak{l})$

within $U_\hbar\big(\mathfrak{sp}(4)\big)$, where $\mathfrak{k} = \mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$. The algebra $U_\hbar(\mathfrak{k})$ is unknown to us at present and will be a subject of our future research.

The stabilizer $U_\hbar(\mathfrak{k})$ will determine the subspace $\mathbb{C}_\hbar[G_{DS}]^{\mathfrak{k}}$ of its invariants that supports the associative multiplication

$$f \otimes g \mapsto f *_\hbar g = \sum_{k,m=0}^{\infty} q^{-m(m-2)-k(k-2)} \frac{(q-q^{-1})^{k+m}}{[2k]_q!![2m]_q!!} \big(\tilde{x}_1^k \tilde{x}_2^m f\big) \cdot_\hbar \big(y_1^k y_2^m g\big), \tag{8.15}$$

for $f, g \in \mathbb{C}_\hbar[G_{DS}]^{\mathfrak{k}}$. This multiplication will be a star-product on $\mathbb{C}[\mathbb{S}^4]$.

Remark that product (8.15) is not perfectly explicit because it is expressed through $\cdot_\hbar$, whose explicit expression through the classical multiplication in $\mathbb{C}[G]$ is unknown. Also, the new multiplication should be isomorphic to $\cdot_\hbar$, because $\mathbb{S}^4$ has a unique structure of Poisson manifold over the Poisson group G. Thus, neither (8.15) nor (7.10) are particularly new with regard to *abstract* quantization. For instance, one can apply the method of characters (which is doable in this special case) and realize the quantized polynomial algebra on $\mathbb{S}^4$ both as a quotient of $\mathbb{C}_\hbar[G]$ and a subalgebra in $\mathbb{C}_\hbar[G_{DS}]$, [DM]. Alternatively, one can quantize $\mathbb{S}^4$ through the quantum plane, along the lines of [FRT]. The novelty of the present work is a realization of $\mathbb{C}[\mathbb{S}^4]$ by operators on a highest weight module. This approach admits far reaching generalization that unites the non-Levi conjugacy classes and the classes with Levi isotropy subgroups in a common quantization context. The general approach to quantization of non-Levi conjugacy classes of simple complex matrix groups will be the subject of our future research.

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