INVARIANT PROLONGATION OF OVERDETERMINED PDE'S IN PROJECTIVE, CONFORMAL AND GRASSMANNIAN GEOMETRY

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Abstract. This is the second in a series of papers on natural modification of the normal tractor connection in a parabolic geometry, which naturally prolongs an underlying overdetermined system of invariant differential equations. We give a short review of the general procedure developed in [5] and then compute the prolongation covariant derivatives for a number of interesting examples in projective, conformal and Grassmannian geometries.

1. Introduction

In this paper we study certain overdetermined linear systems of PDE's that have geometric origin and satisfy strong invariance properties. The goal is to rewrite these systems in a closed form, which for our purposes means to find an extended system described by a covariant derivative in such a way that parallel sections with respect to this covariant derivative are in one to one correspondence with solutions of the original equation. The main advantage of such a prolongation is clear - one immediately obtains a bound on the dimension of the solution space and the curvature of this covariant derivative obstructs the existence of a solution. Moreover, there is a neat relationship between geometry of the underlying manifold and the extended prolongation system, see e.g. [2],[5] and the references therein.

The equations we study appear naturally for parabolic geometries like projective, conformal or Grassmannian structures and include as a special instances the equations describing the infinitesimal symmetries of geometric structures. Special examples of overdetermined linear systems of invariant equations coming from parabolic geometries are discussed in e.g., [2], [13], [7], [10], [18].

In fact, the invariant equations in question appear in the Bernstein-Gelfand-Gelfand (BGG for short) sequences, which are the source of overdetermined invariant operators resp. their prolonged systems in question. The prolongation of the first operator in the BGG sequence is realized by certain commutative square related to BGG operators in the sequence. We are constructing also examples of commutative squares for all operators in the BGG sequence.

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1.1. **The BGG-sequence.** Let G be a semi-simple Lie group and $P \subset G$ a parabolic subgroup. A *parabolic geometry* on a manifold M consists of a P-principal bundle $\mathcal{G} \to M$ together with a *Cartan connection* 1*-form* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, [9]. Here $\mathfrak{g}$ denotes the Lie algebra of G. A major development in the construction of differential invariants of parabolic structure was done in [4], and the construction was subsequently simplified in [3].

Let $\mathbb{V}$ be a finite dimensional G-representation. It is well known that the associated *tractor bundle* $V = \mathcal{G} \times_P \mathbb{V}$ carries the canonical *tractor covariant derivative* ∇ induced by the Cartan connection form ω, see e.g. [1]. The connection uniquely extends to an exterior covariant derivative on the spaces $\mathcal{E}^k(V) := \Omega^k(M, V)$ of k-forms with values in the vector bundle V, denoted $d^\nabla : \mathcal{E}^k(V) \to \mathcal{E}^{k+1}(V)$. The lowest homogenoues part of d^∇ is the G_0-equivariant Lie algebraic differential $\partial_k : \mathcal{E}^k(V) \to \mathcal{E}^{k+1}(V)$ termed the *Kostant differential*, [19]. Here G_0 denotes the Levi part of P. Its adjoint, the *Kostant codifferential* ∂_k^* is P-equivariant and gives rise to a complex

$$\mathcal{E}^{k+1}(V) \overset{\partial_{k+1}^*}{\to} \mathcal{E}^k(V),\ \partial_k^* \circ \partial_{k+1}^* = 0.$$

There are Lie algebra cohomology bundles $H_k = \ker \partial_k^* / \operatorname{im} \partial_{k+1}^*$ due to the P-equivariant projection

$$\Pi_k : \ker \partial_k^* \to H_k.$$

The basic ingredient of the BGG-machinery are the differential *BGG-splitting operators*

$$L_k : H_k \to \ker \partial_k^*,$$

defined uniquely by the property that for every smooth section $\sigma \in \Gamma(H_k)$ one has

$$\partial_{k+1}^*(d^\nabla(L_k(\sigma))) = 0.$$

In particular, one can form the *BGG-operators*

$$D_k : H_k \to H_{k+1},\ D_k := \Pi_{k+1} \circ d^\nabla \circ L_k.$$

It will be usually clear from the context what is the appropriate value for homogeneity k of the form which is acted upon by any of operators, i.e. we usually omit this subscript from the notation.

Let us briefly review the invariant prolongation procedure obtained in [5]:

1.2. **Prolongation of the first BGG operator D_0.** The *first BGG-operator* D_0 associated to $\mathbb{V}$ is overdetermined, and our aim is the construction of invariant prolongation of the corresponding systems $D_0\sigma = 0$ on $\sigma \in \Gamma(H_0)$. Let us recall that the approach of [5] starts by introducing certain class of linear connections on V which are modifications of tractor covariant derivative ∇^V. The first condition on a modification map $\Phi \in \mathcal{E}^1(\operatorname{End} V)$ is that it is homogeneous of degree

≥ 1 with respect to the natural filtrations on TM and V, for which we write $\Phi \in (\mathcal{E}^1(\operatorname{End} V))^1$. This ensures that basic constructions of the BGG-machinery still work. The next condition is that for any section $s \in \Gamma(V)$ we have that $\Phi s \in \mathcal{E}^1(V)$ has values in $\operatorname{im}\partial^*$. As a consequence, the modified covariant derivative is in a suitable sense compatible with the underlying first BGG-operator D_0. The latter condition can be rewritten as $\Phi \in \operatorname{Im}(\partial_V^* \otimes \operatorname{id}_{V^*})$, thus we arrive at a class of *admissible* covariant derivatives

$$\mathcal{C} = \left\{ \widetilde{\nabla} = \nabla + \Phi | \Phi \in Im\,(\partial_V^* \otimes \operatorname{id}_{V^*}) \cap (\mathcal{E}^1(\operatorname{End} V))^1 \right\}.$$

Here ∂_V^* denotes ∂^* acting on $\mathcal{E}^1(V)$ (and *not* on $\mathcal{E}^1(\operatorname{End} V)$) and the same applies for ∂_V^* acting on $\mathcal{E}^k(V)$.

The main theorem of [5] is then

Theorem 1.1. *There exists a unique covariant derivative $\tilde{\nabla} \in \mathcal{C}$ characterized by the property*

$$(\partial_V^* \otimes \operatorname{id}_{V^*})(\widetilde{\Omega}) = 0,$$

where $\widetilde{\Omega}$ is the curvature of $\tilde{\nabla}$.

This implies $\tilde{\nabla} \circ L_0 = L_1 \circ D_0$, which in turn yields

Corollary 1.2. *Consider a tractor bundle V and the covariant derivative $\tilde{\nabla}$ in Theorem 1.1. Then $\tilde{\nabla}$ gives a prolongation of the first BGG operator D_0 in the sense that the restriction of the projection $\Pi_0 : V \to H_0$ to $\tilde{\nabla}$-parallel sections is an isomorphism with the kernel of D_0 acting on smooth sections $\Gamma(H_0)$ and inverted by the differential splitting operator $L_0 : H_0 \to V$.*

We therefore say that $\tilde{\nabla}$ is the *prolongation covariant derivative.*

1.3. **Commutativity for all D_k.** In [5] the authors also obtained the analogue of $\widetilde{\nabla}$ on $\mathcal{E}^k(V)$. Here d^∇ gives rise to the class

$$\mathcal{C}_k := \{\tilde{d}_k = d^\nabla + \Phi \mid \Phi \in A^1, \operatorname{Im}\Phi \subset \operatorname{Im}\partial^*\}$$

where $A := Hom(\mathcal{E}^k(V), \mathcal{E}^{k+1}(V))$ and A^1 denotes homomorphisms homogeneous of the degree ≥ 1. Then it turns out there is a unique $\tilde{d}_k \in \mathcal{C}_k$ such that $\partial_V^* \circ d^\nabla \circ \tilde{d}_k = 0$. This then implies

$$\tilde{d}_k \circ L_k = L_{k+1} \circ D_k$$

and Π_k and L_k restrict to inverse isomorphisms between $\operatorname{Ker}\tilde{d}_k \cap \operatorname{Ker}\partial^*$ and $\operatorname{Ker} D_k$.

1.4. **The guideline for computing examples.** Here is the manual for treating particular examples, which can be used to derive the explicit form of the prolongation covariant derivative. In practice, the normalization procedure for canonical tractor covariant derivative can be summarized as an algorithm based on the following list of steps:

(1) Choose a parabolic geometry $(\mathcal{G}, P, M, \omega)$, where $\mathcal{G} \to M$ is a principal P-bundle on M and $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Choose also a finite dimensional G-module $\mathbb{V}$ and its associated vector bundle V termed tractor bundle. Let us fix the two consecutive vector bundles of k resp. $(k+1)$-forms twisted by V.

(2) Decompose both spaces of k resp. $(k+1)$-forms twisted by V with respect to G_0, the Levi factor of the parabolic subgroup P. Then compute the value of the Laplace-Kostant algebraic operator $\Box$ associated to ∂^* on each irreducible G_0-summand (i.e. G_0-graded components associated to P-equivariant filtration) either by evaluating the action of Casimir operator or from the definition $\Box = \partial^*\partial + \partial\partial^*$.

(3) Choose a Weyl structure, so that there is a well defined splitting of the filtered bundle V into a direct sum of homogeneous components.

(4) Now the procedure splits into two cases:
   - The computation of the prolongation covariant derivative. Check, if $(\partial^*_V \otimes \mathrm{id}_{V^*})(\Omega)$, where Ω is the curvature of ∇, is trivial. In positive case, the procedure ends and we have computed the prolongation covariant derivative.
     If $\alpha := (\partial^*_V \otimes \mathrm{id}_{V^*})(\Omega) \neq 0$, take the lowest nontrivial homogeneous part α_j of α and define
     $$\Phi = -\Box^{-1}\alpha_j; \quad \nabla' = \nabla + \Phi.$$
     Then repeat the procedure with ∇ replaced by ∇'. By construction, the lowest nontrivial component of α in the next step will have degree higher then in the previous step, hence the procedure will terminate in a finite number of steps (bounded by the length of the grading of $\mathbb{V}$).
   - The case of the whole sequence of commuting squares. Here we use another procedure based on the following algorithm. Consider two consecutive squares containing the exterior covariant derivatives $d^\nabla_k : \mathcal{E}^k(V) \mapsto \mathcal{E}^{k+1}(V)$ and $d^\nabla_{k+1} : \mathcal{E}^{k+1}(V) \mapsto \mathcal{E}^{k+2}(V)$. First check, if
     $$(\partial^*_V \otimes \mathrm{id}_{V^*})(d^\nabla_{k+1} \circ d^\nabla_k)$$
     is trivial. If not, the first step is the same as for the construction of prolongation covariant derivative above. Consider $\alpha := (\partial^* \otimes \mathrm{id}_{V^*})(d^\nabla_{k+1} \circ d^\nabla_k) \neq 0$, take the lowest nontrivial homogeneous part α_j of α and define
     $$\Phi = -\Box^{-1}\alpha_j; \quad d'_k = d^\nabla_k + \Phi.$$
     If $\alpha' := (\partial^* \otimes \mathrm{id}_{V^*})(d^\nabla_k \circ d'_k)$ is trivial, the procedure terminates and we define $\tilde{d}_k = d'_k$. If not, take the lowest

nontrivial homogeneous part $\alpha'_{j'}$ of α' and define

$$\Phi' = -\Box^{-1}\alpha'_{j'};\quad d''_k = d'_k + \Phi'.$$

By construction, the degree j' will be bigger than j, hence the procedure will terminate in a finite number of steps (bounded again by the length of the grading of V). Note that iterations φ here are, in general, differential operators and their order rises (in general) by one with each iteration.

The panorama of examples presented in this article follows criterions to be useful, nonelementary, going beyond the examples scattered in the references and at the same time computable by hand while demonstrating the powerful machine developed in [5]. The interested reader will easily recognize the complexity of the computation both in general and specific situations of interest.

2. Notation

In this section we review the basic notation and conventions related to the results of our article.

2.1. **Forms, tensors and tensorial actions.** In order to be explicit and efficient in calculations involving bundles of possibly high rank it is necessary to introduce some further abstract index notation. In the usual abstract index conventions one would write $\mathcal{E}_{[ab\cdots c]}$ (where there are implicitly k-indices skewed over) for the space $\mathcal{E}^k$. To simplify subsequent expressions we use the following conventions. Firstly indices labeled with sequential superscripts which are at the same level (i.e. all contravariant or all covariant) indicate a completely skew set of indices. Formally we set $a^1\cdots a^k = [a^1\cdots a^k]$ and so, for example, $\mathcal{E}_{a^1\cdots a^k}$ is an alternative notation for $\mathcal{E}^k$ while $\mathcal{E}_{a^1\cdots a^{k-1}}$ and $\mathcal{E}_{a^2\cdots a^k}$ both denote $\mathcal{E}^{k-1}$. Next we abbreviate this notation via multi-indices: We will use the form indices

$$\begin{aligned}
\mathbf{a}^k &:= a^1\cdots a^k = [a^1\cdots a^k], \quad k\geq 0,\\
\dot{\mathbf{a}}^k &:= a^2\cdots a^k = [a^2\cdots a^k], \quad k\geq 1,\\
\ddot{\mathbf{a}}^k &:= a^3\cdots a^k = [a^3\cdots a^k], \quad k\geq 2,\\
\dddot{\mathbf{a}}^k &:= a^4\cdots a^k = [a^4\cdots a^k], \quad k\geq 3.
\end{aligned}$$

If, for example, $k=1$ then $\dot{\mathbf{a}}^k$ simply means the index is absent, whereas if $k = 1$ then $\ddot{\mathbf{a}}$ means the term containing the index $\ddot{\mathbf{a}}$ is absent. For example, a 3–form φ can have the following possible equivalent structures of indices:

$$\varphi_{a^1a^2a^3} = \varphi_{[a^1a^2a^3]} = \varphi_{\mathbf{a}^3} = \varphi_{a^1\dot{\mathbf{a}}^3} = \varphi_{[a^1\dot{\mathbf{a}}^3]} = \varphi_{a^1a^2\ddot{\mathbf{a}}^3} \in \mathcal{E}_{\mathbf{a}^3} = \mathcal{E}^3.$$

Note the exterior derivative d on a k-form $f_{\mathbf{a}}$ can be written as $(df)_{a^0\mathbf{a}} = \nabla_{a^0} f_{\mathbf{a}}$ for any torsion–free affine connection ∇.

Later on we define the standard tractor bundle denoted by $\mathcal{E}^A$ and its dual $\mathcal{E}_B$. The form index notation developed above will be used also for skew symmetric powers of these bundles. For example, the bundle of *tractor k–forms* $\mathcal{E}_{[A^1\ldots A^k]}$ will be denoted by $\mathcal{E}_{A^1\ldots A^k}$ or $\mathcal{E}_{\mathbf{A}^k}$.

The bundle of endomorphisms of $\mathcal{E}^A$ (or $\mathcal{E}_A$), $\mathcal{E}^E{}_F$, clearly injects $\mathcal{E}^E{}_F \subseteq \mathrm{End}(\mathcal{T})$ for any tractor bundle $\mathcal{T} \subseteq (\bigotimes \mathcal{E}^A)\otimes(\bigotimes \mathcal{E}_B)$. Consider $\gamma^E{}_F \in \mathcal{E}^E{}_F$ and $f \in \mathcal{T}$. The endomorphism γ acts on $\mathcal{T}$ and we denote this action by $\sharp$. That is, $\gamma\sharp f \in \mathcal{T}$. Using the abstract tractor indices, $\sharp$ is given by the usual tensorial action, i.e. $(\gamma\sharp f)^A = \gamma^A{}_P f^P$ for $f^A \in \mathcal{E}^A$ and $(\gamma\sharp f)_A = -\gamma^P{}_A f_P$ for $f_A \in \mathcal{E}_A$. One then computes $\sharp$ on the tensor products of $\mathcal{E}^A$ and $\mathcal{E}_B$ using the Leibniz rule. We further put $\gamma\sharp$ to be zero on $\mathcal{E}^a$, $\mathcal{E}_b$ and density bundles (which we introduce later) and, using the Leibniz rule, extend $\gamma\sharp$ to the tensor products of $\mathcal{T}$ with latter three bundles. Finally note the action $\sharp$ is denoted $\bullet$ in [9].

2.2. **The adjoint tractor bundle and the Laplace-Kostant operator.** The bundle $\mathcal{A} = \mathcal{G} \times_P \mathfrak{g}$ is called the *adjoint* tractor bundle. By definition, $\mathcal{A} \subseteq \mathcal{E}^A{}_B$ and more generally $\mathcal{A} \hookrightarrow \mathrm{End}(\mathcal{T})$ for any tractor bundle $\mathcal{T}$. We shall use $\sharp$ to denote the action of sections of $\mathcal{A}$ on $\mathcal{T}$ as introduced above. Note the curvature of the normal tractor covariant derivative ∇ is the section of $\mathcal{E}_{a^0a^1} \otimes \mathcal{A}$ and the curvature action is $2(d^\nabla \nabla f)_{a^0a^1} = 2\nabla_{a^0}\nabla_{a^1} f = (\Omega\sharp f)_{a^0a^1} \in \mathcal{E}_{[ab]} \otimes \mathcal{T}$ for each $f \in \mathcal{T}$.

We have identifications $\mathcal{E}_a \cong \mathcal{G} \times_P \mathfrak{g}_-$ and $\mathcal{E}^a \cong \mathcal{A}/\mathcal{A}'$, $\mathcal{A}' := \mathcal{G} \times_P \mathfrak{p}$, which allow to define inclusions $\iota : \mathcal{E}_a \hookrightarrow \mathcal{A}$ and $\bar{\iota} : \mathcal{E}^a \hookrightarrow \mathcal{A}/\mathcal{A}'$. (The latter is just the identity.) We extend these inclusions to

$$\iota : \mathcal{E}_{\mathbf{a}} \hookrightarrow \mathcal{E}_{\dot{\mathbf{a}}} \otimes \mathcal{A} \quad \text{and} \quad \bar{\iota} : \mathcal{E}_{\mathbf{a}} \xrightarrow{\delta_{a^0}{}^b} \mathcal{E}_{a^0\mathbf{a}}{}^b \hookrightarrow \mathcal{E}_{a^0\mathbf{a}} \otimes \mathcal{A}/\mathcal{A}'.$$

Recall that here and below, we use a chosen Weyl structure and the corresponding splittings.

Our aim is to use these tools to express Kostant's differential ∂, codifferential ∂^* and in particular the Laplace-Kostant operator $\Box$ [19] in a form suitable for computations in abstract indices. Defined on $\mathcal{E}_{\mathbf{a}} \otimes \mathcal{T}$, $\mathbf{a} = \mathbf{a}^k$ for any tractor bundle $\mathcal{T}$, they have the form

$$\begin{aligned} &\partial : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \overset{\bar{\iota}}{\hookrightarrow} \mathcal{E}_{a^0\mathbf{a}} \otimes \mathcal{A}/\mathcal{A}' \otimes \mathcal{T} \xrightarrow{\sharp} \mathcal{E}_{a^0\mathbf{a}} \otimes \mathcal{T}, \\ &\partial^* : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \overset{\iota}{\hookrightarrow} \mathcal{E}_{\dot{\mathbf{a}}} \otimes \mathcal{A} \otimes \mathcal{T} \xrightarrow{\sharp} \mathcal{E}_{\dot{\mathbf{a}}} \otimes \mathcal{T} \quad \text{and} \\ &\Box_k = \partial\partial^* + \partial^*\partial : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \longrightarrow \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T}. \end{aligned}$$

Note ∂^* is invariant but ∂ (thus also $\Box_k$) depends on the choice of splitting of the tractor bundles in question. However, $\Box_k$ is invariant on completely reducible subquotients of $\mathcal{E}_{\mathbf{a}} \otimes \mathcal{T}$ and acts by a scalar multiple on each irreducible component of such subquotients. That is, we choose a splitting of the tractor bundle $\mathcal{E}_{\mathbf{a}} \otimes \mathcal{T}$ to compute $\Box_k$ but the value of $\Box_k$ on a given completely reducible subquotient alone is independent of this choice.

The symbol $\oplus$ denotes the composition P-module structure of representations or vector bundles.

Finally note one can compute $\square_k$ from highest weight of bundles concerned, see [19]. We shall use this (less explicit) approach in cases when the abstract index computation is getting too complicated.

Now we are ready to discuss specific geometries. In each case, we first summarize the tractor calculus. We shall particularly need the normal tractor covariant derivative ∇ and the Kostant's differential and codifferential ∂ and ∂^*, respectively. Using these we compute the prolongation covariant derivative $\widetilde{\nabla}$ and/or $\tilde{d}$ on certain bundles.

3. Projective geometry

We follow the notation from [1] here. The projective structure on a smooth manifold M is given by a class $[\nabla]$ of projectively equivalent torsion free connections. That is, connections $\hat{\nabla} \in [\nabla]$ are parametrised by one forms $\Upsilon_a \in \mathcal{E}_a \cong \Gamma(T^*M)$ and have the form

$$\begin{aligned} \hat{\nabla}_a \varphi &= \nabla_a \varphi + w \Upsilon_a \varphi, \quad \varphi \in \mathcal{E}(w), \\ \hat{\nabla}_a f^b &= \nabla_a f^b + \Upsilon_a f^b + \Upsilon_c f^c \delta_a^b, \quad f^b \in \mathcal{E}^b \\ \hat{\nabla}_a \omega_b &= \nabla_a \omega_b - \Upsilon_a \omega_b - \Upsilon_b \omega_a, \quad \omega_a \in \mathcal{E}_a. \end{aligned} \tag{1}$$

The curvature tensor $R_{ab}{}^c{}_d$ of a torsion free ∇ is defined by $(\nabla_a \nabla_b - \nabla_b \nabla_a) f^c = R_{ab}{}^c{}_p f^p$ and it decomposes

$$R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + 2\delta_{[a}{}^c \mathsf{P}_{b]d} + \beta_{ab} \delta^c{}_d, \quad \beta_{ab} = -2\mathsf{P}_{[ab]}.$$

Here $W_{ab}{}^c{}_d$ is projectively invariant (and irreducible) Weyl tensor, P is the Schouten tensor, $\hat{\mathsf{P}}_{ab} = \mathsf{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b$ and $\hat{\beta}_{ab} = \beta_{ab} + 2\nabla_{[a} \Upsilon_{b]}$. We put $A_{abc} := 2\nabla_{[a} \mathsf{P}_{b]c}$. Then the Bianchi identity $\nabla_{[a} R_{bc]}{}^d{}_e = 0$ implies

$$\nabla_c W_{ab}{}^c{}_d = (n-2) A_{abd} \quad \text{and} \quad \nabla_{[a} \beta_{cd]} = 0.$$

The cohomology class $[\beta] \in H^2(M, \mathbb{R})$ is a global invariant of the projective structure. Moreover, $(\nabla_a \nabla_b - \nabla_b \nabla_a)\varphi = w \beta_{ab} \varphi$ for $\varphi \in \mathcal{E}(w)$.

3.1. **Projective tractors.** We shall write sections of the standard projective tractor bundle $\mathcal{E}^A = \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]$, resp. its dual $\mathcal{E}_A = \mathcal{E}[1] \oplus \mathcal{E}_a[1]$ using the injectors Y^A, X^A, resp. Y_A, X_A as

$$\begin{pmatrix} \sigma^a \\ \rho \end{pmatrix} = Y_a^A \sigma^a + X^A \rho \in \mathcal{E}^A, \quad \text{resp.} \quad \begin{pmatrix} \nu \\ \mu_a \end{pmatrix} = Y_A \nu + X_A^a \mu_a \in \mathcal{E}_A.$$

Such splittings of $\mathcal{E}^A$ and $\mathcal{E}_A$ are parametrised by choices of projective connections and we call them *projective splittings*. The change of the splitting under change of the connection parametrised by $\Upsilon_a \in \mathcal{E}_a$ is

$$\widehat{\begin{pmatrix} \sigma^a \\ \rho \end{pmatrix}} = \begin{pmatrix} \sigma^a \\ \rho - \Upsilon_a \sigma^a \end{pmatrix}, \quad \text{i.e. } \hat{Y}^A_a = Y^A_a + X^A \Upsilon_a, \; \hat{X}^A = X^A \quad \text{and}$$

$$\widehat{\begin{pmatrix} \nu \\ \mu_a \end{pmatrix}} = \begin{pmatrix} \nu \\ \mu_a + \Upsilon_a \nu \end{pmatrix}, \quad \text{i.e. } \hat{Y}_A = Y_A - X^a_A \Upsilon_a, \; \hat{X}^a_A = X^a_A.$$

That is, $X^A \in \mathcal{E}^A[1]$, $X^a_A \in \mathcal{E}^a_A[-1]$ are invariant and $Y^A_a \in \mathcal{E}^A_a[1]$, $Y_A \in \mathcal{E}_A[-1]$ depend on the choice of the projective scale. We assume the normalisation of these such that $Y_A X^B + X^c_A Y^B_c = \delta_A{}^B$, i.e. $Y_C X^C = 1$ and $X^a_C Y^C_b = \delta^a{}_b$.

The normal covariant derivative is given by

$$\nabla_c \begin{pmatrix} \sigma^a \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_c \sigma^a + \rho \delta_c{}^a \\ \nabla_c \rho - P_{cp} \sigma^p \end{pmatrix} \quad \text{and} \quad \nabla_c \begin{pmatrix} \nu \\ \mu_a \end{pmatrix} = \begin{pmatrix} \nabla_c \nu - \mu_c \\ \nabla_c \mu_a + P_{ca} \nu \end{pmatrix}, \quad \text{i.e.}$$

$$\nabla_c Y^A_a = -X^A \mathsf{P}_{ca}, \; \nabla_c X^A = Y^A_c \quad \text{and} \quad \nabla_c Y_A = X^a_A \mathsf{P}_{ca}, \; \nabla_c X^a_A = -Y^A \delta^a_c.$$

and its Ω curvature has the form

$$\Omega_{ab}{}^E{}_F = Y^E_e X^f_F W_{ab}{}^e{}_f - X^E X^f_F A_{abf} \in \mathcal{E}_{[ab]} \otimes \mathcal{A}.$$

That is, $\mathcal{A} = \text{trace-free}(\mathcal{E}^E{}_F)$ is the projective adjoint tractor bundle where "trace-free" denotes the trace–free part. Hence the curvature action on $\mathcal{E}_C$ is $(\nabla_a \nabla_b - \nabla_b \nabla_a) F_C = (\Omega \sharp F)_{abC} = -\Omega_{ab}{}^D{}_C F_D$. We shall often write $\Omega_{ab} \sharp F_C$ instead of $(\Omega \sharp F)_{abC}$ to simplify the notation.

Using the notation developed above, the inclusions ι and $\bar{\iota}$ defined in 2.2 have the form $Y^E_{a^0} Y_F : \mathcal{E}_{\mathbf{a}} \xrightarrow{\bar{\iota}} \mathcal{E}_{a^0 \mathbf{a}}{}^E{}_F$ and $X^E X^{a^1}_F : \mathcal{E}_{\mathbf{a}} \xrightarrow{\iota} \mathcal{E}_{\dot{\mathbf{a}}}{}^E{}_F$. Thus

$$\partial : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \ni f_{\mathbf{a}} \mapsto Y^E_{a^0} Y_F f_{\mathbf{a}} \xrightarrow{\sharp} \mathcal{E}_{a^0 \dot{\mathbf{a}}} \otimes \mathcal{T} \quad \text{and}$$

$$\partial^* : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \ni f_{\mathbf{a}} \mapsto X^E X^{a^1}_F f_{\mathbf{a}} \xrightarrow{\sharp} \mathcal{E}_{\dot{\mathbf{a}}} \otimes \mathcal{T}$$

and we can easily compute $\Box_k$ on $\mathcal{E}_{\mathbf{a}} \otimes \mathcal{T}$ using the action $\sharp$ as demonstrated by the following example.

Example 3.1. We shall compute the case $\mathcal{T} = \mathcal{E}^C$ in details. Then $\mathcal{E}_{\mathbf{a}}{}^C = \mathcal{E}_{\mathbf{a}}{}^c[-1] \oplus \mathcal{E}_{\mathbf{a}}[-1]$, where $\mathcal{E}_{\mathbf{a}}$ is irreducible and $\mathcal{E}_{\mathbf{a}}{}^c$ has two irrreducible components (the trace and trace–free parts). We shall compute $\Box_k$ separately for all three irreducible components.

We start with (not necessarily irreducible) section $\sigma_{\mathbf{a}}{}^c \in \mathcal{E}_{\mathbf{a}}{}^c[-1]$. Then ∂ on $f_{\mathbf{a}}{}^C := Y^C_c \sigma_{\mathbf{a}}{}^c$ is zero and $X^E X^{a^1}_F \sharp Y^C_c \sigma_{\mathbf{a}}{}^c = X^C \sigma_{p\dot{\mathbf{a}}}{}^p = (\partial^* f)_{\dot{\mathbf{a}}}{}^C$. Thus $\partial^* f = 0$ for trace–free section $\sigma_{\mathbf{a}}{}^c$. Assume $\sigma_{\mathbf{a}}{}^c = \delta^c_{a^1} \tilde{\sigma}_{\dot{\mathbf{a}}}$. Then $f_{\mathbf{a}}{}^C = Y^C_{a^1} \tilde{\sigma}_{\dot{\mathbf{a}}}$, $(\partial^* f)_{\dot{\mathbf{a}}}{}^C = \frac{n-k+1}{k} X^C \tilde{\sigma}_{\dot{\mathbf{a}}}$ thus $(\Box_k f)_{\mathbf{a}}{}^C = (\partial \partial^* f)_{\mathbf{a}}{}^C = Y^C_{a^1} \tilde{\sigma}_{\dot{\mathbf{a}}}$. Finally if $\bar{f}_{\mathbf{a}}{}^C = X^C \rho_{\mathbf{a}}$ then $(\partial^* \bar{f})_{\dot{\mathbf{a}}}{}^C = 0$, $(\partial \bar{f})_{\mathbf{a}}{}^C = Y^C_{a^0} \rho_{\mathbf{a}}$ and $(\Box_k \bar{f})_{\mathbf{a}}{}^C = (\partial^* \partial \bar{f})_{\mathbf{a}}{}^C = \frac{n-k}{k+1} X^C \rho_{\mathbf{a}}$

Summarizing, $\Box_k$ acts by zero on the trace–free part of $\mathcal{E}_{\mathbf{a}}{}^c[-1] = \mathcal{E}_{\mathbf{a}}{}^C / \mathcal{E}_{\mathbf{a}}[-1]$, by $\frac{n-k+1}{k}$ on the trace part, i.e. on $\mathcal{E}_{\dot{\mathbf{a}}}[-1] \subseteq \mathcal{E}_{\mathbf{a}}{}^C / \mathcal{E}_{\mathbf{a}}[-1]$

and by $\frac{n-k}{k+1}$ on $\mathcal{E}_{\mathbf{a}}[-1] \subseteq \mathcal{E}_{\mathbf{a}}{}^C$. Note the inclusion $\mathcal{E}_{\mathbf{a}}[-1] \hookrightarrow \mathcal{E}_{\mathbf{a}}{}^C$ is realized by $X^C : \mathcal{E}_{\mathbf{a}}[-1] \to \mathcal{E}_{\mathbf{a}}{}^C$.

3.2. **Skew symmetric tractors and tractor forms.** The notation for the standard tractor bundle $\mathcal{E}^C$ developed above can be easily generalised to the products $\bigwedge^\ell \mathcal{E}^C = \mathcal{E}^{\mathbf{C}} = \mathcal{E}^{\mathbf{c}}(-\ell) \oplus \mathcal{E}^{\dot{\mathbf{c}}}(-\ell)$, where $\mathbf{C} = \mathbf{C}^\ell$. Note $\bigwedge^\ell \mathcal{E}^C \cong \bigwedge^{n-\ell+1} \mathcal{E}_D$, hence these products are isomorphic to tractor forms. We put

$$\mathbb{Y}^{\mathbf{C}}_{\mathbf{c}} = Y^{[C^1}_{c^1} \dots Y^{C^\ell]}_{c^\ell} \in \mathcal{E}^{\mathbf{C}}_{\mathbf{c}}(\ell), \quad \mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}} = X^{[C^1} Y^{C^2}_{c^2} \dots Y^{C^\ell]}_{c^\ell} \in \mathcal{E}^{\mathbf{C}}_{\dot{\mathbf{c}}}(\ell),$$

and write the sections of $\mathcal{E}^{\mathbf{C}}$ as

$$\begin{pmatrix} \sigma^{\mathbf{c}} \\ \rho^{\dot{\mathbf{c}}} \end{pmatrix} = \mathbb{Y}^{\mathbf{C}}_{\mathbf{c}} \sigma^{\mathbf{c}} + \mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}} \rho^{\dot{\mathbf{c}}} \in \mathcal{E}^{\mathbf{C}}, \quad \sigma^{\mathbf{c}} \in \mathcal{E}^{\mathbf{c}}(-\ell), \quad \rho^{\dot{\mathbf{c}}} \in \mathcal{E}^{\dot{\mathbf{c}}}(-\ell)$$

where $\mathbf{c} = \mathbf{c}^\ell$. The change of the projective rescaling parametrised by Υ_a is

$$\widehat{\begin{pmatrix} \sigma^{\mathbf{c}} \\ \rho^{\dot{\mathbf{c}}} \end{pmatrix}} = \begin{pmatrix} \sigma^{\mathbf{c}} \\ \rho^{\dot{\mathbf{c}}} - \ell \Upsilon_{c^1} \sigma^{\mathbf{c}} \end{pmatrix}, \quad \text{i.e.} \quad \hat{\mathbb{Y}}^{\mathbf{C}}_{\mathbf{c}} = \mathbb{Y}^{\mathbf{C}}_{\mathbf{c}} + \ell\, \Upsilon_{c^1} \mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}}, \quad \hat{\mathbb{X}}^{\mathbf{C}}_{\dot{\mathbf{c}}} = \mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}}$$

and the normal tractor covariant derivative has the form

$$\nabla_b \begin{pmatrix} \sigma^{\mathbf{c}} \\ \rho^{\dot{\mathbf{c}}} \end{pmatrix} = \begin{pmatrix} \nabla_b \sigma^{\mathbf{c}} + \rho^{\dot{\mathbf{c}}} \delta_b{}^{c^1} \\ \nabla_b \rho^{\dot{\mathbf{c}}} - \ell P_{bc^1} \sigma^{\mathbf{c}} \end{pmatrix}, \quad \text{i.e.} \quad \nabla_b \mathbb{Y}^{\mathbf{C}}_{\mathbf{c}} = -\ell\, P_{bc^1} \mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}}, \ \nabla_b \mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}} = \mathbb{Y}^{\mathbf{C}}_{[b\dot{\mathbf{c}}]}$$

Example 3.2. We shall compute the sequence for the tractor bundle $\mathcal{E}^{\mathbf{C}}$, $\mathbf{C} = \mathbf{C}^\ell$, i.e. $\mathcal{E}^{\mathbf{C}} \xrightarrow{\tilde{d}} \dots \xrightarrow{\tilde{d}} \mathcal{E}_{\mathbf{a}^n}{}^{\mathbf{C}}$. Since the filtration of $\mathcal{E}^{\mathbf{C}}$ has level 2, it follows immediately from the construction of $\tilde{d}$ that $(\tilde{d}F)_{a^0\mathbf{a}} = (d^\nabla F)_{a^0\mathbf{a}}{}^{\mathbf{C}} + (\Box_{k+1})^{-1}(\partial^* d^\nabla d^\nabla F)_{a^0\mathbf{a}}{}^{\mathbf{C}}$ for every $F_{\mathbf{a}}{}^{\mathbf{C}} \in \mathcal{E}_{\mathbf{a}}{}^{\mathbf{C}}$. (In particular, the difference between d^∇ and $\tilde{d}$ is algebraic in this case.)

Let us compute $\tilde{d}$ in details. Assume $F_{\mathbf{a}}{}^{\mathbf{C}} = \mathbb{Y}^{\mathbf{C}}_{\mathbf{c}} \sigma_{\mathbf{a}}{}^{\mathbf{c}} + \mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}} \rho_{\mathbf{a}}{}^{\dot{\mathbf{c}}}$. Then

$$\begin{aligned}(d^\nabla d^\nabla F)_{a^{-1}a^0\mathbf{a}}{}^{\mathbf{C}} &= \frac{1}{2} \Omega_{a^{-1}a^0} \sharp F_{\mathbf{a}}{}^{\mathbf{C}} = \frac{1}{2} \ell \Omega_{a^{-1}a^0}{}^{C^1}{}_P F_{\mathbf{a}}{}^{P\dot{\mathbf{C}}} = \\ &= \frac{1}{2} \ell \mathbb{Y}^{\mathbf{C}}_{\mathbf{c}} W_{a^{-1}a^0}{}^c{}_p \sigma_{\mathbf{a}}{}^{p\dot{\mathbf{c}}} + \mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}} \bar{\rho}_{a^{-1}a^0\mathbf{a}}{}^{\dot{\mathbf{c}}}\end{aligned}$$

for some section $\bar{\rho}$ which we shall not need explicitly. Therefore

$$\begin{aligned}(\partial^* d^\nabla d^\nabla F)_{a^0\mathbf{a}}{}^{\mathbf{C}} &= \frac{\ell^2}{2} X^{C^1} X^r_Q \Omega_{[ra^0}{}^{[Q}{}_{|P|} F_{\mathbf{a}]}{}^{|P|\dot{\mathbf{C}}]} = \frac{\ell^2}{2} \mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}} W_{[ra^0}{}^{[r}{}_{|p|} \sigma_{\mathbf{a}]}{}^{|p|\dot{\mathbf{c}}]} = \\ &= \frac{\ell}{2(k+2)} \mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}} \big[-(\ell-1) W_{pr}{}^{c^2}{}_{a^0} \sigma_{\mathbf{a}}{}^{pr\ddot{\mathbf{c}}} + k W_{a^0a^1}{}^r{}_p \sigma_{r\dot{\mathbf{a}}}{}^{p\dot{\mathbf{c}}} \big].\end{aligned}$$

It remains to apply $(\Box_{k+1})^{-1}$. Note the map $\partial^* d^\nabla d^\nabla : \mathcal{E}_{\mathbf{a}}{}^{\mathbf{C}} \to \mathcal{E}_{a^0\mathbf{a}}{}^{\mathbf{C}}$ has values in the (completely reducible) subbundle $\mathcal{E}_{a^0\mathbf{a}}{}^{\dot{\mathbf{c}}}(-\ell) \subseteq \mathcal{E}_{a^0\mathbf{a}}{}^{\mathbf{C}}$, cf. the precious display. Irreducible components of this subbundle are bundles $\mathrm{tf}[\mathcal{E}_{\mathbf{b}^{k+2-i}}{}^{\mathbf{d}^{\ell-i}}](-\ell)$, $1 \le i \le \min\{\ell, k+2\}$ where the notation tf[..] denotes the trace–free part of the enclosed bundle. The Laplace-Kostant operator $\Box_{k+1}$ on $\mathrm{tf}[\mathcal{E}_{\mathbf{b}^s}{}^{\mathbf{d}^t}](-\ell)$ acts by $A^t_s(\ell) := \frac{1}{s+1}[n-s-t+$

$1+(l-t)(n-s)]$. Note the computation is rather simple if we consider $\mathrm{tf}[\mathcal{E}_{\mathbf{b}^s}{}^{\mathbf{d}^t}](-\ell)$ as the irreducible invariant subbundle of $\mathcal{E}^{\mathbf{D}^t(E_1\ldots E_{l-t})}$ and then follow 3.1. Also note $A^t_s(\ell)$ is always nonzero. This of course follows by general means but can be verified directly since $\mathrm{tf}[\mathcal{E}_{\mathbf{b}^s}{}^{\mathbf{d}^t}] \neq \{0\}$ if and only if $s+t \leq n$.

Proposition 3.3. *The operator $\tilde{d} : \mathcal{E}_{\mathbf{a}}{}^{\mathbf{C}} \to \mathcal{E}_{a^0\mathbf{a}}{}^{\mathbf{C}}$ in the projective geometry has the form*

$$(\tilde{d}F)_{a^0\mathbf{a}}{}^{\mathbf{C}} = (d^\nabla F)_{a^0\mathbf{a}}{}^{\mathbf{C}} - \frac{\ell^2}{2}\sum_{i=1}^{\min\{\ell,k+2\}} \frac{1}{A^{\ell-i}_{k+2-i}(\ell)}\,\mathrm{Proj}^{\ell-i}_{k+2-i}\,\mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}} W_{[ra^0}{}^{[r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{|p|\dot{\mathbf{c}}]}$$

where $\sigma_{\mathbf{a}}{}^{\mathbf{c}} = \mathbb{X}^{\mathbf{c}}_{\mathbf{C}} F_{\mathbf{a}}{}^{\mathbf{C}}$, $\mathbb{X}^{\mathbf{c}}_{\mathbf{C}} = X^{c^1}_{C^1}\ldots X^{c^\ell}_{C^\ell}$ and $\mathrm{Proj}^t_s : \mathcal{E}_{\mathbf{a}^{s+i}}{}^{\mathbf{c}^{t+i}}(\ell) \to$ tf$[\mathcal{E}_{\mathbf{a}^s}{}^{\mathbf{c}^t}](\ell)$, $i \geq 0$ is the projection. □

The operator $\tilde{d}$ simplifies in special cases $\ell = 1$ and $k = 0$. First assume $\ell = 1$. Then $(\partial^* d^\nabla d^\nabla)_{a^0\mathbf{a}}{}^{C} = \frac{k}{2(k+2)} X^{\mathbf{C}} W_{a^0a^1}{}^{r}{}_{p}\sigma_{r\dot{\mathbf{a}}}{}^{p}$ has values in the irreducible subbundle $\mathcal{E}_{a^0\mathbf{a}}(-\ell)$ of $\mathcal{E}_{a^0\mathbf{a}}{}^{C}$. We computed $\Box_{k+1}$ acts by $\frac{n-(k+1)}{k+2}$ on this subbundle. Inverting this scalar, we obtain the result

$$(\tilde{d}F)_{a^0\mathbf{a}}{}^{C} = (d^\nabla F)_{a^0\mathbf{a}} C + \frac{k}{2(n-k-1)} X^C W_{a^0a^1}{}^{r}{}_{p}\sigma_{r\dot{\mathbf{a}}}{}^{p}.$$

Now assume $k = 0$. Then $(\partial^* d^\nabla d^\nabla F)_a{}^{\mathbf{C}} = -\frac{\ell(\ell-1)}{4}\mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}} W_{pr}{}^{c^2}{}_{a}\sigma^{pr\ddot{\mathbf{c}}}$ has values in the trace–free (thus irreducible) part of the subbundle $\mathcal{E}_a{}^{\dot{\mathbf{c}}}(-\ell)$. Since $\Box_{k+1}$ acts on the trace–free part of $\mathcal{E}_a{}^{\dot{\mathbf{c}}}(-\ell) \subseteq \mathcal{E}_a{}^{\mathbf{C}}$ by $\frac{n-\ell}{2}$, the resulting formula is

$$(\tilde{d}F)_a{}^{\mathbf{C}} = (d^\nabla F)_a{}^{\mathbf{C}} + \frac{\ell(\ell-1)}{2(n-\ell)}\mathbb{X}^{\mathbf{C}} W_{pr}{}^{c^2}{}_{a}\sigma^{pr\ddot{\mathbf{c}}}.$$

We claim $\tilde{d}$ actually coicides with the prolongation covariant derivative $\tilde{\nabla}$. To verify this, first observe $((\tilde{\nabla}-\nabla)F)_a{}^{\mathbf{C}} \in \mathrm{Im}\,\partial^*$ by the constru- tion of $\tilde{d} = \tilde{\nabla}$. Thus it remains to verify $(d^{\tilde{\nabla}}\tilde{\nabla}F)_{a^{-1}a^0}{}^{\mathbf{C}} \in \mathrm{Ker}\,\partial^*$. But since $(d^\nabla\tilde{\nabla}F)_{a^{-1}a^0}{}^{\mathbf{C}} \in \mathrm{Ker}\,\partial^*$ (again by the constrution of $\tilde{d} = \tilde{\nabla}$) and $d^{\tilde{\nabla}} - d^\nabla : \mathcal{E}_{a^0} \to \ker\partial^* \subseteq \mathcal{E}_{a^{-1}a^0}{}^{\mathbf{C}}$, cf. the last term in the previous display, the claim follows. Using the matrix notation, $\tilde{\nabla} = \tilde{d}$ has the form

$$\tilde{\nabla}_a \begin{pmatrix}\sigma^{\mathbf{c}}\\ \rho^{\dot{\mathbf{c}}}\end{pmatrix} = \nabla_a \begin{pmatrix}\sigma^{\mathbf{c}}\\ \rho^{\dot{\mathbf{c}}}\end{pmatrix} + \frac{\ell(\ell-1)}{2(n-\ell)}\begin{pmatrix}0\\ W_{pr}{}^{c^2}{}_{a}\sigma^{pr\ddot{\mathbf{c}}}\end{pmatrix}.$$

Finally note $\mathcal{E}^{\mathbf{C}} \cong \mathcal{E}_{\mathbf{D}}$ (using the tractor volume form) for $\mathbf{C} = \mathbf{C}^\ell$ and $\mathbf{D} = \mathbf{D}^{n-\ell+1}$. The case $\ell = n-1$ (i.e. $\mathbf{D} = \mathbf{D}^2$) was solved in [12], where the prolongation of the corresponding BGG operator $\mathcal{E}_a(2) \to \mathcal{E}_{(ab)}$ (explicitly $f_a \mapsto \nabla_{(a}f_{b)}$) is constructed. They construct the prolongation as the tractor covariant derivative $D_a : \mathcal{E}_{\mathbf{D}^2} \to \mathcal{E}_{a\mathbf{D}^2}$, cf. [5]. Since $D_a - \nabla_a : \mathcal{E}_{\mathbf{D}^2} \to \mathrm{im}\,\partial^*$ (this follows from the formula for D_a in p. 9, [12] after a short computation) and the curvature of

$(D_aD_b - D_bD_a) : \mathcal{E}_{\mathbf{D}^2} \to \operatorname{Ker}\partial^*$ (this is obvious form the formula for $D_aD_b - D_bD_a$ on $\mathcal{E}_{\mathbf{D}^2}$ on the same page) we conclude $D_a = \tilde{\nabla}_a$, cf. 1.1.

Example 3.4. Here we discuss the bundle $\mathcal{E}^{(AB)} = \mathcal{E}^{(ab)}(-2) \oplus \mathcal{E}^a(-2) \oplus \mathcal{E}(-2)$. Consider a section $F_{\mathbf{a}}{}^{BC} \in \mathcal{E}_{\mathbf{a}}{}^{(BC)}$, expanded in the basis of injectors as $F_{\mathbf{a}}{}^{BC} = Y^{(B}_b Y^{C)}_c \sigma_{\mathbf{a}}{}^{bc} + X^{(B}Y^{C)}_c \rho_{\mathbf{a}}{}^c + X^BX^C\nu_{\mathbf{a}}$. Then

$$(d^\nabla d^\nabla F)_{a^{-1}a^0\mathbf{a}}{}^{BC} = \frac{1}{2}\Omega_{a^{-1}a^0}\sharp F_{\mathbf{a}}{}^{BC} = \Omega_{a^{-1}a^0}{}^{(B}{}_P F_{\mathbf{a}}{}^{C)P} =$$
$$= Y^{(B}_b Y^{C)}_c W_{a^{-1}a^0}{}^{(b}{}_p \sigma_{\mathbf{a}}{}^{c)p} + X^{(B}Y^{C)}_c\big[\frac{1}{2}W_{a^{-1}a^0}{}^c{}_p\rho_{\mathbf{a}}{}^p - A_{a^{-1}a^0p}\sigma_{\mathbf{a}}{}^{cp}\big] + X^BX^C\bar{\nu}_{\mathbf{a}}$$

for some section $\bar{\nu}$. Applying ∂^* we obtain

$$\begin{aligned}(\partial^* d^\nabla d^\nabla F)_{a^0\mathbf{a}}{}^{BC} =& 2X^{(B}Y^{C)}_c W_{[ra^0}{}^{(r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{c)p} \\ &+ X^BX^C\big[\frac{1}{2}W_{[ra^0}{}^r{}_{|p|}\rho_{\mathbf{a}]}{}^p - A_{[ra^0|p|}\sigma_{\mathbf{a}]}{}^{pr}\big].\end{aligned}$$

The filtration degree of $\mathcal{E}^{(AB)}$ is 3 and so the construction of $\tilde{d}$ will require (at most) 2 steps. In the first step we put $d' := d^\nabla + (\Box^{XY}_{k+1})^{-1}\partial^* d^\nabla d^\nabla : \mathcal{E}_{\mathbf{a}}{}^{BC} \to \mathcal{E}_{a^0\mathbf{a}}{}^{BC}$ where $\Box^{XY}_{k+1}$ denotes $\Box_{k+1}$ restricted to the subquotient $\mathcal{E}_{\mathbf{a}}{}^c(-2)$ of $\mathcal{E}_{\mathbf{a}}{}^{(BC)}$ which corresponds to the injector $X^{(B}Y^{C)}_c : \mathcal{E}_{\mathbf{a}}{}^c(-2) \hookrightarrow \mathcal{E}_{\mathbf{a}}{}^{(BC)}$. Note this subquotient has two irreducible components but we need only the trace–free part since $W_{[ra^0}{}^{(r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{c)p}$ is trace–free. A short computation reveals $\partial^*\partial = \Box_1$ acts on the corresponding subquotient of $\mathcal{E}_a{}^{(BC)}$ by $\frac{n-k}{k+2}$. Hence

$$\begin{aligned}(d'F)_{a^0\mathbf{a}}{}^{BC} =& \nabla_{a^0}F_{\mathbf{a}}{}^{BC} - \frac{k+2}{n-k}\big[2X^{(B}Y^{C)}_c W_{[ra^0}{}^{(r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{c)p} \\ &+ X^BX^C\big(\frac{1}{2}W_{[ra^0}{}^r{}_{|p|}\rho_{\mathbf{a}]}{}^p - A_{[ra^0|p|}\sigma_{\mathbf{a}]}{}^{pr}\big)\big].\end{aligned} \tag{2}$$

Further computation reveals

$$\begin{aligned}(d^\nabla d'F)_{a^{-1}a^0\mathbf{a}}{}^{BC} =& (d^\nabla d^\nabla F)_{a^{-1}a^0\mathbf{a}}{}^{BC} - \frac{k+2}{n-k}\Big[2Y^{(B}_{a^{-1}}Y^{C)}_c W_{[ra^0}{}^{(r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{c)p} \\ &+ 2X^{(B}Y^{C)}_c\big(+\frac{1}{2}\delta^c_{a^{-1}}W_{[ra^0}{}^r{}_{|p|}\rho_{\mathbf{a}]}{}^p - \delta^c_{a^{-1}}A_{[ra^0|p|}\sigma_{\mathbf{a}]}{}^{pr} \\ &+ \nabla_{a^{-1}}W_{[ra^0}{}^{(r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{c)p}\big)\Big] + X^BX^C\gamma_{a^{-1}a^0\mathbf{a}}.\end{aligned}$$

for some section $\gamma_{a^{-1}a^0\mathbf{a}} \in \mathcal{E}_{a^{-1}a^0\mathbf{a}}(-2)$ and

$$\begin{aligned}(\partial^* d^\nabla d'F)_{a^0\mathbf{a}}{}^{BC} =& -\frac{1}{n-k}X^BX^C\Big[2\nabla_s W_{[ra^0}{}^{(r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{s)p} \\ &+ (n-k-2)\big(\frac{1}{2}W_{[ra^0}{}^r{}_{|p|}\rho_{\mathbf{a}]}{}^p - A_{[ra^0|p|}\sigma_{\mathbf{a}]}{}^{pr}\big)\Big].\end{aligned}$$

The previous displays shows that $(\partial^* d^\nabla d'F)_{a^0\mathbf{a}}{}^{BC}$ is the section of the subbundle $\mathcal{E}_{a^0\mathbf{a}}(-2) \subseteq \mathcal{E}_{a^0\mathbf{a}}{}^{BC}$. Since $\Box_{k+1}$ acts on this sunbundle by $\frac{2(n-k-1)}{k+2}$, we obtain the result $\tilde{d} := d' - \frac{k+2}{2(n-k-1)}\partial^* d^\nabla d'$.

Proposition 3.5. *The operator* $\tilde{d} : \mathcal{E}_{\mathbf{a}}{}^{(\mathbf{BC})} \to \mathcal{E}_{a^0\mathbf{a}}{}^{(\mathbf{BC})}$ *in the projective geometry has the form*

$$\begin{aligned}(\tilde{d}F)_{a^0\mathbf{a}}{}^{\mathbf{BC}} =&\nabla_{a^0} F_{\mathbf{a}}{}^{BC} - \frac{k+2}{n-k}\Big[2X^{(B}Y^{C)}_c W_{[ra^0}{}^{(r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{c)p} \\ &- \frac{1}{2(n-k-1)} X^B X^C \big[2\nabla_s W_{[ra^0}{}^{(r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{s)p} \\ &\qquad - (n-k)\big(\frac{1}{2} W_{[ra^0}{}^{r}{}_{|p|}\rho_{\mathbf{a}]}{}^{p} - A_{[ra^0|p|}\sigma_{\mathbf{a}]}{}^{pr}\big)\big]\Big].\end{aligned}$$

where $\sigma_{\mathbf{a}}{}^{bc} = X^b_B X^c_C F_{\mathbf{a}}{}^{BC}$ *and* $\rho_{\mathbf{a}}{}^{b} = 2X^b_B Y_C F_{\mathbf{a}}{}^{BC}$. □

We shall discuss the case $k = 0$ in more details. Then the formula in Proposition 3.5 simplifies to

$$\begin{aligned}(\tilde{d}F)_a{}^{BC} =&\nabla_a F^{BC} - \frac{2}{n} X^{(B}Y^{C)}_c W_{ra}{}^{c}{}_{p}\sigma^{rp} \\ &+ \frac{1}{n} X^B X^C \big(2A_{rap}\sigma^{pr} + \frac{1}{n-1} W_{ra}{}^{s}{}_{p}\nabla_s\sigma^{rp}\big].\end{aligned}$$

This means $\tilde{d}$ is not a covariant derivative on $\mathcal{E}^{(BC)}$ as the term $W_{ra}{}^{s}{}_{p}\nabla_s\sigma^{rp}$ is not algebraic in F^{BC}, i.e. $\tilde{d} \neq \widetilde{\nabla}$ in this case. To compute $\widetilde{\nabla}$ explicitly, assume $k = 0$ and put $\nabla' := d'$ (this is a covariant derivative on $\mathcal{E}^{(BC)}$). That is, $\nabla'_a F^{BC} = \nabla_a F^{BC} - \frac{2}{n}(\Psi F)_a{}^{BC}$, where the homomorphism $\Psi_a : \mathcal{E}^{(BC)} \to \mathcal{E}_a{}^{(BC)}$ is given by the formula (2), i.e. $(\Psi F)_a{}^{BC} = X^{(B}Y^{C)}_c W_{ra}{}^{c}{}_{p}\sigma^{rp} - X^B X^C A_{rap}\sigma^{pr}$. Extending Ψ_{a^0} to an endomorphism $\mathcal{E}_{a^1}{}^{(BC)} \to \mathcal{E}_{a^0a^1}{}^{(BC)}$, an easy computation shows

$$(\Psi\nabla' F)_{a^0a^1}{}^{BC} = X^{(B}Y^{C)}_c \big[W_{ra^0}{}^{c}{}_{p}\nabla_{a^1}\sigma^{rp} - \frac{3}{2} W_{a^0a^1}{}^{c}{}_{p}\rho^{p}\big] + X^B X^C \bar{\nu}$$

for some $\bar{\nu} \in \mathcal{E}(-2)$. Therefore $(\partial^*\Psi\nabla' F)^{BC}_a = -\frac{1}{2} X^B X^C W_{ra}{}^{c}{}_{p}\nabla_c\sigma^{rp}$ and we finally obtain $(\partial^* d^{\nabla'}\nabla' F)_a{}^{BC} = (\partial^* d^{\nabla}\nabla' F)_a{}^{BC} - \frac{2}{n}(\partial^*\Psi\nabla' F)^{BC}_a = 0$. Since the left hand side is the curvature of ∇' (applied to F^{BC}), this curvature is a map $\mathcal{E}^{(BC)} \to \operatorname{Ker}\partial^*$. Thus we verified $\widetilde{\nabla} = \nabla'$, cf. Theorem 1.1. Rewritting $\widetilde{\nabla}$ in the matrix notation, we obtain

$$\widetilde{\nabla}_a \begin{pmatrix}\sigma^{bc}\\ \rho^c\\ \nu\end{pmatrix} = \nabla_a \begin{pmatrix}\sigma^{bc}\\ \rho^c\\ \nu\end{pmatrix} - \frac{2}{n}\begin{pmatrix}0\\ W_{ra}{}^{c}{}_{p}\sigma^{pr}\\ -A_{rap}\sigma^{pr}\end{pmatrix}.$$

Note $\widetilde{\nabla}_a$ provides the prolongation of the corresponding (first order) BGG operator from $\mathcal{E}^{(bc)_0}(-2)$ to the totally trace–free part of $\mathcal{E}_a{}^{(bc)}(-2)$. The same problem was solved in [13] in terms of the connection defined by (3.6) or the left hand side of (5.2) there. Let us denote this connection on $\mathcal{E}^{(BC)}$ by D_a. Note the formula for D_a differs from $\widetilde{\nabla}_a$ in the middle term of the last matrix in the previous display: this term is $-\frac{2}{n} W_{ra}{}^{c}{}_{p}\sigma^{pr}$ for $\widetilde{\nabla}_a$ whereas $\frac{1}{n} W_{ra}{}^{c}{}_{p}\sigma^{pr}$ in the case of D_a, cf. [13, (3.6)]. The reason is purely notational, specifically in the choice of the projectors. If one replaces $X^{(B}Y_c^{C)}$ by $-\frac{1}{2}X^{(B}Y_c^{C)}$ – which means e.g. $F_{\mathbf{a}}{}^{BC} = Y_b^{(B}Y_c^{C)}\sigma^{bc} + (-\frac{1}{2}X^{(B}Y_c^{C)})\rho^c + X^B X^C \nu$ – both terms will coincide. Note also that formulas for ∇_a and the normal covariant derivative defined in the display preceding to [13, Theorem 5.1] coincide after the change of projectors. This confirms the results here coincide with those in [13].

4. Conformal geometry

4.1. **Conformal geometry and tractor calculus.** We summarise here some notation and background. Further details may be found in [15]. Let M be a smooth manifold of dimension $n \geq 3$. Recall that a *conformal structure* of signature (p, q) on M is a smooth ray subbundle $\mathcal{Q} \subset S^2T^*M$ whose fiber over x consists of conformally related signature-(p, q) metrics at the point x. Sections of $\mathcal{Q}$ are metrics g on M. So we may equivalently view the conformal structure as the equivalence class $[g]$ of these conformally related metrics. The principal bundle $\pi : \mathcal{Q} \to M$ has structure group $\mathbb{R}_+$, and so each representation $\mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \mathrm{End}(\mathbb{R})$ induces a natural line bundle on $(M, [g])$ that we term the conformal density bundle $E[w]$. We shall write $\mathcal{E}[w]$ for the space of sections of this bundle. We write $\mathcal{E}^a$ for the space of sections of the tangent bundle TM and $\mathcal{E}_a$ for the space of sections of T^*M. The indices here are abstract in the sense of [6] and we follow the usual conventions from that source. So for example $\mathcal{E}_{ab}$ is the space of sections of $\otimes^2T^*M$. Here and throughout, sections, tensors, and functions are always smooth. When no confusion is likely to arise, we will use the same notation for a bundle and its section space.

We write $\boldsymbol{g}$ for the *conformal metric*, that is the tautological section of $S^2T^*M \otimes E[2]$ determined by the conformal structure. This is used to identify TM with $T^*M[2]$. For many calculations we employ abstract indices in an obvious way. Given a choice of metric g from $[g]$, we write ∇ for the corresponding Levi-Civita connection. With these conventions the Laplacian Δ is given by $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b = \nabla^b\nabla_b$. Here we are raising indices and contracting using the (inverse) conformal metric. Indices will be raised and lowered in this way without further comment. Note $E[w]$ is trivialised by a choice of metric g from the conformal class, and we also write ∇ for the connection corresponding to this trivialisation. The coupled ∇_a preserves the conformal metric.

The curvature $R_{ab}{}^c{}_d$ of the Levi-Civita connection (the Riemannian curvature) is given by $[\nabla_a, \nabla_b]v^c = R_{ab}{}^c{}_d v^d$ ($[\cdot, \cdot]$ indicates the commutator bracket). This can be decomposed into the totally trace-free Weyl curvature C_{abcd} and a remaining part described by the symmetric *Schouten tensor* P_{ab}, according to

$$R_{abcd} = C_{abcd} + 2\boldsymbol{g}_{c[a}\mathsf{P}_{b]d} + 2\boldsymbol{g}_{d[b}\mathsf{P}_{a]c}, \tag{3}$$

where $[\cdots]$ indicates antisymmetrisation over the enclosed indices. The Schouten tensor is a trace modification of the Ricci tensor $\mathrm{Ric}_{ab} = R_{ca}{}^{c}{}_{b}$ and vice versa: $\mathrm{Ric}_{ab} = (n-2)\mathsf{P}_{ab} + \mathsf{J}\boldsymbol{g}_{ab}$, where we write J for the trace $\mathsf{P}_a{}^a$ of P. The *Cotton tensor* is defined by $A_{abc} := 2\nabla_{[a}\mathsf{P}_{b]c}$. Via the Bianchi identity this is related to the divergence of the Weyl tensor as follows:

$$(n-3)A_{abc} = \nabla^d C_{dcab}. \tag{4}$$

Finally we put

$$B_{ab} = \nabla^p A_{pab} + \mathsf{P}^{pq}C_{paqb} \in \mathcal{E}_{(ab)_0}[-2]. \tag{5}$$

In the dimension $n = 4$, this is the conformally invariant *Bach* tensor.

Under a *conformal transformation* we replace a choice of metric g by the metric $\hat{g} = e^{2\Upsilon}g$, where Υ is a smooth function. We recall that, in particular, the Weyl curvature is conformally invariant $\widehat{C}_{abcd} = C_{abcd}$. With $\Upsilon_a := \nabla_a\Upsilon$, the Schouten tensor transforms according to

$$\widehat{\mathsf{P}}_{ab} = \mathsf{P}_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \tfrac{1}{2}\Upsilon^c\Upsilon_c\boldsymbol{g}_{ab}. \tag{6}$$

Explicit formula for the corresponding transformation of the Levi-Civita connection and its curvatures are given in e.g. [1, 15]. From these, one can easily compute the transformation for a general valence (i.e. rank) s section $f_{bc\cdots d} \in \mathcal{E}_{bc\cdots d}[w]$ using the Leibniz rule:

$$\begin{aligned}\hat{\nabla}_{\bar{a}} f_{bc\cdots d} =& \nabla_{\bar{a}} f_{bc\cdots d} + (w-s)\Upsilon_{\bar{a}} f_{bc\cdots d} - \Upsilon_b f_{\bar{a}c\cdots d} \cdots - \Upsilon_d f_{bc\cdots \bar{a}} \\ &+ \Upsilon^p f_{pc\cdots d}\boldsymbol{g}_{b\bar{a}} \cdots + \Upsilon^p f_{bc\cdots p}\boldsymbol{g}_{d\bar{a}}.\end{aligned} \tag{7}$$

We next define the standard tractor bundle over $(M,[g])$. It is a vector bundle of rank $n+2$ defined, for each $g \in [g]$, by $[\mathcal{E}^A]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. If $\widehat{g} = e^{2\Upsilon}g$, we identify $(\alpha, \mu_a, \tau) \in [\mathcal{E}^A]_g$ with $(\widehat{\alpha}, \widehat{\mu}_a, \widehat{\tau}) \in [\mathcal{E}^A]_{\widehat{g}}$ by the transformation

$$\begin{pmatrix}\widehat{\alpha}\\ \widehat{\mu}_a\\ \widehat{\tau}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ \Upsilon_a & \delta_a{}^b & 0\\ -\frac{1}{2}\Upsilon_c\Upsilon^c & -\Upsilon^b & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \mu_b\\ \tau\end{pmatrix}. \tag{8}$$

It is straightforward to verify that these identifications are consistent upon changing to a third metric from the conformal class, and so taking the quotient by this equivalence relation defines the *standard tractor bundle* $\mathcal{E}^A$ over the conformal manifold. On a conformal structure of signature (p,q), the bundle $\mathcal{E}^A$ admits an invariant metric h_{AB} of signature $(p+1, q+1)$ and an invariant connection, which we shall also denote by ∇_a, preserving h_{AB}. Up to an isomorphism this the unique *normal conformal tractor connection* and so induces normal connection on $\bigotimes \mathcal{E}^A$ that will be denoted ∇_a and termed the (normal) tractor connection. In a conformal scale g, the metric h_{AB} and ∇_a on

$\mathcal{E}^A$ are given by

$$(9)\quad h_{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \nabla_a \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a \alpha - \mu_a \\ \nabla_a \mu_b + \boldsymbol{g}_{ab}\tau + \mathsf{P}_{ab}\alpha \\ \nabla_a \tau - \mathsf{P}_{ab}\mu^b \end{pmatrix}.$$

It is readily verified that both of these are conformally well-defined, i.e., independent of the choice of a metric $g \in [g]$. Note that h_{AB} defines a section of $\mathcal{E}_{AB} = \mathcal{E}_A \otimes \mathcal{E}_B$, where $\mathcal{E}_A$ is the dual bundle of $\mathcal{E}^A$. Hence we may use h_{AB} and its inverse h^{AB} to raise or lower indices of $\mathcal{E}_A$, $\mathcal{E}^A$ and their tensor products.

In computations, it is often useful to introduce the 'projectors' from $\mathcal{E}^A$ to the components $\mathcal{E}[1]$, $\mathcal{E}_a[1]$ and $\mathcal{E}[-1]$ which are determined by a choice of scale. They are respectively denoted by $X_A \in \mathcal{E}_A[1]$, $Z_{Aa} \in \mathcal{E}_{Aa}[1]$ and $Y_A \in \mathcal{E}_A[-1]$, where $\mathcal{E}_{Aa}[w] = \mathcal{E}_A \otimes \mathcal{E}_a \otimes \mathcal{E}[w]$, etc. Using the metrics h_{AB} and $\boldsymbol{g}_{ab}$ to raise indices, we define X^A, Z^{Aa}, Y^A. Then we see that $Y_A X^A = 1$, $\; Z_{Ab}Z^A{}_c = \boldsymbol{g}_{bc}$, and all other quadratic combinations that contract the tractor index vanish. In (8) note that $\widehat{\alpha} = \alpha$ and hence X^A is conformally invariant. Reformulating (9), we obtain

$$\nabla_a Y_B = Z^b_B P_{ab}, \quad \nabla_a Z^b_B = -Y_B \delta^b_a - X_B P_a{}^b \quad \text{and} \quad \nabla_a X_B = Z^b_B \boldsymbol{g}_{ab}.$$

Given a choice of $g \in [g]$, the *tractor-D operator* $D_A \colon \mathcal{E}_{B\cdots E}[w] \to \mathcal{E}_{AB\cdots E}[w-1]$ is defined by

$$(10)\quad D_A V := (n+2w-2)wY_A V + (n+2w-2)Z_{Aa}\nabla^a V - X_A \Box V,$$

where $\Box V := \Delta V + w\mathsf{J}V$. This is conformally invariant, as can be checked directly using the formula above.

The curvature Ω of the tractor connection is defined on $\mathcal{E}^C$ by $[\nabla_a, \nabla_b]V^C = \Omega_{ab}{}^C{}_E V^E$. Using (9) and the formulae for the Riemannian curvature yields

$$(11)\qquad \Omega_{abEF} = Z^e_E Z^f_F C_{abef} - 2X_{[E}Z^f_{F]}A_{abf} \in \mathcal{E}_{[ab][EF]} = \mathcal{E}_{[ab]} \otimes \mathcal{A}$$

where $\mathcal{A} = \mathcal{E}_{[EF]}$ is the conformal adjoint tractor bundle. We shall write $\Omega_{ab}\sharp F_C$ or $(\Omega\sharp F)_{abC}$ for the curvature action $(\nabla_a\nabla_b - \nabla_b\nabla_a)F_C = -\Omega_{ab}{}^D{}_C F_D$.

Using the notation developed above, the inclusions ι and $\bar{\iota}$ defined in 2.2 have he form $-2Y_{[E}Z_{F]a^0} : \mathcal{E}_{\mathbf{a}} \xrightarrow{\bar{\iota}} \mathcal{E}_{a^0\mathbf{a}[EF]}$ and $-2X_{[E}X^{a^1}_{F]} : \mathcal{E}_{\mathbf{a}} \xrightarrow{\iota} \mathcal{E}_{\dot{\mathbf{a}}[EF]}$. (The scalar -2 is used for the sake of compatibility of ∂ and ∇, cf. [9].) Thus

$$\partial : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \ni f_{\mathbf{a}} \mapsto -2Y_{[E}Z_{F]a^0} f_{\mathbf{a}} \xrightarrow{\sharp} \mathcal{E}_{a^0\mathbf{a}} \otimes \mathcal{T} \quad \text{and}$$
$$\partial^* : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \ni f_{\mathbf{a}} \mapsto -2X_{[E}Z^{a^1}_{F]} f_{\mathbf{a}} \xrightarrow{\sharp} \mathcal{E}_{\dot{\mathbf{a}}} \otimes \mathcal{T}$$

and we can easily compute $\Box_k$ on $\mathcal{E}_{\mathbf{a}} \otimes \mathcal{T}$ using the tensorial action $\sharp$.

Example 4.1. We shall compute $\tilde{d}$ on forms twisted by $\mathcal{E}_C$. Let $\mathbf{a}=\mathbf{a}^k$ and consider $F_{\mathbf{a}C}=Y_C\sigma_{\mathbf{a}}+Z_C^c\mu_{c\mathbf{a}}+X_C\nu_{\mathbf{a}}\in\mathcal{E}_{\mathbf{a}C}$. Then

$$\begin{aligned}(d^\nabla d^\nabla F)_{a^{-1}a^0\mathbf{a}C}&=\frac{1}{2}\Omega_{a^{-1}a^0}\sharp F_{\mathbf{a}C}=\frac{1}{2}\Omega_{a^{-1}a^0C}{}^PF_{\mathbf{a}P}=\\&=\frac{1}{2}Z_C^c\big[C_{a^{-1}a^0c}{}^p\mu_{\mathbf{a}p}+A_{a^{-1}a^0c}\sigma_{\mathbf{a}}\big]-X_CA_{a^{-1}a^0}{}^p\mu_{\mathbf{a}p}\end{aligned}$$

hence $(\partial^*d^\nabla d^\nabla F)_{a^0\mathbf{a}C}=-\frac{k}{2(k+2)}X_C\big[C_{a^0a^1}{}^{rp}\mu_{r\dot{\mathbf{a}}p}+A_{a^0a^1}{}^r\sigma_{r\dot{\mathbf{a}}}\big]$. This is a section of the subbundle $\mathcal{E}_{a^0\mathbf{a}}[-1]\subseteq\mathcal{E}_{a^0\mathbf{a}C}$ and one easily computes $\Box_k$ acts on this (irreducible) subbundle by $-\frac{n-k-1}{k+2}$. Therefore $(\tilde{d}F)_{a^0\mathbf{a}C}=\nabla_{a^0}F_{\mathbf{a}C}-\frac{k}{2(n-k-1)}X_C\big[C_{a^0a^1}{}^{rp}\mu_{r\dot{\mathbf{a}}p}+A_{a^0a^1}{}^r\sigma_{r\dot{\mathbf{a}}}\big]$ for $0\leq k\leq n-1$ and $\tilde{d}=d^\nabla$ for $k\geq n-1$. Finally note that the prolongation covariant derivative coincides with the normal one for $k=0$, i.e. $\tilde{\nabla}=\nabla$ on $\mathcal{E}_C$.

Example 4.2. The computation of the prolongation covariant derivative is getting rather technical for more complicated bundles. We shall demonstrate it on the prolongation covariant derivative $\widetilde{\nabla}$ on $\mathcal{E}_{(BC)_0}$. (Note $\mathcal{E}_{(BC)_0}$ and $\mathcal{E}^{(BC)_0}$ are isomorphic using the tractor metric.) The computation consists of three steps: we start with ∇ and then define covariant derivatives $\overline{\nabla}$, $\overline{\overline{\nabla}}$ and $\widetilde{\nabla}$. Taking a section $F_{BC}=Y_{(B}Y_{C)}\sigma+Y_{(B}Z_{C)}^c\rho_c+Z_{(B}^bZ_{C)}^c\omega_{bc}+X_{(B}Y_{C)}\nu+X_{(B}Z_{C)}^c\mu_c+X_{(B}X_{C)}\kappa$ we get

$$\begin{aligned}(d^\nabla d^\nabla F)_{a^0a^1BC}&=\frac{1}{2}\Omega_{a^0a^1}\sharp F_{BC}=\frac{1}{2}\Omega'_{a^0a^1BC}{}^{PQ}F_{PQ}=\\&=Y_{(B}Z_{C)}^c\big[\frac{1}{2}C_{a^0a^1c}{}^p\rho_p+A_{a^0a^1c}\sigma\big]+Z_{(B}^bZ_{C)}^c\big[C_{a^0a^1(b}{}^p\omega_{c)p}+\frac{1}{2}A_{a^0a^1(b}\rho_{c)}\big]\\&\quad-\frac{1}{2}X_{(B}Y_{C)}A_{a^0a^1}{}^p\rho_p+X_{(B}Z_{C)}^c\big[\frac{1}{2}C_{a^0a^1c}{}^p\mu_p-A_{a^0a^1}{}^p\omega_{cp}+\frac{1}{2}A_{a^0a^1c}\nu\big]\\&\quad-\frac{1}{2}X_BX_CA_{a^0a^1}{}^p\mu_p.\end{aligned}$$

where $\Omega'_{a^0a^1BC}{}^{PQ}:=2\Omega_{a^0a^1(B}{}^{(P}h_{C)}{}^{Q)}$. Applying ∂^* to the previous display we obtain $(\partial^*d^\nabla d^\nabla F)_{a^1BC}=-2\mathbb{X}_{(B}{}^{Pr}\Omega_{|ra^1|C)}{}^QF_{PQ}$ because $\Omega_{a^0a^1EF}$ is ∂^*-closed (i.e. $\mathbb{X}_{A^0}{}^{Pp}\Omega_{pa^1PA^1}=0$). We put $\Psi_{a^1BC}{}^{PQ}:=-2\mathbb{X}_{(B}{}^{Pr}\Omega_{|ra^1|C)}{}^Q$. Equivalently, $\Psi_{a^1BC}{}^{PQ}$ can be obtained by applying ∂^* to the $\mathcal{E}_{BC}$-factor of $\Omega'_{a^0a^1(BC)}{}^{PQ}$. This is exactly the operator ∂^*_V from [5] since the notation therein means $V=\mathcal{E}_{(BC)_0}$, $V^*=\mathcal{E}^{(PQ)_0}$ and therefore $\Omega'_{a^0a^1BC}{}^{PQ}\in\mathcal{E}_{a^0a^1}\otimes\mathrm{End}(V)$ is the curvature tensor of ∇_a on $V=\mathcal{E}_{(BC)_0}$. We shall denote the operator ∂^*_V by $\partial^*_{BC}:\mathcal{E}_{a^0a^1BC}{}^{PQ}\to\mathcal{E}_{a^1BC}{}^{PQ}$ here. Thus we have $\Psi_{a^1BC}{}^{PQ}=\frac{1}{2}(\partial^*_{BC}\Omega')_{a^1BC}{}^{PQ}$, explicitly

$$\begin{aligned}\Psi_{a^1BC}{}^{PQ}=&-Z_{(B}^bZ_{C)}^c\big[X^{(P}Z^{Q)q}C_{a^1(bc)q}+X^PX^QA_{a^1(bc)}\big]\\&+X_{(B}Z_{C)}^c\big[Z^{p(P}Z^{Q)q}C_{a^1pcq}+2X^{(P}Z^{Q)q}A_{a^1(cq)}\big]\\&+X_{(B}X_{C)}Z^{p(P}Z^{Q)q}A_{pa^1q}.\end{aligned}\tag{12}$$

Since $\frac{1}{2}C_{a^1(bc)}{}^p\rho_p+A_{a^1(bc)}\sigma$ is a section of the Cartan component of the subquotient $\mathcal{E}_{[a^1b]}\otimes\mathcal{E}_c$ of $\mathcal{E}_{a^1(BC)_0}$ and $\Box_1$ acts on this subquotient by $-\frac{3}{2}$, we put $\overline{\nabla}_aF_{BC}=\nabla_aF_{BC}+\frac{2}{3}\Psi_{aBC}{}^{PQ}F_{PQ}$ as the first "approximation" of $\widetilde{\nabla}$. We need to know $\nabla_{a^0}\Psi_{a^1BC}{}^{PQ}$ to compute the curvature $\overline{\Omega}_{a^0a^1BC}{}^{PQ}$ of $\overline{\nabla}$. First, it easily follows from $\Psi_{a^1BC}{}^{PQ}:=-2\mathbb{X}_{(B}{}^{Pr}\Omega_{|ra^1|C)}{}^Q$ that

$$
\begin{aligned}
(d^\nabla\Psi)_{a^0a^1BC}{}^{PQ}&=\nabla_{a^0}\Psi_{a^1BC}{}^{PQ}=-2\nabla_{a^0}\mathbb{X}_{(B}{}^{Pr}\Omega_{|ra^1|C)}{}^Q=\\
&=-2\mathbb{Z}_{(B}^{e^0Pe^1}\boldsymbol{g}_{|a^0e^0}\Omega_{e^1a^1|C)}{}^Q+2\mathbb{W}_{(B}{}^P\Omega_{|a^0a^1|C)}{}^Q-\mathbb{X}_{(B}{}^{Pr}\nabla_{|r}\Omega_{a^0a^1|C)}{}^Q
\end{aligned}
$$

since $\nabla_{a^{-1}}\Omega_{a^0a^1CQ}=0$. Expanding the expressions in the previous display we obtain

$$
\begin{aligned}
&(d^\nabla\Psi)_{a^0a^1BC}{}^{PQ}=-\frac{3}{2}Y_{(B}Z_{C)}^c\big[X^{(P}Z^{Q)q}C_{a^0a^1cq}+X^PX^QA_{a^0a^1c}\big]\\
&+\frac{3}{2}X_{(B}Y_{C)}X^{(P}Z^{Q)q}A_{a^0a^1q}\\
&+Z_{(B}^bZ_{C)}^c\Big[-2Z^{p(P}Z^{Q)q}\boldsymbol{g}_{a^0[b}C_{p]a^1cq}+\frac{1}{2}X^PX^Q\big(\nabla_bA_{a^0a^1c}+P_b{}^rC_{a^0a^1rc}\big)\\
&\qquad\qquad+X^{(P}Z^{Q)q}\big(-2\boldsymbol{g}_{a^0[b}A_{q]a^1c}+\frac{1}{2}\nabla_bC_{a^0a^1cq}-\boldsymbol{g}_{b[c}A_{|a^0a^1|q]}\big)\Big]\\
&+X_{(B}Z_{C)}^c\Big[\frac{3}{2}Y^{(P}Z^{Q)q}C_{a^0a^1cq}-X^{(P}Z^{Q)q}\big(\nabla_{(c}A_{|a^0a^1|q)}+P_{(c}{}^sC_{|a^0a^1s|q)}\big)\\
&\qquad\qquad+Z^{p(P}Z^{Q)q}\big(2\boldsymbol{g}_{a^0[c}A_{p]a^1q}-\frac{1}{2}\nabla_pC_{a^0a^1cq}+\boldsymbol{g}_{p[c}A_{|a^0a^1|q]}\big)\Big]\\
&+X_BX_C\Big[-\frac{3}{2}Y^{(P}Z^{Q)q}A_{a^0a^1q}+\frac{1}{2}Z^{p(P}Z^{Q)q}\big(\nabla_pA_{a^0a^1q}+P_p{}^sC_{a^0a^1sq}\big)\Big]
\end{aligned}
$$

after some computation which uses the differential Bianchi identity, in particular the relation [18, (29)]. Now we need to apply ∂^*_{BC} to the previous display. This yields

$$
\begin{aligned}
(\partial^*_{BC}\,d^\nabla\Psi)_{a^1BC}{}^{PQ}&=\frac{3}{2}Z_{(B}^bZ_{C)}^c\big[X^{(P}Z^{Q)q}C_{a^1(bc)q}+X^PX^QA_{a^1(bc)}\big]\\
&+X_{(B}Z_{C)}^c\big[\frac{1}{2}(n-1)Z^{p(P}Z^{Q)q}C_{a^1(pq)c}-\frac{1}{2}X^PX^QB_{a^1c}\\
&\qquad+X^{(P}Z^{Q)q}\big((n-4)A_{q(a^1c)}-3A_{a^1(qc)}\big)\big]\\
&+X_BX_C\big[\frac{1}{2}(n-1)Z^{p(P}Z^{Q)q}A_{a^1(pq)}+\frac{1}{2}X^{(P}Z^{Q)q}B_{a^1q}\big].
\end{aligned}
$$

We need to compute $\overline{\Psi}_{a_1BC}{}^{PQ}=\frac{1}{2}(\partial^*_{BC}\overline{\Omega})_{a^1BC}{}^{PQ}$ satisfying $\overline{\Psi}_{a^1BC}{}^{PQ}F_{PQ}=(\partial^*d^{\overline{\nabla}}\overline{\nabla}F)_{a^1BC}$. Since $\overline{\nabla}_aF_{BC}=\nabla_a+\frac{2}{3}\Psi_{aBC}{}^{PQ}$ we have

$$
\frac{1}{2}\overline{\Omega}_{a^0a^1BC}{}^{PQ}=\frac{1}{2}\Omega'_{a^0a^1BC}{}^{PQ}+\frac{2}{3}(d^\nabla\Psi)_{a^0a^1BC}{}^{PQ}+\frac{4}{9}(\Psi\wedge\Psi)_{a^0a^1BC}{}^{PQ}
$$

where $(\Psi\wedge\Psi)_{a^0a^1BC}{}^{PQ}=\Psi_{a^0BC}{}^{RS}\Psi_{a^1RS}{}^{PQ}$. Since $\frac{1}{2}(\partial^*_{BC}\Omega')_{a^1BC}{}^{PQ}=\Psi_{a^1BC}{}^{PQ}$ by definition of Ψ, applying ∂^*_{BC} to the previous display yields

(13)

$$\begin{aligned}
\overline{\Psi}_{a_1BC}{}^{PQ} &= \frac{1}{2}(\partial^*_{BC}\overline{\Omega})_{a^1BC}{}^{PQ} = \\
&= \Psi_{a^1BC}{}^{PQ}+\frac{2}{3}(\partial^*_{BC}d^\nabla\Psi)_{a^1BC}{}^{PQ}+\frac{4}{9}(\partial^*(\Psi\wedge\Psi))_{a^1BC}{}^{PQ} = \\
&= \frac{1}{3}X_{(B}Z^c_{C)}\big[(n-4)Z^{p(P}Z^{Q)q}C_{a^1(pq)c}+2(n-4)X^{(P}Z^{Q)q}A_{q(a^1c)}-X^PX^QB_{a^1c}\big] \\
&\quad +\frac{1}{3}X_BX_C\big[(n-4)Z^{p(P}Z^{Q)q}A_{a^1(pq)}+X^{(P}Z^{Q)q}B_{a^1q}\big]+\frac{4}{9}(\partial^*_{BC}(\Psi\wedge\Psi))_{a^1BC}{}^{PQ}.
\end{aligned}$$

where

(14)

$$(\partial^*_{BC}(\Psi\wedge\Psi))_{a^1BC}{}^{PQ}=\frac{1}{2}X_BX_C\big[X^{(P}Z^{Q)q}C_{a^1}{}^{(rs)p}C_{qrsp}+X^PX^QC_{a^1}{}^{(rs)q}A_{qrs}\big].$$

Remark 4.3. The section $(\partial^*d^\nabla\Psi)_{aBC}{}^{PQ}$ is closely related to the conformally invariant curvature quantity

$$\begin{aligned}
W_{\mathbf{EF}}=&(n-4)\mathbb{Z}^{\mathbf{e}}_{\mathbf{E}}\mathbb{Z}^{\mathbf{f}}_{\mathbf{F}}C_{\mathbf{ab}}-2(n-4)\mathbb{Z}^{\mathbf{e}}_{\mathbf{E}}\mathbb{X}^{f}_{\mathbf{F}}A_{\mathbf{e}f}\\
&-2(n-4)\mathbb{X}^{e}_{\mathbf{E}}\mathbb{Z}^{\mathbf{f}}_{\mathbf{F}}A_{\mathbf{f}e}+4\mathbb{X}^{e}_{\mathbf{E}}\mathbb{X}^{f}_{\mathbf{F}}B_{ef},
\end{aligned}$$

cf. [14] where all the form indices $\mathbf{E}$, $\mathbf{F}$, $\mathbf{e}$, $\mathbf{f}$ have the valence 2. In fact, one easily computes $(\partial^*d^\nabla\Psi)_{aBC}{}^{PQ}=-\frac{1}{3}Z^R_aX_{(B}W_{C)}{}^{(P}{}_R{}^{Q)}$. Since $(\partial^*d^\nabla\Psi)_{aBC}{}^{PQ}$ coincides with $\overline{\Psi}_{aBC}{}^{PQ}$ up to the terms involving $C_{a^1}{}^{(rs)p}C_{qrsp}$ and $C_{a^1}{}^{(rs)q}A_{prs}$, cf. (13), conformal invariance of $W_{\mathbf{EF}}$ verifies the invariance of the previous computations.

Looking at the form of $\overline{\Psi}_{a_1BC}{}^{PQ}F_{PQ}$, we see that we need the action of $\square_1$ on the subquotient $\mathcal{E}_{(a^1c)_0}$ of $\mathcal{E}_{a_1BC}$ (corresponding to the injector $X_{(B}Z^c_{C)}$). A short computation reveals this is $-\frac{n}{2}$ hence the next "approximation" of $\widetilde{\nabla}$ will be the covariant derivative

$$\overline{\overline{\nabla}}_a:=\overline{\nabla}_a+\frac{2}{n}\overline{\Psi}_{aBC}{}^{PQ}=\nabla_a+\frac{2}{3}\Psi_{aBC}{}^{PQ}+\frac{2}{n}\overline{\Psi}_{aBC}{}^{PQ}:\mathcal{E}_{(PQ)}\to\mathcal{E}_{a(BC)}.$$

Now we need the curvature $\overline{\overline{\Omega}}_{a^0a^1BC}{}^{PQ}$ of $\overline{\overline{\nabla}}_a$ and then to apply ∂^*_{BC} on $\frac{1}{2}\overline{\overline{\Omega}}_{a^0a^1BC}{}^{PQ}$. It follows from the definition of $\overline{\overline{\nabla}}_a$ that

(15)

$$\frac{1}{2}\overline{\overline{\Omega}}_{a^0a^1BC}{}^{PQ}=\frac{1}{2}\overline{\Omega}_{a^0a^1BC}{}^{PQ}+\frac{2}{n}\nabla_{a^0}\overline{\Psi}_{a^1BC}{}^{PQ}+\frac{4}{3n}\overline{\Psi}_{a^0BC}{}^{RS}\Psi_{a^1RS}{}^{PQ}$$

since $\overline{\Psi}_{a^0BC}{}^{RS}\overline{\Psi}_{a^1RS}{}^{PQ}=\Psi_{a^0BC}{}^{RS}\overline{\Psi}_{a^1RS}{}^{PQ}=0$.

The next step is to compute $\overline{\overline{\Psi}}_{a^1BC}{}^{PQ}:=\frac{1}{2}(\partial^*_{BC}\overline{\overline{\Omega}})_{a^1BC}{}^{PQ}$. We apply ∂^*_{BC} to the three terms on the right hand side of (15). Firstly recall

$\frac{1}{2}(\partial^*_{BC}\overline{\Omega})_{a^1BC}{}^{PQ} = \overline{\Psi}_{a^1BC}{}^{PQ}$ by definition. Secondly, one gets

$$\begin{aligned}
(d^\nabla\overline{\Psi})_{a^0a^1BC}{}^{PQ} = \frac{1}{3}Z_{(B}{}^{b}Z_{C)}{}^{c}\Big[&(n-4)Z^{p(P}Z^{Q)q}\boldsymbol{g}_{ba^0}C_{a^1(pq)c}\\
&+2(n-4)X^{(P}Z^{Q)q}\boldsymbol{g}_{ba^0}A_{q(a^1c)} - X^PX^Q\boldsymbol{g}_{ba^0}B_{a^1c}\Big]\\
+\frac{1}{3}X_{(B}Z_{C)}{}^{c}\Big[&\frac{3}{2}(n-4)Y^{(P}Z^{Q)q}C_{a^0a^1qc} - \frac{3}{2}(n-4)X^{(P}Y^{Q)}A_{a^0a^1c}\\
&+(n-4)Z^{p(P}Z^{Q)q}\big(\nabla_{a^0}C_{a^1(pq)c} + 2\boldsymbol{g}_{a^0(p}A_{q)(a^1c)} + 2\boldsymbol{g}_{a^0c}A_{a^1(pq)}\big)\\
&+2X^{(P}Z^{Q)q}\big((n-4)\nabla_{a^0}A_{q(a^1c)} - (n-4)P_{a^0}{}^{p}C_{a^1(pq)c} + 2\boldsymbol{g}_{a^0[c}B_{q]a^1}\\
&\qquad+\frac{2}{3}\boldsymbol{g}_{ca^0}C_{a^1}{}^{(rs)p}C_{qrsp}\big)\\
&+X^PX^Q\big(-\nabla_{a^0}B_{a^1c} - 2(n-4)P_{a^0}{}^{q}A_{q(a^1c)} + \frac{4}{3}\boldsymbol{g}_{ca^0}C_{a^1}{}^{(rs)p}A_{p(rs)}\big)\Big]\\
+X_BX_C\varphi_{a^1}{}^{PQ}&
\end{aligned}$$

for some $\varphi_{a^1}{}^{PQ} \in \mathcal{E}_{a^1}{}^{PQ}$ after some computation. Using the last display, it is not difficult to verify

$$(\partial^*_{BC}d^\nabla\overline{\Psi})_{a^1BC}{}^{PQ} = -\frac{n}{2}\overline{\Psi}_{a^1BC}{}^{PQ} - \frac{2}{9}(n-2)(\partial^*_{BC}(\Psi\wedge\Psi))_{a^1BC}{}^{PQ}.$$

Thirdly, one easily derives $\overline{\Psi}_{a^0BC}{}^{RS}\Psi_{a^1RS}{}^{PQ} = -\frac{n-4}{3}\Psi_{a^0BC}{}^{RS}\Psi_{a^1RS}{}^{PQ}$. Hence we finally obtain

$$\overline{\overline{\Psi}}_{a^1BC}{}^{PQ} = \frac{1}{2}(\partial^*_{BC}\overline{\overline{\Omega}})_{a^1BC}{}^{PQ} = -\frac{8}{9n}(n-3)(\partial^*_{BC}(\Psi\wedge\Psi))_{a^1BC}{}^{PQ}, \tag{16}$$

where $-\frac{8}{9n}(n-3) = -\frac{4}{9n}(n-2) - \frac{4}{9n}(n-4)$.

In the last step we need the action of $\Box_1$ on the subbundle $\mathcal{E}_{a^1}[-2] \subseteq \mathcal{E}_{a^1(BC)_0}$ corresponding to the injector X_BX_C. This is the scalar $-(n-1)$, so by adding $\frac{1}{n-1}\overline{\overline{\Psi}}_{a^1BC}{}^{PQ}$ to $\overline{\overline{\nabla}}_a$ we obtain the resulting prolongation covariant derivative

$$\widetilde{\nabla}_a := \nabla_a + \frac{2}{3}\Psi_{aBC}{}^{PQ} + \frac{2}{n}\overline{\Psi}_{aBC}{}^{PQ} + \frac{1}{n-1}\overline{\overline{\Psi}}_{aBC}{}^{PQ} : \mathcal{E}_{(PQ)} \to \mathcal{E}_{a(BC)}.$$

Proposition 4.4. *The prolongation connection $\widetilde{\nabla} : \mathcal{E}_{(BC)} \to \mathcal{E}_{a(BC)}$ in the conformal geometry has the form $\widetilde{\nabla}_aF_{BC} = \nabla_aF_{BC} + \frac{2}{3}\widetilde{\Psi}_{aBC}{}^{PQ}F_{PQ}$*

where

$$\begin{aligned}
\widetilde{\Psi}_{aBC}{}^{PQ} &= -Z_{(B}{}^{b} Z_{C)}{}^{c}\Big[X^{(P}Z^{Q)q}C_{a(bc)q} + X^P X^Q A_{a(bc)}\Big] \\
&\quad + X_{(B}Z_{C)}^{c}\Big[-\frac{4}{n}Z^{p(P}Z^{Q)q}C_{a(pq)c} + 2X^{(P}Z^{Q)q}\big(A_{a(cq)} + \frac{n-4}{n}A_{q(ac)}\big) \\
&\qquad\qquad - \frac{1}{n}X^P X^Q B_{ac}\Big] \\
&\quad + X_B X_C\Big[-\frac{4}{n}Z^{p(P}Z^{Q)q}A_{a(pq)} + \frac{1}{n}X^{(P}Z^{Q)q}\big(B_{aq} + \frac{4}{3(n-1)}C_a{}^{(rs)p}C_{qrsp}\big) \\
&\qquad\qquad + \frac{4}{3n(n-1)}X^P X^Q C_a{}^{(rs)p}A_{prs}\Big].
\end{aligned}$$

□

Example 4.5. The prolongation covariant derivative $\widetilde{\nabla}$ on tractor form bundles $\mathcal{E}_{A^0\mathbf{A}}$, $\mathbf{A} = \mathbf{A}^k$ was computed in [11]. Consider a section $F_{A^0\mathbf{A}} = \mathbb{Y}_{A^0\mathbf{A}}{}^{\mathbf{a}}\sigma_{\mathbf{a}} + \frac{1}{k+1}\mathbb{Z}_{A^0\mathbf{A}}^{a^0\mathbf{a}}\mu_{a^0\mathbf{a}} + \mathbb{W}_{A^0\mathbf{A}}{}^{\dot{\mathbf{a}}}\nu_{\dot{\mathbf{a}}} + \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\rho_{\mathbf{a}} \in \mathcal{E}_{A^0\mathbf{A}}$. Then

$$\begin{aligned}
\widetilde{\nabla}_c F_{A^0\mathbf{A}} &= \nabla_c F_{A^0\mathbf{A}} + \frac{1}{2}\mathbb{Z}_{A^0\mathbf{A}}^{a^0\mathbf{a}}\Big[C_c{}^p{}_{a^0a^1}\sigma_{p\dot{\mathbf{a}}} + \frac{k-1}{n}\boldsymbol{g}_{ca^0}C_{a^1a^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}}\Big] \\
&\quad - \frac{k(k-1)}{2n(n-k)}\mathbb{W}_{A^0\mathbf{A}}{}^{\dot{\mathbf{a}}}\Big[(n-2)C_{ca^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} - (k-2)C_{a^2a^3}{}^{pq}\sigma_{cpq\ddot{\mathbf{a}}}\Big] \\
&\quad + \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\Big[-A_c{}^p{}_{a^1}\sigma_{p\dot{\mathbf{a}}} - \frac{(k-1)(k-2)}{2nk}\boldsymbol{g}_{ca^1}C_{a^2a^3}{}^{pq}\nu_{pq\ddot{\mathbf{a}}} \\
&\qquad + \frac{k-1}{2(n-k)}\Big(\frac{n-2k}{2n}(\nabla_c C_{a^1a^2}{}^{pq})\sigma_{pq\ddot{\mathbf{a}}} + \boldsymbol{g}_{ca^1}A^{pq}{}_{a^2}\sigma_{pq\ddot{\mathbf{a}}} \\
&\qquad\qquad - 2A_{ca^1}{}^p\sigma_{p\dot{\mathbf{a}}} - A_{a^1a^2}{}^p\sigma_{cp\ddot{\mathbf{a}}} + C_{ca^1}{}^{pq}\mu_{pq\dot{\mathbf{a}}} \\
&\qquad\qquad + \frac{n(n-k+1)-2k}{nk}C_c{}^p{}_{a^1a^2}\nu_{p\ddot{\mathbf{a}}} - \frac{k}{n}C_{a^1a^2}{}^{pq}\mu_{cpq\ddot{\mathbf{a}}}\Big)\Big],
\end{aligned}$$

cf. [11, Remark 4.2].

The prolongation covariant derivative $\widetilde{\nabla}$ simplifies for $k = 2$ in dimension $n = 4$. Then we have (at least locally) the conformal volume form

$$\epsilon_{\mathbf{c}} \in \mathcal{E}_{\mathbf{c}}[4] \quad \text{such that} \quad \epsilon^{\mathbf{c}}\epsilon_{\mathbf{c}} = 4!, \text{ i.e. } \epsilon^{\mathbf{e}}\epsilon_{\mathbf{c}} = 4!\delta_{c^1}^{e^1}\delta_{c^2}^{e^2}\delta_{c^3}^{e^3}\delta_{c^4}^{e^4}, \tag{17}$$

where $\mathbf{c} = \mathbf{c}^4$, $\mathbf{e} = \mathbf{e}^4$. Recall $\nabla\epsilon = 0$ for any connection ∇ from the conformal class. Then the Hodge–star operator $* : \mathcal{E}_{\mathbf{a}^k} \to \mathcal{E}_{\mathbf{a}^{4-k}}$, $k = 0, \dots, 4$ has the form $(*f)_{\mathbf{a}^k} = \epsilon_{\mathbf{a}^k}{}^{\mathbf{r}^{4-k}}f_{\mathbf{r}^{4-k}}$. The eigenvalues of $*$ for $k = 2$ are ± 2. The induced tractor volume form $E_{\mathbf{C}^6} = -30\mathbb{W}_{\mathbf{C}^6}^{\mathbf{c}^4}\epsilon_{\mathbf{c}^4} \in \mathcal{E}_{\mathbf{C}^6}$ yields analogously the tractor Hodge–star operator $* : \mathcal{E}_{\mathbf{B}^\ell} \to \mathcal{E}_{\mathbf{B}^{6-\ell}}$. The eigenvalues of E for $\ell = 3$ are ± 6.

Henceforth we assume $k = 2$ and $n = 4$ and $*F = 6F$. If not stated otherwise, all form indices will have valence 2, e.g. $\mathbf{A} = \mathbf{A}^2$ or $\mathbf{a} = \mathbf{a}^2$. Our normalization of volume forms E and ϵ means that

$$*\,\sigma = 2\sigma, \quad *\mu = -3\nu, \quad *\nu = 2\mu, \quad *\rho = -2\rho, \tag{18}$$

i.e. $\sigma_{\mathbf{a}}$ is self-adjoint. Using this and (17) one easily verifies

$$\boldsymbol{g}_{ca^0} C_{\mathbf{a}}{}^{\mathbf{r}} \sigma_{\mathbf{r}} = -2 C_c{}^p{}_{\mathbf{a}} \sigma_{pa^0}, \quad C_{\mathbf{a}}{}^{\mathbf{r}} \mu_{c\mathbf{r}} = -2 C_{ca^1}{}^{\mathbf{r}} \mu_{a^2\mathbf{r}}. \tag{19}$$

Thus the prolongation covariant derivative $\tilde{\tilde{\nabla}}$ has the form

$$\begin{aligned}\tilde{\tilde{\nabla}}_c F_{A^0\mathbf{A}} =& \nabla_c F_{A^0\mathbf{A}} + \frac{1}{4}\mathbb{Z}_{A^0\mathbf{A}}^{a^0\mathbf{a}} C_c{}^p{}_{\mathbf{a}} \sigma_{pa^0} - \frac{1}{4}\mathbb{W}_{A^0\mathbf{A}}{}^{a} C_{ca}{}^{\mathbf{r}} \sigma_{\mathbf{r}} \\ &+ \frac{1}{4}\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}} \big[-4A_c{}^p{}_{a^1}\sigma_{pa^2} + \boldsymbol{g}_{ca^1} A^{\mathbf{r}}{}_{a^2}\sigma_{\mathbf{r}} - 2A_{ca^1}{}^p \sigma_{pa^2} \\ &\qquad - A_{\mathbf{a}}{}^p \sigma_{cp} + 2C_{ca^1}{}^{\mathbf{r}} \mu_{a^2\mathbf{r}} + C_c{}^p{}_{\mathbf{a}} \nu_p\big],\end{aligned}$$

The connection $\tilde{\tilde{\nabla}}$ simplifies considerably for half-flat structures, i.e. when

$$\epsilon_{\mathbf{a}}{}^{\mathbf{r}} C_{\mathbf{rb}} + \epsilon_{\mathbf{b}}{}^{\mathbf{r}} C_{\mathbf{ar}} = 4\lambda C_{\mathbf{ab}}, \quad \lambda \in \{+1, -1\}. \tag{20}$$

The self-adjoint structure $\lambda = 1$ equivalently means $C_{\mathbf{a}}{}^{\mathbf{r}} f_{\mathbf{r}} = 0$ for every anti-self-adjoint two form $f_{\mathbf{a}}$ and the anti-self-adjoint structure $\lambda = -1$ analogously means $C_{\mathbf{a}}{}^{\mathbf{r}} f_{\mathbf{r}} = 0$ for every self-adjoint $f_{\mathbf{a}}$. It follows from (20), (18) and (17) that

$$C_c{}^p{}_{\mathbf{a}} \nu_p = \lambda C_{\mathbf{a}}{}^{\mathbf{r}} \mu_{c\mathbf{r}}. \tag{21}$$

We shall discuss the anti-self dual case $\lambda = -1$ in detail. A short computation reveals

$$C_{\mathbf{a}}{}^{\mathbf{r}} \sigma_{\mathbf{r}} = 0, \quad A^{\mathbf{r}}{}_a \sigma_{\mathbf{r}} = 0 \quad \text{and} \quad A_{\mathbf{a}}{}^p \sigma_{cp} = 2A_{a^1c}{}^p \sigma_{a^2p},$$

where the second and the third equally follow by applying ∇^{a^1} and ∇_{a^0}, respectively, to the first one and using $\nabla_{a^0} C_{\mathbf{ar}} = 2\boldsymbol{g}_{a^0r^1} A_{\mathbf{a}r^2}$. (Note the last equality says $A_{[\mathbf{a}}{}^p \sigma_{c]p} = 0$.) From the last display and (21) for $\lambda = -1$ we finally obtain the following:

Proposition 4.6. *Consider an anti-self-dual conformal structure in the dimension 4. Then the prolongation connection* $\tilde{\tilde{\nabla}} : \mathcal{E}^+_{[A^0\mathbf{A}]} \to \mathcal{E}^+_{c[A^0\mathbf{A}]}$, $\mathbf{A} = \mathbf{A}^2$ *on the bundle of self-dual tractor 3-forms* $\mathcal{E}^+_{[A^0\mathbf{A}]} \subseteq \mathcal{E}_{[A^0\mathbf{A}]}$ *has the form*

$$\tilde{\tilde{\nabla}}_c F_{A^0\mathbf{A}} = \nabla_c F_{A^0\mathbf{A}} + \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}} \big[-2A_{c(pa^1)} \sigma^p{}_{a^2} + \frac{1}{2} C_c{}^p{}_{\mathbf{a}} \nu_p\big].$$

for $F_{A^0\mathbf{A}} \in \mathcal{E}^+_{[A^0\mathbf{A}]}$ *where* $\sigma_{\mathbf{a}} = 3\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}} F_{A^0\mathbf{A}}$ *and* $\nu_a = -6\mathbb{W}^{A^0\mathbf{A}}{}_a F_{A^0\mathbf{A}}$.

Note a modification of ∇ on $\mathcal{E}^+_{A^0\mathbf{A}}$ was also obtained in [10, (2.27)] where the spinorial notation is used.

5. Almost Grassmannian geometry

A complex almost Grassmannian (or AG–) structure on a smooth manifold M is given by two auxiliary vector bundles $\mathcal{E}^A$ and $\mathcal{E}_{A'}$ and the identification

$$\mathcal{E}^a = \mathcal{E}_{A'} \otimes \mathcal{E}^A = \mathcal{E}^A_{A'}, \quad \bigwedge^q \mathcal{E}^A \cong \bigwedge^p \mathcal{E}_{A'}, \tag{22}$$

where p is the rank of $\mathcal{E}_{A'}$ and q is the rank of $\mathcal{E}^A$. In fact, all results we obtain hold for all real forms of a given complex geometry, [16]. Motivated by the case $p = q = 2$ when the structure is the spin conformal structure, we shall term $\mathcal{E}^A$ and $\mathcal{E}_{A'}$ spinor bundles.

Following [16] and equation (22), we adopt the convention

$$\mathcal{E}[-1] \cong \mathcal{E}_{\mathbf{A}^q} \cong \mathcal{E}^{\mathbf{B}'^p}, \quad \mathcal{E}[1] \cong \mathcal{E}^{\mathbf{A}^q} \cong \mathcal{E}_{\mathbf{B}'^p}$$

for line bundles. This isomorphism is given explicitly by the tautological section $\boldsymbol{\epsilon}_{\mathbf{A}^q} \in \mathcal{E}_{\mathbf{A}^q}[1]$ as $\mathcal{E}[-1] \ni f \mapsto f\boldsymbol{\epsilon}_{\mathbf{A}^q} \in \mathcal{E}_{\mathbf{A}^q}$. A choice of a *scale* $\xi \in \mathcal{E}[1]$ is equivalent to the choice of spinor volume forms $\boldsymbol{\epsilon}^{\xi}_{\mathbf{A}^q} := \xi^{-1}\boldsymbol{\epsilon}_{\mathbf{A}^q} \in \mathcal{E}_{\mathbf{A}^q}$, and analogously for $\mathcal{E}^{\mathbf{A}'^p}$.

Our convention for the torsion $T_{ab}{}^c$ and the curvature $R_{ab}{}^d{}_c$ of a covariant derivative ∇_a on TM are given by the equation

$$2\nabla_{[a}\nabla_{b]}v^c = T_{ab}{}^d\nabla_d v^c + R_{ab}{}^c{}_d v^d.$$

Summarizing [16, Theorem 2.1], for a scale $\xi \in \mathcal{E}[1]$ on an AG–structure there are unique covariant derivatives on $\mathcal{E}^A$ and $\mathcal{E}_{A'}$ such that the torsion $F_{A}^{A'}{}_{B}^{B'}{}_{C'}^{C}$ of the induced covariant derivative on TM is totally trace-free, the induced covariant derivative preserves (22) and in addition, ξ is parallel. We denote this class of covariant derivatives, parametrized by sections of $\mathcal{E}[1]$, by $[\nabla]$. Changing the scale $\xi \to \hat{\xi} = e^{\Upsilon}\xi \in \mathcal{E}[1]$ with Υ a smooth function, the covariant derivative ∇ changes to $\hat{\nabla}$ in a way that

$$\begin{aligned}
&\hat{\nabla}^{A'}_{A} u^C = \nabla^{A'}_{A} u^C + \delta^C_A \Upsilon^{A'}_{B} u^B, \quad \text{for } u^C \in \mathcal{E}^A,\\
&\hat{\nabla}^{A'}_{A} u_{C'} = \nabla^{A'}_{A} u_{C'} + \delta^{A'}_{C'} \Upsilon^{B'}_{A} u_{B'}, \quad \text{for } u_{C'} \in \mathcal{E}_{C'},\\
&\hat{\nabla}^{A'}_{A} v_B = \nabla^{A'}_{A} v_B - \Upsilon^{A'}_{B} v_A, \quad \text{for } v_B \in \mathcal{E}_B.\\
&\hat{\nabla}^{A'}_{A} v^{B'} = \nabla^{A'}_{A} v^{B'} - \Upsilon^{B'}_{A} v^{A'}, \quad \text{for } v_B \in \mathcal{E}_B \text{ and also}\\
&\hat{\nabla}_a f = \nabla_a f + w\Upsilon_a f, \quad \text{for } f \in \mathcal{E}[w]
\end{aligned} \tag{23}$$

where $\Upsilon_a = \nabla_a \Upsilon$. We use hat sign to denote quantities corresponding to the changed scale $\hat{\xi} = e^{\Upsilon}\xi$ from now on without further notice.

Given $\nabla \in [\nabla]$, we denote all covariant derivatives on tensor products of $\mathcal{E}^A$ ans $\mathcal{E}_{A'}$ also by ∇. The curvature on spinor bundles is given by

$$(2\nabla_{[a}\nabla_{b]} - T_{ab}{}^d\nabla_d)v^C = R_{ab}{}_D^C v^D, \quad (2\nabla_{[a}\nabla_{b]} - T_{ab}{}^d\nabla_d)v_{D'} = -R_{ab}{}_{D'}^{C'} v_{C'}.$$

The curvature of ∇ is $R_{abc}^{\ \ \ d} = R_{abC}^{\ \ \ D}\delta^{C'}_{D'} - R_{abD'}^{\ \ \ C'}\delta^D_C$, where $R_{abC'}^{\ \ \ D'}$ and $R_{abD}^{\ \ \ C}$ are trace–free on the spinor indices displayed. The relations

$$\begin{aligned}
R_{abD}^{\ \ \ C} &= U_{abD}^{\ \ \ C} - \delta^C_B \mathsf{P}_{A}^{A'}{}_{D}^{B'} + \delta^C_A \mathsf{P}_{B}^{B'}{}_{D}^{A'},\\
R_{abD'}^{\ \ \ C'} &= U_{abD'}^{\ \ \ C'} + \delta^{B'}_{D'} \mathsf{P}_{A}^{A'}{}_{B}^{C'} - \delta^{A'}_{D'} \mathsf{P}_{B}^{B'}{}_{A}^{C'},
\end{aligned}$$

together with the condition $U_{R}^{A'}{}_{B}^{B'}{}_{A}^{R} - U_{A}^{R'}{}_{B}^{B'}{}_{R'}^{A'} = 0$ (and the algebraic Bianchi identity) determine $U_{abD}^{\ \ \ C}$, $U_{abD'}^{\ \ \ C'}$ and the *Rho–tensor* P_{ab}. In more details, the curvature on the (co)tangent bundle is

$$R_{abd}{}^{c} = U_{abd}{}^{c} + \delta^{D'}_{C'}\delta^{C}_{A}\mathsf{P}^{B'}_{B}{}^{A'}_{D} - \delta^{D'}_{C'}\delta^{C}_{B}\mathsf{P}^{A'}_{A}{}^{B'}_{D} + \delta^{C}_{D}\delta^{A'}_{C'}\mathsf{P}^{B'}_{B}{}^{D'}_{A} - \delta^{C}_{D}\delta^{B'}_{C'}\mathsf{P}^{A'}_{A}{}^{D'}_{B}$$

where $U_{abd}{}^{c} = U_{ab}{}^{C}_{D}\delta^{D'}_{C'} - U_{ab}{}^{D'}_{C'}\delta^{C}_{D}$. In this form, tensors U are determined by $U_{rba}{}^{r} = U^{A'}_{R}{}^{B'}_{B}{}^{R}_{A} - U^{R'}_{A}{}^{B'}_{B}{}^{A'}_{R'} = 0$. (Note the previous display means the decomposition $U = R + \partial\mathsf{P}$ where U is ∂^*-closed, cf. the theory of Weyl structures in [9].) Furthermore,

$$U_{abC}{}^{C} = -U_{abC'}{}^{C'} = 2\mathsf{P}_{[ab]} \quad \text{and} \quad -2(p+q)\mathsf{P}_{[ab]} = \nabla_c T_{ab}{}^{c} \tag{24}$$

where the last identity follows from the algebraic Bianchi identity.

We will be mostly interested in the case $p = 2$ and $q > 2$. In this case, the only invariants are the trace–free part of $T^{(A'B')C}_{[A\,B]\;C'}$ and the trace–free part of $U^{[A'B']D}_{(A\,B\,C)}$, [16]. That is, if these two vanish, the geometry is locally isomorphic to the homogenous model. Finally note that using the algebraic Bianchi identity we obtain

$$\begin{aligned}
&U^{R'[A'B']}_{(A\,B)R'} = U^{[A'B']R}_{R\,(A\,B)} = U^{R'(A'B')}_{[A\,B]R'} = U^{(A'B')R'}_{R\,[A\,B]} = 0,\\
&U^{R'(A'B')}_{(A\,B)R'} = U^{(A'B')R}_{R\,(A\,B)} = \frac{1}{q}T_{r(A}{}^{(A'|e|}T^{B')}_{B\,)e}{}^{r}\\
&U^{R'[A'B']}_{[A\,B]R'} = U^{[A'B']R}_{R\,[A\,B]} = -\frac{1}{q+4}T_{r[A}{}^{[A'|e|}T^{B']}_{B\,]e}{}^{r}.
\end{aligned} \tag{25}$$

5.1. **Grassmannian tractor calculus.** We follow [16] here. The standard tractor bundle is the (spinor tractor) bundle $\mathcal{E}^{\alpha} = \mathcal{E}^{A} \oplus \mathcal{E}^{A'}$ and we denote its dual by $\mathcal{E}_{\alpha} = \mathcal{E}_{A'} \oplus \mathcal{E}_{A}$. (That is, we use Greek letters for spinor tractor abstract indices.) Using the injectors $Y^{\alpha}_{A} \in \mathcal{E}^{\alpha}_{A}$, $X^{\alpha}_{A'} \in \mathcal{E}^{\alpha}_{A'}$ and $Y^{A'}_{\alpha} \in \mathcal{E}^{A'}_{\alpha}$, $X^{A}_{\alpha} \in \mathcal{E}^{A}_{\alpha}$, sections of $\mathcal{E}^{\alpha}$ and $\mathcal{E}_{\alpha}$ are written conveniently as

$$\begin{pmatrix}\sigma^{A}\\ \rho^{A'}\end{pmatrix} = Y^{\alpha}_{A}\sigma^{A} + X^{\alpha}_{A'}\rho^{A'} \in \mathcal{E}^{\alpha}, \quad \text{resp.} \quad \begin{pmatrix}\nu_{A'}\\ \mu_{A}\end{pmatrix} = Y^{A'}_{\alpha}\nu_{A'} + X^{A}_{\alpha}\mu_{A} \in \mathcal{E}_{\alpha}.$$

Splittings of $\mathcal{E}^{\alpha}$ and $\mathcal{E}_{\alpha}$ are parametrised by choice of the scale $\xi \in \mathcal{E}[1]$. The change of the splitting has the form

$$\widehat{\begin{pmatrix}\sigma^{A}\\ \rho^{A'}\end{pmatrix}} = \begin{pmatrix}\sigma^{A}\\ \rho^{A'} - \Upsilon^{A'}_{B}\sigma^{B}\end{pmatrix}, \quad \text{i.e.} \;\; \hat{Y}^{\alpha}_{A} = Y^{\alpha}_{A} + X^{\alpha}_{B'}\Upsilon^{B'}_{A},\; \hat{X}^{\alpha}_{A'} = X^{\alpha}_{A'} \quad \text{and}$$

$$\widehat{\begin{pmatrix}\nu_{A'}\\ \mu_{A}\end{pmatrix}} = \begin{pmatrix}\nu_{A'}\\ \mu_{A} + \Upsilon^{A'}_{A}\nu_{A'}\end{pmatrix}, \quad \text{i.e.} \;\; \hat{Y}^{A'}_{\alpha} = Y^{A'}_{\alpha} - X^{B}_{\alpha}\Upsilon^{A'}_{B},\; \hat{X}^{A}_{\alpha} = X^{A}_{\alpha}$$

That is, the sections $X^{\alpha}_{A'}$ and X^{A}_{α} are invariant and Y^{α}_{A} and $Y^{A'}_{\alpha}$ depend on the choice of the scale. They are normalized in such a way that $Y^{\beta}_{B}X^{B}_{\alpha} + Y^{B'}_{\alpha}X^{\beta}_{B'} = \delta_{\alpha}{}^{\beta}$, i.e. $X^{B}_{\alpha}Y^{\alpha}_{A} = \delta_{A}{}^{B}$ and $X^{\alpha}_{A'}Y^{B'}_{\alpha} = \delta_{A'}{}^{B'}$.

The normal covariant tractor derivative is given by

$$\nabla^{P'}_{A}\begin{pmatrix}\sigma^{B}\\ \rho^{B'}\end{pmatrix} = \begin{pmatrix}\nabla^{P'}_{A}\sigma^{B} + \rho^{P'}\delta_{A}{}^{B}\\ \nabla^{P'}_{A}\rho^{B'} - \mathsf{P}^{P'B'}_{A\,B}\sigma^{B}\end{pmatrix} \text{ and } \nabla^{P'}_{A}\begin{pmatrix}\nu_{B'}\\ \mu_{B}\end{pmatrix} = \begin{pmatrix}\nabla^{P'}_{A}\nu_{B'} - \delta^{P'}_{B'}\mu_{A}\\ \nabla^{P'}_{A}\mu_{B} + \mathsf{P}^{P'B'}_{A\,B}\nu_{B'}\end{pmatrix}.$$

That is,

$$\nabla_A^{P'} Y_B^\alpha = -X_{B'}^\alpha \mathsf{P}_{A\,B}^{P'B'}, \nabla_A^{P'} X_{B'}^\alpha = Y_A^\alpha \delta_{B'}^{P'} \text{ and}$$
$$\nabla_A^{P'} Y_\alpha^{B'} = X_\alpha^B \mathsf{P}_{A\,B}^{P'B'},\ \nabla_A^{P'} X_\alpha^B = -Y_\alpha^{P'} \delta_A^B.$$

Its curvature $\Omega_{ab}{}^\alpha_\beta$ is trace-free on the spinor tractor bundle and has the explicit form

$$\Omega_{ab}{}^\alpha_\beta = -\, Y_C^\alpha Y_\beta^{C'} T_{abC'}{}^C + Y_C^\alpha X_\beta^D U_{abD}{}^C + X_{C'}^\alpha Y_\beta^{D'} U_{abD'}{}^{C'}$$
$$+ X_{C'}^\alpha X_\beta^C Q_{abC}{}^{C'} \in \mathcal{E}_{[ab]}{}^\alpha_\beta \subseteq \mathcal{E}_{[ab]} \otimes \text{trace-free}(\mathcal{E}_\beta^\alpha)$$

where $Q_{abc} = -2\nabla_{[a}\mathsf{P}_{b]c} + T_{ab}{}^e\mathsf{P}_{ec} \in \mathcal{E}_{[ab]c}$ and trace-free$(\mathcal{E}_\beta^\alpha) = \mathcal{A}$ is the adjoint tractor bundle. That is, $(\nabla_a\nabla_b - \nabla_b\nabla_a - T_{ab}{}^e\nabla_e)f^\alpha = \Omega_{ab}{}^\alpha_\beta f^\beta = (\Omega\sharp f)_{ab}{}^\alpha = \Omega_{ab}\sharp f^\alpha$ in our notation.

The inclusions ι and $\bar\iota$ from 2.2 are of the form $Y_{A^0}^\alpha Y_\beta^{A^{0\prime}} : \mathcal{E}_{\mathbf{a}} \xrightarrow{\bar\iota} \mathcal{E}_{a^0\mathbf{a}}{}^\alpha{}_\beta$ and $X_{A^{1\prime}}^\alpha X_\beta^{A^1} : \mathcal{E}_{\mathbf{a}} \xrightarrow{\iota} \mathcal{E}_{\dot{\mathbf{a}}}{}^\alpha{}_\beta$, where we use the identification $\mathcal{E}_{a^0} = \mathcal{E}_{A^0}^{A^{0\prime}}$ and $\mathcal{E}^{a^1} = \mathcal{E}_{A^{1\prime}}^{A^1}$. Therefore

$$\partial : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \ni f_{\mathbf{a}} \mapsto Y_{A^0}^\alpha Y_\beta^{A^{0\prime}} f_{\mathbf{a}} \xrightarrow{\sharp} \mathcal{E}_{a^0\mathbf{a}} \otimes \mathcal{T} \quad \text{and}$$
$$\partial^* : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \ni f_{\mathbf{a}} \mapsto X_{A^{1\prime}}^\alpha X_\beta^{A^1} f_{\mathbf{a}} \xrightarrow{\sharp} \mathcal{E}_{\dot{\mathbf{a}}} \otimes \mathcal{T}$$

for any subbundle $\mathcal{T}$ of $\bigotimes \mathcal{E}_\alpha \otimes \bigotimes \mathcal{E}^\beta \otimes \mathcal{E}[w]$. This does not cover all tractor bundles but will be sufficient in the examples treated below.

Henceforth we assume $p = 2$, $q > 2$. Note we have the decomposition $\Omega_{ab}{}^\alpha_\beta = \Omega_{(AB)}^{[A'B']}{}^\alpha_\beta + \Omega_{[A\,B]}^{(A'B')}{}^\alpha_\beta$, where the component $\Omega_{[A\,B]}^{(A'B')}{}^\alpha_\beta$ vanishes in the torsion–free case.

5.2. **Skew symmetric tractors and tractor forms.** We shall also need tractor bundles $\bigwedge^\ell \mathcal{E}^\alpha = \mathcal{E}^{\boldsymbol{\alpha}}$ with the notation for the multiindex $\boldsymbol{\alpha} = \boldsymbol{\alpha}^\ell$. Since $\bigwedge^\ell \mathcal{E}^\alpha \cong \bigwedge^{q+2-\ell} \mathcal{E}_\beta$ (we assume orientability here), these are just tractor forms. Specifically, the case $\ell = q+1$ is just the bundle $\mathcal{E}_\beta$.

It follows from the structure of $\mathcal{E}^\alpha$ that

$$\mathcal{E}^{\boldsymbol{\alpha}} = \mathcal{E}^{\mathbf{A}} \mathbin{+\!\!\!\!\!\supset} \mathcal{E}^{B'\dot{\mathbf{A}}} \mathbin{+\!\!\!\!\!\supset} \mathcal{E}^{[B'C']\ddot{\mathbf{A}}}, \qquad \boldsymbol{\alpha} = \boldsymbol{\alpha}^\ell,\ \mathbf{A} = \mathbf{A}^\ell,\ 2 \le \ell \le q.$$

Of course we have the isomorphism $\mathcal{E}^{[B'C']\ddot{\mathbf{A}}} \cong \mathcal{E}^{\ddot{\mathbf{A}}}[-1]$ using the spinor volume form $\boldsymbol{\epsilon}_{B'C'} \in \mathcal{E}_{[B'C']}[-1]$ but it turns out more convenient for the computation to use the form as in the display.

We put

$$\mathbb{Y}_{\mathbf{A}}^{\boldsymbol{\alpha}} = Y_{A^1}^{[\alpha^1} \dots Y_{A^\ell}^{\alpha^\ell]} \in \mathcal{E}_{\mathbf{A}}^{\boldsymbol{\alpha}}, \quad \mathbb{W}_{B'\dot{\mathbf{A}}}^{\boldsymbol{\alpha}} = X_{B'}^{[\alpha^1} Y_{A^2}^{\alpha^2} \dots Y_{A^\ell}^{\alpha^\ell]} \in \mathcal{E}_{B'\dot{\mathbf{A}}}^{\boldsymbol{\alpha}},$$
$$\mathbb{X}_{B'C'\ddot{\mathbf{A}}}^{\boldsymbol{\alpha}} = X_{B'}^{[\alpha^1} X_{C'}^{\alpha^2} Y_{A^3}^{\alpha^3} \dots Y_{A^\ell}^{\alpha^\ell]} \in \mathcal{E}_{[B'C']\ddot{\mathbf{A}}}^{\boldsymbol{\alpha}}$$

where $\mathbb{X}^{\boldsymbol{\alpha}}_{B'C'\dot{\mathbf{A}}}$ is invariant and $\mathbb{Y}^{\boldsymbol{\alpha}}_{\mathbf{A}}$ and $\mathbb{W}^{\boldsymbol{\alpha}}_{B'\dot{\mathbf{A}}}$ are scale dependent. Finally, the normal tractor connection on these section is

$$\begin{aligned}
&\nabla_c \mathbb{Y}^{\boldsymbol{\alpha}}_{\mathbf{A}} = -\ell\, \mathbb{W}^{\boldsymbol{\alpha}}_{B'[\dot{\mathbf{A}}} \mathsf{P}_{|c|A^1]}{}^{B'},\\
&\nabla_c \mathbb{W}^{\boldsymbol{\alpha}}_{B'\dot{\mathbf{A}}} = \mathbb{Y}^{\boldsymbol{\alpha}}_{C\dot{\mathbf{A}}}\, \delta^{C'}_{B'} - (\ell-1)\, \mathbb{X}^{\boldsymbol{\alpha}}_{B'D'[\dot{\mathbf{A}}} \mathsf{P}_{|c|A^2]}{}^{D'}, \quad \text{and}\\
&\nabla_c \mathbb{X}^{\boldsymbol{\alpha}}_{B'D'\dot{\mathbf{A}}} = 2\, \mathbb{W}^{\boldsymbol{\alpha}}_{B'C\ddot{\mathbf{A}}}\, \delta^{C'}_{D'}.
\end{aligned}$$

Example 5.1. We shall demonstrate the prolongation covariant derivative $\widetilde{\nabla}$ for AG–geometries on tractor bundles corresponding to fundamental representations. These are bundles $\bigwedge^{\ell} \mathcal{E}^{\alpha}$ for $1 \le \ell \le q+1$. Since the computation is getting very technical for $1 < \ell < q+1$, we later restrict to torsion–free manifolds.

First we discuss the cases $\mathcal{E}^{\alpha}$ and $\mathcal{E}_{\beta} \cong \bigwedge^{q+1} \mathcal{E}^{\alpha}$. Considering $F^{\alpha} \in \mathcal{E}^{\alpha}$ and $G_{\beta} \in \mathcal{E}_{\beta}$, a short computation gives

$$\begin{aligned}
(\partial^* d^{\nabla} d^{\nabla} F)_c{}^{\alpha} &= \frac{1}{2} X^{\alpha}_{D'} X^{D}_{\omega} S^{D'C'}_{C\,D} F^{\omega} \quad \text{and}\\
(\partial^* d^{\nabla} d^{\nabla} G)_{c\beta} &= -\frac{1}{2} X^{\omega}_{D'} X^{D}_{\beta} S^{D'C'}_{C\,D} G_{\omega},
\end{aligned}$$

where

$$S^{D'C'}_{C\,D} = U^{R'A'B'}_{A\,B\,R'} = U^{A'B'R}_{R\,A\,B} = \frac{1}{q} T_{r(A}{}^{(A'|e|} T_{B)e}{}^{B')\,r} - \frac{1}{q+4} T_{r[A}{}^{[A'|e|} T_{B]e}{}^{B']\,r}.$$

Hence we need the action of the Kostant-Laplace operator $\square$ on $\mathcal{E}^{D'C'}_{C} = \mathcal{E}^{(D'C')}_{C} \oplus \mathcal{E}^{[D'C']}_{C}$. The eigenvalues are, respectively, $\frac{1}{2}(q-1)$ and $\frac{1}{2}(q+1)$. Therefore the prolongation connection $\widetilde{\nabla}$ has the form

$$\widetilde{\nabla}_c F^{\alpha} = \nabla_c F^{\alpha} - X^{\alpha}_{D'} X^{D}_{\omega} \Big[\frac{1}{q-1} S^{(D'C')}_{C\,D} + \frac{1}{q+1} S^{[D'C']}_{C\,D} \Big] F^{\omega} \quad \text{for } F^{\alpha} \in \mathcal{E}^{\alpha},$$

$$\widetilde{\nabla}_c G_{\beta} = \nabla_c G_{\beta} + X^{\omega}_{D'} X^{D}_{\beta} \Big[\frac{1}{q-1} S^{(D'C')}_{C\,D} + \frac{1}{q+1} S^{[D'C']}_{C\,D} \Big] G_{\omega} \quad \text{for } G_{\beta} \in \mathcal{E}_{\beta}.$$

It remains to consider the bundles $\mathcal{E}^{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{\ell}$ for $2 \le \ell \le q$. Consider the section $F^{\boldsymbol{\alpha}} = \mathbb{Y}^{\boldsymbol{\alpha}}_{\mathbf{A}} \sigma^{\mathbf{A}} + \mathbb{W}^{\boldsymbol{\alpha}}_{B'\dot{\mathbf{A}}} \mu^{B'\dot{\mathbf{A}}} + \mathbb{X}^{\boldsymbol{\alpha}}_{B'C'\dot{\mathbf{A}}} \rho^{B'C'\ddot{\mathbf{A}}}$, where $\sigma^{\mathbf{A}} \in \mathcal{E}^{\mathbf{A}}$, $\mu^{B'\dot{\mathbf{A}}} \in \mathcal{E}^{B'\dot{\mathbf{A}}}$ and $\rho^{B'C'\ddot{\mathbf{A}}} \in \mathcal{E}^{[B'C']\ddot{\mathbf{A}}}$. A straightforward computation shows that

$$\begin{aligned}
(d^{\nabla} d^{\nabla} F)_{de}{}^{\boldsymbol{\alpha}} &= \frac{1}{2} \Omega_{de} \sharp F^{\boldsymbol{\alpha}} = \frac{1}{2} \ell\, \Omega_{de}{}^{[\alpha^1}_{\ \ \omega} F^{|\omega|\dot{\boldsymbol{\alpha}}]} =\\
&= \frac{1}{2} \Big\{ \mathbb{Y}^{\boldsymbol{\alpha}}_{\mathbf{A}} \Big[\ell U_{de}{}^{[A^1}_{\ \ Q} \sigma^{|Q|\dot{\mathbf{A}}]} - T_{de}{}^{[A^1}_{\ \ Q'} \mu^{|Q'|\dot{\mathbf{A}}]} \Big]\\
&\qquad + \mathbb{W}^{\boldsymbol{\alpha}}_{B'\dot{\mathbf{A}}} \Big[(\ell-1) U_{de}{}^{[A^2}_{\ \ Q} \mu^{|B'Q|\ddot{\mathbf{A}}]} + \ell Q_{deQ}{}^{B'} \sigma^{Q\dot{\mathbf{A}}} + U_{de}{}^{B'}_{\ \ Q'} \mu^{Q'\dot{\mathbf{A}}}\\
&\qquad\qquad - 2T_{de}{}^{[A^2}_{\ \ Q'} \rho^{|B'Q'|\ddot{\mathbf{A}}]} \Big] + \mathbb{X}^{\boldsymbol{\alpha}}_{B'C'\dot{\mathbf{A}}} \varphi^{B'C'\ddot{\mathbf{A}}} \Big\}
\end{aligned}$$

for a section $\varphi^{B'C'\ddot{\mathbf{A}}} \in \mathcal{E}^{B'C'\ddot{\mathbf{A}}}$. We need to compute ∂^* of the previous display.

It turns out the computation is getting too technical in general, so we compute $\widetilde{\nabla}$ in the torsion–free case only. That is, we assume $T_{ef}{}^{C'}_{C} = 0$ (hence also $S^{D'C'}_{C\,D} = 0$) from now on. Then we obtain

$$(\partial^* d^\nabla d^\nabla F)_e{}^{\boldsymbol{\alpha}} =$$

$$= \frac{1}{2}(\ell-1)\Big\{\ell\, \mathbb{W}^{\boldsymbol{\alpha}}_{B'\dot{\mathbf{A}}} U_{eR}{}^{B'[A^2}_{\ \ Q}\sigma^{|QR|\ddot{\mathbf{A}}]}$$

$$+ \mathbb{X}^{\boldsymbol{\alpha}}_{B'C'\ddot{\mathbf{A}}}\big[(\ell-2)U_{eR}{}^{C'[A^3}_{\ \ Q}\mu^{|B'QR|\ddot{\mathbf{A}}]} - \ell\, Q_{eRQ}{}^{C'B'}\sigma^{QR\dot{\mathbf{A}}} - U_{eRQ'}{}^{C'B'}\mu^{Q'R\ddot{\mathbf{A}}}\big]\Big\}.$$

Since $U^{A'B'D}_{A\,B\,C} = U^{[A'B']D}_{(A\,B\,C)}$ in the torsion–free case, we conclude that $(\partial^* d^\nabla d^\nabla F)_e{}^{\boldsymbol{\alpha}} = 0$. This yields the surprising result $\widetilde{\nabla} = \nabla$ on $\mathcal{E}^{\boldsymbol{\alpha}}$. The same is obviously true also for $\ell = 1$ and $\ell = q+1$. Hence we obtain

Proposition 5.2. *The prolongation connection $\widetilde{\nabla}_c : \mathcal{E}^{\boldsymbol{\alpha}} \to \mathcal{E}_c{}^{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} = \boldsymbol{\alpha}^\ell$ for $1 \le \ell \le q+1$ on torsion–free AG-manifolds is equal to the normal tractor connection, i.e. $\widetilde{\nabla} = \nabla$.*

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