ENERGY FUNCTIONS ON MODULI SPACES OF FLAT SURFACES WITH ERASING FOREST

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Abstract

This paper follows on from [N], in which we study flat surfaces with erasing forest, these surfaces are obtained by deforming the metric structure of translation surfaces, and their moduli space can be viewed as some deformations of the moduli space of translation surfaces. We showed that the moduli spaces of such surfaces are complex orbifolds, and admit a natural volume form μ_{Tr}. The aim of this paper is to show that the volume of those moduli spaces with respect to μ_{Tr} normalized by some energy function involving the area, and the total length of the erasing forest, is finite. Since translation surfaces, and flat surfaces of genus zero can be viewed as special cases of flat surfaces with erasing forest, and on their moduli space, the volume form μ_{Tr} equals the usual ones up to a multiplicative constant, this result allows us to recover some classical results of Masur-Veech, and of Thurston concerning the finiteness of the volume of the moduli space of translation sufaces, and of the moduli space of polyhedral flat surfaces.

1 Introduction

In [N], we have introduced the notion of *flat surface with erasing forest*. An *erasing forest* $\hat{A}$ in a flat surface with conical singularities Σ is a union of disjoint geodesic trees such that

- the vertex set of $\hat{A}$ contains all the singularities of Σ,
- the holonomy of any closed curve which does not intersect the forest $\hat{A}$ is a translation of $\mathbb{R}^2$.

Note that a 'generic' flat surface does not admit any erasing forest.

Recall that a *translation surface* is a flat surface with conical singularities verifying the following property: the holonomy of any closed curve (which does not contain any singularity) is a translation. Given a translation surface Σ, we can construct a flat surface with erasing forest by deforming its metric structure as follows: first, cut off a small disk about a singular point of Σ, note that by the definition of translation surface, the cone angle at any singular point of Σ must belong to $2\pi\mathbb{N}$. We can modify the metric structure inside the small disk to get a flat disk with several singular points, whose cone angles can be chosen arbitrarily, while the boundary stays unchanged. We can then glue the disk back to Σ. If the boundary is convex, then it is not hard to show that there exists a geodesic tree inside the disk whose vertex set is the set of singularities. Carrying out this operation for all the singular points of Σ, we get a new flat surface Σ' together with a family of geodesic trees. By construction, the union of these trees is an erasing forest of Σ'.

A translation surface is a particular flat surface with erasing forest, where each tree in the erasing forest is just a singular point. A flat surface of genus zero can also be viewed as a flat surface with erasing forest, since there always exists a geodesic tree on this surface whose vertex set is the set of singularities, and the complement of such a tree is just a topological disk.

Given a flat surface Σ with an erasing forest $\hat{A}$, a *parallel vector field* on Σ is a vector field defined on the complement of the erasing forest $\hat{A}$ which is invariant by the parallel transport. In a local chart of the flat metric structure, the integral lines of such a field are parallel. On any (connected) flat surface with erasing forest, such vector fields always exist, they are uniquely determined by a tangent vector at a fixed point in the complement of the erasing forest.

Given an integer $g \geqslant 0$, and positive real numbers $\alpha_1, \dots, \alpha_n$, verifying

$$\sum_{i=1}^{n} \alpha_i = (2g + n - 2)2\pi,$$

let us fix a family $\hat{\mathcal{A}} = \{\mathcal{A}_1, \dots, \mathcal{A}_m\}$ of topological trees such that the total number of vertices of the trees in $\hat{\mathcal{A}}$ is n, and choose a numbering on the set of vertices of $\hat{\mathcal{A}}$. Note that we consider an isolated point as a special tree. Let $\underline{\alpha}$ denote the vector $(\alpha_1, \dots, \alpha_n)$, and $\mathcal{M}^{\rm et}(\hat{\mathcal{A}}, \underline{\alpha})$ denote the set of triples $(\Sigma, \hat{A}, \xi)$ where

- Σ is a closed, connected, oriented flat surface of genus g with cone singularities,
- $\hat{A}$ is an erasing forest in Σ consisting of m geodesic trees $A_1, \dots, A_m$, we also suppose that the trees and vertices of $\hat{A}$ are numbered so that A_j is isomorphic to $\mathcal{A}_j$ (as topological trees), and by those isomorphisms, the i-th vertex of $\hat{\mathcal{A}}$ is mapped to a singular point with cone angle α_i.
- ξ is a unitary parallel vector field defined on $\Sigma \setminus \hat{A}$.

In [N], we proved that $\mathcal{M}^{\rm et}(\hat{\mathcal{A}}, \underline{\alpha})$ has a structure of analytic complex orbifold of dimension

$$\begin{cases} 2g + n - 1 & \text{if } \alpha_i \in 2\pi\mathbb{N},\ \forall i = 1, \dots, n, \\ 2g + n - 2 & \text{otherwise,} \end{cases}$$

together with a natural volume form $\mu_{\rm Tr}$. Remark that, as all the trees in the erasing forest shrink to points, a flat surface with erasing forest becomes a translation surface. Therefore, $\mathcal{M}^{\rm et}(\hat{\mathcal{A}}, \underline{\alpha})$ can be viewed as a deformation of some stratum of the moduli space of translation surfaces.

Consider the following function on $\mathcal{M}^{\text{et}}(\hat{\mathcal{A}}, \underline{\alpha})$

$$\begin{array}{rccc} \mathcal{F}^{\text{et}}: & \mathcal{M}^{\text{et}}(\hat{\mathcal{A}}, \underline{\alpha}) & \longrightarrow & \mathbb{R} \\ & (\Sigma, \hat{A}, \xi) & \longmapsto & \exp(-\mathbf{Area}(\Sigma) - \ell^2(\hat{A})) \end{array}$$

where $\ell(\hat{A})$ is the total length of the trees in $\hat{A}$. In what follows, we will call a topological tree which is not a point a *non-trivial tree*. The main result of this paper is the following

Theorem 1.1 *If at least one of the trees in the family $\hat{\mathcal{A}}$ is non-trivial, then the integral of the function $\mathcal{F}^{\text{et}}$ over $\mathcal{M}^{\text{et}}(\hat{\mathcal{A}}, \underline{\alpha})$ with respect to μ_{Tr} is finite:*

$$\int_{\mathcal{M}^{\text{et}}(\hat{\mathcal{A}}, \underline{\alpha})} \mathcal{F}^{\text{et}} d\mu_{\text{Tr}} < \infty. \tag{1}$$

Remark:

- The integral (1) is still finite if we multiply the total length of the erasing forest by a parameter $\epsilon > 0$, that is the statement of Theorem 1.1 is also true for functions $\mathcal{F}^{\text{et}}_{\epsilon} : (\Sigma, \hat{A}, \xi) \mapsto \exp(-\mathbf{Area}(\Sigma) - \epsilon\ell^2(\hat{A}))$, with $\epsilon > 0$.

- Let $e_1, \dots, e_{n-m}$ denote the edges of the trees in the forest $\hat{A}$, and $\ell(e_i)$ denote the length of e_i, then the integral (1) is also finite if we replace $\mathcal{F}^{\text{et}}$ by the function

$$\widetilde{\mathcal{F}}^{\text{et}} : (\Sigma, \hat{A}, \xi) \mapsto \exp(-\mathbf{Area}(\Sigma) - \sum_{i=1}^{n-m} \ell^2(e_i)).$$

  The proofs for $\mathcal{F}^{\text{et}}_{\epsilon}$ and $\widetilde{\mathcal{F}}^{\text{et}}$ are the same as the proof for $\mathcal{F}^{\text{et}}$.

In the case where all the trees in $\hat{\mathcal{A}}$ are points, the space $\mathcal{M}^{\text{et}}(\hat{\mathcal{A}}, \underline{\alpha})$ is identified to a stratum $\mathcal{H}(k_1, \dots, k_n)$ of the moduli space of Abelian differentials on Riemann surfaces of genus g, and we have

$$\mathcal{F}^{\text{et}}(\Sigma, \hat{A}, \xi) = \exp(-\mathbf{Area}(\Sigma)).$$

The similar result for this case can be proved as a consequence of Theorem 1.1, that is

Theorem 1.2 *We have*

$$\int_{\mathcal{H}(k_1,\dots,k_n)} \exp(-\mathbf{Area}(.)) d\mu_{\text{Tr}} < \infty. \tag{2}$$

Note that the assumption that at least one of the trees in the forest is not a point is crucial for the proof of Theorem 1.1, hence, Theorem 1.2 cannot be considered as a particular case of Theorem 1.1.

Let $\mathcal{H}_1(k_1, \dots, k_n)$ denote the subset of $\mathcal{H}(k_1, \dots, k_n)$ consisting of surfaces of unit area. Let μ^1_{Tr} denote the volume form on $\mathcal{H}_1(k_1, \dots, k_n)$ which is induced by μ_{Tr}. A direct consequence of Theorem 1.2 is the following

Corollary 1.3 *The total measure $\mu^1_{\rm Tr}(\mathcal{H}_1(k_1,\dots,k_n))$ is finite.*

Proof: Identifying $\mathcal{H}(k_1,\dots,k_n)$ to $\mathcal{H}_1(k_1,\dots,k_n)\times\mathbb{R}^*_+$, and we can write $d\mu_{\rm Tr}=t^s d\mu^1_{\rm Tr}dt$, where $s=\dim_{\mathbb{R}}\mathcal{H}_1(k_1,\dots,k_n)$ which is odd. Therefore, we have

$$\begin{aligned}\int_{\mathcal{H}(k_1,\dots,k_n)}\exp(-\mathbf{Area}(.))d\mu_{\rm Tr} &= \int_{\mathcal{H}_1(k_1,\dots,k_n)}\int_0^{+\infty}t^se^{-t^2}dtd\mu^1_{\rm Tr},\\ &= \frac{1}{2}(\frac{s-1}{2})!\int_{\mathcal{H}_1(k_1,\dots,k_n)}d\mu^1_{\rm Tr}\end{aligned}$$

and the corollary follows. □

On the space $\mathcal{H}(k_1,\dots,k_n)$, we have (see [MT], [Z2]) a "natural" volume form μ_0 which is defined by the period mapping, let μ^1_0 denote the volume form on $\mathcal{H}_1(k_1,\dots,k_n)$ which is induced by μ_0. In [N], we proved that $\mu_{\rm Tr}=\lambda\mu_0$, where λ is a constant on each connected component of $\mathcal{H}(k_1,\dots,k_n)$. By a well known result of Kontsevich-Zorich [KZ], we know that $\mathcal{H}(k_1,\dots,k_n)$ has at most three connected components, thus Corollary 1.3 is equivalent to the classical result of Masur-Veech stating that the volume of $\mathcal{H}_1(k_1,\dots,k_n)$ with respect to μ^1_0 is finite.

Let us now consider flat surfaces of genus zero, that is flat surfaces homeomorphic to the sphere $\mathbb{S}^2$. Fix n, $n\geqslant 3$, positive real numbers $\alpha_1,\dots,\alpha_n$ verifying

$$\sum_{i=1}^n\alpha_i=(n-2)2\pi.$$

Let $\underline{\alpha}$ denote the n-uple $(\alpha_1,\dots,\alpha_n)$, and $\mathcal{M}(\mathbb{S}^2,\underline{\alpha})^*$ denote the moduli space of flat surfaces of genus zero having exactly n singular points with cone angles $\alpha_1,\dots,\alpha_n$. Let $\mathcal{M}(\mathbb{S}^2,\underline{\alpha})$ denote the product space $\mathcal{M}(\mathbb{S}^2,\underline{\alpha})^*\times\mathbb{S}^1$.

Given a point $(\Sigma,e^{\imath\theta})$ in $\mathcal{M}(\mathbb{S}^2,\underline{\alpha})$, it is not difficult to see that there always exists an erasing forest consisting of only one geodesic tree A in Σ, therefore, a neighborhood of $(\Sigma,e^{\imath\theta})$ in $\mathcal{M}(\mathbb{S}^2,\underline{\alpha})$ can be identified to an open set in $\mathcal{M}^{\rm et}(\hat{\mathcal{A}},\underline{\alpha})$, where the family $\hat{\mathcal{A}}$ contains only one tree which is isomorphic to A. We also get a volume form $\mu_{\hat{\mathcal{A}}}$ on a neighborhood of $(\Sigma,e^{\imath\theta})$ which, *a priori*, depends on a choice of the erasing tree A. In [N], we showed that the volume form $\mu_{\hat{\mathcal{A}}}$ actually does not depend on the choice of the tree A, therefore, we get a well defined volume form $\mu_{\rm Tr}$ on $\mathcal{M}(\mathbb{S}^2,\underline{\alpha})$. Using Theorem 1.1, we will prove

Theorem 1.4 *The integral of the function $(\Sigma,e^{\imath\theta})\longmapsto\exp(-\mathbf{Area}(\Sigma))$ over $\mathcal{M}(\mathbb{S}^2,\underline{\alpha})$ with respect to $\mu_{\rm Tr}$ is finite:*

$$\int_{\mathcal{M}(\mathbb{S}^2,\underline{\alpha})}e^{-\mathbf{Area}}d\mu_{\rm Tr}<\infty. \tag{3}$$

Let $\mathcal{M}_1(\mathbb{S}^2,\underline{\alpha})^*$ denote the subset of $\mathcal{M}(\mathbb{S}^2,\underline{\alpha})^*$ consisting of surfaces of unit area. The volume form μ_{Tr} on $\mathcal{M}(\mathbb{S}^2,\underline{\alpha})$ induces a volume form $\hat{\mu}^1_{\mathrm{Tr}}$ on $\mathcal{M}_1(\mathbb{S}^2,\underline{\alpha})^*$. The same arguments as in Corollary 1.3 show

Corollary 1.5 *The volume of $\mathcal{M}_1(\mathbb{S}^2,\underline{\alpha})^*$ with respect to $\hat{\mu}^1_{\mathrm{Tr}}$ is finite.*

In the case where $\alpha_i < 2\pi$, for $i = 1,\dots,n$, Thurston [Th] showed that $\mathcal{M}_1(\mathbb{S}^2,\underline{\alpha})^*$ can be equipped with a complex hyperbolic metric structure with finite volume. In [N], it is showed that $\hat{\mu}^1_{\mathrm{Tr}} = \lambda\mu_{\mathrm{Hyp}}$, where λ is a constant, and μ_{Hyp} is the volume form induced by the complex hyperbolic metric. Therefore Theorem 1.4 can be considered as a generalization of the Thurston's result. It is also worth noticing that a similar result to Corollary 1.5 has been proved in [V3].

In the next section, we recall the definitions of the local charts for $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$, and the construction of the volume form μ_{Tr}. In Section 3, we will give the proof of Theorem 1.1 in a simple case. The proof of Theorem 1.1 for the general case will be given in Section 4, and subsequently the proof of Theorem 1.2, and Theorem 1.4 will be given in Section 5, and Section 6.

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2 Local charts and volume form on $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$

In this section, we recall the definitions of local charts, and of the volume form μ_{Tr} on $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$ as well as $\mathcal{M}(\mathbb{S}^2,\underline{\alpha})$, details of proofs are given in [N].

Let $(\Sigma,\hat{A},\xi)$ be a point in $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$. A geodesic triangulation T of Σ is said to be *admissible* if its 1-skeleton contains the forest $\hat{A}$. Given such a triangulation, we construct a local chart for $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$ in a neighborhood of $(\Sigma,\hat{A},\xi)$ as follows: first, cut open the surface Σ along the trees of $\hat{A}$, we then get a flat surface $\hat{\Sigma}$ with piecewise geodesic boundary together with a geodesic triangulation $\hat{\mathrm{T}}$.

We choose an orientation for every (geometric) edge in the 1-skeleton of $\hat{\mathrm{T}}$. Map each triangle of $\hat{\mathrm{T}}$ isometrically, and preserving the orientation into $\mathbb{R}^2$ such that the parallel vector field ξ is identified to the constant vertical vector field $(0,1)$ of $\mathbb{R}^2$. We can then associate to each oriented edge e in the 1-skeleton of $\hat{\mathrm{T}}$ a well-defined complex number $z(e)$. The complex numbers associated to edges of $\hat{\mathrm{T}}$ are obviously related, namely

- If e_i, e_j, e_k are the edges of $\hat{\mathrm{T}}$ that bound a triangle then

$$\pm z(e_i) \pm z(e_j) \pm z(e_k) = 0 \tag{4}$$

  where the signs are chosen according to the orientation of e_i, e_j, and e_k.

- If $(e, \bar{e})$ is a pair of edges in the boundary of $\hat{\Sigma}$ which arise from the same edge $\tilde{e}$ of a tree in $\hat{A}$, then

$$\pm z(\bar{e}) \pm e^{\imath\theta} z(e) = 0 \tag{5}$$

where θ is the rotation angle of the holonomy of a closed curve in Σ meeting $\hat{A}$ at only one point in $\tilde{e}$ transversely, θ is determined up to sign by the angles $(\alpha_1, \dots, \alpha_n)$, and the tree that contains $\tilde{e}$.

Let N_1 and N_2 be the number of edges and the number of triangles of $\hat{\mathrm{T}}$ respectively. Simple computations show that

$$N_1 = 3(2g + m - 2) + 4(n - m), \text{ and } N_2 = 2(2g + m - 2) + 2(n - m).$$

The complex numbers associated to the edges of $\hat{\mathrm{T}}$ give us a vector Z in $\mathbb{C}^{N_1}$. The arguments above show that the coordinates of Z satisfy a system $\mathbf{S}_{\mathrm{T}}$ of linear equations consisting of

- N_2 equations of type (4) which will be called *triangle equations*, and
- $n - m$ equations of type (5) which will be called *boundary equations*

Let

$$\mathbf{A}_{\mathrm{T}} : \mathbb{C}^{N_1} \to \mathbb{C}^{N_2+(n-m)}$$

be the complex linear map which is defined in the canonical bases of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2+(n-m)}$ by the matrix whose entries are coefficients of the system $\mathbf{S}_{\mathrm{T}}$. Note that every entry of the matrix of $\mathbf{A}_{\mathrm{T}}$ is either 0, or a complex number of module 1. We then have a map Ψ_{T} defined in a neighborhood of $(\Sigma, \hat{A}, \xi)$ with image in $\ker \mathbf{A}_{\mathrm{T}}$, which associates to any point $(\Sigma', \hat{A}', \xi')$ close to $(\Sigma, \hat{A}, \xi)$ a vector in $\ker \mathbf{A}_{\mathrm{T}}$ whose coordinates arise from an admissible triangulation T' of Σ' isomorphic to T. It turns out that Ψ_{T} is a local chart for $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$, as a consequence $\dim_{\mathbb{C}} \mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha}) = \dim_{\mathbb{C}} \ker \mathbf{A}_{\mathrm{T}}$, and we have

$$\dim_{\mathbb{C}} \mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha}) = N_1 - \mathrm{rk}(\mathbf{S}_{\mathrm{T}}) = \begin{cases} 2g + n - 1 & \text{if } \alpha_i \in 2\pi\mathbb{N}, \ \forall i = 1, \dots, n, \\ 2g + n - 2 & \text{otherwise.} \end{cases}$$

Using $\mathbf{A}_{\mathrm{T}}$, we define a volume form ν_{T} on $\ker \mathbf{A}_{\mathrm{T}}$ as follows:

- If $\dim \mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha}) = 2g + n - 1$, or equivalently $\mathrm{rk}\mathbf{A}_{\mathrm{T}} = N_2 + (n - m) - 1$, then ν_{T} is the volume form on $\ker \mathbf{A}_{\mathrm{T}}$ which is induced by the Lebesgue measures of $\mathbb{C}^{N_1}, \mathbb{C}^{N_2+(m-n)}$, and $\mathbb{C}$ via the following exact sequence

$$0 \longrightarrow \ker \mathbf{A}_{\mathrm{T}} \hookrightarrow \mathbb{C}^{N_1} \xrightarrow{\mathbf{A}_{\mathrm{T}}} \mathbb{C}^{N_2+(n-m)} \xrightarrow{\mathbf{s}} \mathbb{C} \longrightarrow 0 \tag{6}$$

where $\mathbf{s}$ is a linear form on $\mathbb{C}^{N_2+(n-m)}$ of the form

$$\mathbf{s}(z_1, \dots, z_{N_2+n-m}) = \pm z_1 \pm \dots \pm z_{N_2+n-m}.$$

- If $\dim \mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha}) = 2g+n-2$, or equivalently $\mathrm{rk}\mathbf{A}_{\mathrm{T}} = N_2+n-m$, then ν_{T} is the volume form which is induced by the Lebesgue measures of $\mathbb{C}^{N_1}$, and $\mathbb{C}^{N_2+n-m}$ via the exact sequence

$$0 \longrightarrow \ker \mathbf{A}_{\mathrm{T}} \hookrightarrow \mathbb{C}^{N_1} \xrightarrow{\mathbf{A}_{\mathrm{T}}} \mathbb{C}^{N_2+n-m} \longrightarrow 0 \quad (7)$$

Let μ_{T} denote $\Psi_{\mathrm{T}}^*\nu_{\mathrm{T}}$, then μ_{T} is a volume form defined in a neighborhood of $(\Sigma, \hat{A}, \xi)$. It turns out that the volume form μ_{T} does not depend on the choice of the triangulation T, thus we get a well defined volume form on $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$ which is denoted by μ_{Tr}.

Recall that $\mathcal{M}(\mathbb{S}^2, \underline{\alpha})^*$ is the moduli space of flat surfaces of genus zero having exactly n singularities with cone angles given by $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$. Let Σ be a point in $\mathcal{M}(\mathbb{S}^2, \underline{\alpha})^*$, then there exists a geodesic tree A on Σ whose vertex set is the set of singular points, such a tree is by definition an erasing forest of Σ. As a consequence, a neighborhood of a point $(\Sigma, e^{\imath\theta})$ in $\mathcal{M}(\mathbb{S}^2, \underline{\alpha}) = \mathcal{M}(\mathbb{S}^2, \underline{\alpha})^* \times \mathbb{S}^1$ can be identified to a neighborhood of a point (Σ, A, ξ) in $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$, where $\hat{\mathcal{A}}$ contains only one tree, which is isomorphic to A. We can then use the same method as above to define local charts, and the volume form μ_{Tr} for $\mathcal{M}(\mathbb{S}^2, \underline{\alpha})$.

Note that in this case there always exist indices $i \in \{1, \dots, n\}$ such that $\alpha_i \not\in 2\pi\mathbb{N}$, since we must have $\alpha_1 + \dots + \alpha_n = (n-2)2\pi$. It follows that $\dim_{\mathbb{C}} \mathcal{M}(\mathbb{S}^2, \underline{\alpha}) = n-2$, and μ_{Tr} is defined by the exact sequence (7). The fact that μ_{Tr} is well-defined follows from the observation that any two geodesic triangulations of Σ whose vertex sets coincide with the set of singular points of Σ can be transformed, one into the other, by a sequence of elementary moves (see [N], Definition 6.1).

3 Case of flat tori with marked geodesic segments

In this section, we prove Theorem 1.1 for the case $g = 1, n = 2, m = 1, \alpha_1 = \alpha_2 = 2\pi$, and $\hat{\mathcal{A}} = \{\mathcal{I}\}$ where $\mathcal{I}$ is a segment. Via this simple case, we would like to illustrate the strategy of the proof of Theorem 1.1 in the general case. An element of $\mathcal{M}^{\mathrm{et}}(\mathcal{I}, (2\pi, 2\pi))$ is a triple (Σ, I, ξ), where Σ is a flat torus (without singularity), I is an oriented geodesic segment in Σ with distinct endpoints, the orientation of I arises from a numbering of its endpoints, and ξ is a unitary parallel vector field on Σ. Note that

$$\dim_{\mathbb{C}} \mathcal{M}^{\mathrm{et}}(\mathcal{I}, (2\pi, 2\pi)) = 3.$$

Given an element (Σ, I, ξ) in $\mathcal{M}^{\mathrm{et}}(\mathcal{I}, (2\pi, 2\pi))$, let p and q denote the endpoints of I so that the orientation of I is from p to q. Let us start by showing that one can always cut the torus Σ into two cylinders such that one of which contains I. This will allows us to get a domain in $\mathbb{C}^3$ which covers a full measure subset of $\mathcal{M}^{\mathrm{et}}(\mathcal{I}, (2\pi, 2\pi))$.

Lemma 3.1 *There always exists a pair of parallel simple closed geodesic γ_p, γ_q of Σ such that*

$$\gamma_p \cap I = \{p\}, \text{ and } \gamma_q \cap I = \{q\}.$$

Proof: Choose a direction θ which is not parallel to I, and let (ψ_t^θ), $t \in \mathbb{R}$, denote the geodesic flow on Σ in this direction. Observe that there exists $t > 0$ such that

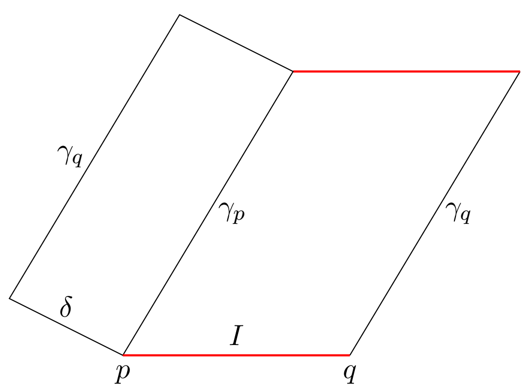
$$\psi_t^\theta(I) \cap I \neq \varnothing \tag{8}$$

since otherwise, the area of the stripe swept out by $(\psi_t^\theta)_{t>0}(I)$ would tend to infinity. Let $t_0 > 0$ be the first time such that (8) holds. By definition, there exists a closed parallelogram P in $\mathbb{R}^2$ with two horizontal sides, and an isometric immersion $\varphi : \mathrm{P} \longrightarrow \Sigma$, whose restriction to int(P) is an embedding, which maps the lower horizontal side of P to I, and the upper horizontal side of P to $\psi_{t_0}^\theta(I)$. Since the segments I and $\psi_{t_0}^\theta(I)$ are parallel, and have the same length, their intersection contains at least one endpoint of I. Without loss of generality, we can assume that

$$p \in I \cap \psi_{t_0}^\theta(I).$$

Consequently, $\varphi^{-1}(p)$ contains exactly two points, one in lower horizontal side, and the other in the upper horizontal side of P.

Let s be the geodesic segment in P joining two points in $\varphi^{-1}(p)$, then $\gamma_p = \varphi(s)$ is a closed geodesic in Σ which intersects I only at p. The closed geodesics parallel to γ_p which intersect I fill out a cylinder whose boundary consists of γ_p, and the closed geodesic parallel to γ_p passing through q, we denote this geodesic by γ_q. By construction, γ_p and γ_q satisfy the required condition of the lemma. $\square$



The closed geodesics γ_p and γ_q cut Σ into two cylinders, the one which contains I will be denoted by C_1, the other one by C_2. Let δ be a geodesic segment joining p and q which is contained in C_2.

The complement in Σ of the set $I \cup \gamma_p \cup \gamma_q \cup \delta$ is the union of two open parallelograms. By an embedding of these two parallelograms into $\mathbb{R}^2$ which sends ξ onto the constant vertical vector field $(0,1)$, we can associate the complex numbers Z, z, w to I, γ_p, δ respectively with a choice of orientation for each of these segments. Recall that I is already oriented, hence Z is well defined, we can choose the orientation of γ_p, and δ so that:

$$\mathbf{Area}(C_1) = \mathrm{Im}(Z\overline{z}) > 0 \text{ and } \mathbf{Area}(C_2) = \mathrm{Im}(z\overline{w}) > 0.$$

We define two functions η_1, η_2 on $\mathbb{C}^3$ by the following formulae

$$\eta_1(Z,z,w) = \mathrm{Im}(Z\overline{z}),\ \eta_2(Z,z,w) = \mathrm{Im}(z\overline{w}).$$

Set

$$\mathcal{D} = \{(Z,z,w) \in \mathbb{C}^3 \mid \eta_1(Z,z,w) > 0, \eta_2(Z,z,w) > 0\}.$$

Remark that, given (Z,z,w) in $\mathcal{D}$, one can construct a flat torus with a marked segment by first constructing two parallelograms in $\mathbb{R}^2$ from the pairs of complex numbers (Z,z) and (z,w), and then gluing these two parallelograms as shown in the above figure. We then get a map:

$$\rho : \mathcal{D} \longrightarrow \mathcal{M}^{\mathrm{et}}(\mathcal{I},(2\pi,2\pi)),$$

which is surjective and locally homeomorphic. The pull-back of the volume form μ_{Tr} on $\mathcal{D}$ is equal to the Lebesgue measure of $\mathbb{C}^3$ up to a multiplicative constant. Clearly, the pull-back of the energy function $\mathcal{F}^{\mathrm{et}}$ on $\mathcal{M}^{\mathrm{et}}(\mathcal{I},(2\pi,2\pi))$ is the following function on $\mathcal{D}$

$$\hat{\mathcal{F}}(Z,z,w) = \exp(-|Z|^2 - (\eta_1(Z,z,w) + \eta_2(Z,z,w))).$$

We say that a triple (Σ, I, ξ) is in *special position* if either I is parallel to ξ, or the trajectory $(\psi_t)_{t>0}(p)$, where (ψ_t) is the flow generated by ξ, returns to p without meeting any other point of I. Let $\mathcal{M}^{\mathrm{et}}(\mathcal{I},(2\pi,2\pi))^{\mathrm{sp}}$ denote the set of triples in special position in $\mathcal{M}^{\mathrm{et}}(\mathcal{I},(2\pi,2\pi))$. We have

Lemma 3.2 *The set $\mathcal{M}^{\mathrm{et}}(\mathcal{I},(2\pi,2\pi))^{\mathrm{sp}}$ is of measure 0 with respect to μ_{Tr}.*

Proof: The lemma follows from the fact that $\mathcal{M}^{\mathrm{et}}(\mathcal{I},(2\pi,2\pi))^{\mathrm{sp}}$ is the image under ρ of the set

$$\{(Z,z,w) \in \mathcal{D} : \mathrm{Re}(Z) = 0 \text{ or } \mathrm{Re}(z) = 0\},$$

which is obviously of measure zero with respect to the Lebesgue measure of $\mathbb{C}^3$. □

Now, let (Σ, I, ξ) be an element in the complement of $\mathcal{M}^{\mathrm{et}}(\mathcal{I},(2\pi,2\pi))^{\mathrm{sp}}$. Let (Z,x,w) be the complex numbers associated to I, γ_p, and δ as above. Set

$$A = \mathrm{Re}(Z), a = \mathrm{Re}(z), b = \mathrm{Re}(w) \text{ and } B = \mathrm{Im}(Z), x = \mathrm{Im}(z), y = \mathrm{Im}(w).$$

Since ξ is not parallel to I, we can take the direction θ in the proof of Lemma 3.1 to be the one determined by ξ. Suppose that γ_p arises from this construction then we have

$$|a| \leqslant |A|.$$

Remark that, since (Σ, I, ξ) is not in special position, we have $|a| > 0$. Since C_2 is a cylinder, we can choose the segment δ so that

$$|b| \leqslant |a|.$$

Now, set

$$\mathcal{D}_0 = \{(Z,z,w) \in \mathcal{D} \ : \ |A| \geqslant |a| \geqslant |b|\}.$$

From the arguments above, we deduce that $\rho(\mathcal{D}_0)$ contains the complement of $\mathcal{M}^{\mathrm{et}}(\mathcal{I},(2\pi,2\pi))^{\mathrm{sp}}$. Hence, the result of Theorem 1.1 for this case will follow from the following proposition:

Proposition 3.3 *We have*

$$\mathcal{J} = \int_{\mathcal{D}_0} \hat{\mathcal{F}}(Z,z,w) dAdBdadbdxdy = \int_{\mathcal{D}_0} \exp(-(A^2+B^2)-(\eta_1+\eta_2))dAdBdadbdxdy < \infty.$$

Proof: From the definition of the domain $\mathcal{D}_0$, we have

$$\mathcal{J} = \int\int \exp(-(A^2+B^2)) \times [\int_{-|A|}^{|A|}[\int_{-|a|}^{|a|}[\int\int \exp(-\eta_1-\eta_2)dxdy]db]da]dAdB.$$

Fix A,B,a,b, and consider the integral

$$\int\int \exp(-\eta_1-\eta_2)dxdy.$$

By definition, we have:

$$\eta_1 = Ba - Ax \text{ and } \eta_2 = xb - ay.$$

Using the change of variables $(x,y) \longmapsto (\eta_1,\eta_2)$, we have

$$d\eta_1 d\eta_2 = |Aa|dxdy.$$

Since $\eta_1(Z,z,w) > 0$, and $\eta_2(Z,z,w) > 0$ for every (Z,z,w) in $\mathcal{D}_0$, it follows

$$\int\int_{(Z,z,w)\in\mathcal{D}_0} \exp(-\eta_1-\eta_2)dxdy = \int_0^{+\infty}\int_0^{+\infty} \frac{e^{-\eta_1}e^{-\eta_2}}{|Aa|}d\eta_1 d\eta_2 = \frac{1}{|Aa|}.$$

Consequently,

$$\mathcal{J} = \int\int \exp(-A^2-B^2)[\int_{-|A|}^{|A|}[\int_{-|a|}^{|a|}\frac{1}{|Aa|}db]da]dAdB = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(A^2+B^2)}dAdB < \infty.$$

This proves the proposition, and hence, Theorem 1.1 is proved for the case of $\mathcal{M}^{\mathrm{et}}(\mathcal{I},(2\pi,2\pi))$. $\square$

4 Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1 for the general case. Our strategy is very similar to the one in the particular case $\mathcal{M}^{\mathrm{et}}(\mathcal{I},(2\pi,2\pi))$, namely, we specify a finite family of open subsets of $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$ which covers a subset of full measure, and show that the integral of the function $\mathcal{F}^{\mathrm{et}}$ on every member of this family is finite. Those open subsets of $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$ are defined by means of special admissible triangulations of surfaces in $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$ which are constructed by using the parallel vector field. Throughout this section, we assume that $m<n$, which means that the family $\hat{\mathcal{A}}=\{\mathcal{A}_1,\dots,\mathcal{A}_m\}$ contains at least a non-trivial tree. Note that the total number of edges of the trees in $\hat{\mathcal{A}}$ is $n-m$.

4.1 Admissible matrix

Set $N_2^*=N_2+(n-m)$, and $N=\dim_{\mathbb{C}}\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$. Recall that we have

$$N=\begin{cases} N_1-N_2^*+1 & \text{if } \alpha_i\in 2\pi\mathbb{N},\ \forall i=1,\dots,n,\\ N_1-N_2^* & \text{otherwise.}\end{cases}$$

Let us define

Definition 4.1 *A matrix $\mathbf{A}$ in $\mathbf{M}_{N_2^*,N_1}(\mathbb{C})$ is called* admissible *if there exists an element $(\Sigma,\hat{A},\xi)$ in $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$, and an admissible triangulation T of Σ such that $\mathbf{A}$ is the coefficient matrix of the linear system associated to T.*

Let a be a row of an admissible matrix. If a corresponds to a triangle equation, then a is called an ordinary row*, otherwise, i.e. when a corresponds to a boundary equation, it is called an* exceptional row*.*

Observe that the set of admissible matrices is finite. To see this, let $(\Sigma,\hat{A},\xi)$ be an element of $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$, T be an admissible triangulation of Σ, and $\mathbf{S}_{\mathrm{T}}$ be the system associated to T. Recall that $\mathbf{S}_{\mathrm{T}}$ consists of N_2 triangle equations, and $(n-m)$ boundary equations. Let $\mathbf{A}_{\mathrm{T}}\in\mathbf{M}_{N_2^*,N_1}(\mathbb{C})$ be the coefficient matrix of $\mathbf{S}_{\mathrm{T}}$. Let a be a row vector of $\mathbf{A}_{\mathrm{T}}$, then either

- a is an ordinary row, in this case, a contains exactly three non-zero entries which belong to $\{\pm 1\}$, or
- a is an exceptional row, in this case a contains exactly two non-zero entries, one of which belongs to $\{\pm 1\}$, the other is of the form $\pm e^{\imath\theta}$.

For any exceptional row, the angle θ belongs to a finite set of $[0;2\pi]$, since it corresponds to an edge of a tree the forest $\hat{A}$, and is determined up to sign by the angles in $\underline{\alpha}$. As a consequence, we see that a belongs to finite set of $\mathbb{C}^{N_1}$. Therefore, $\mathbf{A}_{\mathrm{T}}$ belongs to a finite set of $\mathbf{M}_{N_2^*,N_1}(\mathbb{C})$.

Let a be an exceptional row of an admissible matrix which is associated to an equation of the form

$$\pm z_i \pm e^{\imath\theta} z_j = 0.$$

We will call the operation consisting of multiplying a by $e^{-\imath\theta}$ a *reversing operation*. Recall that a corresponds to an edge of an erasing forest on a flat surface, and the angle θ is the rotation angle of the holonomy of a closed curve which intersects the erasing forest at only one point in the corresponding edge transversely. Reversing the orientation of the closed curve gives rise to the reversing operation on the row a.

Let $(\Sigma, \hat{A}, \xi)$ be an element of $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$. An admissible triangulation T of Σ does not give rise to a unique admissible matrix, since the coefficients of the system $\mathbf{S}_{\mathrm{T}}$ depend on the following data

. a numbering on the set of edges of the triangulation $\hat{\mathrm{T}}$, which is the triangulation induced by T on the surface obtained by slitting open Σ along trees in $\hat{A}$.

. a choice of orientation for each edge of $\hat{\mathrm{T}}$.

. a numbering on the set of triangles of $\hat{\mathrm{T}}$.

. a choice of orientation for the boundary of each triangle of $\hat{\mathrm{T}}$.

. for each edge of the forest $\hat{A}$, a choice of orientation for the closed curve which intersects $\hat{A}$ at only one point in this edge transversely.

Therefore, we have an equivalence relation on the set of admissible matrices defined as follows

Definition 4.2 *Two admissible matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ are said to be equivalent if $\mathbf{A}_2$ can be obtained from $\mathbf{A}_1$ by a sequence of the following operations*

- *interchanging two columns,*
- *interchanging two rows,*
- *changing sign of a columns,*
- *changing sign of a row,*
- *reversing operation on an exceptional row.*

Clearly, two admissible matrices arising from the same admissible triangulation are equivalent.

Let $\mathcal{AD}$ denote the set of equivalence classes of admissible matrices in $\mathbf{M}_{N_2^*,N_1}(\mathbb{C})$. For each s in $\mathcal{AD}$, choose a matrix $\mathbf{A}_s$ in the equivalence class s, we then get a finite family $\{\mathbf{A}_s,\ s \in \mathcal{AD}\}$ of matrices in $\mathbf{M}_{N_2^*,N_1}(\mathbb{C})$. We will associate to each s in $\mathcal{AD}$ an open subset of $\ker \mathbf{A}_s$ on which one can define a map Φ_s with image in $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$ which is locally homeomorphic.

Given s in $\mathcal{AD}$, for any $Z = (z_1, \dots, z_{N_1})$ in $\ker \mathbf{A}_s$, such that $z_i \neq 0$, for $i = 1, \dots, N_1$, let Σ_Z denote the 'surface' obtained from Z by the following construction

1. Construct a triangle in $\mathbb{R}^2$ from z_i, z_j, z_k whenever there is an ordinary row a in $\mathbf{A}_s$ such that

$$a \cdot {}^t Z = \pm z_i \pm z_j \pm z_k.$$

2. Glue the triangles obtained from 1. together by identifying sides corresponding to the same coordinate of Z.

3. Identify the sides corresponding to z_i and z_j whenever there exists an exceptional row a in $\mathbf{A}_s$ such that

$$a \cdot {}^t Z = \pm z_i \pm e^{\imath\theta} z_j.$$

Let $\mathcal{U}_s$ be the open subset of $\ker \mathbf{A}_s$ which is defined by the condition:

$\mathcal{U}_s = \{Z$ in $\ker \mathbf{A}_s$ with non-zero coordinates, such that Σ_Z is a closed, oriented, connected flat surface, having exactly n singularities with cone angles $\alpha_1, \dots, \alpha_n\}$.

We can then define a map Φ_s from $\mathcal{U}_s$ into $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$ by associating to a vector Z in $\mathcal{U}_s$ the triple $(\Sigma_Z, \hat{A}_Z, \xi_Z)$, where $\hat{A}_Z$ is the forest consisting of the segments arising from the exceptional rows in $\mathbf{A}_s$, and ξ_Z is the vector field corresponding to the vertical constant vector field $(0,1)$ of $\mathbb{R}^2$.

By construction, for any point $(\Sigma, \hat{A}, \xi)$ in $\Phi_s(\mathcal{U}_s)$, there is an admissible triangulation T of Σ such that the a local chart Ψ_{T} defined in a neighborhood of $(\Sigma, \hat{A}, \xi)$ verifies $\Phi_s^{-1} = \Psi_{\mathrm{T}}$. It follows that $\Phi_s(\mathcal{U}_s)$ is an open subset of $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$. Since every element of $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$ is contained in the domain of a local chart associated to an admissible triangulation, the following proposition is now clear

Proposition 4.3 *The family $\{\Phi_s(\mathcal{U}_s), s \in \mathcal{AD}\}$ is a finite open cover of the space $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$.*

4.2 Primary, auxiliary systems of indices, and admissible triple

Set

$$K = N - (2g + m - 2) = \begin{cases} n - m + 1 & \text{if } N = 2g + n - 1, \\ n - m & \text{if } N = 2g + n - 2. \end{cases}$$

In what follows, we will identify any matrix in $\mathbf{M}_{N_2^*, N_1}(\mathbb{C})$ (resp. $\mathbf{M}_{N_2, N_1}(\mathbb{C})$) to the linear map from $\mathbb{C}^{N_1}$ to $\mathbb{C}^{N_2^*}$ (resp. to $\mathbb{C}^{N_2}$) which is defined by this matrix in the canonical bases of $\mathbb{C}^{N_1}$, and $\mathbb{C}^{N_2^*}$ (resp. of $\mathbb{C}^{N_1}$, and $\mathbb{C}^{N_2}$).

Definition 4.4 *Given a matrix* $\mathbf{A}$ *in* $\mathbf{M}_{n_1, n_2}(\mathbb{C})$ *with* $n_1 < n_2$, *set* $r = \dim \ker \mathbf{A} = n_2 - \mathrm{rk}\mathbf{A}$. *A* primary system of indices *for* $\mathbf{A}$ *is an ordered subset* $(i_1, \dots, i_r)$ *of* $\{1, \dots, n_2\}$ *such that there exist* n_2 *complex linear functions* $f_i : \mathbb{C}^r \to \mathbb{C}$, $i = 1, \dots, n_2$, *verifying the following condition:*

$$(z_1, \dots, z_{n_2}) \in \ker \mathbf{A} \text{ if and only if } z_i = f_i(z_{i_1}, \dots, z_{i_r}), \text{ for } i = 1, \dots, n_2.$$

Definition 4.5 *Given an s in $\mathcal{AD}$, and a primary system of indices $I = (i_1, \dots, i_N)$ for $\mathbf{A}_s$, an* auxiliary system of indices *for I is an ordered subset $(j_K, \dots, j_N)$ of $\{1, \dots, N_1\}$, which is empty if $K > N$ (that is when $g = 0$, and $m = 1$), such that, for $k = K, \dots, N$*

i) *f_{j_k} depends only on $(z_{i_1}, \dots, z_{i_{k-1}})$,*

ii) *the coefficients of $z_{i_K}, \dots, z_{i_{k-1}}$ in f_{j_k} are all real,*

iii) *There exists an ordinary row in $\mathbf{A}_s$ whose i_k-th and j_k-th entries are both non-zero.*

Convention: Given a matrix $\mathbf{A}$ in $\mathbf{M}_{N_2^*, N_1}(\mathbb{C})$, or in $\mathbf{M}_{N_2, N_1}(\mathbb{C})$, in what follows, we will say that z_j is a linear function of $(z_{i_1}, \dots, z_{i_k})$, or z_j depends linearly on $(z_{i_1}, \dots, z_{i_k})$ as $(z_1, \dots, z_{N_1})$ varies in $\ker \mathbf{A}$ if there exists a vector $(\lambda_1, \dots, \lambda_k)$ in $\mathbb{C}^k$ such that

$$\mathbf{A} \cdot^t (z_1, \dots, z_{N_1}) = 0 \text{ implies } z_j = \lambda_1 z_{i_1} + \dots + \lambda_k z_{i_k}.$$

Remark: If $(j_K, \dots, j_N)$ is an auxiliary system for $(i_1, \dots, i_N)$, then we have

- z_{j_k} can be written as a linear function of $(z_{i_1}, \dots, z_{i_{k-1}})$, for $k = K, \dots, N$, as $Z = (z_1, \dots, z_{N_1})$ varies in $\ker \mathbf{A}_s$.
- Assume that $(\Sigma, \hat{A}, \xi) = \Phi_s(Z)$, and let T be the geodesic triangulation of Σ which is obtained from the construction of Φ_s, then the condition *iii)* of 4.5 implies that z_{i_k} and z_{j_k} are associated to two sides of a triangle in T.

For the case $\mathcal{M}^{\text{et}}(\mathcal{I}, (2\pi, 2\pi))$, let (Σ, I, ξ) be an element of $\mathcal{M}^{\text{et}}(\mathcal{I}, (2\pi, 2\pi))$, and let $p, q, \gamma_p, \gamma_q, \delta$, and Z, z, w be as in Section 3. We can add some geodesic segments whose endpoints are contained in the set $\{p, q\}$ to get a triangulation of Σ. We then get a triangulation of the surface $\hat{\Sigma}$ which is obtained from Σ by slitting along I. This triangulation gives rise to a an admissible matrix $\mathbf{A}$ in $\mathbf{M}_{5,7}(\mathbb{C})$ with $\dim \ker \mathbf{A} = 3$. There exists a linear isomorphism φ from $\mathbb{C}^3$ to $\ker \mathbf{A}$, we can arrange so that, for any $(z_1, \dots, z_7) = \varphi(Z, z, w)$ then $z_1 = Z, z_2 = z, z_3 = w$. In this case, $N = 3, K = 2$, therefore $(1, 2, 3)$ is a primary system of indices for $\mathbf{A}$, and $(1, 2)$ is an auxiliary system of indices for $(1, 2, 3)$.

By Proposition 4.3, we know that $\mathcal{M}^{\text{et}}(\hat{\mathcal{A}}, \underline{\alpha})$ is covered by the family of open subsets $\{\Phi_s(\mathcal{U}_s),\ s \in \mathcal{AD}\}$. Therefore, to prove Theorem 1.1, we only need to show that the integral of the function $\mathcal{F}^{\text{et}}$ on $\Phi_s(\mathcal{U}_s)$ is finite. This would be done if we could show that the integral of $\Phi_s^* \mathcal{F}^{\text{et}}$ on $\mathcal{U}_s$ is finite. However, the domain $\mathcal{U}_s$ is still too large, and this integral can be infinite. The primary and auxiliary systems of indices for $\mathbf{A}_s,\ s \in \mathcal{AD}$, will allow us to specify a finite family of sub-domains of $\mathcal{U}_s$ on which the integral of $\Phi_s^* \mathcal{F}^{\text{et}}$ is finite, and whose images under Φ_s cover a full measure subset of $\Phi_s(\mathcal{U}_s)$.

Consider $\mathbf{A}_s$ for some s in $\mathcal{AD}$. Let $a_1, \dots, a_{N_2}$ denote the ordinary rows, and $b_1, \dots, b_{n-m}$ denote the exceptional rows of $\mathbf{A}_s$. Let $\mathbf{A}_s^* \in \mathbf{M}_{N_2, N_1}(\mathbb{C})$ be the matrix consisting of the ordinary rows of $\mathbf{A}_s$, and set $\tilde{N} = \dim \ker \mathbf{A}_s^*$.

Definition 4.6 *If the i-th column of $\mathbf{A}_s^*$ has only one non-zero entry, we say that i is a* boundary index *of $\mathbf{A}_s$. Two boundary indices i_1 and i_2 are said to be* paired up *if there exists an exceptional row in $\mathbf{A}_s$ whose i_1-th and i_2-th entries are non-zero.*

Fix a vector $Z = (z_1, \dots, z_{N_1})$ in $\mathcal{U}_s$, and let $(\Sigma, \hat{A}, \xi)$ be the image of Z under Φ_s. Recall that Σ comes along with an admissible triangulation T. Let $\hat{\Sigma}$ denote the surface obtained by slitting open Σ along $\hat{A}$, and $\hat{\mathrm{T}}$ denote the triangulation of $\hat{\Sigma}$ which is induced by T. By definition, the coordinates of Z is in bijection with the set of edges of a triangulation $\hat{\mathrm{T}}$, and the rows of $\mathbf{A}_s^*$ is in bijection with the set of triangles of $\hat{\mathrm{T}}$. If i is a boundary index of $\mathbf{A}_s$, then z_i corresponds to an edge of $\hat{\mathrm{T}}$ which is contained in the boundary of $\hat{\Sigma}$. If i_1, i_2 are two boundary indices which are paired up, then the edges corresponding to z_{i_1}, and z_{i_2} arise from the same edge of a tree in the forest $\hat{A}$. Observe that the set of boundary indices of $\mathbf{A}_s$ contains exactly $2(n-m)$ elements divided into $(n-m)$ pairs. First, let us prove the following

Lemma 4.7 *We have* $\mathrm{rk}\mathbf{A}_s^* = N_2$, *or equivalently* $\tilde{N} = N_1 - N_2 = (2g+n-2)+(n-m)$.

Proof: We will show that the row vectors $(a_1, \dots, a_{N_2})$ are linearly independent in $\mathbb{C}^{N_1}$. Assume that there exists $(\lambda_1, \dots, \lambda_{N_2})$ in $\mathbb{C}^{N_2}$ such that

$$\lambda_1 a_1 + \dots + \lambda_{N_2} a_{N_2} = 0.$$

Observe that, if a_{i_1} and a_{i_2} correspond to two adjacent triangles of $\hat{\mathrm{T}}$, then there exists $j \in \{1, \dots, N_1\}$ such that the j-th column of $\mathbf{A}_s^*$ contains exactly two non-zero entries, on the i_1-th and the i_2-th rows. It follows that if $\lambda_{i_1} = 0$, then $\lambda_{i_2} = 0$.

Since $n - m > 0$, the set of boundary indices is non-empty, which means that there exists a column in $\mathbf{A}_s^*$ which contains exactly one non-zero entry. Hence, there exists $j \in \{1, \dots, N_2\}$ such that $\lambda_j = 0$. Since the surface $\hat{\Sigma}$ is connected, the argument above implies that $\lambda_1 = \dots = \lambda_{N_2} = 0$, and the lemma follows. □

The next lemma tells us that a primary system of indices for $\mathbf{A}_s^*$ contains at most $2(n-m)-1$ boundary indices.

Lemma 4.8 *All the boundary indices can not be contained in a primary system of indices for* $\mathbf{A}_s^*$.

Proof: Recall that the sign of a row in $\mathbf{A}_s^*$ is determined by a choice of orientation on the boundary of the corresponding triangle in $\hat{\mathrm{T}}$. Note that we are free to permute, and change sign of rows and columns in $\mathbf{A}_s^*$. Let $I_b \subset \{1, \dots, N_1\}$ be the subset of boundary indices for $\mathbf{A}_s^*$.

For each triangle of $\hat{\mathrm{T}}$, we choose the orientation of its boundary coherently with the orientation of the surface. Since each edge in the interior of $\hat{\Sigma}$ belongs to two distinct triangles, it follows that we have

$$(a_1 + \dots + a_{N_2}) \cdot {}^t Z = \sum_{i \in I_b} \pm z_i.$$

Therefore, for every $(z_1, \dots, z_{N_1})$ in $\ker \mathbf{A}_s^*$, we have

$$\sum_{i \in I_b} \pm z_i = 0 \tag{9}$$

which implies that the family of coordinates $(z_i,\ i \in I_b)$ is not linearly independent as $(z_1, \ldots, z_{N_1})$ varies in $\ker \mathbf{A}_s^*$, and the lemma follows. □

Our goal now is to prove that there exists a primary system of indices for $\mathbf{A}_s$ whose $(K-1)$ first indices are boundary indices. Let us first prove the following

Lemma 4.9 *There exist primary systems of indices for* $\mathbf{A}_s^*$ *whose first* $2(n-m)-1$ *elements are boundary indices.*

Proof: Assume that $I_b = \{1, \ldots, 2(n-m)\}$ is the set of boundary indices of $\mathbf{A}_s^*$. By permuting the rows of $\mathbf{A}_s^*$, we can assume that the only non-zero of the first column is on the first row, that is the first entry of a_1 is ± 1. We will show that, as $(z_1, \ldots, z_{N_1})$ varies in $\ker \mathbf{A}_s^*$, the coordinates $(z_2, \ldots, z_{2(n-m)})$ are linearly independent, that is $\ker \mathbf{A}_s^*$ is not contained in the kernel of any any linear function of the form

$$f : (z_1, \ldots, z_{N_1}) \mapsto \lambda_2 z_2 + \cdots + \lambda_{2(n-m)} z_{2(n-m)}.$$

It follows that we can add $(\tilde{N} - 2(m-n) + 1)$ indices to the family $\{2, \ldots, 2(n-m)\}$ to get a primary system of indices for $\mathbf{A}_s^*$, which proves the lemma.

All we have to show is that, if there exists a vector $\lambda' = (\lambda'_1, \ldots, \lambda'_{N_2}) \in \mathbb{C}^{N_2}$ such that

$$(\sum_{i=1}^{N_2} \lambda'_i a_i) \cdot {}^t(z_1, \ldots, z_{N_1}) = \sum_{i=2}^{2(n-m)} \lambda_i z_i \tag{10}$$

then $\lambda'_i = 0,\ i = 1, \ldots, N_2$.

First, observe that we must have $\lambda'_1 = 0$, since there is only one non-zero entry in the first column of $\mathbf{A}_s^*$. Consider two adjacent triangles Δ_1, Δ_2 of $\hat{\mathrm{T}}$. Each common edge of Δ_1 and Δ_2 correspond to a coordinate z_j of Z, with $j > 2(n-m)$. Let a_{i_1}, a_{i_2} be the rows in $\mathbf{A}_s^*$ which correspond to Δ_1, Δ_2 respectively, then, in the j-th column of $\mathbf{A}_s^*$ there exactly two non-zero entries, on the i_1-th and the i_2-th rows. Now, since the right hand side of (10) does not contain z_j, with $j > 2(n-m)$, we deduce that, if $\lambda'_{i_1} = 0$, then $\lambda'_{i_2} = 0$. We already have $\lambda'_1 = 0$, and since $\hat{\Sigma}$ is connected, it follows that $\lambda'_i = 0$, for $i = 1, \ldots, N_2$. □

Lemma 4.10 *There exist primary systems of indices for* $\mathbf{A}_s$ *whose first* $(K-1)$ *elements are boundary indices.*

Proof: We can assume that the set of boundary indices of $\mathbf{A}_s^*$ is $\{1, \ldots, 2(n-m)\}$, and that i and $(n-m)+i,\ i = 1, \ldots, n-m$, are paired up, which means that any $(z_1, \ldots, z_{N_1})$ in $\ker \mathbf{A}_s$ satisfies $(n-m)$ equations of the form

$$z_i \pm e^{\imath\theta_i} z_{(n-m)+i} = 0,\ i = 1, \ldots, n-m, \tag{11}$$

with some θ_i in a finite set. Since $(z_1, \ldots, z_{N_1})$ also satisfies the equation (9), it follows that we have

$$\sum_{i=1}^{n-m}(1 \pm e^{\imath\theta_i})z_i = 0 \tag{12}$$

By Lemma 4.9, we know that there exists a primary system of indices $\tilde{I}$ for $\mathbf{A}_s^*$ whose $2(n-m)-1$ first elements are boundary indices. We will show that a primary system of indices for $\mathbf{A}_s$ can be obtained by removing some boundary indices in $\tilde{I}$. We have two issues:

- Case 1: $\alpha_i \in 2\pi\mathbb{N}$, $i = 1, \dots, n$. In this case, $N = 2g + n - 1$, $K = (n-m) + 1$, and $\tilde{N} = (2g + n - 2) + (n-m) = N + (n-m) - 1$. Note that in this case, all the angles θ_i are zero, and with appropriate choices of orientation for the edges of $\hat{\mathrm{T}}$ in the boundary of $\hat{\Sigma}$, the equation (12) is trivial (cf. [N]).

  Let I be the ordered subset of $\{1, \dots, N_1\}$ which is obtained by removing the indices $\{(n-m)+1, \dots, 2(n-m)-1\}$ from $\tilde{I}$. The set I contains $n-m = K-1$ boundary indices. Let us show that I is a primary system of indices for $\mathbf{A}_s$. First, observe that, for any $(z_1, \dots, z_{N_1})$ in $\ker \mathbf{A}_s$, $z_i,\ i = 1, \dots, N_1$ can be written as a linear function of $\{z_k,\ k \in \tilde{I}\}$, since $\tilde{I}$ is a primary system of indices for $\mathbf{A}_s^*$. Using the equations (11), we can replace $z_{(n-m)+j}$ by $\pm e^{\imath\theta_j} z_j,\ j = 1, \dots, n-m$. Therefore, $z_i,\ i = 1, \dots, N_1$, can be written as a linear function of $(z_k,\ k \in I)$ as $(z_1, \dots, z_{N_1})$ varies in $\ker \mathbf{A}_s$. Moreover, we have

  $$\mathrm{Card}\{I\} = 2g + n - 1 = \dim\ker \mathbf{A}_s,$$

  which implies that I is a primary system of indices for $\mathbf{A}_s$.

- Case 2: there exist $i \in \{1, \dots, n\}$ such that $\alpha_i \notin 2\pi\mathbb{N}$. In this case, $N = 2g + n - 2$, $K = n - m$, $\tilde{N} - N = n - m$, and the equation (12) is non-trivial (cf. [N]). Without loss of generality, we can assume that the coefficient of z_1 in (12) is non-zero, which means that, as $(z_1, \dots, z_{N_1})$ varies in $\ker \mathbf{A}_s$, z_1 can be written as a linear function of $(z_2, \dots, z_{n-m})$.

  Let I be the ordered subset of $\{1, \dots, N_1\}$ which is obtained by removing the indices $\{1, (n-m)+1, \dots, 2(n-m)-1\}$ from $\tilde{I}$. Clearly, the set I contains $(n-m)-1 = K-1$ boundary indices. Since $\tilde{I}$ is a primary system of indices for $\mathbf{A}_s^*$, using the equations (11), and (12), we see that, as $(z_1, \dots, z_{N_1})$ varies in $\ker \mathbf{A}_s$, for $i = 1, \dots, N_1$, z_i can be written as a linear function of $(z_k,\ k \in I)$. Moreover, we have

  $$\mathrm{Card}\{I\} = N = \dim\ker \mathbf{A}_s,$$

  therefore, I is a primary system of indices for $\mathbf{A}_s$. The proof of the lemma is now complete. $\square$

We can now define

Definition 4.11 *Any triple* $(\mathbf{A}_s; I; J)$*, where* I *is a primary system of indices for* $\mathbf{A}_s$ *whose* $(K-1)$ *first elements are boundary indices, and* J *is an auxiliary system od indices for* I*, will be called an* admissible triple.

Clearly, the number of admissible triples is finite.

4.3 Good triangulation

Throughout this subsection, given a point $(\Sigma, \hat{A}, \xi)$ in $\mathcal{M}^{\rm et}(\hat{\mathcal{A}}, \underline{\alpha})$, we denote by $\hat{\Sigma}$ the flat surface with piecewise geodesic boundary obtained by slitting open Σ along the trees in $\hat{A}$. Let $\hat{V}$ denote the finite subset of $\hat{\Sigma}$ which arises from the vertex set V of $\hat{A}$. The vector field ξ of Σ gives rise to a parallel vector field of $\hat{\Sigma}$ which will be denoted again by ξ. For any admissible triangulation T of $(\Sigma, \hat{A}, \xi)$, let $\hat{\mathrm{T}}$ denote the induced triangulation of $\hat{\Sigma}$.

Let (ψ_t), $t \in \mathbb{R}$, denote the flow generated by ξ on $\hat{\Sigma}$. Given a point p in $\mathrm{int}(\hat{\Sigma}) \setminus \hat{V}$, if there exists $t_0 > 0$ (resp. $t_0 < 0$) such that $\psi_{t_0}(p) \in \hat{V} \cup \partial\hat{\Sigma}$, then, for every $t > t_0$ (resp. $t < t_0$), we consider, by convention, that $\psi_t(p) = \psi_{t_0}(p)$.

Let a be a geodesic segment contained in the boundary of $\hat{\Sigma}$ with endpoints in $\hat{V}$. We can extend the field ξ by continuity to $\mathrm{int}(a)$. Assume that a is not parallel to the field ξ, then we say that a is an *upper* (resp. *lower*) boundary segment, if the field ξ on $\mathrm{int}(a)$ points outward (resp. inward). Observe that in this case, the image of $\mathrm{int}(a)$ by ψ_t is well defined for all $t \in \mathbb{R}$.

Let $(\Sigma, \hat{A}, \xi)$ be a point in $\mathcal{M}^{\rm et}(\hat{\mathcal{A}}, \underline{\alpha})$, and let e be a geodesic segment of $\hat{\Sigma}$ with endpoints in $\hat{V}$, we denote by $h(e)$ the *transversal measure* of e with respect to ξ which is defined as follows: if we choose an isometric embedding of a neighborhood of e into $\mathbb{R}^2$ such that the vector field ξ is mapped to the constant vertical vector field $(0,1)$ of $\mathbb{R}^2$, then $h(e)$ is the length of the horizontal projection of the image of e. We call $h(e)$ the *horizontal length* of e.

A triangle in $\hat{\Sigma}$ whose sides are geodesic segments denoted by e_1, e_2, e_3 is said to be *good* if $h(e_i) > 0$, for $i = 1, 2, 3$. Given a good triangle Δ, we call the unique side of Δ of maximal horizontal length the *base* of Δ. Let $\hat{\mathrm{T}}$ be a triangulation of $\hat{\Sigma}$ which arises from an admissible triangulation of Σ, if all of triangles of $\hat{\mathrm{T}}$ are good, then $\hat{\mathrm{T}}$ is called a *good triangulation*. The following proposition asserts that a 'generic' element always admits a good triangulation.

Proposition 4.12 *Let* $(\Sigma, \hat{A}, \xi)$ *be an element of* $\mathcal{M}^{\rm et}(\hat{\mathcal{A}}, \underline{\alpha})$*. Suppose that there exist no geodesic segments in* $\hat{\Sigma}$ *with endpoints in* $\hat{V}$ *which are parallel to the field* ξ*, then there exists a good triangulation* $\hat{\mathrm{T}}$ *of* $\hat{\Sigma}$ *whose edges are denoted by* $\{e_1, \dots, e_{N_1}\}$ *so that,*

- *The edges of* $\hat{\mathrm{T}}$ *in the boundary of* $\hat{\Sigma}$ *are denoted by* $\{e_1, \dots, e_{2(n-m)}\}$.
- *For every* $i \in \{2(n-m)+1, \dots, N_1\}$*, there exists* $j < i$*, and a triangle* Δ *of* $\hat{\mathrm{T}}$ *whose boundary contains both* e_i, e_j *such that* e_j *is the base of* Δ.

Proof: We construct a geodesic triangulation of $\hat{\Sigma}$ whose vertex set is $\hat{V}$ as follows: let $e_1, \dots, e_{2(n-m)}$ denote the geodesic segments in the boundary of $\hat{\Sigma}$ with endpoints in $\hat{V}$. Assume that the segment e_1 is of maximal horizontal length among the set $\{e_1, \dots, e_{2(n-m)}\}$. By assumption, we have $h(e_1) > 0$. Let p, q denote the endpoints of e_1 (it may happen that $p \equiv q$). Consider the following procedure:

Assume that e_1 is a lower boundary segment, consider the stripe S_t swept by $\{\psi_t(\mathrm{int}(e_1)), t > 0\}$. Since $h(e_1) > 0$, for some t finite, this stripe must meet a point in the set $\hat{V} \cup \partial\hat{\Sigma}$, otherwise its area would tend to infinity as t tends to $+\infty$.

Since the horizontal length of e_1 is maximal among the set $\{h(e_1), \dots, h(e_{2(n-m)})\}$, suppose that, for some $t \in \mathbb{R}^+$, $\psi_t(\mathrm{int}(e_1))$ is contained in a the geodesic segments e_i in the set $\{e_1, \dots, e_{2(n-m)}\}$, then we must have $\overline{\psi_t(\mathrm{int}(e_1))} = e_i$. This implies that there is a geodesic segment parallel to the field ξ joining p to a point in $\hat{V}$, which is a contradiction to the assumption of the lemma. Therefore, there exists a smallest value $t_0 > 0$ such that $\psi_{t_0}(\mathrm{int}(e_1))$ contains a point in $\hat{V}$.

Let r be a point in $\psi_{t_0}(\mathrm{int}(e_1)) \cap \hat{V}$, and let e', e'' denote the two geodesic segments contained in the stripe S_{t_0} which join r to p, and to q. Note that even though p and q may coincide, the two segments e', and e'' are always distinct. It can happen that one of the segments e', e'' already appears in the set $\{e_1, \dots, e_{2(n-m)}\}$ but not both of them, unless $\hat{\Sigma}$ is a triangle. By assumption, we have $h(e') > 0$, and $h(e'') > 0$, and by construction, e_1 is the base of the good triangle bounded by e', e'', and e_1. We will call e_1 the *supporter* of e' and e''.

In the case where e_1 is an upper boundary segment, by considering $\{\psi_t(\mathrm{int}(e_1)),\ t < 0\}$ instead of $\{\psi_t(\mathrm{int}(e_1)),\ t > 0\}$, we also get a similar result.

Cut off the triangle bounded by e_1, e', e'' from the surface $\hat{\Sigma}$ along the segments e' and e''. The remaining surface is a flat surface with piecewise geodesic boundary which is not necessarily connected. On this new surface, we still have a parallel vector field which is the restriction of ξ. We can now reapply the same procedure to the new surface. The assumption of the proposition allows us to continue this procedure until the surface $\hat{\Sigma}$ is cut into triangles with vertices in $\hat{V}$, that is until we get a geodesic triangulation $\hat{\mathrm{T}}$ of $\hat{\Sigma}$ whose vertex set is $\hat{V}$, this triangulation is necessarily a good triangulation.

We number the edges of $\hat{\mathrm{T}}$ which are contained in the interior of $\hat{\Sigma}$ according to their appearing order in the procedure above, the ordering of two edges which appear in the same step is not important. Since every edge of $\hat{\mathrm{T}}$ in the interior of $\hat{\Sigma}$ admits a supporter which appears in the procedure before itself, the proposition is then proved. □

Proposition 4.13 *If $(\Sigma, \hat{A}, \xi)$ is a point in $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$ satisfying the condition of Proposition 4.12, then there exists an admissible triple $(\mathbf{A}_s; I; J)$, where $I = (i_1, \dots, i_N)$, and $J = (j_K, \dots, j_N)$, and a vector $Z^0 = (z^0_1, \dots, z^0_{N_1})$ in $\mathcal{U}_s$ such that*

- $(\Sigma, \hat{A}, \xi) = \Phi_s(Z^0)$.

- $|\mathrm{Re}(z^0_{j_k})| > |\mathrm{Re}(z^0_{i_k})|$ *for any* $k = K, \dots, N$.

Proof: Let $\hat{\mathrm{T}}$ be the good triangulation of $\hat{\Sigma}$ which is obtained from Proposition 4.12. Let $\mathbf{A}_{\mathrm{T}}$ be the matrix in $\mathbf{M}_{N_2^*,N_1}(\mathbb{C})$ associated to $\hat{\mathrm{T}}$. Let $Z^0 = (z^0_1, \dots, z^0_{N_1})$ denote the vector of $\ker \mathbf{A}_{\mathrm{T}}$ whose coordinates are associated to the edges of $\hat{\mathrm{T}}$. In what follows, we consider any vector $Z = (z_1, \dots, z_{N_1})$ in $\mathbb{C}^{N_1}$ as a function from the set of edges of $\hat{\mathrm{T}}$ to $\mathbb{C}$ such that $z_i = Z(e_i)$.

By construction, the set I_b of boundary indices for $\mathbf{A}_{\mathrm{T}}$ is $\{1, \dots, 2(n-m)\}$. Let $\mathbf{A}^*_{\mathrm{T}}$ be the matrix in $\mathbf{M}_{N_2,N_1}(\mathbb{C})$ consisting of all ordinary rows of $\mathbf{A}_{\mathrm{T}}$. Let N, and $\tilde{N}$ denote the dimensions of $\ker \mathbf{A}_{\mathrm{T}}$, and $\ker \mathbf{A}^*_{\mathrm{T}}$ respectively. We first choose a primary system of indices $\tilde{I}$ for $\mathbf{A}^*_{\mathrm{T}}$ as follows:

- The first $2(n-m)-1$ elements of $\tilde{I}$ are $\{2, \dots, 2(n-m)\}$, by Lemma 4.9, we know that, as $Z = (z_1, \dots, z_{N_1})$ varies in $\ker \mathbf{A}^*_{\mathrm{T}}$, the family of coordinates $(z_2, \dots, z_{2(n-m)})$ is linearly independent.

- Assume that we have chosen k indices $(i'_1, \dots, i'_k)$ for $\tilde{I}$, then the index i'_{k+1} of $\tilde{I}$ is the smallest index i' such that, as $(z_1, \dots, z_{N_1})$ varies in $\ker \mathbf{A}^*_{\mathrm{T}}$, $z_{i'}$ can not be written as a linear function of $(z_{i'_1}, \dots, z_{i'_k})$, in other words, the family of coordinates $(z_{i'_1}, \dots, z_{i'_k}, z_{i'})$ is linearly independent.

By Lemma 4.8, we know that, for $k = 2(n-m), \dots, \tilde{N}$, i'_k is not a boundary index, that is $i'_k > 2(n-m)$. For any k in $\{2(n-m), \dots, \tilde{N}\}$, consider the edge $e_{i'_k}$ of $\hat{\mathrm{T}}$. From Proposition 4.12, we know that there exists an edge $e_{j'_k}$ with $j'_k < i'_k$, and a triangle Δ_k of $\hat{\mathrm{T}}$ whose boundary contains both $e_{i'_k}$, and $e_{j'_k}$ such that $e_{j'_k}$ is the base of Δ_k. Consequently, we have

$$|\mathrm{Re}(z^0_{j'_k})| = h(e_{j'_k}) > h(e_{i'_k}) = |\mathrm{Re}(z^0_{i'_k})| \tag{13}$$

Let J denote the ordered subset $(j'_{2(n-m)}, \dots, j'_{\tilde{N}})$ of $\{1, \dots, N_1\}$. From the definition of i'_k, for $k = 2(n-m), \dots, \tilde{N}$, as $(z_1, \dots, z_{N_1})$ varies in $\ker \mathbf{A}^*_{\mathrm{T}}$, we can write

$$z_{j'_k} = \tilde{f}_{j'_k}(z_{i'_1}, \dots, z_{i'_{k-1}}),$$

where $\tilde{f}_{j'_k}$ is some fixed linear function. Since the matrix $\mathbf{A}^*_{\mathrm{T}}$ is real, all the coefficients of $\tilde{f}_{j'_k}$ are also real.

By Lemma 4.10, we know that, by removing $K' = 2(n-m) - K$ boundary indices from $\tilde{I}$, we obtain a primary system of indices I for $\mathbf{A}_{\mathrm{T}}$ whose first $(K-1)$ elements are boundary indices. We will show that J is an auxiliary of I, which, together with (13), will allow us to conclude.

First, observe that we can write $I = (i_1, \dots, i_N)$, where $i_1, \dots, i_{K-1}$ are boundary indices, and for $k = K, \dots, N$, $i_k = i'_{k+K'}$. Since $\tilde{N} - N = 2(n-m) - K = K'$, we can write $J = (j_K, \dots, j_N)$, where $j_k = j'_{k+K'}$. As a consequence, for $k = K, \dots, N$, the condition that there is a triangle in $\hat{\mathrm{T}}$ whose boundary contains both e_{i_k}, and e_{j_k} is satisfied.

We already know that, as $(z_1, \dots, z_{N_1})$ varies in $\ker \mathbf{A}_{\mathrm{T}} \subset \ker \mathbf{A}^*_{\mathrm{T}}$, for $k = K, \dots, N$, we have

$$z_{j_k} = z_{j'_{k+K'}} = \tilde{f}_{j_k}(z_{i'_1}, \dots, z_{i'_{k+K'-1}}),$$

where $\tilde{f}_{j_k}$ is a linear function with real coefficients. We can then transform $\tilde{f}_{j_k}$ into a linear function f_{j_k} of $(z_{i_1}, \dots, z_{i_{k-1}})$ by using equations of the form (11), and (12). Since the equations (11), and (12) involve only boundary indices, we deduce that the coefficients of $z_{i_K}, \dots, z_{i_{k-1}}$ in f_{j_k} are all real, which allows us to conclude that J is an auxiliary system of indices for I.

Clearly, the inequalities (13) can be rewritten as

$$|\mathrm{Re}(z^0_{j_k})| > |\mathrm{Re}(z^0_{i_k})|, \ k = K, \dots, N \tag{14}$$

We know that $\mathbf{A}_{\mathrm{T}}$ is equivalent to a matrix $\mathbf{A}_s$ with s in $\mathcal{AD}$. The transformation of $\mathbf{A}_{\mathrm{T}}$ into $\mathbf{A}_s$ consists of renumbering the coordinates in $\mathbb{C}^{N_1}$, and changing their sign. By this transformation, $(i_1, \dots, i_N)$ becomes a primary system of indices for $\mathbf{A}_s$, and $(j_K, \dots, j_N)$ becomes an auxiliary system of indices for $(i_1, \dots, i_N)$. Therefore, we get an admissible triple $(\mathbf{A}_s; I; J)$, and a vector Z^0 in $\mathcal{U}_s$ which verify the conditions in the statement of the proposition. □

Now, given an admissible triple $(\mathbf{A}_s; I; J)$, where $I = (i_1, \dots, i_N), J = (j_K, \dots, j_N)$, let $\mathcal{U}_s(I; J)$ denote the following subset of $\mathcal{U}_s$

$$\mathcal{U}_s(I; J) = \{(z_1, \dots, z_{N_1}) \in \mathcal{U}_s \ : \ |\mathrm{Re}(z_{i_k})| \leqslant |\mathrm{Re}(z_{j_k})|, \ k = K, \dots, N\}.$$

We say that the element $(\Sigma, \hat{A}, \xi)$ of $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$ is in *special position* if there exists a geodesic segment in $\hat{\Sigma}$ with endpoints in $\hat{V}$ parallel to the field ξ. Let $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})^{\mathrm{sp}}$ denote the subset of $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$ consisting of elements in special position. A direct consequence of Proposition 4.13 is the following

Corollary 4.14 *The finite family* $\{\Phi_s(\mathcal{U}_s(I; J)) : \ (\mathbf{A}_s; I; J)$ *is admissible* $\}$ *covers the complement of* $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})^{\mathrm{sp}}$ *in* $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$.

The next proposition tells us that $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha}) \setminus \mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})^{\mathrm{sp}}$ is a subset of full measure in $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$.

Proposition 4.15 *The set* $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})^{\mathrm{sp}}$ *is a null set in* $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$ *with respect to* μ_{Tr}.

Proof: For every s in $\mathcal{AD}$, let μ_s denote the volume form on $\mathcal{U}_s$ which is the pull-back of μ_{Tr} under Φ_s. Let $(\Sigma, \hat{A}, \xi)$ be a point in $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})^{\mathrm{sp}}$, let e be a geodesic segment of $\hat{\Sigma}$ with endpoint in $\hat{V}$ parallel to the field ξ. There exists an admissible triangulation $\hat{\mathrm{T}}$ of $\hat{\Sigma}$ such that the 1-skeleton of $\hat{\mathrm{T}}$ contains e. Since e is parallel to ξ, the complex number associated to e in the local chart arising from $\hat{\mathrm{T}}$ is purely imaginary. As a consequence, there exist

. $s \in \mathcal{AD}$,

. $i \in \{1, \dots, N_1\}$, and

. $Z \in \{(z_1, \dots, z_{N_1}) \in \mathcal{U}_s | \ \mathrm{Re}(z_i) = 0\}$, such that $(\Sigma, \hat{A}, \xi) = \Phi_s(Z)$. For every $s \in \mathcal{AD}$, and every $i \in \{1, \dots, N_1\}$, set

$$\mathcal{U}_s^i = \mathcal{U}_s \cap \{(z_1, \dots, z_{N_1}) \in \mathbb{C}^{N_1} \mid \mathrm{Re}(z_i) = 0\}.$$

Note that if $Z \in \mathcal{U}_s^i$, then $\Phi_s(Z) \in \mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})^{\mathrm{sp}}$. It follows that

$$\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})^{\mathrm{sp}} = \bigcup_{s \in \mathcal{AD}} \bigcup_{i=1}^{N_1} \Phi_s(\mathcal{U}_s^i).$$

Since $\mathcal{U}_s$ can be identified to an open subset of $\mathbb{C}^N$, and μ_s corresponds to a volume form proportional to the Lebesgue measure, we have $\mu_s(\mathcal{U}_s^i) = 0,\ \forall s \in \mathcal{AD},\ i \in \{1, \dots, N_1\}$. It follows immediately that $\mu_{\mathrm{Tr}}(\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})^{\mathrm{sp}}) = 0$. □

4.4 Proof of Theorem 1.1

From Corollary 4.14, and Proposition 4.15, to prove Theorem 1.1, all we need is the following

Proposition 4.16 *Let $(\mathbf{A}_s; I; J)$, where $I = (i_1, \dots, i_N), J = (j_K, \dots, j_N)$, be an admissible triple. Let $\mathcal{F}_s$, and μ_s denote the pull backs of the function $\mathcal{F}^{\mathrm{et}}$, and the volume form μ_{Tr} onto $\mathcal{U}_s$ by Φ_s. Then we have:*

$$\int_{\mathcal{U}_s(I;J)} \mathcal{F}_s d\mu_s < \infty.$$

Proof: By the definition of primary system of indices, we have a complex linear map

$$\begin{array}{cccc} \mathbf{B}_s : & \mathbb{C}^N & \longrightarrow & \ker \mathbf{A}_s \\ & (z_1, \dots, z_N) & \longmapsto & (f_1(z_1, \dots, z_N), \dots, f_{N_1}(z_1, \dots, z_N)) \end{array}$$

which is an isomorphism, where $f_{i_k}(z_1, \dots, z_N) = z_k$. Consider a vector $(w_1, \dots, w_{N_1})$ in $\mathcal{U}_s$, let $(\Sigma, \hat{A}, \xi)$ denote its image under Φ_s. Let $\mathrm{T}, \hat{\Sigma}, \hat{\mathrm{T}}$ be as in the previous subsection. As usual, we denote the edges of $\hat{\mathrm{T}}$ by $e_i,\ i = 1, \dots, N_1$, so that w_i is the complex number associated to e_i.

By definition, for any $k = K, \dots, N$, the complex numbers w_{i_k} and w_{j_k} correspond to two edges e_{i_k}, and e_{j_k} which are contained in the boundary of a triangle Δ_k of $\hat{\mathrm{T}}$. With appropriate choices of orientation of e_{i_k}, and e_{j_k}, the area of Δ_k is given by the function

$$\hat{\eta}_k = \frac{1}{2}(\mathrm{Re}(w_{i_k})\mathrm{Im}(w_{j_k}) - \mathrm{Im}(w_{i_k})\mathrm{Re}(w_{j_k})).$$

Observe that the triangles $\Delta_k,\ k = K, \dots, N$, are all distinct. Suppose on the contrary that there exist $k < k'$ such that $e_{i_k}, e_{j_k}, e_{i_{k'}}$ are contained in the boundary of the same triangle. Excluding the cases $e_{i_{k'}} = e_{i_k}$, and $e_{i_{k'}} = e_{j_k}$, we see that e_{i_k}, e_{j_k}, and $e_{i_{k'}}$ are three sides of a triangle in $\hat{\mathrm{T}}$, which implies

$$w_{i_{k'}} = \pm w_{i_k} \pm w_{j_k}.$$

Since w_{j_k} is linearly dependent on $(w_{i_1}, \ldots, w_{i_{k-1}})$, it follows that $w_{i_{k'}}$ is linearly dependent on $(w_{i_1}, \ldots, w_{i_k})$ as $(w_1, \ldots, w_{N_1})$ varies in $\ker \mathbf{A}_s$, which is impossible since $(i_1, \ldots, i_N)$ is a primary system of indices for $\mathbf{A}_s$. As a consequence, we have

$$\mathbf{Area}(\Sigma) \geqslant \sum_{k=K}^{N} \hat{\eta}_k \tag{15}$$

Let $\eta_k,\ k = K, \ldots, N$, denote the pull back of the function $\hat{\eta}_k$ by $\mathbf{B}_s$. It follows that $\mathbf{B}_s^{-1}(\mathcal{U}_s(I;J))$ is contained in the following subset of $\mathbb{C}^N$

$$\mathcal{W}_s = \{(z_1, \ldots, z_N) \in \mathbb{C}^N :\ |\mathrm{Re}(z_k)| \leqslant |\mathrm{Re}(f_{j_k}(z_1, \ldots, z_N))|,\ \eta_k > 0,\ k = K, \ldots, N\}.$$

Let $\mathcal{G}_s$ denote the pull back of $\mathcal{F}_s$ by $\mathbf{B}_s$, since the volume form $\mathbf{B}_s^* \mu_s$ equals $\kappa \lambda_{2N}$, where λ_{2N} is the Lebesgue measure of $\mathbb{C}^N$, and κ is a constant, all we need to show is the following

$$\int_{\mathcal{W}_s} \mathcal{G}_s d\lambda_{2N} < \infty \tag{16}$$

To simplify the notations, for $k = 1, \ldots, N$, set $x_k = \mathrm{Re}(z_k),\ y_k = \mathrm{Im}(z_k)$. For $k = K, \ldots, N$, we write f_k instead of f_{j_k}, and set $a_k = \mathrm{Re}(f_k),\ b_k = \mathrm{Im}(f_k)$. Recall that, by definition, f_k depends only on $(z_1, \ldots, z_{k-1})$, and the coefficients of $z_K, \ldots, z_{k-1}$ in f_k are all real. Thus, we deduce that a_k is a function of $(z_1, \ldots, z_{K-1}, x_K, \ldots, x_{k-1})$, and b_k is a function of $(z_1, \ldots, z_{K-1}, y_K, \ldots, y_{k-1})$, for $k = K, \ldots, N$. With these notations, we have

$$\eta_k = \frac{1}{2}(x_k b_k - y_k a_k),\ k = K, \ldots, N. \tag{17}$$

$$|x_k| \leqslant |a_k|,\ k = K, \ldots, N. \tag{18}$$

$$\mathbf{Area}(\Sigma) \geqslant \sum_{k=K}^{N} \eta_k. \tag{19}$$

Recall that, by definition of admissible triple, the complex numbers $z_1, \ldots, z_{K-1}$ correspond to some edges of $\hat{\mathrm{T}}$ in the boundary of $\hat{\Sigma}$, or equivalently to some edges of the forest $\hat{A}$. Therefore, we have

$$\ell^2(\hat{A}) \geqslant \sum_{k=1}^{K-1} |z_k|^2 \tag{20}$$

Consequently, we have

$$\mathcal{G}_s \leqslant \exp(-\sum_{k=1}^{K-1} |z_k|^2 - \sum_{k=K}^{N} \eta_k) \tag{21}$$

Therefore, to prove (16), it suffices to show

Lemma 4.17

$$\mathcal{I} = \int_{\mathcal{W}_s} \exp(-\sum_{k=1}^{K-1} |z_k|^2 - \sum_{k=K}^{N} \eta_k) d\lambda_{2N} < \infty \tag{22}$$

Proof: Fix $(z_1, \ldots, z_{K-1}) \in \mathbb{C}^{K-1}$ and $(x_K, \ldots, x_N) \in \mathbb{R}^{N-K+1}$, and set

$$\begin{aligned} \mathcal{W}_s((z_1, \ldots, z_{K-1}); (x_K, \ldots, x_N)) = \ & \{(y_K, \ldots, y_N) \in \mathbb{R}^{N-K+1} \text{ such that} \\ & (z_1, \ldots, z_{K-1}, x_K + \imath y_K, \ldots, x_N + \imath y_N) \in \mathcal{W}_s\} \end{aligned}$$

Consider the following integral

$$\mathcal{I}((z_1, \ldots, z_{K-1}); (x_K, \ldots, x_N)) = \int_{\mathcal{W}_s((z_1,\ldots,z_{K-1});(x_K,\ldots,x_N))} \exp(-\sum_{k=K}^{N} \eta_k) dy_K \ldots dy_N.$$

Making the change of variables $(y_K, \ldots, y_N) \mapsto (\eta_K, \ldots, \eta_N)$, using (17), and the fact that, with $(z_1, \ldots, z_{K-1}, x_K, \ldots, x_N)$ fixed, a_k is constant, and b_k is an affine function of $(y_K, \ldots, y_{k-1})$, for any $k = K, \ldots, N$, we have:

$$d\eta_K \ldots d\eta_N = \frac{|a_K \ldots a_N|}{2^{N-K+1}} dy_K \ldots dy_N.$$

Since for $k = K, \ldots, N$, the function η_k is positive on $\mathcal{W}_s$, it follows

$$\begin{aligned} \mathcal{I}((z_1, \ldots, z_{K-1}); (x_K, \ldots, x_N)) & \leqslant \frac{2^{N-K+1}}{|a_K \ldots a_N|} \int_0^{+\infty} e^{-\eta_K} d\eta_K \ldots \int_0^{+\infty} e^{-\eta_N} d\eta_N \\ & \leqslant \frac{2^{N-K+1}}{|a_K \ldots a_N|}. \end{aligned}$$

Now, set

$$\mathcal{W}_s^* = \{((z_1, \ldots, z_{K-1}); (x_K, \ldots, x_N)) \in \mathbb{C}^{K-1} \times \mathbb{R}^{N-K+1} \ : \ |x_k| \leqslant |a_k|, \ k = K, \ldots, N\}.$$

We have

$$\begin{aligned} \mathcal{I} & = \int_{\mathcal{W}_s^*} \exp(-\sum_{k=1}^{K-1} |z_k|^2) \mathcal{I}((z_1, \ldots, z_{K-1}); (x_K, \ldots, x_N)) dx_1 dy_1 \ldots dx_{K-1} dy_{K-1} dx_K \ldots dx_N, \\ & \leqslant \int_{\mathcal{W}_s^*} \exp(-\sum_{k=1}^{K-1} |z_k|^2) \frac{2^{N-K+1}}{|a_K \ldots a_N|} dx_1 dy_1 \ldots dx_{K-1} dy_{K-1} dx_K \ldots dx_N, \\ & \leqslant \int_{\mathbb{C}^{K-1}} \exp(-\sum_{k=1}^{K-1} |z_k|^2) [\int_{-|a_K|}^{|a_K|} [\ldots [\int_{-|a_N|}^{|a_N|} \frac{2^{N-K+1}}{|a_K \ldots a_N|} dx_N] \ldots] dx_K] dx_1 dy_1 \ldots dx_{K-1} dy_{K-1}. \end{aligned}$$

Using the fact that a_k does not depend on $x_k, \dots, x_N$ for $k = K, \dots, N$, we see that

$$\int_{-|a_K|}^{|a_K|}[\dots[\int_{-|a_N|}^{|a_N|}\frac{2^{N-K+1}}{|a_K\dots a_N|}dx_N]\dots]dx_K = 4^{N-K+1}.$$

Hence,

$$\mathcal{I} \leqslant 4^{N-K+1}\int_{\mathbb{C}^{K-1}} e^{-(|z_1|^2+\dots+|z_{K-1}|^2)}dx_1dy_1\dots dx_{K-1}dy_{K-1} < \infty.$$

The lemma is then proved. □

The proofs of Proposition 4.16, and of Theorem 1.1 are now complete. □

5 Finiteness of the volume of moduli spaces of translation surfaces

In this section, we prove Theorem 1.2 using Theorem 1.1. Recall that $\mathcal{H}(k_1, \dots, k_n)$ is the moduli space of triples $(\Sigma, \{x_1, \dots, x_n\}, \xi)$, where Σ is a translation surface, $\{x_1, \dots, x_n\}$ is the set of marked singularities of Σ with cone angles $\{(k_1+1)2\pi, \dots, (k_n+1)2\pi\}$ respectively, and ξ is a unitary parallel vector filed on the complement of the set $\{x_1, \dots, x_n\}$. An element of $\mathcal{H}(k_1, \dots, k_n)$ can be identified to a pair (M, ω), where M is a connected, closed Riemann surface, and ω is a holomorphic 1-form on M having exactly n zeros with orders $k_1, \dots, k_n$. Using this identification, one can define a local chart for $\mathcal{H}(k_1, \dots, k_n)$ in a neighborhood of a point (M, ω) as follows: let $p_1, \dots, p_n$ denote the zeros of ω, and let $(\gamma_1, \dots, \gamma_{2g+n-1})$ be a basis of $H_1(M, \{p_1, \dots, p_n\}, \mathbb{Z})$. There exists a neighborhood $\mathcal{U}$ of (M, ω) in $\mathcal{H}(k_1, \dots, k_n)$ such that, for any (M', ω') in $\mathcal{U}$, $(\gamma_1, \dots, \gamma_{2g+n-1})$ gives rise to a basis $(\gamma'_1, \dots, \gamma'_{2g+n-1})$ of $H_1(M', \{p'_1, \dots, p'_n\}, \mathbb{Z})$, where $p'_1, \dots, p'_n$ are the zeros of ω'. It follows that the map

$$\begin{array}{cccc} \Phi: & \mathcal{U} & \longrightarrow & \mathbb{C}^{2g+n-1} \\ & (M', \omega') & \longmapsto & (\int_{\gamma'_1}\omega', \dots, \int_{\gamma'_{2g+n-1}}\omega') \end{array}$$

is a local chart. Let μ_0 denote the pull-back of the Lebesgue measure of $\mathbb{C}^{2g+n-1}$ by Φ, then μ_0 is a well-defined volume form on $\mathcal{H}(k_1, \dots, k_n)$.

Since $\mathcal{H}(k_1, \dots, k_n)$ is a special case of flat surface with erasing forest, where all the trees in the forest are points, on $\mathcal{H}(k_1, \dots, k_n)$, we also have a volume form μ_{Tr}. It turns out that(cf. [N]), on each connected component of $\mathcal{H}(k_1, \dots, k_n)$, we have $\mu_{\mathrm{Tr}} = \lambda\mu_0$, where λ is a constant.

5.1 Translation surface with a marked geodesic segment

To prove Theorem 1.2, we first consider a space $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$ where all the trees in $\hat{\mathcal{A}}$ but one are points, and the remaining one is a segment, together with a projection $\varrho : \mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha}) \longrightarrow \mathcal{H}(k_1, \dots, k_n)$ which is (locally) a fiber bundle. We then use Theorem 1.1, and the fact that the integral of $\mathcal{F}^{\mathrm{et}}$ on the fibers of ϱ is constant to conclude.

Set $\alpha_i = 2(k_i+1)$, $i = 1,\dots,n$. Let $\mathcal{A}_1$ be a topological tree isomorphic to a segment, and for $i = 2,\dots,n$, let $\mathcal{A}_i$ be just a point. Let $\underline{\alpha}$ denote the vector $(2\pi,\alpha_1,\dots,\alpha_n)$, and $\hat{\mathcal{A}}$ denote the family $\{\mathcal{A}_1,\dots,\mathcal{A}_n\}$. Consider the space $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$ with the previous data. In this case, $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$ is the moduli space of triples $(\Sigma,\{I(x_1,x),x_2,\dots,x_n\},\xi)$, where

- Σ is a translation surface,
- $\{x_1,\dots,x_n\}$ is the set of singularities of Σ with cone angles $\{\alpha_1,\dots,\alpha_n\}$ respectively,
- x is a regular point of Σ,
- $I(x_1,x)$ is a geodesic segment joining the singular point x_1 to x,
- and ξ is a unitary parallel vector field on the complement of $I(x_1,x)\cup\{x_2,\dots,x_n\}$.

By definition, we have a natural projection ϱ from $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$ to $\mathcal{H}(k_1,\dots,k_n)$ consisting of forgetting the segment $I(x_1,x)$, that is

$$\varrho : (\Sigma,\{I(x_1,x),x_2,\dots,x_n\},\xi) \longmapsto (\Sigma,\{x_1,\dots,x_n\},\xi).$$

Let $N = \dim_{\mathbb{C}} \mathcal{H}(k_1,\dots,k_n)$, clearly, $\dim_{\mathbb{C}} \mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha}) = N+1$. Let $\hat{\mu}_{\mathrm{Tr}}$, and μ_{Tr} denote the volume forms on $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$ and $\mathcal{H}(k_1,\dots,k_n)$ arising from admissible triangulations respectively.

Let Φ denote the period mapping defining a local chart of $\mathcal{H}(k_1,\dots,k_n)$ in an open set $\mathcal{U}$. We can then define some local charts $\hat{\Phi}$ for $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$ whose domains cover $\varrho^{-1}(\mathcal{U})$ as follows: first, we identify any $(\Sigma,\{x_1,\dots,x_n\},\xi)$ in $\mathcal{U}$ to a pair (M,ω), if $\Phi(\Sigma,\{x_1,\dots,x_n\},\xi) = (z_1,\dots,z_N)$, then $\hat{\Phi}(\Sigma,\{I(x_1,x),x_2,\dots,x_n\},\xi) = (z_1,\dots,z_N,z_{N+1})$, where

$$z_{N+1} = \int_{x_1}^{x} \omega,$$

and the integral is taken along $I(x_1,x)$. In the local charts $\hat{\Phi}$, and Φ, the map ϱ can be written as

$$\varrho(z_1,\dots,z_{N+1}) = (z_1,\dots,z_N).$$

Let λ_{2N}, and $\lambda_{2(N+1)}$ denote the Lebesgue measures of $\mathbb{C}^N$, and $\mathbb{C}^{N+1}$. Up to some multiplicative constants depending on the connected component of $(\Sigma,\{x_1,\dots,x_n\},\xi)$ in $\mathcal{H}(k_1,\dots,k_n)$, we can write

$$\mu_{\mathrm{Tr}} = \Phi^*\lambda_{2N} \text{ and } \hat{\mu}_{\mathrm{Tr}} = \hat{\Phi}^*\lambda_{2(N+1)}. \tag{23}$$

5.2 Proof of Theorem 1.2

Consider a point $(\Sigma,\{x_1,\dots,x_n\},\xi)$ in $\mathcal{H}(k_1,\dots,k_n)$. Fix a unitary tangent vector $v_1 \in T_{x_1}\Sigma$, we can then identify the set of unitary tangent vectors of $T_{x_1}\Sigma$ to $\mathbb{R}/(\alpha_1\mathbb{Z})$. Any geodesic segment in Σ which contains x_1 as an endpoint is uniquely determined by its tangent vector at x_1, and its length. Hence, we have an injective map:

$$\varphi : \varrho^{-1}\{(\Sigma, \{x_1, \dots, x_n\}, \xi)\} \longrightarrow (\mathbb{R}/(\alpha_1\mathbb{Z})) \times \mathbb{R}^+.$$

Let $\mathcal{U}$ be a neighborhood of $(\Sigma, \{x_1, \dots, x_n\}, \xi)$ in $\mathcal{H}(k_1, \dots, k_n)$ on which a local chart can be defined by some period mapping Φ. For each point $(\Sigma', \{x'_1, \dots, x'_n\}, \xi')$ in $\mathcal{U}$, we choose a tangent vector v'_1 in $T_{x'_1}\Sigma'$ to be the reference vector, we can assume that v'_1 varies continuously as $(\Sigma', \{x'_1, \dots, x'_n\}, \xi')$ varies in $\mathcal{U}$ so that the map φ can be extended into an injective, continuous map:

$$\varphi : \varrho^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times (\mathbb{R}/\alpha_1\mathbb{Z}) \times \mathbb{R}^+.$$

Using the local charts $\hat{\Phi}$ on $\varrho^{-1}(\mathcal{U})$, we can write

$$\varphi(z_1, \dots, z_{N+1}) = ((z_1, \dots, z_N), \arg(z_{N+1}) + c, |z_{N+1}|), \quad \text{where } c \text{ is some constant} \tag{24}$$

Let $d\theta$, and dr denote the standard measures on $\mathbb{R}/(\alpha_1\mathbb{Z})$, and $\mathbb{R}^+$ respectively. From (23), and (24), we have

$$\varphi_* d\hat{\mu}_{\mathrm{Tr}} = r d\mu_{\mathrm{Tr}} d\theta dr.$$

Consequently,

$$\int_{\varrho^{-1}(\mathcal{U})} e^{-\mathbf{Area}(\Sigma) - \ell^2(I)} d\hat{\mu}_{\mathrm{Tr}} = \int_{\varphi(\varrho^{-1}(\mathcal{U}))} e^{-\mathbf{Area}(\Sigma) - r^2} r d\mu_{\mathrm{Tr}} d\theta dr. \tag{25}$$

By a well known result (for example, see [MT], Theorem 1.8), we know that, on a translation surface, there exists a countable subset Θ of $\mathbb{R}/\alpha_1\mathbb{Z}$ such that if θ is not in Θ, then the geodesic ray starting from x_1 in the direction θ can be extended infinitely. It follows immediately that $\varphi(\varrho^{-1}(\mathcal{U}))$ is an open dense subset, hence of full measure, of $\mathcal{U} \times (\mathbb{R}/\alpha_1\mathbb{Z}) \times \mathbb{R}^+$. Therefore, we have

$$\begin{aligned}
\int_{\varphi(\varrho^{-1}(\mathcal{U}))} e^{-\mathbf{Area}(\Sigma) - r^2} r d\mu_{\mathrm{Tr}} d\theta dr &= \int_{\mathcal{U} \times (\mathbb{R}/\alpha_1\mathbb{Z}) \times \mathbb{R}^+} e^{-\mathbf{Area}(\Sigma) - r^2} r d\mu_{\mathrm{Tr}} d\theta dr, \\
&= \int_0^{+\infty} e^{-r^2} r dr \int_0^{\alpha_1} d\theta \int_{\mathcal{U}} e^{-\mathbf{Area}(\Sigma)} d\mu_{\mathrm{Tr}}, \\
&= \frac{\alpha_1}{2} \int_{\mathcal{U}} e^{-\mathbf{Area}(\Sigma)} d\mu_{\mathrm{Tr}}.
\end{aligned}$$

It follows from (25) that

$$\int_{\varrho^{-1}(\mathcal{U})} e^{-\mathbf{Area}(\Sigma) - \ell^2(I)} d\hat{\mu}_{\mathrm{Tr}} = \frac{\alpha_1}{2} \int_{\mathcal{U}} e^{-\mathbf{Area}(\Sigma)} d\mu_{\mathrm{Tr}} \tag{26}$$

Since (26) is true for any small neighborhood $\mathcal{U}$ in $\mathcal{H}(k_1, \dots, k_n)$, we deduce that

$$\int_{\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})} e^{-\mathbf{Area}(\Sigma) - \ell^2(I)} d\hat{\mu}_{\mathrm{Tr}} = \frac{\alpha_1}{2} \int_{\mathcal{H}(k_1, \dots, k_n)} e^{-\mathbf{Area}(\Sigma)} d\mu_{\mathrm{Tr}}.$$

By Theorem 1.1, we know that

$$\int_{\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})} e^{-\mathbf{Area}(\Sigma)-\ell^2(I)} d\hat{\mu}_{\mathrm{Tr}} < \infty.$$

Therefore,

$$\int_{\mathcal{H}(k_1,\dots,k_n)} e^{-\mathbf{Area}(\Sigma)} d\mu_{\mathrm{Tr}} < \infty,$$

and Theorem 1.2 is then proved. □

6 Finiteness of $\mu^1_{\mathrm{Tr}}(\mathcal{M}_1(\mathbb{S}^2,\underline{\alpha}))$

In this section, we are interested in the moduli space of flat surfaces of genus zero with prescribed cone angles. Let $\mathcal{M}(\mathbb{S}^2,\underline{\alpha})^*$ denote the moduli space of flat surfaces having n singularities, which are numbered, with cone angles given by $\underline{\alpha} = (\alpha_1,\dots,\alpha_n)$. Recall that we have a volume form μ_{Tr} on the space $\mathcal{M}(\mathbb{S}^2,\underline{\alpha}) = \mathcal{M}(\mathbb{S}^2,\underline{\alpha})^* \times \mathbb{S}^1$, which is defined by identifying locally $\mathcal{M}(\mathbb{S}^2,\underline{\alpha})$ to $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\underline{\alpha})$, with some appropriate choice of $\hat{\mathcal{A}}$. Let $\mathcal{M}_1(\mathbb{S}^2,\underline{\alpha})^*$ de the set of surfaces having unit area in $\mathcal{M}(\mathbb{S}^2,\underline{\alpha})^*$, and $\mathcal{M}_1(\mathbb{S}^2,\underline{\alpha})$ denote the product space $\mathcal{M}_1(\mathbb{S}^2,\underline{\alpha})^* \times \mathbb{S}^1$. The space $\mathcal{M}_1(\mathbb{S}^2,\underline{\alpha})^*$ can be considered as the moduli space of the configurations of n marked points on the sphere $\mathbb{S}^2$ up to Möbius transformations.

The volume form μ_{Tr} induces naturally a volume form μ^1_{Tr} on the space $\mathcal{M}_1(\mathbb{S}^2,\underline{\alpha}) = \mathbf{Area}^{-1}(\{1\})$, and hence, a volume form $\hat{\mu}^1_{\mathrm{Tr}}$ on $\mathcal{M}_1(\mathbb{S}^2,\underline{\alpha})^*$ by pushing forward. As we have seen in the introduction, Theorem 1.4 is equivalent to the finiteness of $\mu^1_{\mathrm{Tr}}(\mathcal{M}_1(\mathbb{S}^2,\underline{\alpha}))$, and of $\hat{\mu}^1_{\mathrm{Tr}}\mathcal{M}_1(\mathbb{S}^2,\underline{\alpha})^*$. Our aim in this section is to prove Theorem 1.4 using Theorem 1.1.

6.1 The function δ

Let Σ be an element of $\mathcal{M}(\mathbb{S}^2,\underline{\alpha})^*$. Let $x_1,\dots,x_n$ denote the singular points of Σ so that the cone angle at x_i is α_i. Let $\mathbf{d}$ denote the distance induced by the flat metric on Σ. For any subset I of $\{1,\dots,n\}$, let $\mathbf{diam}_I(\Sigma)$ denote the diameter of the set $\{x_i,\ i\in I\}$. We define

$$\delta_I(\Sigma) = \min\{\mathbf{d}(x_i,x_j) :\ i\in I,\ j\notin I\},$$

and

$$\delta^+_I(\Sigma) = \begin{cases} \delta_I(\Sigma) & \text{if } \delta_I(\Sigma) \geqslant 3\mathbf{diam}_I(\Sigma)\ , \\ 0 & \text{otherwise.}\end{cases}$$

A subset I of $\{1,\dots,n\}$ is called *essential* if we have

$$\sum_{i\in I}\alpha_i \notin 2\pi\mathbb{N}.$$

We define a function δ on the space $\mathcal{M}(\mathbb{S}^2,\underline{\alpha})^*$ as follows

$$\text{for every } \Sigma \in \mathcal{M}(\mathbb{S}^2, \underline{\alpha})^*, \ \delta(\Sigma) = \max\{\delta^+_I(\Sigma) : \ I \subset \{1, \dots, n\}, \ I \text{ is essential } \}.$$

Remark: The function δ is always positive, since when $I = \{i\}$, we have

$$\delta^+_{\{i\}}(\Sigma) = \min\{\mathbf{d}(x_i, x_j), \ j \neq i\} > 0,$$

and there always exists $i \in \{1, \dots, n\}$ such that $\alpha_i \notin 2\pi\mathbb{N}$, which means that $\{i\}$ is essential. To simplify the notations, we also denote by δ the composition of δ with the natural projection from $\mathcal{M}(\mathbb{S}^2, \underline{\alpha}) = \mathcal{M}(\mathbb{S}^2, \underline{\alpha})^* \times \mathbb{S}^1$ onto $\mathcal{M}(\mathbb{S}^2, \underline{\alpha})^*$.

6.2 Good tree and good forest

Fix a surface Σ in $\mathcal{M}(\mathbb{S}^2, \underline{\alpha})^*$, and let $x_1, \dots, x_n$ denote the singular points of Σ so that the cone angle at x_i is α_i. Let V denote the set $\{x_1, \dots, x_n\}$, and set $\delta = \delta(\Sigma)$. For any geodesic tree A on Σ, we denote by $\mathrm{Ver}(A)$ the vertex set of A, $\max(A)$ the length of the longest edge of A, and by $\mathrm{R}(A)$ the distance from $\mathrm{Ver}(A)$ to the set $\mathrm{V} \setminus \mathrm{Ver}(A)$.

Definition 6.1 *Let A be a geodesic tree in Σ whose vertex set is a subset of V. Let k be the number of edges of A. The tree A is said to be* good*, if either A is a singular point with cone angle in $2\pi\mathbb{N}$, or $k \geqslant 1$ and we have*

- $\max(A) \leqslant 4^{k-1}\delta$,
- $\mathbf{diam}(\mathrm{Ver}(A)) \leqslant 4^{k-1}\delta$,
- *The set of indices corresponding to the vertex set of A is non essential, that is the sum of all cone angles at the vertices of A belongs to $2\pi\mathbb{N}$.*
- *Either* $\mathrm{Ver}(A) = \mathrm{V}$*, or* $\mathrm{R}(A) \geqslant 3.4^{k-1}\delta$.

A union of disjoint good trees such that the union of the vertex sets is V is called a good forest*.*

We have

Lemma 6.2 *There always exists a good forest in Σ.*

The proof of this lemma is given in Appendices, Section A.

Corollary 6.3 *There exists a constant κ depending only on n such that for any Σ in $\mathcal{M}(\mathbb{S}^2, \underline{\alpha})^*$, there exists an erasing forest $\hat{A}$ in Σ which verifies*

$$\ell(\hat{A}) \leqslant \kappa\delta.$$

Proof: By Lemma 6.2, we know that there exists a good forest $\hat{A} = \sqcup_{j=1}^{m} A_j$ in Σ. By definition, for every $j \in \{1, \dots, n\}$, the sum of the cone angles at the vertices of A_j belongs to $2\pi\mathbb{N}$, therefore, $\hat{A}$ is an erasing forest. Since every tree A_j in $\hat{A}$ is good, we have

$$\ell(A_j) \leqslant k_j 4^{k_j-1}\delta,$$

where k_j is the number of edges of A_j. Observe that $k_1 + \cdots + k_m = n - m \leqslant n-1$. Therefore, we have

$$\ell(\hat{A}) = \sum_{j=1}^{m} \ell(A_j) \leqslant (n-1)4^{n-1}\delta,$$

and the corollary follows. □

6.3 Proof of Theorem 1.4

Theorem 1.4 is a consequence of two following propositions:

Proposition 6.4 *We have*

$$\int_{\mathcal{M}(\mathbb{S}^2,\underline{\alpha})} \exp(-\mathbf{Area} - \delta^2) d\mu_{\mathrm{Tr}} < \infty.$$

and

Proposition 6.5 *There exists a constant $C(\underline{\alpha})$ depending on $\underline{\alpha}$ such that for any surface Σ in $\mathcal{M}(\mathbb{S}^2, \underline{\alpha})^*$ we have*

$$\delta^2(\Sigma) < C(\underline{\alpha})\mathbf{Area}(\Sigma).$$

The proof of Proposition 6.5 is rather straight forward but quite lengthy, it will be given in Appendices, Section B. Here below, we give the proof of Proposition 6.4 using Corollary 6.3.

Proof: (of Proposition 6.4) Let $\mathcal{A}_{\mathrm{ad}}(\underline{\alpha})$ denote the set of all families $\hat{\mathcal{A}} = \{\mathcal{A}_1, \dots, \mathcal{A}_m\}$ $(0 < m < n)$ of topological trees, whose vertices are labelled by $\{1, \dots, n\}$, up to isomorphism, verifying the following condition: if I_j, $j = 1, \dots, m$, is the subset of $\{1, \dots, n\}$ in bijection with the vertices of the tree $\mathcal{A}_j$, then

$$\sum_{i \in I_j} \alpha_i \in 2\pi\mathbb{N}.$$

For each $\hat{\mathcal{A}} = \{\mathcal{A}_1, \dots, \mathcal{A}_m\} \in \mathcal{A}_{\mathrm{ad}}(\underline{\alpha})$, let $\mathcal{U}_{\hat{\mathcal{A}}}$ denote the subset of $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \underline{\alpha})$ consisting of all triples $(\Sigma, \hat{A}, \xi)$ satisfying the following condition:

$$\ell(\hat{A}) \leqslant \kappa\delta(\Sigma),$$

where κ is the constant in Corollary 6.3. Let $\rho_{\hat{A}}$ denote the map from $\mathcal{M}^{\text{et}}(\hat{\mathcal{A}}, \underline{\alpha})$ onto $\mathcal{M}(\mathbb{S}^2, \underline{\alpha})^*$, which associates to every triple $(\Sigma, \hat{A}, \xi)$ the surface Σ. From Corollary 6.3, we know that the family

$$\{\mathcal{V}_{\hat{A}} = \rho_{\hat{A}}(\mathcal{U}_{\hat{A}}) : \hat{A} \in \mathcal{A}_{\text{ad}}(\underline{\alpha})\}$$

covers the space $\mathcal{M}(\mathbb{S}^2, \underline{\alpha})^*$. Let ρ_1 be the natural projection from $\mathcal{M}(\mathbb{S}^2, \underline{\alpha})$ onto $\mathcal{M}(\mathbb{S}^2, \underline{\alpha})^*$, it follows that the family

$$\{\rho_1^{-1}(\mathcal{V}_{\hat{A}}) : \hat{A} \in \mathcal{A}_{\text{ad}}(\underline{\alpha})\}$$

covers the space $\mathcal{M}(\mathbb{S}^2, \underline{\alpha})$. Since the set $\mathcal{A}_{\text{ad}}(\underline{\alpha})$ is finite, it is enough to show that, for every $\hat{A}$ in $\mathcal{A}_{\text{ad}}(\underline{\alpha})$, we have

$$\int_{\rho_1^{-1}(\mathcal{V}_{\hat{A}})} \exp(-\mathbf{Area} - \delta^2) d\mu_{\text{Tr}} < \infty. \tag{27}$$

Since the space $\mathcal{M}(\mathbb{S}^2, \underline{\alpha})$ can be locally identified to $\mathcal{M}^{\text{et}}(\hat{\mathcal{A}}, \underline{\alpha})$, we have

$$\int_{\rho_1^{-1}(\mathcal{V}_{\hat{A}})} \exp(-\mathbf{Area} - \delta^2) d\mu_{\text{Tr}} = \int_{\mathcal{U}_{\hat{A}}} \exp(-\mathbf{Area} - \delta^2) d\mu_{\text{Tr}}$$

By definition, for every $(\Sigma, \hat{A}, \xi)$ in $\mathcal{U}_{\hat{A}}$, we have $\ell(\hat{A}) \leqslant \kappa\delta(\Sigma)$. It follows

$$\int_{\mathcal{U}_{\hat{A}}} \exp(-\mathbf{Area} - \delta^2) d\mu_{\text{Tr}} \leqslant \int_{\mathcal{U}_{\hat{A}}} \exp(-\mathbf{Area} - \frac{1}{\kappa^2}\ell^2) d\mu_{\text{Tr}} \tag{28}$$

By Theorem 1.1, we know that the right hand side of (28) is finite. Consequently, (27) is true, and the proposition follows. □

Proposition 6.4, and Proposition 6.5 imply that

$$\int_{\mathcal{M}(\mathbb{S}^2, \underline{\alpha})} \exp(-(1 + C(\underline{\alpha}))\mathbf{Area}(.)) d\mu_{\text{Tr}} < \infty \tag{29}$$

which is equivalent to

$$\int_{\mathcal{M}(\mathbb{S}^2, \underline{\alpha})} \exp(-\mathbf{Area}(.)) d\mu_{\text{Tr}} < \infty \tag{30}$$

since both (29), and (30) are equivalent to the fact that the volume of $\mathcal{M}_1(\mathbb{S}^2, \underline{\alpha})$ is finite. The proof of Theorem 1.4 is now complete. □

Appendices

A Existence of good forest

A.1 Existence of good tree

Let $\Sigma, x_1, \dots, x_n, \mathrm{V}, \delta$ be as in Section 6.2. Let $\mathbf{d}$ denote the distance induced by the metric of Σ. Let us start by proving the following

Lemma A.1 *For any Σ in $\mathcal{M}(\mathbb{S}^2, \underline{\alpha})^*$, there always exists a good tree on Σ.*

Proof: First, let e be a geodesic segment which realizes the distance

$$\min\{\mathbf{d}(x_i, x_j),\ \alpha_i \not\in 2\pi\mathbb{N} \text{ and } i \neq j\}.$$

By definition, we have

$$\mathbf{leng}(e) = \min\{\delta^+_{\{i\}}(\Sigma),\ \alpha_i \not\in 2\pi\mathbb{N}\} \leqslant \delta.$$

Let A^1 denote the tree which contains only the segment e. By assumption, we have

$$\max(A^1) = \mathbf{diam}(\mathrm{Ver}(A^1)) = \mathbf{leng}(e_1) \leqslant \delta.$$

Consider the following procedure, which will be called the *vertex adding procedure*: suppose that we already have a geodesic tree A^k, $k \geqslant 1$, connecting $k+1$ points in $\{x_1, \dots, x_n\}$ verifying the following condition:

$$(*) \left\{ \begin{array}{lcl} \max(A^k) & \leqslant & 4^{k-1}\delta, \\ \mathbf{diam}(\mathrm{Ver}(A^k)) & \leqslant & 4^{k-1}\delta. \end{array} \right.$$

Let I be the subset of $\{1, \dots, n\}$ corresponding to the vertex set of A^k. We have two cases:

- Case 1: I is essential. In this case, let e_{k+1} be a segment realizing the distance $\delta_I(\Sigma)$, and let x_j be the endpoint of e_{k+1} which does not belong to $\mathrm{Ver}(A^k)$. By definition, we have either

  . $\mathbf{leng}(e_{k+1}) \leqslant 3\mathbf{diam}(\mathrm{Ver}(A^k))$ or,

  . $\mathbf{leng}(e_{k+1}) \leqslant \delta$.

  Since $\mathbf{diam}(\mathrm{Ver}(A^k)) \leqslant 4^{k-1}\delta$, it follows that $\mathbf{leng}(e_{k+1}) \leqslant 3.4^{k-1}\delta$, in both cases

  Slit open the surface Σ along the tree A^k, and let Σ' denote the new surface. The vertex set $\mathrm{Ver}(A^k)$ of A^k gives rise to a finite subset V^k of the boundary of Σ'. Let us prove that the distance in Σ' from x_j to V^k is at most $4^k\delta$.

  Consider e_{k+1} as a ray exiting from x_j, and let y be the first intersection point between e_{k+1} and the tree A^k. Since we have $\max(A^k) \leqslant 4^{k-1}\delta$, there exists a path in Σ joining x_j to an endpoint of the edge containing y without crossing any edge of A^k, whose length is at most

$$3.4^{k-1}\delta + 4^{k-1}\delta = 4^k\delta.$$

Because this path does not cross any edge of the tree A^k, it represents a path on Σ' joining x_j to a point in V^k. Thus, we deduce that the distance between x_j and V^k in Σ' is at most $4^k\delta$.

The path realizing the distance from x_j to V^k in Σ' corresponds to a path a in Σ which is piecewise geodesic with endpoints in V, joining x_j to a vertex of the tree A^k. Note that we have

$$\mathbf{leng}(a) \leqslant 4^k\delta.$$

Adding a to A^k, we get a new tree which contains $k + r$ edges, and will be denoted by A^{k+r}, where r is the number of geodesic segments with endpoints in V contained in a. Let us prove that this new tree also verifies the condition $(*)$.

- If $r = 1$, then $\mathrm{Ver}(A^{k+1}) = \mathrm{Ver}(A^k) \cup \{x_j\}$. Since $\mathbf{diam}(A^k) \leqslant 4^{k-1}\delta$, and the distance from x_j to $\mathrm{Ver}(A^k)$ is at most $3.4^{k-1}\delta$, we deduce that

  $$\mathbf{diam}(\mathrm{Ver}(A^{k+1})) \leqslant 4^{k-1}\delta + 3.4^{k-1}\delta = 4^k\delta.$$

  By assumption, we know that $\max(A^k) \leqslant 4^{k-1}\delta$, and we have proved that the length of the added edge is at most $4^k\delta$, hence, we have $\max(A^{k+1}) \leqslant 4^k\delta$.

- If $r > 1$, it means that the path a contains some points of V in its interior. The distance from these points to the set $\mathrm{Ver}(A^k)$ is bounded by the length of a which is at most $4^k\delta$. Hence, the diameter of the set $\mathrm{Ver}(A^{k+r})$ is at most

  $$4^{k-1}\delta + 4^k\delta \leqslant 4^{k+r-1}\delta.$$

  As for $\max(A^{k+r})$, we have

  $$\max(A^{k+r}) \leqslant \max\{\max(A^k), \mathbf{leng}(a)\} \leqslant 4^k\delta.$$

We can now restart the procedure with A^{k+r} in the place of A^k.

- Case 2: I is non-essential. In this case, if $\mathrm{Ver}(A^k) = \mathrm{V}$, or $\mathrm{R}(\mathrm{Ver}(A^k)) \geqslant 3.4^{k-1}\delta$, then the procedure stops since we already get a good tree. Otherwise, there exist x_i in $\mathrm{Ver}(A^k)$, x_j in $\mathrm{V} \setminus \mathrm{Ver}(A^k)$, and a geodesic segment e joining x_i to x_j with

  $$\mathbf{leng}(e) < 3.4^{k-1}\delta.$$

  Using the same arguments as in Case 1, we can add to A^k some edges so that the new tree also verifies the condition $(*)$, and repeat the procedure.

Since we only have finitely many singular points in Σ, the vertex adding procedure must stop, and we obtain a good tree. □

A.2 Proof of Lemma 6.2

By Lemma A.1, we know that there exists a good tree A_1 in Σ. If $\mathrm{Ver}(A_1) = \mathrm{V}$, or every point in the set $\mathrm{V}\setminus\mathrm{Ver}(A_1)$ has cone angle in $2\pi\mathbb{N}$, then we are done. Otherwise, there exists a point x_i in $\mathrm{V}\setminus\mathrm{Ver}(A_1)$, with cone angle not in the set $2\pi\mathbb{N}$. In this case, we would like to construct a good tree A_2 containing x_j by the vertex adding procedure. However, this procedure can not be carried out straightly because of the presence of the tree A_1. Namely, it may happen that we have $\mathrm{R}(\mathrm{Ver}(A_2)) \leqslant 3.4^{k_2-1}\delta$, where k_2 is the number of edges of A_2, but the segment realizing the distance $\mathbf{d}(\mathrm{Ver}(A_2), \mathrm{V}\setminus\mathrm{Ver}(A_2))$ intersects the tree A_1.

To fix this problem, let us consider the following procedure, which will be called the *tree joining procedure*: let $A_1, \dots, A_l$ be a family of disjoint geodesic trees whose vertex sets are contained in V. Let $k_1, \dots, k_l,\ k_i > 0$, be the numbers of edges of $A_1, \dots, A_l$ respectively. Assume that the family $\{A_1, \dots, A_l\}$ verifies the following properties:

$$(**)\begin{cases} a) & A_1, \dots, A_{l-1} \text{ are good trees,} \\ b) & A_l \text{ satisfies the condition } (*), \\ c) & \mathbf{d}(A_l, \sqcup_{j=1}^{l-1} A_j) \leqslant 3.4^{k_l-1}\delta. \end{cases}$$

Let s be a path of length at most $3.4^{k_l-1}\delta$ joining a point of A_l to a point of $\sqcup_{j=1}^{l-1} A_j$. Without loss of generality, we can assume that s joins a point in A_l to a point in A_{l-1}. Since both A_{l-1} and A_l verify the condition $(*)$, in particular, we have

$$\max(A_l) \leqslant 4^{k_l-1}\delta, \text{ and } \max(A_{l-1}) \leqslant 4^{k_{l-1}-1}\delta.$$

It follows that there exists a path c joining a vertex of A_{l-1} to a vertex of A_l without crossing any edge of the family $\{A_1, \dots, A_l\}$ such that

$$\mathbf{leng}(c) \leqslant 4^{k_l-1}\delta + 3.4^{k_l-1}\delta + 4^{k_{l-1}-1}\delta \leqslant 4^{k_l+k_{l-1}}\delta.$$

Consider the surface with boundary Σ' obtained by slitting open Σ along the trees $A_1, \dots, A_l$. Let $C_j,\ j = 1, \dots, l$, denote the connected component of $\partial\Sigma'$ arising from A_j, and V'_j denote the finite subset of C_j corresponding to the vertices of A_j. We denote by V' the finite subset of Σ' arising from V, note that $V'_j = \mathrm{V}' \cap C_j$. Let $\mathbf{d}'$ denote the distance induced by the metric structure of Σ'.

The path c represents then a path c' in Σ' joining a point x'_l in V'_l to a point x'_{l-1} in V'_{l-1}. Since $\mathbf{leng}(c') = \mathbf{leng}(c) \leqslant 4^{k_l+k_{l-1}}\delta$, we deduces that

$$\mathbf{d}'(x'_l, x'_{l-1}) \leqslant 4^{k_l+k_{l-1}}\delta.$$

Let c'_0 be a path realizing the distance from x'_{l-1} to x'_l in Σ', then c'_0 is a union of geodesic segments with endpoints in V', and $\mathbf{leng}(c'_0) \leqslant 4^{k_l+k_{l-1}}\delta$. Now, the path c'_0 corresponds to a path c_0 in Σ, joining a vertex of A_l to a vertex of A_{l-1}. By construction, c_0 is a union of geodesic segments with endpoints in V, each of which is either an edge of a tree in $\{A_1, \dots, A_l\}$, or a geodesic segment which does not cross any edge of the trees in the family $\{A_1, \dots, A_l\}$. As a consequence, the union of c_0 and all the trees in $\{A_1, \dots, A_l\}$ which have at least a common point with c_0 is a geodesic tree. This new tree contains obviously A_{l-1} and A_l as subtrees, hence it contains at least $k_l + k_{l-1} + 1$ edges. We denote by $A'_{l'}$ this new tree, and by $A'_1, \dots, A'_{l'-1}$ the remaining trees in the family $\{A_1, \dots, A_l\}$.

It is a routine to verify that the family $\{A'_1, \dots, A'_{l'}\}$ also satisfies the conditions a), and b) of $(**)$. If the condition c) still holds, then we can restart the procedure. Since the number of singularities of Σ is finite, the procedure can be repeated until we get either

. a single geodesic tree A verifying the property $(*)$ or,

. a family $\{\tilde{A}_1, \dots, \tilde{A}_{\tilde{l}}\}$ of disjoint geodesic trees, verifying a), and b) of the condition $(**)$, and in addition, we have:

$$\mathbf{d}(\tilde{A}_{\tilde{l}}, \tilde{A}_1 \sqcup \cdots \sqcup \tilde{A}_{\tilde{l}-1}) \geqslant 3.4^{k_{\tilde{l}}-1}\delta,$$

where $k_{\tilde{l}}$ is the number of edges of $\tilde{A}_{\tilde{l}}$.

Now, let us show that the tree joining procedure, and the vertex adding procedure in Lemma A.1 will allow us to construct a good forest in Σ. First, by Lemma A.1, we know that, there exists a good tree A_1. We will proceed by induction. Assume that we already have a family $\{A_1, \dots, A_l\}$ of disjoint good trees. If the union of the vertex sets of $A_1, \dots, A_l$ is V, or all the remaining singularities have cone angle in $2\pi\mathbb{N}$, then we are done. Otherwise, we can start a vertex adding procedure with a singular point which is not a vertex of the family $\{A_1, \dots, A_l\}$.

The vertex adding procedure can be carried out until we get a new good tree A_{l+1} disjoint from $A_1 \sqcup \cdots \sqcup A_l$, or until we get a geodesic tree A such that

. A satisfies the condition $(*)$,

. the segment realizing the distance $\mathbf{d}(\mathrm{Ver}(A), \mathrm{V}\backslash\mathrm{Ver}(A))$ intersects a tree in the family $\{A_1, \dots, A_l\}$.

In the latter case, we see that the family $\{A_1, \dots, A_l, A\}$ satisfies the condition $(**)$, therefore we can start the tree joining procedure. When this procedures terminates, we get a family of disjoint geodesic trees $\{\tilde{A}_1, \dots, \tilde{A}_{\tilde{l}}\}$, it may happen that $\tilde{l} = 1$, where $\tilde{A}_1, \dots, \tilde{A}_{\tilde{l}-1}$ are good, $\tilde{A}_{\tilde{l}}$ verifies the condition $(*)$, and we can carry out the vertex adding procedure on $\tilde{A}_{\tilde{l}}$. Since the number of singularities of Σ is finite, this algorithm must terminate, and we obtain a good forest for Σ. □

B Proof of Proposition 6.5

Let I_0 be a subset of $\{1,\dots,n\}$ such that $\delta^+_{I_0}(\Sigma)=\delta(\Sigma)=\delta$. Let s be a geodesic segment joining a point x_{i_0} with $i_0\in I_0$ and a point x_{i_1} with $i_1\notin I_0$ such that $\mathbf{leng}(s)=\delta$. Let p denote the midpoint of s. As usual, we denote by $\mathbf{d}$ the distance induced by the flat metric of Σ. First, we have

Lemma B.1 *$B(p,\delta/2)=\{x\in\Sigma:\mathbf{d}(p,x)<\delta/2\}$ does not contain any singular point of Σ.*

Proof: Suppose on the contrary that a singular point x_k, with $k\notin\{i_0,i_1\}$, is contained in $B(p,\delta/2)$, then we have $\mathbf{d}(x_{i_0},x_k)<\delta$, and $\mathbf{d}(x_{i_1},x_k)<\delta$, but this would imply that $\delta_{I_0}(\Sigma)<\delta$, and we have a contradiction. □

Let $D(\delta/2)$ denote the open disk with center $(0,0)$ and radius $\delta/2$ in the Euclidean plane $\mathbb{E}^2=\mathbb{R}^2$. Let f be the isometric immersion from $D(\delta/2)$ to Σ, which maps the horizontal diameter of $D(\delta/2)$ to the segment s, and the origin $(0,0)$ to the point p. The immersion f exists because the smallest distance from p to a singular point of Σ is $\delta/2$.

Let ϵ be the maximal value such that the restriction of f on the disk $D(\epsilon\delta)$ with center $(0,0)$ and radius $\epsilon\delta$ is an embedding. If $\epsilon\geqslant 1/4$ then there is an embedded Euclidean disk of radius $\delta/4$ in Σ, which means that $\mathbf{Area}(\Sigma)\geqslant(\pi\delta^2)/16$. In what follows, we will suppose that $\epsilon<1/4$, consequently, the set $f^{-1}(\{p\})$ contains points other than $(0,0)$. Let p_1 be the point in $f^{-1}(\{p\})\backslash\{(0,0)\}$ closest to $(0,0)$.

For any subset I of $\{1,\dots,n\}$, we denote by α_I the sum $\sum_{i\in I}\alpha_i$, and $\|\alpha_I\|$ the distance from α_I to the set $\pi\mathbb{Z}$ in $\mathbb{R}$. Set

$$\alpha_0=\min\{\|\alpha_I\|:\ I\subset\{1,\dots,n\},\|\alpha_I\|\neq 0\}.$$

Choose a number ϵ_0 such that

$$\epsilon_0<\min\{1/6,\sin(\alpha_0)/4\}.$$

We will prove that there exists an embedded disk of radius $\epsilon_0\delta$ in Σ, which is enough to prove the proposition.

Let d_0 denote the horizontal diameter of $D(\delta/2)$, and d_1 denote the lift of s passing through p_1. Let c_1 denote the segment joining $(0,0)$ to p_1 in $D(\delta/2)$, and c denote the image of c_1 under f, c is then a geodesic loop in Σ with base point p. Let θ be angle between d_0 and d_1, by this we mean the angle in $[0;\pi/2]$ between the two lines supporting d_0 and d_1. First, let us prove

Lemma B.2 *We have either $\theta=0$, or $\epsilon>\epsilon_0$.*

Proof: Remark that θ equals the rotation angle of the holonomy of c, which is the sum of some angles in $\{\alpha_1,\dots,\alpha_n\}$ modulo π. Suppose that $\theta\neq 0$, then, by the definition of α_0, we have $\theta\geqslant\alpha_0$.

If $\epsilon < \epsilon_0$, then the distance from $(0, 0)$ to d_1 is less than $2\epsilon_0\delta < \sin(\alpha_0)\delta/2$. Together with the fact that $\theta \geqslant \alpha_0$, this implies that d_1 intersects d_0, in other words, the segment s has self-intersection, which is impossible. Therefore, we can conclude that either $\theta = 0$, or $\epsilon > \epsilon_0$. □

If $\epsilon > \epsilon_0$, then we are done. Therefore, we only have to consider the case $\theta = 0$, and we have

Lemma B.3 *If $\theta = 0$, then the rotation angle of the holonomy of c is 0 modulo 2π.*

Proof: If it is not the case, then this angle equals π modulo 2π, and hence, the holonomy of c is the composition of a rotation of angle π and a translation which maps $(0, 0)$ to p_1. Such a transformation must fix the midpoint q_1 of the segment joining $(0, 0)$ to p_1. It follows that q_1 is mapped by f into a singular point of Σ, which is impossible because q_1 is contained in the disk $D(\delta/2)$. □

From Lemma B.3, we deduce that the image of $D(\delta)$ under f contains a cylinder C with length $(1 - 2\epsilon)\delta$ and width bounded by $2\epsilon\delta$. Remark that c is then a closed geodesic in C which cuts Σ into two flat surfaces with geodesic boundary, each of which is homeomorphic to a topological closed disk. We denote by Σ_0 the flat disk that contains x_{i_0}.

Lemma B.4 *For any i in I_0, x_i is contained in Σ_0.*

Proof: Recall that by the definition of δ, we have

$$\mathbf{diam}\{x_i,\ i \in I_0\} < \delta/3,$$

which implies that $\mathbf{d}(x_{i_0}, x_i) < \delta/3$, for any i in I_0. If there exists $i \in I_0$ such that $x_i \notin \Sigma_0$, then the path realizing the distance $\mathbf{d}(x_{i_0}, x_i)$ must intersect the closed geodesic c, therefore it crosses C. Consequently,

$$\mathbf{d}(x_{i_0}, x_i) \geqslant (1 - 2\epsilon)\delta > 2/3\delta,$$

which is impossible. □

The rotation angle of the holonomy of c equals the sum of all cone angles at singular points in Σ_0 modulo 2π. By assumption, we know that $\alpha_{I_0} \notin 2\pi\mathbb{Z}$, it means that Σ_0 contains singular points which do not belong to $\{x_i,\ i \in I_0\}$. Note that we have

$$\min\{\mathbf{d}(x_i, x_j\},\ i \in I_0,\ j \notin I_0,\ x_j \in \Sigma_0\} \geqslant \delta_{I_0}(\Sigma) = \delta.$$

Since Σ_0 is a flat surface with geodesic boundary which contains no singularities on the boundary, we can restrict ourselves into Σ_0 and restart the whole procedure. This procedure can be continued as long as the rotation angle of the loop c is zero.

Since we only have finitely many singular points in Σ, the procedure must stop, and we get a point in Σ whose injectivity radius is at least $\epsilon_0\delta$. Proposition 6.5 is then proved. □

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