

ELLIPSES IN TRANSLATION SURFACES

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ABSTRACT. We study the topology and geometry of the moduli space of immersions of ellipses into a translation surface. The frontier of this space is naturally stratified by the number of ‘cone points’ that an ellipse meets. The stratum consisting of ellipses that meet three cone points is naturally a two dimensional (non-manifold) polygonal cell complex. We show that the topology of this cell-complex together with the eccentricity and direction of each of its vertices determines the translation surface up to homothety. As a corollary we characterize the Veech group of the translation surface in terms of automorphisms of this polygonal cell complex.

1. INTRODUCTION

A translation structure μ on a (connected) topological surface X is an equivalence class of atlases whose transition functions are translations. Translation surfaces are fundamental objects in Teichmüller theory, the study of polygonal billiards, and the study of interval exchange maps.

Cylinders that are isometrically embedded in a translation surface play a central role in the theory. In Teichmüller theory, they appear as solutions to moduli problems [Strebel]. In rational billiards and interval exchange maps, cylinders correspond to periodic orbits [HMSZ06] [MsrTbc02] [Smillie00].

Each periodic geodesic γ on a translation surface belongs to a unique ‘maximal’ cylinder that is foliated by the geodesics that are both parallel and homotopic to γ . One method for producing such periodic geodesics implicitly uses ellipses. Indeed, if X admits an isometric immersion of an ellipse with area greater than that of X , then the image of the immersion contains a cylinder, and hence a periodic geodesic [MsrSml91] [Smillie00].

Ellipses interiors also serve to interpolate between maximal cylinders. The set, $\mathcal{E}(X, \mu)$, of ellipse interiors isometrically immersed in X has a natural geometry coming from the space of quadratic forms on \mathbb{R}^3 (see §6). The set of maximal cylinders is a discrete set lying in the frontier of the path connected space $\mathcal{E}(X, \mu)$.

If the frontier of a translation surface X is finite, then each point in the frontier may be naturally regarded as a *cone point* with angle equal to an integral multiple of 2π . If the frontier of an immersed ellipse interior U contains a cone point c , then we will say that U *meets* c . If an ellipse interior meets a cone point, then the ellipse interior belongs to the frontier of $\mathcal{E}(X, \mu)$. The remainder of the frontier consists of cylinders.

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The number of cone points met by an ellipse interior determines a natural stratification of $\mathcal{E}(X, \mu)$.¹ In §8 we demonstrate that $\mathcal{E}(X, \mu)$ is homotopy equivalent to the stratum consisting of ellipse interiors that meet at least three cone points. In §9, we prove that the completion of this stratum is naturally a (non-manifold) 2-dimensional cell complex whose 2-cells are convex polygons.

We show that the topology of this polygonal complex and the geometry of the immersed ellipses and cylinders that serve as its vertices together encode the geometry of (X, μ) up to homothety.

Theorem 1.1. *Suppose that there is a homeomorphism Φ that maps the polygonal complex associated to (X, μ) onto the polygonal complex associated to (X', μ') . If for each vertex U , the ellipses (or strips) U and $\Phi(U)$ differ by a homothety, then (X, μ) and (X', μ') are equivalent up to homothety.*

Affine mappings naturally act on planar ellipses, and hence the group of (orientation preserving) affine homeomorphisms of (X, μ) acts on $\mathcal{E}(X, \mu)$. Because μ is a translation structure, the differential of an affine homeomorphism is a well-defined 2×2 matrix of determinant 1 [Vch89]. The set of all differentials is a discrete subgroup of $SL_2(\mathbb{R})$ that is sometimes called the *Veech group* and is denoted $\Gamma(X, \mu)$. Using Theorem 1.1, one can characterize $\Gamma(X, \mu)$.

Theorem 1.2. *The group $\Gamma(X, \mu)$ consists of the $g \in SL_2(\mathbb{R})$ for which there exists an orientation preserving self homeomorphism of the polygonal complex associated to (X, μ) such that for each vertex U there exist a homothety h_U such that U differs from $\Phi(U)$ by $h_U \circ g$.*

The group $\Gamma(X, \mu)$ is closely related to the subgroup of the mapping class group of X that stabilizes the Teichmüller disc associated to (X, μ) [Vch89]. To be precise, each mapping class in the stabilizer has a unique representative that is affine with respect to μ . The Veech group is the set of differentials of these affine maps, and is isomorphic to the stabilizer modulo automorphisms. In particular, if there are no nontrivial automorphisms in the stabilizer, then the quotient of the hyperbolic plane by a lattice Veech group is isometric to a Teichmüller curve.

There is a natural map that sends each ellipse interior $U \subset \mathbb{R}^2$ to the coset of $SO(2) \setminus SL_2(\mathbb{R})$ consisting of g such that $g(U)$ is a disc. This map naturally determines a map from $\mathcal{E}(X, \mu)$ onto the Poincaré disc. The image of the 1-skeleton of the polygonal cell complex determines a tessellation of the upper half-plane that coincides with a tessellation defined by William Veech [Vch99] [Vch04] and independently Joshua Bowman [Bwm08]. Indeed, our work began with a reading of a preprint of [Vch04]. In a companion paper [BrtJdg11], we will discuss these connections in more detail.

To develop the deformation theory of immersed ellipses, we use quadratic forms on \mathbb{R}^3 . Each ellipse interior in an affine plane $P \subset \mathbb{R}^3$ is a sublevel set, $\{\vec{x} \in P \mid q(\vec{x}) < 0\}$, of a quadratic form q . The space of quadratic forms on \mathbb{R}^3 is a six dimensional real vector space, and two quadratic forms determine the same ellipse interior if and only if they differ by a positive scalar.

A cylinder is the image of an isometric immersion of a *strip*, the interior of the convex hull of two parallel lines in \mathbb{R}^2 . A strip is also a sublevel set of a quadratic form q restricted to an affine plane P .

¹To be precise one must lift to the universal cover before counting. See §6.

The language of quadratic forms unifies the treatment of immersed ellipses and cylinders. We define a planar *subconic* to be a sublevel set $\{(x, y) \mid q(x, y, 1) < 0\}$ where q is a quadratic form on \mathbb{R}^3 . The set of planar subconics includes ellipse interiors and strips, but also includes ellipse exteriors, parabola interiors, etc.

We will let $\mathcal{S}(X, \mu)$ denote the space of immersions of subconics into a translation surface (X, μ) . We let $\mathcal{S}_n(X, \mu)$ denote the stratum consisting of subconics that meet at least n cone points.

The translation structure μ determines a canonical Euclidean metric on the surface X . The completion of X with respect to this metric will be denoted by \bar{X} . If A is a subset of X , then ∂A will denote the complement of the interior of A in its closure in \bar{X} . In particular, ∂X we will denote the set of cone points.

Other notation can be found in the table that is located at the end of this introduction.

Outline of paper. In sections §2 through §5 we establish notation and introduce basic tools. In §2 we recall the basic theory of translation surfaces including the developing map, and in §3 we recall some basic facts about quadratic forms. In §4, we collect elementary facts about the subconics in the plane. In §5 we consider subconics in the plane determined by finite sets of points.

In §6 we define the geometry of the space of immersed subconics in a translation surface. We show, for example, that $\mathcal{S}(X, \mu)$ is naturally a 5-dimensional real-projective manifold.

Beginning in §7 we restrict attention to the translation surfaces that cover a precompact translation surface. For such surfaces, the only possible subconics are ellipse interiors and (immersed) strips. We show that $\mathcal{S}_5(X, \mu)$ is discrete and that only finitely many members of $\mathcal{S}_5(X, \mu)$ contain a given nonempty open subset of X .

In §8 we show that the subspace $\mathcal{E}_3(X) \subset \mathcal{S}_3(X)$ consisting of ellipse interiors in homotopy equivalent to X . As a consequence, $\mathcal{E}_3(\tilde{X}, \tilde{\mu})$ is the universal cover of $\mathcal{E}_3(X, \mu)$. This fact is used crucially in our proof of Theorem 1.1.

In §9 we define the cell structure on $\mathcal{S}_3(X, \mu)$. In particular, each 2-cell \mathcal{S}_Z corresponds to a triple $Z \subset \partial X$ that defines a triangle in \bar{X} . We show that $\bar{\mathcal{S}}_Z$ may be regarded as the convex hull of all subconics that lie in $\bar{\mathcal{S}}_Z \cap \mathcal{S}_5(X, \mu)$. In §10, we characterize those triples (resp. quadruples) in ∂X that determine a 2-cell (resp. 1-cell) in $\mathcal{S}_3(X, \mu)$.

The set $\bar{\mathcal{S}}_Z \cap \mathcal{S}_5(X, \mu)$ determines an orientation of each 2-cell. In §11, we relate this orientation to the ordering of $\partial U \cap \partial X$ where $\bar{\mathcal{S}}_Z \cap \mathcal{S}_5(X, \mu)$. In §12 we study the oriented link of each vertex in the 2-skeleton. For example, we show that the link of U is determined up to isomorphism by the cardinality of $\partial U \cap \partial X$. In §13, we use the analysis of the oriented links to show that a homeomorphism $\Phi : \mathcal{S}(X, \mu) \rightarrow \mathcal{S}(X', \mu')$ determines a unique bijection $\beta : \partial X \rightarrow \partial X'$ such that for each $Z \subset \partial U \cap \partial X$, we have $\Phi(\mathcal{S}_Z) = \mathcal{S}_{\beta Z}$. Note that the results of these three sections are purely combinatorial.

In §14 we prove the main geometric lemma. Roughly speaking, the sides of a cyclic polygon are determined up to translation by the directions of the bisectors of successive diagonals in the polygon. In §15, we combine this geometric Lemma with the combinatorial results to prove Theorem 1.1. In §16, we prove Theorem 1.2.

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Notation. For the convenience of the reader, we include the following list of notation.

X	§2	topological surface
\tilde{X}	§2	universal cover of X
$\text{Gal}(\tilde{X}/X)$	§6	group of covering transformations
μ	§2	translation atlas/structure
dev_μ	§2	developing map associated to μ
\bar{X}	§2	metric completion of X
∂X	§2	frontier of X
$\text{Card}(F)$	§3	cardinality of a set F
$Q(\mathbb{R}^n)$	§3	vector space of real quadratic forms on \mathbb{R}^n
Q_F	§3	quadratic forms that vanish on F
\underline{q}	§4	restriction of quadratic form q to first two coordinates
$\widehat{(x_1, x_2)}$	§4	$(x_1, x_2, 1)$
U_q	§4	subconic, set of x such that $q(\hat{x}) < 0$
∂U_q	§4	boundary of subconic
\mathcal{S}_Z	§5	set of subconics U such that $Z \subset \partial U \cap \partial X$
\mathcal{E}_Z	§5	set of ellipse interiors U such that $Z \subset \partial U \cap \partial X$
$\mathcal{S}(X, \mu)$	§6	set of subconics in a translation surface
$\mathcal{S}_n(X, \mu)$	§6	stratum of $\mathcal{S}(X, \mu)$
$\mathcal{E}(X, \mu)$	§8	set of ellipse interiors in a translation surface
$\mathcal{E}_n(X, \mu)$	§8	stratum of $\mathcal{E}(X, \mu)$
s_U	§11	successor function of $\partial U \cap \partial X$
$Z_U(x, x')$	§11	$\{x, s_U(x), x', s_U(x')\}$
$\text{Lk}(U)$	§12	link of the vertex U in $\mathcal{S}_3(X, \mu)$
Φ	§13	isomorphism of polygonal cell complexes
β_U	§13	map on $\partial U \cap \partial X$
$[U]$	§14	orbit of U under the group generated by homotheties and translations.

2. PRELIMINARIES ON TRANSLATION SURFACES

Recall that a *translation atlas* on a surface X is an atlas $\{\mu_\alpha : U_\alpha \rightarrow \mathbb{R}^2\}$ such that each transition map $\mu_\beta \circ \mu_\alpha^{-1}$ is a translation. A translation structure μ on X is an equivalence class of translation atlases where two atlases are equivalent if they agree on their common refinement up to translations.

Let (X, μ) and (Y, ν) be translation surfaces. A mapping $f : X \rightarrow Y$ is called a translation map with respect to μ and ν —or simply a *translation*—iff for each chart μ_α in μ and ν_β in ν , the map $\nu_\beta \circ f \circ \mu_\alpha^{-1}$ is a translation.

One may define a metric structure on (X, μ) in several equivalent ways. One approach is to simply pull back the Riemannian metric tensor $dx^2 + dy^2$ on the Euclidean plane via a translation atlas. Since translations preserve $dx^2 + dy^2$, the

resulting metric tensor on X is well-defined. Thus, μ determines a (Riemannian) distance function $d_\mu : X \times X \rightarrow \mathbb{R}$.

Let \overline{X} denote the metric completion of X with respect to d_μ .

Definition 2.1. The *frontier* of X with respect to μ is the set $\partial X := \overline{X} \setminus X$.

A translation structure pulls-back under a covering map. In particular, one associates to each translation structure μ on X a translation structure $\tilde{\mu}$ on the universal cover \tilde{X} .

The following assumption will be implicit throughout this paper.

Assumption 2.2. *The frontier ∂X is a discrete subset of \overline{X} . Moreover, the frontier $\partial \tilde{X}$ of the universal cover \tilde{X} is a discrete subset of the completion of \tilde{X} .*

Assumption 2.2 implies that the frontier of a precompact² translation surface is finite. In addition, a precompact translation surface has finite genus.

A translation structure is an example of a geometric structure in the sense of [Thurston]. If X is simply connected, for each translation structure μ on X , there exists a map $\mathbf{dev}_\mu : X \rightarrow \mathbb{R}^2$ such that the set of restrictions to convex sets is an atlas that represents μ . This *developing map* is unique up to post-composition by a translation. Note that \mathbf{dev}_μ extends uniquely to a continuous function on \overline{X} . We will abusively use \mathbf{dev}_μ to denote this extension.

In general, the developing map is not injective. But the restriction of \mathbf{dev}_μ to a ‘star convex’ subset is injective. To be precise, a set $Y \subset X$ will be said to be *star convex* with respect to x if and only if for each $x' \in Y$, there exists a geodesic segment in Y joining x' to x . A subset $Y \subset X$ will be called *convex* if and only if it is star convex with respect to each of its points.

It will be convenient to extend the notions of convexity and star convexity to subsets of the completion \overline{X} . We will say that $Y \subset \overline{X}$ is star convex with respect to y if and only if for each $y' \in Y$, there exists a convex subset $C \subset X$ such that $y, y' \in \overline{C}$. A set $Y \subset \overline{X}$ will be called *convex* if and only if it is star convex with respect to each $y \in Y$.

Remark 2.3. Our notions of convexity for subset of \overline{X} are defined in a way that assures that the restriction of the developing map is injective. Note that the definition differs significantly from the one that is natural if \overline{X} is regarded as a length space. If ∂X is discrete, then the geodesics in the length space \overline{X} are finite concatenations of completions of geodesic segments in X . From the perspective of length spaces, each geodesic is (star) convex, but from our viewpoint many geodesics are not (star) convex.

Let (X, μ) be a simply connected translation surface. Given $A \subset X$, define the *inradius* of A with respect to μ to be the supremum of the radii of disks that can be embedded into A .

Proposition 2.4. *If (X, μ) is the universal cover of a precompact translation surface with nonempty discrete frontier, then the inradius of X is finite.*

Proof. By assumption we have a precompact translation surface (Y, ν) and a covering $p : X \rightarrow Y$ such that $p^*(\nu) = \mu$. Let $d_X : \overline{X} \times \overline{X} \rightarrow \mathbb{R}$ and $d_Y : \overline{Y} \times \overline{Y} \rightarrow \mathbb{R}$

²Here we use ‘precompact’ as a synonym for totally bounded so that completing the space gives a compact space.

denote the respective distance functions on the respective completions of X and Y . Note that p is a local isometry with respect to these distance functions.

The function $y \mapsto d_Y(y, \partial Y)$ is continuous. Thus since \bar{Y} is compact, there exists K_Y such that for all $y \in Y$, we have $d_Y(y, \partial Y) < K_Y$.

For each $x \in X$, we have $d_X(x, \partial X) = d_Y(p(x), \partial Y)$. Indeed, since ∂Y is finite, there exists a geodesic segment γ joining $p(x)$ to ∂Y whose length equals $d_Y(p(x), \partial Y)$. This geodesic segment lifts under p to a unique path $\tilde{\gamma}$ that joins x to ∂X . Since one endpoint of $\tilde{\gamma}$ lies in ∂X and (X, μ) is a translation surface, the restriction of p to $\tilde{\gamma}$ is an isometry onto γ . In particular, the length of $\tilde{\gamma}$ equals the length of γ . Thus, $d_X(x, \partial X) = d_Y(p(x), \partial Y)$.

Therefore, for each $x \in X$, we have $d_X(x, \partial X) < K_Y$. It follows that if D is a Euclidean disk embedded in X , then the radius of D is less than K_Y . \square

3. QUADRATIC FORMS ON \mathbb{R}^3

We recall some basic facts about quadratic forms defined over the real numbers. See chapter 14 in [Berger].

Let $Q(V)$ denote the vector space of all real-valued quadratic forms on a real vector space V . Given $q \in Q(V)$ we let $q(\cdot, \cdot)$ denote its polarization. There is a basis for V such that the matrix associated to the polarization is diagonal with entries belonging to $\{+1, -1, 0\}$ (Sylvester's law). Let $n_+(q), n_-(q)$, and $n_0(q)$ denote, respectively, the number of diagonal entries that are $+1$, -1 , and 0 . We say that q has *signature* $(n_+(q), n_-(q))$. If $n_0(q) \neq 0$, then we say that q is *degenerate*.

The *radical*, $\text{rad}(q)$, of (the polarization of) the quadratic form q is the set of v such that $q(v, w) = 0$ for all $w \in \mathbb{R}^3$. The form q is nondegenerate iff only $\text{rad}(q) = \{0\}$. If $\text{rad}(q) \neq \{0\}$, then 0 is an eigenvalue and $\text{rad}(q)$ is the associated eigenspace.

We will be interested in quadratic forms that vanish on a prescribed subset $F \subset \mathbb{R}^3$. Define

$$Q_F = \{q \in Q(\mathbb{R}^3) \mid q(F) = \{0\}\}.$$

Note that Q_F is a vector subspace.

To each point $v \in \mathbb{R}^3 \setminus \{0\}$ we may associate the unique line $\ell(v)$ containing v and the origin. We say that a set of points $F \subset \mathbb{R}^3$ is *in general position* if no three of the lines in $\ell(F)$ are coplanar.

To construct quadratic forms that vanish on subsets of $F \subset \mathbb{R}^3$, one can use the collection, $(\mathbb{R}^3)^*$ of linear forms on \mathbb{R}^3 . Given linearly independent vectors $v, w \in \mathbb{R}^3$, let $L_{vw} \subset (\mathbb{R}^3)^*$ be the collection of linear forms that vanish on the plane spanned by v and w . Note that a product $\eta \cdot \eta'$ of linear forms $\eta \in L_{xy}$ and $\eta' \in L_{zw}$ is a quadratic form in $Q_{\{x,y,z,w\}}$.

Let $\text{Card}(F)$ denote the cardinality of a set F .

Proposition 3.1. *If $\text{Card}(F) \leq 5$ and F is general position, then $\dim(Q_F) = 6 - \text{Card}(F)$.*

Proof. Let $F = \{x_1, \dots, x_n\}$ where $n = \text{Card}(F)$. Note that for each i , the map $q \mapsto q(v_i)$ is a linear functional on $Q(\mathbb{R}^3)$. The set Q_F is the kernel of the homomorphism $\phi : Q(\mathbb{R}^3) \rightarrow \mathbb{R}^n$ defined by

$$\phi(q) = (q(v_1), \dots, q(v_n)).$$

Since $\dim(Q(\mathbb{R}^3)) = 6$, it suffices to show that ϕ has full rank. If $n = 1$, then this is true. If $1 < n \leq 5$, then suppose that there exist $(a_1, \dots, a_n) \in \mathbb{R}^n$ such that for each $q \in Q(\mathbb{R}^3)$ we have $\sum_i a_i \cdot q(v_i) = 0$.

Since F is in general position, for each j there exists $q_j \in Q(\mathbb{R}^3)$ such that $q_j(v_j) \neq 0$ and $q_j(v_i) = 0$ if $i \neq j$. For example, if $n = 5$ and $j = j_1$, then choose $\eta \in L_{v_{j_2}, v_{j_3}}, \eta' \in L_{v_{j_4}, v_{j_5}}$, and set $q = \eta \cdot \eta'$.

Hence $a_j \cdot q_j(v_j) = 0$ and therefore $a_j = 0$ for each j . Thus, ϕ has full rank. \square

We also have the following variant of the preceding proposition that will be used in §10. Let $dq_x : \mathbb{R}^3 \rightarrow \mathbb{R}$ denote the differential of q at x .

Proposition 3.2. *Let $n = 3$ or 4 , and let $\{v_1, \dots, v_n\} \subset \mathbb{R}^3 \setminus \{0\}$ be in general position. Let $\{w_1 \dots w_{5-n}\}$ be a subset of $\mathbb{R}^3 \setminus \{0\}$ such that for each i, k , the vector w_k does not belong to the plane spanned by v_k and v_i . Then the set of quadratic forms q with $q(v_i) = 0$ and $dq_{v_i}(w_i) = 0$ is a 1-dimensional vector space.*

Proof. The set in question is the kernel of the homomorphism $\phi : Q(\mathbb{R}^3) \rightarrow \mathbb{R}^5$ defined by

$$\phi(q) = (q(v_1), \dots, q(v_n), dq_{v_1}(w_1), \dots, dq_{v_{5-n}}(w_{5-n})).$$

It suffices to show that ϕ has full rank.

Suppose that there exist $(a_1, \dots, a_n) \in \mathbb{R}^n$ and (b_1, \dots, b_{5-n}) such that for each $q \in Q(\mathbb{R}^3)$ we have

$$\sum_i a_i \cdot q(v_i) + \sum_k b_k \cdot dq_{v_k}(w_k) = 0.$$

Since F is in general position and each w_k does not belong to the plane spanned by v_k and v_i , for each $i = 1, \dots, n$, there exists $q_i \in Q(\mathbb{R}^3)$ such that $q_i(v_i) \neq 0$ but $q_i(v_j) = 0$ for each $j \neq i$ and $d(q_i)_{v_k}(w_k) = 0$ for each k . Similarly, for each $k = 1, \dots, 5 - n$, there exists q'_k so that $d(q'_k)_{v_k}(w_k) \neq 0$ but $q'_k(v_j) = 0$ for each j and $dq_{v_j}(w_j) = 0$ for each $j \neq k$.

It follows that $a_i = 0$ for each i and $b_k = 0$ for each k . Thus, ϕ has full rank. \square

Let $F \subset \mathbb{R}^3$ be a triple. For each $x \in F$, define $D_x \subset Q_F$ to be the set of quadratic forms d such that $d(x, v) = 0$ for all $v \in \mathbb{R}^3$. Note that the set D_x is a 1-dimensional subspace. If $d \in D_x$ and $d \neq 0$, then

$$d^{-1}\{0\} = \bigcup_{y \in F \setminus \{x\}} \langle x, y \rangle.$$

where $\langle x, y \rangle$ is the plane spanned by x and y .

Lemma 3.3. *Let $F = \{v_1, v_2, v_3\}$ and let $d_i \in D_{v_i}$. If the triple F is in general position, then $\{d_1, d_2, d_3\}$ is a basis for Q_F .*

We will call $\{d_1, d_2, d_3\}$ a *natural basis* for Q_F .

Proof. Suppose that there exist $a_i \in \mathbb{R}$ such that $\sum a_i \cdot d_i(v) = 0$ for all $v \in \mathbb{R}^3$. Let $\{i, j, k\}$ be distinct indices. Note that both d_i and d_j vanish identically on the span of $\{v_i, v_j\}$, but since $\ell(F)$ is not coplanar, the form d_k does not. It follows that $a_k = 0$. Hence the forms $\{d_1, d_2, d_3\}$ are independent, and the claim follows from Proposition 3.1. \square

Corollary 3.4. *If the triple F is in general position and $q \in Q_F \setminus \{0\}$ is degenerate, then q is of type $(1, 1)$.*

If F is a triple in general position, then the vector space Q_F has a canonical orientation. Indeed, note that since F is in general position, then F has a canonical cyclic ordering: If $F = \{x, y, z\}$, then we say that y follows x if and only if $(y-x, z-x, x)$ is an oriented basis for \mathbb{R}^3 . We say that a natural basis $\{q_x \in Q_x \mid x \in F\}$ is *negative* if and only if for each p in the interior of the convex hull of F and each $x \in F$ we have $q_x(p) < 0$. We say that an ordered negative natural basis (d_1, d_2, d_3) is *oriented* if and only if the corresponding ordered triple $(x_1, x_2, x_3) \in F^3$ is cyclically ordered. In general, a basis of Q_F is oriented if and only if it differs from an oriented negative basis by a linear transformation having positive determinant.

We now turn to quadruples $F \subset \mathbb{R}^3$.

Lemma 3.5. *If the quadruple F is in general position, then the set of degenerate quadratic forms in Q_F is a union of three distinct lines.*

Proof. Let $F = \{v_1, v_2, v_3, v_4\}$. For $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$, define $Q_{ij,kl}$ to be the set of pairwise products of linear forms from $L_{v_i v_j}$ and $L_{v_k v_\ell}$. Each $Q_{ij,kl}$ is a line in Q_F , and there are three distinct such lines.

Let $q \in Q_F$ be degenerate. It follows from Corollary 3.4 that q is of type $(1, 1)$. Thus, there exist linear forms $\eta, \eta' \in (\mathbb{R}^3)^*$ such that $q = \eta \cdot \eta'$. Since $q \in Q_F$, for each $i = 1, 2, 3$, either $\eta(v_i) = 0$ or $\eta'(v_i) = 0$. Since F is in general position, η and η' each vanish on exactly two points in F and since $q \in Q_F$ these pairs are distinct. Thus, q belongs to one of the distinct lines. \square

4. SUBCONICS IN THE PLANE

Recall that a conic section is the intersection of a hyperplane in \mathbb{R}^3 and the zero level set of a quadratic form $q \in Q(\mathbb{R}^3)$. In the context of translation surfaces, we are mainly interested in the intersection of the sublevel set $q^{-1}\{(-\infty, 0)\}$ and a hyperplane. For simplicity, we choose this hyperplane to be $\{(x_1, x_2, 1) : (x_1, x_2) \in \mathbb{R}^2\}$. The following notation will be convenient.

Notation 4.1. Given a point $x = (x_1, x_2) \in \mathbb{R}^2$, let $\hat{x} = (x_1, x_2, 1)$. Given a subset $Z \subset \mathbb{R}^2$, let $\hat{Z} = \{\hat{x} \mid x \in Z\}$.

Definition 4.2. The *subconic associated to q* is the set

$$U_q = \{x \in \mathbb{R}^2 \mid q(\hat{x}) < 0\}.$$

The subconic U_q is said to be *nontrivial* if and only if U_q is a proper subset of \mathbb{R}^2 . The *boundary* of U_q is the set

$$\partial U_q = \{x \in \mathbb{R}^2 \mid q(\hat{x}) = 0\}.$$

Given a quadratic form $q \in Q(\mathbb{R}^3)$, let \underline{q} denote the restriction of q to the plane $\{(x, y, 0) \mid (x, y) \in \mathbb{R}^2\}$. Subconics can be classified according to the signatures of q and \underline{q} .

Definition 4.3. A subconic U_q is called an *ellipse interior* if and only if q has signature $(2, 1)$ and \underline{q} has signature $(2, 0)$.

Definition 4.4. A subconic U_q is called a *strip* if and only if q has signature $(1, 1)$ and \underline{q} has signature $(0, 0)$.

Definition 4.5. A subconic U_q is called a *half-plane* if and only if q has signature $(1, 1)$ and \underline{q} has signature $(0, 0)$.

Definition 4.6. A subconic U_q is called a *parabola interior* if and only if q has signature $(2, 1)$ and \underline{q} has signature $(1, 0)$.

Remark 4.7. In this paper, we will be almost exclusively concerned with ellipse interiors and strips. See Proposition 7.2.

The following propositions are elementary.

Proposition 4.8. *A non-trivial subconic is convex if and only if it is an ellipse interior, a parabola interior, a strip, or a half-plane.*

Proposition 4.9. *A nontrivial subconic is bounded if and only if it is an ellipse interior.*

Let $\mathcal{S}(\mathbb{R}^2)$ denote the set of all nontrivial subconics. The natural topology on $Q(\mathbb{R}^3)$ induces a topology on $\mathcal{S}(\mathbb{R}^2)$. Namely, a collection of subconics is said to be open if and only if the set of corresponding quadratic forms is open.

The following propositions are elementary.

Proposition 4.10. *The set of ellipse interiors is open in $\mathcal{S}(\mathbb{R}^2)$, and its frontier equals the set of parabola interiors, strips, and half-planes.*

Proposition 4.11. *The set of quadratic forms associated to ellipse interiors is convex and is preserved by the \mathbb{R}^+ action.*

5. SUBCONICS DETERMINED BY A FINITE SET OF POINTS

Given a finite set $Z \subset \mathbb{R}^2$, let \mathcal{S}_Z denote the set of nontrivial subconics whose boundary contains Z , and let Q_Z denote the set of quadratic forms q such that $q(\hat{Z}) = \{0\}$.

Note that $\mathcal{S}_Z \subset \{U_q \mid q \in Q_Z\}$. The opposite inclusion does not hold in general. For example, if Z lies on a line, then let η be a nontrivial linear form that vanishes on a plane containing \hat{Z} . The quadratic form η^2 belongs to Q_Z but U_{η^2} is the empty set.

Lemma 5.1. *Suppose that $Z \subset \mathbb{R}^2$ is not collinear. If U_q is empty for some $q \in Q_Z$, then $q \equiv 0$. In particular, $\mathcal{S}_Z = \{U_q \mid q \in Q_Z \setminus \{0\}\}$.*

Proof. Suppose that $U_q = \emptyset$ for some $q \in Q_Z$. Given a line $\ell \subset \mathbb{R}^2$, the restriction of q to $\hat{\ell}$ is a nonnegative quadratic function. Thus, if x and y are distinct points in ℓ such that $q(\hat{x}) = 0 = q(\hat{y})$, then q vanishes identically on $\hat{\ell}$.

Let z_1, z_2 , and z_3 be three noncollinear points in Z . Since $q \in Q_Z$, the form q vanishes on the lines determined by the pairs z_i, z_j . It follows that q vanishes on the plane $\{(x_1, x_2, 1) \mid x_1, x_2 \in \mathbb{R}\}$. Hence $q \equiv 0$. \square

Let $Z \subset \mathbb{R}^2$ be a noncollinear triple. Then \hat{Z} is in general position. Let $\vec{d} = (d_1, d_2, d_3)$ be an oriented negative natural basis for Q_Z . (See §3). Let $T_{\vec{d}} \subset Q_Z$ denote the plane

$$T_{\vec{d}} = \left\{ \sum_i t_i \cdot d_i \mid \sum_i t_i = 1 \right\}.$$

Let $r_{\vec{d}}: T_{\vec{d}} \rightarrow \mathcal{S}_Z$ denote the restriction of the map $q \mapsto U_q$ to $T_{\vec{d}}$.

Proposition 5.2. *$r_{\vec{d}}$ is a homeomorphism from $T_{\vec{d}}$ onto \mathcal{S}_Z .*

Proof. Since $\vec{d} = \{d_1, d_2, d_3\}$ is a basis for Q_Z , Lemma 5.1 implies that map $q \mapsto U_q$ is onto \mathcal{S}_Z . Note that $U_q = U_{q'}$ if and only if there exists $\lambda \in \mathbb{R}^+$ such that $\lambda \cdot q = q'$. It follows that $r_{\vec{d}}$ is injective. The continuity of $q \mapsto U_q$ and the inverse of its restriction follow from linearity and the definitions of the various topologies. \square

The plane $T_{\vec{d}} \subset \mathbb{R}^3$ has a canonical affine structure and a canonical outward normal orientation. In particular, the ordered set $(d_2 - d_1, d_3 - d_1)$ is an oriented basis for the tangent space to $T_{\vec{d}}$ at d_1 .

Proposition 5.3. *Let \vec{d} and \vec{d}' be oriented negative natural bases for Q_Z . The map $r_{\vec{d}}^{-1} \circ r_{\vec{d}'} : T_{\vec{d}'} \rightarrow T_{\vec{d}}$ is an orientation preserving affine map.*

Proof. Since $\vec{d} = (d_1, d_2, d_3)$ and $\vec{d}' = (d'_1, d'_2, d'_3)$ are both oriented negative degenerate bases, there exists there exists a 3-cycle $\sigma \in S(\{1, 2, 3\})$ and $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{R}^3)^+$ such that for $i = 1, 2, 3$

$$d_i = \lambda_i \cdot d_{\sigma(i)}.$$

Define a linear map $A : Q_Z \rightarrow Q_Z$ by setting $A(d_i) = \lambda_i \cdot d'_{\sigma(i)}$. Since $\lambda_i > 0$ and σ is a 3-cycle, the map A is orientation preserving. The map $r_{\vec{d}'} \circ r_{\vec{d}}^{-1}$ is the restriction of the A to the plane $T_{\vec{d}'}$. The claim follows. \square

Let \mathcal{E}_Z denote the collection of ellipse interiors whose boundaries contain Z .

Proposition 5.4. *Let $Z \subset \mathbb{R}^2$ be a noncollinear triple, let \vec{d} be a natural basis for Q_Z , and let $T = T_{\vec{d}}$. The set $\{q \in T \mid U_q \in \mathcal{E}_Z\}$ is a bounded, convex subset of the plane T . In particular, $\overline{\mathcal{E}_Z}$ is compact, and $\{q \in T \mid U_q \in \overline{\mathcal{E}_Z}\}$ is a compact convex subset of T .*

Proof. The set Q_Z is a vector subspace, and hence it follows from Proposition 4.11 that the set of all q such that $U_q \in \mathcal{E}_Z$ is a convex subset of $Q(\mathbb{R}^3)$. Thus, $\{q \in T \mid U_q \in \mathcal{E}_Z\}$ is convex.

Let $(\hat{z}_1, \hat{z}_2, \hat{z}_3)$ be the cyclic ordering of the elements of \hat{Z} with respect to the orientation of \mathbb{R}^3 . Let $\vec{d} = (d_1, d_2, d_3)$ be a corresponding oriented negative natural basis for Q_Z . To show that $\{q \in T \mid U_q \in \overline{\mathcal{E}_Z}\}$ is bounded, it suffices to show that if $U_q \in \mathcal{E}_Z$ and $q \in T_{\vec{d}}$, then each coordinate of q with respect to \vec{d} is positive.

For $\{i, j, k\} = \{1, 2, 3\}$, let σ_i denote the segment joining z_j and z_k . Since $U_q \in \mathcal{E}_Z$ is strictly convex and $Z \subset \partial U_q$, we have $\sigma_i \subset \overline{U_q}$ and $\sigma_i \cap \partial U_q = \{x_j, x_k\}$. In particular, for each $y \in \sigma_i \setminus \hat{Z}$ we have $q(\hat{y}) < 0$.

Let (t_1, t_2, t_3) be the coordinates of q with respect to \vec{d} . Since \vec{d} is a negative natural base, for each $y \in \sigma_i \setminus Z$ we have $q(\hat{y}) < 0$. It follows that $t_i > 0$ for each $i = 1, 2, 3$. \square

6. SUBCONICS IN A TRANSLATION SURFACE

In this section, we make precise the notion of subconic in a translations surface (X, μ) , we define the space of subconics in (X, μ) and its stratification, and we derive some basic properties.

Let $\text{Gal}(\tilde{X}/X)$ denote the group of covering transformations associated to the universal covering $p : \tilde{X} \rightarrow X$. Each covering transformation is a translation mapping of $(\tilde{X}, \tilde{\mu})$ and hence $\text{Gal}(\tilde{X}/X)$ acts on the set of convex subsets $\tilde{U} \subset \tilde{X}$ such that $\mathbf{dev}_\mu(\tilde{U})$ is a subconic in \mathbb{R}^2 .

Definition 6.1. A *subconic* in X with respect to μ is an orbit of the action of $\text{Gal}(\tilde{X}/X)$ on the collection of convex subsets \tilde{U} of \tilde{X} such that $\mathbf{dev}_\mu(\tilde{U})$ is a subconic in \mathbb{R}^2 .

There is a natural one-to-one correspondence between the subconics U in X and equivalence³ classes of isometric immersions from a subconic $U' \subset \mathbb{R}^2$ into X . Indeed, each isometric immersion $f : U' \rightarrow X$ lifts to an immersion $\tilde{f} : U' \rightarrow \tilde{X}$. The lift is unique up to the action of $\text{Gal}(\tilde{X}/X)$. On the other hand, given a subconic $\tilde{U} \subset \tilde{X}$, the map $p \circ \mathbf{dev}_\mu^{-1}$ restricted to $\mathbf{dev}_\mu(\tilde{U})$ gives an immersion.

Each representative \tilde{U} of a subconic U will be called a *lift* of U . In most cases, a subconic in X is determined by the image a lift under p . For example, if X is simply connected, then each subconic has a unique lift. In this case, we will identify the singleton set with the element that it contains.

On the other hand, there are situations in which one must keep track of the orbit in the universal cover (or, equivalently, the immersion). For example, if X is the once-punctured torus, $(\mathbb{R}^2 \setminus \mathbb{Z}^2)/\mathbb{Z}^2$, then the set X is itself the image of two ellipses that do not differ by an element of $\text{Gal}(\tilde{X}/X)$.⁴

In practice, we will be working with a particular lift \tilde{U} in the universal cover, and we will regard the image $p(\tilde{U})$ as a subconic in X .

We will use the terminology used in the classification of subconics in the plane to describe subconics in X with respect to μ .

Remark 6.2. An isometric embedding of a cylinder, $[a, b] \times (\mathbb{R}/c \cdot \mathbb{Z})$, corresponds to a strip in a translation surface (X, μ) .

If X is simply connected, the developing map furnishes a natural topology on $\mathcal{S}(\tilde{X}, \mu)$. A set $\tilde{\mathcal{U}} \subset \mathcal{S}(\tilde{X}, \mu)$ is said to be open if and only if $\mathbf{dev}_\mu(\mathcal{U})$ is open in $\mathcal{S}(\mathbb{R}^2)$. In other words, $\tilde{\mathcal{U}}$ is open if and only if the associated classes of quadratic forms constitute an open set in $Q(\mathbb{R}^3)/\mathbb{R}^+$.

More generally, we endow $\mathcal{S}(X, \mu) = \mathcal{S}(\tilde{X}, \mu)/\text{Gal}(\tilde{X}/X)$ with the quotient topology. The action of the deck group $\text{Gal}(\tilde{X}/X)$ on $\mathcal{S}(\tilde{X}, \mu)$ is discontinuous, and hence the quotient map $p_* : \mathcal{S}(\tilde{X}, \mu) \rightarrow \mathcal{S}(X, \mu)$ is a covering map with deck group $\text{Gal}(\tilde{X}/X)$.

Roughly speaking, the space $\mathcal{S}(X, \mu)$ has a natural stratification determined by the number of points in the boundary, $\partial\tilde{X}$, that are met by a lift \tilde{U} . To be more precise we make the following definition.

Definition 6.3. We say that a subset $A \subset \mathbb{R}^d$ has *maximal span* iff there exists a subset $B \subset A$ such that the span of $\{b - b' \mid b, b' \in B\}$ has dimension at least as large as either d or $\text{Card}(A) - 1$.

If $\text{Card}(A) \leq d$, then A has maximal span iff A is in general position. If $\text{Card}(A) > d$, then A has maximal span iff the set of differences $\{a - a' \mid a, a' \in \mathbb{R}^d\}$ spans \mathbb{R}^d .

Definition 6.4. For $n \in \mathbb{Z}^+$, define the *n-stratum*, $\mathcal{S}_n(x, \mu)$, to be the collection of all subconics $U \subset X$ such that $\text{Card}(\partial\tilde{U} \cap \partial\tilde{X}) \geq n$ and $\mathbf{dev}_\mu(\partial\tilde{U} \cap \partial\tilde{X})$ has maximal span in \mathbb{R}^2 .

³Two immersions $f : K \rightarrow X$, $f' : K' \rightarrow X$ are equivalent iff there exists isometry $g : K \rightarrow K'$ so that $f' \circ g = f$.

⁴We thank the referee for pointing out a similar example.

If one component of the boundary of a strip U contains at least three points in ∂X while the other other component contains none, then the strip does not belong to $\mathcal{S}_n(X, \mu)$ for any n . Indeed, in this case, the span of $\mathbf{dev}_\mu(\partial\tilde{U} \cap \partial\tilde{X})$ is 1-dimensional.

Proposition 6.5. *The space $\mathcal{S}(X, \mu) \setminus \mathcal{S}_1(X, \mu)$ is a five dimensional real projective manifold.*

Proof. Since \mathbf{dev}_μ is a local embedding, it follows that the map $U \mapsto \mathbf{dev}_\mu$ is a local embedding from $\mathcal{S}(\tilde{X}, \mu)$ to $\mathcal{S}(\mathbb{R}^2)$. The set $\mathcal{S}(\tilde{X}, \mu) \setminus \mathcal{S}_1(\tilde{X}, \mu)$ is mapped to an open subset of $\mathcal{S}(\mathbb{R}^2)$. Define $f : \mathcal{S}(\mathbb{R}^2) \rightarrow PQ(\mathbb{R}^3)$ by $f(U) = \{q \mid U_{\pm q} = \mathbf{dev}_\mu(U)\}$. Note that f is a local embedding.

The action of the translations on X is \mathbf{dev}_μ -equivariant action of a subgroup of translations on \mathbb{R}^2 . In turn this action is f -equivariant to an action of a subgroup of $PGL(Q(\mathbb{R}^3))$. \square

7. SUBCONICS IN COVERINGS OF PRECOMPACT SURFACES WITH FINITE FRONTIER

In this section and the ones that follow, we make the following assumption.

Assumption 7.1. *We consider only translation surfaces (X, μ) that cover a precompact translation surface with nonempty frontier.*

This assumption limits the types of subconics that can be immersed in X . Recall that an embedded cylinder is a strip. See Remark 6.2.

Proposition 7.2. *Each subconic $U \subset X$ is either a strip or an ellipse interior.*

Proof. Let \tilde{U} be a lift of U to the universal cover \tilde{X} . It follows from Proposition 2.4 and Assumption 7.1 that the inradius of \tilde{X} is finite. The only nontrivial subconics that have finite inradius are strips and ellipse interiors. \square

In order to deal with a strip $U \subset X$, it will prove convenient to construct a canonical ‘maximal’ neighborhood M of U such that the restriction $\mathbf{dev}_\mu|_M$ to M is an injective map.

Definition 7.3. Let M be the collection of all $x \in X$ such that there exists $y \in U$ and a geodesic segment α joining x to y such that the vector $\mathbf{dev}_\mu(x) - \mathbf{dev}_\mu(y)$ is orthogonal to the radical of q where $U_q = \mathbf{dev}_\mu(U)$. We will call M the *canonical neighborhood* of U .

The following is elementary.

Proposition 7.4. *If $U \subset X$ is a strip, then there exists a unique strip that contains every strip that contains U . If $\tau : X \rightarrow X$ is a nontrivial translation and $\tau(U) = U$, then for each connected component C of ∂U , the set $C \cap \partial X$ is infinite and $\tau(C \cap \partial X) = C \cap \partial X$.*

We will call the strip described in Proposition 7.4 the *maximal strip* containing U .

Proposition 7.5. *If (X, μ) is simply connected and $U \subset X$ is a maximal strip, then there exists a nontrivial translation $\tau : X \rightarrow X$ such that $\tau(U) = U$.*

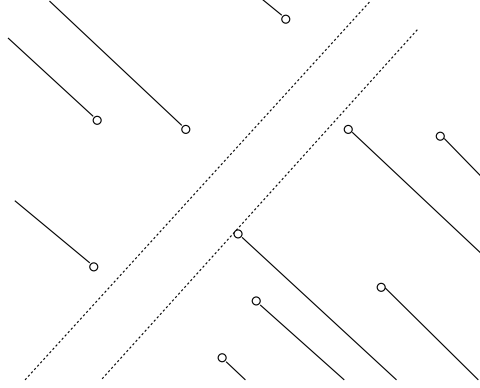


FIGURE 1. The \mathbf{dev}_μ image of a canonical neighborhood of a strip. The circles indicate points that belong to $\mathbf{dev}_\mu(\partial X \cap \overline{M})$.

Proof. Let $p : X \rightarrow Y$ be as in Assumption 7.1. Since X is simply connected, p is the universal covering. Since ∂Y is finite, Y is precompact, and hence $p(U)$ is precompact. Since U is a strip, it is not precompact (Proposition 4.9). Therefore, there exists $x, x' \in U$ and a nontrivial deck transformation $\tau : X \rightarrow X$ such that $\tau(x) = x'$. Because U is maximal and $\tau(\partial X) = \partial X$, we find that $\tau(U)$ is also maximal.

The intersection $\tau(U) \cap U$ is a strip. Since $U \cap \tau(U)$ belongs to both U and $\tau(U)$, it follows from the uniqueness of maximal strips—Proposition 7.4—that $U = \tau(U)$. \square

By combining Proposition 7.4 and 7.5 we obtain the following.

Corollary 7.6. *Each maximal strip (resp. cylinder) belongs to $\bigcap_n S_n(X, \mu)$.*

Proposition 7.7. $\mathcal{S}_5(X, \mu)$ is a discrete and countable subset of $\mathcal{S}(X, \mu)$.

Proof. By assumption there exists a precompact translation surface (X, ν) with finite frontier and a covering $p : X \rightarrow Y$ such that $\mu = p \circ \nu$. It suffices to assume that p is a universal covering. Note that p extends uniquely to a continuous map from \overline{X} to \overline{Y} that we will abusively call p .

Let U be a subconic such that $\partial U \cap \partial X$ has at least five points. Given δ , define the δ -neighborhood of U by

$$N_\delta = \{x \in \overline{X} \mid \text{dist}(y, U) < \delta\}.$$

Since ∂Y is finite, there exists $\epsilon > 0$ such that $p(N_\epsilon) \cap \partial Y = p(\overline{U}) \cap \partial Y$. Thus, we find that

$$(1) \quad N_\epsilon \cap \partial Y = \partial U \cap \partial X.$$

Let $q \in Q(\mathbb{R}^3)$ be such that $\mathbf{dev}_\mu(U) = U_q$. Let $M \subset \mathbb{R}^2$ denote the image of N_ϵ under the developing map. Since \mathbf{dev}_μ is an open mapping, the set M is open. By Proposition 7.2, U is either an ellipse interior or a strip. In each of these cases, we will show that U is isolated in $\mathcal{S}_5(X, \mu)$.

Suppose that U , and hence U_q , is an ellipse interior. Let $x_0 \in U_q$. Let A be the set of all $r \in Q(\mathbb{R}^3)$ such that either $r(\widehat{x_0}) \geq 0$ or there exists $x \in \partial M$ such that

$r(\hat{x}) \leq 0$. Since ∂M is compact, the set A is closed. Since $q(x_0) < 0$ and $U_q \subset M$, the form q does not belong to A .

Let U' be a subconic in X such that $\partial U \cap \partial X$ has at least five elements. Let $q' \in Q(\mathbb{R}^3)$ so that $U_{q'} = \mathbf{dev}_\mu(U')$. We claim that if $U' \neq U$, then $q' \in A$. Note that if $q'(x_0) \geq 0$ held, then we would have $q' \in A$. Thus, we assume, without loss of generality, that $x_0 \in U'$.

It follows from (1) that each element of $\mathbf{dev}_\mu(\partial U' \cap \partial X)$ either belongs to $\mathbf{dev}_\mu(\partial U \cap \partial X)$ or to the complement of M . If $\mathbf{dev}_\mu(\partial U' \cap \partial X)$ were a subset of $\mathbf{dev}_\mu(\partial U \cap \partial X)$, then since $\mathbf{dev}_\mu(\partial U' \cap \partial X)$ contains at least five points, Proposition 3.1 would imply that $U' = U$. In other words, if $U' \neq U$, then there exists $x \in \partial U' \cap \partial X$ that does not belong to M . In particular, $x \in \overline{U}_{q'}$.

Since x and x_0 both belong to the convex set $\overline{U}_{q'}$, there exists a path $\alpha : [0, 1] \rightarrow \mathbb{R}^2$ with $\alpha(0) = x_0$ and $\alpha(1) = x$. Let $t_0 = \sup\{t \mid \alpha(t) \in M\}$. Because $x_0 \in M$ and $x \notin M$, we have that $\alpha(t_0) \in \partial M$. But $q(\alpha(t_0)) \leq 0$ and hence $q \in A'$ as desired.

The image of A under the map $q \rightarrow U_q$ is a closed subset of $\mathcal{S}(\mathbb{R}^2)$ that contains $\mathbf{dev}_\mu(\mathcal{S}_5(X, \mu) \setminus \{U\})$ but does not contain $\mathbf{dev}_\mu(U)$. Hence, if $U \in \mathcal{S}_5(X, \mu)$ is an ellipse interior, then it is an isolated point of $\mathcal{S}_5(X, \mu)$.

If $U \in \mathcal{S}_5(X, \mu)$ is a strip, then it follows from Propositions 7.4 and 7.5 that there exists a nontrivial planar translation τ such that $\tau(Z) = Z$ where $Z = \mathbf{dev}_\mu(\partial U \cap \partial X)$. Since $\tau(Z \cap \partial U) = Z \cap \partial U$, the length of each component of $\partial U \setminus Z$ is at most the translation distance $|\tau|$ of τ .

Let w denote the distance between the boundary components of the strip U_q . Let ℓ be the ‘center-line’ of U_q , namely $\ell := \{x \in U_q \mid \text{dist}(x, \partial U_q) = w/2\}$. Let σ be a compact connected subset of ℓ with length

$$|\sigma| > \frac{\epsilon + w}{\epsilon} \cdot |\tau|.$$

Let $A \subset Q(\mathbb{R}^3)$ be the set consisting of all r such that for some $x \in \sigma$, we $q(\hat{x}) \geq 0$. Note that A is a closed subset of $Q(\mathbb{R}^3)$. Indeed, given a sequence $\{r_n\} \subset A$, let $x_n \in \sigma$ such that $r_n(x_n) \geq 0$. Since σ is compact, a subsequence of x_n converges to some $x \in \sigma$. Thus, if r_n converges to r , then $r(x) \geq 0$.

For each $x \in \sigma$, we have $q(x) < 0$, and hence $q \notin A$. Let $U' \neq U$ be a subconic such that $\partial U \cap \partial X$ contains at least five elements. Let $q' \in Q(\mathbb{R}^3)$ be such that $U_{q'} = \mathbf{dev}_\mu(U')$. It suffices to show that $q' \in A$. Indeed, A is closed and $q \notin A$.

An argument similar to the one given in the case of ellipse interiors shows that since $q \neq q'$ there exists x' such that $\text{dist}(x', U) = \epsilon$ and such that $q(x) = 0$. Since $U_{q'}$ is convex, the set $\sigma' = \ell \cap \overline{U}_{q'}$ is a compact connected set. If $\sigma' = \emptyset$, then $q'(x) = 0$ for all $x \in \ell$, and so $q' \in A$. Therefore, it suffices to assume that $\sigma' \neq \emptyset$.

Let T be the convex hull of x' and the endpoints of σ' . Observe that T is a (perhaps degenerate) triangle. Since $\overline{U}_{q'}$ is convex, the triangle T is a subset of $\overline{U}_{q'}$. Since σ' is parallel to ∂U , we may use a similar triangles argument to show that the segment $\alpha = T \cap \partial U$ has length

$$|\alpha| = |\sigma'| \cdot \frac{\epsilon}{\epsilon + w}.$$

Since $U \cap Z = \emptyset$, the interior of T does not intersect Z . It follows that $|\alpha| < |\tau|$, and hence $|\sigma'| < |\sigma|$. Thus, there exists $x \in \sigma$ such that $q'(x) \geq 0$. Therefore $q' \in A$ as desired. \square

Proposition 7.7 leads us to make the following definition.

Definition 7.8. The elements of $\mathcal{S}_5(X, \mu)$ will be called *rigid subconics*.

The following proposition ensures that each polygonal 2-cell in the cellulation of $\mathcal{S}_3(X, \mu)$ has finitely many vertices. (See §9).

Proposition 7.9. *Let $V \subset X$ be open and nonempty. The set of all rigid subconics that contain V is finite.*

Proof. By Proposition 7.7, it suffices to show that the set of all rigid subconics that contain V is compact. Without loss of generality, we may assume that (X, μ) is a universal covering of a precompact translation surface that has finite frontier.

Let U_n be a sequence of rigid subconics containing V . Since V is nonempty and each U_n is convex, the union $W = \bigcup_n U_n$ is star convex. Thus, the restriction of \mathbf{dev}_μ to W is injective.

Let $\|\cdot\|$ be a norm on the finite dimensional vector space $Q(\mathbb{R}^3)$. For each n , let q_n be such that $U_{q_n} = \mathbf{dev}_\mu(U_n)$ and $\|q\| = 1$. Since the unit sphere is compact, the sequence q_n has a limit point q . Without loss of generality, the sequence q_n converges to q for otherwise take a subsequence. We have $U_q \subset \mathbf{dev}_\mu(W)$, and hence $\mathbf{dev}_\mu|_{\overline{W}}^{-1}(U_q)$ is a subconic in X . It suffices to show that U is rigid.

We first claim that U_q is either an ellipse interior or a strip. To prove this, we will eliminate the other possibilities: U_q is empty, is the complement of a line, is a parabola, or is the plane. For each n , we have $\mathbf{dev}_\mu(V) \subset U_{q_n}$. Thus, since V is open and $q_n \rightarrow q$, we have $V \subset U_q$ and hence U_q is nonempty. By Proposition 2.4, X has finite inradius R . Thus for each n , the inradius of U_{q_n} is at most R , and hence the inradius of U_q is at most R . This eliminates the remaining cases.

Define

$$M_\epsilon = \{x \mid q(\hat{x}) < \epsilon\}.$$

Since $q_n \rightarrow q$, for each $\epsilon > 0$ there exists N_ϵ such that if $n > N_\epsilon$, then $\overline{U_{q_n}} \subset M_\epsilon$. In particular, if $n > N_\epsilon$, then the set $Z_n = \mathbf{dev}_\mu(\partial U_n \cap \partial X)$ is a subset of M_ϵ .

Suppose that U_q is an ellipse interior. Since $q_n \rightarrow q$, Proposition 4.10 implies that—by passing to a tail if necessary—we may assume that for each n , the subconic U_{q_n} is an ellipse that lies in M_1 . In particular, $\mathbf{dev}_\mu(W) \subset M_1$. Since $\overline{M_1}$ is compact, ∂X is discrete, and $\mathbf{dev}_\mu|_W$ is a homeomorphism onto its image, the set $F = \mathbf{dev}_\mu(\overline{W} \cap \partial X)$ is finite.

For each n , let $x_n^1 \in Z_n$. Since $Z_n \subset Z$, there exists an infinite set $A^1 \subset \mathbb{N}$ and $c^1 \in F$ such that if $n \in A^1$, then $x_n^1 = c^1$. For each $\epsilon > 0$, we have $c^1 \in M_\epsilon$ and hence $c^1 \in \mathbf{dev}_\mu(\partial U \cap \partial X)$. Since $\text{Card}(Z_n) > 1$, for each $n \in A^1$, there exists $x_n^2 \in Z_n$ such that $x_n^2 \neq c^1$. Since F is finite, there exists an infinite set $A^2 \subset A^1$ and $c^2 \in F$ such that if $n \in A^2$, then $x_n^2 = c^2$. We have $c^2 \in \mathbf{dev}_\mu(\partial U \cap \partial X)$. By continuing in this way, we find $\{c^1, c^2, c^3, c^4, c^5\} \subset \mathbf{dev}_\mu(\partial U \cap \partial X)$. Hence $U \in \mathcal{S}_5(X, \mu)$.

If U is a strip, then by Proposition 7.4, U belongs to a maximal strip U' . For each $\epsilon > 0$, we have $Z_n \subset M_\epsilon$. Thus for each $\epsilon > 0$, the set $\mathbf{dev}_\mu(U') \subset M_\epsilon$. Hence $\mathbf{dev}_\mu(U') \subset \mathbf{dev}_\mu(U)$ and thus $U' = U$. Thus, by Corollary 7.6, U is a rigid conic. \square

8. THE HOMOTOPY TYPE OF THE SPACE OF ELLIPSE INTERIORS

Let $\mathcal{E}(X, \mu)$ denote the space of ellipse interiors in a translation surface (X, μ) .

Proposition 8.1. *The space $\mathcal{E}(X, \mu)$ is homotopy equivalent to X .*

Proof. Given an ellipse interior U in X , let $\tilde{c}(U)$ be the center of a lift of U to \tilde{X} . Define a map $c : \mathcal{E}(X, \mu) \rightarrow X$ by setting $c(U) = p(\tilde{c}(u))$. Given a point $x \in X$, let $\tilde{D}(x)$ be largest disk contained in \tilde{X} with center at some $\tilde{x} \in p^{-1}(x)$. (Since X is an open manifold, a disk exists, and by Assumption 7.1 and Proposition 2.4 there is a largest such disk). Let $D(x)$ be the orbit of $\tilde{D}(x)$ under $\text{Gal}(\tilde{X}/X)$. It is straightforward to show that the maps $D : X \rightarrow \mathcal{E}(X, \mu)$ and c determine a homotopy equivalence. \square

Proposition 8.2. *There is a deformation retraction from $\mathcal{E}(X, \mu)$ onto $\mathcal{E}_3(\tilde{X}, \tilde{\mu})$.*

Proof. We first define the retraction under the assumption that X is simply connected.

Let X be simply connected. Let $\mathcal{E}_n(X, \mu)$ denote the collection of ellipses $U \subset X$ such that $\partial X \cap \partial U$ contains at least n points. The desired retraction is a concatenation of three retractions: from $\mathcal{E}(X, \mu)$ to $\mathcal{E}_1(X, \mu)$, from $\mathcal{E}_1(X, \mu)$ to $\mathcal{E}_2(X, \mu)$, and from $\mathcal{E}_2(X, \mu)$ to $\mathcal{E}_3(X, \mu)$.

The first retraction consists of dilating the ellipse interior until the boundary meets ∂X . To be precise, let τ be the planar translation that sends the center of $\mathbf{dev}(U)$ to the origin. Let q be the quadratic form such that $U_q = \tau(\mathbf{dev}(U))$. Since the center of U_q is the origin, we have

$$q(v) = a \cdot v_1^2 + \underline{q}(v_2, v_3).$$

where $a < 0$ and \underline{q} is a positive definite form on \mathbb{R}^2 . Define

$$q_t(v) = a \cdot v_1^2 + (1-t) \cdot \underline{q}(v_2, v_3).$$

Note that the area of U_{q_t} increases to infinity as t tends to 1. By Assumption 7.1 and Proposition 2.4, the supremum, t_0 , of the set of t such that $\tau_U^{-1}(U_{q_t})$ is the \mathbf{dev} -image of a lift of a subconic is less than one. By setting

$$(2) \quad f_1(s, U) = \mathbf{dev}_\mu^{-1} \left(\tau^{-1} \left(U_{q_{(s/t_0)}} \right) \right)$$

we obtain a retraction $f_1 : [0, 1] \times \mathcal{E}(X, \mu) \rightarrow \mathcal{E}_1(X, \mu)$.

The deformation retract from $\mathcal{E}_1(X, \mu)$ onto the $\mathcal{E}_2(X, \mu)$ is defined by dilating and translating the center while maintaining contact with ∂X . To be precise, given an ellipse U and $x \in \partial U \cap \partial X$, let τ and q be as before. Let η be a linear form on \mathbb{R}^3 defined as the differential of q at $\widehat{\mathbf{dev}_\mu(\tau(x))}$. In particular, $\eta \left(\widehat{\mathbf{dev}_\mu(\tau(x))} \right) = 0$. Define

$$(3) \quad q_t = q - t \cdot \eta^2.$$

It follows from Assumption 7.1 and Proposition 2.4 that the supremum, t_0 , of the set of t such that $\tau_U^{-1}(U_{q_t})$ is the \mathbf{dev} -image of a lift of a subconic is finite. By defining f_3 as in (2), we obtain a retraction $f_2 : [0, 1] \times \mathcal{E}_1(X, \mu) \rightarrow \mathcal{E}_2(X, \mu)$.

The deformation retract from $\mathcal{E}_2(X, \mu)$ onto the $\mathcal{E}_3(X, \mu)$ is defined by dilating and translating the center while maintaining contact with two points in ∂X . To be precise, given an ellipse U and $x_1, x_2 \in \partial U \cap \partial X$, let τ and q be as before. Let η be a linear form on \mathbb{R}^3 whose kernel is spanned by $\widehat{\mathbf{dev}_\mu(\tau(x_1))}$ and $\widehat{\mathbf{dev}_\mu(\tau(x_2))}$ and so that the value at the midpoint of these two vectors equals 1. Define q_t as in (3). It follows from Assumption 7.1 and Proposition 2.4 that the supremum, t_0 , of the set of t such that $\tau_U^{-1}(U_{q_t})$ is the \mathbf{dev} -image of a lift of a subconic is finite. By defining f_3 as in (2), we obtain a retraction $f_3 : [0, 1] \times \mathcal{E}_1(X, \mu) \rightarrow \mathcal{E}_2(X, \mu)$.

By concatenating f_1 , f_2 and f_3 , we obtain a retract $f : \mathcal{E}(X, \mu) \rightarrow \mathcal{E}_3(X, \mu)$.

Suppose that \tilde{X} is the universal cover of a translation surface (X, μ) . Let $\sigma : \tilde{X} \rightarrow \tilde{X}$ be a translation mapping. Since the translation τ associated to U differs from the translation associated to $\sigma(U)$ by σ , we find that $f_i(s, \sigma(U)) = f_i(s, U)$ for each $U \in \mathcal{E}_{i-1}$. Since each covering transformation is a translation, the retraction f descends to a homotopy retraction from $\mathcal{E}(X, \mu)$ onto $\mathcal{E}_3(X, \mu)$. \square

Corollary 8.3. *If X is connected, then $\mathcal{S}_3(X, \mu)$ is connected.*

Corollary 8.4. *$\mathcal{E}(X, \mu)$ is simply connected if and only if X is simply connected.*

The deck group $\text{Gal}(\tilde{X}/X)$ acts on \tilde{X} as translations. In particular, $\text{Gal}(\tilde{X}/X)$ acts on each stratum $\mathcal{S}_n(\tilde{X}, \tilde{\mu})$. The action is free and discontinuous, and hence the natural projection map $p_* : \mathcal{S}_n(\tilde{X}, \tilde{\mu}) \rightarrow \mathcal{S}_n(X, \mu)$ is a covering.

Corollary 8.5. *The map $p_* : \mathcal{S}_n(\tilde{X}, \tilde{\mu}) \rightarrow \mathcal{S}_n(X, \mu)$ is a universal covering.*

From henceforth we assume that X is connected.

9. A TWO DIMENSIONAL CELL COMPLEX

In this section we will assume that (X, μ) is the universal cover of a precompact translation surface with finite frontier. Here we show that the set, $\mathcal{S}_3(X, \mu)$ is naturally a 2-dimensional polyhedral complex. Moreover, $\mathcal{S}_4(X, \mu)$ is the 1-skeleton and $\mathcal{S}_5(X, \mu)$ is the 0-skeleton. To simplify the exposition, we will sometimes abbreviate $\mathcal{S}_n(X, \mu)$ by \mathcal{S}_n .

Given $Z \subset \partial X$, let $\mathcal{S}_Z(X, \mu)$ denote the set of all subconics U such that $Z = \partial U \cap \partial X$ and such that Z intersects each component of ∂U . Note that $\overline{\mathcal{S}}_Z(X, \mu)$ is the set of all subconics such that $Z \subset \partial U \cap \partial X$. We will abbreviate $\mathcal{S}_Z(X, \mu)$ by \mathcal{S}_Z if the context makes the choice of (X, μ) clear.

Abusing notation slightly, we will let \mathbf{dev}_μ denote the map from $\mathcal{S}(X, \mu)$ to $\mathcal{S}(\mathbb{R}^2)$ defined by $U \mapsto \mathbf{dev}_\mu(U)$.

Lemma 9.1. *If $\text{Card}(Z) \geq 2$, then the restriction of \mathbf{dev}_μ to $\overline{\mathcal{S}}_Z$ is a homeomorphism onto its image in $\mathcal{S}(\mathbb{R}^2)$.*

Proof. Suppose that $\mathcal{S}_Z \neq \emptyset$ and let $\{z, z'\} \subset Z$. Each $U \in \mathcal{S}_Z$ contains the segment σ joining z and z' . It follows that the union, W , of all $U \in \mathcal{S}_Z$ is star convex with respect to σ , for example, the midpoint of σ . In particular, the restriction of the developing map to W is injective. Thus, \mathbf{dev}_μ determines an injection from \mathcal{S}_Z into $\overline{\mathcal{E}(\mathbb{R}^2)}$. Hence since \mathbf{dev}_μ defines the topology of $\mathcal{S}(X, \mu)$, the restriction is a homeomorphism onto its image. \square

Let $Z \subset \partial X$. If there exists a convex subset $P \subset \overline{X}$ with nonempty interior such that $\mathbf{dev}_\mu(P)$ equals the convex hull of $\mathbf{dev}_\mu(Z)$, then we will say that Z defines the polygon P .⁵

We will say that a set of points $A \subset \overline{X}$ is *noncollinear* if and only if $\mathbf{dev}_\mu(A)$ is not a subset of some line.

Proposition 9.2. *If $Z \subset \partial X$ contains three noncollinear points and $\mathcal{S}_Z(X, \mu) \neq \emptyset$, then Z defines a polygon.*

⁵One may naïvely regard the set P as the convex hull of Z . See Remark 2.3.

Proof. Let U be a subconic such that $Z = \partial U \cap \partial X$. By Proposition 7.2, the set \overline{U} is convex. Thus, $\mathbf{dev}_\mu|_{\overline{U}}$ is injective. Since Z contains three noncollinear points, the convex hull H of $\mathbf{dev}_\mu(Z)$ has nonempty interior. The set $P = \mathbf{dev}_\mu|_{\overline{U}}^{-1}(H)$ is the desired set. \square

Remark 9.3. Note that $\mathcal{S}_Z \neq \emptyset$ does not imply that Z defines a polygon. For example, let Z be the intersection of ∂X and a single boundary component of a strip. On the other hand, the assumption that Z defines a polygon does not imply that $\mathcal{S}_Z \neq \emptyset$. For example, suppose that U is a subconic such that $\mathbf{dev}_\mu(Z)$ is the unit disc and $\mathbf{dev}_\mu(\partial U \cap \partial X)$ is the set of roots of $z^6 - 1$. (Here we have made the usual identification of \mathbb{R}^2 with the complex plane). If Z is the set of roots of $z^3 - 1$, then $\mathcal{S}_Z = \emptyset$.

Proposition 9.4. *If Z defines a polygon, then the closure $\overline{\mathcal{S}_Z(X, \mu)}$ is compact.*

Proof. By Proposition 7.2, we have

$$\mathbf{dev}_\mu\left(\overline{\mathcal{S}_Z(X, \mu)}\right) \subset \overline{\mathcal{E}_{\widehat{\mathbf{dev}_\mu(Z)}}(\mathbb{R}^2)}.$$

By Proposition 5.4, the set $\overline{\mathcal{E}_{\widehat{\mathbf{dev}_\mu(Z)}}(\mathbb{R}^2)}$ is compact. Thus, by Proposition 9.1 it suffices to show that $\mathbf{dev}_\mu\left(\overline{\mathcal{S}_Z(X, \mu)}\right)$ is closed in $\mathcal{S}_{\mathbf{dev}_\mu(Z)}(\mathbb{R}^2)$.

Let $\{U_n\} \subset \overline{\mathcal{S}_Z(X, \mu)}$ be a sequence such that $\{\mathbf{dev}_\mu(U_n)\}$ converges to $V \in \mathcal{S}_{\mathbf{dev}_\mu(Z)}(\mathbb{R}^2)$. Each U_n contains the interior of the polygon P defined by Z . In particular, the intersection $\bigcap_n U_n$ is nonempty, and therefore the union $W = \bigcup_n U_n$ is star convex. Therefore, the restriction $\mathbf{dev}_\mu|_W$ is an injection and the image is the union $\mathbf{dev}_\mu(W) = \bigcup_n \mathbf{dev}_\mu(U_n)$.

Let $q \in Q(\mathbb{R}^3)$ such that $U_q = V$. Since $\mathbf{dev}_\mu(U_n)$ converges to V , there exists sequence q_n converging to q such that $U_{q_n} = \mathbf{dev}_\mu(U_n)$. If $x \in \mathbb{R}^2$ and $q(\hat{x}) < 0$, then there exists N such that for all $n > N$, we have $q_n(\hat{x}) < 0$. Thus, $V \subset \mathbf{dev}_\mu(W)$ and $U = \mathbf{dev}_\mu|^{-1}(V)$ is a subconic belonging to $\mathcal{S}_Z(X, \mu)$. \square

If Z defines a triangle, then $W = \mathbf{dev}_\mu(Z)$ is a noncollinear triple. Thus we can apply the results of §4. In particular, let (d_1, d_2, d_3) be an oriented negative natural basis for $Q_{\widehat{\mathbf{dev}_\mu(Z)}}$ and let T be the plane consisting of linear combinations $\sum_i t_i \cdot d_i$ with $\sum t_i = 1$. By Lemma 5.2 the restriction of $q \mapsto U_q$ to T is a homeomorphism onto $\mathcal{S}_{\mathbf{dev}_\mu(Z)}$. Let $h : \mathcal{S}_{\mathbf{dev}_\mu(Z)} \rightarrow T$ denote the inverse of this homeomorphism.

Define $f : \overline{\mathcal{S}_Z(X, \mu)} \rightarrow T$ by

$$f = h \circ \mathbf{dev}_\mu.$$

Since \mathbf{dev}_μ and h are homeomorphisms onto their respective images, the map f is a homeomorphism onto its image.

In general, if Z defines a polygon, then there exists a triple $Z' \subset Z$ that defines a triangle. Note that $\mathcal{S}_Z \subset \mathcal{S}_{Z'}$ and hence one may restrict the map f associated to Z' to the set \mathcal{S}_Z .

Proposition 9.5. *If Z defines a triangle, then $f(\mathcal{S}_Z(X, \mu))$ is open in T .*

Proof. Let x belong to the interior of the triangle P defined by Z . Let M be the maximal star convex neighborhood of x , and let $D \subset \mathbb{R}^2$ be the set $\mathbf{dev}_\mu(\overline{M_x} \cap \partial X)$. Since ∂X is discrete, the set D is discrete. It follows that the set

$$W = \left\{ q \in Q_{\widehat{\mathbf{dev}_\mu(Z)}} \mid q\left(\hat{D} \setminus \widehat{\mathbf{dev}_\mu(Z)}\right) \subset (0, \infty) \right\}$$

is open in $Q_{\widehat{\mathbf{dev}_\mu(Z)}}$. By Proposition 4.10 and Lemma 5.1,

$$W' = \left\{ q \in Q_{\widehat{\mathbf{dev}_\mu(Z)}} \mid U_q \text{ is an ellipse interior} \right\}$$

is open in $Q_{\widehat{\mathbf{dev}_\mu(Z)}}$. Thus, it suffices to show that $f(\mathcal{S}_Z(X, \mu)) = W \cap W' \cap T$.

Suppose that $U \in \mathcal{S}_Z(X, \mu)$. Since $\bar{U} \cap \partial X = Z$, the set $\mathbf{dev}_\mu(\bar{U})$ does not intersect $D \setminus \mathbf{dev}_\mu(Z)$. It follows that $f(U) \in W$. Since Z contains only three points, it follows from Proposition 7.6 that $f(U) \in W'$. In sum, $f(\mathcal{S}_Z(X, \mu)) \subset W \cap W' \cap T$.

Let $q \in W \cap W' \cap T$. Since $q \in W'$, the set \bar{U}_q is convex, and therefore contains P . It follows that U_q contains $\mathbf{dev}_\mu(x)$. In other words, $q(\widehat{\mathbf{dev}_\mu(x)}) < 0$. Since $q \in W$ and $W \subset Q_{\widehat{\mathbf{dev}_\mu(Z)}}$, we have $q(\hat{z}) \geq 0$ for each $z \in D$.

Define $\gamma_z : [0, \infty) \rightarrow \mathbb{R}$ by

$$\gamma_z(t) = q\left((1-t) \cdot \widehat{\mathbf{dev}_\mu(x)} + t \cdot \hat{z}\right).$$

Since the signature of \underline{q} is $(2, 0)$, the function γ_z has a unique global minimum t_0 and is strictly increasing for $t > t_0$. Thus, since $q(\widehat{\mathbf{dev}_\mu(x)}) < 0$ and $q(\hat{z}) \geq 0$, we have $\gamma_z(t) \geq 0$ for each $t \geq 1$.

We have

$$\mathbf{dev}_\mu(M) = \mathbb{R}^2 \setminus \bigcup_{z \in D} \gamma_z([1, \infty)).$$

Hence $U_q \subset \mathbf{dev}_\mu(M)$. Thus, $q = f(U)$ for $U = \mathbf{dev}_\mu|_M^{-1}(U_q)$. Therefore $W \cap W' \cap T \subset f(\mathcal{S}_Z(X, \mu))$. \square

Proposition 9.6. *If $Z \subset \partial X$ defines a polygon, then $f(\mathcal{S}_Z(X, \mu))$ is convex.*

Proof. Let $U, U' \in \mathcal{S}_Z(X, \mu)$. Since U (resp. U') is convex, U (resp. U') contains the interior of the polygon defined by Z . In particular, $U \cap U' \neq \emptyset$. Therefore, the union $U \cup U'$ is star convex, and hence the restriction of \mathbf{dev}_μ to $U \cup U'$ is injective.

For $t \in [0, 1]$, consider $q_t = t \cdot f(U') + (1-t) \cdot f(U)$. If $q_t(x, y, 1) < 0$, then either $q(x, y, 1) < 0$ or $q'(x, y, 1) < 0$. Thus, for each $t \in [0, 1]$, we have $U_{q_t} \subset \mathbf{dev}_\mu(U) \cup \mathbf{dev}_\mu(U')$, and hence $\mathbf{dev}_\mu|_{U \cup U'}^{-1}(U_{q_t})$ is a subconic in $\mathcal{S}_Z(X, \mu)$. \square

Proposition 9.7. *Let Z define a polygon. The point $f(U)$ is an extreme point of $f(\overline{\mathcal{S}_Z(X, \mu)})$ if and only if U is a rigid subconic that belongs to $\overline{\mathcal{S}_Z(X, \mu)}$.*

Proof. Let U be a rigid subconic in $\overline{\mathcal{S}_Z}$. Suppose to the contrary that $f(U)$ is not an extreme point. That is, assume that there exists V and $V' \in \overline{\mathcal{S}_Z}$ such that $f(U) = \frac{1}{2}(f(V) + f(V'))$. Since $\overline{\mathcal{S}_Z}$ is convex and the set, \mathcal{S}_5 , of rigid subconics is discrete, we may assume that neither $f(V)$ nor $f(V')$ is a rigid subconic. It follows that there exists $x \in \partial U \cap \partial X$ such that $x \notin \bar{V} \cup \bar{V}'$. Thus, $f(V)(x) > 0$ and $f(V')(x) > 0$, and therefore $f(U)(x) > 0$. This contradicts the fact that $x \in \partial U \cap \partial X$.

Suppose that $U \in \overline{\mathcal{S}_Z(X, \mu)}$ is not a rigid subconic. In particular, letting $Z' = \partial U \cap \partial X$, we have $\text{Card}(Z') = 3$ or 4 . By Proposition 9.2, the set \hat{Z}' is in general position. Thus, by Proposition 3.1, the vector space $Q_{\widehat{\mathbf{dev}_\mu(Z)}}$ has dimension 2 or 3. Thus, by Lemma 5.1, there exists a nontrivial linear family $t \mapsto q_t$ of quadratic forms in T such that $U_{q_0} = \mathbf{dev}_\mu(U)$ and $\mathbf{dev}_\mu(Z) \subset \partial U_{q_t}$. By Lemma 9.5, there

exists $\delta > 0$ such that if $|t| < \delta$, then $U_{qt} \in \mathbf{dev}_\mu(\mathcal{S}_Z(X, \mu))$. Therefore, $f(U)$ is not an extreme point. \square

The following theorem summarizes the preceding material and provides the basis for the cell complex. In particular, if Z defines a triangle and $\mathcal{S}_Z \neq \emptyset$, then \mathcal{S}_Z is homeomorphic to a cell.

Theorem 9.8. *Let $Z \subset \partial X$ define a triangle. The map f is a homeomorphism from $\mathcal{S}_Z(X, \mu)$ onto a compact convex planar polygon with finitely many sides. Each side of $f(\mathcal{S}_Z(X, \mu))$ equals $f(\overline{\mathcal{S}_{Z'}(X, \mu)})$ where $Z' \subset \partial X$ defines a quadrilateral and $Z \subset Z' \subset \partial X$. Each vertex of $f(\mathcal{S}_Z(X, \mu))$ equals $f(U)$ where U is a rigid conic with $Z \subset \partial U \cap \partial X$. A vertex $f(U)$ belongs to a side $f(\overline{\mathcal{S}_{Z'}(X, \mu)})$ if and only if $Z' \subset \partial U \cap \partial X$.*

Proof. By Proposition 9.6, the image $f(\overline{\mathcal{S}_Z})$ is convex. By Proposition 9.4 the set $f(\overline{\mathcal{S}_Z})$ is compact and hence closed. Therefore, by the Krein-Milman theorem and Proposition 9.7, the set $f(\overline{\mathcal{S}_Z})$ is the convex hull of the f -images of the rigid subconics that belong to $\overline{\mathcal{S}_Z}$. Each such rigid subconic contains the nonempty interior of the polygon P_Z . Therefore, by Corollary 7.9, the set of such subconics is finite. In sum, $f(\overline{\mathcal{S}_Z})$ is the convex hull of finitely many points in the 2-dimensional plane T .

Since f is a homeomorphism onto its image, Proposition 9.5 implies that $f(\mathcal{S}_Z)$ is the interior of the polygon $f(\overline{\mathcal{S}_Z})$. In particular, the boundary $\partial \mathcal{S}_Z$ consist of those subconics U with $\text{Card}(\partial U \cap \partial X) \geq 4$. Since the rigid subconics correspond to extreme points, each side corresponds to a subset $Z' \subset X$ with $\text{Card}(\partial U \cap \partial X) = 4$. \square

Corollary 9.9. *The space $\mathcal{S}_3(X, \mu)$ and the collection of cells*

$$\{\mathcal{S}_Z(X, \mu) \mid \mathcal{S}_Z(X, \mu) \neq \emptyset \text{ and } \text{Card}(Z) \geq 3\}$$

constitute a 2-dimensional cell complex.

Remark 9.10. The cell complex $\mathcal{S}_3(X, \mu)$ is not a CW -complex. Indeed, the topology on $\mathcal{S}_3(X, \mu)$ does not give the weak topology with respect to the cells. If U is a rigid strip, then there exists a sequence of 1-cells \mathcal{S}_{Z_n} and points $U_n \in \mathcal{S}_{Z_n}$ such that U_n converges U . The set $\{U_n\}$ is closed in the weak topology but not in the natural topology chosen here. On the other hand, Theorem 7.9 implies that the cell complex $\mathcal{S}_3(X, \mu)$ has the closure finiteness property.

Proposition 9.11. *The set $\mathcal{S}_4(X, \mu)$ is path connected.*

Proof. Let $U, V \in \mathcal{S}_4(X, \mu)$. By Corollary there exists a path $\gamma : [0, 1] \rightarrow \mathcal{S}_3(X, \mu)$ with $\gamma(0) = U$ and $\gamma(1) = V$. Since each 2-cell is bounded by finitely many 1-cells in $\mathcal{S}_\mu(X, \mu)$ the path γ can be homotoped to lie entirely in $\mathcal{S}_4(X, \mu)$. \square

10. REALIZABILITY AND ADJACENCY

In this section (X, μ) is the universal cover of a precompact translation surface with finite and nonempty frontier.

The cells of $\mathcal{S}_3(X, \mu)$ are indexed by the subsets $Z \subset \partial X$ such that $\mathcal{S}_Z(X, \mu) \neq \emptyset$ and $\text{Card}(Z) \geq 3$. We will call such a set Z a *realizable* set. By Theorem 9.8, the inclusion of closed cells corresponds to the inclusion of realizable subsets of ∂X .

In this section, we define the notion of ‘adjacency’ and use it to characterize the subsets of a realizable set that are realizable.

Definition 10.1. Let U be a subconic. We say that $x, y \in \partial U \cap \partial X$ are *adjacent* in $\partial U \cap \partial X$ if and only if there exists a connected component of $\partial U \setminus \{x, y\}$ that does not intersect ∂X .

Proposition 10.2. *Let U be a subconic with $\text{Card}(\partial U \cap \partial X) \geq 3$. Each $x \in \partial U \cap \partial X$ is adjacent to exactly two points in $\partial U \cap \partial X$.*

Proof. Let C be a connected component of ∂U . Let $\alpha : \mathbb{R} \rightarrow C$ denote a universal covering with $\alpha(0) = x$. Since $\partial U \cap \partial X$ is discrete and α is a covering, $\alpha^{-1}(\partial X)$ is discrete. If U is a strip, then α is a homeomorphism. It follows from Corollary 7.6, both $A_- = \{s < 0 \mid \alpha(s) \in \partial X\}$ and $A_+ = \{s > 0 \mid \alpha(s) \in \partial X\}$ are nonempty. If U is an ellipse interior, then $\pi_1(\partial U) \cong \mathbb{Z}$, and it follows that A_+ and A_- are nonempty. Let $x_- = \alpha(\sup(A_-))$ and $x_+ = \alpha(\inf(A_+))$. Then x_{\pm} is adjacent to x , and since $\text{Card}(\partial U \cap \partial X) \geq 3$, we have $x_+ \neq x_-$. \square

Suppose that Z is realizable with $\text{Card}(Z) \geq 5$. If $Z' \subset Z$ is also realizable, then it follows from Proposition 3.1 that $\text{Card}(Z') = 3$ or 4 .

Proposition 10.3. *Let $Z \subset \partial X$ be realizable and let $Z' \subset Z$ with $\text{Card}(Z') = 4$. The set Z' is realizable if and only if there exists $U \in \mathcal{S}_Z$ such that Z' intersects each component of ∂U and there exists a partition of Z' into pairs $\{x_-, y_-\}$, $\{x_+, y_+\}$ such that each pair $\{x_{\pm}, y_{\pm}\}$ is adjacent in $\partial U \cap \partial X$.*

Proof. We first note that the claim is true if $Z = Z'$. Indeed, if $Z = Z'$, then $\text{Card}(Z) = 4$, and hence each subconic $U \in \mathcal{S}_Z$ is an ellipse interior. Thus, ∂U is homeomorphic to the unit circle, and each point in $Z' = \partial X \cap \partial U$ is adjacent to exactly two other points in Z' . In particular, there exists a partition of Z' into adjacent pairs. Conversely, if $Z = Z'$, then since Z is realizable, Z' is realizable.

Thus, we may assume that $Z' \neq Z$.

(\Rightarrow) If Z' is realizable, then there exists $U' \in \mathcal{S}_{Z'}$. Since $Z \neq Z'$, we have $U \neq U'$. Let $q \in Q(\mathbb{R}^3)$ (resp. $q' \in Q(\mathbb{R}^3)$) such that $U_q = \mathbf{dev}_{\mu}(U)$ and $U_{q'} = \mathbf{dev}_{\mu}(U')$. Since $U \neq U'$, the forms q and q' do not belong to the same line. Let $F = \mathbf{dev}_{\mu}(Z)$. Consider the components of the complement $\partial U_q \setminus F$. Since ∂U is a 1-manifold, the intersection $\overline{C} \cap \overline{C}'$ of the closures of two components, C, C' , is either empty or is a singleton. In the latter case, we will say that the components are ‘adjacent’.

By Proposition 3.1, the zero locus of the restriction $q'|_{\partial U_q}$ equals F . (Indeed, if it were not the case, then q and q' would both vanish at 5 common points.) Thus, since $q'|_{\partial U_q}$ is continuous, the restriction to each component $q'|_C$ is either positive or negative. We claim that if C and C' are adjacent, then the restrictions $q'|_C$ and $q'|_{C'}$ have opposite signs.

Indeed, suppose to the contrary that the signs of $q'|_C$ and $q'|_{C'}$ are the same. Let $\alpha : (-\epsilon, \epsilon) \rightarrow \partial C_q$ be a differentiable path with $\alpha(0)$ equal to the point in $\overline{C} \cap \overline{C}'$ and $|\alpha'(0)| \neq 0$. Note that $t \mapsto q \circ \hat{\alpha}(t)$ is constant, and by assumption $t \mapsto q' \circ \hat{\alpha}(t)$ has a critical point at $t = 0$. (Recall that $\widehat{(v_1, v_2)} = (v_1, v_2, 1)$.) If we let $v_1 = \hat{\alpha}(0)$ and let $w_1 = \hat{\alpha}'(0)$, then $dq_{v_1}(w_1) = 0$ and $d(q')_{v_1}(w_1) = 0$. Thus, by Proposition 3.2, the forms q and q' belong to the same line. This is a contradiction.

Since Z' is realizable, Z' intersects each component of ∂U . We now identify the desired points $\{x_{\pm}, y_{\pm}\}$.

If U is a strip, then let ℓ_+ , ℓ_- denote the two components. Since $U \in S_{Z'}$ and $\text{Card}(Z) = 4$, we have $\text{Card}(Z' \cap \ell_{\pm}) = 2$. Indeed, otherwise U' would contain three collinear points, and U' would be a strip. But if U' were a strip, then by Proposition 7.6, the set $Z' = \partial U' \cap \partial X$ would be infinite. Since ℓ_{\pm} is homeomorphic to a line, the complement $\ell_{\pm} \setminus Z$ has exactly two unbounded components and one bounded component. The frontier of each unbounded component C is a singleton and hence $U \cap C = \emptyset$. Thus, q is positive on the corresponding ray. Therefore, the form q' is negative on the segment that corresponds to the bounded component, $B_{\pm} \subset \ell_{\pm}$. Thus $U' \supset B_{\pm}$, and we define $\{x_{\pm}, y_{\pm}\}$ to be the frontier points of B_{\pm} .

If U is an ellipse interior, then ∂U is homeomorphic to the unit circle. Thus, $\partial U \setminus Z$ has four components that we index with $\mathbb{Z}/4\mathbb{Z}$ so that C_i and C_{i+1} are adjacent for each $i \in \mathbb{Z}/4\mathbb{Z}$. For some $i \in \mathbb{Z}/4\mathbb{Z}$, we have that the restriction of q' to arc corresponding to C_i is negative, and hence the restriction of q' to the arc corresponding to C_{i+2} is negative. In particular, $U' \supset C_i \cup C_{i+2}$. Set $\{x_-, y_-\}$ to be the frontier points of C_i , and set $\{x_+, y_+\}$ equal to the frontier points of C_{i+2} .

(\Leftarrow) Suppose that there exists $U \in S_Z$ such that Z' admits a partition into two pairs of points adjacent in Z and such that Z' intersects each component of ∂U . Let $F = \mathbf{dev}_{\mu}(Z)$ and $F' = \mathbf{dev}_{\mu}(Z')$. Let $q \in Q_F$ so that $\mathbf{dev}_{\mu}(U) = U_q$.

Since Z' intersects each component of ∂U , the set Z' defines a polygon P . Since $F' = \mathbf{dev}_{\mu}(Z')$ is the set of extreme points of $K = \mathbf{dev}_{\mu}(P)$, the set K is the intersection of four closed half planes. Thus $\mathbb{R}^2 \setminus K$ is the union of four open half planes. Since $F' \neq F$ and $F \subset \mathbb{R}^2 \setminus K$, one of these open half planes, H_1 , intersects F . Let σ_1 be the side of K that is contained in the closure of $\overline{H_1}$. Let σ_3 be the side of P opposite to σ , and let H_3 be the open half plane whose closure \overline{H} contains σ_3 . Let H_2 and H_4 be the other two half-planes.

Let p_- and p_+ be the endpoints of σ_1 . Then $p_-, p_+ \in F'$. We claim that the corresponding points $x_- = \mathbf{dev}_{\mu}|_{\overline{U}}^{-1}(p_-)$ and $x_+ = \mathbf{dev}_{\mu}|_{\overline{U}}^{-1}(p_+)$ are not adjacent in $Z = \partial U \cap \partial X$.

If U_q is a strip and x_- and x_+ were adjacent, then p_- and p_+ would belong to the same component ℓ of ∂U_q . The component ℓ would coincide with the boundary of H_1 , and ∂U_q would then intersect both components of $\mathbb{R}^2 \setminus \ell$. Since U_q is a strip, ℓ could not be the boundary of U_q . Thus, in this case, x_- and x_+ are not adjacent.

If U_q is an ellipse interior, then the intersection $\partial H_i \cap \partial U_q$ consists of two points. Since $H_1 \cap \partial U_q \neq \emptyset$, the component C of $\partial U_q \setminus F$ that joins $p_-, p_+ \in \partial H_1 \cap \partial U_q$ belongs to H_1 . But $C \cap F \neq \emptyset$ and so x_- and x_+ are not adjacent in this case.

Since Z' has a partition into two adjacent pairs, x_+ (resp. x_-) is adjacent to an endpoint y_+ (resp. y_-) of σ_3 . Let C^+ and C^- denote the arcs with endpoints $\{x_{\pm}, y_{\pm}\}$ such that $\partial X \cap C^{\pm} = \emptyset$.

Let η_i be a linear 1-form whose kernel contains the span of $\widehat{\partial H_i}$ and such that η_i is negative on \widehat{K}° where K° is the interior of K . Then the quadratic form $q' = -\eta_1 \cdot \eta_3$ belongs to $Q_{F'}$. Note that $H_1 \cup H_3 = (q')^{-1}((0, \infty))$, and q' is negative on C^{\pm} .

For $t \in \mathbb{R}$, define

$$q_t = t \cdot q' + (1 - t) \cdot q.$$

If $t \in [0, 1]$, then $U_{q_t} \subset U_q \cap U_{q'}$. Indeed, if $q_t(x) < 0$, then either $q(x) < 0$ or $q'(x) < 0$.

Let $x \in K^{\circ}$, and let $M \subset X$ be the maximal star convex neighborhood of x . Since U is convex and $U_q = \mathbf{dev}_{\mu}(U)$, we have $U_q \subset \mathbf{dev}_{\mu}(M)$.

Let $S = \mathbf{dev}_\mu|_M^{-1}(U_{q'})$ and let $D = \mathbf{dev}_\mu(\overline{S} \cap \partial X)$. Since $\partial X \cap C^\pm = \emptyset$, we have $D \cap \mathbf{dev}_\mu(C^\pm) = \emptyset$. Since ∂X is discrete, the set D is discrete. Since $C^\pm \cap D = \emptyset$, we have $q_0^{-1}((-\infty, 0]) \cap D = F'$. Therefore since $t \mapsto q_t$ is continuous and D is discrete, there exists $\epsilon \in (0, 1)$ such that if $|t| < \epsilon$, then $q_t^{-1}((-\infty, 0]) \cap D = F'$.

We claim that there exists $\epsilon' > 0$ such that if $t \in [0, \epsilon')$, then U_{q_t} is an ellipse interior. If U_q is an ellipse, then this follows from Proposition 4.11 and the continuity of $t \mapsto q_t$.

Suppose that U_q is a strip. Then $q|_{U_q}$ is bounded from below. Indeed, $q = \eta_+ \cdot \eta_-$ where η_- are linear 1-forms. Since kernel, ℓ_\pm , of η_\pm is parallel to the kernel, ℓ_\mp , of ℓ_\pm , it follows that the form $\eta_\pm|_{U_q}$ is bounded. Hence $q|_{U_q} > -N$ for some $N > 0$. Let v belong to the kernel of η_\pm . For each $p \in \mathbb{R}^2$ define $f_p(s) = q'(\hat{p} + s \cdot v)$. Note that f_p is a quadratic polynomial and we claim that f_p is nontrivial with leading coefficient positive. Since $\text{Card}(U_{q'} \cap U_q) = 4$, each line in $q^{-1}\{0\}$ intersects $(q')^{-1}\{0\}$. It follows that $\eta_i(v) \neq 0$ for $i = 1$ or 3 , and hence $f_p(s) = (\eta_1(p) + s \cdot \eta_1(v)) \cdot (\eta_3(p) + s \cdot \eta_3(v))$ is nonconstant. Since $q'|_{\sigma_i} \equiv 0$ and $q'|_{K^0} \leq 0$, the leading coefficient of the quadratic polynomial f_p is positive for each $p \in \sigma_1$. Since σ_1 is compact for each t , there exists N'_t such that if $|s| > N'_t$ and $p \in \sigma_1$, then $q_t(p + s \cdot v) > (N + 1)/t$. It follows that $U_{q_t} \cap U_q$ is bounded for each $t > 0$. Since $U_q \cap U_{q'}$ is bounded, there exists $\epsilon' \in (0, 1)$ so that if $|t| < \epsilon$, then $U_{q_t} \cap U_{q'}$ is bounded. Thus, for $t \in (0, \epsilon')$, we have that U_{q_t} is bounded, and hence by Proposition 4.9, the subconic U_q is an ellipse interior.

Fix $t \in (0, \min\{\epsilon, \epsilon'\})$. We have $U_{q_t} \setminus U_{q'} \subset U_q \subset \mathbf{dev}_\mu(M)$. Thus, it suffices to show that $U_{q_t} \cap U_{q'} \subset \mathbf{dev}_\mu(M)$. For then $\mathbf{dev}_\mu|_M^{-1}(U_{q_{t_0}}) \in \mathcal{S}_{Z'}$.

If $y \in U_{q_t} \cap U_{q'}$, then $y \notin D$. Since U_{q_t} is convex, the line segment σ joining y and $\mathbf{dev}_\mu(x)$ belongs to U_{q_t} . Therefore, $D \cap \sigma = \emptyset$, and it follows that $y \in \mathbf{dev}_\mu(M)$. \square

Proposition 10.4. *Let $Z \subset \partial X$ be realizable and $Z' \subset Z$ with $\text{Card}(Z') = 3$. The triple Z' is realizable if and only if there exists $U \in \mathcal{S}_Z$ and a pair $\{x, x'\} \subset Z'$ that is adjacent in $\partial U \cap \partial X$.*

Proof. The proof is similar to the proof of Proposition 10.3. We leave it to the reader. \square

11. ORIENTATION AND SUCCESSION

In this section, we first observe that each 2-cell of $\mathcal{S}(X, \mu)$ has a canonical orientation. We then reinterpret this orientation in terms of *succession*, the natural refinement of adjacency. We will assume that (X, μ) is the universal covering of a precompact translation surface with nonempty and finite frontier.

Let $Z \subset X$ be a realizable triple, and thus, in particular, $\mathbf{dev}_\mu(Z)$ is noncollinear. Let \vec{d} is an oriented negative natural basis for $Q_{\widehat{\mathbf{dev}_\mu(Z)}}$, and let $r_{\vec{d}}: T_{\vec{d}} \rightarrow \mathcal{S}_{\mathbf{dev}_\mu(Z)}$ be the homeomorphism defined in §5. Recall that the plane $T_{\vec{d}}$ has a canonical (outward normal) orientation, and hence the homeomorphism $r_{\vec{d}}$ induces an orientation on \mathcal{S}_Z . By Proposition 5.3, this orientation does not depend on the choice of oriented negative natural basis. As a result, the cell \mathcal{S}_Z has a canonical orientation.

The orientation of \mathcal{S}_Z induces an orientation of its boundary $\partial \mathcal{S}_Z$. In particular, it induces a cyclic ordering of the 1-cells lying in the boundary of \mathcal{S}_Z . These 1-cells are in one-to-one correspondence with the set, ∂Z , of realizable quadruples Z' that contain Z .

Definition 11.1. Let $Z', Z'' \in \partial Z$. We will say Z' follows Z'' if and only if $\mathcal{S}_{Z'}$ immediately follows $\mathcal{S}_{Z''}$ in the canonical ordering of ∂Z .

The cyclic ordering has an alternate description that uses the oriented refinement of adjacency. In particular, if Z'' follows Z' , then Z'' and Z' share a vertex and this vertex corresponds to the unique rigid conic U such that $Z'' \cup Z' \subset \partial U \cap \partial X$. In Proposition 11.6 below, we reinterpret ‘following’ in terms of an ordering of $\partial U \cap \partial X$.

Definition 11.2. Suppose that $\text{Card}(\partial U \cap \partial X) \geq 3$. Given adjacent points $x, y \in \partial U \cap \partial X$, let C be the unique component of $\partial U \setminus \partial X$ such that $\{x, y\} \subset \overline{C}$. If there exists an oriented path $\alpha : [-1, 1] \rightarrow \overline{C}$ such that $\alpha(-1) = x$ and $\alpha(+1) = y$, then we say that y is the *successor* of x in $\partial U \cap \partial X$, and we write $s_U(x) = y$.

Proposition 11.3. *The map s_U is a permutation of $\partial U \cap \partial X$.*

Proof. Straightforward. □

In the remainder of this section, we will assume that U is a rigid subconic. That is, we assume that $\text{Card}(\partial U \cap \partial X) \geq 5$.

Notation 11.4. Given $x, x' \in \partial U \cap \partial X$, set

$$Z_U(x, x') := \{x, s_U(x), x', s_U(x')\}.$$

If $x \neq x'$ and $\{x, x'\}$ are nonadjacent, then $Z(x, x')$ is a quadruple, and hence by Proposition 10.3, this quadruple is realizable. Conversely, if Z is realizable and $U \in \overline{\mathcal{S}_Z}$, then Z can be partitioned into adjacent pairs $\{x_-, y_-\}$ and $\{x_+, y_+\}$. For each pair either $s_U(x_\pm) = y_\pm$ or $s_U(y_\pm) = x_\pm$, but not both. Thus, $(x, y) \mapsto Z_U(x, y)$ is a bijection from the set of nonadjacent pairs in $\partial U \cap \partial X$ onto the realizable quadruples $Z \subset \partial U \cap \partial X$.

Definition 11.5. We will say that $Z \subset \partial U \cap \partial X$ is *consecutive* in $\partial U \cap \partial X$ if and only if there exists $x \in Z$ and $k \in \mathbb{Z}^+$ such that $Z = \{x, s_U(x), \dots, s_U^k(x)\}$. Let $x \in Z$. If for all $y \in Z$, we have $s_U(y) \neq x$ and $s_U(x) \neq y$, then we will say that x is *Z -isolated* in $\partial U \cap \partial X$.

Let $Z \subset \partial U \cap \partial X$ be a realizable triple. Then either Z is consecutive, namely, $Z = \{x, s_U^2(x), s_U^3(x)\}$ for a unique $x \in Z$, or there exists $y \in Z$ that is isolated, that is $Z = \{x, s_U(x), y\}$ for a unique $x \in Z$.

Proposition 11.6. *Let U be a rigid subconic and let $Z = \partial U \cap \partial X$ be a realizable triple. If $Z = \{x, s_U(x), s_U^2(x)\}$ is consecutive in $\partial U \cap \partial X$, then $Z_U(x, s_U^2(x))$ follows $Z_U(s_U^{-1}(x), s_U(x))$ in ∂Z . If $Z = \{x, s_U(x), y\}$ is nonconsecutive, then $Z_U(x, y)$ follows $Z_U(x, s_U^{-1}(y))$ in ∂Z .*

Proof. Suppose that $Z = \{x, s_U(x), s_U^2(x)\}$ is consecutive in $\partial U \cap \partial X$. Let $v_i = \mathbf{dev}_\mu(s_U^i(x))$ for $i \in \mathbb{Z}$. We first note that the ordered triple (v_0, v_1, v_2) is cyclically ordered with respect to the standard orientation of \mathbb{R}^2 . Indeed, Z belongs to a single component of ∂U , and thus, since Z is realizable, ∂U has only one component. In particular, U is an ellipse interior. It is then straightforward to construct a homotopy of piecewise smooth embeddings $f_t : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^2$ such that f_0 is an oriented parametrization of $\mathbf{dev}_\mu(\partial U)$, f_1 is a parametrization of the boundary of the convex hull of $\{v_0, v_1, v_2\}$, and $f_t([i/3]) = v_i$ for all $t \in [0, 1]$ and $i = 0, 1, 2$.

Let $\vec{d} = (d_0, d_1, d_2)$ be an oriented negative natural basis associated to the cyclically ordered noncollinear triple (v_0, v_1, v_2) . In particular, for $\{i, j, k\} = \{1, 2, 3\}$, let η_{ij} be the linear form with kernel $\langle \hat{v}_i, \hat{v}_j \rangle$ such that $\eta_{ij}(\hat{v}_k) = 1$, and set

$$d_i = -\eta_{i,k} \cdot \eta_{i,j}.$$

Let $T = \{(t_0, t_1, t_2) \mid \sum t_i = 1\}$, let $r : T \rightarrow \mathcal{S}_{\{v_1, v_2, v_3\}}$ be the homeomorphism $r(t_0, t_1, t_2) = \sum t_i \cdot d_i$, and let $f = r^{-1} \circ \mathbf{dev}_\mu$.

For $i = 0, 1$, let $V_i = \{v_{i-1}, v_i, v_{i+1}, v_{i+2}\}$, and let ℓ_i denote the set

$$r^{-1}(\mathcal{S}_{V_i}) = Q_{\hat{V}_i} \cap T.$$

By Proposition 3.1, the subspace \hat{Q}_{V_i} is 2-dimensional, and thus it follows that ℓ_i is an affine line. Since $\mathbf{dev}_\mu(Z(s^{-1}(x), s(x))) = V_0$ (resp. $\mathbf{dev}_\mu(Z(x, s^2(x))) = V_1$), we have $f(\mathcal{S}_{Z(s^{-1}(x), s(x))}) \subset \ell_0$ (resp. $f(\mathbf{dev}_\mu(Z(x, s^2(x)))) \subset \ell_1$).

The $\ell_0 \cap \ell_1$ meet at the unique point $q \in T$ such that if $j = -1, \dots, 3$, then $q(\hat{v}_j) = 0$. In particular, $\mathbf{dev}_\mu(U) = U_q$. Since, for example, U is an extreme point, $\ell_0 \cap \ell_1 = \{q\}$.

Let $\ell'_i \subset T$ be the line containing d_i and d_{i+1} . The lines ℓ_i and ℓ'_i intersect at one point a_i . It will be convenient to give an explicit construction of this point. If $i = 0$, the construction runs as follows: Note that $\eta_{1,2}(\hat{v}_{-1}) < 0$ and $\eta_{0,2}(\hat{v}_{-1}) > 0$, and hence $-\eta_{1,2}(\hat{v}_{-1})/\eta_{0,2}(\hat{v}_{-1}) > 0$. Thus, there exists a unique $t_0 \in (0, 1)$ such that

$$(4) \quad \frac{t_0}{1-t_0} = -\frac{\eta_{1,2}(\hat{v}_{-1})}{\eta_{0,2}(\hat{v}_{-1})}.$$

Define

$$\eta_{-1,2} = t_0 \cdot \eta_{0,2} + (1-t_0) \cdot \eta_{1,2}$$

and

$$a_0 = -\eta_{-1,2} \cdot \eta_{0,1}.$$

In other words,

$$a_0 = t_0 \cdot d_0 + (1-t_0) \cdot d_1.$$

It follows from (4) that $\eta_{-1,2}(v_{-1}) = 0$, and hence $a_0 \in Q_{\hat{V}_0}$.

Note that $a_0(\hat{v}_3) > 0$. Indeed, since $\{v_{-1}, v_0, v_1, v_2\}$ are consecutive \hat{v}_0 and \hat{v}_3 lie in distinct components of $\mathbb{R}^3 \setminus \langle \hat{v}_{-1}, \hat{v}_2 \rangle$. Thus, since $\eta_{-1,2}(\hat{v}_0) = (1-t_0) > 0$ we have $\eta_{-1,2}(\hat{v}_3) < 0$. Since \hat{v}_2 and \hat{v}_3 lie in the same component of $\mathbb{R}^3 \setminus \langle \hat{v}_{-1}, \hat{v}_2 \rangle$ and $\eta_{0,1}(\hat{v}_2) = 1 > 0$, we have $\eta_{0,1}(\hat{v}_3) > 0$. Thus,

$$a_0(\hat{v}_3) = -\eta_{0,1}(\hat{v}_3) \cdot \eta_{-1,2}(\hat{v}_3) > 0.$$

A similar construction produces $t_1 \in (0, 1)$ such that

$$a_1 = t_1 \cdot d_1 + (1-t_1) \cdot d_2,$$

$a_1 \in \ell_1$, and $a_1(\hat{v}_{-1}) > 0$.

We claim that $a_i \neq q$ for $i = 1, 2$. Since the set $\{x, s(x), s^2(x)\}$ is consecutive, this set is a subset of one boundary component of ∂U . Thus, since $\{x, s(x), s^2(x)\}$ is realizable, U is an ellipse interior and q is nondegenerate. Therefore since a_i is degenerate, $a_i \neq q$.

We claim that $f(\mathcal{S}_{Z(x, s^2(x))})$ is contained in the ray $\overrightarrow{qa_0}$ and that $f(\mathcal{S}_{Z(s^{-1}(x), s(x))})$ is contained in the ray $\overrightarrow{qa_1}$. Indeed, for $i = 0, 1$, let

$$q_i^t = (1-t) \cdot q + t \cdot a_i,$$

Since $a_i \neq q$, the affine function $t \mapsto q_t$ maps \mathbb{R} onto ℓ_i . Since $a_0(\hat{v}_3) > 0$ (resp. $a_1(\hat{v}_{-1}) > 0$) and $q(\hat{v}_3) = 0$ (resp. $q(\hat{v}_{-1}) = 0$), the affine function $t \mapsto q_t^0(\hat{v}_3)$ (resp. $t \mapsto q_t^1(\hat{v}_{-1})$) is increasing. Therefore, for $t < 0$, we have $q^0(\hat{v}_3) < 0$ (resp. $q^1(\hat{v}_{-1}) < 0$) and hence $q_t \notin f(\mathcal{S}_{Z(s^{-1}(x), s(x))})$ (resp. $q_t \notin f(\mathcal{S}_{Z(x, s^2(x))})$).

Let ℓ be the line containing a_0 and a_1 . We claim that if $q' \in \ell$, then either $q'(\hat{v}_{-1}) > 0$ or $q'(\hat{v}_3) > 0$. Let $a_t = (1-t) \cdot a_0 + t \cdot a_1$. We have $a_0(\hat{v}_3) > 0$ and $a_1(\hat{v}_3) = 0$. Therefore, if $t > 0$, then $a_t(\hat{v}_3) > 0$. We have $a_1(\hat{v}_{-1}) > 0$ and $a_0(\hat{v}_{-1}) = 0$. Therefore, if $t < 1$, then $a_t(\hat{v}_{-1}) > 0$.

Define $q_t = (1-t) \cdot q + t \cdot d_1$. We claim that for all $t \geq 0$, the quadratic form q_t does not lie in ℓ . Recall that $d_1 = -\eta_{0,1} \cdot \eta_{1,2}$. The points \hat{v}_2 , \hat{v}_3 , and \hat{v}_{-1} lie in the same component of $\mathbb{R}^2 \setminus \langle \hat{v}_0, \hat{v}_1 \rangle$, and hence $\eta_{0,1}(\hat{v}_3) > 0$ and $\eta_{0,1}(\hat{v}_{-1}) > 0$. The points \hat{v}_0 , \hat{v}_3 , and \hat{v}_{-1} lie in the same component of $\mathbb{R}^2 \setminus \langle \hat{v}_1, \hat{v}_2 \rangle$, and hence $\eta_{1,2}(\hat{v}_3) > 0$ and $\eta_{1,2}(\hat{v}_{-1}) > 0$. It follows that $d_1(\hat{v}_3) < 0$ and $d_1(\hat{v}_{-1}) < 0$. Therefore, since $q(\hat{v}_3) = 0 = q(\hat{v}_{-1})$, if $t \geq 0$, then $q_t(\hat{v}_3) \leq 0$ and $q_t(\hat{v}_{-1}) \leq 0$.

By the choice of orientation of T , we have that $(d_2 - d_1, d_0 - d_1)$ is an oriented basis for \mathbb{R}^2 . Since $t_0, t_1 \in (0, 1)$, the ordered pair $(a_1 - d_1, a_0 - d_1)$ is an oriented basis for \mathbb{R}^2 . If $t \geq 0$, the quadratic form q_t does not lie in the line ℓ containing a_0 and a_1 . Therefore, for $t \geq 0$, the set $\{a_1 - q_t, a_0 - q_t\}$ is a basis and by continuity, $(a_1 - q, a_0 - q)$ is an oriented basis. Thus, $f(\mathcal{S}_{Z(x, s^2(x))})$ follows $f(\mathcal{S}_{Z(s^{-1}(x), s(x))})$.

The proof of the other claim is similar. \square

12. THE LINK OF A VERTEX

Recall that if v is a vertex in a 2-dimensional cell complex, then the link of v , $\text{Lk}(v)$, is the abstract graph defined as follows: The vertices are the 1-cells that contain v . Two 1-cells $C, C' \subset \partial U \cap \partial X$ are joined by an edge iff there exists a 2-cell D such that $v \in \partial C \cap \partial C'$ and $C \cup C' \subset \partial D$.

In this section, we study the link $\text{Lk}(U)$ of a rigid conic U in $\mathcal{S}_3(X, \mu)$. The vertices of $\text{Lk}(U)$ may be regarded as realizable quadruples $Z \subset \partial U \cap \partial X$. Two quadruples Z and Z' are joined by an edge if and only if $Z \cap Z'$ is a realizable triple. The orientation of the 2-cell $\mathcal{S}_{Z \cap Z'}$ determines a direction of this edge. Hence, $\text{Lk}(U)$ is naturally a directed graph.

We will assume throughout this section that (X, μ) is the universal covering of a precompact translation surface with nonempty and finite frontier.

By Proposition 10.3, the set of 1-cells that contain U is in bijection with the set of unordered pairs $\{x, y\}$ of non-adjacent points in $\partial U \cap \partial X$. If U is an ellipse, then the convex hull of $\partial U \cap \partial X$ is an n -gon with $n = \text{Card}(\partial U \cap \partial X) < \infty$. The non-adjacent vertices define ‘diagonals’ in the n -gon. The classic count of diagonals gives the following.

Proposition 12.1. *If U is a rigid ellipse and $\text{Card}(\partial U \cap \partial X) = n$, then $\text{Lk}(U)$ has $n(n-3)/2$ vertices.*

To simplify the notation in what follows, we will write $x+k$ in place of $s_U^k(x)$.

Proposition 12.2. *Each vertex of $\text{Lk}(U)$ has degree 4. If the quadruple $Z(x, y)$ is consecutive, then the neighbours of $Z(x, y)$ are*

$$(5) \quad Z(x-1, x+2), \quad Z(x-1, x+1).$$

If the quadruple $Z(x, y)$ is not consecutive, then the neighbours of $Z(x, y)$ are

$$(6) \quad Z(x, y-1), \quad Z(x, y+1), \quad Z(x+1, y), \quad Z(x-1, y).$$

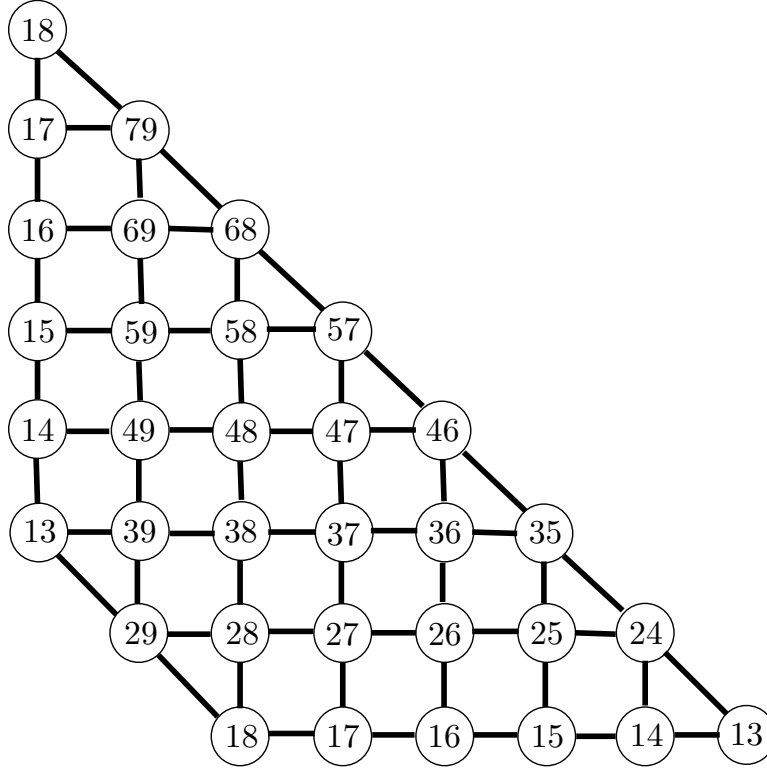


FIGURE 2. The link of U when $\text{Card}(\partial U \cap \partial X) = 9$. The vertex corresponding to $Z_U(x_i, x_j)$ is labeled ij . Note that the vertices labeled 13, 14, 15, 16, 17, and 18 appear twice. These pairs should be identified to obtain a graph that can be embedded in the Möbius band.

Proof. A quadruple Z' is a neighbour of $Z_U(x, y)$ if and only if the intersection $Z' \cap Z_U(x, y)$ consists of three points.

If $|x - y| = 2$, then either $Z_U(x, y) = \{x, x + 1, x + 2, x + 3\}$ or $Z_U(x, y) = \{y, y + 1, y + 2, y + 3\}$. Without loss of generality, the former holds. In this case, the possible intersections are $\{x, x + 1, x + 2\}$, $\{x + 1, x + 2, x + 3\}$, $\{x, x + 2, x + 3\}$, and $\{x, x + 1, x + 3\}$. By Proposition 11.6, the only other quadruple that contain these triples are listed in (5).

If $|x - y| > 2$, then $Z_U(x, y) = \{x, x + 1, y, y + 1\}$. The possible intersections are $\{x, x + 1, y\}$, $\{x, x + 2, y + 1\}$, $\{x, y, y + 1\}$, and $\{x + 1, y, y + 1\}$. By Proposition 11.6, the only other quadruples that contains these are triples are listed in (6). \square

We next consider the 3-cycles in the undirected graph $\text{Lk}(U)$.

Lemma 12.3. *The vertex $Z_U(x, x + 2)$ belongs to exactly two 3-cycles:*

$$\{Z(x, x + 2), Z(x - 1, x + 2), Z(x - 1, x + 1)\}$$

and

$$\{Z(x, x + 2), Z(x, x + 2), Z(x + 1, x + 3)\}.$$

Suppose that $Z_U(x, y)$ is not consecutive with respect to U . If a 3-cycle contains $Z_U(x, y)$, then either $x = y + 3$ and the cycle is

$$\{Z(x, y), Z(x, y + 1), Z(x - 1, y)\}$$

or $y = x + 3$ and the cycle is

$$\{Z(x, y), Z(x + 1, y), Z(x, y - 1)\}$$

Proof. Suppose that $Z = \{x, x + 1, x + 2, x + 3\}$ is consecutive with respect to U . Then by Proposition 12.2, the neighbours of Z are $Z(x - 1, x + 2)$, $Z(x - 1, x + 1)$, $Z(x, x + 3)$, and $Z(x + 1, x + 3)$. Inspection shows that the first two intersect in 3 points and the last two intersect in three points. No other pairs intersect in three points. Thus, we have exactly two 3-cycles containing A .

Suppose that $Z = Z(x, y)$ is not consecutive. Then by Proposition 12.2 the neighbours of $Z(x, y)$ are $Z(x + 1, y)$, $Z(x, y + 1)$, $Z(x - 1, y)$ and $Z(x, y - 1)$. Since $Z(x, y)$ is not consecutive, we have $x \neq y - 2, y - 1, y, y + 1, y + 2$. Since $\text{Card}(\partial U \cap \partial X) \geq 5$ we have $x \neq x - 2, x - 1, x, x + 1, x + 2$. Inspection shows that

$$\begin{aligned} Z(x + 1, y) \cap Z(x, y + 1) &= \{x + 1, y + 1\} \\ Z(x + 1, y) \cap Z(x - 1, y) &= \{y, y + 1\} \\ Z(x, y + 1) \cap Z(x, y - 1) &= \{x, x + 1\} \\ Z(x - 1, y) \cap Z(x, y - 1) &= \{x, y\} \end{aligned}$$

If $y + 2 = x - 1$, then $Z(x, y + 1) \cap Z(x - 1, y) = \{x, y + 1, y + 2\}$ and if $y + 2 \neq x - 1$, then $Z(x, y + 1) \cap Z(x - 1, y) = \{x, y + 1\}$. If $x + 2 = y - 1$, then $Z(x + 1, y) \cap Z(x, y - 1) = \{x + 1, x + 2, y\}$ and if $y + 2 \neq x - 1$, then $Z(x, y + 1) \cap Z(x - 1, y) = \{x + 1, y\}$.

In sum, if $y + 2 \neq x - 1$ and $y + 2 \neq x - 1$, then $Z(x, y)$ does not belong to a 3-cycle. If $y + 2 \neq x - 1$ or $y + 2 \neq x - 1$ but not both, then $Z(x, y)$ belongs to exactly one 3-cycle. Finally, $y + 2 = x - 1$ and $y + 2 = x - 1$ if and only if $Z(x, y)$ belongs to exactly two 3-cycles.

In particular, if $Z(x, y)$ is nonconsecutive and belongs to two 3-cycles, then $x = x + 6$ and hence $\text{Card}(\partial U \cap \partial X) = 6$. \square

Corollary 12.4. *Each 3-cycle contains at most one non-consecutive vertex.*

Corollary 12.5. *If U is a maximal strip, then $\text{Lk}(U)$ has no 3-cycles.*

Proof. Indeed, if a quadruple Z is consecutive, then Z is a subset of one component of the boundary of the strip. Hence Z is collinear and is not realizable. \square

Proposition 12.6. *Let $\{Z, Z', Z''\}$ be a 3-cycle. Then Z is consecutive if and only if either both Z'' and Z' follow Z or Z follows both Z' and Z'' .*

Proof. If $Z = Z(x, x + 2)$ is consecutive, then by Proposition 12.3, the 3-cycles given by $\{Z(x, x + 2), Z(x - 1, x + 2), Z(x - 1, x + 1)\}$ and $\{Z(x, x + 2), Z(x, x + 2), Z(x + 1, x + 3)\}$. In the former case $Z(x, x + 2)$ follows both $Z(x - 1, x + 2)$ and $Z(x - 1, x + 1)$. In the latter, $Z(x, x + 2)$ and $Z(x + 1, x + 3)$ both follow $Z(x, x + 2)$.

If $Z = Z(x, y)$ is nonconsecutive and belongs to a 3-cycle, then by Proposition 12.3, either $x = y + 3$ and

$$\{Z(x, y), Z(x, y + 1), Z(x - 1, y)\}$$

or $y = x + 3$ and the cycle is

$$\{Z(x, y), Z(x + 1, y), Z(x, y - 1)\}.$$

In the former case, $Z(x, y)$ follows $Z(x - 1, y)$ and $Z(x, y + 1)$ follows $Z(x, y)$. In the latter case, $Z(x, y)$ follows $Z(x, y - 1)$ and $Z(x + 1, y)$ follows $Z(x, y)$. \square

Next we turn to the analysis of $\text{Lk}(U)$ in the case that U is a maximal strip. First recall that the n -dimensional infinite grid, Grid^n , is the graph whose vertex set is \mathbb{Z}^n and two vertices \vec{m} and \vec{n} are joined by an edge iff \vec{m} and \vec{n} differ in exactly one coordinate and this difference has absolute value 1. This graph can be geometrically realized as the set of points in \mathbb{R}^n all but one of whose coordinates equals an integer. Any graph that is isomorphic to the n -dimensional infinite grid will be called an n -dimensional grid graph.

Proposition 12.7. *If $U \subset X$ is a maximal strip, then $\text{Lk}(U)$ is a 2-dimensional grid graph. In particular, if x and y are points in the frontier of X that belong to distinct components of ∂U , then*

$$f(m, n) = Z(x + m, y + n)$$

defines an isomorphism $f : \text{Grid}^2 \rightarrow \text{Lk}(U)$.

Proof. Let ℓ_x (resp. ℓ_y) denote the component of ∂U containing x (resp. y .) For each $x' \in \ell_x$ (resp. $y' \in \ell_y$) there exists a unique m (resp. n) such that $x' = x + m$ (resp. $y' = y + n$). It follows that f is a bijection. Since U is a strip, each 1-cell is non-consecutive. Therefore (6) implies that f is a graph isomorphism. \square

We will say that a subset L of a grid graph G is a *line* if and only if L is isomorphic to a 1-dimensional grid and no two consecutive edges of L belong to a cycle of order 4. Let $\mathcal{L}(G)$ denote the set of lines in G . The lines in the infinite grid correspond exactly to the lines in \mathbb{R}^n that are contained within its geometric realization.

13. THE MAP OF FRONTIERS

Let (X, μ) and (X', μ') be universal covers of precompact translation surfaces with ∂X finite and nonempty. Let $\Phi : \mathcal{S}_3(X, \mu) \rightarrow \mathcal{S}_3(X', \mu')$ be an isomorphism of 2-complexes. In this section, we will assume that Φ preserves the natural orientation of each 2-cell. Such an isomorphism will be called *orientation preserving*.

The goal of this section is to prove the existence of a bijection $\beta : \partial X \rightarrow \partial X'$ so that for each cell in the 2-complex $\mathcal{S}_3(X, \mu)$ we have

$$(7) \quad \Phi(\mathcal{S}_Z) = \mathcal{S}_{\beta(Z)}.$$

We will reduce the construction of β to a ‘local’ problem associated to a rigid subconic U . In particular, we say that a bijection $\beta_U : \partial U \cap \partial X \rightarrow \partial \Phi(U) \cap \partial X'$ is *adapted to Φ* if and only if for each realizable $Z \subset \partial U \cap \partial X$ we have

$$(8) \quad \Phi(\mathcal{S}_Z) = \mathcal{S}_{\beta_U(Z)}.$$

The following proposition tells us that the ‘global’ construction of β is equivalent to the ‘local’ problem of constructing β_U adapted to Φ for each rigid conic U .

Proposition 13.1. *Let U and $U' \in \mathcal{S}_5(X, \mu)$. If β_U and $\beta_{U'}$ are adapted to Φ , then for each $x \in \partial U \cap \partial U' \cap \partial X$ we have $\beta_U(x) = \beta_{U'}(x)$.*

Proof. By Proposition 9.11, it suffices to suppose that U and U' are endpoints of a 1-cell in $\mathcal{S}_3(X, \mu)$. In particular, $Z = \partial U \cap \partial U'$ is a quadruple $\{x_1, x_2, x_3, x_4\}$. Let Z_i be the triple $Z \setminus \{z_i\}$. The triple Z_i is realizable, and hence

$$\mathcal{S}_{\beta_U(Z_i)} = \Phi(\mathcal{S}_{Z_i}) = \mathcal{S}_{\beta_{U'}(Z_i)}.$$

Hence, $\beta_U(Z_i) = \beta_{U'}(Z_i)$ for each i . Therefore,

$$\begin{aligned} \beta_U(\{x_j\}) &= \beta_U\left(\bigcap_{i \neq j} Z_i\right) \\ &= \bigcap_{i \neq j} \beta_U(Z_i) \\ &= \bigcap_{i \neq j} \beta_{U'}(Z_i) \\ &= \beta_{U'}\left(\bigcap_{i \neq j} Z_i\right) \\ &= \beta_{U'}(\{x_j\}). \end{aligned}$$

□

Notation 13.2. Since each cell is determined by a subset of the frontier, the map Φ may be regarded as a map from certain subsets of ∂X to subsets of $\partial X'$. Abusing notation slightly, we will sometimes use Φ to denote this mapping of subsets. For example, if $Z \subset \partial X$ is a realizable triple and \mathcal{S}_Z is the associated 2-cell, then we will let $\Phi(Z)$ denote the triple that determines the image 2-cell $\Phi(\mathcal{S}_Z)$.

We first define β_U in the case that U is a maximal strip. Let $x \in \partial U \cap \partial X$. The isomorphism Φ determines a graph isomorphism of $\text{Lk}(U)$ onto $\text{Lk}(\Phi(U))$. Since $\text{Lk}(U)$ is infinite, $\text{Lk}(\Phi(U))$ is infinite, and hence $\Phi(U)$ is a maximal strip. Since U and $\Phi(U)$ are maximal strips, each quadruple $Z \subset \partial U \cap \partial X$ with $\mathcal{S}_Z \neq \emptyset$ is of the form $Z_U(x, y)$ where x and y belong to different components of ∂U . Let ℓ_x denote the component containing x and let ℓ_x^\perp denote the component that does not contain x .

Given $x \in \partial U \cap \partial X$, let L_x be the line in $\text{Lk}(U)$ consisting of 1-cells $Z_U(x, y)$ with $y \in \ell_x^\perp$. The image of L_x is a line in $\text{Lk}(\Phi(U))$ and hence there exists unique $\beta(x) \in \Phi(U) \cap \partial X$ such that

$$(9) \quad \Phi(L_x) = L_{\beta(x)}.$$

Since Φ is invertible, the map $\beta : \partial U \cap \partial X \rightarrow \partial \Phi(U) \cap \partial X'$ is invertible. Indeed, given $x' \in \partial \Phi(U) \cap \partial X'$, define $\beta^{-1}(x')$ by the identity $L_{\beta^{-1}(x')} = \Phi^{-1}(L_{x'})$.

Proposition 13.3. *Let U be a maximal strip. The map β_U defined in (9) is adapted to Φ . Moreover,*

$$(10) \quad \beta_U \circ s_U = s_{\Phi(U)} \circ \beta_U.$$

Proof. In this proof, we will suppress the subscripts ‘ U ’ and ‘ $\Phi(U)$ ’ since they will be clear from the context.

Let $y \in \ell_x^\perp$. Then $Z(x, y)$ is a 1-cell that belongs to L_x , and hence $\Phi(Z(x, y))$ belongs to the line $L_{\beta(x)}$. In particular, there exists $y' \in \ell_{\beta(x)}^\perp$ such that $\Phi(Z(x, y)) =$

$Z(\beta(x), y')$. Note that $Z(x, y) = Z(y, x)$ and hence $\Phi(Z(y, x)) = Z(\beta(y), x)$ for some $x' \in \ell_{\beta(y)}^\perp$. Hence since $Z(\beta(x), y') = Z(y', \beta(x))$, we have

$$(11) \quad Z(\beta(y), x') = \Phi(Z(y, x)) = Z(y', \beta(x)).$$

Since $L_x \cap L_y = Z(x, y)$, we have $L_{\beta(x)} \neq L_{\beta(y)}$. In particular, $\beta(x) \neq \beta(y)$. Hence it follows from (11) that

$$(12) \quad \Phi(Z(x, y)) = Z(\beta(x), \beta(y)).$$

By Proposition 11.6, the 1-cell $Z(s_U(x), y)$ follows $Z(x, y)$ in $\text{Lk}(U)$. Thus, since Φ is orientation preserving, the 1-cell $\Phi(Z(s_U(x), y))$ follows $\Phi(Z(x, y))$ in $\text{Lk}(\Phi(U))$. Or, equivalently by (12), the 1-cell $Z(\beta(s_U(x)), \beta(y))$ follows $Z(\beta(x), \beta(y))$. But by Proposition 11.6, the 1-cell $Z(s(\beta(x)), \beta(y))$ follows $Z(\beta(x), \beta(y))$. Hence

$$Z(s(\beta(x)), \beta(y)) = Z(\beta(s(x)), \beta(y))$$

and therefore (10) holds as desired.

If $Z \subset \partial U \cap \partial X$ is realizable, then since U is a strip, then either $Z = \partial U \cap \partial X$, $Z = Z(x, y)$ for x, y on different components of ∂U , or $Z = Z(x, y) \cap Z(x, s(y))$ for x, y on different components. Since β is a surjection, in the first case we have $\Phi(\mathcal{S}_{\partial U \cap \partial X}) = \mathcal{S}_{\beta(\partial U \cap \partial X)}$. Suppose that $x, y \in \partial X$ belong to different components of ∂U . By (12) and (10), we have

$$\begin{aligned} \Phi(\{x, s(x), y, s(y)\}) &= \{\beta(x), s(\beta(x)), \beta(y), s(\beta(y))\} \\ &= \{\beta(x), \beta(s(x)), \beta(y), \beta(s(y))\}. \end{aligned}$$

Hence

$$\Phi(\{x, s(x), s(y), s^2(y)\}) = \{\beta(x), \beta(s(x)), \beta(s(y)), \beta(s^2(y))\}$$

and thus since $\Phi(Z(x, y) \cap Z(x, s(y))) = \Phi(Z(x, y)) \cap \Phi(Z(x, s(y)))$,

$$\Phi(\{x, s(x), s(y)\}) = \{\beta(x), \beta(s(x)), \beta(s(y))\}.$$

□

Lemma 13.4. *A quadruple Z is consecutive with respect to the rigid conic U if and only if $\Phi(Z)$ is consecutive with respect to $\Phi(U)$.*

Proof. If Z is consecutive, then exactly two 3-cycles contain Z . Let $\{Z, Z', Z''\}$ be such a 3-cycle. By Proposition 12.6, either both Z' and Z'' follow Z or Z follows Z' and Z'' . Since Φ is orientation preserving, we have that either both $\Phi(Z')$ and $\Phi(Z'')$ follow $\Phi(Z)$ or $\Phi(Z)$ follows $\Phi(Z')$ and $\Phi(Z'')$. Hence by applying Proposition 12.6 to $\{\Phi(Z), \Phi(Z'), \Phi(Z'')\}$ we find that $\Phi(Z)$ is consecutive.

The converse follows by considering Φ^{-1} . □

We now define β_U in the case that U is a rigid ellipse interior. If $\text{Card}(\partial U \cap \partial X) = n$, then the set of consecutive vertices constitute a directed n -cycle in $\text{Lk}(U)$. Lemma 13.4 implies that Φ maps this directed n -cycle bijectively onto the directed n -cycle of consecutive vertices in $\text{Lk}(\Phi(U))$. For each $x \in \partial U \cap \partial X$, define $\beta_U(x)$ be the unique element of $\partial\Phi(U) \cap \partial X'$ such that

$$(13) \quad \Phi(Z_U(x, s_U^2(x))) = Z_{\Phi(U)}(\beta(x), s_{\Phi(U)}^2(\beta(x))).$$

Proposition 13.5. *Let U be a rigid ellipse interior. The map β_U defined by (13) is adapted to Φ . Moreover,*

$$(14) \quad \beta_U \circ s_U = s_{\Phi(U)} \circ \beta_U.$$

Proof. Since $Z(s(x), s^3(x))$ follows $Z(x, s^2(x))$ and Φ is orientation preserving, $Z(\beta(s(x)), \beta(s^3(x)))$ follows $Z(\beta(x), \beta(s^2(x)))$. But $Z(s(\beta(x)), s^3(\beta(x)))$ follows $Z(\beta(x), \beta(s^2(x)))$ and so $s(\beta(x)) = \beta(s(x))$.

Each realizable quadruple $Z \subset \partial U \cap \partial X$ is of the form $Z(x, s^{n+1}(x))$ for some $x \in \partial X$ and $n \geq 1$. If $n = 1$, then it follows from (13) and (14) that

$$(15) \quad \Phi(Z(x, s^{n+1}(x))) = \beta(Z(x, s^{n+1}(x))).$$

If (15) holds for some n , then by using (14), the fact that $Z(x, s^{n+2}(x))$ follows $Z(x, s^{n+1}(x))$, the fact that $Z(\beta(x), s^{n+2}(\beta(x)))$ follows $Z(\beta(x), s^{n+1}(\beta(x)))$, and the fact that Φ is orientation preserving, we find that (15) holds for $n + 1$. Hence for each quadruple Z we have $\Phi(Z) = \beta(Z)$. The claim for triples follows by considering intersections and the claim for rigid conics is clear. \square

We summarize this section with the following.

Theorem 13.6. *If $\Phi : \mathcal{S}_3(X, \mu) \rightarrow \mathcal{S}_3(X', \mu')$ is an orientation preserving cell complex isomorphism, then there exists a unique bijection $\beta : \partial X \rightarrow \partial X'$ such that for each realizable $Z \subset \partial X$ we have*

$$\Phi(\mathcal{S}_Z) = \mathcal{S}_{\beta(Z)}.$$

Moreover, for each rigid conic $U \subset X$, we have

$$\beta \circ s_U = s_{\Phi(U)} \circ \beta.$$

14. CONFIGURATIONS, ORIENTED BISECTORS, AND EIGENLINES

In this section we will prove a geometric lemma that will be used in the following section to prove Theorem 1.1. We will assume throughout that (X, μ) is the universal covering of a precompact translation surface with nonempty and finite frontier.

Definition 14.1. We will say that two subconics in the plane are *equivalent up to homothety* if and only if they differ by a homothety and/or a translation. Let $[U]$ denote the equivalence class of the subconic $[U]$.

Two (non-circular) ellipse interiors are equivalent if and only if their major axes are parallel and they have equal eccentricities.⁶

A *configuration of subconics about an ellipse* consists of an ellipse interior U , a finite subset $Z \subset \partial U$ with $\text{Card}(Z) \geq 5$, and for each nonconsecutive pair $x, y \in Z$, a subconic $U_{x,y} \in \mathcal{E}(\mathbb{R}^2)$ such that $Z \cap \partial U_{x,y} = \{x, y\}$.

Lemma 14.2 (Geometric Lemma). *Let $(U, Z, U_{x,y})$ and $(U', Z', U'_{x,y})$ be two configurations of subconics about an ellipse. If $[U] = [U']$ and there exists a bijection $\beta : Z \rightarrow Z'$ such that $\beta \circ s_U = s_{U'} \circ \beta$, and for each nonconsecutive pair $x, y \in Z$ we have*

$$[U_{x,y}] = [U'_{\beta(x), \beta(y)}],$$

then for each $x \in Z$, the line $\ell(x, s(x))$ is parallel to the line $\ell(\beta(x), \beta(s(x)))$.

⁶The referee has informed us that some refer to this data as the ‘complex dilatation’.

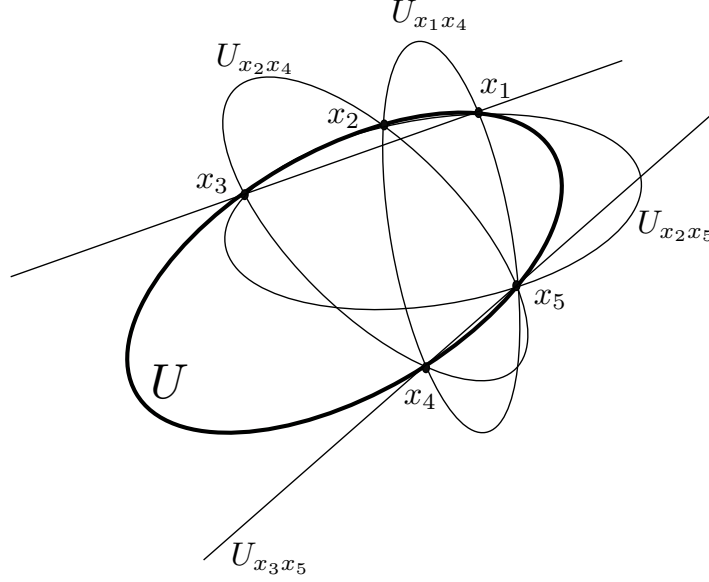


FIGURE 3. A configuration of subconics about the the ellipse U .

The remainder of this section is dedicated to proving Lemma 14.2. We first discuss the geometry of eigenvectors for degenerate quadratic forms in \mathbb{R}^2 . Then at the end of the section, we use this discussion to prove Lemma 14.2.

Let $q \in Q(\mathbb{R}^2)$ be a quadratic form of signature $(2, 0)$. Let $SO(q)$ denote the group of orientation preserving linear transformations that preserve q . This group acts transitively and freely on the ‘ellipse’ $q^{-1}(1)$, and thus $SO(q)$ is homeomorphic to a circle. There exists a unique orientation preserving universal covering map $e : \mathbb{R} \rightarrow SO(q)$ such that e is a group homomorphism and $2\pi = \inf\{\theta > 0 \mid e(\theta) = \text{Id}\}$. Of course, one should be thinking of the example $q(x_1, x_2) = x_1^2 + x_2^2$ in which case

$$e(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Lemma 14.3. *Let $\theta_- \neq \theta_+ \in SO(q)$ and let $x \in \mathbb{R}^2$. The point*

$$e\left(\frac{\theta_- + \theta_+}{2}\right) \cdot x$$

is q -orthogonal to $e(\theta_+) \cdot x - e(\theta_-) \cdot x$.

Proof. Let $v_{\pm} = e(\theta_{\pm}) \cdot x$. The orthogonal complement of $v_+ - v_-$ is the fixed point set of the involution $\iota : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\iota(w) = w - \frac{2 \cdot q(w, v_+ - v_-)}{q(v_+ - v_-, v_+ - v_-)} \cdot (v_+ - v_-).$$

We also have the involution $\iota' : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\iota'(\theta) = \theta - (\theta_+ - \theta_-)/2$. One checks that $e \circ \iota' = \iota \circ e$. Since $(\theta_- + \theta_+)/2$ is a fixed point of ι' , the claim follows. \square

Let (x, y) be an ordered pair of distinct vectors belonging to $q^{-1}(r^2)$, the q -circle of radius r . Let

$$\theta = \inf\{\theta' > 0 \mid e(\theta') \cdot x = y\},$$

and define

$$v_{xy} = e\left(\frac{\theta}{2}\right) \cdot x,$$

Note that $v_{xy} = -v_{yx}$ and hence $\alpha_{vw} = -\alpha_{wv}$. We will refer to v_{xy} as the *oriented q -bisector* of (x, y) .

Definition 14.4. Let $F = (x_0, \dots, x_n) \subset q^{-1}(r^2)$ be an ordered n -tuple of distinct points. We will say that the (*ordering of*) F is *compatible with e* if and only if there exist $0 = \theta_0 < \dots < \theta_n \leq 2\pi$ such that $x_i = e(\theta_i) \cdot x_0$.

For example, let x, y be distinct points in $q^{-1}(r^2)$. The ordered triple (x, v_{xy}, y) is compatible with e , but (x, v_{yx}, y) is not.

Let $q(\cdot, \cdot)$ denote the polarization of the quadratic form q . For each ordered pair (x, y) of distinct points in $q^{-1}(r^2)$, define the linear form α_{xy} that is q -dual to v_{xy} by

$$(16) \quad \alpha_{xy}(w) = q(w, v_{xy}).$$

For each ordered quadruple $Q = (x_0, x_1, x_2, x_3)$ of distinct points, define a quadratic form q_Q by

$$(17) \quad q_Q(v) = -\alpha_{x_0x_1}(v) \cdot \alpha_{x_2x_3}(v).$$

The signature of q_Q is either $(1, 0)$, $(0, 1)$, or $(1, 1)$. Hence, the form q_Q has two distinct eigenvalues, $\lambda^- \leq 0 \leq \lambda^+$, with respect to q . Let L_Q^\pm denote the 1-dimensional eigenspace associated to λ^\pm .

Proposition 14.5. *The vector $v_{x_2x_3} - v_{x_0x_1}$ belongs to L_Q^+ .*

Proof. The polarization of q_Q is given by

$$q_Q(v, w) = \alpha_{x_0x_1}(v) \cdot \alpha_{x_2x_3}(w) + \alpha_{x_0x_1}(w) \cdot \alpha_{x_2x_3}(v).$$

Thus, using (16) and the fact that $q(v_{x_0x_1}) = r^2 = q(v_{x_2x_3})$, we find that for each $w \in \mathbb{R}^2$

$$q_Q(v_{x_2x_3} - v_{x_0x_1}, w) = (r^2 - q(v_{x_2x_3}, v_{x_0x_1})) \cdot q(v_{x_2x_3} - v_{x_0x_1}, w).$$

Thus, by definition, $v_{x_2x_3} - v_{x_0x_1}$ is an eigenvector with eigenvalue $r^2 - q(v_{x_2x_3} - v_{x_0x_1})$. Since, by assumption, $v_{x_2x_3} \neq v_{x_0x_1}$, this eigenvalue is positive. \square

Proposition 14.6. *Let $0 \leq \theta_0 < \theta_1 < \theta_2 < \theta_3 < 2\pi$ and set $x_i = e(\theta_i) \cdot x$. The vector*

$$u_Q := e\left(\frac{\theta_0 + \theta_1 + \theta_2 + \theta_3}{4}\right) \cdot x$$

belongs to the eigenline L_Q^- where $Q = (x_0, x_1, x_2, x_3)$.

We will call u_Q , the *eigenvector associated* to the ordered quadruple Q .

Proof. Since $0 < \theta_1 - \theta_0 < 2\pi$, we have $\theta_1 - \theta_0 = \inf\{\theta > 0 \mid e(\theta) \cdot x_0 = x_1\}$. Thus, $v_{x_0x_1} = e((\theta_0 + \theta_1)/2) \cdot x$. Similarly, $v_{x_2x_3} = e((\theta_2 + \theta_3)/2) \cdot x$. Thus, by Lemma 14.3, u_Q is orthogonal to $v_{x_2x_3} - v_{x_0x_1}$. Therefore, since L^+ and L^- are orthogonal, the claim follows from Proposition 14.5. \square

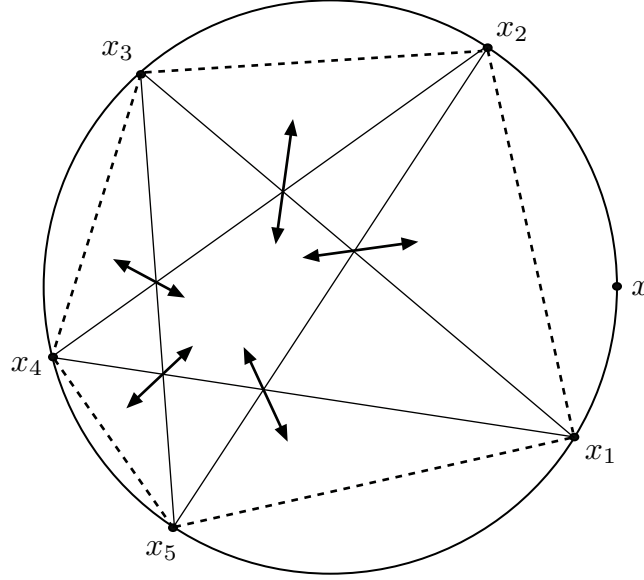


FIGURE 4. The geometric content of Proposition 14.7: The directions of the bisectors (\leftrightarrow) of successive diagonals determine the directions of the dotted lines. The direction of the bisector of $\overline{x_i x_{i+2}}$ and $\overline{x_{i+1} x_{i+3}}$ is determined by the average of the four angles, $(\theta_i + \theta_{i+1} + \theta_{i+2} + \theta_{i+3})/4$. The direction of the dotted line $\overline{x_i x_{i+1}}$ is determined by $(\theta_i + \theta_{i+1})/2$.

We now apply the preceding discussion to obtain an intermediate form of the geometric lemma. Let Z be a finite subset of $q^{-1}(r^2)$ that contains at least five points. Let $s : Z \rightarrow Z$ be the successor function associated to the counter-clockwise orientation. In particular, $s(x) = y$ if and only if there exists $\theta > 0$ such that $e(\theta \cdot x) = y$ and $e((0, \theta') \cdot x) \cap Z = \emptyset$.

Recall that, given two nonsuccessive points in Z , the symbol $Z(x, y)$ denotes the ordered quadruple $(x, s(x), y, s(y))$.

Proposition 14.7. *Let Z and Z' be finite subsets of the unit circle ∂U_{q_0} each with cardinality at least five. If there exists a bijection $\beta : Z \rightarrow Z'$ such that $\beta \circ s = s \circ \beta$ and for each pair of nonsuccessive vertices $x, y \in Z$, we have*

$$u_{Z(x,y)} = u_{Z(\beta(x),\beta(y))},$$

then for each $x \in Z$, we have that

$$v_{xs(x)} = v_{\beta(x)s(\beta(x))}.$$

Proof. Let $x \in Z$. For $-2 < i \leq 3$, define

$$\theta_i = \inf \{ \theta > 0 \mid e(\theta_i) \cdot s^{-2}(x) = s^i(x) \}.$$

Since $\text{Card}(Z) \geq 5$, we have

$$0 = \theta_{-2} < \theta_{-1} < \theta_0 < \theta_1 < \theta_2 \leq \theta_3 \leq 2\pi.$$

Similarly, we obtain

$$0 = \theta'_{-2} < \theta'_{-1} < \theta'_0 < \theta'_1 < \theta'_2 \leq \theta'_3 \leq 2\pi.$$

so that $e(\theta_i) = s^i(\beta(x))$.

It follows from the hypothesis that

$$\begin{aligned} \frac{\theta_0 + \theta_1 + \theta_{-2} + \theta_{-1}}{4} &= \frac{\theta'_0 + \theta'_1 + \theta'_{-2} + \theta'_{-1}}{4} \\ \frac{\theta_{-2} + \theta_{-1} + \theta_1 + \theta_2}{4} &= \frac{\theta'_{-2} + \theta'_{-1} + \theta'_1 + \theta'_2}{4} \\ \frac{\theta_0 + \theta_1 + \theta_2 + \theta_3}{4} &= \frac{\theta'_0 + \theta'_1 + \theta'_2 + \theta'_3}{4} \\ \frac{\theta_2 + \theta_3 + \theta_{-1} + \theta_0}{4} &= \frac{\theta'_2 + \theta'_3 + \theta'_{-1} + \theta'_0}{4} \\ \frac{\theta_1 + \theta_2 + \theta_{-1} + \theta_0}{4} &= \frac{\theta'_1 + \theta'_2 + \theta'_{-1} + \theta'_0}{4} \end{aligned}$$

By taking the alternating sum of the left hand sides of these equations and the alternating sum of the right hand sides, we find that

$$\frac{\theta_0 + \theta_1}{2} = \frac{\theta'_0 + \theta'_1}{2}.$$

The claim follows. \square

To prove the main geometric lemma, we shift the context to quadratic forms on \mathbb{R}^3 . In particular, q will denote an element of $Q(\mathbb{R}^3)$ and \underline{q} will denote the ‘restriction’ to the first two coordinates

$$\underline{q}(x_1, x_2) = q(x_1, x_2, 0).$$

Proposition 14.8. *Let $q \in Q(\mathbb{R}^3)$. If q has signature $(2, 1)$ and \underline{q} has signature $(2, 0)$, then there exists a (unique) homothety h and a (unique) translation τ so that $h \circ \tau(\partial U_q) = \underline{q}^{-1}(1)$.*

Proof. Let $A = (a_{ij})$ be the Gram determinant of q with respect to the standard basis for \mathbb{R}^3 . We have

$$q(x_1, x_2, x_3) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3.$$

We have

$$\underline{q}(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2.$$

Let $\vec{x} = (x_1, x_2)$ and $\vec{t} = (t_1, t_2)$. A straightforward calculation shows that

$$(18) \quad q(x_1 - t_1, x_2 - t_2, 1) = \underline{q}(\vec{x}) + 2(\vec{a} - \vec{t} \cdot \underline{A}) \cdot \vec{x} + \underline{q}(\vec{t}) - 2\vec{t} \cdot \vec{a} + a_{33}.$$

where $\vec{a} = (a_{13}, a_{23})$ and

$$\underline{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}.$$

Set

$$\vec{t} = \vec{a} \cdot A^{-1}.$$

One computes that

$$\underline{q}(\vec{t}) - 2\vec{t} \cdot \vec{a} + a_{33} = \frac{\det(A)}{\det(\underline{A})},$$

and hence from (18) it follows that $\vec{x} - \vec{t} \in \partial U_q$ iff

$$(19) \quad \underline{q}(\vec{x}) = -\frac{\det(A)}{\det(\underline{A})}.$$

Since q has signature $(2, 1)$, we have $\det(A) < 0$, and since \underline{q} has signature $(2, 0)$ we have that $\det(\underline{A}) > 0$. Therefore, the right hand side of (19) is positive.

Let $\tau(\vec{x}) = \vec{x} + \vec{t}$ and $h(\vec{x}) = (-\det(A)/\det(\underline{A}))^{-\frac{1}{2}} \cdot \vec{x}$. \square

Proof of Lemma 14.2. Note that the set of weights on a configuration is invariant under homotheties and translations.⁷ Also note that parallelism is invariant under homotheties and translations. In particular, without loss of generality, we have $U = U'$.

Since U is an ellipse interior, there exists a quadratic form $q \in Q(\mathbb{R}^3)$ of signature $(2, 1)$ such that $U_q = U$ and such that \underline{q} has signature $(2, 0)$. By Proposition 14.8, there exists a homothety h and a translation τ such that $h \circ \tau(U) = \underline{q}^{-1}(1)$. It follows that without loss of generality, $U = \underline{q}^{-1}(1)$.

Let $x, y \in Z$ be distinct, nonsuccessive points. We will consider three quadratic forms that belong to the plane, $Q_{\widehat{Z(x,y)}}$, consisting of quadratic forms that vanish on the quadruple $\widehat{Z(x,y)}$.

The first form is q .

The second form is a quadratic form r_{xy} such that $U_{r_{xy}} = U_{xy}$. Note that since $\partial U_{xy} \cap Z = Z(x, y)$, we have $r_{xy} \in Q_{\widehat{Z(x,y)}}$. Since $\text{Card}(Z) \geq 5$ and $U_{xy} \neq U$, the restriction of r_{xy} is negative on the arc in $\partial \widehat{U}$ that joins \hat{x} to $\widehat{s(x)}$.

The third quadratic form is a degenerate form defined as follows. Let $\eta_{xy} \in (\mathbb{R}^3)^*$ be a linear form such that $\eta_{xy}(\hat{x}) = 0 = \eta_{xy}(\widehat{s(x)})$ and η_{xy} is negative on the interior of the convex hull of $Z(x, y)$. Let $\eta'_{xy} \in (\mathbb{R}^3)^*$ be such that $\eta'_{xy}(\hat{y}) = 0 = \eta'_{xy}(\widehat{s(y)})$ and η'_{xy} is negative on the interior of the convex hull of $\widehat{Z(x,y)}$. The third quadratic form is defined by $p_{xy} = -\eta_{xy} \cdot \eta'_{xy}$. Note that $p_{x,y} \in Q_{\widehat{Z(x,y)}}$ and p_{xy} is positive on the arc in $\partial \widehat{U}$ that joins \hat{x} to $\widehat{s(x)}$.

Note that all three forms are nonpositive on the convex hull of $\widehat{Z(X,Y)}$. Thus, since q vanishes on the arc in $\partial \widehat{U}$ that joins \hat{x} to $\widehat{s(x)}$ and $\dim(Q_{\widehat{Z(x,y)}}) = 2$, there exist positive $a, b \in \mathbb{R}^+$ such that

$$(20) \quad q = a \cdot r_{x,y} + b \cdot p_{x,y}.$$

Since r_{xy} is not a multiple of q (resp. $p_{x,y}$ is not a multiple of q), the form $\underline{r_{x,y}}$ (resp. $\underline{p_{x,y}}$) has two distinct eigenvalues $\mu_- < \mu_+$ (resp. $\nu_+ < \nu_-$) with respect to \underline{q} . Let $M_{x,y}^\pm$ (resp. $N_{x,y}^\pm$) denote the eigenspace of $\underline{r_{x,y}}$ (resp. $\underline{p_{x,y}}$) associated to μ_\pm (resp. ν_\mp). It follows from (20) that $M_{x,y}^\pm = N_{x,y}^\mp$ and

$$b \cdot \nu_\pm = 1 - a \cdot \mu_\mp.$$

The discussion above applies equally well to a pair of nonadjacent distinct points $x', y' \in Z'$. We define forms, $r_{x',y'}$, $p_{x',y'}$, and eigenspaces, $M_{x',y'}^\pm = N_{x',y'}^\mp$, in an analogous fashion.

⁷If a translation or homothety is applied to one element of a configuration, then it should be applied to all members of the configuration.

Since, by hypothesis, $[U_{x,y}] = [U'_{\beta(x),\beta(y)}]$, the forms $\underline{r_{x,y}}$ and $\underline{r_{\beta(x),\beta(y)}}$ differ by a positive constant multiple. In particular, $M_{x,y}^+ = M_{\beta(x),\beta(y)}^+$, and thus $N_{x,y}^- = N_{\beta(x),\beta(y)}^-$.

Let $q_{Z(x,y)} \in Q(\mathbb{R}^2)$ be defined as in (17). Note that the form $\underline{r_{x,y}}$ is a positive multiple of $q_{Z(x,y)}$. In particular, the eigenvector $u_{Z(x,y)}$ belongs to $N_{x,y}^-$. Similarly, the eigenvector $u_{Z'(\beta(x),\beta(y))}$ belongs to $N_{\beta(x),\beta(y)}^-$. Since $N_{x,y}^- = N_{\beta(x),\beta(y)}^-$ is one dimensional and $q(u_{Z(x,y)}) = 1 = q(u_{Z'(\beta(x),\beta(y))})$, we have $u_{Z(x,y)} = \pm u_{Z'(\beta(x),\beta(y))}$. It then follows from the relation $s' \circ \beta = \beta \circ s$ that we have equality.

Thus, by Proposition 14.7, for each $x \in Z$ we have $v_{xs(x)} = v_{\beta(x)\beta(s(x))}$. Note that $v_{xs(x)}$ (resp. $v_{\beta(x)\beta(s(x))}$) is perpendicular to the line $\ell(s, s(x))$ (resp. $\ell(\beta(x), \beta(s(x)))$). \square

15. ISOMORPHISMS AND AFFINE HOMEOMORPHISMS

In this section, we prove Theorem 1.1. We begin with a lemma that reduces the construction of a homeomorphism to a local problem.

Lemma 15.1. *Let (X, μ) and (X', μ') be simply connected translation surfaces that cover a precompact translation surface with finite and nonempty frontier. Let $\Phi : \mathcal{S}_3(X, \mu) \rightarrow \mathcal{S}_3(X', \mu')$ be an orientation preserving isomorphism and let $\beta : \partial X \rightarrow \partial X'$ be the compatible bijection. If for each realizable quadruple $Z \subset \partial X$, there exists $\psi_Z \in \text{Aff}^+(\mathbb{R}^2)$ such that for each $z \in Z$, we have*

$$\psi_Z(\mathbf{dev}_\mu(z)) = \mathbf{dev}_{\mu'}(\beta(z)),$$

then there exists $g \in GL^+(\mathbb{R}^2)$ and a homeomorphism $\phi : X \rightarrow X'$ such that $g \circ \mu = \mu' \circ \phi$ and $\phi(U) = \Phi(U)$ for each $U \in \mathcal{S}_3(X, \mu)$.

Proof. Let $Z, Z' \subset \partial X$ be realizable quadruples such that $\text{Card}(Z \cap Z') = 3$. For each $x \in \mathbf{dev}_\mu(Z \cap Z')$, we have $\psi_Z(x) = \psi_{Z'}(x)$. Since each element of $\text{Aff}^+(\mathbb{R}^2)$ is determined by its values at three points, we have $\psi_Z = \psi_{Z'}$. By Proposition 9.11, the graph $\mathcal{S}_4(X, \mu)$ is connected, and hence the function $Z \mapsto \psi_Z$ is constant. Let $\psi \in \text{Aff}^+(\mathbb{R}^2)$ denote this constant value.

Let $U \in \mathcal{S}_5(X, \mu)$. The map ψ sends $\mathbf{dev}_\mu(\partial U \cap \partial X)$ onto $\mathbf{dev}_{\mu'}(\partial \Phi(U) \cap \partial X')$. Thus, since $\text{Card}(\partial U \cap \partial X) \geq 5$ and $\text{Card}(\partial \Phi(U) \cap \partial X') \geq 5$, we have $\psi(\mathbf{dev}_\mu(U)) = \mathbf{dev}_{\mu'}(\Phi(U))$. In particular, we may define a homeomorphism $\phi_U : U \rightarrow \Phi(U)$ by

$$\phi_U = \mathbf{dev}_\mu|_{\Phi(U)}^{-1} \circ \psi \circ \mathbf{dev}_\mu(x).$$

Let Z be a realizable quadruple, and let $U_-, U_+ \in \mathcal{S}_5(X, \mu)$ be the endpoints of \mathcal{S}_Z . For each $x \in \mathbf{dev}_{\mu'}(\Phi(U_-) \cap \Phi(U_+))$, we have

$$\mathbf{dev}_\mu|_{\Phi(U_-)}^{-1}(x) = \mathbf{dev}_\mu|_{\Phi(U_+)}^{-1}(x).$$

Therefore, it follows that $\phi_{U_+}|_{U_+ \cap U_-} = \phi_{U_-}|_{U_+ \cap U_-}$. Since $\mathcal{S}_4(X, \mu)$ is a connected graph, for any $U, V \in \mathcal{S}_5(X, \mu)$, we have

$$\phi_U|_{U \cap V} = \phi_V|_{U \cap V}.$$

Define $\phi : X \rightarrow X'$ by setting

$$\phi(x) = \phi_U(x)$$

if $x \in U \in \mathcal{S}_4(X, \mu)$. By considering the same construction for Φ^{-1} and β^{-1} , we obtain an inverse for ϕ . Let g denote the differential of ψ . Then $g \cdot \mu = \mu' \circ \phi$. \square

As a first application, we have the following.

Proposition 15.2. *Let (X, μ) and (X', μ') be simply connected translation surfaces that cover a precompact translation surface with finite and nonempty frontier. Let $\Phi : \mathcal{S}_3(X, \mu) \rightarrow \mathcal{S}_3(X', \mu')$ be an orientation preserving isomorphism. If each rigid conic $U \in \mathcal{S}_5(X, \mu)$ is a strip, then there exist $g \in GL_2^+(\mathbb{R})$ and a homeomorphism $\phi : X \rightarrow X'$ such that $g \cdot \mu = \mu' \circ \phi$.*

Proof. Let $\beta : \partial X \rightarrow \partial X'$ be the bijection that is compatible with Φ .

Each realizable quadruple $Z \subset \partial X$ is the intersection of the boundaries of two strips. In particular, $\mathbf{dev}_\mu(Z)$ is the vertex set of a parallelogram. Since Φ is an isomorphism, each rigid subconic $U \in \mathcal{S}_5(X, \mu)$ is also a strip. Thus, it follows that $\mathbf{dev}_\mu(\beta(Z))$ is also the vertex set of a parallelogram. Since β is orientation preserving, there exists a $\psi_Z \in \text{Aff}^+(\mathbb{R}^2)$ such that $\psi_Z(\mathbf{dev}_\mu(z)) = \mathbf{dev}_{\mu'}(\beta(z))$. The claim now follows from Lemma 15.1. \square

Recall that the developing map \mathbf{dev}_μ is determined up to post-composition by translations. In particular, each subset $A \subset X$ determines a unique class $[A] := [\mathbf{dev}_\mu(A)]$.

We will say that a pair $\{x, y\} \in \partial X$ is realizable if and only if there exists a subconic U such that $\{x, y\} = \partial U \cap \partial X$. Note that since ∂X is discrete, realizability of a pair is equivalent to the existence of a geodesic segment joining x and y . Indeed, given a geodesic segment σ joining x and y and $\epsilon > 0$, there exists an ellipse interior U such that $\{x, y\} \subset \partial U$ and the Hausdorff distance between σ and \bar{U} is less than ϵ .

Proposition 15.3. *Let (X, μ) and (X', μ') be simply connected translation surfaces that cover a precompact translation surface with finite and nonempty frontier. Let $\Phi : \mathcal{S}_3(X, \mu) \rightarrow \mathcal{S}_3(X', \mu')$ be an orientation preserving isomorphism. If for each rigid subconic $U \in \mathcal{S}_5(X, \mu)$ we have*

$$(21) \quad [\Phi(U)] = [U],$$

then for each realizable pair x, y , we have

$$(22) \quad \ell_\mu(x, y) = \ell_{\mu'}(\beta(x), \beta(y)).$$

Proof. If x, y realizable, then there exists a subconic $U \in \mathcal{S}_5(X, \mu)$ such that $\{x, y\} \subset \partial U \cap \partial X$ and either $s_U(x) = y$ or $s_U(y, x)$. Without loss of generality $y = s_U(x)$.

If U is a strip, then $\Phi(U)$ is a strip. Since $[\Phi(U)] = [U]$, each boundary component of $\mathbf{dev}_\mu(U)$ is parallel to each boundary component of $\mathbf{dev}_{\mu'}(\Phi(U))$. It follows that $\ell_\mu(x, y) = \ell_{\mu'}(\beta(x), \beta(y))$.

If U is an ellipse, then $\Phi(U)$ is an ellipse, and the claim then follows from Lemma 14.2. \square

Theorem 15.4. *Let (X, μ) and (X', μ') be translations surfaces that each cover precompact translation surfaces with finite and nonempty frontier. Let $\Phi : \mathcal{S}_3(X, \mu) \rightarrow \mathcal{S}_3(X', \mu')$ be an isomorphism. If for each rigid subconic $U \in \mathcal{S}_5(X, \mu)$ we have*

$$(23) \quad [\Phi(U)] = [U],$$

then there exists $h \in H$ and a homeomorphism $\phi : X \rightarrow X'$ such that $h \cdot \mu = \mu' \circ \phi$ and $\phi(U) = \Phi(U)$ for each $U \in \mathcal{S}_3(X, \mu)$.

Proof. We first suppose that X and X' are simply connected. Let $\beta : \partial X \rightarrow \partial X'$ be the bijection compatible with Φ .

Let $Z \subset \partial X$ be a realizable quadruple. Let $U \in \mathcal{S}_Z$ and let $z \in Z = \partial U \cap \partial X$. By Proposition 15.3, for each $z, z' \in Z$, we have $\ell(\beta(z), \beta(z')) = \ell(z, z')$.

Let $\{z_1, z_2, z_3\} \subset Z$. Since Z is in general position, there exists a unique $\psi_{z_1, z_2, z_3} \in H(\mathbb{R}^2) \cdot T(\mathbb{R}^2)$ such that $\psi(\mathbf{dev}_\mu(z_i)) = \mathbf{dev}_{\mu'}(\beta(z_i))$ for each $i = 1, 2, 3$. Since each element of $H(\mathbb{R}^2) \cdot T(\mathbb{R}^2)$ is determined by its values on two points, ψ_{z_1, z_2, z_3} does not depend on the choice of triple $\{z_1, z_2, z_3\} \subset Z$. Let ψ_Z denote the common value of ψ_{z_1, z_2, z_3} . The case of simply connected surfaces then follows from Lemma 15.1.

If X and X' are not simply connected, then we can reduce to the simply connected case by considering the universal coverings $p : \tilde{X} \rightarrow X$ and $p' : \tilde{X}' \rightarrow X'$.

The subspaces of ellipses, $\mathcal{E}_3(X, \mu)$ and $\mathcal{E}_3(X', \mu')$, are obtained by removing strip vertices from $\mathcal{S}_3(X, \mu)$ and $\mathcal{S}_3(X', \mu')$ respectively. Since the link of a strip is not homeomorphic to link of any ellipse, the homeomorphism Φ restricts to a homeomorphism from $\mathcal{E}_3(X, \mu)$ onto $\mathcal{E}_3(X', \mu')$.

By Corollary 8.5, the restriction of p (resp. p') to $\mathcal{E}_3(\tilde{X}, \tilde{\mu})$ (resp. $\mathcal{E}_3(\tilde{X}', \tilde{\mu}')$) is a covering onto $\mathcal{E}_3(X, \mu)$ (resp. $\mathcal{E}_3(X', \mu')$). It follows that the homeomorphism Φ on ellipses lifts to a homeomorphism $\tilde{\Phi} : \mathcal{E}_3(\tilde{X}, \tilde{\mu}) \rightarrow \mathcal{E}_3(\tilde{X}', \tilde{\mu}')$. Moreover, the map $\gamma \rightarrow \Phi \circ \gamma \circ \Phi^{-1}$ defines an isomorphism from $\text{Gal}(\tilde{X}/X)$ to $\text{Gal}(\tilde{X}'/X')$.

We claim that $\tilde{\Phi}$ extends to a homeomorphism from $\mathcal{S}_3(\tilde{X}, \tilde{\mu})$ to $\mathcal{S}_3(\tilde{X}', \tilde{\mu}')$. Indeed, a given strip vertex U is an extreme point of a convex polygonal 2-cell Z in $\mathcal{S}_3(\tilde{X}, \tilde{\mu})$. The map $\tilde{\Phi}$ maps Z homeomorphically onto a 2-cell Z' in $\mathcal{S}_3(\tilde{X}', \tilde{\mu}')$. Moreover each 1-cell in the boundary of Z is mapped homeomorphically onto the corresponding 1-cell in the boundary of Z' . In sum, we have a homeomorphism between two closed convex polygons each with finitely many vertices removed that maps each edge of one polygon homeomorphically onto an edge of the other. An elementary argument provides an extension to a homeomorphism from \bar{Z} to \bar{Z}' . Since the link of the vertex U is connected, an inductive argument shows that the extension to U does not depend on the choice of Z . An inverse for the extension can be constructed by extending $\tilde{\Phi}^{-1}$.

If (23) holds for each subconic U , then it holds for each lifted subconic $\tilde{U} \subset \tilde{X}$. Hence Theorem 15.4 provides a homeomorphism $\tilde{\phi} : \tilde{X} \rightarrow \tilde{X}'$ and a homothety h such that $h \cdot \mu = \mu' \circ \tilde{\phi}$. Moreover $\tilde{\phi}(U) = \Phi(U)$ for each $U \in \mathcal{S}_3(\tilde{X}, \tilde{\mu})$, and hence the map $\gamma \mapsto \tilde{\phi} \circ \gamma \circ \tilde{\phi}^{-1}$ defines an isomorphism of H onto H' . In particular, $\tilde{\phi}$ descends to a homeomorphism $\phi : X \rightarrow X'$ so that $h \cdot \mu = \mu' \circ \phi$. \square

16. CHARACTERIZING THE VEECH GROUP

Let $SL(\mathbb{R}^2)$ denote the multiplicative group of 2×2 matrices with unit determinant. Recall that the Veech group $\Gamma(X, \mu)$ consists of those $g \in SL(\mathbb{R}^2)$ such that there exists a homeomorphism $\phi : X \rightarrow X$ with $g \circ \mu = \mu \circ \phi$ [Vch89] [GtKJdg00].

Theorem 16.1. *$\Gamma(X, \mu)$ consists of those $g \in SL(\mathbb{R}^2)$ such that there exists an orientation preserving homeomorphism $\Phi : \mathcal{S}_3(X, \mu) \rightarrow \mathcal{S}_3(X, \mu)$ such that for each $U \in \mathcal{S}_5(X, \mu)$ we have $[\Phi(U)] = [g'(U)]$.*

Proof. Let $g \in SL(\mathbb{R}^2)$ and let $\phi : X \rightarrow X$ be a homeomorphism such that $g \circ \mu = \mu \circ \phi$. The homeomorphism ϕ acts on subsets of X . Since $g \circ \mu = \mu \circ \phi$, we have

$$(24) \quad [\mathbf{dev}_\mu(\phi(U))] = [g \cdot \mathbf{dev}_\mu(U)],$$

In particular, if U is a subconic then the set $\mathbf{dev}_\mu(\phi(U))$ is also a subconic. Since $g \circ \mu = \mu \circ \phi$, the homeomorphism ϕ extends to homeomorphism $\bar{\phi} : \bar{X} \rightarrow \bar{X}$. If $\text{Card}(\partial U \cap \partial X) = n$, then we have $\text{Card}(\bar{\phi}(\partial U \cap \partial X)) = n$. Thus ϕ defines a homeomorphism from $\mathcal{S}_3(X, \mu) \rightarrow \mathcal{S}_3(X, \mu)$ that satisfies (24).

Conversely, let $\Phi : \mathcal{S}_3(X, \mu) \rightarrow \mathcal{S}_3(X, \mu)$ and $g' \in SL(\mathbb{R}^2)$ such that for each $U \in \mathcal{S}_5(X, \mu)$ we have $[\mathbf{dev}_\mu(\Phi(U))] = [g'(\mathbf{dev}_\mu(U))]$. If we let $\mu' = g \circ \mu$, then Φ defines a homeomorphism from $\mathcal{S}_3(X, \mu) \rightarrow \mathcal{S}_3(X, \mu')$ such that $[\mathbf{dev}_\mu(\Phi(U))] = [\mathbf{dev}_{\mu'}(U)]$. Hence we can apply Theorem 15.4 to obtain a homeomorphism $\phi : X \rightarrow X$ and a homothety h such that $h \circ \mu' = \mu \circ \phi$. In particular, $h \circ g \circ \mu = \mu \circ \phi$. Since $\det(g) = 1$, ϕ is a homeomorphism and (X, μ) has finite area, the homothety h is the identity. \square

REFERENCES

- [Berger] Marcel Berger, *Geometry*. Vol I, II. Translated from the French by M. Cole and S. Levy. Universitext. Springer-Verlag, Berlin, (1987).
- [Bwm08] Joshua P. Bowman, *Teichmüller geodesics, Delaunay triangulations, and Veech groups*. To appear in ‘Proceedings of the International Workshop on Teichmüller theory and moduli problems’.
- [BrtJdg11] S. Allen Broughton and Chris Judge, *Subconics and tessellations of Teichmüller discs*. In preparation.
- [GtgJdg00] Eugene Gutkin and Chris Judge, *Affine mappings of translation surfaces: geometry and arithmetic*. Duke Math. J. 103 (2000), no. 2, 191–213.
- [HMSZ06] Pascal Hubert, Howard Masur, Thomas Schmidt, and Anton Zorich, *Problems on billiards, flat surfaces and translation surfaces*. Problems on mapping class groups and related topics, 233–243, Proc. Sympos. Pure Math., **74**, Amer. Math. Soc., Providence, RI, 2006.
- [MsrSml91] Howard Masur and John Smillie, *Hausdorff dimension of sets of nonergodic measured foliations*. Annals of Math. (2) **134** (1991), no. 3, 455–543.
- [MsrTbc02] Howard Masur and Serge Tabachnikov, *Rational billiards and flat structures* in Handbook of dynamical systems, Vol. **1A**, 1015–1089, North-Holland, Amsterdam, 2002.
- [Strebel] Kurt Strebel, *Quadratic differentials*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **5**. Springer-Verlag, Berlin, 1984.
- [Smillie00] John Smillie, *The dynamics of billiard Flows in rational polygons in Dynamical systems, ergodic theory and applications*. Edited and with a preface by Sinai. Translated from the Russian. Second, expanded and revised edition. Encyclopaedia of Mathematical Sciences, 100. Mathematical Physics, I. Springer-Verlag, Berlin, (2000), 360–382.
- [Thr98] William P. Thurston, *Shapes of polyhedra and triangulations of the sphere* in ‘The Epstein birthday schrift’, 511–549 (electronic), Geom. Topol. Monogr., 1, Geom. Topol. Publ., Coventry, 1998.
- [Thurston] William P. Thurston, *Three-dimensional Geometry and Topology* Princeton University Press, 1997.
- [Vch89] William Veech *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*. Invent. Math. 97 (1989), no. 3, 553–583.
- [Vch99] William Veech *A tessellation associated to a quadratic differential. Preliminary report*. Abstracts of the AMS 946-37-61.
- [Vch04] William Veech, *Bicuspid F-structures and Hecke groups* (preprint).

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