A characterization of the Maass space on $O(2, m+2)$ by symmetries

Bernhard Heim and Atsushi Murase

In this paper, we define certain symmetries for automorphic forms on $O(2, m+2)$ and show that the space of automorphic forms satisfying these symmetries coincides with the Maass space, the image of Saito-Kurokawa lifting.

Keywords: Automorphic forms; Theta lifting; Maass space

Mathematics Subject Classification (2000): 11F27, 11F30, 11F50

1 Introduction

In [3], the first named author introduced certain symmetries for Siegel modular forms of even degree and showed that the space of Siegel modular forms of degree two satisfying the symmetries coincides with the so-called *Maass Spezialschar*, which is the space of Siegel modular forms whose Fourier coefficients satisfy Maass relations. Note that this space coincides with the image of Saito-Kurokawa lifting. Later Bringmann and Heim proved certain symmetries for Jacobi Eisenstein series of degree two ([1]).

On the other hand, Oda ([5]) and Rallis-Schiffmann ([6]) independently studied a theta lifting from elliptic modular forms of integral or half-integral weight to automorphic forms on the orthogonal group $O(2, m+2)$. Note that, if $m = 1$, the theta lifting coincides with the Saito-Kurokawa lifting. Later Gritsenko ([2]) and Sugano ([7]) studied the theta lifting in terms of Jacobi forms of degree 1. It is known that the image of the theta lifting coincides with the space of holomorphic automorphic forms whose Fourier coefficients satisfy the Maass relation ([7]). Thus it is natural to ask whether certain symmetries characterizes the Maass space in the general orthogonal group case. The object of the paper is to give an affirmative answer to this question.

In this paper, we introduce symmetries of automorphic forms on $G = O(2, m+2)$ arising from two embeddings of SL_2 into G, and show that the space of holomorphic automorphic forms on G satisfying the symmetries coincides with the Maass space.

The paper is organized as follows. In Section 2, we first recall the definitions of automorphic forms on $G = O(2, m+2)$ and the Maass space. After defining certain symmetries for automorphic forms on G, we state the main result of the paper (Theorem 2.2): The space of automorphic forms satisfying these symmetries coincides with the space of those satisfying Maass relations. As a direct consequence of the characterizaion of the Maass space by symmetries, we show that the restriction mapping induced by an embedding $G' = O(2, m+1) \hookrightarrow G = O(2, m+2)$ maps the Maass space on G to that on G'. The proof of Theorem 2.2 is carried out in Section 3. By using some combinatorics, we prove an algebraic result (Proposition 3.2), from which Theorem 2.2 follows.

Notation

The upper half plane is denoted by $\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. For a real symmetric matrix R of degree n, we put $R(x, y) = {}^t x R y$ and $R[x] = {}^t x R x$ for $x, y \in \mathbb{C}^n$. For a condition P, we put

$$\delta(P) = \begin{cases} 1 & \text{if } P \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $\mathbb{N}$ the set of natural numbers. We put $\mathbf{e}[z] = \exp(2\pi\sqrt{-1}z)$ for $z \in \mathbb{C}$.

2 Main results

2.1 The orthogonal group G

Let S be a positive definite even integral symmetric matrix of degree m. We put

$$Q_1 = \begin{pmatrix} & & 1 \\ & -S & \\ 1 & & \end{pmatrix}, \quad Q = \begin{pmatrix} & & 1 \\ & Q_1 & \\ 1 & & \end{pmatrix}.$$

In the following, we include the case of $m = 0$. Note that the signatures of Q_1 and Q_2 are $(1, m+1)$ and $(2, m+2)$, respectively.

Let

$$\begin{aligned} L_0 &= \mathbb{Z}^m, L_0^* = S^{-1}L_1, V_0 = L_0 \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}^m, \\ L_1 &= \mathbb{Z}^{m+2}, L_1^* = Q_1^{-1}L_1, V_1 = L_1 \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}^{m+2}, \\ L &= \mathbb{Z}^{m+4}, L^* = Q^{-1}L, V = L \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}^{m+4}. \end{aligned}$$

Let $G = O(Q)$ be the orthogonal group of Q and G_∞^+ the identity component of $G_\infty = G(\mathbb{R})$. Let

$$\mathcal{D} = \left\{ Z = \begin{pmatrix} \tau \\ w \\ z \end{pmatrix} \in \mathbb{C}^{m+2} \mid \tau, z \in \mathfrak{H}, w \in \mathbb{C}^m, Q_1[\mathrm{Im}(Z)] = \mathrm{Im}(\tau)\mathrm{Im}(z) - \frac{1}{2}S[\mathrm{Im}(w)] > 0 \right\}.$$

As is well-known, $\mathcal{D}$ is a hermitian symmetric domain of type (IV). We often write (τ, w, z) for $\begin{pmatrix} \tau \\ w \\ z \end{pmatrix} \in \mathcal{D}$. We define an action of G_∞^+ on $\mathcal{D}$ and an automorphic factor $J \colon G_\infty^+ \times \mathcal{D} \to \mathbb{C}^\times$ by $g\widetilde{Z} = \widetilde{g\langle Z\rangle}\, J(g, Z)$ for $g \in G_\infty^+$ and $Z \in \mathcal{D}$, where

$$\widetilde{Z} = \begin{pmatrix} -2^{-1}Q_1[Z] \\ Z \\ 1 \end{pmatrix} \in \mathbb{C}^{m+4}.$$

Let k be an integer and F a function on $\mathcal{D}$. For $g \in G_\infty^+$, we define the Petersson slash operator by $(F|_k g)(Z) = J(g, Z)^{-k} F(g\langle Z \rangle)$.

2.2 Embeddings of SL_2 into G

Let $H = SL_2$. For $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_\infty$ and $z \in \mathfrak{H}$, let $h\langle z \rangle = (az+b)(cz+d)^{-1}$ and $j(h, z) = cz + d$ as usual. We define two embeddings $\iota^\uparrow$ and $\iota^\downarrow$ of H into G by

$$\iota^\uparrow(h) = \begin{pmatrix} a & & & -b & \\ & a & & & b \\ & & 1_m & & \\ -c & & & d & \\ & c & & & d \end{pmatrix},$$

$$\iota^\downarrow(h) = \begin{pmatrix} a & -b & & & \\ -c & d & & & \\ & & 1_m & & \\ & & & a & b \\ & & & c & d \end{pmatrix}$$

for $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$, respectively. It is easily verified that $\iota^\uparrow(h)$ and $\iota^\downarrow(h)$ commute each other and that $\iota^\uparrow(H_\infty), \iota^\downarrow(H_\infty) \subset G_\infty^+$. A straightforward calculation shows the following.

Lemma 2.1. *For $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_\infty$ and $Z = (\tau, w, z) \in \mathcal{D}$, we have*

$$\iota^\uparrow(h)\langle Z \rangle = \begin{pmatrix} h\langle \tau \rangle \\ j(h,\tau)^{-1} w \\ z - \dfrac{c}{2j(h,\tau)} S[w] \end{pmatrix}, \quad J(\iota^\uparrow(h), Z) = j(h, \tau)$$

and

$$\iota^\downarrow(h)\langle Z \rangle = \begin{pmatrix} \tau - \dfrac{c}{2j(h,z)} S[w] \\ j(h,z)^{-1} w \\ h\langle z \rangle \end{pmatrix}, \quad J(\iota^\downarrow(h), Z) = j(h, z).$$

2.3 Automorphic forms

Let Γ be a discrete subgroup of G_∞ commensurable with $\Gamma(L) = \{\gamma \in G_\infty^+ \mid \gamma L = L\}$. We assume that

$$\begin{pmatrix} 1 & -{}^t x Q_1 & -2^{-1} Q_1[x] \\ 0 & 1_{m+2} & x \\ 0 & 0 & 1 \end{pmatrix}, \ \iota^\uparrow(\gamma), \ \iota^\downarrow(\gamma') \in \Gamma \qquad (x \in \mathbb{Z}^{m+2}, \gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})). \tag{2.1}$$

Note that $\Gamma(L)$ and $\Gamma^*(L) = \{\gamma \in \Gamma(L) \mid \gamma l \equiv l \pmod{L} \text{ for any } l \in L^*\}$ satisfy this condition.

For a positive integer k, let $M_k(\Gamma)$ denote the space of holomorphic functions F on $\mathcal{D}$ satisfying the following two conditions:

$$F|_k\gamma = F \quad \text{for any } \gamma \in \Gamma. \tag{2.2}$$

$$\text{If } m = 0,\ F \text{ is holomorphic at any cusp of } \Gamma. \tag{2.3}$$

Let

$$\Lambda = \mathbb{Z} \times L_0^* \times \mathbb{Z}, \tag{2.4}$$

$$\Lambda^+ = \{(a, \alpha, b) \in \Lambda \mid a, b, 2ab - S[\alpha] \geq 0\}. \tag{2.5}$$

An automorphic form $F \in M_k(\Gamma)$ admits the Fourier expansion

$$F(\tau, w, z) = \sum_{(a,\alpha,b)\in\Lambda^+} A_F(a, \alpha, b)\mathbf{e}[az - S(\alpha, w) + b\tau].$$

We say that $\lambda = (a, \alpha, b) \in \Lambda$ is *primitive* if $(n^{-1}a, n^{-1}\alpha, n^{-1}b) \notin \Lambda$ for any $n \in \mathbb{N}, n > 1$. Denote by Λ_{prm} (respectively Λ^+_{prm}) the set of primitive elements of Λ (respectively Λ^+).

2.4 The Maass space and symmetries

We now define two subspaces of $M_k(\Gamma)$.

Let $M_k^{\mathcal{M}}(\Gamma)$ be the space of $F \in M_k(\Gamma)$ satisfying

$$A_F(la, l\alpha, lb) = \sum_{r|l} r^{k-1} A_F((r^{-1}l)^2 ab, (r^{-1}l)\alpha, 1) \tag{2.6}$$

for any $l \in \mathbb{N}$ and $(a, \alpha, b) \in \Lambda^+_{\mathrm{prm}}$, where r runs over the positive divisors of l. Note that

$$A_F(a, \alpha, b) = \sum_{d\in\mathbb{Z}_{>0},\, d^{-1}(a,\alpha,b)\in\Lambda} d^{k-1} A_F\left(\frac{ab}{d^2}, \frac{\alpha}{d}, 1\right)$$

for $F \in M_k^{\mathcal{M}}(\Gamma)$. When $\Gamma = \Gamma^*(L)$, this space coincides with the Maass space introduced by Maass ([4]) when $m = 1$ and Sugano ([7]) when $m > 1$.

To define symmetries, let

$$T_n = \{\xi \in M_2(\mathbb{Z}) \mid \det \xi = n\} = \bigcup_j SL_2(\mathbb{Z})\xi_j \quad \text{(a disjoint union)}$$

for $n \in \mathbb{N}$. Define

$$F|_k T_n^{\uparrow} = n^{k/2-1} \sum_j F|_k \iota^{\uparrow}(n^{-1/2}\xi_j),$$
$$F|_k T_n^{\downarrow} = n^{k/2-1} \sum_j F|_k \iota^{\downarrow}(n^{-1/2}\xi_j)$$

for $F \in M_k(\Gamma)$. Note that $F|_k T_n^{\uparrow}$ and $F|_k T_n^{\downarrow}$ are not in $M_k(\Gamma)$ in general. We define the space $M_k^{\mathcal{S}}(\Gamma)$ to be the space of $F \in M_k(\Gamma)$ satisfying $F|_k T_n^{\uparrow} = F|_k T_n^{\downarrow}$ for any $n \in \mathbb{N}$. It is easy to see that $F \in M_k^{\mathcal{S}}(\Gamma)$ if and only if $F|_k T_p^{\uparrow} = F|_k T_p^{\downarrow}$ for any prime number p. Observe that, for a prime number p, we have

$$(F|_k T_p^{\uparrow})(\tau, w, z) = p^{k-1} F\left(p\tau, \sqrt{p} w, z\right) + p^{-1} \sum_{c=0}^{p-1} F\left(p^{-1}(\tau + c), \sqrt{p}^{-1} w, z\right)$$

$$(F|_k T_p^{\downarrow})(\tau, w, z) = p^{k-1} F\left(\tau, \sqrt{p} w, pz\right) + p^{-1} \sum_{c=0}^{p-1} F\left(\tau, \sqrt{p}^{-1} w, p^{-1}(z + c)\right).$$

The main result of the paper is stated as follows.

Theorem 2.2. *The Maass space $M_k^{\mathcal{M}}(\Gamma)$ coincides with $M_k^{\mathcal{S}}(\Gamma)$.*

2.5 The compatibility with restrictions

Let (L_0', S') be a quadratic sub-lattice of (L_0, S). Let Q_1', Q', G' and $\mathcal{D}'$ be as in 2.1 corresponding to S'. We assume that the inverse image Γ' of Γ by the embedding $G' \subset G$ satisfies a condition similar to (2.1). Then the restriction of $F \in M_k(\Gamma)$ to $\mathcal{D}'$ gives rise to a linear mapping $j \colon M_k(\Gamma) \to M_k(\Gamma')$. Since the symmetry is compatible with j, we have proved the following.

Theorem 2.3. *We have $j(M_k^{\mathcal{S}}(\Gamma)) \subset M_k^{\mathcal{S}}(\Gamma')$ and hence $j(M_k^{\mathcal{M}}(\Gamma)) \subset M_k^{\mathcal{M}}(\Gamma')$.*

3 Proof of Theorem 2.2

3.1 Symmetries and Fourier expansion

Lemma 3.1. *Let $F \in M_k(\Gamma)$. Then $F \in M_k^{\mathcal{S}}(\Gamma)$ if and only if the following holds for any $(a, b, \alpha) \in \Lambda^+$ and any prime number p:*

$$p^{k-1} A_F(a, p^{-1}\alpha, p^{-1}b) - p^{k-1} A_F(p^{-1}a, p^{-1}\alpha, b) + A_F(a, \alpha, pb) - A_F(pa, \alpha, b) = 0.$$

Here we make a convention that $A_F(a, \alpha, b) = 0$ if $(a, \alpha, b) \notin \Lambda^+$.

Proof. We have

$$\begin{aligned}(F|_k T_p^{\uparrow})(\tau, \sqrt{p} w, z) &= p^{k-1} F(p\tau, pw, z) + p^{-1} \sum_{c=0}^{p-1} F(p^{-1}(\tau + c), w, z) \\ &= p^{k-1} \sum_{(a,\alpha,b)\in\Lambda^+} A_F(a, p^{-1}\alpha, p^{-1}b) \mathbf{e}[b\tau + az - S(\alpha, w)] \\ &\quad + \sum_{(a,\alpha,b)\in\Lambda^+} A_F(a, \alpha, pb) \mathbf{e}[b\tau + az - S(\alpha, w)]\end{aligned}$$

and

$$(F|_k T_p^{\downarrow})(\tau, \sqrt{p}w, z) = p^{k-1} \sum_{(a,\alpha,b)\in\Lambda^+} A_F(p^{-1}a, p^{-1}\alpha, b)\mathbf{e}[b\tau + az - S(\alpha, w)]$$
$$+ \sum_{(a,\alpha,b)\in\Lambda^+} A_F(pa, \alpha, b)\mathbf{e}[b\tau + az - S(\alpha, w)],$$

from which the lemma immediately follows. □

3.2 The spaces $\mathcal{F}^{\mathcal{M}}$ and $\mathcal{F}^{\mathcal{S}}$

For a function f on $\mathcal{V} = \mathbb{Q} \times V_0 \times \mathbb{Q}$ and $r \in \mathbb{N}$, we put

$$M(r)f(a, \alpha, b) = f(r^2ab, r\alpha, 1),$$
$$N(r)f(a, \alpha, b) = f(ra, r\alpha, rb) \qquad ((a, \alpha, b) \in \mathcal{V}).$$

It is easy to see that

$$M(r)f(a, \alpha, b) = M(r)f(ma, \alpha, m^{-1}b), \tag{3.1}$$
$$M(mr)f(a, \alpha, b) = M(r)f(m^2a, m\alpha, b) = M(r)f(a, m\alpha, m^2b) \tag{3.2}$$

for $r, m \in \mathbb{N}$. Let $\mathcal{F}$ be the space of functions on $\mathcal{V}$ whose support is contained in Λ. For $f \in \mathcal{F}$, a prime number p and $(a, \alpha, b) \in \mathcal{V}$, we set

$$I_pf(a, \alpha, b) = p^{k-1}f(a, p^{-1}\alpha, p^{-1}b) - p^{k-1}f(p^{-1}a, p^{-1}\alpha, b) + f(a, \alpha, pb) - f(pa, \alpha, b).$$

We define two subspaces of $\mathcal{F}$ as follows:

$$\mathcal{F}^{\mathcal{M}} = \{f \in \mathcal{F} \mid N(l)f(X) = \sum_{r|l} r^{k-1}M(r^{-1}l)f(X) \text{ for any } l \in \mathbb{N} \text{ and } X \in \Lambda_{\mathrm{prm}}\},$$
$$\mathcal{F}^{\mathcal{S}} = \{f \in \mathcal{F} \mid I_pf(X) = 0 \text{ for any prime number } p \text{ and } X \in \Lambda\}.$$

Let $F \in M_k(\Gamma)$ and consider A_F as an element of $\mathcal{F}$. In view of (2.6) and Lemma 3.1, we see that $F \in M_k^{\mathcal{M}}(\Gamma)$ if and only if $A_F \in \mathcal{F}^{\mathcal{M}}$ and that $F \in M_k^{\mathcal{S}}(\Gamma)$ if and only if $A_F \in \mathcal{F}^{\mathcal{S}}$. Thus the proof of Theorem 2.2 is now reduced to that of the following result.

Proposition 3.2. *We have* $\mathcal{F}^{\mathcal{M}} = \mathcal{F}^{\mathcal{S}}$.

3.3 Proof of $\mathcal{F}^{\mathcal{M}} \subset \mathcal{F}^{\mathcal{S}}$

In this subsection, we let p be a prime number and $f \in \mathcal{F}^{\mathcal{M}}$, and show that $I_p(X) = 0$ for any $X \in \Lambda$.

Lemma 3.3. *Let* $l = p^s n \in \mathbb{N}$ *with* $s \geq 0, n \in \mathbb{N}, p \nmid n$. *For* $X \in \Lambda_{\mathrm{prm}}$, *we have*

$$N(pl)f(X) - p^{k-1}N(l)f(X) = \sum_{r|n} r^{k-1}M(pr^{-1}l)f(X). \tag{3.3}$$

PROOF. The left-hand side of (3.3) is equal to

$$
\begin{aligned}
&\sum_{r|p^{s+1}n} M(r^{-1}p^{s+1}n)f(X) - p^{k-1}\sum_{r|p^s n} M(r^{-1}p^s n)f(X) \\
&= \sum_{j=0}^{s+1}\sum_{r|n}(p^j r)^{k-1}M(p^{s-j+1}r^{-1}n)f(X) - \sum_{j=0}^{s}\sum_{r|n}(p^j r)^{k-1}M(p^{s-j}r^{-1}n)f(X) \\
&= \sum_{r|n} r^{k-1}M(p^{s+1}r^{-1}n)f(X),
\end{aligned}
$$

which proves the lemma. □

Let $X \in \Lambda$. Then $X = (lc, l\beta, ld)$ with $l = p^s n \in \mathbb{N}$ ($s \geq 0, n \in \mathbb{N}, p \nmid n$) and $(c, \beta, d) \in \Lambda_{\mathrm{prm}}$. To simplify the notation, we write I for $I_p f(X)$. We have

$$
\begin{aligned}
I = {} & p^{k-1}N(l)f(c, p^{-1}\beta, p^{-1}d) - N(l)f(pc, \beta, d) \\
& - p^{k-1}N(l)f(p^{-1}c, p^{-1}\beta, d) + N(l)f(c, \beta, pd).
\end{aligned}
$$

First consider the case where $\beta \in L_0^* - pL_0^*$. Then $(pc, \beta, d), (c, \beta, pd) \in \Lambda_{\mathrm{prm}}$ and

$$M(r^{-1}l)f(pc, \beta, d) = M(r^{-1}l)f(c, \beta, pd)$$

for $r|l$. Suppose that $s = 0$. Then we have

$$N(l)f(c, p^{-1}\beta, p^{-1}d) = N(l)f(p^{-1}c, p^{-1}\beta, d) = 0,$$

since $p^{-1}l\beta \notin L_0^*$. It follows that

$$
\begin{aligned}
I &= -N(l)f(pc, \beta, d) + N(l)f(c, \beta, pd) \\
&= -\sum_{r|l} r^{k-1}M(r^{-1}l)f(pc, \beta, d) + \sum_{r|l} r^{k-1}M(r^{-1}l)f(c, \beta, pd) \\
&= 0.
\end{aligned}
$$

Next suppose that $s > 0$. By Lemma 3.3, we have

$$
\begin{aligned}
I &= p^{k-1}N(p^{-1}l)f(pc, \beta, d) - N(l)(pc, \beta, d) - p^{k-1}N(p^{-1}l)(c, \beta, pd) + N(l)f(c, \beta, pd) \\
&= -\sum_{r|n} r^{k-1}M(r^{-1}l)f(pc, \beta, d) + \sum_{r|n} r^{k-1}M(r^{-1}l)f(c, \beta, pd) \\
&= 0.
\end{aligned}
$$

Next consider the case where $\beta \in pL_0^*, p|c$ and $p \nmid d$. Then $(pc, \beta, d), (p^{-1}c, p^{-1}\beta, d) \in \Lambda_{\mathrm{prm}}$ and

$$M(r^{-1}l)f(pc, \beta, d) = M(pr^{-1}l)f(p^{-1}c, p^{-1}\beta, d).$$

First suppose that $s=0$. Then $N(l)f(c,p^{-1}\beta,p^{-1}d)=0$ since $p^{-1}ld\notin\mathbb{Z}$. By Lemma 3.3, we have

$$\begin{aligned} I &= -N(l)f(pc,\beta,d)-p^{k-1}N(l)f(p^{-1}c,p^{-1}\beta,d)+N(pl)f(p^{-1}c,p^{-1}\beta,d) \\ &= -\sum_{r|l} r^{k-1}M(r^{-1}l)f(pc,\beta,d)+\sum_{r|l} r^{k-1}M(pr^{-1}l)f(p^{-1}c,p^{-1}\beta,d) \\ &= 0. \end{aligned}$$

If $s>0$, we have

$$\begin{aligned} I &= p^{k-1}N(p^{-1}l)(pc,\beta,d)-N(l)f(pc,\beta,d) \\ &\quad -p^{k-1}N(l)(p^{-1}c,p^{-1}\beta,d)+N(pl)f(p^{-1}c,p^{-1}\beta,d) \\ &= -\sum_{r|n} r^{k-1}M(r^{-1}l)f(pc,\beta,d)+\sum_{r|n} r^{k-1}M(pr^{-1}l)f(p^{-1}c,p^{-1}\beta,d) \\ &= 0 \end{aligned}$$

by Lemma 3.4.

We can show that $I=0$ in the other cases in a similar way.

3.4 Proof of $\mathcal{F}^{\mathcal{S}}\subset\mathcal{F}^{\mathcal{M}}$

Let $f\in\mathcal{F}^{\mathcal{S}}$. We will show that

$$N(l)f(X)=\sum_{r|l} r^{k-1}M(r^{-1}l)f(X) \tag{3.4}$$

holds for $l\in\mathbb{N}$ and $X=(a,\alpha,b)\in\Lambda_{\mathrm{prm}}$ by induction on bl. If $bl=1$, both sides of (3.4) are equal to $f(X)$. Suppose that $bl>1$ and let p be a prime factor of bl. Then we have

$$N(l)f(X)=N(l)f(pa,\alpha,p^{-1}b)+p^{k-1}N(p^{-1}l)f(a,\alpha,b)-p^{k-1}N(l)f(a,p^{-1}\alpha,p^{-2}b).$$

First consider the case where $p\nmid l$. Then b is divisible by p. Since $(p^{-1}la,p^{-1}l\alpha,p^{-1}lb)\notin\Lambda$, we have $N(p^{-1}l)f(a,\alpha,b)=0$. Suppose that $\alpha\in L_0^*-pL_0^*$. Then $(pa,\alpha,p^{-1}b)\in\Lambda_{\mathrm{prm}}$ and $N(l)f(a,p^{-1}\alpha,p^{-2}b)=0$. By induction, we have

$$N(l)f(X)=N(l)f(pa,\alpha,p^{-1}b)=\sum_{r|l} r^{k-1}M(r^{-1}l)f(pa,\alpha,p^{-1}b)=\sum_{r|l} r^{k-1}M(r^{-1}l)f(a,\alpha,b),$$

which proves the claim (3.4). Next suppose that $\alpha\in pL_0^*$ and $\mathrm{ord}_p\, b=1$. A similar argument as above shows that

$$N(l)f(X)=N(l)f(pa,\alpha,p^{-1}b)=\sum_{r|l} r^{k-1}M(r^{-1}l)f(pa,\alpha,p^{-1}b)=\sum_{r|l} r^{k-1}M(r^{-1}l)f(a,\alpha,b).$$

Suppose that $\alpha \in pL_0^*$ and $\mathrm{ord}_p\, b \geq 2$. Then $p \nmid a$ and $(a, p^{-1}\alpha, p^{-2}b) \in \Lambda_{\mathrm{prm}}$. By induction, we have

$$\begin{aligned}
N(l)f(X) &= N(pl)f(a, p^{-1}\alpha, p^{-2}b) - p^{k-1}N(l)f(a, p^{-1}\alpha, p^{-2}b) \\
&= \sum_{r|l} r^{k-1}M(pr^{-1}l)f(a, p^{-1}\alpha, p^{-2}b) + \sum_{r|l}(pr)^{k-1}M(p(pr)^{-1}l)f(a, p^{-1}\alpha, p^{-2}b) \\
&\quad - p^{k-1}\sum_{r|l} r^{k-1}M(r^{-1}l)f(a, p^{-1}\alpha, p^{-2}b) \\
&= \sum_{r|l} r^{k-1}M(pr^{-1}l)f(a, p^{-1}\alpha, p^{-2}b) \\
&= \sum_{r|l} r^{k-1}M(r^{-1}l)f(a, \alpha, b).
\end{aligned}$$

Next consider the case where $p|l$. We let $l = p^s n$ with $s \geq 1, s \in \mathbb{N}$ and $p \nmid n$. First suppose that $p \nmid b$. Then $(p^2a, p\alpha, b) \in \Lambda_{\mathrm{prm}}$. By induction, $N(l)f(X)$ ie equal to

$$\begin{aligned}
& N(p^{-1}l)f(p^2a, p\alpha, b) + p^{k-1}N(p^{-1}l)f(a, \alpha, b) - p^{k-1}\delta(s \geq 2)N(p^{-2}l)f(p^2a, p\alpha, b) \\
&= \sum_{j=0}^{s-1}\sum_{r|n}(p^j r)^{k-1}M(p^{s-j-1}r^{-1}n)f(p^2a, p\alpha, b) + \sum_{j=0}^{s-1}\sum_{r|n}(p^{j+1}r)^{k-1}M(p^{s-j-1}r^{-1}n)f(a, \alpha, b) \\
&\quad - \delta(s \geq 2)\sum_{j=0}^{s-2}\sum_{r|n}(p^{j+1}r)^{k-1}M(p^{s-j-2}r^{-1}n)f(p^2a, p\alpha, b) \\
&= \sum_{j=0}^{s-1}\sum_{r|n}(p^j r)^{k-1}M(p^{s-j}r^{-1}n)f(a, \alpha, b) + (p^s r)^{k-1}M(r^{-1}n)f(a, \alpha, b) \\
&= \sum_{j=0}^{s}\sum_{r|n}(p^j r)^{k-1}M(p^{s-j}r^{-1}n)f(a, \alpha, b) \\
&= \sum_{r|l} r^{k-1}M(r^{-1}l)f(a, \alpha, b).
\end{aligned}$$

Suppose that either $\mathrm{ord}_p\, b = 1$ or "$\mathrm{ord}_p\, b \geq 2$ and $\alpha \in L_0^* - pL_0^*$" holds. Then $(pa, \alpha, p^{-1}b) \in \Lambda_{\mathrm{prm}}$. By induction, $N(l)f(X)$ is equal to

$$\begin{aligned}
& N(l)f(pa, \alpha, p^{-1}b) + p^{k-1}N(p^{-1}l)f(a, \alpha, b) - p^{k-1}N(p^{-1}l)f(pa, \alpha, p^{-1}b) \\
&= \sum_{r|l} r^{k-1}M(r^{-1}l)f(pa, \alpha, p^{-1}b) + \sum_{r|p^{-1}l}(pr)^{k-1}M(p^{-1}r^{-1}l)f(a, \alpha, b) \\
&\quad - \sum_{r|p^{-1}l}(pr)^{k-1}M(p^{-1}r^{-1}l)f(pa, \alpha, p^{-1}b) \\
&= \sum_{r|l} r^{k-1}M(r^{-1}l)f(a, \alpha, b).
\end{aligned}$$

Finally suppose that $\mathrm{ord}_p\, b \geq 2$ and $\alpha \in pL_0^*$. Then $p \nmid a$ and $(a, p^{-1}\alpha, p^{-2}b) \in \Lambda_{\mathrm{prm}}$. By induction, $N(l)f(X)$ is equal to

$$
\begin{aligned}
&N(pl)f(a,p^{-1}\alpha,p^{-2}b)+p^{k-1}N(p^{-1}l)f(a,\alpha,b)-p^{k-1}N(l)f(a,p^{-1}\alpha,p^{-2}b)\\
&=\sum_{r|pl} r^{k-1}M(pr^{-1}l)f(a,p^{-1}\alpha,p^{-2}b)+\sum_{r|p^{-1}l}(pr)^{k-1}M(p^{-1}r^{-1}l)f(a,\alpha,b)\\
&\quad-\sum_{r|l}(pr)^{k-1}M(r^{-1}l)f(a,p^{-1}\alpha,p^{-2}b)\\
&=\sum_{j=0}^{s+1}\sum_{r|n}(p^jr)^{k-1}M(p^{s-j+1}r^{-1}n)f(a,p^{-1}\alpha,p^{-2}b)+\sum_{j=0}^{s-1}\sum_{r|n}(p^{j+1}r)^{k-1}M(p^{s-j-1}r^{-1}n)f(a,\alpha,b)\\
&\quad-\sum_{j=0}^{s}\sum_{r|n}(p^{j+1}r)^{k-1}M(p^{s-j}r^{-1}n)f(a,p^{-1}\alpha,p^{-2}b)\\
&=\sum_{r|n} r^{k-1}M(p^{s+1}r^{-1}n)f(a,p^{-1}\alpha,p^{-2}b)+\sum_{j=0}^{s-1}\sum_{r|n}(p^{j+1}r)^{k-1}M(p^{s-j-1}r^{-1}n)f(a,\alpha,b)\\
&=\sum_{j=0}^{s}\sum_{r|n}(p^jr)^{k-1}M(p^{s-j}r^{-1}n)f(a,\alpha,b)\\
&=\sum_{r|l} r^{k-1}M(r^{-1}l)f(a,\alpha,b),
\end{aligned}
$$

which completes the proof of (3.4).

Acknowledgments
The authors gratefully acknowledges partial support from Max-Planck-Institut für Mathematik in Bonn and Mathematisches Forschungsinstitut Oberwolfach. The second named author was partially supported by Grants-in-Aids from JSPS (20540031).

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Bernhard Heim
German University of Technology in Oman (GUtech), Department of Applied Information Technology, PO Box 1816 Athaibah, PC 130, Sultanate of Oman, Corner of Beach and Wadi Athaibah Way
e-mail: bernhard.heim@gutech.edu.om

Atsushi Murase
Department of Mathematics, Faculty of Science, Kyoto Sangyo University, Motoyama, Kamigamo, Kita-ku, Kyoto 603-8555, Japan
e-mail: murase@cc.kyoto-su.ac.jp