ON A CONJECTURE OF HIVERT AND THIÉRY ABOUT STEENROD OPERATORS

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Abstract. We prove some results related to a conjecture of Hivert and Thiéry about the dimension of the space of q-harmonics ([HT]). In the process we compute the actions of the involved operators on symmetric and alternating functions, which have some independent interest. We then use these computations to prove other results related to the same conjecture.

1. Introduction

The so called *harmonics polynomials* (or $\mathfrak{S}_n$-*harmonics*) are a classical object in invariant and representation theory. They are the polynomial solutions to the system of partial differential equations

$$\nabla_k f(\mathbf{x}) = 0 \quad \text{for } k \geq 1,$$

where $\mathbf{x} = x_1, x_2, \ldots, x_n$ and the operators

$$\nabla_k := \sum_{i=1}^{n} \frac{\partial^k}{\partial x_i^k}$$

are generalized laplacians. Since the ∇_k's are symmetric, we have an action of the symmetric group $\mathfrak{S}_n$ by permutation of the variables. Hence the space of harmonic polynomials is a representation of $\mathfrak{S}_n$, that turns out to be a regular representation, whose Frobenius characteristic is (see [M])

$$F_n(t) = F_{n;0}(t) = \sum_{\lambda \vdash n} s_\lambda \sum_{T \in ST(n)} t^{co(T)},$$

where $\lambda \vdash n$ indicates that λ is a partition of n, s_λ is the Schur function indexed by λ, $ST(\lambda)$ denotes the set of standard tableaux of shape λ, and $co(T)$ denote the cocharge of the tableau T.

Recently many authors have studied various generalizations of the operators ∇_k's, looking at similar spaces of polynomials. It turns out that in many situations these spaces have conjecturally the same Hilbert series (or the Frobenius characteristic when the operators are symmetric) of the classical harmonic polynomials.

In [W97, W98, W01] Wood raised several questions about the *rational Steenrod algebra* (twisted by the algebraic Thom map), which is the subalgebra of the Weyl algebra generated by the *Steenrod operators*

$$D_k^* = \sum_{i=1}^{n} x_i^k \left(1 + x_i \frac{\partial}{\partial x_i}\right),$$

for $k \geq 1$. Let's call 1-*harmonic polynomials* the ones killed by the duals of the D_k^*'s with respect to the scalar product defined by

$$\langle f(\mathbf{x}), g(\mathbf{x})\rangle := f(\partial)g(\mathbf{x})\big|_{\mathbf{x}=0},$$

where $f(\partial)$ denote the differential operator obtained from $f(\mathbf{x})$ by substituting the variables x_i with the operators $\frac{\partial}{\partial x_i}$. Among other things, Wood asked (in a different language) if the space of 1-harmonic polynomials is a graded regular representation of the symmetric group $\mathfrak{S}_n$ (*Rational hit conjecture*). We refer to the works of Wood for motivations in Algebraic Topology.

In [HT] Hivert and Thiéry considered a deformed version of those operators (and their duals), introducing the *q-Steenrod algebra*. They investigated questions similar to the ones that Wood asked, finding interesting phenomena: consider the operators

$$D_{k;q} := q\widetilde{D}_k + \nabla_k,$$

with $\widetilde{D}_k := \sum_{i=1}^n x_i\partial_i^{k+1}$ and $\nabla_k := \sum_{i=1}^n \partial_i^k$, where $\partial_j := \frac{\partial}{\partial x_j}$, acting on $\mathbb{C}(q)[\mathbf{x}] := \mathbb{C}(q)[x_1, \ldots, x_n]$, and q is an indeterminate or a complex number.

We put

$$\mathcal{H}_{\mathbf{x};q} := \{g \in \mathbb{C}(q)[\mathbf{x}] \mid D_{k;q}f = 0 \text{ for all } k \geq 1\},$$

and we call its elements *q-harmonics*. Also, we denote by

$$\sum_{d\geq 0} \dim \pi_d(\mathcal{H}_{\mathbf{x};q})t^d$$

its Hilbert series.

Notice that the group $\mathfrak{S}_n$ acts on these spaces by permutation of the variables, since the operators involved are symmetric.

Remark. Observe that for $q = 0$ we retrieve the $\mathfrak{S}_n$-harmonics, while for $q = 1$ the $D_{k;1}$'s are the dual of the Steenrod operators. In fact the idea of Hivert and Thiéry was to "interpolate" the two situations via the coefficient q.

In [HT] Hivert and Thiéry proved the following theorem and stated the following conjecture.

Theorem 1.1 ([HT]). *When q in an indeterminate, if we denote by $[n]_t!$ the usual t-analogue of n-factorial, we have*

$$\sum_{d\geq 0} \dim \pi_d(\mathcal{H}_{\mathbf{x};q})t^d << [n]_t!$$

with '$<<$' denoting coefficient-wise inequality.

In fact from this theorem it follows (see [BGW]) that in this case $\mathcal{H}_{\mathbf{x};q}$ is isomorphic to a graded $\mathfrak{S}_n$-submodule of the $\mathfrak{S}_n$-harmonics.

Conjecture 1. *In the case where q is a variable or a complex number not of the form $-a/b$ where $a \in \{1, 2, \ldots, n\}$ and $b \in \mathbb{N}$, we have the equality*

$$\sum_{d\geq 0} \dim \pi_d(\mathcal{H}_{\mathbf{x};q})t^d = [n]_t!.$$

In particular, in the case where q is a variable, $\mathcal{H}_{\mathbf{x};q}$ is isomorphic as a graded $\mathfrak{S}_n$-module to the $\mathfrak{S}_n$-harmonics.

Notice that in the case where q is a complex number the same inequality of Theorem 1.1 is not known even for generic values of q.

After this work, in [BGW] Bergeron, Garsia and Wallach investigated even more general operators, bringing new insights in this subject. Among other things, using commutative algebra, they proved the following theorem.

Theorem 1.2 ([BGW]). *For any value of $q \in \mathbb{C}$ the dimension of the space of q-harmonics in n variables does not exceed $(n+1)!$.*

Notice that of course the conjectured dimension for generic values of $q \in \mathbb{C}$ is $n!$.

The common feature of all these works is the appearance of a graded representations of $\mathfrak{S}_n$ which is conjecturally isomorphic to the classical $\mathfrak{S}_n$-harmonics.

The present work arose from an attempt to make some progress on Conjecture 1.

1.1. **First reductions.** *Unless otherwise stated, q will always be an indeterminate. We will discuss the case $q \in \mathbb{C}$ mainly in the last section of the present work.*

We start with a general remark. Let $f \in \mathcal{H}_{\mathbf{x};q}$. By multiplying by an element of $\mathbb{C}(q)$, we can always assume that

$$f = f_0 + f_1 q + f_2 q^2 + \cdots + f_m q^m$$

with $f_i \in \mathbb{C}[\mathbf{x}]$ for all $i = 1, \ldots, m$ and $f_0 \neq 0 \neq f_m$. It's easy to see (cf. [BGW] or see later) that f_0 is necessarily an $\mathfrak{S}_n$-harmonic.

Lemma 1.3. *Conjecture 1 is true if and only if for any $\mathfrak{S}_n$-harmonic g we have a q-harmonic f with $f_0 = g$.*

Proof. Suppose that the conjecture is true, and fix a basis $g_1, \ldots, g_{n!}$. We can assume that each g_i is of the form

$$g_i = \sum_{j=0}^{m_i} g_{i,j} q^j,$$

with $g_{i,j} \in \mathbb{C}[\mathbf{x}]$ and $g_{i,0} \neq 0 \neq g_{i,m_i}$ for all i. We can also assume that the sequence $\underline{m} := (m_1, m_2, \ldots, m_{n!})$ is in increasing order. Choose a basis with minimal $\underline{m}$ with respect to the lexicographic order. We claim that $\{g_{1,0}, g_{2,0}, \ldots, g_{n!,0}\}$ is a basis for the $\mathfrak{S}_n$-harmonics. If not, then we can find a non-trivial linear combination

$$\sum_{i=1}^{n!} \alpha_i g_{i,0} = 0.$$

But then we can replace $g_{n!}$ by the linear combination

$$\sum_{i=1}^{n!} \alpha_i g_i,$$

and after dividing by a suitable power of q we get a new basis, with a smaller $\underline{m}$, which gives a contradiction. From this the "only if" part follows.

The other implication is similar: choose a basis $\{g_{1,0}, g_{2,0}, \ldots, g_{n!,0}\}$ of the $\mathfrak{S}_n$-harmonics, and by using the hypothesis we can find q-harmonics $g_1, \ldots, g_{n!}$ such that

$$g_i = \sum_{j=0}^{m_i} g_{i,j} q^j.$$

I claim that these are independent over $\mathbb{C}(q)$. If not, we would have a nontrivial combination

$$\sum_{i=1}^{n!} \alpha_i(q) g_i = 0,$$

with $\alpha_i(q) \in \mathbb{C}(q)$. Of course we can normalize these coefficients so that they are all polynomials, and at least one non-zero coefficient has non-zero constant term. But then the constant term of this linear combination would give a non-trivial linear relation among the $g_{i,0}$'s, which gives a contradiction. □

From the easy relations

$$[D_{k;q}, D_{h;q}] = q(k-h) D_{k+h;q},$$

it follows that a polynomial f is in $\mathcal{H}_{\mathbf{x};q}$ if and only if

$$D_{1;q} f = D_{2;q} f = 0.$$

This is clearly true even for $q \in \mathbb{C}$, $q \neq 0$.

It's easy to show (see [BGW]) that the previous two equations are equivalent to the following system of equations:

$$\begin{array}{rcll} \nabla_k f_0 & = & 0, & \\ \nabla_k f_i & = & -\widetilde{D}_k f_{i-1} & \text{for } i = 1, 2, \ldots, m, \\ \widetilde{D}_k f_m & = & 0, & \end{array} \tag{1}$$

for $k = 1, 2$. Notice in particular that f_0 is an $\mathfrak{S}_n$-harmonic.

Together with the previous lemma, this shows that if for any $\mathfrak{S}_n$-harmonic f_0 we are able to find $f_1, f_2, \ldots$ that satisfy those equations, then the conjecture is true.

In this work we try to attack Conjecture 1 using these observations. The idea would be to construct the entire sequence $f_1, f_2, \ldots$ for any f_0. We only succeeded in constructing an f_1 for any f_0, and the corresponding f_2 for some special $\mathfrak{S}_n$-harmonic. We found two methods to achieve this, one computationally heavier than the other, that provide different solutions. We present both of them, since the hope is to eventually find the entire sequence $f_1, f_2, \ldots$.

Along the way we determine the action of the operators ∇_1, ∇_2, $\widetilde{D}_1$ and $\widetilde{D}_2$ on symmetric and alternating polynomials, which is of independent interest.

In fact in the last part we will use these actions to prove some results related to the conjecture in the case $q \in \mathbb{C}$.

1.2. **Further reductions.** The first goal is to show how to construct an f_1 for any $\mathfrak{S}_n$-harmonic. Before doing that we want to show that it's enough to construct an f_1 for $f_0 = \partial_1 \Delta$, where Δ denotes the Vandermonde determinant in the variables $x_1, \ldots, x_n$.

Remark 1. In what follows we will repeatedly use the observation that any symmetric homogeneous differential operators that lower the degree kills the Vandermonde determinant. This is true since when we act on Δ with such an operator we still get an alternant, but of a lower degree. This forces it to be zero since the Vandermonde determinant is the alternant of smallest possible degree.

Suppose that we know how to construct such an f_1. By permuting its variables, it's clear how to construct an f_1 for $\partial_i\Delta$ for all i's. Let's call it $f_1^{(i)}$.

Also, remember that the partial derivatives of Δ span the space of $\mathfrak{S}_n$-harmonics. Hence by linearity it's enough to find an f_1 for any of those derivatives.

We set for any multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$

$$\partial^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\ldots\partial_n^{\alpha_n}.$$

Remark 2. We have

$$[\widetilde{D}_k, \partial^\alpha] = -\sum_{\alpha_i\neq 0} \alpha_i \partial^{\alpha+kv_i},$$

where $v_i \in \mathbb{N}^n$ is the vector with 1 in the i-th position and 0 elsewhere. Since clearly $\tilde{D}_k\Delta = 0$ (cf. Remark 1), it follows that

$$-\widetilde{D}_k\partial^\alpha\Delta = \sum_{\alpha_i\neq 0} \alpha_i\partial^{\alpha+kv_i}\Delta = \sum_{\alpha_i\neq 0} \alpha_i\partial^{\alpha-v_i}(\partial_i^{k+1}\Delta).$$

Hence if we set

$$f_1^\alpha := \sum_{\alpha_i\neq 0} \alpha_i\partial^{\alpha-v_i} f_1^{(i)},$$

we have

$$\nabla_k f_1^\alpha = -\widetilde{D}_k\partial^\alpha\Delta$$

for all multi-indices α and $k = 1, 2$.

We are left with the task of computing $f_1^{(1)}$.

1.3. **Organization of the paper.** The rest of the paper is organized in the following way:

- In the second section we find an f_1 for $\partial_1\Delta$.
- In the third section we find an entire family of f_1's, which include the previous one as a special case. For one member of this family we find an f_2 also, but we relegated the computations in the appendix.
- In the fourth section we show another method of finding an f_1 and an f_2 for $\partial_1\Delta$.
- In the fifth section we compute systematically the action of the operators ∇_1, ∇_2, $\widetilde{D}_1$ and $\widetilde{D}_2$ on symmetric and alternating polynomials.
- In the sixth section we discuss the case $q \in \mathbb{C}$. We apply our formulae to investigate what we will call "singular" values of q. We prove that most of the values excluded in Conjecture 1 are indeed singular, and we finally state a new conjecture on these singular values.

2. Computation of f_1 for $\partial_1\Delta$

We want to construct an $f_1 = f_1^{(1)}$ for $f_0 = \partial_1\Delta$. We can of course assume that f_1 is homogeneous.

We want

$$\nabla_k f_1 = -\tilde{D}_k \partial_1 \Delta$$

for $k = 1, 2$. We already noticed that $[\tilde{D}_k, \partial_1] = -\partial_1^{k+1}$, so we can rewrite those equations as

$$\nabla_k f_1 = \partial_1^{k+1}\Delta.$$

We now assume that f_1 is of the form $\Delta^{(1)}g$, where $\Delta^{(1)}$ is the Vandermonde in the variables $x_2, \ldots, x_n$ and g is a polynomial of the form

$$g = \sum_{j=1}^{n-2} g_j x_1^j$$

where each g_j is a symmetric polynomial in $x_2, \ldots, x_n$ homogeneous of degree $n-2-j$. In this case we get

$$\begin{aligned}
\nabla_1 f_1 &= (\nabla_1 \Delta^{(1)})g + \Delta^{(1)}(\nabla_1 g) \\
&= \Delta^{(1)} \left(\sum_{s=0}^{n-2} (\nabla_1 g_s) x_1^s + \sum_{j=1}^{n-2} j g_j x_1^{j-1} \right) \\
&= \Delta^{(1)} \left(\sum_{s=0}^{n-3} (\nabla_1 g_s + (s+1) g_{s+1}) \, x_1^s \right)
\end{aligned}$$

where the second equality holds since $\nabla_1 \Delta^{(1)} = 0$.

We fix the notation $e_k := e_k(x_2, \ldots, x_n)$, which will be used also in the following sections, except the last one. We start by recording some easy identities:

$$\begin{aligned}
\nabla_1 e_k &= (n-k)e_{k-1}; \\
\nabla_1^s e_k &= (n-k)(n-k+1)\cdots(n-k+s-1)e_{k-s} \quad \text{for } s \geq 1; \\
\nabla_1 e_k^a &= a(n-k)e_k^{a-1}e_{k-1}; \\
\nabla_h e_k &= 0 \qquad \text{for all } h \geq 2.
\end{aligned} \tag{2}$$

We have

$$\begin{aligned}
\partial_1^2\Delta &= \Delta^{(1)}\left(\partial_1^2\prod_{j=2}^{n}(x_1-x_j)\right)\\
&= \Delta^{(1)}\left(\partial_1^2\sum_{j=0}^{n-1}(-1)^j e_j x_1^{n-1-j}\right)\\
&= \Delta^{(1)}\left(\sum_{j=0}^{n-3}(-1)^j(n-1-j)(n-2-j)e_j x_1^{n-3-j}\right)\\
&= \Delta^{(1)}\left(\sum_{s=0}^{n-3}(-1)^{n-3-s}(s+2)(s+1)e_{n-3-s}x_1^s\right).
\end{aligned}$$

Equating the coefficients we get the system of equations

Lemma 2.1.

$$(\mathbf{C}1)\quad (-1)^{n-3-s}(s+2)(s+1)e_{n-3-s} = \nabla_1 g_s + (s+1)g_{s+1} \quad \textit{for } s = 0,1,\ldots,n-3.$$

This system can be integrated in many ways. We now use these equations to write all the g_j's for $j \geq 1$ in terms of $\nabla_1^h g_0$ for $h \geq 0$.

Lemma 2.2. *For $s = 1,2,\ldots,n-2$ we have the following formula:*

$$(\bullet)\qquad g_s = (-1)^{n-2-s}(s+1)s\, e_{n-2-s} + \frac{(-1)^s}{s!}\nabla_1^s g_0.$$

Proof. First of all notice that for $s \geq 0$ we can write the equations $(\mathbf{C}1)$ as

$$g_{s+1} = (-1)^{n-3-s}(s+2)e_{n-3-s} - \frac{1}{s+1}\nabla_1 g_s.$$

We proceed by induction on s, the case $s = 1$ being just equation $(\mathbf{C}1)$. Assume that the result is true for $s \geq 1$. Then we have

$$\begin{aligned}
g_{s+1} &= (-1)^{n-3-s}(s+2)e_{n-3-s} - \frac{1}{s+1}\nabla_1 g_s\\
&= (-1)^{n-3-s}(s+2)e_{n-3-s} - \frac{1}{s+1}\nabla_1\left((-1)^{n-2-s}(s+1)s\, e_{n-2-s} + \frac{(-1)^s}{s!}\nabla_1^s g_0\right)\\
&= (-1)^{n-3-s}\left((s+2)+s(s+2)\right)e_{n-3-s} + \frac{(-1)^{s+1}}{(s+1)!}\nabla_1^{s+1}g_0\\
&= (-1)^{n-3-s}(s+2)(s+1)\, e_{n-3-s} + \frac{(-1)^{s+1}}{(s+1)!}\nabla_1^{s+1}g_0.
\end{aligned}$$

□

Of course to find what we want, we need to take into account the other set of equations coming from $\nabla_2 f_1 = \partial_1^3 \Delta$. We have

$$\begin{aligned}
\sum_{i=1}^{n} \partial_i^2 f_1 &= \sum_{i=1}^{n} \Big((\partial_i^2 \Delta^{(1)}) g + \Delta^{(1)} (\partial_i^2 g) + 2(\partial_i \Delta^{(1)} \cdot \partial_i g) \Big) \\
&= (\nabla_2 \Delta^{(1)}) g + \Delta^{(1)} (\nabla_2 g) + \sum_{i=1}^{n} 2(\partial_i \Delta^{(1)} \cdot \partial_i g) \\
&= \Delta^{(1)} (\nabla_2 g) + \sum_{i=2}^{n} 2(\partial_i \Delta^{(1)} \cdot \partial_i g).
\end{aligned}$$

Dividing by $\Delta^{(1)}$ we get

$$\begin{aligned}
\frac{1}{\Delta^{(1)}} \sum_{i=1}^{n} \partial_i^2 f_1 &= \nabla_2 g + 2 \sum_{i=2}^{n} \left(\frac{\partial_i \Delta^{(1)}}{\Delta^{(1)}} \cdot \partial_i g \right) \\
&= \nabla_2 g + 2 \sum_{i=2}^{n} \Big(\partial_i \log \Delta^{(1)} \cdot \partial_i g \Big) \\
&= \nabla_2 g + 2 \sum_{i=2}^{n} \left(\sum_{2 \le j \le n,\ j \ne i} \frac{(-1)^{\chi(i>j)}}{x_i - x_j} \cdot \partial_i g \right) \\
&= \nabla_2 g + 2 \sum_{2 \le i < j \le n} \frac{1}{x_i - x_j} \left(\partial_i - \partial_j \right) g,
\end{aligned}$$

where $\chi(\mathcal{P})$ is equal to 1 if the proposition $\mathcal{P}$ is true, 0 otherwise.

Setting

$$P_2 := \sum_{2 \le i < j \le n} \frac{1}{x_i - x_j} \left(\partial_i - \partial_j \right),$$

we have

$$\begin{aligned}
\frac{1}{\Delta^{(1)}} \sum_{i=1}^{n} \partial_i^2 f_1 &= (\nabla_2 + 2P_2) g \\
&= \sum_{s=0}^{n-4} \left((\nabla_2 + 2P_2) g_s + (s+2)(s+1) g_{s+2} \right) x_1^s.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\partial_1^3 \Delta &= \Delta^{(1)} \left(\partial_1^3 \prod_{j=2}^{n} (x_1 - x_j) \right) \\
&= \Delta^{(1)} \left(\partial_1^3 \sum_{j=0}^{n-1} (-1)^j e_j x_1^{n-1-j} \right) \\
&= \Delta^{(1)} \cdot \sum_{j=0}^{n-4} (-1)^j (n-1-j)(n-2-j)(n-3-j) e_j x_1^{n-4-j} \\
&= \Delta^{(1)} \cdot \sum_{s=0}^{n-4} (-1)^{n-4-s} (s+3)(s+2)(s+1) e_{n-4-s}\, x_1^s.
\end{aligned}$$

Equating the coefficients we get the following system of equalities:

Lemma 2.3.

(**C**2) $\quad (-1)^{n-4-s}(s+3)(s+2)(s+1)e_{n-4-s} = (\nabla_2 + 2P_2)g_s + (s+2)(s+1)g_{s+2},$

for $s = 0, 1, \ldots, n-4$.

We study now some properties of the operator P_2.

Lemma 2.4. *We have the following identities:*

(3)
$$\begin{aligned}
P_2 e_k &= -\binom{n-k+1}{2} e_{k-2}; \\
P_2 e_k^h &= h e_k^{h-1} P_2 e_k = -h \binom{n-k+1}{2} e_k^{h-1} e_{k-2}.
\end{aligned}$$

Proof. If we denote by $e_k^{(i)}$ the elementary symmetric function of degree k in the variables $\{x_2, \ldots, x_n\} \setminus \{x_i\}$, we have

$$\partial_i e_k = e_{k-1}^{(i)}.$$

Consider the difference

$$\partial_i e_k - \partial_j e_k = e_{k-1}^{(i)} - e_{k-1}^{(j)}.$$

The monomials in $e_{k-1}^{(i)}$ that don't involve x_j are cancelled by the ones in $e_{k-1}^{(j)}$ that don't contain i; while the monomials in $e_{k-1}^{(i)}$ that involve x_j can be paired with the ones in $e_{k-1}^{(j)}$ that involve x_i, to get a factor $x_j - x_i$, so that when we divide by $x_i - x_j$ we are left only with the negative of a multiple of e_{k-2}.

To see what this multiple is, it's enough to count how many times the monomial $x_2 x_3 \cdots x_{k-1}$ appears: this number is the number of ways of choosing i and j in $\{k, k+1, \ldots, n\}$, which is what we wanted.

The second identity follows from the first one and Leibniz rule. $\square$

Lemma 2.5. *If g is a symmetric polynomial, then*

$$[\nabla_1, P_2] g = 0.$$

Proof. It's enough to check this relation on the monomials e_λ, where λ denotes as usual a partition, since they form a basis of symmetric polynomials. Using repeatedly Leibniz rule we reduce ourselves to check the identity on the e_k's. But this follows immediately from the identities (2) and (3). $\square$

Substituting ($\bullet$) in (**C**2) and using the previous lemmas we get

$$(-1)^{n-4-s}(s+3)(s+2)(s+1)e_{n-4-s} =$$

$$\begin{aligned}
&= (\nabla_2 + 2P_2)\left((-1)^{n-2-s}(s+1)s\, e_{n-2-s} + \frac{(-1)^s}{s!}\nabla_1^s g_0\right)\\
&+ (s+2)(s+1)\left((-1)^{n-4-s}(s+3)(s+2)\, e_{n-4-s} + \frac{(-1)^{s+2}}{(s+2)!}\nabla_1^{s+2} g_0\right)\\
&= 2(-1)^{n-3-s}(s+1)s\binom{s+3}{2} e_{n-4+s} + \frac{(-1)^s}{s!}(\nabla_2 + 2P_2)\nabla_1^s g_0\\
&+ (-1)^{n-4-s}(s+3)(s+2)^2(s+1)\, e_{n-4-s} + \frac{(-1)^s}{s!}\nabla_1^{s+2} g_0
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{n-3-s}(s+3)(s+2)(s+1)s\, e_{n-4+s} + \frac{(-1)^s}{s!}(\nabla_2 + 2P_2)\nabla_1^s g_0\\
&+ (-1)^{n-4-s}(s+3)(s+2)^2(s+1)\, e_{n-4-s} + \frac{(-1)^s}{s!}\nabla_1^{s+2} g_0\\
&= 2(-1)^{n-4-s}(s+3)(s+2)(s+1)e_{n-4-s} + \frac{(-1)^s}{s!}\nabla_1^s(\nabla_2 + 2P_2 + \nabla_1^2)g_0,
\end{aligned}$$

from which we get the following system of identities:

$$(-1)^{n-4-s}(s+3)(s+2)(s+1)\, e_{n-4-s} + \frac{(-1)^s}{s!}\nabla_1^s(\nabla_2 + 2P_2 + \nabla_1^2)g_0 = 0$$

for $s = 0, 1, \ldots, n-4$.

These equations can be rewritten in the following form:

Lemma 2.6.

$$\nabla_1^s(\nabla_2 + 2P_2 + \nabla_1^2)g_0 = (-1)^{n-1}(s+3)!\, e_{n-4-s}$$

for $s = 0, 1, \ldots, n-4$.

Notice that by (2) we have

$$\nabla_1^s e_{n-4} = 4 \cdot 5 \cdot \cdots \cdot (s+3)e_{n-4-s} = \frac{1}{6}(s+3)! e_{n-4-s},$$

hence

$$(\nabla_2 + 2P_2 + \nabla_1^2)g_0 = (-1)^{n-1} 6\, e_{n-4}$$

would give a solution to all our systems.

Remark 3. It's straightforward to check that

$$(\nabla_1^2 + 2P_2)e_k = 0 \quad \text{for all } k.$$

Since also $\nabla_2 e_k = 0$ for all k, we must look for a g_0 that involves e_λ with partitions λ consisting of at least two parts.

In the following calculations we will use identities (2) and (3); remember that the e_k's are in the $n-1$ variables $x_2, \dots, x_n$.

$$2P_2(e_{n-3}e_1) = 2P_2(e_{n-3})e_1 + 2e_{n-3}P_2(e_1) = -12e_{n-5}e_1;$$

$$\begin{aligned}\nabla_1^2(e_{n-3}e_1) &= (\nabla_1^2 e_{n-3})e_1 + 2\nabla_1 e_{n-3}\nabla_1 e_1 + e_{n-3}(\nabla_1^2 e_1)\\ &= 12e_{n-5}e_1 + 6(n-1)e_{n-4};\end{aligned}$$

$$\begin{aligned}\nabla_2(e_{n-3}e_1) &= \sum_{i=2}^{n}(\partial_i^2 e_{n-3})e_1 + 2\sum_{i=2}^{n}\partial_i e_{n-3}\partial_i e_1 + \sum_{i=2}^{n} e_{n-3}(\partial_i^2 e_1)\\ &= 2\nabla_1 e_{n-3} = 6e_{n-4}.\end{aligned}$$

From these we get

$$(\nabla_2 + 2P_2 + \nabla_1^2)e_{n-3}e_1 = 6ne_{n-4}.$$

Hence our solution will be

$$g_0 := \frac{(-1)^{n-1}}{n}e_{n-3}e_1.$$

Now we want to make formula $(\bullet)$ more explicit.

Lemma 2.7. *For $s \geq 1$ we have*

$$\nabla_1^s(e_{n-3}e_1) = \frac{(s+2)!}{2}e_{n-3-s}e_1 + \frac{(s+1)!}{2}s(n-1)e_{n-2-s}.$$

Proof. By induction on s, the case $s=1$ being clear. We assume the formula true for $s \geq 1$. We have

$$\begin{aligned}\nabla_1^{s+1}(e_{n-3}e_1) &= \nabla_1\left(\frac{(s+2)!}{2}e_{n-3-s}e_1 + \frac{(s+1)!}{2}s(n-1)e_{n-2-s}\right)\\ &= \frac{(s+2)!}{2}((\nabla_1 e_{n-3-s})e_1 + e_{n-3-s}(\nabla_1 e_1)) + \frac{(s+1)!}{2}s(n-1)\nabla_1 e_{n-2-s}\\ &= \frac{(s+3)!}{2}e_{n-4-s}e_1 + \frac{(s+2)!}{2}(n-1)e_{n-3-s} + \frac{(s+2)!}{2}s(n-1)e_{n-3-s}\\ &= \frac{(s+3)!}{2}e_{n-4-s}e_1 + \frac{(s+2)!}{2}(s+1)(n-1)e_{n-3-s}.\end{aligned}$$

□

Plugging these formulae into $(\bullet)$ we get for all $s \geq 1$

$$\begin{aligned}
g_s &= (-1)^{n-2-s}(s+1)s\, e_{n-2-s} + \frac{(-1)^s}{s!}\nabla_1^s g_0 \\
&= (-1)^{n-2-s}(s+1)s\, e_{n-2-s} + \frac{(-1)^s}{s!}\nabla_1^s \left(\frac{(-1)^{n-1}}{n} e_{n-3}e_1\right) \\
&= (-1)^{n-2-s}(s+1)s\, e_{n-2-s} + \\
&+ \frac{(-1)^{n+s-1}}{s!\, n}\left(\frac{(s+2)!}{2} e_{n-3-s}e_1 + \frac{(s+1)!}{2} s(n-1)e_{n-2-s}\right) \\
&= \left(\frac{n+1}{2n}\right)(-1)^{n-2-s}(s+1)s\, e_{n-2-s} + \left(\frac{1}{2n}\right)(-1)^{n-1-s}(s+2)(s+1)\, e_{n-3-s}e_1 \\
&= \frac{(-1)^{n-2-s}}{n}\left((n+1)\binom{s+1}{2} e_{n-2-s} - \binom{s+2}{2} e_{n-3-s}e_1\right).
\end{aligned}$$

We follow the convention that the binomial "n choose k" is 0 when $n < k$, hence this formula works for $s \geq 0$.

Putting everything together, we get the formula

$$f_1 = f_1^{(1)} = \Delta^{(1)} \sum_{s=0}^{n-2} \frac{(-1)^{n-2-s}}{n}\left((n+1)\binom{s+1}{2} e_{n-2-s} - \binom{s+2}{2} e_{n-3-s}e_1\right) x_1^s.$$

Encouraged by this promising first step, we tried to pursue our methods to compute an f_2 for our f_1. Notice that this f_2 would work only for $\partial_j \Delta$, and not for a general $\mathfrak{S}_n$-harmonic, since the other reduction that we did for f_1 doesn't work for f_2.

With some patience and stamina we went trough our computations, to finally realize that we couldn't find an f_2 for all values of n in this way. But not all efforts were lost: some of those computations are now part of the fifth section!

Looking back at the work in the present section, we realized that something more general could be done.

3. A family of f_1's for $\partial_1 \Delta$

When we constructed our explicit f_1 we had to solve the system of equations

$$(\nabla_2 + 2P_2 + \nabla_1^2)g_0 = (-1)^{n-1} 6\, e_{n-4}.$$

Of course the solution that we had found was not unique. In fact there are infinitely many solutions to this system. In this section we construct a whole family of solutions. Of course we are going to use much of what we did in the last section.

We need the following identities:

Lemma 3.1. *For $k \geq h$ we have*

$$\begin{aligned}
\nabla_2(e_k e_2) &= 2(n-k)e_{k-1}e_1 - 2ke_k; \\
(\nabla_1^2 + 2P_2)e_k e_2 &= 2(n-k)(n-2)e_{k-1}e_1; \\
(\nabla_1^2 + 2P_2 + \nabla_2)e_k e_2 &= 2(n-k)(n-1)e_{k-1}e_1 - 2ke_k; \\
(\nabla_1^2 + 2P_2 + \nabla_2)e_k e_1^2 &= 4n(n-k)e_{k-1}e_1 + 2n(n-1)e_k.
\end{aligned}$$

Proof. The first identity is a special case of a more general formula that can be found in the fifth section with its proof. The second one follows easily from remark (3). The third one follows from the previous two. The last one is a special case of previous identities. $\square$

We can now look for a solution of our system. We assume that g_0 is of the form

$$g_0 = a\, e_{n-4}e_2 + b\, e_{n-4}e_1^2 + c\, e_{n-3}e_1,$$

where $a = a(n)$, $b = b(n)$ and $c = c(n)$ are indeterminate coefficients.

We have

$$\begin{aligned}(\nabla_1^2 + 2P_2 + \nabla_2)g_0 &= a(8(n-1)e_{n-5}e_1 - 2(n-4)e_{n-4}) \\ &+ b(16n\, e_{n-5}e_1 + 2n(n-1)e_{n-4}) \\ &+ c\,6n\, e_{n-4},\end{aligned}$$

from which we get the two equations

$$\begin{aligned} a\,8(n-1) + b\,16n &= 0; \\ -a\,2(n-4) + b\,2n(n-1) + c\,6n &= (-1)^{n-1}6. \end{aligned}$$

Solving for a and b we get

$$a = -\frac{6((-1)^{n-1} - c\,n)}{n^2-7}, \qquad b = \frac{3(n-1)((-1)^{n-1} - c\,n)}{n(n^2-7)},$$

where c can be any number. Hence we get the family of solutions

$$g_{0;c} = -\frac{6((-1)^{n-1} - c\,n)}{n^2-7} e_{n-4}e_2 + \frac{3(n-1)((-1)^{n-1} - c\,n)}{n(n^2-7)} e_{n-4}e_1^2 + c\, e_{n-3}e_1.$$

Observe that in the previous section we got $g_{0;c}$ for $c = (-1)^{n-1}/n$.

We record the following two identities, which are just consequences of the identities that we already established and Leibniz rule:

Lemma 3.2.

$$\begin{aligned}\nabla_1^s e_{n-4}e_2 &= \frac{(s+3)!}{3!} e_{n-s-4}e_2 + s(n-2)\frac{(s+2)!}{3!} e_{n-s-3}e_1 \\ &+ \frac{s(s-1)}{2}(n-1)(n-2)\frac{(s+1)!}{3!} e_{n-s-2}; \\ \nabla_1^s e_{n-4}e_1^2 &= \frac{(s+3)!}{3!} e_{n-s-4}e_1^2 + 2s(n-1)\frac{(s+2)!}{3!} e_{n-s-3}e_1 \\ &+ s(s-1)(n-1)^2\frac{(s+1)!}{3!} e_{n-s-2}.\end{aligned}$$

Hence we have

$$
\begin{aligned}
\nabla_1^s g_{0;c} &= -\frac{6((-1)^{n-1}-c\,n)}{n^2-7}\left(\frac{(s+3)!}{3!}e_{n-s-4}e_2 + s(n-2)\frac{(s+2)!}{3!}e_{n-s-3}e_1\right.\\
&+ \left.\frac{s(s-1)}{2}(n-1)(n-2)\frac{(s+1)!}{3!}e_{n-s-2}\right)\\
&+ \frac{3(n-1)((-1)^{n-1}-c\,n)}{n(n^2-7)}\left(\frac{(s+3)!}{3!}e_{n-s-4}e_1^2 + 2s(n-1)\frac{(s+2)!}{3!}e_{n-s-3}e_1\right.\\
&+ \left. s(s-1)(n-1)^2\frac{(s+1)!}{3!}e_{n-s-2}\right)\\
&+ c\left(\frac{(s+2)!}{2}e_{n-3-s}e_1 + \frac{(s+1)!}{2}s(n-1)e_{n-s-2}\right)\\
&= -\frac{6((-1)^{n-1}-c\,n)}{n^2-7}\frac{(s+3)!}{3!}e_{n-s-4}e_2 + \frac{3(n-1)((-1)^{n-1}-c\,n)}{n(n^2-7)}\frac{(s+3)!}{3!}e_{n-s-4}e_1^2\\
&+ \left(-\frac{((-1)^{n-1}-c\,n)}{n^2-7}s(n-2) + \frac{(n-1)((-1)^{n-1}-c\,n)}{n(n^2-7)}s(n-1) + \frac{c}{2}\right)(s+2)!e_{n-s-3}e_1\\
&+ \left(-\frac{((-1)^{n-1}-c\,n)}{n^2-7}\frac{s(s-1)}{2}(n-1)(n-2)\right.\\
&+ \left.\frac{(n-1)((-1)^{n-1}-c\,n)}{n(n^2-7)}\frac{s(s-1)}{2}(n-1)^2 + c\frac{s(n-1)}{2}\right)(s+1)!e_{n-s-2}\\
&= -\frac{((-1)^{n-1}-c\,n)}{n^2-7}(s+3)!e_{n-s-4}e_2 + \frac{(n-1)((-1)^{n-1}-c\,n)}{2n(n^2-7)}(s+3)!e_{n-s-4}e_1^2\\
&+ \left(\frac{s((-1)^{n-1}-c\,n)}{n(n^2-7)} + \frac{c}{2}\right)(s+2)!e_{n-s-3}e_1\\
&+ \left(\frac{((-1)^{n-1}-c\,n)}{n^2-7}\frac{s(s-1)}{2}\frac{n-1}{n} + c\frac{s(n-1)}{2}\right)(s+1)!e_{n-s-2}.
\end{aligned}
$$

Finally for $s \geq 1$ we have

$$
\begin{aligned}
g_{s;c} &= (-1)^{n-2-s}(s+1)s\,e_{n-s-2} + \frac{(-1)^s}{s!}\nabla_1^s g_{0;c}\\
&= -(-1)^s\frac{((-1)^{n-1}-c\,n)}{n^2-7}(s+3)(s+2)(s+1)e_{n-s-4}e_2\\
&+ (-1)^s\frac{(n-1)((-1)^{n-1}-c\,n)}{2n(n^2-7)}(s+3)(s+2)(s+1)e_{n-s-4}e_1^2\\
&+ (-1)^s\left(\frac{s((-1)^{n-1}-c\,n)}{n(n^2-7)} + \frac{c}{2}\right)(s+2)(s+1)e_{n-s-3}e_1\\
&+ (-1)^s\left(\frac{((-1)^{n-1}-c\,n)}{n^2-7}\frac{s(s-1)}{2}\frac{n-1}{n} + c\frac{s(n-1)}{2} + s(-1)^n\right)(s+1)e_{n-s-2}.
\end{aligned}
$$

From this we could write a formula for $f_{1;c}$.

At this point we looked for a value of c for which we could find an f_2. In the end we found exactly one for each value of n:

$$c = (-1)^{n-1} \frac{2(2n^3 - 2n - 3)}{3n(n-1)(n^2+n+2)}.$$

We relegated the derivation of the value of c and the computation of the corresponding f_2 in the appendix, since the calculation is quite long. Reading the appendix should make clear that these methods can't be pushed much further without a tremendous stamina.

In the next section we show instead a different method to get other f_1's.

4. Another computation of f_1

We want to find an f_1 for $\partial_j \Delta$. In fact we will prove something more. First of all we make the following simple observation: from the obvious $\nabla_1 \Delta = 0$ (see Remark 1) we get

$$\nabla_1^{(j)} \Delta = -\partial_j \Delta,$$

where $\nabla_1^{(j)}$ denotes the sum of the partial derivatives with ∂_j omitted.

We assume that f_1 is of the form

$$f_1 = (ax_j + be_1^{(j)}) \partial^\alpha \Delta,$$

with a and b coefficients to be determined. Applying ∇_1 we get

$$\nabla_1 f_1 = (a + (n-1)b) \partial^\alpha \Delta,$$

while applying ∇_2 we get

$$\begin{aligned} \nabla_2 f_1 &= 2 \sum_{i=1}^{n} \partial_i (ax_j + be_1^{(j)}) \partial_i \partial^\alpha \Delta \\ &= 2a \partial_j \partial^\alpha \Delta + 2b \partial^\alpha \nabla_1^{(j)} \Delta \\ &= 2(a-b) \partial_j \partial^\alpha \Delta. \end{aligned}$$

Since the matrix

$$\begin{pmatrix} 1 & n-1 \\ 1 & -1 \end{pmatrix}$$

is invertible for every $n \geq 1$, we just showed how to construct a solution of the system of equations

$$\begin{aligned} \nabla_1 f_1 &= c \partial^\alpha \Delta, \\ \nabla_2 f_1 &= d \partial_j \partial^\alpha \Delta \end{aligned}$$

for any coefficients c and d and any j. All this together with the observations in the first section takes care of the f_1's.

We indicate here how one could proceed to get an f_2 such that

$$\begin{aligned} \nabla_1 f_2 &= -\widetilde{D}_1 f_1 \\ \nabla_2 f_2 &= -\widetilde{D}_2 f_1, \end{aligned}$$

for $f_1^{(1)} = (ax_1 + be_1^{(1)})\partial_1^2\Delta$. We have

$$\begin{aligned} -\widetilde{D}_1 f_1^{(1)} &= -2\sum_{i=1}^{n} x_i\partial_i(ax_1 + be_1^{(1)})\partial_i\partial_1^2\Delta \\ &= -2ax_1\partial_1^3\Delta - 2b\sum_{i=2}^{n} x_i\partial_i\partial_1^2\Delta. \end{aligned}$$

Now if we set $g = \partial_1^2 \prod_{i=2}^{n}(x_1 - x_i)$ we have

$$\begin{aligned} \sum_{i=2}^{n} x_i\partial_i\partial_1^2\Delta &= \left(\sum_{i=2}^{n} x_i\partial_i\Delta^{(1)}\right) g + \Delta^{(1)}\sum_{i=2}^{n} x_i\partial_i g \\ &= \binom{n-1}{2}\partial_1^2\Delta + (n-3)\partial_1^2\Delta + x_1\partial_1^3\Delta \\ &= \frac{n^2-n-4}{2}\partial_1^2\Delta + x_1\partial_1^3\Delta. \end{aligned}$$

Hence

$$-\widetilde{D}_1 f_1^{(1)} = -2(a+b)x_1\partial_1^3\Delta - b(n^2-n-4)\partial_1^2\Delta.$$

Also

$$\begin{aligned} -\widetilde{D}_2 f_1^{(1)} &= -3\sum_{i=1}^{n} x_i\partial_i(ax_1 + be_1^{(1)})\partial_i^2\partial_1^2\Delta \\ &= -3ax_1\partial_1^4\Delta - 3b\partial_1^2\left(\sum_{i=2}^{n} x_i\partial_i^2\Delta\right). \end{aligned}$$

Now

$$\begin{aligned} \sum_{i=2}^{n} x_i\partial_i^2\Delta &= 2\sum_{i=2}^{n}\partial_i\Delta^{(1)}x_i\partial_i g \\ &= \Delta^{(1)}2P_1 g = x_1\partial_1^2\Delta, \end{aligned}$$

hence

$$-\widetilde{D}_2 f_1^{(1)} = -3(a+b)x_1\partial_1^4\Delta - 6b\partial_1^3\Delta.$$

Since we already know how to take care of the terms $-b(n^2-n-4)\partial_1^2\Delta$ and $-6b\partial_1^3\Delta$, it will be more than enough to solve the following more general problem:

$$\begin{aligned} \nabla_1 f_2 &= \tilde{a}x_1\partial_1^k\Delta + \tilde{b}e_1^{(1)}\partial_1^k\Delta \\ \nabla_2 f_2 &= \tilde{c}x_1\partial_1^{k+1}\Delta + \tilde{d}e_1^{(1)}\partial_1^{k+1}\Delta, \end{aligned}$$

where $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ are coefficients, and $k \geq 0$ is an integer.

Assume that f_2 is of the form

$$f_2 = (ax_1^2 + bx_1e_1^{(1)} + c(e_1^{(1)})^2 + de_2^{(1)})\partial_1^k\Delta + (\hat{a}x_1 + \hat{b}e_1^{(1)})\partial_1^{k-1}\Delta,$$

where $a, b, c, d, \hat{a}$ and $\hat{b}$ are coefficients to be determined.

Now

$$\nabla_1 f_2 = ((2a + (n-1)b)x_1 + (b + 2(n-1)c + (n-2)d)e_1^{(1)})\partial_1^k\Delta + (\hat{a} + (n-1)\hat{b})\partial_1^{k-1}\Delta,$$

while

$$\begin{aligned}
\nabla_2 f_2 &= (2a+2(n-1)c)\partial_1^k\Delta + 2(\hat{a}-\hat{b})\partial_1^k\Delta \\
&+ (4ax_1 + 2be_1^{(1)})\partial_1^{k+1}\Delta \\
&+ (2bx_1 + 4ce_1^{(1)})\nabla_1^{(1)}\partial_1^k\Delta \\
&+ d\partial_1^k \sum_{s=0}^{n-1}(-1)^{s+n-1}\nabla_2(e_2^{(1)}e_{n-1-s}^{(1)})x_1^s.
\end{aligned}$$

Notice that the formula for $\nabla_2(e_k e_2)$ works also for $k=1$. Hence the last term is

$$d\left(2e_1^{(1)}\partial_1^{k+1}\Delta - 2(n-1)\partial_1^k\Delta + 2x_1\partial_1^{k+1}\Delta\right).$$

Finally we have

$$\begin{aligned}
\nabla_2 f_2 &= (2a+2(n-1)c-2(n-1)d+2(\hat{a}-\hat{b}))\partial_1^k\Delta \\
&+ (4a-2b+2d)x_1\partial_1^{k+1}\Delta \\
&+ (2b-4c+2d)e_1^{(1)}\partial_1^{k+1}\Delta.
\end{aligned}$$

We already observed that with the coefficients $\hat{a}$ and $\hat{b}$ we can get anything, hence we can disregard the terms with $\partial_1^{k-1}\Delta$ and $\partial_1^k\Delta$. What's left gives rise to a linear system with matrix

$$\begin{pmatrix} 2 & n-1 & 0 & 0 \\ 0 & 1 & 2(n-1) & (n-2) \\ 4 & -2 & 0 & 2 \\ 0 & 2 & -4 & 2 \end{pmatrix},$$

whose determinant is $32(n^2-n)$. Hence for $n \geq 2$ this matrix is non-singular, and this allows us to solve the system for all values of $\widetilde{a}, \widetilde{b}, \widetilde{c}$ and $\widetilde{d}$, and of course for any $k \geq 0$.

Using the Remark 2, we can easily see that in order to get an f_2 for any of the f_1 we found, we still need to solve the system of equations

$$\begin{aligned}
\nabla_1 f_2 &= f_1 \\
\nabla_2 f_2 &= \partial_j f_1.
\end{aligned}$$

We have for $j \neq 1$

$$\begin{aligned}
\partial_j f_1 &= b\partial_1^2\Delta + (ax_1 + be_1^{(1)})\partial_j\partial_1^2\Delta \\
&= b\partial_1^2\Delta - (ax_1 + be_1^{(1)})\nabla_1^{(j)}\partial_1^2\Delta \\
&= b\partial_1^2\Delta - \nabla_1^{(j)}\left((ax_1 + be_1^{(1)})\partial_1^2\Delta\right) + (a+(n-2)b)\partial_1^2\Delta.
\end{aligned}$$

Also,

$$\nabla_1^{(j)}\left((ax_1 + be_1^{(1)})\partial_1^2\Delta\right) = \frac{1}{2}\nabla_2\left(e_1^{(j)}\cdot(ax_1 + be_1^{(1)})\partial_1^2\Delta\right) - \frac{1}{2}e_1^{(j)}\partial_1^3\Delta.$$

On the other hand,

$$\nabla_1\left(e_1^{(j)}\cdot(ax_1 + be_1^{(1)})\partial_1^2\Delta\right) = (n-1)(ax_1 + be_1^{(1)})\partial_1^2\Delta + e_1^{(j)}\partial_1^2\Delta.$$

Using what we have proved above, it's now clear that it's more than enough to solve the system

$$\begin{aligned}\nabla_1 f_2 &= \tilde{a} e_1^{(j)} \partial_1^k \Delta \\ \nabla_2 f_2 &= \tilde{b} e_1^{(j)} \partial_1^{k+1} \Delta,\end{aligned}$$

where $\tilde{a}$ and $\tilde{b}$ are arbitrary coefficients, and $k \geq 0$ is an integer.

We leave the problem of finding a solution to this system open.

5. Actions on alternating and symmetric polynomials

We stick to the notation $e_k := e_k(x_2, x_3, \ldots, x_n)$, while $e_k^{(i_1,i_2,\ldots,i_r)}$ indicates the elementary symmetric function of degree k in the variables $\{x_2, x_3, \ldots, x_n\} \setminus \{i_1, i_2, \ldots, i_r\}$. We recall also the obvious relations

$$\partial_j e_k^{(i_1,i_2,\ldots,i_r)} = e_{k-1}^{(i_1,i_2,\ldots,i_r,j)}, \quad \text{and} \quad e_k^{(i_1,i_2,\ldots,i_r)} = e_k^{(i_1,i_2,\ldots,i_r,j)} + x_j e_{k-1}^{(i_1,i_2,\ldots,i_r,j)}$$

for $j \in \{x_2, x_3, \ldots, x_n\} \setminus \{i_1, i_2, \ldots, i_r\}$.

We remark also that all the identities that we are going to prove will remain valid for elementary functions in any subset of the variables involved, as long as we replace n by the number of variables involved plus one.

Another basic observation is that the elementary symmetric functions e_λ's and the $\Delta \cdot e_\lambda$'s, where λ runs over all partitions, form a basis of symmetric and alternating polynomials respectively.

We are going to use all this without mentioning it anymore along the way. Note also that we leave without proof the identities that have been already proved in the previous sections.

In what follows g will be a symmetric functions in the variables $x_2, x_3, \ldots, x_n$.

The action of ∇_1 on symmetric functions is described by the identity

$$\nabla_1 e_k = (n-k) e_{k-1}$$

together with Leibniz rule.

The action on alternating functions now follows immediately from this one and Leibniz rule:

$$\nabla_1(\Delta^{(1)} g) = (\nabla \Delta^{(1)}) g + \Delta^{(1)}(\nabla_1 g) = \Delta^{(1)}(\nabla_1 g).$$

The following identity together with Leibniz rule describes the action of the laplacian on symmetric functions.

Lemma 5.1. *For $k \geq h$ we have*

$$\nabla_2(e_k e_h) = 2(n-k) e_{k-1} e_{h-1} - 2 \sum_{i=1}^{h-1} (k-h+2i) e_{k+i-1} e_{h-i-1}.$$

Proof. We proceed by multiple induction on k, h and n.

$$
\begin{aligned}
\nabla_2(e_k e_h) &= 2\sum_{j=1}^{n} \partial_j e_k \partial_j e_h \\
&= 2\sum_{j=1}^{n} \partial_j (e_k^{(n)} + x_n e_{k-1}^{(n)}) \cdot \partial_j (e_h^{(n)} + x_n e_{h-1}^{(n)}) \\
&= 2\partial_n (e_k^{(n)} + x_n e_{k-1}^{(n)}) \cdot \partial_n (e_h^{(n)} + x_n e_{h-1}^{(n)}) \\
&+ 2\sum_{j=1}^{n-1} \partial_j (e_k^{(n)} + x_n e_{k-1}^{(n)}) \cdot \partial_j (e_h^{(n)} + x_n e_{h-1}^{(n)}) \\
&= 2\, e_{k-1}^{(n)} e_{h-1}^{(n)} + 2\sum_{j=1}^{n-1} \partial_j e_k^{(n)} \partial_j e_h^{(n)} + 2\sum_{j=1}^{n-1} \partial_j x_n e_k^{(n)} \partial_j e_{h-1}^{(n)} \\
&+ 2\sum_{j=1}^{n-1} x_n \partial_j e_{k-1}^{(n)} \partial_j e_h^{(n)} + 2\sum_{j=1}^{n-1} x_n^2 \partial_j e_{k-1}^{(n)} \partial_j e_{h-1}^{(n)} \\
&= 2\, e_{k-1}^{(n)} e_{h-1}^{(n)} + 2(n-k-1) e_{k-1}^{(n)} e_{h-1}^{(n)} - 2\sum_{i=1}^{h-1} (k-h+2i) e_{k+i-1}^{(n)} e_{h-i-1}^{(n)} \\
&+ x_n \left(2(n-k-1) e_{k-1}^{(n)} e_{h-2}^{(n)} - 2\sum_{i=1}^{h-2} (k-h+1+2i) e_{k+i-1}^{(n)} e_{h-i-2}^{(n)} \right. \\
&+ \left. 2(n-k) e_{k-2}^{(n)} e_{h-1}^{(n)} - 2\sum_{i=1}^{h-1} (k-h-1+2i) e_{k+i-2}^{(n)} e_{h-i-1}^{(n)} \right) \\
&+ x_n^2 \left(2(n-k) e_{k-2}^{(n)} e_{h-2}^{(n)} - 2\sum_{i=1}^{h-2} (k-h+2i) e_{k+i-2}^{(n)} e_{h-i-2}^{(n)} \right) \\
&= 2(n-k) e_{k-1}^{(n)} e_{h-1}^{(n)} - 2\sum_{i=1}^{h-1} (k-h+2i) e_{k+i-1}^{(n)} e_{h-i-1}^{(n)} \\
&+ x_n \left(2(n-k) \left(e_{k-1}^{(n)} e_{h-2}^{(n)} + e_{k-2}^{(n)} e_{h-1}^{(n)} \right) \right. \\
&- \left. 2\sum_{i=1}^{h-1} (k-h+2i) \left(e_{k+i-1}^{(n)} e_{h-i-2}^{(n)} + e_{k+i-2}^{(n)} e_{h-i-1}^{(n)} \right) \right) \\
&+ x_n^2 \left(2(n-k) e_{k-2}^{(n)} e_{h-2}^{(n)} - 2\sum_{i=1}^{h-2} (k-h+2i) e_{k+i-2}^{(n)} e_{h-i-2}^{(n)} \right) \\
&= 2(n-k) e_{k-1} e_{h-1} - 2\sum_{i=1}^{h-1} (k-h+2i) e_{k+i-1} e_{h-i-1}.
\end{aligned}
$$

The base cases are trivial. $\square$

The action of the laplacian on alternating functions now follows from

$$\frac{1}{\Delta^{(1)}}\nabla_2(\Delta^{(1)}g) = (\nabla_2 + 2P_2)g,$$

where

$$P_2 := \sum_{2\le i<j\le n} \frac{1}{x_i - x_j}(\partial_i - \partial_j),$$

the formula

$$P_2 e_k = -\binom{n-k+1}{2} e_{k-2},$$

and Leibniz rule.

The following identity together with Leibniz rule describes the action of the operator $\widetilde{D}_1$ on symmetric functions.

Lemma 5.2. *For $k \ge h$,*

$$\widetilde{D}_1(e_k e_h) = 2\sum_{i=0}^{h-1}(k-h+1+2i)e_{k+i}e_{h-1-i}.$$

Proof. We proceed by multiple induction on k, h and n.

$$\begin{aligned}
\widetilde{D}_1(e_k e_h) &= 2\sum_{i=2}^{n} x_i \partial_i e_k \partial_i e_h \\
&= 2\sum_{i=2}^{n} x_i \partial_i (e_k^{(n)} + x_n e_{k-1}^{(n)})\partial_i (e_h^{(n)} + x_n e_{h-1}^{(n)}) \\
&= 2\, x_n e_{k-1}^{(n)} e_{h-1}^{(n)} + 2\sum_{i=2}^{n-1} x_i \partial_i (e_k^{(n)} + x_n e_{k-1}^{(n)})\partial_i (e_h^{(n)} + x_n e_{h-1}^{(n)}) \\
&= 2\, x_n e_{k-1}^{(n)} e_{h-1}^{(n)} + 2\sum_{i=2}^{n-1} x_i \partial_i e_k^{(n)} \partial_i e_h^{(n)} \\
&+ 2\, x_n \sum_{i=2}^{n-1} x_i \left(\partial_i e_k^{(n)} \partial_i e_{h-1}^{(n)} + \partial_i e_{k-1}^{(n)} \partial_i e_h^{(n)}\right) \\
&+ 2\, x_n^2 \sum_{i=2}^{n-1} x_i \partial_i e_{k-1}^{(n)} \partial_i e_{h-1}^{(n)} \\
&= 2\, x_n e_{k-1}^{(n)} e_{h-1}^{(n)} + 2\sum_{i=0}^{h-1}(k-h+1+2i) e_{k+i}^{(n)} e_{h-1-i}^{(n)} \\
&+ 2\, x_n \left(\sum_{i=0}^{h-2}(k-h+2+2i) e_{k+i}^{(n)} e_{h-2-i}^{(n)} + \sum_{i=0}^{h-1}(k-h+2i) e_{k-1+i}^{(n)} e_{h-1-i}^{(n)}\right) \\
&+ 2\, x_n^2 \sum_{i=0}^{h-2}(k-h+1+2i) e_{k-1+i}^{(n)} e_{h-2-i}^{(n)}
\end{aligned}$$

$$\begin{aligned}
&= 2\sum_{i=0}^{h-1}(k-h+1+2i)e^{(n)}_{k+i}e^{(n)}_{h-1-i} \\
&+ 2\,x_n\left(\sum_{i=0}^{h-2}(k-h+1+2i)e^{(n)}_{k+i}e^{(n)}_{h-2-i}+\sum_{i=0}^{h-1}(k-h+1+2i)e^{(n)}_{k-1+i}e^{(n)}_{h-1-i}\right) \\
&+ 2\,x_n^2\sum_{i=0}^{h-2}(k-h+1+2i)e^{(n)}_{k-1+i}e^{(n)}_{h-2-i} \\
&= 2\sum_{i=0}^{h-1}(k-h+1+2i)e_{k+i}e_{h-1-i}.
\end{aligned}$$

The base cases are trivial. □

We have

$$\frac{1}{\Delta^{(1)}}\widetilde{D}_1(\Delta^{(1)}g)=(2P_1+\widetilde{D}_1)g,$$

where

$$P_1 := \sum_{2\le i<j\le n}\frac{1}{x_i-x_j}(x_i\partial_i - x_j\partial_j).$$

We have the following identity, whose proof is analogous to the one of the identities (3):

$$P_1 e_k = \binom{n-k}{2}e_{k-1}.$$

All this together with Leibniz rule describes the action of $\widetilde{D}_1$ on alternating polynomials.

The following identity together with Leibniz rule describes the action of the operator $\widetilde{D}_2$ on symmetric functions.

Lemma 5.3. *For $k\ge h\ge l$,*

$$\begin{aligned}
\widetilde{D}_2(e_ke_he_l) = 6\Bigg(&\sum_{j=0}^{l-1}\sum_{i=0}^{h-1}(k-h+1+j+2i)e_{k+i+j}e_{h-1-i}e_{l-1-j} \\
&-\sum_{j=0}^{l-2}\sum_{i=1}^{l-1-j}(h-l+j+2i)e_{k+j}e_{h-1+i}e_{l-1-i-j}\Bigg).
\end{aligned}$$

Proof. We proceed by multiple induction on k,h,l and n.

$$\begin{aligned}
\frac{1}{6}\widetilde{D}_2(e_ke_he_l) &= \sum_{i=2}^{n}x_i\partial_ie_k\partial_ie_h\partial_ie_l \\
&= \sum_{i=2}^{n}x_i\partial_i(e^{(n)}_k+x_ne^{(n)}_{k-1})\partial_i(e^{(n)}_h+x_ne^{(n)}_{h-1})\partial_i(e^{(n)}_l+x_ne^{(n)}_{l-1})
\end{aligned}$$

$$
\begin{aligned}
&= x_n e^{(n)}_{k-1} e^{(n)}_{h-1} e^{(n)}_{l-1} + \sum_{i=2}^{n-1} x_i \partial_i e^{(n)}_k \partial_i e^{(n)}_h \partial_i e^{(n)}_l \\
&+ x_n \left(\sum_{i=2}^{n-1} x_i \left(\partial_i e^{(n)}_{k-1} \partial_i e^{(n)}_h \partial_i e^{(n)}_l + \partial_i e^{(n)}_k \partial_i e^{(n)}_{h-1} \partial_i e^{(n)}_l + \partial_i e^{(n)}_k \partial_i e^{(n)}_h \partial_i e^{(n)}_{l-1} \right) \right) \\
&+ x_n^2 \left(\sum_{i=2}^{n-1} x_i \left(\partial_i e^{(n)}_k \partial_i e^{(n)}_{h-1} \partial_i e^{(n)}_{l-1} + \partial_i e^{(n)}_{k-1} \partial_i e^{(n)}_h \partial_i e^{(n)}_{l-1} + \partial_i e^{(n)}_{k-1} \partial_i e^{(n)}_{h-1} \partial_i e^{(n)}_l \right) \right) \\
&+ x_n^3 \sum_{i=2}^{n-1} x_i \partial_i e^{(n)}_{k-1} \partial_i e^{(n)}_{h-1} \partial_i e^{(n)}_{l-1}.
\end{aligned}
$$

At this point we use induction, replacing the suitable terms by our formula. To be more efficient, we analyze the expansion with respect to powers of x_n.

For the factor of x_n we get

$$
\begin{aligned}
e^{(n)}_{k-1} e^{(n)}_{h-1} e^{(n)}_{l-1} &+ \sum_{j=0}^{l-1} \sum_{i=0}^{h-1} (k-h+j+2i) e^{(n)}_{k-1+i+j} e^{(n)}_{h-1-i} e^{(n)}_{l-1-j} \\
&+ \sum_{j=0}^{l-1} \sum_{i=0}^{h-2} (k-h+2+j+2i) e^{(n)}_{k+i+j} e^{(n)}_{h-2-i} e^{(n)}_{l-1-j} \\
&+ \sum_{j=0}^{l-2} \sum_{i=0}^{h-1} (k-h+1+j+2i) e^{(n)}_{k+i+j} e^{(n)}_{h-1-i} e^{(n)}_{l-2-j} \\
&- \sum_{j=0}^{l-2} \sum_{i=1}^{l-1-j} (h-l+j+2i) e^{(n)}_{k-1+j} e^{(n)}_{h-1+i} e^{(n)}_{l-1-i-j} \\
&- \sum_{j=0}^{l-2} \sum_{i=1}^{l-1-j} (h-l-1+j+2i) e^{(n)}_{k+j} e^{(n)}_{h-2+i} e^{(n)}_{l-1-i-j} \\
&- \sum_{j=0}^{l-3} \sum_{i=1}^{l-2-j} (h-l+1+j+2i) e^{(n)}_{k+j} e^{(n)}_{h-1+i} e^{(n)}_{l-2-i-j}.
\end{aligned}
$$

Rearranging the terms we get what we want:

$$
\sum_{j=0}^{l-2} \sum_{i=0}^{h-1} (k-h+1+j+2i) \left(e^{(n)}_{k+i+j} e^{(n)}_{h-2-i} e^{(n)}_{l-1-j} + e^{(n)}_{k+i+j} e^{(n)}_{h-1-i} e^{(n)}_{l-2-j} + e^{(n)}_{k-1+i+j} e^{(n)}_{h-1-i} e^{(n)}_{l-1-j} \right) +
$$

$$
- \sum_{j=0}^{l-2} \sum_{i=1}^{l-1-j} (h-l+j+2i) \left(e^{(n)}_{k-1+j} e^{(n)}_{h-1+i} e^{(n)}_{l-1-i-j} + e^{(n)}_{k+j} e^{(n)}_{h-2+i} e^{(n)}_{l-1-i-j} + e^{(n)}_{k+j} e^{(n)}_{h-1+i} e^{(n)}_{l-2-i-j} \right)
$$

Analogously for the factor of x_n^2. What is left is already what we want.

The base cases are trivial. □

We have
$$\frac{1}{\Delta^{(1)}}\widetilde{D}_2(\Delta^{(1)}g) = (6Q_2 + 3\widetilde{P}_2 + \widetilde{D}_2)g,$$
where
$$Q_2 := \sum_{j=1}^{n}\sum_{i<k}^{(j)} \frac{1}{(x_j - x_i)(x_j - x_k)} x_j\partial_j,$$
and
$$\widetilde{P}_2 := \sum_{1\leq i<j\leq n} \frac{1}{x_i - x_j}(x_i\partial_i^2 - x_j\partial_j^2).$$
The following Lemma together with Leibniz rule describes the action of Q_2 on symmetric polynomials.

Lemma 5.4. *We have*
$$Q_2 e_k = -\binom{n-k+1}{3} e_{k-2}.$$

Proof. It's clear that we have the following relations:
$$e_m^{(i_1,i_2,\ldots,i_r)} = e_m^{(i_1,i_2,\ldots,i_r,j)} + x_j e_{m-1}^{(i_1,i_2,\ldots,i_r,j)},$$
for all $j \notin \{i_1,\ldots,i_r\}$. We are going to use them repeatedly without mentioning it.

For $2 \leq i < j < k \leq n$ we have
$$\frac{x_j\partial_j e_m}{(x_j - x_i)(x_j - x_k)} + \frac{x_i\partial_i e_m}{(x_i - x_j)(x_i - x_k)} + \frac{x_k\partial_k e_m}{(x_k - x_i)(x_k - x_j)} =$$
$$\begin{aligned} &= \frac{-x_j(x_i - x_k)e_{m-1}^{(j)} + x_i(x_j - x_k)e_{m-1}^{(i)} + x_k(x_i - x_j)e_{m-1}^{(k)}}{(x_i - x_j)(x_j - x_k)(x_i - x_k)} \\ &= -\left(\frac{x_j(x_k - x_i)e_{m-1}^{(j)} + x_i(x_j - x_k)e_{m-1}^{(i)} + x_k(x_i - x_j)e_{m-1}^{(k)}}{(x_i - x_j)(x_j - x_k)(x_k - x_i)}\right).\end{aligned}$$
Clearly the denominator divides the numerator, but we want to compute the quotient. The numerator is equal to
$$(x_k - x_i)(x_j e_{m-1}^{(j)}) + x_i x_k(e_{m-1}^{(k)} - e_{m-1}^{(i)}) + x_j(x_i e_{m-1}^{(i)} - x_k e_{m-1}^{(k)}) =$$
$$\begin{aligned} &= (x_k - x_i)(x_j e_{m-1}^{(j)}) + x_i x_k(x_i - x_k)e_{m-2}^{(i,k)} + x_j(x_i - x_k)e_{m-1}^{(i,k)} \\ &= (x_k - x_i)(x_j e_{m-1}^{(j)} - x_i x_k e_{m-2}^{(i,k)} - x_j e_{m-1}^{(i,k)}).\end{aligned}$$
The second factor of the last term is equal to
$$x_j e_{m-1}^{(i,j,k)} + x_j x_i e_{m-2}^{(i,j,k)} + x_j x_k e_{m-2}^{(i,j,k)} + x_i x_j x_k e_{m-3}^{(i,j,k)} +$$
$$\begin{aligned} &- x_i x_k e_{m-2}^{(i,j,k)} - x_i x_j x_k e_{m-3}^{(i,j,k)} - x_j e_{m-1}^{(i,j,k)} - x_j^2 e_{m-2}^{(i,j,k)} \\ &= (x_i x_j + x_j x_k - x_i x_k - x_j^2)e_{m-2}^{(i,j,k)} \\ &= (x_i - x_j)(x_j - x_k)e_{m-2}^{(i,j,k)}.\end{aligned}$$

In conclusion we get

$$Q_2 e_m = - \sum_{2\le i<j<k\le n} e^{(i,j,k)}_{m-2} = - \binom{n-m+1}{3} e_{m-2},$$

where the last equality comes from counting how many times the monomial $x_2 x_3 \cdots x_{m-1}$ shows up. $\square$

The following identity together with Leibniz rule describes the action of the operator $\widetilde{P}_2$ on symmetric functions.

Lemma 5.5. *For $k \ge h$,*

$$\widetilde{P}_2(e_k e_h) = (n-k)(n-k-1)e_{k-1}e_{h-1} - (2n-h-k-1)\left(\sum_{i=1}^{h-1}(k-h+2i)e_{k-1+i}e_{h-1-i}\right).$$

Proof. By induction on n:

$$\begin{aligned}
\widetilde{P}_2(e_k e_h) =&\ 2 \sum_{2\le i<j\le n} \frac{1}{x_i - x_j}\left(x_i \partial_i e_k \partial_i e_h - x_j \partial_j e_k \partial_j e_h\right) \\
=&\ 2 \sum_{2\le i<j\le n} \frac{1}{x_i - x_j}\left(x_i e^{(i)}_{k-1} e^{(i)}_{h-1} - x_j e^{(j)}_{k-1} e^{(j)}_{h-1}\right) \\
=&\ 2 \sum_{2\le i<j\le n} \frac{1}{x_i - x_j}\left(x_i \left(e^{(i,j)}_{k-1} + x_j e^{(i,j)}_{k-2}\right)\left(e^{(i,j)}_{h-1} + x_j e^{(i,j)}_{h-2}\right)\right. \\
-&\ \left. x_j \left(e^{(i,j)}_{k-1} + x_i e^{(i,j)}_{k-2}\right)\left(e^{(i,j)}_{h-1} + x_i e^{(i,j)}_{h-2}\right)\right) \\
=&\ 2 \sum_{2\le i<j\le n} \frac{1}{x_i - x_j}\left((x_i - x_j) e^{(i,j)}_{k-1} e^{(i,j)}_{h-1} + x_i x_j (x_j - x_i) e^{(i,j)}_{k-2} e^{(i,j)}_{h-2}\right) \\
=&\ 2 \sum_{2\le i<j\le n} \left(e^{(i,j)}_{k-1} e^{(i,j)}_{h-1} - x_i x_j e^{(i,j)}_{k-2} e^{(i,j)}_{h-2}\right) \\
=&\ 2 \sum_{2\le i<n} \left(e^{(i,n)}_{k-1} e^{(i,n)}_{h-1} - x_i x_n e^{(i,n)}_{k-2} e^{(i,n)}_{h-2}\right) \\
+&\ 2 \sum_{2\le i<j\le n-1} \left(e^{(i,j,n)}_{k-1} e^{(i,j,n)}_{h-1} - x_i x_j e^{(i,j,n)}_{k-2} e^{(i,j,n)}_{h-2}\right) \\
+&\ 2\, x_n \sum_{2\le i<j\le n-1} \left(\left(e^{(i,j,n)}_{k-1} e^{(i,j,n)}_{h-2} + e^{(i,j,n)}_{k-2} e^{(i,j,n)}_{h-1}\right) - x_i x_j \left(e^{(i,j,n)}_{k-2} e^{(i,j,n)}_{h-3} + e^{(i,j,n)}_{k-3} e^{(i,j,n)}_{h-2}\right)\right) \\
+&\ 2\, x_n^2 \sum_{2\le i<j\le n-1} \left(e^{(i,j,n)}_{k-2} e^{(i,j,n)}_{h-2} - x_i x_j e^{(i,j,n)}_{k-3} e^{(i,j,n)}_{h-3}\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{2\le i<n}\partial_i e_k^{(n)}\partial_i e_h^{(n)} - 2\,x_n\sum_{2\le i<n} x_i\partial_i e_{k-1}^{(n)}\partial_i e_{h-1}^{(n)}\\
&+ (n-k-1)(n-k-2)e_{k-1}^{(n)}e_{h-1}^{(n)} - (2n-h-k-3)\left(\sum_{i=1}^{h-1}(k-h+2i)e_{k-1+i}^{(n)}e_{h-1-i}^{(n)}\right)\\
&+ x_n\left((n-k-1)(n-k-2)e_{k-1}^{(n)}e_{h-2}^{(n)} - (2n-h-k-2)\left(\sum_{i=1}^{h-2}(k-h+1+2i)e_{k-1+i}^{(n)}e_{h-2-i}^{(n)}\right)\right)\\
&+ x_n\left((n-k)(n-k-1)e_{k-2}^{(n)}e_{h-1}^{(n)} - (2n-h-k-2)\left(\sum_{i=1}^{h-1}(k-h-1+2i)e_{k-2+i}^{(n)}e_{h-1-i}^{(n)}\right)\right)\\
&+ x_n^2\left((n-k)(n-k-1)e_{k-2}^{(n)}e_{h-2}^{(n)} - (2n-h-k-1)\left(\sum_{i=1}^{h-2}(k-h+2i)e_{k-2+i}^{(n)}e_{h-2-i}^{(n)}\right)\right).
\end{aligned}
$$

We have

$$
2\sum_{2\le i<n}\partial_i e_k^{(n)}\partial_i e_h^{(n)} - 2\,x_n\sum_{2\le i<n} x_i\partial_i e_{k-1}^{(n)}\partial_i e_{h-1}^{(n)} = \nabla_2(e_k^{(n)}e_h^{(n)}) - x_n\widetilde{D}_1(e_{k-1}^{(n)}e_{h-1}^{(n)}),
$$

Hence

$$
\begin{aligned}
\widetilde{P}_2(e_ke_h) = &\ 2(n-k-1)e_{k-1}^{(n)}e_{h-1}^{(n)} - 2\sum_{i=1}^{h-1}(k-h+2i)e_{k+i-1}^{(n)}e_{h-i-1}^{(n)}\\
&- x_n\left(2\sum_{i=0}^{h-2}(k-h+1+2i)e_{k-1+i}^{(n)}e_{h-2-i}^{(n)}\right)\\
&+ (n-k-1)(n-k-2)e_{k-1}^{(n)}e_{h-1}^{(n)} - (2n-h-k-3)\left(\sum_{i=1}^{h-1}(k-h+2i)e_{k-1+i}^{(n)}e_{h-1-i}^{(n)}\right)\\
&+ x_n\left((n-k-1)(n-k-2)e_{k-1}^{(n)}e_{h-2}^{(n)} - (2n-h-k-2)\left(\sum_{i=1}^{h-2}(k-h+1+2i)e_{k-1+i}^{(n)}e_{h-2-i}^{(n)}\right)\right)\\
&+ x_n\left((n-k)(n-k-1)e_{k-2}^{(n)}e_{h-1}^{(n)} - (2n-h-k-2)\left(\sum_{i=1}^{h-1}(k-h-1+2i)e_{k-2+i}^{(n)}e_{h-1-i}^{(n)}\right)\right)\\
&+ x_n^2\left((n-k)(n-k-1)e_{k-2}^{(n)}e_{h-2}^{(n)} - (2n-h-k-1)\left(\sum_{i=1}^{h-2}(k-h+2i)e_{k-2+i}^{(n)}e_{h-2-i}^{(n)}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (n-k)(n-k-1)e^{(n)}_{k-1}e^{(n)}_{h-1} - (2n-h-k-1)\left(\sum_{i=1}^{h-1}(k-h+2i)e^{(n)}_{k-1+i}e^{(n)}_{h-1-i}\right)\\
&+ x_n\left((n-k)(n-k-1)e^{(n)}_{k-1}e^{(n)}_{h-2} - (2n-h-k-1)\left(\sum_{i=1}^{h-2}(k-h+1+2i)e^{(n)}_{k-1+i}e^{(n)}_{h-2-i}\right)\right)\\
&+ x_n\left((n-k)(n-k-1)e^{(n)}_{k-2}e^{(n)}_{h-1} - (2n-h-k-1)\left(\sum_{i=1}^{h-1}(k-h-1+2i)e^{(n)}_{k-2+i}e^{(n)}_{h-1-i}\right)\right.\\
&+ \left.((2n-h-k-1)-2(n-k-1)-(k-h+1))e^{(n)}_{k-1}e^{(n)}_{h-2}\right)\\
&+ x_n^2\left((n-k)(n-k-1)e^{(n)}_{k-2}e^{(n)}_{h-2} - (2n-h-k-1)\left(\sum_{i=1}^{h-2}(k-h+2i)e^{(n)}_{k-2+i}e^{(n)}_{h-2-i}\right)\right)\\
&= (n-k)(n-k-1)e_{k-1}e_{h-1} - (2n-h-k-1)\left(\sum_{i=1}^{h-1}(k-h+2i)e_{k-1+i}e_{h-1-i}\right).
\end{aligned}
$$

The base cases are trivial. □

All this together with Leibniz rule describes the action of $\widetilde{D}_2$ on alternating polynomials.

5.1. **List of Formulae.** For convenience and for future reference, we give a list of the formulae that we found along the way. ***In this subsection*** we state them in terms of the variables $x_1, x_2, \ldots, x_n$, adapting the definitions accordingly.

Here e_k will be the elementary symmetric function in n variables of degree k, and g a symmetric function in the variables $x_1, x_2, \ldots, x_n$.

5.1.1. *Action of ∇_1.*

$$\frac{1}{\Delta}\nabla_1(\Delta g) = \nabla_1 g.$$

$$\nabla_1 e_k = (n-k+1)e_{k-1}.$$

5.1.2. *Action of ∇_2.*

$$\frac{1}{\Delta}\nabla_2(\Delta g) = (\nabla_2 + 2P_2)g,$$

where

$$P_2 := \sum_{1\leq i<j\leq n}\frac{1}{x_i - x_j}(\partial_i - \partial_j).$$

$$P_2 e_k = -\binom{n-k+2}{2}e_{k-2}.$$

For $k \geq h$ we have

$$\nabla_2(e_k e_h) = 2(n-k+1)e_{k-1}e_{h-1} - 2\sum_{i=1}^{h-1}(k-h+2i)e_{k+i-1}e_{h-i-1}.$$

If g is a symmetric function,

$$[\nabla_1, P_2]g = 0.$$

5.1.3. *Action of $\widetilde{D}_1$.*

$$\frac{1}{\Delta}\widetilde{D}_1(\Delta g) = (2P_1 + \widetilde{D}_1)g,$$

where

$$P_1 := \sum_{1\le i<j\le n} \frac{1}{x_i - x_j}(x_i\partial_i - x_j\partial_j).$$

$$P_1 e_k = \binom{n-k+1}{2} e_{k-1}.$$

For $k \ge h$,

$$\widetilde{D}_1(e_k e_h) = 2\sum_{i=0}^{h-1}(k-h+1+2i)e_{k+i}e_{h-1-i}.$$

5.1.4. *Action of $\widetilde{D}_2$.*

$$\frac{1}{\Delta}\widetilde{D}_2(\Delta g) = (6Q_2 + 3\widetilde{P}_2 + \widetilde{D}_2)g,$$

where

$$Q_2 := \sum_{j=1}^{n}\sum_{i<k}^{(j)} \frac{1}{(x_j - x_i)(x_j - x_k)} x_j\partial_j,$$

and

$$\widetilde{P}_2 := \sum_{1\le i<j\le n} \frac{1}{x_i - x_j}(x_i\partial_i^2 - x_j\partial_j^2).$$

$$Q_2 e_k = -\binom{n-k+2}{3} e_{k-2}.$$

For $k \ge h$,

$$\widetilde{P}_2(e_k e_h) = (n-k+1)(n-k)e_{k-1}e_{h-1} - (2n-h-k+1)\left(\sum_{i=1}^{h-1}(k-h+2i)e_{k-1+i}e_{h-1-i}\right).$$

For $k \ge h \ge l$,

$$\widetilde{D}_2(e_k e_h e_l) = 6\left(\sum_{j=0}^{l-1}\sum_{i=0}^{h-1}(k-h+1+j+2i)e_{k+i+j}e_{h-1-i}e_{l-1-j}\right.$$
$$\left. - \sum_{j=0}^{l-2}\sum_{i=1}^{l-1-j}(h-l+j+2i)e_{k+j}e_{h-1+i}e_{l-1-i-j}\right).$$

6. Singular q_0-harmonics

Warning: *in this section we use the notations of section 5.1*

Recall here that in [HT] Thiéry and Hivert stated the conjecture that in the case where q is a complex number not of the form $-a/b$ where $a \in \{1, 2, \dots, n\}$ and $b \in \mathbb{N}$, we have the equality

$$\sum_{d \geq 0} \dim \pi_d(\mathcal{H}_{\mathbf{x};q}) t^d = [n]_t!.$$

Inspired by a similar definition in [HT], we define a complex number q_0 *singular* if the Frobenius characteristic $F_{n;q_0}(t)$ of the q_0-harmonics is different from the Frobenius characteristic $F_n(t) = F_{n;0}(t)$ of the classical harmonics, which is (see [M])

$$F_n(t) = \sum_{\lambda \vdash n} s_\lambda \sum_{T \in ST(n)} t^{co(T)},$$

where $\lambda \vdash n$ indicates that λ is a partition of n, s_λ is the Schur function indexed by λ, $ST(\lambda)$ denotes the set of standard tableaux of shape λ, and $co(T)$ denote the cocharge of the tableau T.

One of the main result of this section is the following theorem.

Theorem 6.1. *The values of q_0 of the form $-a/b$ where $a \in \{1, 2, \dots, n\}$, $b \in \mathbb{N}$ and $b \geq n$ are singular.*

Remark. Notice that in the statement we don't require that a and b are coprime. For example if $n = 6$, then we will show that $-2/3$ is singular, since it can be written as $-4/6$.

More generally, in this appendix we will investigate the q_0-harmonics for singular values of q_0.

Remark. Since the case $q_0 = 0$ reduces to the well known case of classical $\mathfrak{S}_n$-harmonics, in this section we will always assume $q_0 \neq 0$. Recall also, from the easy relations

$$[D_{k;q_0}, D_{h;q_0}] = q_0(k - h) D_{k+h;q_0},$$

it follows that a polynomial f is in $\mathcal{H}_{\mathbf{x};q_0}$ if and only if

$$D_{1;q_0} f = D_{2;q_0} f = 0.$$

We will use repeatedly this observation without mentioning it anymore.

In our computer investigations we realized that polynomials of certain forms are q_0-harmonics for special values of q_0. Using the formulae of the previous section we are now able to prove that this is the case.

First of all we prove that for $1 \leq k < n$ and $q_0 = -1/(n-k)$ the alternant Δe_k is in $\mathcal{H}_{\mathbf{x};q_0}$. This shows immediately that these values of q_0 are singular, since in the classical case the only alternant is Δ in degree $\binom{n}{2}$.

Theorem 6.2. *The polynomial Δe_k is q_0-harmonic if and only if $k < n$ and $q_0 = -1/(n-k)$.*

Proof. Let's look at the action of $D_{1;q_0} = \nabla_1 + q_0\widetilde{D}_1$ on Δe_k. Using the formulae listed in the previous section, we have

$$\begin{aligned} D_{1;q_0}\Delta e_k &= (\nabla_1 + q_0\widetilde{D}_1)\Delta e_k = \Delta(\nabla_1 + q_0(2P_1 + \widetilde{D}_1))e_k \\ &= \left((n-k+1) + 2q_0\binom{n-k+1}{2}\right)\Delta e_k. \end{aligned}$$

Hence to have $D_{1;q_0}\Delta e_k = 0$ we need to have $k < n$ and

$$q_0 = -\frac{1}{n-k}.$$

Let's now look at $D_{2;q_0}\Delta e_k$. We have

$$\begin{aligned} D_{2;q_0}\Delta e_k &= \Delta((\nabla_2 + 2P_2) + q_0(6Q_2 + 3\widetilde{P}_2 + \widetilde{D}_2))e_k \\ &= \left(-2\binom{n-k+2}{2} - q_0 6\binom{n-k+2}{3}\right)\Delta e_k, \end{aligned}$$

which is 0 for $q_0 = -1/(n-k)$. □

We determine now another class of q_0-harmonics which will imply the singularity of many values of q_0.

Recall that we work in $n \geq 2$ variables.

Theorem 6.3. *The polynomial $e_1^m(x_1, x_2, \ldots, x_k)(x_1 - x_2)$, with $2 \leq k \leq n$ and $m \geq 1$ is a q_0-harmonic if and only if $q_0 = -\frac{k}{m+1}$.*

Proof. Let's look at the action of $D_{1;q_0} = \nabla_1 + q_0\widetilde{D}_1$. We have

$$\nabla_1 e_1^m(x_1, x_2, \ldots, x_k)(x_1 - x_2) = m\,k\,e_1^{m-1}(x_1, x_2, \ldots, x_k)(x_1 - x_2),$$

while

$$\begin{aligned} &\widetilde{D}_1 e_1^m(x_1, x_2, \ldots, x_k)(x_1 - x_2) = \\ &= \left(2\binom{m}{2} + 2m\right) e_1^{m-1}(x_1, x_2, \ldots, x_k)(x_1 - x_2) \\ &= m(m+1)\,e_1^{m-1}(x_1, x_2, \ldots, x_k)(x_1 - x_2). \end{aligned}$$

Therefore

$$D_{1;q_0} e_1^m(x_1, x_2, \ldots, x_k)(x_1 - x_2) = (k\,m + q_0 m(m+1))\,e_1^{m-1}(x_1, x_2, \ldots, x_k)(x_1 - x_2),$$

and this is equal to 0 if and only if $q_0 = -\frac{k}{m+1}$.

We are left to check that also $D_{2;q_0}$ kills our polynomial. We have

$$\nabla_2 e_1^m(x_1, x_2, \ldots, x_k)(x_1 - x_2) = \binom{m}{2} 2k\,e_1^{m-2}(x_1, x_2, \ldots, x_k)(x_1 - x_2),$$

while

$$\begin{aligned} \widetilde{D}_2 e_1^m(x_1, x_2, \ldots, x_k)(x_1 - x_2) &= \left(\binom{m}{3} 6 + 3m(m-1)\right) e_1^{m-2}(x_1, x_2, \ldots, x_k)(x_1 - x_2) \\ &= (m+1)m(m-1)e_1^{m-2}(x_1, x_2, \ldots, x_k)(x_1 - x_2). \end{aligned}$$

Therefore

$$D_{2;q_0} e_1^m(x_1, x_2, \ldots, x_k)(x_1 - x_2) =$$

$$= (m(m-1)k + q_0(m+1)m(m-1)) e_1^{m-2}(x_1, x_2, \ldots, x_k)(x_1 - x_2) = 0.$$

$\square$

Notice that the degree of the polynomial $e_1^m(x_1, x_2, \ldots, x_k)(x_1 - x_2)$ is $m+1$, hence whenever $m+1 > \binom{n}{2}$, by the previous theorem the value $q_0 = -\frac{k}{m+1}$ with $2 \leq k \leq n$ is singular. This shows that for each n, all but finitely many of the numbers of the form $-a/b$ with $a \in \{1, 2, \ldots, n\}$ and $b \in \mathbb{N}$ (the ones that show up in Conjecture 1) are in fact singular.

We are now in a position to proof Theorem 6.1.

proof of Theorem 6.1. For every integer $d \geq 1$ and every partition μ of d we denote by V_μ the irreducible $\mathfrak{S}_d$-representation corresponding to μ.

Given $m \geq 1$ and $n \geq k \geq 2$, for $1 \leq i, j \leq n$, $i \neq j$, we set

$$p_{i,j} := \sum_{h=1}^{n} \left(\sum_{\substack{\{i,h\} \subseteq S \subseteq \{1,2,\ldots,n\} \\ |S|=k}} e_1^m(\mathbf{x}_S)(x_i - x_h) - \sum_{\substack{\{j,h\} \subseteq S \subseteq \{1,2,\ldots,n\} \\ |S|=k}} e_1^m(\mathbf{x}_S)(x_j - x_h) \right),$$

where $\mathbf{x}_S$ indicates the set of variables indexed by the elements of S. It's easy to see that the map $p_{i,j} \mapsto x_i - x_j$ is an isomorphism of representations of $\mathfrak{S}_n$. Since clearly the $p_{i,j}$'s are in the $\mathfrak{S}_n$-module generated by $e_1^m(x_1, x_2, \ldots, x_k)(x_1 - x_2)$, we have just showed that this module contains a submodule isomorphic to $V_{(n-1,1)}$.

All this implies the singularity of $q_0 = -a/b$ with $a \in \{1, 2, \ldots, n\}$, $b \in \mathbb{N}$ and $b \geq n$, since in the Frobenius characteristic of the classical harmonics $s_{(n-1,1)}$ shows up only up to degree $n-1$. This proves the theorem. $\square$

During our computer investigations we realized that we couldn't find an example of singular value of q_0 which is not in the form of Theorem 6.1.

We risk the following conjecture.

Conjecture 2. *The numbers of the form $-a/b$ where $a, b \in \mathbb{N}$ and $b \geq n \geq a \geq 1$ are the only singular values of q_0.*

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Appendix: an f_1 and an f_2 for $\partial_1\Delta$

Notice that in this section we use notations and results from all the previous sections.

We want to find now a value of c for which we can find an f_2 for $f_{1;c}$, if any exists. First we proceed as we did with our first solution. We write

$$f_{1;c} = \Delta^{(1)} g$$

and

$$g_{s,c} = a_s e_{n-s-2} + b_s e_{n-s-3} e_1 + c_s e_{n-s-4} e_1^2 + d_s e_{n-s-4} e_2,$$

where the coefficients are determined by the previous equations, and of course they depend not only on s, but also on n and $c = c(n)$. Again we get

$$x_1 \partial_1^2 f_{1;c} = \Delta^{(1)} \sum_{s=0}^{n-3} (s+1)s(a_{s+1}e_{n-s-3} + b_{s+1}e_{n-s-4}e_1 + c_{s+1}e_{n-s-5}e_1^2 + d_{s+1}e_{n-s-5}e_2),$$

and

$$\begin{aligned}\sum_{j=2}^{n} x_j \partial_j^2 f_{1;c} &= 2\sum_{j=2}^{n} (\partial_j \Delta^{(1)}) x_j \partial_j g + \Delta^{(1)} \sum_{j=2}^{n} x_j \partial_j^2 g \\ &= \Delta^{(1)} \left(2\sum_{s=0}^{n-2} (P_1 g_{s;c}) x_1^s + \sum_{s=0}^{n-2} (\widetilde{D}_1 g_{s;c}) x_1^s \right).\end{aligned}$$

Now

$$\begin{aligned}\widetilde{D}_1 g_{s;c} &= 2(n-s-3)b_s e_{n-s-3} + (4(n-s-4)+2)c_s e_{n-s-4} e_1 \\ &+ 2(n-s-5)d_s e_{n-s-4} e_1 + 2(n-s-3)d_s e_{n-s-3},\end{aligned}$$

and

$$\begin{aligned}2P_1 g_{s;c} &= (s+2)(s+1)a_s e_{n-s-3} + (s+3)(s+2)b_s e_{n-s-4}e_1 + (n-1)(n-2)b_s e_{n-s-3} \\ &+ (s+4)(s+3)c_s e_{n-s-5}e_1^2 + 2(n-1)(n-2)c_s e_{n-s-4}e_1 \\ &+ (s+4)(s+3)d_s e_{n-s-5}e_2 + (n-2)(n-3)d_s e_{n-s-4}e_1.\end{aligned}$$

Hence we can write

$$-\widetilde{D}_1 f_{1;c} = \Delta^{(1)} \sum_{s=0}^{n-2} (\tilde{a}_s\, e_{n-s-3} + \tilde{b}_s\, e_{n-s-4}e_1 + \tilde{c}_s\, e_{n-s-5}e_1^2 + \tilde{d}_s\, e_{n-s-5}e_2) x_1^s,$$

where

$$\begin{aligned}\tilde{a}_s &:= -(s+1)s\, a_{s+1} - 2(n-s-3)b_s - 2(n-s-3)d_s - (s+2)(s+1)a_s - (n-1)(n-2)b_s \\ &= -(s+1)s(-1)^{s+1} \left(\frac{((-1)^{n-1} - c\,n)}{n^2-7} \frac{(s+1)s}{2} \frac{n-1}{n} + c\frac{(s+1)(n-1)}{2} + (s+1)(-1)^n \right)(s+2) \\ &- (2(n-s-3) + (n-1)(n-2))(-1)^s \left(\frac{s((-1)^{n-1} - c\,n)}{n(n^2-7)} + \frac{c}{2} \right)(s+2)(s+1)\end{aligned}$$

$$\begin{aligned}
&- \quad 2(n-s-3)(-1)^{s+1}\frac{((-1)^{n-1}-c\,n)}{n^2-7}(s+3)(s+2)(s+1)\\
&- \quad (s+2)(s+1)(-1)^s\left(\frac{((-1)^{n-1}-c\,n)}{n^2-7}\frac{s(s-1)}{2}\frac{n-1}{n}+c\frac{s(n-1)}{2}+s(-1)^n\right)(s+1)\\
&= \quad (-1)^s\frac{(s+2)(s+1)}{2n(n^2-7)}\Big(3(n-1)(cn+(-1)^n)s^2\\
&- \quad (-1)^n(21n((-1)^n-n)c+2n^2-21n+7))s\\
&- \quad n(n-1)(n^3+n-28)c+(-1)^n 12n(n-3)\Big),
\end{aligned}$$

$$\begin{aligned}
\tilde{b}_s \quad &:= \quad -(s+1)s\,b_{s+1}-(4(n-s-4)+2)c_s-2(n-s-5)d_s-(s+3)(s+2)b_s\\
&- \quad 2(n-1)(n-2)c_s-(n-2)(n-3)d_s\\
&= \quad -(s+1)s(-1)^{s+1}\left(\frac{(s+1)((-1)^{n-1}-c\,n)}{n(n^2-7)}+\frac{c}{2}\right)(s+3)(s+2)\\
&- \quad ((4(n-s-4)+2)+2(n-1)(n-2))(-1)^s\frac{(n-1)((-1)^{n-1}-c\,n)}{2n(n^2-7)}(s+3)(s+2)(s+1)\\
&- \quad (2(n-s-5)+(n-2)(n-3))(-1)^{s+1}\frac{((-1)^{n-1}-c\,n)}{n^2-7}(s+3)(s+2)(s+1)\\
&- \quad (s+3)(s+2)(-1)^s\left(\frac{s((-1)^{n-1}-c\,n)}{n(n^2-7)}+\frac{c}{2}\right)(s+2)(s+1)\\
&= \quad (-1)^s\frac{(s+3)(s+2)(s+1)}{2n(n^2-7)}\left(6(cn+(-1)^n)s+(24cn+2(-1)^n n^2+10(-1)^n)\right),
\end{aligned}$$

$$\begin{aligned}
\tilde{c}_s \quad &:= \quad -(s+1)s\,c_{s+1}-(s+4)(s+3)c_s\\
&= \quad -(s+1)s(-1)^{s+1}\frac{(n-1)((-1)^{n-1}-c\,n)}{2n(n^2-7)}(s+4)(s+3)(s+2)\\
&- \quad (s+4)(s+3)(-1)^s\frac{(n-1)((-1)^{n-1}-c\,n)}{2n(n^2-7)}(s+3)(s+2)(s+1)\\
&= \quad (-1)^s\frac{(s+4)(s+3)(s+2)(s+1)}{2n(n^2-7)}(3(n-1)(cn+(-1)^n)),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{d}_s \quad &:= \quad -(s+1)s\,d_{s+1}-(s+4)(s+3)d_s\\
&= \quad -(s+1)s(-1)^{s+2}\frac{((-1)^{n-1}-c\,n)}{n^2-7}(s+4)(s+3)(s+2)\\
&- \quad (s+4)(s+3)(-1)^{s+1}\frac{((-1)^{n-1}-c\,n)}{n^2-7}(s+3)(s+2)(s+1)\\
&= \quad (-1)^s\frac{(s+4)(s+3)(s+2)(s+1)}{(n^2-7)}(-3(cn+(-1)^n))
\end{aligned}$$

To compute $\widetilde{D}_2 f_{1;c}$, first we have

$$x_1\partial_1^3 f_{1;c} = \Delta^{(1)}\sum_{s=0}^{n-4}(s+2)(s+1)s(a_{s+2}\, e_{n-4-s}+b_{s+2}\, e_{n-5-s}e_1+c_{s+2}\, e_{n-6-s}e_1^2+d_{s+2}\, e_{n-6-s}e_2)x_1^s.$$

Then

$$\begin{aligned}\sum_{j=2}^{n} x_j\partial_j^3 f_{1;c} &= 3\sum_{j=2}^{n}(\partial_j^2\Delta^{(1)})x_j\partial_j g + 3\sum_{j=2}^{n}(\partial_j\Delta^{(1)})x_j\partial_j^2 g + \Delta^{(1)}\sum_{j=2}^{n} x_j\partial_j^3 g \\ &= \Delta^{(1)}\sum_{j=2}^{n}(6\, Q_2 g_{s;c} + 3\widetilde{P}_2 g_{s;c} + \widetilde{D}_2 g_{s:c})x_1^s,\end{aligned}$$

where

$$\widetilde{P}_2 := \sum_{2\le i<j\le n}\frac{1}{x_i-x_j}(x_i\partial_i^2 - x_j\partial_j^2).$$

Now

$$\begin{aligned}6\, Q_2 g_{s;c} &= -(s+3)(s+2)(s+1)a_s e_{n-s-4} - (s+4)(s+3)(s+2)b_s e_{n-s-5}e_1 \\ &- (s+5)(s+4)(s+3)c_s e_{n-s-6}e_1^2 - (s+5)(s+4)(s+3)d_s e_{n-s-6}e_2 \\ &- (n-1)(n-2)(n-3)d_s e_{n-s-4},\end{aligned}$$

$$\begin{aligned}3\,\widetilde{P}_2 g_{s;c} &= 3(s+3)(s+2)b_s e_{n-s-4} \\ &+ 6(s+4)(s+3)c_s e_{n-s-5}e_1 + 3(n-1)(n-2)c_s e_{n-s-4} \\ &+ 3(s+4)(s+3)d_s e_{n-s-5}e_1 - 3(n-s-4)(n+s+1)d_s e_{n-s-4},\end{aligned}$$

while clearly $\widetilde{D}_2 g_{s;c} = 0$.

Hence we have

$$-\widetilde{D}_2 f_{1;c} = \Delta^{(1)}\sum_{s=0}^{n-4}(\hat{a}_s e_{n-s-4} + \hat{b}_s e_{n-s-5}e_1 + \hat{c}_s e_{n-s-6}e_1^2 + \hat{d}_s e_{n-s-6}e_2)x_1^s,$$

where

$$\begin{aligned}\hat{a}_s &:= -(s+2)(s+1)s\, a_{s+2} + (s+3)(s+2)(s+1)a_s + (n-1)(n-2)(n-3)d_s \\ &- 3(s+3)(s+2)b_s - 3(n-1)(n-2)c_s + 3(n-s-4)(n+s+1)d_s \\ &= -(s+2)(s+1)s(-1)^{s+2}(s+2)\left(\frac{((-1)^{n-1}-c\,n)}{n^2-7}\frac{(s+1)}{2}\frac{n-1}{n} + c\frac{(n-1)}{2} + (-1)^n\right)(s+3) \\ &+ (s+3)(s+2)(s+1)(-1)^s\left(\frac{((-1)^{n-1}-c\,n)}{n^2-7}\frac{s(s-1)}{2}\frac{n-1}{n} + c\frac{s(n-1)}{2} + s(-1)^n\right)(s+1)\end{aligned}$$

$$
\begin{aligned}
&+ \quad (n-1)(n-2)(n-3)(-1)^{s+1}\frac{((-1)^{n-1}-c\,n)}{n^2-7}(s+3)(s+2)(s+1)\\
&- \quad 3(s+3)(s+2)(-1)^s\left(\frac{s((-1)^{n-1}-c\,n)}{n(n^2-7)}+\frac{c}{2}\right)(s+2)(s+1)\\
&- \quad 3(n-1)(n-2)(-1)^s\frac{(n-1)((-1)^{n-1}-c\,n)}{2n(n^2-7)}(s+3)(s+2)(s+1)\\
&+ \quad 3(n-s-4)(n+s+1)(-1)^{s+1}\frac{((-1)^{n-1}-c\,n)}{n^2-7}(s+3)(s+2)(s+1)\\
&= \quad (-1)^s\frac{(s+3)(s+2)(s+1)}{2n(n^2-7)}\Big((n-1)(cn+(-1)^n)s^2+\\
&+ \quad ((-n(n-1)(n^2+3n+23))c+(-1)^n(-13n-2n^3+9))s\\
&+ \quad (n(n-1)(2n^3-n^2-15n-36))c\\
&+ \quad (-1)^n(-3n^3-8n^2-6+2n^4-21n)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\hat{b}_s \quad &:= \quad -(s+2)(s+1)s\,b_{s+2}+(s+4)(s+3)(s+2)b_s-6(s+4)(s+3)c_s-3(s+4)(s+3)d_s\\
&= \quad (-1)^s\frac{(s+4)(s+3)(s+2)(s+1)}{2n(n^2-7)}\left(-(6(cn+(-1)^n))s+18(-1)^{1+n}+2cn^3-32cn\right),
\end{aligned}
$$

$$
\begin{aligned}
\hat{c}_s \quad &:= \quad -(s+2)(s+1)s\,c_{s+2}+(s+5)(s+4)(s+3)c_s\\
&= \quad (-1)^s\frac{(s+5)(s+4)(s+3)(s+2)(s+1)}{2n(n^2-7)}\left(-(3(n-1))(c\,n+(-1)^n)\right)
\end{aligned}
$$

$$
\begin{aligned}
\hat{d}_s \quad &:= \quad -(s+2)(s+1)s\,d_{s+2}+(s+5)(s+4)(s+3)d_s\\
&= \quad (-1)^s\frac{(s+5)(s+4)(s+3)(s+2)(s+1)}{n^2-7}3(c\,n+(-1)^n)
\end{aligned}
$$

Again, we set

$$
A_s := \tilde{a}_s\,e_{n-s-3}+\tilde{b}_s\,e_{n-s-4}e_1+\tilde{c}_s\,e_{n-s-5}e_1^2+\tilde{d}_s\,e_{n-s-5}e_2
$$

and

$$
B_s := \hat{a}_s e_{n-s-4}+\hat{b}_s e_{n-s-5}e_1+\hat{c}_s e_{n-s-6}e_1^2+\hat{d}_s e_{n-s-6}e_2.
$$

Again, we get

$$
\begin{aligned}
\frac{(-1)^s}{s!}\nabla_1^s(\nabla_2+2P_2+\nabla_1^2)g_{0;c} \quad &= \quad (B_s+\nabla_1A_s-(s+1)A_{s+1})+\\
&- \quad \left(\sum_{j=0}^{s-1}(-1)^{s-1-j}\frac{j!}{s!}\nabla_1^{s-1-j}(\nabla_2+2P_2+\nabla_1^2)A_j\right). \qquad (4)
\end{aligned}
$$

Now,

$$
B_s+\nabla_1A_s-(s+1)A_{s+1}=
$$

$$
\begin{aligned}
&= \hat{a}_s e_{n-s-4} + \hat{b}_s e_{n-s-5} e_1 + \hat{c}_s e_{n-s-6} e_1^2 + \hat{d}_s e_{n-s-6} e_2 \\
&+ (s+3)\tilde{a}_s\, e_{n-s-4} + (s+4)\tilde{b}_s\, e_{n-s-5} e_1 + (n-1)\tilde{b}_s\, e_{n-s-4} \\
&+ (s+5)\tilde{c}_s\, e_{n-s-6} e_1^2 + 2(n-1)\tilde{c}_s\, e_{n-s-5} e_1 + (s+5)\tilde{d}_s\, e_{n-s-6} e_2 + (n-2)\tilde{d}_s\, e_{n-s-5} e_1 \\
&- (s+1)(\tilde{a}_{s+1}\, e_{n-s-4} + \tilde{b}_{s+1}\, e_{n-s-5} e_1 + \tilde{c}_{s+1}\, e_{n-s-6} e_1^2 + \tilde{d}_{s+1}\, e_{n-s-6} e_2) \\
&= (\hat{a}_s + (s+3)\tilde{a}_s + (n-1)\tilde{b}_s - (s+1)\tilde{a}_{s+1}) e_{n-s-4} \\
&+ (\hat{b}_s + (s+4)\tilde{b}_s + 2(n-1)\tilde{c}_s + (n-2)\tilde{d}_s - (s+1)\tilde{b}_{s+1}) e_{n-s-5} e_1 \\
&+ (\hat{c}_s + (s+5)\tilde{c}_s - (s+1)\tilde{c}_{s+1}) e_{n-s-6} e_1^2 \\
&+ (\hat{d}_s + (s+5)\tilde{d}_s - (s+1)\tilde{d}_{s+1}) e_{n-s-6} e_2 \\
&= (-1)^s \frac{(s+3)(s+2)(s+1)}{2n(n^2-7)} (-n(n-1)(n^2+3n-31)c + (-1)^n(-4n^2 - 17 + 41n - 2n^3)) e_{n-s-4} \\
&+ (-1)^s \frac{(s+4)(s+3)(s+2)(s+1)}{2n(n^2-7)} (6(cn + (-1)^n)s \\
&+ (28n + 2n^3)c + (-1)^n(14 + 4n^2)) e_{n-s-5} e_1 \\
&+ (-1)^s \frac{(s+5)(s+4)(s+3)(s+2)(s+1)}{2n(n^2-7)} (3(n-1)(cn + (-1)^n)) e_{n-s-6} e_1^2 \\
&+ (-1)^s \frac{(s+5)(s+4)(s+3)(s+2)(s+1)}{(n^2-7)} (-3(cn + (-1)^n)) e_{n-s-6} e_2.
\end{aligned}
$$

Since

$$
(\nabla_2 + 2P_2 + \nabla_1^2)(e_k e_2) = 2(n-k)(n-1) e_{k-1} e_1 - 2k e_k,
$$

we have

$$
(\nabla_2 + 2P_2 + \nabla_1^2) A_j
$$

$$
\begin{aligned}
&= 2n(j+4)\tilde{b}_j e_{n-j-5} + \\
&+ 4n(j+5)\tilde{c}_j e_{n-j-6} e_1 + 2n(n-1)\tilde{c}_j e_{n-j-5} \\
&+ 2(j+5)(n-1)\tilde{d}_j e_{n-j-6} e_1 - 2(n-j-5)\tilde{d}_j e_{n-j-5} \\
&= (-1)^j \frac{(j+4)(j+3)(j+2)(j+1)}{(n^2-7)} ((3n^3 - 3n)c + (-1)^n(5n^2 - 17)) e_{n-j-5}.
\end{aligned}
$$

The second term of the RHS of (4) is

$$
-\sum_{j=0}^{s-1} (-1)^{s-1-j} \frac{j!}{s!} \nabla_1^{s-1-j} (\nabla_2 + 2P_2 + \nabla_1^2) A_j =
$$

$$
\begin{aligned}
&= -\sum_{j=0}^{s-1}(-1)^{s-1-j}\frac{j!}{s!}\nabla_1^{s-1-j}\left((-1)^j\frac{(j+4)(j+3)(j+2)(j+1)}{(n^2-7)}((3n^3-3n)c\right.\\
&+ \left.(-1)^n(5n^2-17))e_{n-j-5}\right)\\
&= -\sum_{j=0}^{s-1}(-1)^{s-1-j}\frac{j!}{s!}\left((-1)^j\frac{(j+4)(j+3)(j+2)(j+1)}{(n^2-7)}((3n^3-3n)c\right.\\
&+ \left.(-1)^n(5n^2-17))\frac{(s+3)!}{(j+4)!}e_{n-s-4}\right)\\
&= (-1)^s(s+3)(s+2)(s+1)\left(\frac{1}{(n^2-7)}((3n^3-3n)c\right.\\
&+ \left.(-1)^n(5n^2-17))\right)\left(\sum_{j=0}^{s-1}1\right)e_{n-s-4}\\
&= (-1)^s\frac{(s+3)(s+2)(s+1)}{(n^2-7)}\left((3n^3-3n)c+(-1)^n(5n^2-17)\right)s\,e_{n-s-4}.
\end{aligned}
$$

Finally, we can write (4) as

$\nabla_1^s(\nabla_2+2P_2+\nabla_1^2)g_{0;c} =$

$$
\begin{aligned}
&= \frac{(s+3)!}{2n(n^2-7)}\left(3(n-1)(cn+(-1)^n)s^2\right.\\
&+ (n(n-1)(5n^2+3n+31)c+(-1)^n(7n+8n^3-17-4n^2))s\\
&+ \left.(-n(n-1)(n^2+17n-68))c+(-1)^n(85n-n^3-26-36n^2+2n^4)\right)e_{n-s-4}\\
&+ \frac{(s+4)!}{2n(n^2-7)}\left(6(cn+(-1)^n)s+(28n+2n^3)c+(-1)^n(14+4n^2)\right)e_{n-s-5}e_1\\
&+ \frac{(s+5)!}{2n(n^2-7)}(3(n-1)(cn+(-1)^n))e_{n-s-6}e_1^2\\
&+ \frac{(s+5)!}{(n^2-7)}(-3(cn+(-1)^n))e_{n-s-6}e_2.
\end{aligned}
$$

We reduced ourselves to solve this system of equations. We assume that we can find a solution of the form:

$$(5)\qquad (\nabla_2+2P_2+\nabla_1^2)g_{0;c} = 3!\alpha e_{n-4}+4!\beta e_{n-5}e_1+5!\gamma e_{n-6}e_1^2+5!\delta e_{n-6}e_2,$$

where α, β, γ and δ are coefficients depending only on n, and the normalization with the factorials is made for convenience in the following computations.

We have

$$\nabla_1^s\left(3!\alpha e_{n-4}+4!\beta e_{n-5}e_1+5!\gamma e_{n-6}e_1^2+5!\delta e_{n-6}e_2\right) =$$

$$
\begin{aligned}
= \quad & \left(\alpha + \beta\, s(n-1) + \gamma\, s(s-1)(n-1)^2 + \delta\, \frac{s(s-1)}{2}(n-1)(n-2)\right)(s+3)!e_{n-s-4} \\
+ \quad & (\beta + \gamma\, 2s(n-1) + \delta\, s(n-2))(s+4)!e_{n-s-5}e_1 \\
+ \quad & \gamma(s+5)!e_{n-s-6}e_1^2 + \delta(s+5)!e_{n-s-6}e_2.
\end{aligned}
$$

Now we have to equate the unknown coefficients to the one we have in the system. First we get

$$
\gamma = \frac{1}{2n(n^2-7)}(3(n-1)(cn+(-1)^n)) \quad \text{and} \quad \delta = \frac{1}{(n^2-7)}(-3(cn+(-1)^n)).
$$

Replacing them in the second coefficient we have

$$
\begin{aligned}
\beta + &\left(\frac{1}{2n(n^2-7)}(3(n-1)(cn+(-1)^n))\right) 2s(n-1) + \left(\frac{1}{(n^2-7)}(-3(cn+(-1)^n))\right) s(n-2) = \\
&= \frac{1}{2n(n^2-7)}\left(6(cn+(-1)^n)s + (28n+2n^3)c + (-1)^n(14+4n^2)\right),
\end{aligned}
$$

from which we get

$$
\beta = \frac{1}{n(n^2-7)}((n^3+14n)c + (-1)^n(2n^2+7)).
$$

From the other equation we get

$$
\begin{aligned}
\alpha \quad = \quad & -\left(\frac{1}{n(n^2-7)}((n^3+14n)c + (-1)^n(2n^2+7))\right) s(n-1) \\
- \quad & \frac{s(s-1)(n-1)^2}{2n(n^2-7)}(3(n-1)(cn+(-1)^n)) \\
- \quad & \frac{s(s-1)(n-1)(n-2)}{2(n^2-7)}(-3(cn+(-1)^n)) \\
+ \quad & \frac{1}{2n(n^2-7)}\Big(3(n-1)(cn+(-1)^n)s^2 \\
+ \quad & (n(n-1)(5n^2+3n+31)c + (-1)^n(7n+8n^3-17-4n^2))s \\
+ \quad & (-n(n-1)(n^2+17n-68))c + (-1)^n(85n-n^3-26-36n^2+2n^4)\Big) \\
= \quad & \frac{1}{2n(n^2-7)}\Big((3n(n-1)(n^2+n+2)c + (-1)^n(-4n+4n^3-6))s \\
+ \quad & (-n(n-1)(n^2+17n-68))c + (-1)^n(-36n^2+85n+2n^4-26-n^3)\Big).
\end{aligned}
$$

Since we want α depending only on n, we must have

$$
3n(n-1)(n^2+n+2)c + (-1)^n(-4n+4n^3-6) = 0,
$$

so

$$
c = (-1)^{n-1}\frac{2(2n^3-2n-3)}{3n(n-1)(n^2+n+2)}.
$$

We determined a value of c for which we reduced all the system to the single equation (5).

Before computing the solution of the equation, we compute the explicit formula for the $f_{1;c}$ for this value of c:

$$\begin{aligned}
g_{s;c} &= (-1)^{s+n}\frac{(s+1)}{6n(n^2+n+2)}(n\,s^2+(2n^3+6n^2+15n+6)s)e_{n-s-2} \\
&+ (-1)^{s+n}\frac{(s+2)(s+1)}{3n(n-1)(n^2+n+2)}(n\,s+(-2n^3+2n+3))e_{n-s-3}e_1 \\
&+ (-1)^{s+n}\frac{(s+3)(s+2)(s+1)}{6(n^2+n+2)}e_{n-s-4}e_1^2 \\
&- (-1)^{s+n}\frac{(s+3)(s+2)(s+1)n}{3(n-1)(n^2+n+2)}e_{n-s-4}e_2,
\end{aligned}$$

and from this we can write a formula for f_1.

Now we substitute the value of c that we have found into the coefficients:

$$\begin{aligned}
\alpha &= (-1)^n\frac{(6n^4+7n^3+11n^2-86n-36)}{6n(n^2+n+2)}, \\
\beta &= (-1)^n\frac{(n+2)(2n^2-4n-3)}{3n(n-1)(n^2+n+2)}, \\
\gamma &= -(-1)^n\frac{1}{2(n^2+n+2)}, \\
\delta &= (-1)^n\frac{n}{(n-1)(n^2+n+2)}.
\end{aligned}$$

We now assume that g_0 is of the form

$$u\,e_{n-3}e_1+v\,e_{n-4}e_1^2+w\,e_{n-4}e_2+y\,e_{n-5}e_1^3+z\,e_{n-5}e_2e_1,$$

where u, v, w and z are coefficients depending on n which are to be determined.

For convenience we record the following identities:

$$\begin{aligned}
(\nabla_2+2P_2+\nabla_1^2)e_{n-5}e_2e_1 &= 10(n-1)e_{n-6}e_1^2+10ne_{n-6}e_2+(2n(n-2)-2(n-5))e_{n-5}e_1; \\
(\nabla_2+2P_2+\nabla_1^2)e_{n-5}e_1^3 &= 30ne_{n-6}e_1^2+6n(n-1)e_{n-5}e_1; \\
(\nabla_2+2P_2+\nabla_1^2)e_{n-4}e_2 &= 8(n-1)e_{n-5}e_1-2(n-4)e_{n-4}; \\
(\nabla_2+2P_2+\nabla_1^2)e_{n-4}e_1^2 &= 16ne_{n-5}e_1+2n(n-1)e_{n-4}; \\
(\nabla_2+2P_2+\nabla_1^2)e_{n-3}e_1 &= 6ne_{n-4}.
\end{aligned}$$

we get

$$\begin{aligned}
(\nabla_2+2P_2+\nabla_1^2)g_0 &= 10n\,z\,e_{n-6}e_2+(10(n-1)\,z+30n\,y)\,e_{n-6}e_1^2 \\
&+ ((2n(n-2)-2(n-5))z+6n(n-1)y+8(n-1)w+16n\,v)\,e_{n-5}e_1 \\
&+ (-2(n-4)w+2n(n-1)v+6n\,u)\,e_{n-4}.
\end{aligned}$$

Equating coefficients we have:

$$\begin{aligned}
z &= (-1)^n \frac{-6}{n(n^2+n+2)};\\
y &= (-1)^n \frac{-2}{n^2(n^2+n+2)};\\
w &= -\frac{1}{2(n^2+n+2)(n^2-7)}\left((-12n^2(n^2+n+2))u + (12n^4+7n^3+31n^2-168n-48)\right);\\
v &= \frac{1}{2}\frac{1}{(n-1)(n^2-7)(n^2+n+2)n^2}\left((-6n^2(n^2+n+2)(n-1)^2)u\right.\\
&+ \left.(6n^6-5n^5+10n^4-138n^3+179n^2-22n+60)\right),
\end{aligned}$$

and u is arbitrary. Hence we got a family $g_{0;u}$ of solutions.

References

[BGW] **Bergeron, F., Garsia, A., Wallach, N.** *Harmonics for Deformed Steenrod Operators*, preprint.

[HT] **Hivert, F., Thiéry, N.** *Non commutative deformation of symmetric functions and the integral Steenrod algebra*, in Invariant theory in all characteristics, CRM Proc. Lecture Notes, Amer. Math. Soc., Providence, RI, vol. **35** (2004), 879-895.

[M] **Macdonald, I. G.** *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs , The Clarendon Press Oxford University Press, New York, (1995). With contributions by A. Zelevinsky, Oxford Science Publications.

[W97] **Wood, R. M. W.** *Differential operators and the Steenrod algebra*, Proc. London Math. Soc. (3), **75** (1), (1997), 194-220.

[W98] **Wood, R. M. W.** *Problems in the Steenrod algebra*, Bull. London Math. Soc., **30** (5), (1998), 449-517.

[W01] **Wood, R. M. W.** *Hit problems and the Steenrod algebra*, In Proceedings of the summer school Interactions between Algebraic Topology and Invariant Theory, University of Ioannina, Greece (2001), University of Ioannina reports, June 2001.

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