EXTENDED EQUIVARIANT PICARD COMPLEXES AND HOMOGENEOUS SPACES

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ABSTRACT. Let k be a field of characteristic 0 and let $\bar{k}$ be a fixed algebraic closure of k. Let X be a smooth geometrically integral k-variety; we set $\overline{X} = X \times_k \bar{k}$ and denote by $\mathscr{K}(\overline{X})$ the field of rational functions on $\overline{X}$. In [BvH2] we defined the *extended Picard complex of* X as the complex of $\mathrm{Gal}(\bar{k}/k)$-modules

$$\mathrm{UPic}(\overline{X}) := \left(\mathscr{K}(\overline{X})^\times/\bar{k}^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\overline{X})\right),$$

where $\mathscr{K}(\overline{X})^\times/\bar{k}^\times$ is in degree 0 and $\mathrm{Div}(\overline{X})$ is in degree 1. We computed the isomorphism class of $\mathrm{UPic}(\overline{G})$ in the derived category of Galois modules for a connected linear k-group G.

Here we compute the isomorphism class of $\mathrm{UPic}(\overline{X})$ in the derived category of Galois modules when X is a homogeneous space of a connected linear k-group G with $\mathrm{Pic}(\overline{G}) = 0$. Let $\bar{x} \in X(\bar{k})$ and let $\overline{H}$ denote the stabilizer of $\bar{x}$ in $\overline{G}$. It is well known that the character group $\mathbb{X}(\overline{H})$ of $\overline{H}$ has a natural structure of a Galois module. We prove that

$$\mathrm{UPic}(\overline{X}) \cong \left(\mathbb{X}(\overline{G}) \xrightarrow{\mathrm{res}} \mathbb{X}(\overline{H})\right)$$

in the derived category, where res is the restriction homomorphism. The proof is based on the notion of the extended equivariant Picard complex of a G-variety.

INTRODUCTION

In this paper k is always a field of characteristic 0, and $\bar{k}$ is a fixed algebraic closure of k. A k-variety X is always a geometrically integral k-variety, we set $\overline{X} = X \times_k \bar{k}$. We write $\mathscr{O}(X)$ for the ring of regular functions on X, and $\mathscr{K}(X)$ for the field of rational functions on X. By $\mathrm{Div}(X)$ we denote the group of Cartier divisors on X, and by $\mathrm{Pic}(X)$ the Picard group of X. By a k-group we mean a *linear* algebraic group over k.

The *extended Picard complex* $\mathrm{UPic}(\overline{X})$ of a smooth geometrically integral variety X over a field k was introduced in the research announcement [BvH1] and the paper [BvH2]. It is an object of the derived category of discrete Galois modules

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(i.e. a complex of Galois modules) with cohomology

$$\mathscr{H}^0(\mathrm{UPic}(\overline{X})) = U(\overline{X}) := \mathscr{O}(\overline{X})^\times/\bar{k}^\times,$$
$$\mathscr{H}^1(\mathrm{UPic}(\overline{X})) = \mathrm{Pic}(\overline{X}),$$
$$\mathscr{H}^i(\mathrm{UPic}(\overline{X})) = 0 \text{ if } i \neq 0,1.$$

This object $\mathrm{UPic}(\overline{X})$ is given by the complex of Galois modules

$$\left[\mathscr{K}(\overline{X})^\times/\bar{k}^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\overline{X})\right\rangle,$$

that is, the complex of length 2 with $\mathscr{K}(\overline{X})^\times/\bar{k}^\times$ in degree 0 and $\mathrm{Div}(\overline{X})$ in degree 1; the differential div is the divisor map, associating to the class $[f]$ of a rational function $f \in \mathscr{K}(\overline{X})^\times$ the divisor $\mathrm{div}(f) \in \mathrm{Div}(\overline{X})$. This complex plays an important role in understanding arithmetic invariants such as the so-called *algebraic part* $\mathrm{Br_a}(X)$ of the Brauer group $\mathrm{Br}(X)$ and the elementary obstruction of Colliot-Thélène and Sansuc [CS] to the existence of k-points in X.

In [BvH2, Cor. 3] it was shown that when X is a k-torsor under a connected k-group G, there is a canonical isomorphism (in the derived category of Galois modules)

$$\mathrm{UPic}(\overline{X}) \cong [\mathbb{X}(\overline{T}) \to \mathbb{X}(\overline{T}^{\mathrm{sc}})\rangle, \tag{1}$$

where $[\mathbb{X}(\overline{T}) \to \mathbb{X}(\overline{T}^{\mathrm{sc}})\rangle$ is a certain complex constructed from G using a group-theoretic construction (see Theorem 6.4 below for the precise statement). In this way, a straightforward explanation was given why this complex $[\mathbb{X}(\overline{T}) \to \mathbb{X}(\overline{T}^{\mathrm{sc}})\rangle$ played such an important role in the study of Picard groups and Brauer groups and also of the Brauer-Manin obstruction for torsors under linear algebraic groups.

The main result of the present paper is the following theorem announced in [BvH1, Theorem 3.1] concerning not necessarily principal homogeneous spaces under a connected k-group over a field k of characteristic 0.

Main Theorem 1 (Theorem 5.8)**.** *Let X be a right homogeneous space under a connected (linear) k-group G with $\mathrm{Pic}(\overline{G}) = 0$. Let $\overline{H} \subset \overline{G}$ be the stabilizer of a geometric point $\bar{x} \in X(\bar{k})$ in $\overline{G}$ (we do not assume that $\overline{H}$ is connected). Then we have a canonical isomorphism in the derived category of complexes of Galois modules*

$$\mathrm{UPic}(\overline{X}) \cong [\mathbb{X}(\overline{G}) \to \mathbb{X}(\overline{H})\rangle,$$

which is functorial in G and X.

Here $\mathbb{X}(\overline{G})$ and $\mathbb{X}(\overline{H})$ are the character groups of $\overline{G}$ and $\overline{H}$, resp. It is well-known that $\mathbb{X}(\overline{H})$ has a natural structure of a Galois module. In the present paper we actually get this Galois structure in a natural way by using V.L. Popov's identification $\mathbb{X}(\overline{H}) = \mathrm{Pic}_G(\overline{X})$, where the right hand side is the *equivariant Picard group* of $\overline{X}$. By $[\mathbb{X}(\overline{G}) \to \mathbb{X}(\overline{H})\rangle$ we denote the complex of Galois modules with $\mathbb{X}(\overline{G})$ in degree 0 and with $\mathbb{X}(\overline{H})$ in degree 1; the differential $\mathbb{X}(\overline{G}) \to \mathbb{X}(\overline{H})$ is the restriction of characters from $\overline{G}$ to $\overline{H}$.

From Main Theorem 1 we derive the main result of [BvH2], i.e the isomorphism (1) for a principal homogeneous space X of a connected k-group G. Thus we obtain a new, more conceptual, and less computational proof of this result.

We mention some applications and further results arising out of the main theorem.

Picard and Brauer groups

Theorem 2 (Theorem 7.1)**.** *Let X be a homogeneous space under a connected k-group G with $\mathrm{Pic}(\overline{G}) = 0$. Let $\overline{H}$ be the stabilizer of a geometric point $\overline{x} \in X(\overline{k})$ (we do not assume that $\overline{H}$ is connected). Then there is a canonical injection*

$$\mathrm{Pic}(X) \hookrightarrow H^1(k, [\mathbb{X}(\overline{G}) \to \mathbb{X}(\overline{H})\rangle),$$

which is an isomorphism if $X(k) \neq \emptyset$ or $\mathrm{Br}(k) = 0$.

Here H^1 denotes the first hypercohomology.

Theorem 3 (Theorem 7.2)**.** *Let X, G, and $\overline{H}$ be as in Theorem 2. Then there is a canonical injection*

$$\mathrm{Br}_{\mathrm{a}}(X) \hookrightarrow H^2(k, [\mathbb{X}(\overline{G}) \to \mathbb{X}(\overline{H})\rangle),$$

which is an isomorphism if $X(k) \neq \emptyset$ or $H^3(k, \mathbf{G}_{\mathrm{m}}) = 0$ (e.g., when k is a number field or a $\mathfrak{p}$-adic field).

This proves the conjecture [Bo3, Conj. 3.2] of the first-named author about the subquotient $\mathrm{Br}_{\mathrm{a}}(X) := \ker[\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})]/\mathrm{im}\ [\mathrm{Br}(k) \to \mathrm{Br}(X)]$ of the Brauer group $\mathrm{Br}(X)$ of a homogeneous space X.

Remarks. (1) C. Demarche [De] computed the group $\mathrm{Br}_{\mathrm{a}}(X,G)$ (introduced in [BD]) for a homogeneous space X with *connected* geometric stabilizers of a connected k-group G with $\mathrm{Pic}(\overline{G}) = 0$, when $X(k) \neq \emptyset$ or $H^3(k, \mathbf{G}_{\mathrm{m}}) = 0$. Here $\mathrm{Br}_{\mathrm{a}}(X,G) = \mathrm{Br}_1(X,G)/\mathrm{Br}(k)$, where $\mathrm{Br}_1(X,G)$ is the kernel of the composed homomorphism $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X}) \to \mathrm{Br}(\overline{G})$.

(2) In [Bo4] the first-named author uses Theorem 3 of the present paper in order to find sufficient conditions for the Hasse principle and weak approximation for a homogeneous space of a *quasi-trivial* k-group over a number field k, with connected or abelian geometric stabilizer $\overline{H}$, in terms of the Galois module $\mathbb{X}(\overline{H})$, see [Bo4, Corollaries 2.2 and 2.6]. As a consequence, he finds sufficient conditions for the Hasse principle and weak approximation for *principal* homogeneous spaces of a connected linear algebraic k-group G (where G is not assumed to be quasi-trivial or such that $\mathrm{Pic}(\overline{G}) = 0$) in terms of the Galois module $\pi_1(\overline{G})$, see [Bo4, Corollaries 2.9 and 2.10]. Here $\pi_1(\overline{G})$ is the algebraic fundamental group of G introduced in [Bo2].

Comparison with topological invariants. In [BvH2] the complex $\mathrm{UPic}(\overline{G})$ of a connected k-group G was shown to be dual (in the derived sense) to the algebraic fundamental group of G introduced in [Bo2]. In particular, if we fix an embedding $\overline{k} \hookrightarrow \mathbf{C}$ and an isomorphism $\pi_1(\mathbf{C}^\times) \xrightarrow{\sim} \mathbf{Z}$, then we have a canonical isomorphism in the derived category

$$\mathrm{UPic}(\overline{G})^D \cong \pi_1(G(\mathbf{C})),$$

where we denote by $\mathrm{UPic}(\overline{G})^D$ the dual object (in the derived sense) to $\mathrm{UPic}(\overline{G})$. Also for homogeneous spaces we have a topological interpretation of $\mathrm{UPic}(\overline{X})^D$.

Theorem 4 (Theorem 8.5)**.** *Let X be a homogeneous space under a connected (linear) k-group G with connected geometric stabilizers. Let us fix an embedding $\overline{k} \hookrightarrow \mathbf{C}$ and an isomorphism $\pi_1(\mathbf{C}^\times) \xrightarrow{\sim} \mathbf{Z}$.*

(i) *We have a canonical isomorphism of groups*

$$\mathscr{H}^0(\mathrm{UPic}(\overline{X})^D) \cong \pi_1(X(\mathbf{C})).$$

(ii) *We have a canonical isomorphism of abelian groups*

$$\mathscr{H}^{-1}(\mathrm{UPic}(\overline{X})^D) \cong \pi_2(X(\mathbf{C}))/\pi_2(X(\mathbf{C}))_{\mathrm{tors}},$$

where $\pi_2(X(\mathbf{C}))_{\mathrm{tors}}$ *is the torsion subgroup of* $\pi_2(X(\mathbf{C}))$.

Remarks. (1) We see from Theorem 4(i) that for a homogeneous space X with connected geometric stabilizers, the topological fundamental group $\pi_1(X(\mathbf{C}))$ has a canonical structure of a Galois module. This is closely related to the fact that any element of $\mathrm{Hom}(\pi_1(\mathbf{C}^\times), \pi_1(X(\mathbf{C}))$ is *algebraic*, i.e. can be represented by a regular map $\mathbf{G}_{\mathrm{m},\bar{k}} \to \overline{X}$, cf. [Bo2, Rem. 1.14].
(2) The assumption in Theorem 4(ii) that the geometric stabilizers are connected can be somewhat relaxed, see Proposition 8.6, but some condition must definitely be imposed, see Example 8.7.

The elementary obstruction. In [BvH2] it was shown that for any smooth geometrically integral k-variety X the *elementary obstruction* (to the existence of a k-point in X) as defined by Colliot-Thélène and Sansuc [CS, Déf. 2.2.1] (see also [BCS, Introduction]) may be identified with a class $e(X) \in \mathrm{Ext}^1(\mathrm{UPic}(\overline{X}), \bar{k}^\times)$ naturally arising from the construction of $\mathrm{UPic}(\overline{X})$. With X, G and $\overline{H}$ as above, we have for every integer i a canonical isomorphism

$$\mathrm{Ext}^i(\mathrm{UPic}(\overline{X}), \bar{k}^\times) \cong H^i\left(k, \langle H^{\mathrm{m}}(\bar{k}) \to G^{\mathrm{tor}}(\bar{k})]\right)$$

(Corollary 9.3). Here G^{tor} and H^{m} are the k-groups of multiplicative type with character groups $\mathbb{X}(\overline{G})$ and $\mathbb{X}(\overline{H})$, resp., and $\langle H^{\mathrm{m}}(\bar{k}) \to G^{\mathrm{tor}}(\bar{k})]$ is the complex of Galois modules with $H^{\mathrm{m}}(\bar{k})$ in degree -1 and $G^{\mathrm{tor}}(\bar{k})$ in degree 0, dual to the complex $[\mathbb{X}(\overline{G}) \to \mathbb{X}(\overline{H})\rangle$.

The complex $\langle H^{\mathrm{m}}(\bar{k}) \to G^{\mathrm{tor}}(\bar{k})]$ already appeared in earlier work by the first-named author. In [Bo3] the first-named author defined (by means of explicit cocycles) an obstruction class $\eta(G,X) \in H^1\left(k, \langle H^{\mathrm{m}}(\bar{k}) \to G^{\mathrm{tor}}(\bar{k})]\right)$ to the existence of a rational point on X, see also 9.5 below. The results of the present paper enable us to show that this obstruction class $\eta(G,X)$ coincides, up to sign, with the elementary obstruction.

Theorem 5 (Theorem 9.6)**.** *Let X be a homogeneous space under a connected (linear) k-group G with* $\mathrm{Pic}(\overline{G}) = 0$. *Let $\overline{H}$ be the stabilizer of a geometric point $\bar{x} \in X(\bar{k})$ (we do not assume that $\overline{H}$ is connected). Then* $e(X) \in \mathrm{Ext}^1(\mathrm{UPic}(\overline{X}), \bar{k}^\times)$ *coincides with* $-\eta(G,X) \in H^1(k, \langle H^{\mathrm{m}} \to G^{\mathrm{tor}}])$ *under the canonical identification*

$$\mathrm{Ext}^1(\mathrm{UPic}(\overline{X}), \bar{k}^\times) \cong H^1(k, \langle H^{\mathrm{m}} \to G^{\mathrm{tor}}]).$$

The results of the present paper were partially announced in the research announcement [BvH1].

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1. PRELIMINARIES

1.1. Let X be a geometrically integral algebraic variety over a field k of characteristic 0. We denote by $\mathscr{O}(X)$ the ring of regular functions on X, and by $\mathscr{K}(X)$ the field of rational functions on X. We denote by $\mathscr{O}(X)^\times$ and $\mathscr{K}(X)^\times$ the corresponding multiplicative groups. We denote by $\mathrm{Div}(X)$ the group of Cartier divisors on X, and by $\mathrm{Pic}(X)$ the Picard group of X (i.e. the group of isomorphism classes of invertible sheaves on X). We have an exact sequence

$$0 \to \mathscr{O}(X)^\times/k^\times \to \mathscr{K}(X)^\times/k^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0.$$

We set

$$U(X) = \mathscr{O}(X)^\times/k^\times.$$

We set $\overline{X} = X \times_k \bar{k}$. In [BvH2] we defined an object $\mathrm{UPic}(\overline{X})$ in the derived category of discrete $\mathrm{Gal}(\bar{k}/k)$-modules with the property that

$$\mathscr{H}^0(\mathrm{UPic}(\overline{X})) = U(\overline{X}),\ \mathscr{H}^1(\mathrm{UPic}(\overline{X})) = \mathrm{Pic}(\overline{X}),\ \mathscr{H}^i(\mathrm{UPic}(\overline{X})) = 0 \text{ if } i \neq 0,1.$$

This object is given by the complex

$$\mathscr{K}(\overline{X})^\times/\bar{k}^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\overline{X}) \tag{2}$$

with $\mathscr{K}(\overline{X})^\times/\bar{k}^\times$ in degree 0 and $\mathrm{Div}(\overline{X})$ in degree 1. (cf. [BvH2, Cor. 2.5, Rem. 2.6]).

1.2. Let X be a geometrically integral k-variety, G a connected linear algebraic k-group, and $w\colon X \times_k G \to X$ a right action of G on X: $(x,g) \mapsto xg$, where $x \in X$, $g \in G$. For $g \in G(\bar{k})$ we write $w_g\colon \overline{X} \to \overline{X}$ for the map $x \mapsto xg$. We denote by $\mathbb{X}(G)$ the character group of G, i.e. $\mathbb{X}(G) = \mathrm{Hom}_k(G, \mathbf{G}_\mathrm{m})$.

We denote by $\mathrm{Pic}_G(X)$ the group of isomorphism classes of G-equivariant invertible sheaves $(\mathscr{L}, \beta)$ on X, where $\mathscr{L}$ is an invertible sheaf on X and β is a *G-linearization* of $\mathscr{L}$, see [MFK, Ch. 1, §3, Def. 1.6] or Definition 3.2 below. We say that two G-linearizations β and β' of an invertible sheaf $\mathscr{L}$ are *equivalent* if $(\mathscr{L}, \beta)$ and $(\mathscr{L}, \beta')$ are G-equivariantly isomorphic. The group structure on $\mathrm{Pic}_G(X)$ is given by the tensor product. We have a canonical homomorphism of abelian groups

$$\mathrm{Pic}_G(X) \to \mathrm{Pic}(X)$$

taking the class $[\mathscr{L}, \beta]$ of $(\mathscr{L}, \beta)$ to the class $[\mathscr{L}]$ of $\mathscr{L}$.

Note that the set of G-linearizations of an invertible sheaf may be empty. In other words, the homomorphism $\mathrm{Pic}_G(X) \to \mathrm{Pic}(X)$ need not be surjective. The set of equivalence classes of G-linearizations of an invertible sheaf is either empty or a coset in $\mathrm{Pic}_G(X)$ of the subgroup of equivalence classes of G-linearizations of the trivial invertible sheaf $\mathscr{O}_X$ (i.e. a coset of the kernel of the map $\mathrm{Pic}_G(X) \to \mathrm{Pic}(X)$).

We define

$$\mathscr{O}_G(X) = \mathscr{O}(X)^G,$$
$$U_G(X) = (\mathscr{O}(X)^G)^\times/k^\times,$$

where $\mathscr{O}(X)^G$ denotes the ring of G-invariant regular functions on X.

1.3. *Cones and fibres.* Let $f\colon P \to Q$ be a morphism of complexes of objects of an abelian category $\mathscr{A}$. We denote by $\langle P \to Q]$ the *cone* of f, and by $[P \to Q\rangle$ the *fibre* (or co-cone) of f, see [BvH2, § 1.1]. Note that $[P \to Q\rangle = \langle P \to Q][-1]$. Note also that if $f\colon P \to Q$ is a morphism of *objects* of our abelian category, then $[P \to Q\rangle$ is the complex $P \xrightarrow{f} Q$ with P in degree 0, and $\langle P \to Q]$ is the complex $P \xrightarrow{-f} Q$ with Q in degree 0.

2. THE EXTENDED EQUIVARIANT PICARD COMPLEX

2.1. Let X and G be as in §1.2. For $n \geq 0$ we write $C^n_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times)$ for the Galois module $\mathscr{O}(\overline{X} \times \overline{G}^n)^\times$ of invertible regular functions on $\overline{X} \times \overline{G}^n$. For $f \in C^n_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times)$ we write $f_{g_1,\cdots,g_n} = f|_{X\times(g_1,\dots,g_n)} \in \mathscr{O}(\overline{X})^\times$ for the restriction to the fibre over $(g_1,\dots,g_n) \in G^n(\overline{k})$. As the notation suggests, we regard $C^n_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times)$ as the group of *algebraic n-cochains of G with coefficients in* $\mathscr{O}(\overline{X})^\times$. The assignment

$$f \mapsto (\ (g_1,\dots,g_n) \mapsto f_{g_1,\cdots,g_n}\ )$$

defines a canonical injection of $C^n_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times)$ into the group $C^n(G(\overline{k}), \mathscr{O}(\overline{X})^\times)$ of ordinary cochains of the abstract group $G(\overline{k})$ with coefficients in $\mathscr{O}(\overline{X})^\times$. The usual construction gives rise to a cochain complex

$$C^0_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times) \xrightarrow{d^0} C^1_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times) \xrightarrow{d^1} C^2_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times) \xrightarrow{d^2} \cdots$$

(which we may identify with a subcomplex of the standard cochain complex $C^n(G(\overline{k}), \mathscr{O}(\overline{X})^\times)$). We denote the corresponding cocycles by $Z^*_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times)$, the coboundaries by $B^*_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times)$ and the cohomology by $H^*_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times)$. In fact, we shall mainly be interested in the first two degrees of the complex. Here $d^0\colon \mathscr{O}(\overline{X})^\times \to C^1_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times)$ is given by

$$f \mapsto (f \circ w)/(f \circ p_X), \text{ i.e. } f \mapsto (\ (x,g) \mapsto f(xg)/f(x)\ ),$$

where $w\colon X \times_k G \to X$ is the action of G on X, and $p_X\colon X \times_k G \to X$ is the first projection. Note that $Z^1_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times) \subset C^1_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times)$ consists of those invertible regular functions c on $\overline{X} \times G$ for which

$$c_{g_1g_2}(x) = c_{g_1}(x) \cdot c_{g_2}(xg_1). \tag{3}$$

We also define the complexes $C^*_{\mathrm{alg}}(\overline{G}, \mathscr{K}(\overline{X})^\times)$ and $C^*_{\mathrm{alg}}(\overline{G}, \mathrm{Div}(\overline{X}))$ by taking $C^n_{\mathrm{alg}}(\overline{G}, \mathscr{K}(\overline{X})^\times)$ (resp. $C^*_{\mathrm{alg}}(\overline{G}, \mathrm{Div}(\overline{X}))$ to be the group of invertible rational functions (resp. divisors) on $X \times G^n$. The differentials of the complexes are defined as above.

2.2. We write $\mathrm{KDiv}(\overline{X})$ for the 2-term complex $\left[\mathscr{K}(\overline{X})^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\overline{X})\right\rangle$ (as in [BvH2]). Recall that $\mathscr{K}(\overline{X})$ is in degree 0 and $\mathrm{Div}(\overline{X})$ is in degree 1. We define

$C^*_{\mathrm{alg}}(\overline{G},\mathrm{KDiv}(\overline{X}))$ to be the total complex associated to the double complex

$$\begin{array}{ccc} \cdots & & \cdots \\ \uparrow{\scriptstyle d^1_{\mathscr{K}}} & & \uparrow{\scriptstyle d^1_{\mathrm{Div}}} \\ C^1_{\mathrm{alg}}(\overline{G},\mathscr{K}(\overline{X})^\times) & \xrightarrow{\mathrm{div}^1} & C^1_{\mathrm{alg}}(\overline{G},\mathrm{Div}(\overline{X})) \\ \uparrow{\scriptstyle d^0_{\mathscr{K}}} & & \uparrow{\scriptstyle d^0_{\mathrm{Div}}} \\ C^0_{\mathrm{alg}}(\overline{G},\mathscr{K}(\overline{X})^\times) & \xrightarrow{\mathrm{div}^0} & C^0_{\mathrm{alg}}(\overline{G},\mathrm{Div}(\overline{X})). \end{array}$$

In other words, $C^*_{\mathrm{alg}}(\overline{G},\mathrm{KDiv}(\overline{X}))$ is the total complex associated to the double complex

$$\begin{array}{ccc} \cdots & & \cdots \\ \uparrow{\scriptstyle d^1_{\mathscr{K}}} & & \uparrow{\scriptstyle d^1_{\mathrm{Div}}} \\ \mathscr{K}(\overline{X}\times\overline{G})^\times & \xrightarrow{\mathrm{div}^1} & \mathrm{Div}(\overline{X}\times\overline{G}) \\ \uparrow{\scriptstyle d^0_{\mathscr{K}}} & & \uparrow{\scriptstyle d^0_{\mathrm{Div}}} \\ \mathscr{K}(\overline{X})^\times & \xrightarrow{\mathrm{div}^0} & \mathrm{Div}(\overline{X}). \end{array}$$

Here $\mathscr{K}(\overline{X})^\times$ is in bidegree $(0,0)$. We write $H^i_{\mathrm{alg}}(\overline{G},\mathrm{KDiv}(\overline{X}))$ for $\mathscr{H}^i(C^*_{\mathrm{alg}}(\overline{G},\mathrm{KDiv}(\overline{X})))$.

Definition 2.3. Let X and G be as in §1.2. We define the *extended equivariant Picard complex of G and X* to be the complex

$$\mathrm{UPic}_G(\overline{X}) = \tau_{\le 1}C^*_{\mathrm{alg}}(\overline{G},\mathrm{KDiv}(\overline{X}))/\bar{k}^\times.$$

In other words, $\mathrm{UPic}_G(\overline{X})$ is the complex

$$\mathscr{K}(\overline{X})^\times/\bar{k}^\times \xrightarrow{\begin{pmatrix} d^0_{\mathscr{K}} \\ \mathrm{div}^0 \end{pmatrix}} \left\{(z,D)\in Z^1_{\mathrm{alg}}(\overline{G},\mathscr{K}(\overline{X})^\times)\oplus\mathrm{Div}(\overline{X})\colon\ \mathrm{div}^1(z)=d^0_{\mathrm{Div}}(D)\right\},$$

where $d^0_{\mathscr{K}}([f])=(f\circ w)/(f\circ p_X)$ and $d^0_{\mathrm{Div}}(D)=w^*D-p_X^*D$. Clearly we have

$$\mathscr{H}^0(\mathrm{UPic}_G(\overline{X}))=H^0_{\mathrm{alg}}(\overline{G},\mathrm{KDiv}(\overline{X}))/\bar{k}^\times,\quad \mathscr{H}^1(\mathrm{UPic}_G(\overline{X}))=H^1_{\mathrm{alg}}(\overline{G},\mathrm{KDiv}(\overline{X})).$$

We have

$$H^0_{\mathrm{alg}}(\overline{G},\mathrm{KDiv}(\overline{X}))=\{f\in\mathscr{K}(\overline{X})^\times\colon\ \mathrm{div}(f)=0 \text{ and } f\circ w=f\circ p_X\}=(\mathscr{O}(\overline{X})^\times)^G,$$

whence

$$\mathscr{H}^0(\mathrm{UPic}_G(\overline{X}))=(\mathscr{O}(\overline{X})^\times)^G/\bar{k}^\times=U_G(\overline{X}).$$

It turns out that

$$\mathscr{H}^1(\mathrm{UPic}_G(\overline{X}))=\mathrm{Pic}_G(\overline{X}),$$

see Corollary 4.1 below.

Note that by slight abuse of notation we write $\mathrm{UPic}_G(\overline{X})$, $U_G(\overline{X})$, $\mathrm{Pic}_G(\overline{X})$ rather than $\mathrm{UPic}_{\overline{G}}(\overline{X})$, $U_{\overline{G}}(\overline{X})$, $\mathrm{Pic}_{\overline{G}}(\overline{X})$.

2.4. The complex $\mathrm{UPic}_G(\overline{X})$ can be regarded as an equivariant version of the extended Picard complex $\mathrm{UPic}(\overline{X})=\left[\mathscr{K}(\overline{X})^\times/\bar{k}^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\overline{X})\right\rangle$ (indeed, if $G=1$, then $Z^1_{\mathrm{alg}}(\overline{G},\mathscr{K}(\overline{X})^\times)=1$ and $\mathrm{UPic}_G(\overline{X})=\mathrm{UPic}(\overline{X})$). We have an obvious natural morphism of complexes of Galois modules

$$\nu\colon\ \mathrm{UPic}_G(\overline{X})\to\mathrm{UPic}(\overline{X}),$$

where $\nu^0=\mathrm{id}$ and

$$\nu^1(z,D)=D \text{ for } (z,D)\in\mathrm{UPic}_G(\overline{X})^1\subset Z^1_{\mathrm{alg}}(\overline{G},\mathscr{K}(\overline{X})^\times)\oplus\mathrm{Div}(\overline{X}).$$

2.5. *Functoriality.*

Let X and G be as in §1.2. It is clear that a homomorphism $G'\to G$ of linear algebraic groups over k induces a pull-back homomorphism

$$C^*_{\mathrm{alg}}(\overline{G},-)\to C^*_{\mathrm{alg}}(\overline{G}',-)$$

for any of the coefficients considered, and also a pull-back homomorphism

$$\mathrm{UPic}_G(\overline{X})\to\mathrm{UPic}_{G'}(\overline{X}).$$

Functoriality in X is a bit more subtle. For a *dominant* G-equivariant morphism

$$f\colon X'\to X,$$

from another G-variety X' as in Section 1.2 to X, we clearly have a pull-back morphism of complexes $\mathrm{UPic}(\overline{X})\to\mathrm{UPic}(\overline{X}')$ and a pull-back morphism of complexes $\mathrm{UPic}_G(\overline{X})\to\mathrm{UPic}_G(\overline{X}')$.

However, for a G-equivariant morphism f as above that is not dominant, we need to modify our complexes. We assume that both X and X' are smooth. We choose an arbitrary G-invariant scheme-theoretic point $x'\in X'$, which need not be closed. In more geometric terms this amounts to taking the generic point of an irreducible, but not necessarily geometrically irreducible G-orbit on X'. Taking $x=f(x')$, we then consider the subcomplex $\mathscr{O}_x\mathrm{Div}(\overline{X})\subset\mathrm{KDiv}(\overline{X})$ given by

$$(\mathscr{O}_{X,x}\underset{k}{\otimes}\bar{k})^\times\xrightarrow{\mathrm{div}}\mathrm{Div}(\overline{X})_x\,,$$

where $\mathscr{O}_{X,x}\subset\mathscr{K}(X)$ is the local ring at x and $\mathrm{Div}(\overline{X})_x$ consists of the divisors whose support does not contain x. Using a moving lemma for divisors on a smooth variety (cf. [Sh], Vol. 1, III.1.3, Thm. 1 and the remark after the proof), we see that the inclusion $\mathscr{O}_x\mathrm{Div}(\overline{X})\hookrightarrow\mathrm{KDiv}(\overline{X})$ is a quasi-isomorphism, and so is the induced inclusion $\mathscr{O}_x\mathrm{Div}(\overline{X})/\bar{k}^\times\hookrightarrow\mathrm{UPic}(\overline{X})$. We shall denote $\mathscr{O}_x\mathrm{Div}(\overline{X})/\bar{k}^\times$ by $\mathrm{UPic}(\overline{X})_x$. Similarly, we define

$$\mathrm{UPic}_G(\overline{X})_x=\tau_{\le1}C^*_{\mathrm{alg}}(\overline{G},\mathscr{O}_x\mathrm{Div}(\overline{X}))/\bar{k}^\times$$

and we see that the canonical inclusion $\mathrm{UPic}_G(\overline{X})_x\hookrightarrow\mathrm{UPic}_G(\overline{X})$ is a quasi-isomorphism.

By construction, a G-equivariant morphism f as above now induces pull-back homomorphisms of complexes

$$f_x^*\colon\ \mathrm{UPic}(\overline{X})_x\to\mathrm{UPic}(\overline{X}')_{x'}$$

and

$$f_x^*\colon\ \mathrm{UPic}_G(\overline{X})_x\to\mathrm{UPic}_G(\overline{X}')_{x'},$$

hence morphisms in the derived category

$$f^*\colon\ \mathrm{UPic}(\overline{X})\to\mathrm{UPic}(\overline{X}')$$

and

$$f^*\colon\ \mathrm{UPic}_G(\overline{X}) \to \mathrm{UPic}_G(\overline{X}'). \tag{4}$$

For $\mathrm{UPic}(-)$ this morphism coincides with the morphism we obtain from the derived functor construction of UPic in [BvH2, 2.1], so it is independent of the choice of $x' \in X'$. In the case of UPic_G we shall have to verify this directly by an auxiliary construction.

For this, we generalize the above construction from a point $x \in X$ to a finite set $S = \{x_1,\dots,x_n\}$ of G-equivariant points in X by replacing $\mathscr{O}_{X,x}\otimes\overline{k}$ by

$$\mathscr{O}_{X,S}\otimes\overline{k} = \bigcap_{x\in S}\mathscr{O}_{X,x}\otimes\overline{k} \subset \mathscr{K}(\overline{X})$$

and replacing $\mathrm{Div}(\overline{X})_x$ by

$$\mathrm{Div}(\overline{X})_S = \cap_{x\in S}\mathrm{Div}(\overline{X})_x \subset \mathrm{Div}(\overline{X}),$$

etc. We then see from the diagram below, in which every injection of complexes is a quasi-isomorphism, that the morphism (4) in the derived category does not depend on the choice of $x' \in X'$:

$$\begin{array}{ccccc}
 & & \mathrm{UPic}_G(\overline{X})_{x_1} & \xrightarrow{f^*_{x_1}} & \mathrm{UPic}_G(\overline{X}')_{x'_1} \\
 & \nearrow & & \nearrow & \\
\mathrm{UPic}_G(\overline{X})_{\{x_1,x_2\}} & \xrightarrow{f^*_S} & \mathrm{UPic}_G(\overline{X}')_{\{x'_1,x'_2\}} & & \\
 & \searrow & & \searrow & \\
 & & \mathrm{UPic}_G(\overline{X})_{x_2} & \xrightarrow{f^*_{x_2}} & \mathrm{UPic}_G(\overline{X}')_{x'_2}.
\end{array}$$

We shall call any modification where we replace $\mathscr{K}(\overline{X})^\times$ by $(\mathscr{O}_{X,x}\otimes_k\overline{k})^\times$ and $\mathrm{Div}(\overline{X})$ by $\mathrm{Div}(\overline{X})_x$, etc. a *local modification*.

3. G-LINEARIZATIONS OF INVERTIBLE SHEAVES

3.1. Let G be an algebraic group (not necessarily linear) over an algebraically closed field k of characteristic 0. Let X be a k-variety with a right action w of G. This means that we are given a morphism of varieties

$$w\colon X\times G\to X,\quad (x,g)\mapsto w_g(x) = xg,$$

satisfying the usual conditions.

Definition 3.2 ([MFK, Ch. 1, §3, Def. 1.6])**.** Let $\mathscr{L}$ be an invertible sheaf on a G-variety X. A G-linearization of $\mathscr{L}$ is an isomorphism

$$\beta\colon w^*\mathscr{L}\to p_X^*\mathscr{L}$$

of invertible sheaves on $X\times G$ satisfying the following cocycle condition:

Let $m\colon G\times G\to G$ be the group law. Consider the projection $p_X\colon X\times G\to X$. We have morphisms w and p_X from $X\times G$ to X. Consider the projection $p_{X,G1}\colon X\times G\times G\to X\times G$ taking (x,g_1,g_2) to (x,g_1). We have morphisms $p_{X,G1},\ 1_X\times$

m, and $w \times 1_G$ from $X \times G \times G$ to $X \times G$. The cocycle condition is the commutativity of the following diagram:

$$\begin{array}{ccccc}
[w \circ (w \times 1_G)]^* \mathscr{L} & \xrightarrow{(w\times 1_G)^*\beta} & [p_X \circ (w \times 1_G)]^* \mathscr{L} & & \\
\| & & \| & & \\
 & & [w \circ p_{X,G1}]^* \mathscr{L} & \xrightarrow{(p_{X,G1})^*\beta} & [p_X \circ p_{X,G_1}]^* \mathscr{L} \\
\| & & & & \| \\
[w \circ (1_X \times m)]^* \mathscr{L} & & \xrightarrow{(1_X\times m)^*\beta} & & [p_X \circ (1_X \times m)]^* \mathscr{L}.
\end{array}$$

This is the same as to say that for each $g_1, g_2 \in G(k)$ we have the cocycle condition

$$\beta_{g_1 g_2} = \beta_{g_1} \circ w_{g_1}^*(\beta_{g_2}), \tag{5}$$

where for $g \in G(k)$ we write β_g for the inverse image of β under the map $X \to X \times G$, $x \mapsto (x, g)$.

Definition 3.3. Let G be a k-group and X be a G-variety over k. By an invertible G-sheaf we mean a pair $(\mathscr{L}, \beta)$, where $\mathscr{L}$ is an invertible sheaf and β is a G-linearization of $\mathscr{L}$. We denote by $\mathrm{Pic}_G(X)$ the group of isomorphism classes of invertible G-sheaves on a G-variety X. We denote by $[\mathscr{L}, \beta]$ the class of the pair $(\mathscr{L}, \beta)$ in $\mathrm{Pic}_G(X)$.

We wish to compute this group $\mathrm{Pic}_G(X)$ in terms of divisors and rational functions (see Theorem 3.13 below).

3.4. From now on we assume that G is a *connected linear* k-group and X is an *integral* G-variety. We denote by $\mathscr{O}_X$ the structure sheaf on X, and by $\mathscr{K}_X$ the sheaf of total quotient rings of $\mathscr{O}_X$. Then $\mathscr{O}(X) = \Gamma(X, \mathscr{O}_X)$ and $\mathscr{K}(X) = \Gamma(X, \mathscr{K}_X)$.

By an invertible $\mathscr{K}_X$-sheaf $\mathscr{R}$ on X we mean a locally free sheaf of modules of rank one over the sheaf of rings $\mathscr{K}_X$. Note that if an invertible $\mathscr{K}_X$-sheaf $\mathscr{R}$ has a non-zero global section s, then $\mathscr{R}$ is isomorphic to $\mathscr{K}_X$ as a $\mathscr{K}_X$-module.

Let $\mathscr{L}$ be an invertible sheaf on X. We set $\mathscr{L}^{\mathscr{K}} = \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{K}_X$. Note that $\mathscr{L}^{\mathscr{K}}$ has a non-zero global section, because $\mathscr{L}$ has a non-zero rational section. If $\psi\colon \mathscr{L}_1 \to \mathscr{L}_2$ is a morphism of invertible sheaves on X, then we have an induced morphism

$$\psi^{\mathscr{K}} : \mathscr{L}_1^{\mathscr{K}} \to \mathscr{L}_2^{\mathscr{K}}.$$

Let $f\colon X \to Y$ be a *dominant* morphism of integral k-varieties. Then we have a morphism of ringed spaces

$$(X, \mathscr{K}_X) \to (Y, \mathscr{K}_Y).$$

If $\mathscr{R}_Y$ is a sheaf of $\mathscr{K}_Y$-modules on Y, then we define

$$f^* \mathscr{R}_Y = f^{-1} \mathscr{R}_Y \otimes_{f^{-1}\mathscr{K}_Y} \mathscr{K}_X,$$

cf. [Ha, Ch. II, Section 5, p. 110]. If $\mathscr{L}_Y$ is a sheaf of $\mathscr{O}_Y$-modules on Y, then

$$f^*(\mathscr{L}_Y \otimes_{\mathscr{O}_Y} \mathscr{K}_Y) = (f^* \mathscr{L}_Y) \otimes_{\mathscr{O}_X} \mathscr{K}_X.$$

Definition 3.5. A G-linearization of an invertible $\mathscr{K}_X$-sheaf $\mathscr{R}$ on a G-variety X is an isomorphism of invertible $\mathscr{K}_{X\times G}$-sheaves on $X \times G$

$$\gamma\colon w^* \mathscr{R} \to p_X^* \mathscr{R}$$

such that the following diagram commutes:

$$\begin{array}{ccccc}
[w\circ(w\times 1_G)]^*\mathscr{R} & \xrightarrow{(w\times 1_G)^*\gamma} & [p_X\circ(w\times 1_G)]^*\mathscr{R} & & \\
\| & & \| & & \\
\| & & [w\circ p_{X,G1}]^*\mathscr{R} & \xrightarrow{(p_{X,G1})^*\gamma} & [p_X\circ p_{X,G_1}]^*\mathscr{R} \\
\| & & & & \| \\
[w\circ(1_X\times m)]^*\mathscr{R} & & \xrightarrow{\quad (1_X\times m)^*\gamma \quad} & & [p_X\circ(1_X\times m)]^*\mathscr{R}.
\end{array}$$

Note that the diagram of Definition 3.5 is just the diagram of Definition 3.2 with $\mathscr{R}$ instead of $\mathscr{L}$ and with γ instead of β.

Lemma 3.6. *Let $\mathscr{L}$ be an invertible sheaf on X, and let $\beta\colon w^*\mathscr{L}\to p_X^*\mathscr{L}$ be an isomorphism. Then β is a G-linearization of $\mathscr{L}$ if and only if*

$$\beta^{\mathscr{K}}: w^*\mathscr{L}^{\mathscr{K}}\to p_X^*\mathscr{L}^{\mathscr{K}}$$

is a G-linearization of $\mathscr{L}^{\mathscr{K}}$.

Proof. It is clear that if β is a G-linearization of an invertible sheaf $\mathscr{L}$ on X, then

$$\beta^{\mathscr{K}}: w^*\mathscr{L}^{\mathscr{K}}\to p_X^*\mathscr{L}^{\mathscr{K}}$$

is a G-linearization of $\mathscr{L}^{\mathscr{K}}$. Conversely, assume that $\gamma:=\beta^{\mathscr{K}}$ is a G-linearization of $\mathscr{R}:=\mathscr{L}^{\mathscr{K}}$. We compare the isomorphisms

$$(1_X\times m)^*\beta,\ (p_{X,G1})^*\beta\circ(w\times 1_G)^*\beta\colon\ [w\circ(1_X\times m)]^*\mathscr{L}\to[p_X\circ(1_X\times m)]^*\mathscr{L}$$

from the diagram of Definition 3.2. We may write

$$(1_X\times m)^*\beta=\psi\cdot(p_{X,G1})^*\beta\circ(w\times 1_G)^*\beta$$

for some $\psi\in\mathscr{O}(X\times G\times G)^\times$. We substitute $\mathscr{R}=\mathscr{L}^{\mathscr{K}}$ and $\gamma=\beta^{\mathscr{K}}$ in the diagram of Definition 3.5, and we obtain that

$$(1_X\times m)^*\gamma=\psi\cdot(p_{X,G1})^*\gamma\circ(w\times 1_G)^*\gamma.$$

But by assumption γ is a G-linearization of $\mathscr{L}^{\mathscr{K}}$, hence γ makes commutative the diagram of Definition 3.5, i.e.

$$(1_X\times m)^*\gamma=(p_{X,G1})^*\gamma\circ(w\times 1_G)^*\gamma.$$

We see that $\psi=1$, therefore

$$(1_X\times m)^*\beta=(p_{X,G1})^*\beta\circ(w\times 1_G)^*\beta,$$

hence β makes commutative the diagram of Definition 3.2. Thus β is a G-linearization of $\mathscr{L}$. □

Definition 3.7. Let X be a G-variety. As in 2.1, we define $Z^1_{\mathrm{alg}}(G,\mathscr{K}(X)^\times)$ to be the group of nonzero rational functions $z\in\mathscr{K}(X\times G)^\times$ satisfying the cocycle condition

$$(1_X\times m)^*z=(p_{X,G1})^*z\cdot(w\times 1_G)^*z. \tag{6}$$

Of course, this is the same as to say that

$$z_{g_1g_2}(x)=z_{g_1}(x)\cdot z_{g_2}(xg_1)$$

for all triples $(x,g_1,g_2)\in X(k)\times G(k)\times G(k)$ for which all the three values $z_{g_1g_2}(x),\ z_{g_1}(x)$, and $z_{g_2}(xg_1)$ are different from 0 and ∞.

Lemma 3.8. *Let $\mathscr{R}$ be an invertible $\mathscr{K}_X$-sheaf on a G-variety X having a nonzero section s. Then there exists a canonical bijection*

$$\begin{array}{ccc}\{G\text{-linearizations of } \mathscr{R}\} & \to & Z^1_{\mathrm{alg}}(G, \mathscr{K}(X)^\times) \\ \gamma & \mapsto & z\end{array}$$

such that $\gamma(w^*s) = z \cdot p_X^* s$.

Proof. Let $\alpha\colon \mathscr{K}_X \to \mathscr{R}$ be the isomorphism of sheaves of $\mathscr{K}_X$-modules such that $\alpha(1) = s$, where $1 \in \Gamma(X, \mathscr{K}_X) = \mathscr{K}(X)$ is the unit element. The isomorphism

$$\gamma\colon w^*\mathscr{R} \to p_X^*\mathscr{R}$$

gives, via α, an automorphism

$$\gamma'\colon \mathscr{K}_{X\times G} \to \mathscr{K}_{X\times G}$$

and this automorphism γ' is given by multiplication by a rational function $z \in \mathscr{K}(X \times G)^\times$. The cocycle condition of commutativity of the diagram of Definition 3.5 writes then as

$$(1_X \times m)^* z = (p_{X,G1})^* z \cdot (w \times 1_G)^* z.$$

We see that $z \in Z^1_{\mathrm{alg}}(G, \mathscr{K}(X)^\times)$. Note that $\gamma'(1) = z$, and we can write it as $\gamma'(w^*1) = z \cdot p_X^* 1$. Returning to our original sheaf $\mathscr{R}$ and section s, we obtain that

$$\gamma(w^*s) = z \cdot p_X^* s.$$

It is easy to see that our map $\gamma \mapsto z$ is bijective. □

Lemma 3.9 (cf. [KKV], 2.1)**.** *Consider the trivial invertible sheaf $\mathscr{L} = \mathscr{O}_X$ on a G-variety X. There exists a canonical isomorphism of abelian groups*

$$\begin{array}{ccc}\{G\text{-linearizations of } \mathscr{O}_X\} & \to & Z^1_{\mathrm{alg}}(G, \mathscr{O}(X)^\times) \\ \beta & \mapsto & c\end{array}$$

such that $\beta(1) = c$.

Proof. Similar to that of Lemma 3.8. □

3.10. Let $\mathscr{L}_1, \mathscr{L}_2$ be two invertible sheaves on a variety X, and let s_i be a nonzero rational section of $\mathscr{L}_i$ ($i = 1, 2$). Let $\gamma\colon \mathscr{L}_1^{\mathscr{K}} \to \mathscr{L}_2^{\mathscr{K}}$ be an isomorphism. Then there exists a unique rational function $f_\gamma \in \mathscr{K}(X)^\times$ such that

$$\gamma(s_1) = f_\gamma \cdot s_2.$$

Conversely, it is clear that for any $f \in \mathscr{K}(X)^\times$ there exists an isomorphism $\gamma\colon \mathscr{L}_1^{\mathscr{K}} \to \mathscr{L}_2^{\mathscr{K}}$ such that $f_\gamma = f$.

Lemma 3.11. *An isomorphism $\gamma\colon \mathscr{L}_1^{\mathscr{K}} \to \mathscr{L}_2^{\mathscr{K}}$ as above comes from some isomorphism of invertible sheaves $\beta\colon \mathscr{L}_1 \to \mathscr{L}_2$ if and only if*

$$\mathrm{div}(f_\gamma) = \mathrm{div}(s_1) - \mathrm{div}(s_2).$$

Proof. Assume that $\gamma = \beta^{\mathscr{K}}$ for some isomorphism (of sheaves of $\mathscr{O}_X$-modules) $\beta\colon \mathscr{L}_1 \to \mathscr{L}_2$. Then

$$\beta(s_1) = f_\gamma \cdot s_2.$$

Since β is an isomorphism of sheaves of $\mathscr{O}_X$-modules, we have

$$\mathrm{div}(\beta(s_1)) = \mathrm{div}(s_1).$$

Thus

$$\mathrm{div}(f_\gamma) = \mathrm{div}(\beta(s_1)) - \mathrm{div}(s_2) = \mathrm{div}(s_1) - \mathrm{div}(s_2).$$

Conversely, assume that

$$\mathrm{div}(f_\gamma) = \mathrm{div}(s_1) - \mathrm{div}(s_2).$$

Then $\mathrm{div}(s_1) - \mathrm{div}(s_2)$ is a principal divisor, hence the classes of $\mathscr{L}_1$ and $\mathscr{L}_2$ in $\mathrm{Pic}(X)$ are equal. It follows that there exists an isomorphism $\beta'\colon \mathscr{L}_1 \to \mathscr{L}_2$. We obtain as above a rational function $f_{\beta'}$ such that

$$\beta'(s_1) = f_{\beta'} \cdot s_2.$$

As above, we have

$$\mathrm{div}(f_{\beta'}) = \mathrm{div}(s_1) - \mathrm{div}(s_2).$$

We see that $\mathrm{div}(f_{\beta'}) = \mathrm{div}(f_\gamma)$, hence $f_\gamma = \varphi f_{\beta'}$ for some $\varphi \in \mathscr{O}(X)^\times$. Set $\beta = \varphi\beta'\colon \mathscr{L}_1 \to \mathscr{L}_2$, then $f_\beta = \varphi f_{\beta'} = f_\gamma$, hence $\gamma = \beta^{\mathscr{K}}$. Thus γ comes from the isomorphism β of sheaves of $\mathscr{O}_X$-modules. □

Definition 3.12. Let X be a G-variety, where G is connected and X is integral. We define

$$Z^1_{\mathrm{alg}}(G,\mathrm{KDiv}(X)) = \{(z,D) \mid z \in Z^1_{\mathrm{alg}}(G,\mathscr{K}(X)),\ D \in \mathrm{Div}(X),\ \mathrm{div}(z) = w^*D - p_X^*D\}.$$

We define a homomorphism

$$d\colon \mathscr{K}(X)^\times \to Z^1_{\mathrm{alg}}(G,\mathrm{KDiv}(X)),\ f \mapsto (w^*(f)/p_X^*(f), \mathrm{div}(f)).$$

We set $B^1_{\mathrm{alg}}(G,\mathrm{KDiv}(X)) = \mathrm{im}\, d$ and

$$H^1_{\mathrm{alg}}(G,\mathrm{KDiv}(X)) = Z^1_{\mathrm{alg}}(G,\mathrm{KDiv}(X))/B^1_{\mathrm{alg}}(G,\mathrm{KDiv}(X)),$$

as in 2.2.

Theorem 3.13. *There is a canonical isomorphism*

$$\mathrm{Pic}_G(X) \xrightarrow{\sim} H^1_{\mathrm{alg}}(G,\mathrm{KDiv}(X)).$$

Proof. We construct a map

$$\varkappa\colon \mathrm{Pic}_G(X) \to H^1_{\mathrm{alg}}(G,\mathrm{KDiv}(X)).$$

Let $[\mathscr{L},\beta] \in \mathrm{Pic}_G(X)$, where $\mathscr{L}$ is an invertible sheaf on X and

$$\beta\colon w^*\mathscr{L} \to p_X^*\mathscr{L}$$

is a G-linearization. Tensoring with $\mathscr{K}_X$ we obtain a G-linearization of $\mathscr{L}^{\mathscr{K}}$

$$\beta^{\mathscr{K}}\colon w^*\mathscr{L}^{\mathscr{K}} \to p_X^*\mathscr{L}^{\mathscr{K}}.$$

Choose a rational section s of $\mathscr{L}$, i.e. a section of $\mathscr{L}^{\mathscr{K}}$. By Lemma 3.8 $\beta^{\mathscr{K}}$ corresponds to a cocycle $z \in Z^1_{\mathrm{alg}}(G,\mathscr{K}(X)^\times)$ such that

$$\beta^{\mathscr{K}}(w^*s) = z \cdot p_X^*s.$$

Since $\beta^{\mathscr{K}}$ comes from an isomorphism of sheaves of $\mathscr{O}_{X\times G}$-modules $\beta\colon w^*\mathscr{L} \to p_X^*\mathscr{L}$, by Lemma 3.11 we have

$$\mathrm{div}(z) = \mathrm{div}(w^*s) - \mathrm{div}(p_X^*s).$$

Set $D = \mathrm{div}(s)$, then

$$\mathrm{div}(z) = w^*D - p_X^*D.$$

We see that $(z,D)\in Z^1_{\mathrm{alg}}(G,\mathrm{KDiv}(X))$. We set

$$\varkappa([\mathscr{L},\beta])=[z,D]\in H^1_{\mathrm{alg}}(G,\mathrm{KDiv}(X)).$$

An easy calculation shows that $\varkappa$ is a well defined homomorphism.

We construct a map

$$\lambda\colon H^1_{\mathrm{alg}}(G,\mathrm{KDiv}(X))\to \mathrm{Pic}_G(X).$$

Let $(z,D)\in Z^1_{\mathrm{alg}}(G,\mathrm{KDiv}(X))$. Then the divisor D defines an invertible sheaf $\mathscr{L}$ together with a nonzero rational section s of $\mathscr{L}$ (i.e a section of $\mathscr{L}^{\mathscr{K}}$) such that $\mathrm{div}(s)=D$. By Lemma 3.8 z defines a G-linearization

$$\gamma\colon w^*\mathscr{L}^{\mathscr{K}}\to p_X^*\mathscr{L}^{\mathscr{K}}$$

such that

$$\gamma(w^*s)=z\cdot p_X^*s. \tag{7}$$

Since $(z,D)\in Z^1_{\mathrm{alg}}(G,\mathrm{KDiv}(X))$, we have $\mathrm{div}(z)=w^*D-p_X^*D$, hence

$$\mathrm{div}(z)=\mathrm{div}(w^*s)-\mathrm{div}(p_X^*s). \tag{8}$$

By Lemma 3.11 it follows from (7) and (8) that γ comes from an isomorphism of $\mathscr{O}_{X\times G}$-modules

$$\beta\colon w^*\mathscr{L}\to p_X^*\mathscr{L},$$

that is, $\gamma=\beta^{\mathscr{K}}$. Since γ is a G-linearization of $\mathscr{L}^{\mathscr{K}}$, by Lemma 3.6 β is a G-linearization of $\mathscr{L}$. We set

$$\lambda([z,D])=[\mathscr{L},\beta]\in\mathrm{Pic}_G(X).$$

Easy calculations show that λ is a well defined homomorphism and that $\varkappa$ and λ are mutually inverse. Thus $\varkappa$ is an isomorphism. □

4. Relations between $\mathrm{UPic}_G(\overline{X})$ and $\mathrm{UPic}(\overline{X})$

We return to the assumptions and notation of §1.2. Theorem 3.13 says that we have a canonical isomorphism of Galois modules

$$H^1_{\mathrm{alg}}(\overline{G},\mathrm{KDiv}(\overline{X}))\cong\mathrm{Pic}_G(\overline{X}).$$

Corollary 4.1. *Let X and G be as in §1.2. We have a canonical isomorphism of Galois modules*

$$\mathscr{H}^1(\mathrm{UPic}_G(\overline{X}))\cong\mathrm{Pic}_G(\overline{X}).$$

Proof. This follows immediately from Theorem 3.13, since $\mathscr{H}^1(\mathrm{UPic}_G(\overline{X}))=H^1_{\mathrm{alg}}(\overline{G},\mathrm{KDiv}(\overline{X}))$. □

Question 4.2. *What is a geometric interpretation of the cohomology groups $H^n_{\mathrm{alg}}(\overline{G},\mathrm{KDiv}(\overline{X}))$ for $n>1$?*

Lemma 4.3 ([KKV, Lemma 2.2])**.** *Let X and G be as in §1.2 and assume that X is normal. Then the canonical homomorphism $\mathrm{Pic}_G(\overline{X})\to\mathrm{Pic}(\overline{X})$ fits into an exact sequence*

$$0\to H^1_{\mathrm{alg}}(\overline{G},\mathscr{O}(\overline{X})^\times)\to\mathrm{Pic}_G(\overline{X})\to\mathrm{Pic}(\overline{X})\to\mathrm{Pic}(\overline{G}). \tag{9}$$

□

From now on we shall assume that the natural map $\mathrm{Pic}_G(\overline{X}) \to \mathrm{Pic}(\overline{X})$ is surjective. By Lemma 4.3 this condition is satisfied when X is normal and $\mathrm{Pic}(\overline{G}) = 0$ (which can be forced for homogeneous spaces of connected groups, as we shall see below). Examples of k-groups that satisfy $\mathrm{Pic}(\overline{G}) = 0$ are algebraic tori, simply connected semisimple groups, and quasi-trivial groups.

Lemma 4.4. *Let X and G be as in §1.2. If $\mathrm{Pic}_G(\overline{X}) \to \mathrm{Pic}(\overline{X})$ is surjective, we have a short exact sequence of complexes*

$$0 \to Z^1_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times)[-1] \xrightarrow{\ \mu\ } \mathrm{UPic}_G(\overline{X}) \xrightarrow{\ \nu\ } \mathrm{UPic}(\overline{X}) \to 0,$$

which is functorial in X and G. Here ν is the morphism of §2.4, and for $c \in Z^1_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times)$ we set

$$\mu^1(c) = (c^{-1}, 0) \in \mathrm{UPic}_G(\overline{X})^1 \subset Z^1_{\mathrm{alg}}(\overline{G}, \mathscr{K}(\overline{X})^\times) \oplus \mathrm{Div}(\overline{X}).$$

Proof. Consider the canonical morphism of complexes $\nu\colon \mathrm{UPic}_G(\overline{X}) \to \mathrm{UPic}(\overline{X})$ of §2.4. An easy diagram chase in the commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathscr{K}(\overline{X})^\times/\bar{k}^\times & \longrightarrow & \mathrm{UPic}_G(\overline{X})^1 & \longrightarrow & \mathrm{Pic}_G(\overline{X}) & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle \nu^1} & & \downarrow & & \\
\mathscr{K}(\overline{X})^\times/\bar{k}^\times & \longrightarrow & \mathrm{Div}(\overline{X}) & \longrightarrow & \mathrm{Pic}(\overline{X}) & \longrightarrow & 0
\end{array}$$

shows that if the map $\mathrm{Pic}_G(\overline{X}) \to \mathrm{Pic}(\overline{X})$ is surjective, then the map $\nu^1\colon \mathrm{UPic}_G(\overline{X})^1 \to \mathrm{Div}(\overline{X})$ is surjective and hence the morphism of complexes $\nu\colon \mathrm{UPic}_G(\overline{X}) \to \mathrm{UPic}(\overline{X})$ is surjective. On the other hand, the kernel of the morphism ν coincides with the complex $Z^1_{\mathrm{alg}}(\overline{G}, \mathscr{O}(\overline{X})^\times)[-1] = \mathrm{im}\,\mu$. □

In the rest of this section we derive consequences of Lemma 4.4. We consider a more general setting.

4.5. Let

$$0 \to A^\bullet \xrightarrow{\ \mu\ } B^\bullet \xrightarrow{\ \nu\ } C^\bullet \to 0 \qquad (10)$$

be a short exact sequence of complexes in an abelian category, e.g. in the category of discrete $\mathfrak{g}$-modules where $\mathfrak{g}$ is a profinite group. By a morphism φ of exact sequences we mean a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle \varphi_B} & & \downarrow{\scriptstyle \varphi_C} & & \\
0 & \longrightarrow & A'^\bullet & \longrightarrow & B'^\bullet & \longrightarrow & C'^\bullet & \longrightarrow & 0.
\end{array}$$

We say that φ is a quasi-isomorphism of exact sequences if φ_A, φ_B and φ_C are quasi-isomorphisms.

Lemma 4.6 (well known)**.** *Consider a short exact sequence of complexes as in* (10)*. Define a morphism of complexes*

$$\lambda\colon \langle A^\bullet \to B^\bullet] \to C^\bullet \quad \textit{by} \quad \lambda^i(a^{i+1}, b^i) = \nu^i(b^i).$$

Then λ is indeed a morphism of complexes and is a quasi-isomorphism, functorial in the exact sequence (10)*.*

Proof. See [GM, Ch. III, §3, Proof of Prop. 5] or [We, 1.5.8] for a proof that λ is a morphism of complexes and a quasi-isomorphism. The functoriality is obvious. □

Lemma 4.7 (well known)**.** *Assume we have a commutative square of complexes*

$$\begin{array}{ccc} P & \longrightarrow & Q \\ {\scriptstyle\gamma_P}\downarrow & & \downarrow{\scriptstyle\gamma_Q} \\ P' & \longrightarrow & Q' \end{array}$$

where all the arrows are morphisms of complexes and the two vertical arrows γ_P and γ_Q are quasi-isomorphisms. Then the induced morphism of cones

$$\langle P \to Q] \xrightarrow{\gamma_*} \langle P' \to Q']$$

is a quasi-isomorphism.

Proof. The lemma follows easily from the five-lemma. □

Construction 4.8. Let

(S) $$0 \to [0 \to A^1\rangle \xrightarrow{\mu} [B^0 \to B^1\rangle \xrightarrow{\nu} [C^0 \to C^1\rangle \to 0$$

be a short exact sequence of complexes in an abelian category. We write

$$A^\bullet = [0 \to A^1\rangle, \quad B^\bullet = [B^0 \xrightarrow{d_B} B^1\rangle, \quad C^\bullet = [C^0 \xrightarrow{d_C} C^1\rangle,$$

so we have an exact sequence of complexes

$$0 \to A^\bullet \xrightarrow{\mu} B^\bullet \xrightarrow{\nu} C^\bullet \to 0.$$

Assume that the following condition is satisfied:

($\mathscr{H}^0$) $$\mathscr{H}^0(B^\bullet) = 0.$$

Then we have a canonical quasi-isomorphism $B^\bullet \to \mathscr{H}^1(B^\bullet)[-1]$. By Lemma 4.7 the induced morphism of cones

$$\langle A^\bullet \to B^\bullet] \to \langle A^\bullet \to \mathscr{H}^1(B^\bullet)[-1]]$$

is a quasi-isomorphism. Since $A^\bullet = A^1[-1]$, we obtain a quasi-isomorphism

$$\varepsilon\colon \langle A^\bullet \to B^\bullet] \to [A^1 \xrightarrow{\sigma} \mathscr{H}^1(B^\bullet)\rangle,$$

where $\sigma(a^1) = -\mu^1(a^1) + \mathrm{im}\ d_B \in \mathscr{H}^1(B^\bullet)$ for $a^1 \in A^1$.

We write formulae for ε:

$$\varepsilon^0(a^1, b^0) = a^1, \quad \varepsilon^1(b^1) = b^1 + \mathrm{im}\ d_B.$$

Corollary 4.9. *(a) Let (S) be an exact sequence of complexes as in Construction 4.8, satisfying condition $(\mathscr{H}^0)$. Then there is a canonical, functorial in (S) isomorphism in the derived category*

$$[A^1 \xrightarrow{\sigma} \mathscr{H}^1(B^\bullet)\rangle \xrightarrow{\sim} C^\bullet$$

given by the diagram

$$[A^1 \xrightarrow{\sigma} \mathscr{H}^1(B^\bullet)\rangle \xleftarrow{\varepsilon} \langle A^\bullet \to B^\bullet] \xrightarrow{\lambda} C^\bullet$$

where ε is the quasi-isomorphism of Construction 4.8 and λ is the quasi-isomorphism of Lemma 4.6.

(b) Let $\varphi\colon (S)\to(S')$ be a quasi-isomorphism of exact sequences as in Construction 4.8, and assume that both (S) and (S') satisfy the condition $(\mathscr{H}^0)$. Then in the commutative diagram

$$\begin{array}{ccccc}
\Big[A^1 \xrightarrow{\sigma} \mathscr{H}^1(B^\bullet)\Big\rangle & \xleftarrow{\varepsilon} & \langle A^\bullet \to B^\bullet] & \xrightarrow{\lambda} & C^\bullet \\
\downarrow & & \downarrow & & \downarrow \\
\Big[A'^1 \xrightarrow{\sigma'} \mathscr{H}^1(B'^\bullet)\Big\rangle & \xleftarrow{\varepsilon'} & \langle A'^\bullet \to B'^\bullet] & \xrightarrow{\lambda'} & C'^\bullet
\end{array}$$

all the vertical arrows are quasi-isomorphisms.

Proof. The assertion (a) follows from Lemma 4.6 and Construction 4.8, and the assertion (b) follows from Lemma 4.7. □

Now we apply Corollary 4.9 to the exact sequence of Lemma 4.4.

Theorem 4.10. *Let X and G be as in §1.2. Assume that the map $\mathrm{Pic}_G(\overline{X})\to\mathrm{Pic}(\overline{X})$ is surjective and that $U_G(\overline{X})=0$. Then there is a canonical isomorphism in the derived category*

$$\Big[Z^1_{\mathrm{alg}}(\overline{G},\mathscr{O}(\overline{X})^\times) \xrightarrow{\sigma} \mathrm{Pic}_G(\overline{X})\Big\rangle \xrightarrow{\sim} \mathrm{UPic}(\overline{X}). \tag{11}$$

This isomorphism is functorial in X and G.

We specify the map σ and the isomorphism in the derived category. The map σ takes a cocycle $c\in Z^1_{\mathrm{alg}}(\overline{G},\mathscr{O}(\overline{X})^\times)$ to the class of the trivial invertible sheaf $\mathscr{O}_{\overline{X}}$ on $\overline{X}$ with the G-linearization given by c, see Lemma 3.9. The isomorphism (11) is given by the commutative diagram

$$\begin{array}{ccccc}
Z^1_{\mathrm{alg}}(\overline{G},\mathscr{O}(\overline{X})^\times) & \longleftarrow & Z^1_{\mathrm{alg}}(\overline{G},\mathscr{O}(\overline{X})^\times)\oplus\mathscr{K}(\overline{X})^\times/\bar{k}^\times & \longrightarrow & \mathscr{K}(\overline{X})^\times/\bar{k}^\times \\
\downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\mathrm{div}} \\
\mathrm{Pic}_G(\overline{X}) & \longleftarrow & \mathrm{UPic}_G(\overline{X})^1 & \longrightarrow & \mathrm{Div}(\overline{X})
\end{array} \tag{12}$$

where the arrow ψ is given by

$$\psi(c,[f]) = (c\cdot d^0_{\mathscr{K}}(f),\mathrm{div}(f)) \in \mathrm{UPic}_G(\overline{X})^1 \subset Z^1_{\mathrm{alg}}(\overline{G},\mathscr{K}(\overline{X})^\times)\oplus\mathrm{Div}(\overline{X}),$$

and all the unlabeled arrows are the obvious ones.

Proof. The isomorphism of the theorem is the isomorphism of Corollary 4.9(a) applied to the exact sequence of Lemma 4.4. Functoriality in the case of a dominant morphism $f\colon X'\to X$ is evident. In the case of a non-dominant G-morphism $f\colon X'\to X$ we use local modifications as in §2.5. □

5. Extended Picard complex of a homogeneous space

Let X be a homogeneous space under a connected k-group G. In general X may have no k-rational points, hence not be of the form G/H. The results of Section 4 give a nice description of $\mathrm{UPic}(\overline{X})$ as long as $\mathrm{Pic}(\overline{G})=0$. Fortunately, the latter assumption does not give any serious loss of generality by virtue of Lemma 6.6 below.

First let G be a connected k-group acting on a geometrically integral k-variety X. The character group $\mathbb{X}(\overline{G})$ canonically embeds into $Z^1_{\mathrm{alg}}(\overline{G},\mathscr{O}(\overline{X})^\times)$. We show that $Z^1_{\mathrm{alg}}(\overline{G},\mathscr{O}(\overline{X})^\times)=\mathbb{X}(\overline{G})$ using Rosenlicht's lemma.

Lemma 5.1. *Let X be a geometrically integral variety over an arbitrary field k, and G be a connected k-group. Then every $u\in\mathscr{O}(\overline{X}\times_{\bar{k}}\overline{G})^\times$ can be uniquely written in the form $u(x,g)=v(x)\chi(g)$, where $v\in\mathscr{O}(\overline{X})^\times$ and $\chi\in\mathbb{X}(\overline{G})$.* □

Proof. The lemma follows easily from Rosenlicht's lemma, see e.g. [Sa, Lemme 6.5]. □

Proposition 5.2. *Let G be a connected k-group acting on a geometrically integral k-variety X. Then $Z^1_{\mathrm{alg}}(\overline{G},\mathscr{O}(\overline{X})^\times)=\mathbb{X}(\overline{G})$.*

Proof. Let $c\in Z^1_{\mathrm{alg}}(\overline{G},\mathscr{O}(\overline{X})^\times)$. By Lemma 5.1 we may write

$$c_g(x)=v(x)\chi(g)\text{ for some }v\in\mathscr{O}(\overline{X})^\times,\ \chi\in\mathbb{X}(\overline{G}).$$

Writing the cocycle condition (3) of §2.1, we obtain

$$v(x)\chi(g_1g_2)=v(x)\chi(g_1)v(xg_1)\chi(g_2),$$

whence $v(xg_1)=1$. Substituting $g_1=1$, we obtain $v(x)=1$. Thus $c_g(x)=\chi(g)$ and $c=\chi$. □

Now let X be a homogeneous space under a connected k-group G. We compute $U_G(\overline{X})$ and (following Popov) $\mathrm{Pic}_G(\overline{X})$.

Lemma 5.3. *Let X be a homogeneous space under a connected k-group G. Then $U_G(\overline{X})=0$.*

Proof. By definition, $U_G(\overline{X})=(\mathscr{O}(\overline{X})^\times)^G/\bar{k}^\times$. Since G acts transitively, the only G-invariant functions are the constant functions, and the lemma follows. □

5.4. Let X be a homogeneous space under a connected k-group G, and let $\bar{x}\in X(\bar{k})$ be a $\bar{k}$-point. Let $\overline{H}\subset\overline{G}$ be the stabilizer of $\bar{x}$ in $\overline{G}$. We define a homomorphism $\pi_{\bar{x}}\colon\mathrm{Pic}_G(\overline{X})\to\mathbb{X}(\overline{H})$ as follows. Let $(\mathscr{L},\beta)$ be an invertible sheaf on $\overline{X}$ with a $\overline{G}$-linearization. Consider the embedding $i\colon\overline{H}\hookrightarrow\overline{X}\times\overline{G},\ h\mapsto(\bar{x},h)$. Then $w\circ i=p_X\circ i\colon\overline{H}\hookrightarrow\overline{X}$, hence

$$(w\circ i)^*\mathscr{L}=(p_X\circ i)^*\mathscr{L}.$$

We see that the G-linearization $\beta\colon w^*\mathscr{L}\to p_X^*\mathscr{L}$ gives an automorphism of the (trivial) invertible sheaf $(p_X\circ i)^*\mathscr{L}$ on $\overline{H}$, and this automorphism is given by an invertible regular function χ on $\overline{H}$. The cocycle condition (5) of Definition 3.2 readily gives $\chi(h_1h_2)=\chi(h_1)\chi(h_2)$, hence χ is a character of $\overline{H}$. We obtain a homomorphism $\pi_{\bar{x}}\colon\mathrm{Pic}_G(\overline{X})\to\mathbb{X}(\overline{H})$, $\pi_{\bar{x}}([L,\beta])=\chi$.

Let $\bar{x}'\in X(\bar{k})$ be another point and $\overline{H}'$ its stabilizer. We may write $\bar{x}'=\bar{x}g$ for some $g\in G(\bar{k})$. Then $\overline{H}'=g^{-1}\overline{H}g$, and we obtain an isomorphism $\overline{H}\to\overline{H}'\colon h\mapsto g^{-1}hg$. The induced isomorphism $g_*\colon\mathbb{X}(\overline{H}')\to\mathbb{X}(\overline{H})$ does not depend on the choice of g, and so we obtain a canonical identification of $\mathbb{X}(\overline{H})$ with $\mathbb{X}(\overline{H}')$. By Lemma 5.5 below, under this identification the homomorphism $\pi_{\bar{x}}\colon\mathrm{Pic}_G(\overline{X})\to\mathbb{X}(\overline{H})$ does not depend on $\bar{x}$.

Lemma 5.5. *Let G, X, $\bar{x}$, $\bar{x}'$, and g be as above, in particular, $\bar{x}'=\bar{x}g$. Then for $h\in\overline{H}(\bar{k})$ and $\beta\in\mathrm{Pic}_G(\overline{X})$ we have $\pi_{\bar{x}'}(\beta)(g^{-1}hg)=\pi_{\bar{x}}(\beta)(h)$.*

Proof. Omitted (it uses only the cocycle condition (5) of Definition 3.2). □

Proposition 5.6 ([Po, Thm. 4]). *Let X be a homogeneous space under a connected k-group G, and let $\overline{x} \in X(\bar{k})$ be a $\bar{k}$-point. Let $\overline{H} \subset \overline{G}$ be the stabilizer of $\overline{x}$. Then the canonical homomorphism of abelian groups $\pi_{\overline{x}}\colon \mathrm{Pic}_G(\overline{X}) \to \mathbb{X}(\overline{H})$ of §5.4 is an isomorphism.*

Sketch of proof. Let $\chi \in \mathbb{X}(\overline{H})$. We define an embedding

$$\theta\colon \overline{H} \hookrightarrow \overline{G} \times \mathbf{G}_{\mathrm{m},\bar{k}}, \quad h \mapsto (h, \chi(h)^{-1}).$$

Set $\overline{Y} = \theta(\overline{H})\backslash(\overline{G} \times \mathbf{G}_{\mathrm{m},\bar{k}})$, this is a quotient space of a linear algebraic group by an algebraic subgroup. The group $\overline{G}$ acts on $\overline{Y}$ on the right. By Hilbert's Theorem 90 the principal $\mathbf{G}_{\mathrm{m},\bar{k}}$-bundle $\overline{Y} \to \overline{X}$ admits a rational section, and, since $\overline{G}$ acts transitively on $\overline{X}$, this bundle is locally trivial in the Zariski topology. Using transition functions for the bundle $\overline{Y} \to \overline{X}$, we define an invertible sheaf $\mathscr{L}$ on $\overline{X}$. The action of $\overline{G}$ on $\overline{Y}$ defines a $\overline{G}$-linearization β of $\mathscr{L}$. We set $\pi'(\chi) = [\mathscr{L}, \beta] \in \mathrm{Pic}_G(\overline{X})$. We obtain a homomorphism $\pi'\colon \mathbb{X}(\overline{H}) \to \mathrm{Pic}_G(\overline{X})$, which is inverse to $\pi_{\overline{x}}$, hence $\pi_{\overline{x}}$ is an isomorphism. □

Remarks 5.7. (1) Since the Galois group acts on $\mathrm{Pic}_G(\overline{X})$, using Popov's isomorphism $\pi_{\overline{x}}$ of Proposition 5.6, we can endow $\mathbb{X}(\overline{H})$ with a canonical structure of a Galois module. This structure was earlier constructed by hand in [Bo1].

(2) We have a canonical homomorphism $\mathbb{X}(\overline{G}) \to \mathrm{Pic}_G(\overline{X})$ taking a character $\chi \in \mathbb{X}(\overline{G})$ to the trivial invertible sheaf $\mathscr{L} = \mathscr{O}_{\overline{X}}$ with the $\overline{G}$-linearization $\beta_g(x) = \chi(g)\colon \mathscr{O}_{\overline{X}\times\overline{G}} \to \mathscr{O}_{\overline{X}\times\overline{G}}$. Clearly this homomorphism is a morphism of Galois modules. On the other hand, under Popov's identification $\mathrm{Pic}_G(\overline{X}) = \mathbb{X}(\overline{H})$ this homomorphism corresponds to the restriction map $\mathrm{res}\colon \mathbb{X}(\overline{G}) \to \mathbb{X}(\overline{H})$ taking a character χ of $\overline{G}$ to its restriction to $\overline{H}$. We see that the restriction map res is a morphism of Galois modules.

(3) Let $\overline{x}' \in X(\bar{k})$ and $\overline{H}' \subset \overline{G}$ be as in §5.4. If we identify $\mathbb{X}(\overline{H})$ and $\mathbb{X}(\overline{H}')$ as in §5.4, then the map $\mathrm{res}\colon \mathbb{X}(\overline{G}) \to \mathbb{X}(\overline{H})$ does not depend on the choice of the base point $\overline{x}$.

The following theorem is the main result of this paper.

Theorem 5.8. *Let X be a homogeneous space under a connected k-group G with $\mathrm{Pic}(\overline{G}) = 0$. Let $\overline{H}$ be a geometric stabilizer. We have a canonical isomorphism in the derived category of Galois modules*

$$\mathrm{UPic}(\overline{X}) \cong \Big[\mathbb{X}(\overline{G}) \xrightarrow{\mathrm{res}} \mathbb{X}(\overline{H})\Big\rangle$$

which is functorial in G and X. Here $\mathbb{X}(\overline{H})$ has the Galois module structure given by the Popov's isomorphism, and res *is the restriction map.*

Note that the complex of Galois modules $[\mathbb{X}(\overline{G}) \xrightarrow{\mathrm{res}} \mathbb{X}(\overline{H})\rangle$ and the isomorphism of Theorem 5.8 do not depend on the choice of the base point $\overline{x}$ (up to the canonical identification of §5.4). Indeed, $\mathbb{X}(\overline{H})$ does not depend on $\overline{x}$, see §5.4. By Remark 5.7(3) neither does the homomorphism $\mathrm{res}\colon \mathbb{X}(\overline{G}) \to \mathbb{X}(\overline{H})$. The Galois action on $\mathbb{X}(\overline{H})$ is given by Popov's isomorphism, hence does not depend on $\overline{x}$ either. One can easily see from the proof of Theorem 5.8 that the isomorphism of this theorem does not depend on $\overline{x}$ (we use Lemma 5.5).

Proof of Theorem 5.8. By Lemma 5.3 $U_G(\overline{X})=0$. By Theorem 4.10 there is a canonical isomorphism in the derived category of Galois modules

$$\left[Z^1_{\mathrm{alg}}(\overline{G},\mathscr{O}(\overline{X})^\times)\to \mathrm{Pic}_G(\overline{X})\right\rangle \xrightarrow{\sim} \mathrm{UPic}(\overline{X}).$$

By Proposition 5.2 $Z^1_{\mathrm{alg}}(\overline{G},\mathscr{O}(\overline{X})^\times)=\mathbb{X}(\overline{G})$. By Proposition 5.6 $\mathrm{Pic}_G(\overline{X})\cong\mathbb{X}(\overline{H})$. The map σ of Theorem 4.10 corresponds to the homomorphism $\mathrm{res}\colon [\mathbb{X}(\overline{G})\to\mathbb{X}(\overline{H})\rangle$, and the theorem follows. ☐

6. EXTENDED PICARD COMPLEX OF A k-GROUP

Notation 6.1. Let G be a connected k-group. Then:

G^{u} is the unipotent radical of G;
$G^{\mathrm{red}}=G/G^{\mathrm{u}}$, it is a reductive group;
G^{ss} is the derived group of G^{red}, it is semisimple.
$G^{\mathrm{tor}}=G^{\mathrm{red}}/G^{\mathrm{ss}}$, it is a torus;
G^{sc} is the universal covering of G^{ss}, it is simply connected.

Remark 6.2. It is easy to see that $\mathrm{Pic}(\overline{G})=\mathrm{Pic}(\overline{G}^{\mathrm{ss}})$. We have $\mathrm{Pic}(\overline{G}^{\mathrm{ss}})=\mathbb{X}(\ker[\overline{G}^{\mathrm{sc}}\to\overline{G}^{\mathrm{ss}}])$, see [Po, Theorem 3] or [FI, Corollary 4.6]. It follows that $\mathrm{Pic}(\overline{G})=0$ if and only if G^{ss} is simply connected.

6.3. We have a canonical homomorphism

$$\rho\colon G^{\mathrm{sc}}\twoheadrightarrow G^{\mathrm{ss}}\hookrightarrow G^{\mathrm{red}}$$

(Deligne's homomorphism), which in general is neither injective nor surjective. Let $T\subset G^{\mathrm{red}}$ be a maximal torus. Let $T^{\mathrm{sc}}=\rho^{-1}(T)$ be the corresponding maximal torus of T^{sc}. By abuse of notation we denote the restriction of ρ to T^{sc} again by $\rho\colon T^{\mathrm{sc}}\to T$. We have a pullback morphism

$$\rho^*\colon\mathbb{X}(\overline{T})\to\mathbb{X}(\overline{T}^{\mathrm{sc}}).$$

The following theorem is the main result of our paper [BvH2].

Theorem 6.4 ([BvH2, Thm. 1])**.** *Let G be a connected k-group. Let $T\subset G^{\mathrm{red}}$ be a maximal torus. Then there is a canonical isomorphism in the derived category*

$$\mathrm{UPic}(\overline{G})\xrightarrow{\sim}\left[\mathbb{X}(\overline{T})\xrightarrow{\rho^*}\mathbb{X}(\overline{T}^{\mathrm{sc}})\right\rangle,$$

which is functorial in the pair (G,T).

At the core of the proof in [BvH2] were concrete but rather tedious calculations. Our present results allow us to give a new, more conceptual proof of this theorem.

Definition 6.5. An m-extension of a connected k-group G is a central extension

$$1\to M\to G'\to G\to 1$$

where $\mathrm{Pic}(\overline{G}')=0$ and M is a k-group of multiplicative type.

Lemma 6.6 (well known)**.** *Any connected k-group G admits an m-extension.*

Proof. We give a simple proof. We use Notation 6.1. Let $Z(G^{\mathrm{red}})^\circ$ denote the identity component of the center $Z(G^{\mathrm{red}})$ of G^{red}; it is a k-torus. Set $G''=G^{\mathrm{sc}}\times Z(G^{\mathrm{red}})^\circ$. We have an epimorphism

$$\varsigma\colon G''\twoheadrightarrow G^{\mathrm{red}}\colon (g,z)\mapsto\rho(g)z\quad\text{for}\quad g\in G^{\mathrm{sc}},z\in Z(G^{\mathrm{red}})^\circ.$$

Since k is a field of characteristic 0, there exists a splitting of the extension

$$1 \to G^{\mathrm{u}} \to G \to G^{\mathrm{red}} \to 1$$

(see [Mo, Theorem 7.1]). Such a splitting defines an action of G^{red} on G^{u}, and we have an isomorphism $G \xrightarrow{\sim} G^{\mathrm{u}} \rtimes G^{\mathrm{red}}$ (Levi decomposition). The group G'' acts on G^{u} via G^{red}. Set $G' = G^{\mathrm{u}} \rtimes G''$. We have an epimorphism

$$\mathrm{id} \times \varsigma \colon G' \twoheadrightarrow G.$$

Since $(G')^{\mathrm{ss}} = (G'')^{\mathrm{ss}} = G^{\mathrm{sc}}$, we see that $(G')^{\mathrm{ss}}$ is simply connected, hence $\mathrm{Pic}(\overline{G}') = 0$. Clearly $M := \ker[G' \to G]$ is a central, finite subgroup of multiplicative type in G'. □

Note that a much stronger assertion than Lemma 6.6 is known, see Lemma 6.7 below. According to Colliot-Thélène [CT, Prop. 2.2], a quasi-trivial k-group is an extension of a quasi-trivial k-torus by a simply connected k-group. Recall that a simply connected k-group is an extension of a simply connected semisimple k-group by a (connected) unipotent k-group.

Lemma 6.7 (Colliot-Thélène)**.** *Let G be a connected k-group. Then there exists a central extension*

$$1 \to T \to G' \to G \to 1,$$

where G' is a quasi-trivial k-group and T is a k-torus.

Proof. See [CT, Prop.-Déf. 3.1]. □

Lemma 6.8. *Let $G_1 \to G_2$ be a homomorphism of connected k-groups, and let $G'_i \to G_i$ $(i = 1, 2)$ be m-extensions. Then there exists a commutative diagram*

$$\begin{array}{ccccc} G'_1 & \longleftarrow & G'_3 & \longrightarrow & G'_2 \\ \downarrow & & \downarrow & & \downarrow \\ G_1 & \xleftarrow{\mathrm{id}} & G_1 & \longrightarrow & G_2 \end{array}$$

in which $G'_3 \to G_1$ is an m-extension.

Proof. Similar to [Ko, Proof of Lemma 2.4.4]. □

The following theorem gives an alternative description of $\mathrm{UPic}(\overline{G})$ for a connected k-group G.

Theorem 6.9. *Let G be a connected k-group and let X be a k-torsor (a principal homogeneous space) under G. Let*

$$1 \to M \to G' \to G \to 1$$

be an m-extension. Then there is a canonical isomorphism, functorial in the triple (G, G', X),

$$\mathrm{UPic}(\overline{X}) \xrightarrow{\sim} \left[\mathbb{X}(\overline{G}') \xrightarrow{\mathrm{res}} \mathbb{X}(\overline{M}) \right\rangle$$

in the derived category of discrete Galois modules. Here $\mathrm{res} \colon \mathbb{X}(\overline{G}') \to \mathbb{X}(\overline{M})$ *is the restriction homomorphism.*

Proof. The group G' acts on X via G, and X is a homogeneous space of G' with geometric stabilizer $\overline{M}$. Since $\mathrm{Pic}(\overline{G}') = 0$, the theorem follows from Theorem 5.8. □

6.10. *New proof of Theorem 6.4.* By [BvH2, Lemma 4.1] the canonical epimorphism $r\colon G\to G^{\mathrm{red}}$ induces a canonical isomorphism in the derived category $r^*\colon \mathrm{UPic}(\overline{G}^{\mathrm{red}}) \xrightarrow{\sim} \mathrm{UPic}(\overline{G})$, and therefore we may assume that G is reductive. Choose an m-extension

$$1\to M\to G'\to G\to 1,$$

then G' is reductive as well. By Theorem 6.9 there is a canonical, functorial in G and G' isomorphism

$$\mathrm{UPic}(\overline{G}) \xrightarrow{\sim} [\mathbb{X}(\overline{G}')\to\mathbb{X}(\overline{M})\rangle.$$

Let $T\subset G$ be a maximal torus. Let T' denote the preimage of T in G', then T' is a maximal torus in the reductive group G'. Consider the commutative diagram

$$\begin{array}{ccc}
\mathbb{X}(\overline{G}') & \longrightarrow & \mathbb{X}(\overline{M}) \\
\downarrow & & \downarrow \\
\mathbb{X}(\overline{T}') & \longrightarrow & \mathbb{X}(\overline{M})\oplus\mathbb{X}(\overline{T}^{\mathrm{sc}}) \\
\uparrow & & \uparrow \\
\mathbb{X}(\overline{T}) & \longrightarrow & \mathbb{X}(\overline{T}^{\mathrm{sc}})
\end{array} \tag{13}$$

with obvious arrows. It is easy to check that this diagram gives quasi-isomorphisms

$$[\mathbb{X}(\overline{G}')\to\mathbb{X}(\overline{M})\rangle \longrightarrow [\mathbb{X}(\overline{T}')\to\mathbb{X}(\overline{M})\oplus\mathbb{X}(\overline{T}^{\mathrm{sc}})\rangle$$
$$[\mathbb{X}(\overline{T})\to\mathbb{X}(\overline{T}^{\mathrm{sc}})\rangle \longrightarrow [\mathbb{X}(\overline{T}')\to\mathbb{X}(\overline{M})\oplus\mathbb{X}(\overline{T}^{\mathrm{sc}})\rangle,$$

hence we obtain an isomorphism

$$[\mathbb{X}(\overline{G}')\to\mathbb{X}(\overline{M})\rangle \xrightarrow{\sim} [\mathbb{X}(\overline{T})\to\mathbb{X}(\overline{T}^{\mathrm{sc}})\rangle$$

in the derived category. Thus we obtain an isomorphism

$$\mathrm{UPic}(\overline{G}) \xrightarrow{\sim} [\mathbb{X}(\overline{T})\to\mathbb{X}(\overline{T}^{\mathrm{sc}})\rangle$$

in the derived category. Using Lemma 6.8, one can easily see that this isomorphism does not depend on the choice of the m-extension G' of G and is functorial in (G,T). $\square$

7. PICARD AND BRAUER GROUPS

Theorem 7.1. *Let X be a homogeneous space under a connected k-group G with* $\mathrm{Pic}(\overline{G})=0$. *Let $\overline{H}$ be the stabilizer of a geometric point $\overline{x}\in X(\overline{k})$ (we do not assume that $\overline{H}$ is connected). Then there is a canonical injection*

$$\mathrm{Pic}(X)\hookrightarrow H^1(k,[\mathbb{X}(\overline{G})\to\mathbb{X}(\overline{H})\rangle),$$

which is an isomorphism if $X(k)\neq\emptyset$ or $\mathrm{Br}(k)=0$.

Proof. This follows immediately from Theorem 5.8 and the exact sequence

(14)
$$0\to\mathrm{Pic}(X)\to H^1(k,\mathrm{UPic}(\overline{X}))\to\mathrm{Br}(k)\to\mathrm{Br}_1(X)\to H^2(k,\mathrm{UPic}(\overline{X}))\to H^3(k,\mathbf{G}_{\mathrm{m}})$$

established in [BvH2, Prop. 2.19] for an arbitrary smooth geometrically integral variety. Note that the homomorphisms $H^1(k,\mathrm{UPic}(\overline{X}))\to\mathrm{Br}(k)$ and $H^2(k,\mathrm{UPic}(\overline{X}))\to H^3(k,\mathbf{G}_{\mathrm{m}})$ in this exact sequence are zero if $X(k)\neq\emptyset$, see [BvH2, Prop. 2.19]. $\square$

We also prove a conjecture [Bo3, Conj. 3.2] of the first-named author concerning the subquotient $\mathrm{Br_a}(X) = \ker[\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})]/\mathrm{im}\ [\mathrm{Br}(k) \to \mathrm{Br}(X)]$ of the Brauer group of a homogeneous space X.

Theorem 7.2. *Let X, G, and $\overline{H}$ be as in Theorem 7.1. Then there is a canonical injection*

$$\mathrm{Br_a}(X) \hookrightarrow H^2(k, [\mathbb{X}(\overline{G}) \to \mathbb{X}(\overline{H})\rangle),$$

which is an isomorphism if $X(k) \neq \emptyset$ or $H^3(k, \mathbf{G}_\mathrm{m}) = 0$ (e.g., when k is a number field or a $\mathfrak{p}$-adic field).

Proof. This follows immediately from Theorem 5.8 and exact sequence (14). □

7.3. We consider the special case when X is a *principal* homogeneous space of a connected k-group G (we do not assume that $\mathrm{Pic}(\overline{G}) = 0$). Let

$$1 \to M \to G' \to G \to 1$$

be an m-extension. Then, assuming that either X has a k-point or $\mathrm{Br}(k) = 0$, we obtain from Theorem 7.1 that

$$\mathrm{Pic}(X) \cong H^1(k, [\mathbb{X}(\overline{G}') \to \mathbb{X}(\overline{M})\rangle). \tag{15}$$

Let $T \subset G^{\mathrm{red}}$ and $T^{\mathrm{sc}} \subset G^{\mathrm{sc}}$ be as in Notation 6.1 Using the isomorphism in the derived category given by diagram (13) in §6.10, we obtain from (15) that

$$\mathrm{Pic}(X) \cong H^1(k, [\mathbb{X}(\overline{T}) \to \mathbb{X}(\overline{T}^{\mathrm{sc}})\rangle). \tag{16}$$

Similarly, assuming that either X has a k-point or $H^3(k, \mathbf{G}_\mathrm{m}) = 0$ (and not assuming that $\mathrm{Br}(k) = 0$), we obtain from Theorem 7.2 that

$$\mathrm{Br_a}(X) \cong H^2(k, [\mathbb{X}(\overline{G}') \to \mathbb{X}(\overline{M})\rangle) \tag{17}$$

and

$$\mathrm{Br_a}(X) \cong H^2(k, [\mathbb{X}(\overline{T}) \to \mathbb{X}(\overline{T}^{\mathrm{sc}})\rangle). \tag{18}$$

The formulae (16) and (18) are versions of our previous results [BvH2, Cor. 5 and Cor. 7] and results of Kottwitz [Ko, 2.4]. Note that when G is a k-torus or a semisimple k-group, formulae for $\mathrm{Pic}(G)$ and $\mathrm{Br_a}(G)$ were earlier given by Sansuc [Sa, Lemme 6.9].

8. Comparison with topological invariants

8.1. For a k-group G, the main result of [BvH2] and the comparison between the algebraic and the topological fundamental group of complex linear algebraic groups in [Bo2], imply that the derived dual object

$$\mathrm{UPic}(\overline{G})^D := R\mathrm{Hom}_{\mathbf{Z}}(\mathrm{UPic}(\overline{G}), \mathbf{Z})$$

is represented by a finitely generated Galois module concentrated in degree 0 which, as an abelian group, is isomorphic to the fundamental group of the complex analytic space $G(\mathbf{C})$ for any embedding $\overline{k} \hookrightarrow \mathbf{C}$ of $\overline{k}$ into the complex numbers. For later use, let us make the further observation that in fact

$$U(\overline{G})^D = \mathbb{X}(\overline{G})^D = \pi_1(G(\mathbf{C}))/\pi_1(G(\mathbf{C}))_{\mathrm{tors}},$$
$$(\mathrm{Pic}(\overline{G})[-1])^D = \mathrm{Ext}^1_{\mathbf{Z}}(\mathrm{Pic}(\overline{G}), \mathbf{Z}) = \mathrm{Hom}(\mathrm{Pic}(\overline{G}), \mathbf{Q}/\mathbf{Z}) = \pi_1(G(\mathbf{C}))_{\mathrm{tors}},$$

see (20) below, where M_{tors} denotes the torsion subgroup of a finitely generated abelian group M.

For a homogeneous space X under a k-group G, the derived dual object in general is not representable by a single group concentrated in degree 0, as we see from the following lemma.

Lemma 8.2. *Let X be a homogeneous space with geometric stabilizer $\overline{H}$ under a connected k-group G with $\mathrm{Pic}(\overline{G})=0$. We have a long exact sequence of finitely generated Galois modules*

$$(19)\quad 0\to \mathscr{H}^{-1}(\mathrm{UPic}(\overline{X})^D)\to \mathrm{Hom}_{\mathbf{Z}}(\mathbb{X}(\overline{H}),\mathbf{Z})\to \mathrm{Hom}_{\mathbf{Z}}(\mathbb{X}(\overline{G}),\mathbf{Z})\to \mathscr{H}^{0}(\mathrm{UPic}(\overline{X})^D)\to \mathrm{Hom}_{\mathbf{Z}}(\mathbb{X}(\overline{H})_{\mathrm{tors}},\mathbf{Q}/\mathbf{Z})\to 0$$

and $\mathscr{H}^{i}(\mathrm{UPic}(\overline{X})^D)=0$ *for* $i\neq 0,-1$.

Proof. It follows from Theorem 5.8 that we have an exact triangle

$$\mathbb{X}(\overline{H})[-1]\to \mathrm{UPic}(\overline{X})\to \mathbb{X}(\overline{G})\to \mathbb{X}(\overline{H})$$

in the derived category of Galois modules. Applying $R\mathrm{Hom}_{\mathbf{Z}}(-,\mathbf{Z})$, we get the dual triangle

$$R\mathrm{Hom}_{\mathbf{Z}}(\mathbb{X}(\overline{H}),\mathbf{Z})\to R\mathrm{Hom}_{\mathbf{Z}}(\mathbb{X}(\overline{G}),\mathbf{Z})\to \mathrm{UPic}(\overline{X})^D\to R\mathrm{Hom}_{\mathbf{Z}}(\mathbb{X}(\overline{H}),\mathbf{Z})[1].$$

Taking cohomology yields the exact sequence

$$0\to \mathscr{H}^{-1}(\mathrm{UPic}(\overline{X})^D)\to R^0\mathrm{Hom}_{\mathbf{Z}}(\mathbb{X}(\overline{H}),\mathbf{Z})\to R^0\mathrm{Hom}_{\mathbf{Z}}(\mathbb{X}(\overline{G}),\mathbf{Z})\to \mathscr{H}^{0}(\mathrm{UPic}(\overline{X})^D)\to R^1\mathrm{Hom}_{\mathbf{Z}}(\mathbb{X}(\overline{H}),\mathbf{Z})\to R^1\mathrm{Hom}_{\mathbf{Z}}(\mathbb{X}(\overline{G}),\mathbf{Z}),$$

which coincides with exact sequence (19), since

$$(20)\qquad R^i\mathrm{Hom}_{\mathbf{Z}}(M,\mathbf{Z})=\begin{cases}\mathrm{Hom}_{\mathbf{Z}}(M,\mathbf{Z}) & \text{if } i=0,\\ \mathrm{Ext}^1_{\mathbf{Z}}(M,\mathbf{Z}) & \text{if } i=1,\\ 0 & \text{otherwise}\end{cases}$$

and $\mathrm{Ext}^1_{\mathbf{Z}}(M,\mathbf{Z})=\mathrm{Hom}_{\mathbf{Z}}(M_{\mathrm{tors}},\mathbf{Q}/\mathbf{Z})$ for any finitely generated abelian group M. ☐

8.3. Let $\overline{H}$ be a $\bar{k}$-group, not necessarily connected. We denote by $\overline{H}^{\mathrm{mult}}$ the largest quotient group of multiplicative type of $\overline{H}$, then $\mathbb{X}(\overline{H}^{\mathrm{mult}})=\mathbb{X}(\overline{H})$. We set $\overline{H}_1=\ker[\overline{H}\to\overline{H}^{\mathrm{mult}}]$, then $\overline{H}_1=\bigcap_{\chi\in\mathbb{X}(\overline{H})}\ker\chi$. We consider the following condition on $\overline{H}$:

$$(21)\qquad \overline{H}_1 \text{ is connected and } \mathbb{X}(\overline{H}_1)=0.$$

Note that this condition is satisfied if $\overline{H}$ is connected.

Lemma 8.4. *Let H be a* $\mathbf{C}$*-group satisfying the condition* (21)*. Then there is a canonical, functorial in H epimorphism*

$$\pi_1(H^\circ(\mathbf{C}))\to \mathrm{Hom}(\mathbb{X}(H),\mathbf{Z})$$

inducing an isomorphism

$$\pi_1(H^\circ(\mathbf{C}))/\pi_1(H^\circ(\mathbf{C}))_{\mathrm{tors}}\xrightarrow{\sim}\mathrm{Hom}(\mathbb{X}(H),\mathbf{Z}),$$

where we denote by H° the identity component of H.

Proof. We have

$$\mathrm{Hom}(\mathbb{X}(H),\mathbf{Z}) = \mathrm{Hom}(\mathbb{X}(H^{\mathrm{mult}}),\mathbf{Z}) = \mathrm{Hom}(\mathbb{X}((H^{\mathrm{mult}})^\circ),\mathbf{Z}) = \pi_1((H^{\mathrm{mult}})^\circ(\mathbf{C})).$$

Since H satisfies the condition (21), we have an exact sequence of connected $\mathbf{C}$-groups

$$1 \to H_1 \to H^\circ \to (H^{\mathrm{mult}})^\circ \to 1$$

and a homotopy exact sequence

$$\pi_1(H_1(\mathbf{C})) \to \pi_1(H^\circ(\mathbf{C})) \to \pi_1((H^{\mathrm{mult}})^\circ(\mathbf{C})) \to \pi_0(H_1(\mathbf{C})) = 1.$$

From this exact sequence we obtain an epimorphism

$$\pi_1(H^\circ(\mathbf{C})) \to \pi_1((H^{\mathrm{mult}})^\circ(\mathbf{C})) = \mathrm{Hom}(\mathbb{X}(H),\mathbf{Z}).$$

Since $\pi_1(H_1(\mathbf{C}))$ is a finite group and $\mathrm{Hom}(\mathbb{X}(H),\mathbf{Z})$ is torsion free, we obtain an isomorphism

$$\pi_1(H^\circ(\mathbf{C}))/\pi_1(H^\circ(\mathbf{C}))_{\mathrm{tors}} \xrightarrow{\sim} \mathrm{Hom}(\mathbb{X}(H),\mathbf{Z}).$$

□

Theorem 8.5. *Let X be a homogeneous space under a connected k-group G with connected geometric stabilizers. Let us fix an embedding $\bar{k} \hookrightarrow \mathbf{C}$ and an isomorphism $\pi_1(\mathbf{C}^\times) \to \mathbf{Z}$.*

(i) *We have an isomorphism of groups*

$$\mathscr{H}^0(\mathrm{UPic}(\overline{X})^D) \cong \pi_1(X(\mathbf{C})).$$

(ii) *We have an isomorphism of abelian groups*

$$\mathscr{H}^{-1}(\mathrm{UPic}(\overline{X})^D) \cong \pi_2(X(\mathbf{C}))/\pi_2(X(\mathbf{C}))_{\mathrm{tors}}.$$

Proof. By Lemma 6.7 we can represent X as a homogeneous space of a connected k-group G' with $\mathrm{Pic}(\overline{G}') = 0$ with connected geometric stabilizers, see also [Bo1, Lemma 5.2]. Therefore we may and shall assume that $\mathrm{Pic}(\overline{G}) = 0$. After base change to $\mathbf{C}$ we may assume $X = G/H$. Since $\pi_2(G(\mathbf{C})) = 0$ (Élie Cartan, for the case of compact Lie groups see [Brl]), the long exact sequence of homotopy groups of a fibration gives us an exact sequence

$$0 \to \pi_2(X(\mathbf{C})) \to \pi_1(H(\mathbf{C})) \to \pi_1(G(\mathbf{C})) \to \pi_1(X(\mathbf{C})) \to 1.$$

Proof of (i). We have a commutative diagram with exact top row

$$\begin{array}{ccccccc} \pi_1(H(\mathbf{C})) & \longrightarrow & \pi_1(G(\mathbf{C})) & \longrightarrow & \pi_1(X(\mathbf{C})) & \longrightarrow & 1 \\ \downarrow & & \downarrow & & & & \\ \mathrm{Hom}(\mathbb{X}(H),\mathbf{Z}) & \longrightarrow & \mathrm{Hom}(\mathbb{X}(G),\mathbf{Z}) & & & & \end{array}$$

where by Lemma 8.4 both vertical arrows are epimorphisms. Since $\mathrm{Pic}(\overline{G}) = 0$, the group $\pi_1(G(\mathbf{C}))$ is torsion free, and by Lemma 8.4 the right vertical arrow is an isomorphism. Thus the diagram gives an exact sequence

$$\mathrm{Hom}(\mathbb{X}(H),\mathbf{Z}) \to \mathrm{Hom}(\mathbb{X}(G),\mathbf{Z}) \to \pi_1(X(\mathbf{C})) \to 1.$$

From this exact sequence and Lemma 8.2 we obtain an isomorphism

$$\pi_1(X(\mathbf{C})) \cong \mathscr{H}^0(\mathrm{UPic}(\overline{X})^D)$$

because $\mathrm{Hom}(\mathbb{X}(\overline{H})_{\mathrm{tors}}, \mathbf{Q}/\mathbf{Z}) = 0$ for a connected group $\overline{H}$.

Proof of (ii). See Proposition 8.6 below. □

Proposition 8.6. *Let X be a homogeneous space under a connected k-group G with* $\mathrm{Pic}(\overline{G})=0$. *Let $\overline{H}$ be a geometric stabilizer, and assume that $\overline{H}$ satisfies the condition* (21). *Let us fix an embedding $\bar{k}\hookrightarrow \mathbf{C}$ and an isomorphism $\pi_1(\mathbf{C}^\times)\to \mathbf{Z}$.*

(i) *We have an isomorphism of abelian groups*

$$\mathscr{H}^{-1}(\mathrm{UPic}(\overline{X})^D)\cong \pi_2(X(\mathbf{C}))/\pi_2(X(\mathbf{C}))_{\mathrm{tors}}.$$

(ii) *If, moreover,* $\mathrm{Pic}(\overline{H}^\circ)=0$, *then we have*

$$\mathscr{H}^{-1}(\mathrm{UPic}(\overline{X})^D)\cong \pi_2(X(\mathbf{C})).$$

Proof. (i) By Lemma 8.4 we have

$$\mathrm{Hom}(\mathbb{X}(\overline{H}),\mathbf{Z})=\pi_1(H^\circ(\mathbf{C}))/\pi_1(H^\circ(\mathbf{C}))_{\mathrm{tors}}.$$

Since G is connected and $\mathrm{Pic}(\overline{G})=0$, we have

$$\mathrm{Hom}(\mathbb{X}(\overline{G}),\mathbf{Z})=\pi_1(G(\mathbf{C})).$$

We obtain from the fibration exact sequence

$$(22)\quad 0\to\pi_2(X(\mathbf{C}))\to\pi_1(H^\circ(\mathbf{C}))\to\pi_1(G(\mathbf{C}))\to\pi_1(X(\mathbf{C}))\to\pi_0(H(\mathbf{C}))\to 1$$

that

$$\pi_2(X(\mathbf{C}))/\pi_2(X(\mathbf{C}))_{\mathrm{tors}}=\ker[\mathrm{Hom}(\mathbb{X}(\overline{H}),\mathbf{Z})\to\mathrm{Hom}(\mathbb{X}(\overline{G}),\mathbf{Z})],$$

hence $\mathscr{H}^{-1}(\mathrm{UPic}(\overline{X})^D)\cong \pi_2(X(\mathbf{C}))/\pi_2(X(\mathbf{C}))_{\mathrm{tors}}$ by Lemma 8.2.

(ii) If $\mathrm{Pic}(\overline{H}^\circ)=0$, then $\pi_1(H^\circ(\mathbf{C}))$ is torsion free, hence so is $\pi_2(X(\mathbf{C}))$ by the fibration exact sequence (22). □

Note that some condition on stabilizers in Theorem 8.5 must be imposed, as shown by the following example:

Example 8.7. Take $G=\mathrm{SL}_{n,k}$, take T to be the group of diagonal matrices in G, and take $H=N$ to be the normalizer of T in G. We have $H^\circ=T$. Set $X=H\backslash G$. Then

$$\pi_2(X(\mathbf{C}))/\pi_2(X(\mathbf{C}))_{\mathrm{tors}}=\pi_1(H^\circ(\mathbf{C}))/\pi_1(H^\circ(\mathbf{C}))_{\mathrm{tors}}=\pi_1(T(\mathbf{C}))\cong\mathbf{Z}^{n-1}.$$

On the other hand, it follows from the next Lemma 8.8 that $\mathrm{Hom}(\mathbb{X}(\overline{H}),\mathbf{Z})=0$, hence $\mathscr{H}^{-1}(\mathrm{UPic}(\overline{X})^D)=0$.

Lemma 8.8 (well-known)**.** *Let $G=\mathrm{SL}_{n,k}$, where $n\geq 2$, let T be the group of diagonal matrices in G, and let N to be the normalizer of T in G. Then $\mathbb{X}(\overline{N})\cong\mathbf{Z}/2\mathbf{Z}$.*

Proof. Let $t=\mathrm{diag}(z,z^{-1},1,\dots,1)\in T(\bar{k})$ where $z\in\bar{k}^\times$, and let $s\in N(\bar{k})$ be a representative of the transposition $(1,2)\in S_n=N/T$. Then $sts^{-1}=t^{-1}$, hence $tst^{-1}s^{-1}=t^2=\mathrm{diag}(z^2,z^{-2},1,\dots,1)$. It follows easily that $T\subset N^{\mathrm{der}}$, where N^{der} denotes the derived group of N. We see that $\mathbb{X}(\overline{N})=\mathbb{X}(\overline{N}/\overline{T})=\mathbb{X}(S_n)$. Since $(S_n)^{\mathrm{der}}=A_n$, we see that $\mathbb{X}(S_n)=\mathbb{X}(S_n/A_n)\cong\mathbf{Z}/2\mathbf{Z}$, hence $\mathbb{X}(\overline{N})\cong\mathbf{Z}/2\mathbf{Z}$. □

9. THE ELEMENTARY OBSTRUCTION

9.1. Recall that in [BvH2, Def. 2.10] the *elementary obstruction* (to the existence of a k-point in a smooth geometrically integral k-variety X) was defined as the class $e(X)\in \mathrm{Ext}^1(\mathrm{UPic}(\overline{X}),\overline{k}^\times)$ associated to the extension of complexes of Galois modules

$$0\to \overline{k}^\times \to \Big(\mathscr{K}(\overline{X})^\times \to \mathrm{Div}(\overline{X})\Big)\to \Big(\mathscr{K}(\overline{X})^\times/\overline{k}^\times \to \mathrm{Div}(\overline{X})\Big)\to 0.$$

It is a variant of the original elementary obstruction $\mathrm{ob}(X)$ of Colliot-Thélène and Sansuc [CS, Déf. 2.2.1] which lives in $\mathrm{Ext}^1(\mathscr{K}(\overline{X})^\times/\overline{k}^\times,\overline{k}^\times)$. In fact, we have a canonical injection $\mathrm{Ext}^1(\mathrm{UPic}(\overline{X}),\overline{k}^\times)\to \mathrm{Ext}^1(\mathscr{K}(\overline{X})^\times/\overline{k}^\times,\overline{k}^\times)$ which sends $e(X)$ to $\mathrm{ob}(X)$ (see [BvH2, Lemma 2.12]).

9.2. Recall that $\overline{H}^{\mathrm{mult}}$ denotes the largest quotient group of $\overline{H}$ that is a group of multiplicative type. We have $\mathbb{X}(\overline{H}^{\mathrm{mult}})=\mathbb{X}(\overline{H})$. The Galois action on $\mathbb{X}(\overline{H})$ defines a k-form H^{m} of $\overline{H}^{\mathrm{mult}}$. Since we have a morphism of Galois modules $\mathbb{X}(\overline{G})\to \mathbb{X}(\overline{H})$, we obtain a k-homomorphism $i_*\colon H^{\mathrm{m}}\to G^{\mathrm{tor}}$ (related to the embedding $i\colon \overline{H}\hookrightarrow\overline{G}$).

Set $G^{\mathrm{ssu}}=\ker[G\to G^{\mathrm{tor}}]$, it is an extension of a semisimple group by a unipotent group. Set $\overline{H}_1=\ker[\overline{H}\to\overline{H}^{\mathrm{mult}}]$. The embedding $\overline{H}\hookrightarrow\overline{G}$ gives a homomorphism $\overline{H}\to\overline{G}^{\mathrm{tor}}$, and $\overline{H}_1$ is contained in the kernel of this homomorphism (because the restriction of any character of $\overline{H}$ to $\overline{H}_1$ is trivial). Thus $\overline{H}_1\subset\overline{G}^{\mathrm{ssu}}$. It is clear that $i_*\colon H^{\mathrm{m}}\to G^{\mathrm{tor}}$ is an embedding if and only if $\overline{H}_1=\overline{H}\cap\overline{G}^{\mathrm{ssu}}$.

Proposition 9.3. *Let X be a homogeneous space under a connected k-group G with $\mathrm{Pic}(\overline{G})=0$. Let $\overline{H}$ be a geometric stabilizer. For every integer i we have a canonical isomorphism*

$$\mathrm{Ext}^i(\mathrm{UPic}(\overline{X}),\overline{k}^\times)\cong H^i(k,\langle H^{\mathrm{m}}\to G^{\mathrm{tor}}]),$$

which is functorial in G and X.

We need a lemma.

Lemma 9.4. *Let $\mathfrak{g}$ be a profinite group. Let $Y^\bullet$ be a bounded complex of discrete $\mathfrak{g}$-modules with finitely generated (over $\mathbf{Z}$) cohomology.*

(i) There exists a quasi-isomorphism $\psi\colon M^\bullet\to Y^\bullet$, where $M^\bullet$ is a bounded complex of finitely generated (over $\mathbf{Z}$) torsion free $\mathfrak{g}$-modules.

(ii) For any two such resolutions (quasi-isomorphisms) $\psi_1\colon M_1^\bullet\to Y^\bullet$ and $\psi_2\colon M_2^\bullet\to Y^\bullet$ of $Y^\bullet$, there exists a third such complex $M_3^\bullet$ and quasi-isomorphisms $\mu_j\colon M_3^\bullet\to M_j^\bullet$ ($j=1,2$), such that the following diagram commutes up to a homotopy:

$$\begin{array}{ccc} M_3^\bullet & \xrightarrow{\mu_1} & M_1^\bullet \\ {\scriptstyle\mu_2}\downarrow & & \downarrow{\scriptstyle\psi_1} \\ M_2^\bullet & \xrightarrow{\psi_2} & Y^\bullet. \end{array}$$

Proof. Assume that $Y^i=0$ for $i>n$. We choose a finite set of generators $h_1,\dots,h_\kappa$ (over $\mathbf{Z}$) of $\mathscr{H}^n(Y^\bullet)$. We lift each h_j for $j=1,\dots,\kappa$ to $y_j\subset\ker[Y^n\to Y^{n+1}]$. Let $\mathfrak{h}_j\subset\mathfrak{g}$ denote the stabilizer of y_j in $\mathfrak{g}$, it is an open subgroup (hence of finite index) in $\mathfrak{g}$. We have a canonical $\mathfrak{g}$-morphism $\mathbf{Z}[\mathfrak{g}/\mathfrak{h}_j]\to\ker[Y^n\to Y^{n+1}]$ taking the image of the unit element $e \in \mathfrak{g}$ in $\mathfrak{g}/\mathfrak{h}_j$ to y_j. Set $A^n = \bigoplus_j \mathbf{Z}[\mathfrak{g}/\mathfrak{h}_j]$, then A^n is a finitely generated (over $\mathbf{Z}$) torsion free $\mathfrak{g}$-module. We have a morphism of $\mathfrak{g}$-modules $A^n \to \ker[Y^n \to Y^{n+1}]$ such that the induced morphism $A^n \to \mathscr{H}^n(Y^\bullet)$ is surjective. We consider the complex $A^n[-n]$ (with one $\mathfrak{g}$-module A^n in degree n). We have a morphism of complexes $\varphi\colon A^n[-n] \to Y^\bullet$. We set $Y^\bullet_{(1)} = \langle A^n[-n] \to Y^\bullet]$ (the cone of φ). It is easy to see that $\mathscr{H}^n(Y^\bullet_{(1)}) = 0$. Then we apply this procedure to $Y^\bullet_{(1)}$ for $n-1$ to obtain $Y^\bullet_{(2)}$ with $\mathscr{H}^{n-1}(Y^\bullet_{(2)}) = 0$, and so on.

Assume that $Y^i = 0$ for $i \le n-m$ for some integer $m > 0$. Then $Y^i_{(m)} = 0$ for $i < n-m$, and $Y^{n-m}_{(m)}$ is finitely generated and torsion free. By construction $\mathscr{H}^i(Y^\bullet_{(m)}) = 0$ for $i > n-m$. Moreover, since $Y^{n-m-1}_{(m)} = 0$, we have

$$\mathscr{H}^{n-m}(Y^\bullet_{(m)}) = \ker[Y^{n-m}_{(m)} \to Y^{n-m+1}_{(m)}].$$

Set $A' = \mathscr{H}^{n-m}(Y^\bullet_{(m)})$, then A' is finitely generated and torsion free, because it is a subgroup of the finitely generated torsion free abelian group $Y^{n-m}_{(m)}$. We have an *injective* morphism of $\mathfrak{g}$-modules $A' \hookrightarrow Y^{n-m}_{(m)}$ and a morphism of complexes $\varphi'\colon A'[n-m] \to Y^\bullet_{(m)}$. As before, set $Y^\bullet_{(m+1)} = \langle A'[n-m] \to Y^\bullet_{(m)}]$ (the cone of φ'). One can easily see that the complex $Y^\bullet_{(m+1)}$ is acyclic.

One can check that $Y^\bullet_{(m+1)}$ is the cone of some morphism of complexes $\psi\colon M^\bullet \to Y^\bullet$, where $M^\bullet$ is a bounded complex of finitely generated torsion free $\mathfrak{g}$-modules. Since the cone $Y^\bullet_{(m+1)}$ of ψ is acyclic, we see that ψ is a quasi-isomorphism.

(ii) Let $N^\bullet := [M_1^\bullet \oplus M_2^\bullet \xrightarrow{\psi_1 - \psi_2} Y^\bullet\rangle$ (the fibre of $\psi_1 - \psi_2$), then we have morphisms

$$\lambda_j\colon N^\bullet \to M_1^\bullet \oplus M_2^\bullet \to M_j^\bullet,\ j = 1,2.$$

From the short exact sequence of complexes

$$0 \to [M_2^\bullet \to Y^\bullet\rangle \to N^\bullet \xrightarrow{\lambda_1} M_1^\bullet \to 0$$

where $[M_2^\bullet \to Y^\bullet\rangle$ is acyclic because $\psi_2\colon M_2^\bullet \to Y^\bullet$ is a quasi-isomorphism, we see that λ_1 is a quasi-isomorphism, and similarly λ_2 is a quasi-isomorphism. An easy calculation shows that the following diagram commutes up to a homotopy:

$$\begin{array}{ccc} N^\bullet & \xrightarrow{\lambda_1} & M_1^\bullet \\ {\scriptstyle\lambda_2}\downarrow & & \downarrow{\scriptstyle\psi_1} \\ M_2^\bullet & \xrightarrow{\psi_2} & Y^\bullet. \end{array}$$

Now we apply (i) to the complex $N^\bullet$ and obtain a quasi-isomorphism $\varkappa\colon M_3^\bullet \to N^\bullet$, where $M_3^\bullet$ is a bounded complex of finitely generated (over $\mathbf{Z}$) torsion free $\mathfrak{g}$-modules. We set $\mu_j = \lambda_j \circ \varkappa\colon M_3^\bullet \to M_j^\bullet$. □

Proof of Proposition 9.3. Since the complex $\mathrm{UPic}(\overline{X})$ is bounded and, for X as in the proposition, has finitely generated cohomology, by Lemma 9.4(i) there is a bounded resolution $\psi\colon M^\bullet \to \mathrm{UPic}(\overline{X})$ consisting of finitely generated torsion free Galois modules. We have a canonical isomorphism

$$\psi^*\colon\ \mathrm{Ext}^i(\mathrm{UPic}(\overline{X}), \bar{k}^\times) \xrightarrow{\sim} \mathrm{Ext}^i(M^\bullet, \bar{k}^\times).$$

It is well known (see for example [BvH2, Lem. 1.5]) that there is a canonical isomorphism

$$\mathrm{Ext}^i(M^\bullet,\bar{k}^\times)\xrightarrow{\sim} H^i(k,\mathrm{Hom}^\bullet_{\mathbf{Z}}(M^\bullet,\bar{k}^\times)),$$

hence we obtain an isomorphism

$$\mathrm{Ext}^i(\mathrm{UPic}(\overline{X}),\bar{k}^\times)\xrightarrow{\sim} H^i(k,R\mathscr{H}om_{\mathbf{Z}}(\mathrm{UPic}(\overline{X}),\bar{k}^\times)),$$

which does not depend on the choice of the resolution $\psi\colon M^\bullet\to\mathrm{UPic}(\overline{X})$ by Lemma 9.4(ii). It now follows from Theorem 5.8 that we have a canonical isomorphism

$$\mathrm{Ext}^i(\mathrm{UPic}(\overline{X}),\bar{k}^\times)\cong H^i(k,R\mathscr{H}om_{\mathbf{Z}}([\mathbb{X}(\overline{G})\to\mathbb{X}(\overline{H})\rangle,\bar{k}^\times)).$$

Since $\bar{k}^\times$ is divisible,

$$\begin{aligned}R\mathscr{H}om_{\mathbf{Z}}([\mathbb{X}(\overline{G})\to\mathbb{X}(\overline{H})\rangle,\bar{k}^\times)&=\langle\mathrm{Hom}_{\mathbf{Z}}(\mathbb{X}(\overline{H}),\bar{k}^\times)\to\mathrm{Hom}_{\mathbf{Z}}(\mathbb{X}(\overline{G}),\bar{k}^\times)]\\&=\langle H^{\mathrm{m}}(\bar{k})\to G^{\mathrm{tor}}(\bar{k})].\end{aligned}$$

Thus we obtain a canonical isomorphism

$$\mathrm{Ext}^i(\mathrm{UPic}(\overline{X}),\bar{k}^\times)\cong H^i(k,\langle H^{\mathrm{m}}\to G^{\mathrm{tor}}]).$$

□

9.5. In [Bo3] the first-named author defined (by means of explicit cocycles) an obstruction class to the existence of a rational point on X. We recall the definition.

Fix $\bar{x}\in X(\bar{k})$. Let $\overline{H}\subset G_{\bar{k}}$ be the stabilizer of $\bar{x}$. For each $\sigma\in\mathrm{Gal}(\bar{k}/k)$ choose $g_\sigma\in G(\bar{k})$ such that ${}^\sigma\bar{x}=\bar{x}\cdot g_\sigma$. We may assume that the map $\sigma\mapsto g_\sigma$ is continuous (locally constant) on $\mathrm{Gal}(\bar{k}/k)$. We denote by $\hat{g}_\sigma\in G^{\mathrm{tor}}(\bar{k})$ the image of g_σ in $G^{\mathrm{tor}}(\bar{k})$. We obtain a continuous map $\hat{g}\colon\sigma\mapsto\hat{g}_\sigma$.

Let $\sigma,\tau\in\mathrm{Gal}(\bar{k}/k)$. Set $u_{\sigma,\tau}=g_{\sigma\tau}(g_\sigma{}^\sigma g_\tau)^{-1}\in G(\bar{k})$, then it is easy to check that $u_{\sigma,\tau}\in\overline{H}(\bar{k})$. We denote by $\hat{u}_{\sigma,\tau}\in H^{\mathrm{m}}(\bar{k})$ the image of $u_{\sigma,\tau}$ in $H^{\mathrm{m}}(\bar{k})$. We obtain a continuous map $\hat{u}\colon(\sigma,\tau)\mapsto\hat{u}_{\sigma,\tau}$. One can check that $(\hat{u},\hat{g})\in Z^1(k,H^{\mathrm{m}}\to G^{\mathrm{tor}})$. We denote by $\eta(G,X)\in H^1(H^{\mathrm{m}}\to G^{\mathrm{tor}})$ the hypercohomology class of the hypercocycle $(\hat{u},\hat{g})$, see [Bo3] for details.

Theorem 9.6. *Let X be a homogeneous space under a connected k-group G with* $\mathrm{Pic}(\overline{G})=0$. *Let $\overline{H}$ be a geometric stabilizer. Then $e(X)\in\mathrm{Ext}^1(\mathrm{UPic}(\overline{X}),\bar{k}^\times)$ coincides with $-\eta(G,X)\in H^1(k,\langle H^{\mathrm{tor}}\to G^{\mathrm{tor}}])$ under the identification*

$$\mathrm{Ext}^1(\mathrm{UPic}(\overline{X}),\bar{k}^\times)\cong H^1(k,\langle H^{\mathrm{m}}\to G^{\mathrm{tor}}])$$

of Proposition 9.3.

We shall prove Theorem 9.6 by *dévissage* in three steps, using an idea of [Bo1].

9.7. We prove Theorem 9.6 when G is a torus. The geometric stabilizer of a point $\bar{x}\in X(\bar{k})$ does not depend on $\bar{x}$ and is defined over k; we denote the corresponding k-group by H. We have $G^{\mathrm{tor}}=G$, $H^{\mathrm{m}}=H$. Set $T=G/H$, it is a k-torus, and X is a torsor under T. We have a canonical morphism of complexes of abelian k-groups $\lambda\colon\langle H\to G]\to T$, which is a quasi-isomorphism.

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1(\mathrm{UPic}(\overline{X}),\overline{k}^\times) & \xrightarrow{\beta_{G,X}} & H^1(k,\langle H\to G]) \\ \| & & \downarrow{\lambda_*} \\ \mathrm{Ext}^1(\mathrm{UPic}(\overline{X}),\overline{k}^\times) & \xrightarrow{\beta_{T,X}} & H^1(k,T) \end{array}$$

in which all the arrows are isomorphisms. Since $\eta(G,X)$ is functorial in (G,X), we see that $\lambda_*(\eta(G,X)) = \eta(T,X)$. By [BvH2, Thm. 5.5] $\beta_{T,X}(e(X)) = -\eta(T,X)$. It follows that $\beta_{G,X}(e(X)) = -\eta(G,X)$ (because λ_* is an isomorphism). □

9.8. Now we prove Theorem 9.6 assuming that the pair (G,X) satisfies the following condition:

$$H^{\mathrm{m}} \text{ embeds into } G^{\mathrm{tor}}. \tag{23}$$

Set $Y = X/G^{\mathrm{ssu}}$, see [Bo1, Lemma 3.1] for a proof that this quotient exists. Then the torus G^{tor} acts transitively on Y, and it follows from our condition (23) that the stabilizer in $\overline{G}^{\mathrm{tor}}$ of any point $\overline{y}\in\overline{Y}$ is $\overline{H}^{\mathrm{mult}}$. We have a morphism of pairs $t\colon (G,X)\to(G^{\mathrm{tor}},Y)$, which gives rise to a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1(\mathrm{UPic}(\overline{X}),\overline{k}^\times) & \xrightarrow{\beta_{G,X}} & H^1(k,\langle H^{\mathrm{m}}\to G^{\mathrm{tor}}]) \\ \downarrow{t_*} & & \downarrow{t_*} \\ \mathrm{Ext}^1(\mathrm{UPic}(\overline{Y}),\overline{k}^\times) & \xrightarrow{\beta_{G^{\mathrm{tor}},Y}} & H^1(k,\langle H^{\mathrm{m}}\to G^{\mathrm{tor}}]). \end{array}$$

Since G^{tor} is a torus, by 9.7 we have $\beta_{G^{\mathrm{tor}},Y}(e(Y)) = -\eta(G^{\mathrm{tor}},Y)$. The right hand vertical map t_* is the identity map. It follows from the functoriality of $e(X)$ and $\eta(G,X)$ that $\beta_{G,X}(e(X)) = -\eta(G,X)$. □

In order to complete the proof we need a construction of [Bo1, Section 4], see also [BCS, Proof of Thm. 3.5].

Proposition 9.9. *Let X be a homogeneous space of a connected k-group G over a field k of characteristic 0. Then there exists a morphism of pairs $(\varphi,\pi)\colon (F,Z)\to(G,X)$, where Z is a homogeneous space of a k-group F, with the following properties:*

(a) $F = G\times P$, and $\varphi\colon G\times P\to G$ is the projection, where P is a quasi-trivial k-torus;

(b) $\pi\colon Z\to X$ is a torsor under P, where P acts on Z through the injection $P\hookrightarrow G\times P = F$;

(c) the pair (F,Z) satisfies the condition (23) *of* §*9.8.*

Proof. (cf. [BCS, Proof of Thm. 3.5].) Let $\overline{x}\in X(\overline{k})$ be a $\overline{k}$-point with stabilizer $\overline{H}$. Let $\mu\colon \overline{H}\to\overline{H}^{\mathrm{mult}}$ be the canonical surjection. Let $\xi\in H^2(k,H^{\mathrm{m}})$ be the image of $\eta(G,X)\in H^1(k,\langle H^{\mathrm{m}}\to G^{\mathrm{tor}}])$. By [BCS, Lemma 3.7] there exists an embedding $j\colon H^{\mathrm{m}}\hookrightarrow P$ of H^{m} into a quasi-trivial torus P such that $j_*(\xi)=0$.

Consider the k-group $F = G\times P$, and the embedding

$$\overline{H}\hookrightarrow\overline{F} \quad \text{given by} \quad h\mapsto (h, j(\mu(h))).$$

Set $\overline{Z} = \overline{H}\backslash\overline{F}$. We have a right action $\overline{a}\colon \overline{Z}\times\overline{F}\to\overline{Z}$ and an $\overline{F}$-equivariant map

$$\overline{\pi}\colon \overline{Z}\to\overline{X}, \quad \overline{H}\cdot(g,p)\mapsto\overline{H}\cdot g, \quad \text{where } g\in\overline{G},\ p\in\overline{P}.$$

Then $\overline{Z}$ is a homogeneous space of $\overline{F}$ with respect to the action $\overline{a}$, and the map $\overline{\pi}\colon \overline{Z} \to \overline{X}$ is a torsor under $\overline{P}$. The homomorphism $M \to F^{\mathrm{tor}}$ is injective.

In [Bo1, 4.7] it was proved that $\mathrm{Aut}_{\overline{F},\overline{X}}(\overline{Z}) = P(\overline{k})$. By [Bo1, Lemma 4.8] the element $j_*(\xi) \in H^2(k,P)$ is the only obstruction to the existence of a k-form (Z,a,π) of the triple $(\overline{Z},\overline{a},\overline{\pi})$: there exists such a k-form if and only if $j_*(\xi) = 0$. In our case by construction we have $j_*(\xi) = 0$, hence there exists a k-form (Z,a,π) of $(\overline{Z},\overline{a},\overline{\pi})$. This completes the proof of the proposition.

Note that in [Bo1] and [BCS] we always assumed that $\overline{H}_1$ was connected and had no nontrivial characters, but this assumption was not used in the cited constructions. □

9.10. We prove Theorem 9.6 in the general case. Consider a morphism of pairs

$$(\varphi,\pi)\colon (F,Z) \to (G,X)$$

as in Proposition 9.9. Since $\mathrm{Pic}(\overline{G}) = 0$ and $F = G \times P$, we have $\mathrm{Pic}(\overline{F}) = 0$. We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^1(\mathrm{UPic}(\overline{Z}),\overline{k}^\times) & \xrightarrow{\beta_{F,Z}} & H^1(k,\langle H^{\mathrm{m}} \to F^{\mathrm{tor}}]) \\
\downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \varphi_*} \\
\mathrm{Ext}^1(\mathrm{UPic}(\overline{X}),\overline{k}^\times) & \xrightarrow{\beta_{G,X}} & H^1(k,\langle H^{\mathrm{m}} \to G^{\mathrm{tor}}]).
\end{array}$$

Since the pair (F,Z) satisfies the condition (23), by 9.8 $\beta_{F,Z}(e(Z)) = -\eta(F,Z)$. It follows easily from the functoriality of $e(X)$ and $\eta(G,X)$ that $\beta_{G,X}(e(X)) = -\eta(G,X)$. This completes the proof of Theorem 9.6. □

Using Theorem 9.6 we can give new proofs for the following results of [BCS].

Corollary 9.11 ([BCS, Theorem 3.5])**.** *Let k be a p-adic field. Let X be a homogeneous space under a connected (linear) k-group G with connected geometric stabilizers. Then $X(k) \neq \emptyset$ if and only if $e(X) = 0$.*

Proof. By [Bo1, Lemma 5.2] we may assume that $\mathrm{Pic}(\overline{G}) = 0$. If $X(k) \neq \emptyset$, then clearly $e(X) = 0$. Conversely, if $e(X) = 0$, then by Theorem 9.6 $\eta(G,X) = 0$, hence by [Bo3, Thm. 2.1] $X(k) \neq \emptyset$. □

Corollary 9.12 ([BCS, Theorem 3.10])**.** *Let k be a number field. Let X be a homogeneous space under a connected (linear) k-group G with connected geometric stabilizers. Assume $X(k_v) \neq \emptyset$ for every real place v of k. Then $X(k) \neq \emptyset$ if and only if $e(X) = 0$.*

Proof. Similar to the proof of Corollary 9.11, but using [Bo3, Cor. 2.3] instead of [Bo3, Thm. 2.1]. □

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