MODULI OF SYMPLECTIC INSTANTON VECTOR BUNDLES OF HIGHER RANK ON PROJECTIVE SPACE $\mathbb{P}^3$

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ABSTRACT. Symplectic instanton vector bundles on the projective space $\mathbb{P}^3$ constitute a natural generalization of mathematical instantons of rank 2. We study the moduli space $I_{n,r}$ of rank-$2r$ symplectic instanton vector bundles on $\mathbb{P}^3$ with $r \geq 2$ and second Chern class $n \geq r$, $n \equiv r(\mathrm{mod}2)$. We give an explicit construction of an irreducible component $I^*_{n,r}$ of this space for each such value of n and show that $I^*_{n,r}$ has the expected dimension $4n(r+1) - r(2r+1)$.

1. Introduction

By a *symplectic instanton vector bundle* of rank $2r$ and charge n (shortly, a *symplectic* (n,r)-*instanton*) on the 3-dimensional projective space $\mathbb{P}^3$ we understand an algebraic vector bundle $E = E_{2r}$ of rank $2r$ on $\mathbb{P}^3$ with Chern classes

$$c_1(E) = 0, \tag{1}$$

$$c_2(E) = n, \quad n \geq 1, \tag{2}$$

supplied with a symplectic structure and satisfying the vanishing conditions

$$h^0(E) = h^1(E \otimes \mathcal{O}_{\mathbb{P}^3}(-2)) = 0. \tag{3}$$

By a symplectic structure we mean an anti-self-dual isomorphism

$$\phi : \ E \xrightarrow{\simeq} E^\vee, \quad \phi^\vee = -\phi, \tag{4}$$

considered modulo proportionality. The vanishing of the first Chern class (1) follows from the existence of a symplectic structure (4), and if $r = 1$, then the two conditions are equivalent. We will denote the moduli space of symplectic (n,r)-instantons by $I_{n,r}$.

For $r = 1$ these bundles relate, via the so-called Atiyah-Ward correspondence, to rank-2 "physical" instantons over the 4-sphere S^4, these being anti-self-dual connections with structure group $SU(2) = \mathbf{Sp}(1)$ [AW]. Important results on the moduli spaces $I_n = I_{n,1}$ of rank-2 instantons have been obtained recently: smoothness [JV] for all n, irreducibility [T] for odd n.

Much less is known about the moduli spaces $I_{n,r}$ for $r > 1$. In fact the symplectic instantons with $r > 1$ are as natural as those with $r = 1$, for they are related, via the same Atiyah-Ward correspondence, to the anti-self-dual connections over S^4 with structure group $\mathbf{Sp}(r)$, see [A]. As far as we know, the present paper is the first one addressing the properties of the corresponding spaces $I_{n,r}$. The tool we use to construct $I_{n,r}$ is the monad method; it originates in the work of Horrocks [H] and is known as the ADHM construction of instantons since [ADHM]. It was further sharpened in the work of Barth [B], Barth and Hulek [BH] and Tyurin [Tju1], [Tju2]. This method permits to encode the instantons, usual ones or symplectic of higher rank, by hyperwebs of quadrics.

For a sample of the physical literature about symplectic instantons, see e.g. [Mc].

We fix basic terminology and notation in Section 2 and introduce the hyperwebs of quadrics in Section 3. We prove that, for any $r \geq 2$ and for any $n \geq r$ such that $n \equiv r (\mathrm{mod}\, 2)$, the moduli space $I_{n,r}$ is nonempty and is realized as a free quotient $MI_{n,r}/(GL(n)/\pm \mathrm{id})$, where $MI_{n,r}$ is a Zariski locally closed subset of an affine space (see Theorem 3.1). Thus $MI_{n,r}$ carries a natural structure of a reduced scheme, and $I_{n,r}$ is an algebraic space. In Section 4 we give an explicit construction of vector bundles from $I_{n,r}$ for the above values of n and r and introduce a component $I^*_{n,r}$ of $I_{n,r}$ characterized by a certain open condition (*), see Definition 4.6. In Section 5 we prove Theorem 5.3 on the irreducibility of $I^*_{n,r}$, the main result of this paper.

Acknowledgements. D. M. was partially supported by the grant VHSMOD-2009 No. ANR-09-BLAN-0104-01, and U. B. by PRIN "Geometria delle varietà algebriche e dei loro spazi di moduli". U. B. and A. S. T. acknowledge support and hospitality of the Max Planck Institute for Mathematics in Bonn, where they started the work on this paper during their stay in winter 2011. U. B. is a member of the VBAC group.

2. Notation and conventions

In many respects, we follow the exposition of [T], and we stick to the notation introduced therein. The base field $\mathbf{k}$ is assumed to be algebraically closed of characteristic 0. We identify vector bundles with locally free sheaves. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on an algebraic variety or a scheme X, then $n\mathcal{F}$ denotes a direct sum of n copies of $\mathcal{F}$, $H^i(\mathcal{F})$ denotes the i^{th} cohomology group of $\mathcal{F}$, $h^i(\mathcal{F}) := \dim H^i(\mathcal{F})$, and $\mathcal{F}^\vee$ denotes the dual of $\mathcal{F}$, that is, $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. If $X = \mathbb{P}^r$ and t is an integer, then by $\mathcal{F}(t)$ we denote the sheaf $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r}(t)$. $[\mathcal{F}]$ will denote the isomorphism class of a sheaf $\mathcal{F}$. For any morphism of $\mathcal{O}_X$-sheaves $f : \mathcal{F} \to \mathcal{F}'$ and any $\mathbf{k}$-vector space U (respectively, for any homomorphism $f : U \to U'$ of $\mathbf{k}$-vector spaces) we will denote, for short, by the same letter f the induced morphism of sheaves $id \otimes f : U \otimes \mathcal{F} \to U \otimes \mathcal{F}'$ (respectively, the induced morphism $f \otimes id : U \otimes \mathcal{F} \to U' \otimes \mathcal{F}$).

We fix an integer $n \geq 1$ and denote by H_n a fixed n-dimensional vector space over $\mathbf{k}$. Throughout the paper, V will be a fixed vector space of dimension 4 over $\mathbf{k}$, and we set $\mathbb{P}^3 := P(V)$. We reserve the letters u and v for denoting the two morphisms in the Euler exact sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{u} V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{v} T_{\mathbb{P}^3}(-1) \to 0$. For any $\mathbf{k}$-vector spaces U and W and any vector $\phi \in \mathrm{Hom}(U, W \otimes \wedge^2 V^\vee) \subset \mathrm{Hom}(U \otimes V, W \otimes V^\vee)$ understood as a linear map $\phi : U \otimes V \to W \otimes V^\vee$ or, equivalently, as a map ${}^\sharp\phi : U \to W \otimes \wedge^2 V^\vee$, we will denote by $\widetilde{\phi}$ the composition $U \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{{}^\sharp\phi} W \otimes \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} W \otimes \Omega_{\mathbb{P}^3}(2)$, where ϵ is the induced morphism in the exact triple $0 \to \wedge^2 \Omega_{\mathbb{P}^3}(2) \xrightarrow{\wedge^2 v^\vee} \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} \Omega_{\mathbb{P}^3}(2) \to 0$ obtained by taking the second wedge power of the dual Euler exact sequence.

Given an integer $m \geq 1$, we denote by $\mathbf{S}_m$ (resp. $\mathbf{\Sigma}_{m+1}$) the vector space $S^2 H_m^\vee \otimes \wedge^2 V^\vee$ (resp. $\mathrm{Hom}(H_m, H_{m+1}^\vee \otimes \wedge^2 V^\vee)$). Abusing notation, we will denote by the same symbol a $\mathbf{k}$-vector space, say U, and the associated affine space $\mathbf{V}(U^\vee) = \mathrm{Spec}(Sym^* U^\vee)$.

All the schemes considered in the paper are Noetherian. By a general point of an irreducible (but not necessarily reduced) scheme $\mathcal{X}$ we mean any closed point of some dense open subset of $\mathcal{X}$. An irreducible scheme is called generically reduced if it is reduced at a general point.

3. Generalities on symplectic instantons and definition of $MI_{n,r}$

In this section we enumerate some facts about symplectic instantons which are completely parallel to those for rank-2 usual instantons, see [T, Section 3].

For a given symplectic (n,r)-instanton E, the first condition (3) yields $h^0(E(-i))=0, i\geq 0$, which, together with the exact triple $0\to E(-j-1)\to E(-j)\to E(-j)|_{\mathbb{P}^2}\to 0$ for $j=0$ and (3), implies that $h^0(E(-1)|_{\mathbb{P}^2})=0$, hence also $h^0(E(-i)|_{\mathbb{P}^2})=0,\ i\geq 1$. The last equality for $i=2$, together with (3) and the above triple for $j=2$, gives $h^1(E(-3))=0$, hence also $h^1(E(-4))=0$. Then, from Serre duality and (4), we deduce:

(5) $h^i(E)=h^i(E(-1))=h^{3-i}(E(-3))=h^{3-i}(E(-4))=0,\quad i\neq 1,\quad h^i(E(-2))=0,\quad i\geq 0.$

By Riemann-Roch and (3), (5), we have

(6) $$h^1(E(-1))=h^2(E(-3))=n,\ h^1(E)=h^2(E(-4))=2n-2r.$$

By tensoring the dual Euler sequence by E we also obtain

(7) $$h^1(E\otimes\Omega^1_{\mathbb{P}^3})=h^2(E\otimes\Omega^2_{\mathbb{P}^3})=2n+2r,$$

Consider a triple (E,f,ϕ) where E is a (n,r)-instanton, $f:H_n\overset{\simeq}{\to}H^2(E(-3))$ an isomorphism and $\phi:E\overset{\simeq}{\to}E^\vee$ a symplectic structure on E. Two triples (E,f,ϕ) and $(E'f',\phi')$ are called equivalent if there is an isomorphism $g:E\overset{\simeq}{\to}E'$ such that $g_*\circ f=\lambda f'$ with $\lambda\in\{1,-1\}$ and $\phi=g^\vee\circ\phi'\circ g$, where $g_*:H^2(E(-3))\overset{\simeq}{\to}H^2(E'(-3))$ is the induced isomorphism. We denote by $[E,f,\phi]$ the equivalence class of a triple (E,f,ϕ). It follows from this definition that the set $F_{[E]}$ of all equivalence classes $[E,f,\phi]$ with given $[E]$ is a homogeneous space of the group $GL(H_n)/\{\pm\mathrm{id}\}$.

Each class $[E,f,\phi]$ defines a point

(8) $$A=A([E,f,\phi])\in S^2H_n^\vee\otimes\wedge^2V^\vee$$

in the following way. Consider the exact sequences

(9) $$0\to\Omega^1_{\mathbb{P}^3}\overset{i_1}{\to}V^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\to\mathcal{O}_{\mathbb{P}^3}\to 0,$$
$$0\to\Omega^2_{\mathbb{P}^3}\to\wedge^2V^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(-2)\to\Omega^1_{\mathbb{P}^3}\to 0,$$
$$0\to\wedge^4V^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(-4)\to\wedge^3V^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(-3)\overset{i_2}{\to}\Omega^2_{\mathbb{P}^3}\to 0,$$

induced by the Koszul complex of $V^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\overset{ev}{\twoheadrightarrow}\mathcal{O}_{\mathbb{P}^3}$. Twisting these sequences by E and taking into account (3), (5)-(7), we obtain the vanishing

(10) $$h^0(E\otimes\Omega_{\mathbb{P}^3})=h^3(E\otimes\Omega^2_{\mathbb{P}^3})=h^2(E\otimes\Omega_{\mathbb{P}^3})=0$$

and the diagram with exact rows

(11) $$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2(E(-4))\otimes\wedge^4V^\vee & \longrightarrow & H^2(E(-3))\otimes\wedge^3V^\vee & \xrightarrow{i_2} & H^2(E\otimes\Omega^2_{\mathbb{P}^3}) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle A'} & & \cong\uparrow{\scriptstyle\partial} & & \\ 0 & \longleftarrow & H^1(E)) & \longleftarrow & H^1(E(-1))\otimes V^\vee & \xleftarrow{i_1} & H^1(E\otimes\Omega_{\mathbb{P}^3}) & \longleftarrow & 0, \end{array}$$

where $A':=i_1\circ\partial^{-1}\circ i_2$. The Euler exact sequence (9) yields the canonical isomorphism $\omega_{\mathbb{P}^3}\overset{\simeq}{\to}\wedge^4V^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(-4)$, and fixing an isomorphism $\tau:\mathbf{k}\overset{\simeq}{\to}\wedge^4V^\vee$ we have the isomorphisms $\tilde{\tau}:V\overset{\simeq}{\to}\wedge^3V^\vee$ and $\hat{\tau}:\omega_{\mathbb{P}^3}\overset{\simeq}{\to}\mathcal{O}_{\mathbb{P}^3}(-4)$. We define A in (8) as the composition

(12) $$A:H_n\otimes V\overset{\tilde{\tau}}{\underset{\simeq}{\to}}H_n\otimes\wedge^3V^\vee\overset{f}{\underset{\simeq}{\to}}H^2(E(-3))\otimes\wedge^3V^\vee\overset{A'}{\to}H^1(E(-1))\otimes V^\vee\overset{\phi}{\underset{\simeq}{\to}}$$

$$\overset{\phi}{\overset{\simeq}{\to}} H^1(E^\vee(-1))\otimes V^\vee \overset{SD}{\overset{\simeq}{\to}} H^2(E(1)\otimes\omega_{\mathbb{P}^3})^\vee\otimes V^\vee \overset{\hat\tau}{\overset{\simeq}{\to}} H^2(E(-3))^\vee\otimes V^\vee \overset{f^\vee}{\overset{\simeq}{\to}} H_n^\vee\otimes V^\vee,$$

where SD is the Serre duality isomorphism. One can verify that A is a skew symmetric map which depends only on the class $[E,f,\phi]$, but does not depend on the choice of τ, and that $A\in\wedge^2(H_n^\vee\otimes V^\vee)$ lies in the direct summand $\mathbf{S}_n = S^2H_n^\vee\otimes\wedge^2V^\vee$ of the canonical decomposition

$$\wedge^2\,(H_n^\vee\otimes V^\vee) = S^2H_n^\vee\otimes\wedge^2V^\vee\oplus\wedge^2H_n^\vee\otimes S^2V^\vee. \tag{13}$$

Here $\mathbf{S}_n$ is the space of hyperwebs of quadrics in H_n. For this reason we call A the (n,r)-*instanton hyperweb of quadrics* corresponding to the data $[E,f,\phi]$.

Denote $W_A := H_n\otimes V/\ker A$. Using the above chain of isomorphisms we can rewrite the diagram (11) as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker A & \longrightarrow & H_n\otimes V & \xrightarrow{c_A} & W_A & \longrightarrow & 0\\ & & & & \downarrow A & & \cong\downarrow q_A & & \\ 0 & \longleftarrow & \ker A^\vee & \longleftarrow & H_n^\vee\otimes V^\vee & \xleftarrow{c_A^\vee} & W_A^\vee & \longleftarrow & 0. \end{array} \tag{14}$$

In view of (7), $\dim W_A = 2n+2r$ and $q_A : W_A \overset{\simeq}{\to} W_A^\vee$ is a skew-symmetric isomorphism. An important property of $A = A([E,f,\phi])$ is that the induced morphism of sheaves

$$a_A^\vee : W_A^\vee\otimes\mathcal{O}_{\mathbb{P}^3} \overset{c_A^\vee}{\to} H_n^\vee\otimes V^\vee\otimes\mathcal{O}_{\mathbb{P}^3} \overset{ev}{\to} H_n^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1) \tag{15}$$

is surjective and the composition $H_n\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\overset{a_A}{\to} W_A\otimes\mathcal{O}_{\mathbb{P}^3}\overset{q_A}{\to} W_A^\vee\otimes\mathcal{O}_{\mathbb{P}^3}\overset{a_A^\vee}{\to} H_n^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1)$ is zero. Applying Beilinson spectral sequence [Bei] to $E(-1)$, we see that $E\simeq\ker(a_A^\vee\circ q_A)/\operatorname{Im} a_A$. Thus A defines a monad

$$\mathcal{M}_A:\quad 0\to H_n\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\overset{a_A}{\to} W_A\otimes\mathcal{O}_{\mathbb{P}^3}\overset{a_A^\vee\circ q_A}{\to} H_n^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1)\to 0\ , \tag{16}$$

whose cohomology sheaf

$$E_{2r}(A) := \ker(a_A^\vee\circ q_A)/\operatorname{Im} a_A. \tag{17}$$

is isomorphic to E. Twisting $\mathcal{M}_A$ by $\mathcal{O}_{\mathbb{P}^3}(-3)$ and using (17), we obtain the isomorphism $f : H_n\overset{\simeq}{\to} H^2(E(-3))$. Furthermore, the fact that q_A is symplectic implies that there is a canonical isomorphism of $\mathcal{M}_A$ with its dual which induces the symplectic isomorphism $\phi : E\overset{\simeq}{\to} E^\vee$. Thus, the data $[E,f,\phi]$ are recovered from A. This leads to the following description of the moduli space $I_{n,r}$. Consider the *set of (n,r)-instanton hyperwebs of quadrics*

$$MI_{n,r} := \left\{ A\in\mathbf{S}_n \;\middle|\; \begin{array}{rl} \text{(i)} & rk(A: H_n\otimes V\to H_n^\vee\otimes V^\vee) = 2n+2r,\\ \text{(ii)} & \text{the morphism } a_A^\vee : W_A^\vee\otimes\mathcal{O}_{\mathbb{P}^3}\to H_n^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1) \text{ defined}\\ & \text{by } A \text{ in (15) is surjective,}\\ \text{(iii)} & h^0(E_{2r}(A)) = 0, \text{ where } E_{2r}(A) = \ker(a_A^\vee\circ q_A)/\operatorname{Im} a_A\\ & \text{and } q_A : W_A\overset{\simeq}{\to} W_A^\vee \text{ is a symplectic isomorphism}\\ & \text{associated to } A \text{ by (14).} \end{array} \right\} \tag{18}$$

It is a locally closed subscheme of the affine space $\mathbf{S}_n$.

Theorem 3.1. *The natural morphism*

$$\pi_{n,r} : MI_{n,r}\to I_{n,r},\ A\mapsto [E_{2r}(A)], \tag{19}$$

is a principal $GL(H_n)/\{\pm\mathrm{id}\}$-bundle in the étale topology. Hence $I_{n,r}$ is a quotient stack $MI_{n,r}/(GL(H_n)/\{\pm\mathrm{id}\})$, making it an algebraic space.

Proof. See [T, Section 3]. □

Each fibre $F_{[E]} = \pi_n^{-1}([E])$ over an arbitrary point $[E] \in I_{n,r}$ is a principal homogeneous space of the group $GL(H_n)/\{\pm \mathrm{id}\}$. Hence the irreducibility of $(I_{n,r})_{red}$ is equivalent to the irreducibility of the scheme $(MI_{n,r})_{red}$.

We can also state:

Theorem 3.2. *For each $n \geq 1$, the space $MI_{n,r}$ of (n,r)-instanton hyperwebs of quadrics is a locally closed subscheme of the vector space $\mathbf{S}_n$ given locally at any point $A \in MI_{n,r}$ by*

$$\binom{2n-2r}{2} = 2n^2 - n(4r+1) + r(2r+1) \tag{20}$$

equations obtained as the rank condition (i) in (18).

Note that from (20) it follows that

$$\dim_{[A]} MI_{n,r} \geq \dim \mathbf{S}_n - (2n^2 - n(4r+1) + r(2r+1)) = n^2 + 4n(r+1) - r(2r+1) \tag{21}$$

at any point $A \in MI_{n,r}$. Hence,

$$\dim_{[E]} I_{n,r} \geq 4n(r+1) - r(2r+1) \tag{22}$$

at any point $[E] \in I_{n,r}$, since $MI_{n,r} \to I_{n,r}$ is a principal $GL(H_n)/\{\pm \mathrm{id}\}$-bundle in the étale topology.

4. Explicit construction of symplectic instantons

4.1. Example: symplectic (n,n)-instantons. In this subsection we recall some known facts about symplectic (n,n)-instantons and their relation to usual rank-2 instantons, see [T, Sections 5-6]. We first show that each invertible hyperweb of quadrics $A \in \mathbf{S}_n$ naturally leads to a construction of a symplectic (n,n)-instanton $E_{2n}(A)$ on $\mathbb{P}^3$. Given an integer $n \geq 1$, set

$$\mathbf{S}_n^0 := \{A \in \mathbf{S}_n \mid A : H_n \otimes V \to H_n^\vee \otimes V^\vee \text{ is an invertible map}\}. \tag{23}$$

Then $\mathbf{S}_n^0$ is a dense open subset of $\mathbf{S}_n$, and it is easy to see that for any $A \in \mathbf{S}_n^0$ the following conditions are satisfied.

(1) The morphism $\widetilde{A} : H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to H_n^\vee \otimes \Omega_{\mathbb{P}^3}(1)$ induced by A is a subbundle embedding, and

$$E_{2n}(A) := \mathrm{coker}(\widetilde{A}) \tag{24}$$

is a symplectic (n,n)-instanton, that is,

$$[E_{2n}(A)] \in I_{n,n}. \tag{25}$$

(2) For all $i \geq 0$,

$$h^i(E_{2n}(A)) = h^i(E_{2n}(A)(-2)) = 0. \tag{26}$$

This follows from the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & H_n\otimes\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\widetilde{A}} & H_n^\vee\otimes\Omega_{\mathbb{P}^3}(1) & \xrightarrow{e} & E_{2n}(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v^\vee} & & & & \\
 & & H_n\otimes V\otimes\mathcal{O}_{\mathbb{P}^3} & \xrightarrow[\simeq]{A} & H_n^\vee\otimes V^\vee\otimes\mathcal{O}_{\mathbb{P}^3} & & & & \\
 & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u^\vee} & & & & \\
0\to E_{2n}(A)^\vee & \longrightarrow & H_n\otimes T_{\mathbb{P}^3}(-1) & \xrightarrow{\widetilde{A}^\vee} & H_n^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{27}
$$

Thus $\mathbf{S}_n^0\subset MI_{n,n}$. In fact, the following result is true.

Proposition 4.1. $\mathbf{S}_n^0=MI_{n,n}$. *In particular,* $MI_{n,n}$ *is irreducible of dimension* $3n^2+3n$, *and hence* $I_{n,n}$ *is irreducible of dimension* $2n^2+3n$.

Proof. We have to show that $MI_{n,n}\subset\mathbf{S}_n^0$. Let $A\in MI_{n,n}$. Since $n=r$, by condition (i) from (18) the rank of the hyperweb of quadrics $A:H_n\otimes V\to H_n^\vee\otimes V^\vee$ is $2n+2r=4n=\dim H_n^\vee\otimes V^\vee$, hence A is invertible. By (23), this means that $A\in\mathbf{S}_n^0$. $\square$

Now we proceed to spell out the relation between symplectic (n,n)-instantons and usual rank-2 instantons with second Chern class $2n-1$. This relation is given at the level of spaces of hyperwebs of quadrics $MI_{n,n}$ and $MI_{2n-1,1}$ interpreted as spaces of monads.

We need some more notation. Let $B\in\mathbf{S}_n^0$. By definition, B is an invertible anti-self-dual map $H_n\otimes V\to H_n^\vee\otimes V^\vee$. Then the inverse

$$B^{-1}:H_n^\vee\otimes V^\vee\to H_n\otimes V \tag{28}$$

is also anti-self-dual. Consider the vector space $\mathbf{\Sigma}_n=H_n^\vee\otimes H_{n-1}^\vee\otimes\wedge^2V^\vee$. An element $C\in\mathbf{\Sigma}_n$ can be viewed as a linear map $C:H_{n-1}\otimes V\to H_n^\vee\otimes V^\vee$, and its transpose $C^\vee$ as a map $C^\vee:H_n\otimes V\to H_{n-1}^\vee\otimes V^\vee$. As the composition $C^\vee\circ B^{-1}\circ C$ is anti-self-dual, we can consider it as an element of $\wedge^2(H_{n-1}^\vee\otimes V^\vee)\simeq\mathbf{S}_{n-1}\oplus\wedge^2H_{n-1}^\vee\otimes S^2V^\vee$ (cf. (13)). Thus the condition

$$C^\vee\circ B^{-1}\circ C\in\mathbf{S}_{n-1} \tag{29}$$

makes sense.

Next, consider the upper horizontal triple in (27) with $A=B$. Twisting it by $\mathcal{O}_{\mathbb{P}^3}(1)$ and passing to global sections we obtain the exact triple

$$0\to H_n\overset{\sharp B}{\to}H_n^\vee\otimes\wedge^2V^\vee\overset{\epsilon(B)}{\to}H^0(E_{2n}(B)(1))\to 0 \tag{30}$$

Besides, interpreting $C\in\mathbf{\Sigma}_n$ as a map $^\sharp C:H_{n-1}\to H_n^\vee\otimes\wedge^2V^\vee$, we obtain the composition $H_{n-1}\overset{\sharp C}{\to}H_n^\vee\otimes\wedge^2V^\vee\overset{\epsilon(B)}{\to}H^0(E_{2n}(B)(1))$ which induces the morphism of sheaves

$$\rho_{B,C}:\ H_{n-1}\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\to E_{2n}(B). \tag{31}$$

Note also that the maps $B:H_n\otimes V\to H_n^\vee\otimes V^\vee$ and $C:H_{n-1}\otimes V\to H_n^\vee\otimes V^\vee$ provide a map $(H_n\oplus H_{n-1})\otimes V\to H_n^\vee\otimes V^\vee$, which induces the morphism of sheaves

$$\tau_{B,C}:\ (H_n\oplus H_{n-1})\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\to H_n^\vee\otimes V^\vee\otimes\mathcal{O}_{\mathbb{P}^3}. \tag{32}$$

Now set

$$X_n := \left\{ (B, C) \in \mathbf{S}_n^0 \times \mathbf{\Sigma}_n \;\middle|\; \begin{array}{r l} \text{(i)} & \text{the condition (29) is satisfied,} \\ \text{(ii)} & \rho_{B,C} \text{ in (31) is a subbundle inclusion,} \\ \text{(iii)} & \tau_{B,C} \text{ in (32) is a subbundle inclusion.} \end{array} \right\} \tag{33}$$

By definition, X_n is a locally closed subset of $\mathbf{S}_n^0 \times \mathbf{\Sigma}_n$. Hence it is naturally endowed with a structure of a reduced scheme.

Now for any direct sum decomposition

$$\xi : H_{2n-1} \overset{\simeq}{\to} H_n \oplus H_{n-1}, \tag{34}$$

we obtain the corresponding decomposition

$$\tilde{\xi} : \mathbf{S}_{2n-1} \overset{\simeq}{\to} \mathbf{S}_n \oplus \mathbf{\Sigma}_n \oplus \mathbf{S}_{n-1} : A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)). \tag{35}$$

Thus, considering the set $MI_{2n-1,1}$ of $(2n-1)$-instanton hyperwebs of quadrics as a subset of $\mathbf{S}_{2n-1}$, we obtain a natural projection

$$f_n : MI_{2n-1,1} \to \mathbf{S}_n \oplus \mathbf{\Sigma}_n : A \mapsto (A_1(\xi), A_2(\xi)). \tag{36}$$

The following result is proved in [T, Theorems 1.1, 6.1 and Remark 7.6].

Proposition 4.2. *For a general decomposition ξ in (34), there exists a dense open subset $MI_{2n-1,1}(\xi)$ of $MI_{2n-1,1}$ such that the projection f_n in (36) induces an isomorphism or integral schemes*

$$f_n : MI_{2n-1,1}(\xi) \overset{\simeq}{\to} X_n : A \mapsto (A_1(\xi), A_2(\xi)). \tag{37}$$

The inverse isomorphism is given by the formula

$$f_n^{-1} : \; X_n \overset{\simeq}{\to} MI_{2n-1,1}(\xi) : \; (B, C) \mapsto \; \tilde{\xi}^{-1}(B, \; C, \; -C^\vee \circ B^{-1} \circ C). \tag{38}$$

Besides, the projection

$$pr_1 : \; X_n \to \mathbf{S}_n^0 : \; (B, C) \mapsto B \tag{39}$$

is dominant.

It is not hard to check that the morphism $\rho_{B,C} : H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to E_{2n}(B)$ defined in (31) satisfies the condition ${}^t\rho_{B,C} \circ \rho_{B,C} = 0$, where ${}^t\rho_{B,C}$ is the composition

$$ {}^t\rho_{B,C} : E_{2n}(B) \underset{\simeq}{\overset{\phi}{\to}} E_{2n}(B)^\vee \xrightarrow{\rho_{B,C}^\vee} H_{n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) $$

and ϕ is a symplectic structure on $E_{2n}(B)$ (cf. [T, formulas (71)-(72)]). In other words, we obtain an anti-self-dual monad

$$0 \to H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\rho_{B,C}} E_{2n}(B) \underset{\simeq}{\overset{\phi}{\to}} E_{2n}(B)^\vee \xrightarrow{\rho_{B,C}^\vee} H_{n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \tag{40}$$

with cohomology sheaf

$$E_2(A) = E_2(B, C) := \ker {}^t\rho_{B,C} / \operatorname{im} \rho_{B,C}, \quad A = f_n^{-1}(B, C). \tag{41}$$

Next, by (19) we have the natural projection

$$\pi_{2n-1,1} : MI_{2n-1,1} \to I_{2n-1,1} : \; A \mapsto [E_2(A)]. \tag{42}$$

We have the following interpretation of the isomorphism (38) on the level of vector bundles:

$$[E_2(B, C)] = \pi_{2n-1,1}(f_n^{-1}(B, C)). \tag{43}$$

Remark 4.3. Note that, according to the definitions (16)-(18) of $MI_{2n-1,1}$ and $MI_{n,n}$, for any $A \in MI_{2n-1,1}$, if $B = A_1(\xi)$ is defined by the direct sum decomposition (35), one has two other anti-self-dual monads

$$\mathcal{M}_A : \quad 0 \to H_{2n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{a_A}{\to} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \overset{a_A^\vee \circ q_A}{\to} H_{2n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \tag{44}$$

$$\mathcal{M}_B : \quad 0 \to H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{a_B}{\to} W_B \otimes \mathcal{O}_{\mathbb{P}^3} \overset{a_B^\vee \circ q_B}{\to} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \tag{45}$$

with cohomology sheaves

$$E_2(A) = \ker(a_A^\vee \circ q_A)/\mathrm{im}\, a_A, \;\; E_{2n}(B) = \ker(a_B^\vee \circ q_B)/\mathrm{im}\, a_B \tag{46}$$

respectively. Moreover, (40) and (41) provide an isomorphism $w: \; W_B = H^2(E_2(B) \otimes \Omega_{\mathbb{P}^3}) \overset{\simeq}{\to} H^2(E_{2n}(A) \otimes \Omega_{\mathbb{P}^3}) = W_A$. We thus obtain a commutative anti-self-dual diagram relating these monads:
(47)

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{a_B} & W_B \otimes \mathcal{O}_{\mathbb{P}^3} & \underset{\cong}{\xrightarrow{q_B}} & W_B^\vee \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{a_B^\vee} & H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0 \\
 & & \big\downarrow i_\xi & & \cong \big\downarrow w & & w^\vee \big\uparrow \cong & & i_\xi^\vee \big\uparrow & & \\
0 & \longrightarrow & H_{2n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{a_A} & W_A \otimes \mathcal{O}_{\mathbb{P}^3} & \underset{\cong}{\xrightarrow{q_A}} & W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{a_A^\vee} & H_{2n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0,
\end{array}$$

where $i_\xi : H_n \hookrightarrow H_{2n-1}$ is the embedding induced by the decomposition (34). In view of (46) and the canonical isomorphism $H_{2n-1}/i_\xi(H_n) \simeq H_{n-1}$, from this diagram we obtain the monad

$$\mathcal{M}_{A,B} : \quad 0 \to H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{a_{A,B}}{\to} E_{2n}(B) \underset{\simeq}{\overset{\phi}{\to}} E_{2n}(B)^\vee \overset{a_{A,B}^\vee}{\to} H_{2n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \tag{48}$$

with cohomology sheaf

$$E_2(A) = \ker(a_{A,B}^\vee \circ \phi)/\mathrm{im}\, a_A. \tag{49}$$

We call (48) the *quotient monad* of the monads (44) and (45).

Remark 4.4. Note that, by Proposition 4.2, the set of all diagrams (47) is parametrized by the irreducible variety $I_{2n-1,1}(\xi)$.

4.2. Example: a special family of symplectic (n, r)-instantons. Now assume $n \geq 2$ and, for any integer r, $2 \leq r \leq n-1$, consider an inclusion

$$\tau : H_{2n-r} \hookrightarrow H_{2n-1} \tag{50}$$

such that

$$\tau(H_{2n-r}) \supset i_\xi(H_n). \tag{51}$$

We obtain a hyperweb of quadrics

$$A_\tau \in S^2 H_{2n-r}^\vee \otimes \wedge^2 V^\vee$$

as the image of A under the map $S^2 H_{2n-1}^\vee \otimes \wedge^2 V^\vee \to S^2 H_{2n-r}^\vee \otimes \wedge^2 V^\vee$ induced by τ. The corresponding monad

$$\mathcal{M}_\tau : \quad 0 \to H_{2n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{a_\tau}{\to} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \overset{a_\tau^\vee \circ q_A}{\to} H_{2n-r}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{52}$$

has a rank-$2r$ cohomology bundle

$$E_{2r}(A_\tau) = \ker(a_\tau^\vee \circ q_A)/\mathrm{im}\, a_\tau. \tag{53}$$

where $a_\tau := a_A \circ \tau$. By construction, $E_{2r}(A_\tau)$ inherits a natural symplectic structure

$$\phi_r : \ E_{2r}(A_\tau) \xrightarrow{\simeq} E_{2r}(A_\tau)^\vee. \tag{54}$$

Besides, in view of (51), the monad (52) can be inserted as a midle row into the diagram (47), extending it to a three-row commutative anti-self-dual diagram. Arguing as in Remark 4.3 we obtain, in addition to the quotient monad (48), two more quotient monads:

$$\mathcal{M}'_\tau : \quad 0 \to H_{n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a'_\tau} E_{2n}(B) \underset{\simeq}{\xrightarrow{\phi}} E_{2n}(B)^\vee \xrightarrow{a'^\vee_\tau} H^\vee_{n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{55}$$

$$E_{2r}(A_\tau) = \ker(a'^\vee_\tau \circ \phi)/\mathrm{im}\, a'_\tau,$$

$$\mathcal{M}''_\tau : \quad 0 \to H_{r-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a''_\tau} E_{2r}(B) \underset{\simeq}{\xrightarrow{\phi_\tau}} E_{2r}(B)^\vee \xrightarrow{a''^\vee_\tau} H^\vee_{r-1} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{56}$$

$$E_2(A) = \ker(a'^\vee_\tau \circ \phi_\tau)/\mathrm{im}\, a_A.$$

From (26) and (55) we easily deduce:

$$h^0(E_{2r}(A_\tau)) = h^i(E_{2r}(A_\tau)(-2)) = 0, \quad i \geq 0, \quad c_2(E_{2r}(A_\tau)) = 2n - r. \tag{57}$$

By definition, this together with (52)-(54) means that

$$[E_{2r}(A_\tau)] \in I_{2n-r,r}. \tag{58}$$

Remark 4.5. Observe that, in view of (50), the maps τ belong to the set

$$N_{n,r} := \{\tau \in \mathrm{Hom}(H_{2n-r}, H_{2n-1}) |\ \tau \ \text{ is injective and } \mathrm{im}\,\tau \supset \mathrm{im}\, i_\xi\}.$$

When $A \in MI_{2n-1,1}(\xi)$ is fixed,$N_{n,r}$ parametrizes some family of hyperwebs A_τ from $MI_{2n-r,r}$. Since $N_{n,r}$ is a principal $GL(H_{2n-r})$-bundle over an open subset of the Grassmannian $Gr(n-r, n-1)$, it it is irreducible. Thus, by Remark 4.4, the family of the three-row extensions of the diagrams (47) can be parametrized by the irreducible variety $MI_{2n-1,1}(\xi) \times N_{n,r}$. Hence the family $D_{n,r}$ of isomorphism classes of symplectic rank-$2r$ bundles obtained from these diagrams by formula (53) is an irreducible locally closed subset of $I_{2n-r,r}$.

Note that it is a priori not clear whether the closure of $D_{n,r}$ in $I_{2n-r,r}$ is an irreducible component of $I_{2n-r,r}$.

Definition 4.6. Let $2 \leq r \leq n-1$. We say that $A \in MI_{2n-r,r}$ *satisfies property* (*) if there exists a monomorphism $i : H_n \hookrightarrow H_{2n-r}$ such that the image B of A under the surjection $\mathbf{S}_{2n-r} \twoheadrightarrow \mathbf{S}_n$ induced by i is invertible as a homomorphism $B : H_n \otimes V \to H_n^\vee \otimes V^\vee$.

The property (*) is clearly an open condition on A. Moreover, since $\pi_{2n-r,r} : MI_{2n-r,r} \to I_{2n-r,r}$ is a principal bundle (Theorem 3.1), if an element $A \in \pi^{-1}_{2n-r,r}([E_{2r}])$ satisfies (*), then any other point $A' \in \pi^{-1}_{2n-r,r}([E_{2r}])$ satisfies (*). We thus say that $[E_{2r}] \in I_{2n-r,r}$ *satisfies property* (*) if some (hence any) $A \in \pi^{-1}_{2n-r,r}([E_{2r}])$ satisfies property (*). It is obviously an open condition on $[E_{2r}] \in I_{2n-r,r}$.

Remark 4.7. By Proposition 4.2 and using (51), we see that any $[E_{2r}] \in D_{n,r}$, as well as any $A \in f_n^{-1}(D_{n,r})$ satisfies property (*). We define

$$I^*_{2n-r,r} := I_{(1)} \cup \ldots \cup I_{(k)}, \tag{59}$$

where $I_{(1)}, \ldots, I_{(k)}$ are all the irreducible components of $I_{2n-r,r}$ whose general points satisfy property (*). By definition, $D_{n,r} \subset I^*_{2n-r,r}$, hence $I^*_{2n-r,r}$ is nonempty. We also set $MI^*_{2n-r,r} = \pi^{-1}_{2n-r,r}(I^*_{2n-r,r})$, so that the map $\pi_{2n-r,r} : MI^*_{2n-r,r} \to I^*_{2n-r,r}$ is a principal bundle with structure group $GL(H_{2n-r})/\{\pm 1\}$.

5. Irreducibility of $I^*_{2n-r,r}$

5.1. A dense open subset $X_{n,r}$ of $MI^*_{2n-r,r}$. Reduction of the irreducibility of $I^*_{n,r}$ to that of $X_{n,r}$. In this section we prove the irreducibility of the component $I^*_{2n-r,r}$ of $I_{2n-r,r}$ defined in (59), see Theorem 5.3. The explicit construction of symplectic instantons in Section 4 gives us a hint to the proof. We proceed along the lines of Subsection 4.1.

Take any $B \in \mathbf{S}^0_n$ and consider it as an invertible anti-self-dual linear map $H_n \otimes V \to H^\vee_n \otimes V^\vee$. Then B^{-1} is also anti-self-dual. Let

$$\mathbf{\Sigma}_{n,r} := H^\vee_{n-r} \otimes H^\vee_n \otimes \wedge^2 V^\vee. \tag{60}$$

An element $C \in \mathbf{\Sigma}_n$ can be understood as a map $C : H_{n-r} \otimes V \to H^\vee_n \otimes V^\vee$, and its transpose $C^\vee$ is a map $H_n \otimes V \to H^\vee_{n-r} \otimes V^\vee$. The composition $C^\vee \circ B^{-1} \circ C$ is anti-self-dual, i.e., it is an element of $\wedge^2(H^\vee_{n-r} \otimes V^\vee) \simeq \mathbf{S}_{n-r} \oplus \wedge^2 H^\vee_{n-r} \otimes S^2 V^\vee$ (cf. (13)). We will later impose the condition

$$C^\vee \circ B^{-1} \circ C \in \mathbf{S}_{n-r}. \tag{61}$$

Next, as in (30), we have a well defined epimorphism $\epsilon(B) : H^\vee_n \otimes \wedge^2 V^\vee \twoheadrightarrow H^0(E_{2n}(B)(1))$. Besides, interpreting the above element $C \in \mathbf{\Sigma}_{n,r}$ as a map ${}^\sharp C : H_{n-r} \to H^\vee_n \otimes \wedge^2 V^\vee$, we obtain the composition $H_{n-r} \stackrel{\sharp C}{\to} H^\vee_n \otimes \wedge^2 V^\vee \stackrel{\epsilon(B)}{\to} H^0(E_{2n}(B)(1))$ which induces the morphism of sheaves

$$\rho_{B,C} : \ H_{n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to E_{2n}(B). \tag{62}$$

Note also that $B : H_n \otimes V \to H^\vee_n \otimes V^\vee$ and $C : H_{n-r} \otimes V \to H^\vee_n \otimes V^\vee$ define a map $(H_n \oplus H_{n-r}) \otimes V \to H^\vee_n \otimes V^\vee$ which induces the morphism of sheaves

$$\tau_{B,C} : \ (H_n \oplus H_{n-r}) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to H^\vee_n \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}. \tag{63}$$

Now set

$$X_{n,r} := \left\{ (B, C) \in \mathbf{S}^0_n \times \mathbf{\Sigma}_{n,r} \;\middle|\; \begin{array}{rl} \text{(i)} & \text{the condition (61) is satisfied,} \\ \text{(ii)} & \rho_{B,C} \text{ in (62) is a subbundle inclusion,} \\ \text{(iii)} & \tau_{B,C} \text{ in (63) is a subbundle inclusion.} \end{array} \right\} \tag{64}$$

By definition, $X_{n,r}$ is a locally closed subset of $\mathbf{S}^0_n \times \mathbf{\Sigma}_{n,r}$. Hence it has a natural structure of reduced scheme.

Now for an arbitrary direct sum decomposition

$$\xi : H_{2n-r} \stackrel{\simeq}{\to} H_n \oplus H_{n-r} \tag{65}$$

we obtain the corresponding decomposition

$$\tilde{\xi} : \mathbf{S}_{2n-r} \stackrel{\simeq}{\to} \mathbf{S}_n \oplus \mathbf{\Sigma}_{n,r} \oplus \mathbf{S}_{n-r} : A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)). \tag{66}$$

Thus, considering the set $MI_{2n-r,r}$ of symplectic $(2n-r, r)$-instanton hyperwebs of quadrics as a subset of $\mathbf{S}_{2n-r}$, we obtain a natural projection

$$f_{n,r} : MI_{2n-r,r} \to \mathbf{S}_n \oplus \mathbf{\Sigma}_{n,r} : A \mapsto (A_1(\xi), A_2(\xi)). \tag{67}$$

We now prove the following result parallel to Proposition 4.2.

Theorem 5.1. *Let $n \geq 3$ and $2 \leq r \leq n-1$.*

(i) For a general decomposition ξ in (65) there is an open dense subset $MI^*_{2n-r,r}(\xi)$ of $MI^*_{2n-r,r}$ and an isomorphism of reduced schemes

$$f_{n,r}:\ MI^*_{2n-r,r}(\xi) \xrightarrow{\simeq} X_{n,r}:\ A \mapsto (A_1(\xi), A_2(\xi)), \tag{68}$$

where $A_1(\xi)$ and $A_2(\xi)$ are defined by (66).

(ii) The inverse isomorphism is given by the formula

$$f^{-1}_{n,r}:\ X_{n,r} \xrightarrow{\simeq} MI^*_{2n-r,r}(\xi):\ (B,C) \mapsto\ \widetilde{\xi}^{-1}(B,\ C,\ -C^\vee \circ B^{-1} \circ C), \tag{69}$$

where $\widetilde{\xi}$ is defined by (66).

Proof. Set $MI^*_{2n-r,r}(\xi) := \{A \in MI^*_{2n-r,r} \mid A$ satisfies property (*) for the monomorphism $i: H_n \hookrightarrow H_{2n-r}$ defined by $\xi\}$. It follows from Definition 4.6and Remark 4.7 that, for a general decomposition ξ in (65), $MI^*_{2n-r,r}(\xi)$ is a dense open subset of $MI^*_{2n-r,r}$. Then, for this choice of ξ, the proof of this Theorem essentially mimics the proof of [T, Proposition 6.1] in which we make the substitution $m+1 \mapsto n,\ m \mapsto n-r$ and change the notation accordingly. □

The proof of the following theorem will be given in Subsection 5.2.

Theorem 5.2. *$X_{n,r}$ is irreducible of dimension $(2n-r)^2 + 4(2n-r)(r+1) - r(2r+1)$.*

From Theorems 5.1 and 5.2 it follows that $MI^*_{2n-r,r}$ is irreducible of dimension $(2n-r)^2 + 4(2n-r)(r+1) - r(2r+1)$ for any $n \leq 3$ and $2 \leq r \leq n-1$. Hence $I^*_{2n-r,r}$ is irreducible of dimension $4(2n-r)(r+1)-r(2r+1)$ for these values of n and r. Note that the irreducibility of $I^*_{2n-r,r}$ is also true when $r=n$, and in this case $I^*_{n,n}$ coincides with $I_{n,n}$. Substituting $2n-1 \mapsto n$, we obtain the following main result of the paper.

Theorem 5.3. *For any integer $r \geq 2$ and for any integer $n \geq r$ such that $n \equiv r (\mathrm{mod} 2)$, $I^*_{n,r}$ is an irreducible component of $I_{n,r}$ of dimension $4n(r+1) - r(2r+1)$.*

5.2. Proof of the irreducibility of $X_{n,r}$. In this subsection we give the proof of Theorem 5.2. Define

$$\widetilde{X}_{n,r} := \{(D,C) \in (\mathbf{S}^\vee_n)^0 \times \mathbf{\Sigma}_{n,r} \mid\ (C^\vee \circ D \circ C : H_{n-r} \otimes V \to H^\vee_{n-r} \otimes V^\vee) \in \mathbf{S}_{n-r}\}, \tag{70}$$

a closed subscheme of $(\mathbf{S}^\vee_m)^0 \times \mathbf{\Sigma}_{n,r}$ defined by the equations

$$C^\vee \circ D \circ C \in \mathbf{S}_{n-r}. \tag{71}$$

Since the conditions (ii) and (iii) in the definition (33) of $X_{n,r}$ are open and $X_{n,r}$ is nonempty (see Theorem 5.1), the isomorphism

$$\mathbf{S}^0_n \xrightarrow{\simeq} (\mathbf{S}^\vee_n)^0:\ B \mapsto B^{-1}$$

implies that $X_{n,r}$ is a nonempty open subset of $(\widetilde{X}_{n,r})_{red}$,

$$\varnothing \neq X_{n,r} \xhookrightarrow{\text{open}} (\widetilde{X}_{n,r})_{red}. \tag{72}$$

Fix a direct sum decomposition

$$H_n \xrightarrow{\simeq} H_{n-r} \oplus H_r.$$

Then any linear map

$$C \in \mathbf{\Sigma}_{n,r} = \mathrm{Hom}(H_{n-r}, H^\vee_n \otimes \wedge^2 V^\vee), \quad C: H_{n-r} \otimes V \to H^\vee_n \otimes V^\vee, \tag{73}$$

can be represented as a map

$$C : H_{n-r} \otimes V \to H_{n-r}^\vee \otimes V^\vee \ \oplus \ H_r^\vee \otimes V^\vee, \tag{74}$$

or else as a block matrix

$$C = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{75}$$

where

$$\phi \in \mathrm{Hom}(H_{n-r}, H_{n-r}^\vee) \otimes \wedge^2 V^\vee = \mathbf{\Phi}_{n-r}, \quad \psi \in \mathbf{\Psi}_{n,r} := \mathrm{Hom}(H_{n-r}, H_r^\vee) \otimes \wedge^2 V^\vee. \tag{76}$$

Similarly, any $D \in (\mathbf{S}_n^\vee)^0 \subset \mathbf{S}_n^\vee = S^2 H_n \otimes \wedge^2 V \subset \mathrm{Hom}(H_n^\vee \otimes V^\vee, H_n \otimes V)$ can be represented in the form

$$D = \begin{pmatrix} D_1 & \lambda \\ -\lambda^\vee & \mu \end{pmatrix}, \tag{77}$$

where

$$D_1 \in \mathbf{S}_{n-r}^\vee \subset \mathrm{Hom}(H_{n-r}^\vee \otimes V^\vee, H_{n-r} \otimes V), \tag{78}$$

$$\lambda \in \mathbf{L}_{n,r} := \mathrm{Hom}(H_r^\vee, H_{n-r}) \otimes \wedge^2 V, \quad \mu \in \mathbf{M}_r := S^2 H_r \otimes \wedge^2 V.$$

By (75) and (77) the composition

$$C^\vee \circ D \circ C : H_{n-r} \otimes V \to H_{n-r}^\vee \otimes V^\vee \ \ (C^\vee \circ D \circ C \in \wedge^2(H_{n-r}^\vee \otimes V^\vee))$$

can be written in the form

$$C^\vee \circ D \circ C = \phi^\vee \circ D_1 \circ \phi + \phi^\vee \circ \lambda \circ \psi - \psi^\vee \circ \lambda^\vee \circ \phi + \psi^\vee \circ \mu \circ \psi. \tag{79}$$

By (75)-(78) we have

$$\mathbf{S}_n^\vee \times \mathbf{\Sigma}_{n,r} = \mathbf{S}_{n-r}^\vee \times \mathbf{\Phi}_{n-r} \times \mathbf{\Psi}_{n,r} \times \mathbf{L}_{n,r} \times \mathbf{M}_r,$$

and there are well defined morphisms

$$\tilde{p} : \widetilde{X}_{n,r} \to \mathbf{L}_{n,r} \times \mathbf{M}_r : (D_1, \phi, \psi, \lambda, \mu) \mapsto (\lambda, \mu).$$

and

$$p := \tilde{p}|\overline{X}_{n,r} : \overline{X}_{n,r} \to \mathbf{L}_{n,r} \oplus \mathbf{M}_r,$$

where $\overline{X}_{n,r}$ is the closure of $X_{n,r}$ in $(\mathbf{S}_n^\vee)^0 \times \mathbf{\Sigma}_{n,r}$. We now invoke the following result from [T]:

Proposition 5.4. *Let $n \geq 2$. Then for any $D \in (\mathbf{S}_n^\vee)^0$ and for a general choice of the decomposition $H_n \overset{\sim}{\to} H_{n-r} \oplus H_r$, the block D_1 of D in (77) is nondegenerate.*

Proof. See [T, Proposition 7.3]. By repeatedly applying this proposition r times, we can find a decomposition $H_n \overset{\sim}{\to} H_{n-r} \oplus H_r$ such that $D_1 : H_{n-r}^\vee \otimes V^\vee \to H_{n-r} \otimes V$ in (77) is nondegenerate, i.e., $D_1 \in (\mathbf{S}_{n-r}^\vee)^0$. □

Let $\mathcal{X}$ be any irreducible component of $X_{n,r}$ and let $\overline{\mathcal{X}}$ be its closure in $\overline{X}_{n,r}$. Fix a point $z = (D_1, \phi, \psi, \lambda, \mu) \in \mathcal{X}$ not lying in the components of $X_{n,r}$ different from $\mathcal{X}$. Consider the morphism

$$f : \ \mathbb{A}^1 \to \overline{\mathcal{X}} : \ t \mapsto (D_1, t^2\phi, t\psi, t\lambda, t^2\mu), \quad f(1) = z, \tag{80}$$

which is well defined by (79). By definition, the point $f(0)=(D_1,0,0,0,0)$ lies in the fibre $p^{-1}(0,0)$. Hence, $p^{-1}(0,0)\cap\overline{\mathcal{X}}\neq\varnothing$. In other words,

$$\rho^{-1}(0,0)\neq\varnothing, \quad \text{where} \quad \rho:=p|\overline{\mathcal{X}}. \tag{81}$$

Now, it follows from (79) and the definition of $\widetilde{X}_{n,r}$ that

$$\tilde{p}^{-1}(0,0)=\{(D_1,\phi,\psi)\in(\mathbf{S}^\vee_{n-r})^0\times\mathbf{\Phi}_{n-r}\times\mathbf{\Psi}_{n,r}\mid\phi^\vee\circ D_1\circ\phi\in\mathbf{S}_{n-r}\}. \tag{82}$$

Consider the set

$$Z_{n-r}=\{(D,\phi)\in(\mathbf{S}^\vee_{n-r})^0\times\mathbf{\Phi}_{n-r}\mid\phi^\vee\circ D\circ\phi\in\mathbf{S}_{n-r}\}.$$

It carries a natural scheme structure, where it is a closed subscheme of $(\mathbf{S}^\vee_{n-r})^0\times\mathbf{\Phi}_{n-r}$. Comparing the definition of Z_{n-r} with (82) we see that there are scheme-theoretic inclusions of schemes

$$\rho^{-1}(0,0)\subset p^{-1}(0,0)\subset\tilde{p}^{-1}(0,0)=Z_{n-r}\times\mathbf{\Psi}_{n,r}. \tag{83}$$

By [T, Theorem 7.2], Z_{n-r} is an integral scheme of dimension $4(n-r)(n-r+2)$. This together with (83) implies that

$$\begin{aligned}\dim\rho^{-1}(0,0)\leq\dim p^{-1}(0,0)\leq\dim Z_{n-r}+\dim\mathbf{\Psi}_{n,r}&=4(n-r)(n-r+2)+6r(n-r)=\\&=(n-r)(4n+2r+8).\end{aligned} \tag{84}$$

Hence in view of (81)

$$\begin{aligned}\dim\overline{\mathcal{X}}\leq\dim\rho^{-1}(0,0)+\dim\mathbf{L}_{n,r}+\dim\mathbf{M}_r&\leq(n-r)(4n+2r+8)+6r(n-r)+3r(r+1)=\\&=(2n-r)^2+4(2n-r)(r+1)-r(2r+1).\end{aligned} \tag{85}$$

On the other hand, formula (21), with $2n-r$ substituted for n, and Theorem 5.1(ii) show that, for any point $x\in\mathcal{X}$ such that $A:=f^{-1}_{n,r}(x)\in MI^0_{2n-r,r}(\xi)$,

$$(2n-r)^2+4(2n-r)(r+1)-r(2r+1)\leq\dim_A MI^0_{2n-r,r}(\xi)=\dim\overline{\mathcal{X}}. \tag{86}$$

Comparing (85) with (86), we see that all the inequalities in (84)-(86) are equalities. In particular,

$$\dim\rho^{-1}(0,0)=\dim(Z_{n-r}\times\mathbf{\Psi}_{n,r})=\dim\overline{\mathcal{X}}-\dim(\mathbf{L}_{n,r}\times\mathbf{M}_r). \tag{87}$$

Since by Theorem [T, Theorem 7.2] the scheme Z_{n-r} is integral and so $Z_{n-r}\times\mathbf{\Psi}_{n,r}$ is integral as well, (83) and (87) yield the equalities of integral schemes

$$\rho^{-1}(0,0)=p^{-1}(0,0)=\tilde{p}^{-1}(0,0)=Z_{n-r}\times\mathbf{\Psi}_{n,r}. \tag{88}$$

Now we invoke one auxiliary result from [T].

Lemma 5.5. *Let $f:X\to Y$ be a morphism of reduced schemes, where Y is a smooth integral scheme. Assume that there exists a closed point $y\in Y$ such that for any irreducible component X' of X the following conditions are satisfied:*

(a) $\dim f^{-1}(y)=\dim X'-\dim Y$,

(b) the scheme-theoretic inclusion of fibres $(f|_{X'})^{-1}(y)\subset f^{-1}(y)$ is an isomorphism of integral schemes.

Then

(i) there exists an open subset U of Y containing the point y such that the morphism $f|_{f^{-1}(U)}:f^{-1}(U)\to U$ is flat, and

(ii) X is integral.

Proof. See [T, Lemma 7.4]. □

Applying assertions (i)-(ii) of this lemma to $X = X_{n,r}$, $X' = \mathcal{X}$, $Y = \mathbf{L}_{n,r} \times \mathbf{M}_r$, $y = (0,0)$, $f = p$, and using (87) and (88), we obtain that $X_{n,r}$ is integral of dimension $(2n-r)^2 + 4(2n-r)(r+1) - r(2r+1)$. Theorem 5.2 is proved.

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