INTEGRABILITY OF WEIGHT MODULE OF DEGREE 1

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Abstract. The aim of this article is to find all weight modules of degree 1 of a simple complex Lie algebra that integrate to a continuous representation of a simply-connected real Lie group on some Hilbert space. Weight modules and Representations of Lie groups and Gelfand-Kirillov dimension

1. Introduction

Let $\mathfrak{g}$ denote a simple Lie algebra over the field $\mathbb{C}$ of complex numbers. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. A weight module is a $\mathfrak{g}$-module, $\mathfrak{h}$-diagonalizable, having finite dimensional weight spaces. The set of all weight $\mathfrak{g}$-modules is a category containing the BGG category $\mathcal{O}$. This category of weight modules has been much studied in recent years (e.g. [5, 6, 3, 12, 2, 4, 7, 8, 14, 13]). It is then a natural question to find those weight modules that integrate to continuous (resp. unitary) representations of simply-connected real Lie groups. They should form a small but interesting class of representations, with small Gelfand-Kirillov dimension. In this paper, we treat the case of weight modules with weight multiplicities equal to 1.

Let us explain our strategy. Let G be a simply-connected real Lie group. Assume the $\mathfrak{g}$-module V integrates to a continuous representation π of G in some Hilbert space $\mathcal{H}$. Let K be a compact subgroup of G and denote by $(\pi_{|K}, \mathcal{H}_{|K})$ the representation $(\pi, \mathcal{H})$ restricted to K. Then it is well known that the representation $(\pi_{|K}, \mathcal{H}_{|K})$ is unitarizable. Therefore, we can express $\mathcal{H}_{|K}$ as a direct sum of simple finite dimensional unitary representation of K. A consequence is that the (complexified) Lie algebra of K should act nicely on V: the module V should be a $\mathfrak{k}$-finite $\mathfrak{g}$-module. If K is big enough, this condition is strong enough to imply that V should be a highest (or lowest) weight module. We then use a classification result due to Benkart, Britten and Lemire to describe the possible modules. Then it remains to check whether or not these modules can be integrated. Our results also make use of a theorem of Jørgensen and Moore and classical results about discrete series.

2. Some facts about weight modules

2.1. **Weight modules of degree 1.** Let $\mathfrak{g}$ denote a simple Lie algebra over the field $\mathbb{C}$ of complex numbers. Fix a Cartan subalgebra $\mathfrak{h}$. A $\mathfrak{g}$-module V is called a *weight module* if

(1) The module V is finitely generated,
(2) We have the following decomposition of V:

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \quad V_\lambda := \{m \in V \mid \forall\ H \in \mathfrak{h},\ H \cdot m = \lambda(H)m\},$$

(3) The weight spaces V_λ are all finite dimensional.

We call *degree* of a weight module the supremum of the dimension of the weight spaces:

$$deg(V) = \sup_{\lambda} \{dim(V_\lambda)\} \in \mathbb{Z}_{\geq 0} \cup \infty.$$

When $deg(V) \in \mathbb{Z}_{\geq 0}$, we call V a *bounded module*. In particular, if V is a weight $\mathfrak{g}$-module of degree 1, then all the non zero weight spaces are 1-dimensional. The weight modules of degree 1 have been studied by Benkart, Britten and Lemire [1]. In particular, they constructed weight modules of degree 1, $N(a)$ (with $a \in \mathbb{C}^n$) for $\mathfrak{sl}(n+1,\mathbb{C})$ and $M(b)$ (with $b \in \mathbb{C}^n$) for $\mathfrak{sp}(n,\mathbb{C})$ (see [1] for the explicit construction of these modules). Moreover, they proved the following:

Theorem 2.1 (Benkart, Britten, Lemire [1])**.** *Let V be a simple infinite dimensional weight $\mathfrak{g}$-module of degree 1. Then*

(1) The Lie algebra $\mathfrak{g}$ is isomorphic to either $\mathfrak{sl}(n+1,\mathbb{C})$ or to $\mathfrak{sp}(n,\mathbb{C})$.
(2) The Gelfand-Kirillov dimension of V equals the rank of $\mathfrak{g}$.
(3) If $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$, then there is $a \in \mathbb{C}^n$ such that $V \cong N(a)$.
(4) If $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{C})$, then there is $b \in \mathbb{C}^n$ such that $V \cong M(b)$.

For our purpose we shall need another notion. Let $\mathfrak{l}$ be a subalgebra of $\mathfrak{g}$. A $\mathfrak{g}$-module V is a $(\mathfrak{g},\mathfrak{l})$-*module of finite type* if as an $\mathfrak{l}$-module, V splits into a direct sum of simple finite dimensional $\mathfrak{l}$-modules, with finite multiplicities. For instance, a weight module is a $(\mathfrak{g},\mathfrak{h})$-module of finite type. The general notion of $(\mathfrak{g},\mathfrak{k})$-module has been studied in details by Penkov, Serganova and Zuckerman in [15, 16, 17, 18, 19].

2.2. **Classification of $(\mathfrak{g},\mathfrak{l}_j)$-module of finite type and of degree** 1**.** Let n be a positive integer greater than 1. Let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{sl}(n+1,\mathbb{C})$. Let $\mathfrak{h}$ denote the standard Cartan subalgebra of $\mathfrak{g}$, consisting of diagonal matrices. Denote by $H_0, H_1, \ldots, H_{n-1}$ its canonical basis. Denote by E_j (resp. F_j) the vector in $\mathfrak{g}$ corresponding to the elementary matrix $E_{j+1,j+2}$ (resp. $E_{j+2,j+1}$) for $0 \leq j \leq n-1$. Then $\mathfrak{g}$ is Lie-generated by the vectors $\{H_j, E_j, F_j\}_{0\leq j\leq n-1}$. Denote by $\mathfrak{l}_j$ the maximal standard Levi subalgebra of $\mathfrak{g}$ Lie-generated by $\mathfrak{h}$ and the vectors $\{E_k, F_k\}_{k\neq j}$.

For future use, we shall find those infinite dimensional weight $\mathfrak{g}$-modules of degree 1 whose restriction to some $\mathfrak{l}_j$ is a direct sum of finite dimensional $\mathfrak{l}_j$-modules. To this aim, we need the following general result:

Lemma 2.2 (Fernando [5], Benkart-Britten-Lemire [1])**.** *Let $\mathfrak{a}$ be a simple Lie algebra over $\mathbb{C}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{a}$. Let V be a simple weight $\mathfrak{a}$-module. Let $\mathcal{R}$ denote the root system of $(\mathfrak{a},\mathfrak{t})$. Then*

(1) For any $\alpha \in \mathcal{R}$, and any $X \in \mathfrak{a}_\alpha \setminus \{0\}$, the action of X on V is either locally finite or injective.
(2) Let $\alpha, \beta \in \mathcal{R}$ be such that there are $X^\pm \in \mathfrak{a}_{\pm\alpha} \setminus \{0\}$ and $Y^\pm \in \mathfrak{a}_{\pm\beta} \setminus \{0\}$ satisfying $X^\pm$ both act locally finitely on V and $Y^\pm$ both act injectively on V. Then $\alpha+\beta \notin \mathcal{R}$.

Proof. See [1, Section4]. □

Corollary 2.3. *Let V be a simple weight $\mathfrak{g}$-module. Let $0 \leq j \leq n-1$. Assume that V is a $(\mathfrak{g},\mathfrak{l}_j)$-module of finite type. Then V is a highest weight or a lowest weight module.*

Proof. From lemma 2.2, the vectors E_k and F_k act locally finitely or injectively on V. By hypothesis, they act locally finitely for $k \neq j$. Now, lemma 2.2 applied to E_j, F_j and either E_{j-1}, F_{j-1} or E_{j+1}, F_{j+1} shows that at least one of the vectors E_j and F_j acts locally finitely on V. In the first case, the module is a highest weight module. In the second case, it is a lowest weight module. □

Denote by $\{\omega_i\}_{i=1..n}$ the fundamental weights for $\mathfrak{g}$. Recall now the following:

Proposition 2.4 (Benkart-Britten-Lemire [1, Proposition 3.4])**.** *Up to isomorphism, the only highest weight $\mathfrak{g}$-module of degree 1 are the modules with highest weight $a\omega_1$, $a\omega_i - (1+a)\omega_{i+1}$, and $a\omega_n$, for $a \in \mathbb{C}$.*

In the notation of theorem 2.1, these modules correspond to $N(a, 0, \dots, 0)$, $N(\underbrace{-1, \dots, -1}_{i}, -1-a, 0, \dots, 0)$, and $N(-1, \dots, -1, -1-a)$, for $a \in \mathbb{C}$.

In the sequel we will often work with the $\mathfrak{sl}(n+1, \mathbb{C})$-module $N(a, 0, \dots, 0)$. For the convenience of the reader we write down here the action of the vectors $\{H_j, E_j, F_j\}_{0 \leq j \leq n-1}$ on a basis. The module $N(a, 0, \dots, 0)$ has a basis $x(k)$ indexed by $k \in \mathbb{Z}^n_{\geq 0}$. If $k = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geq 0}$, we set $|k| := k_1 + \cdots + k_n$. The action is given by:

$$H_0 \cdot x(k) = (a - k_1 - |k|)x(k) \tag{1a}$$

$$H_j \cdot x(k) = (k_j - k_{j+1})x(k) \tag{1b}$$

$$E_0 \cdot x(k) = k_1 x(k - \epsilon_1) \tag{1c}$$

$$F_0 \cdot x(k) = (a - |k|)x(k + \epsilon_1) \tag{1d}$$

$$E_j \cdot x(k) = k_{j+1} x(k - \epsilon_{j+1} + \epsilon_j) \tag{1e}$$

$$F_j \cdot x(k) = k_j x(k + \epsilon_{j+1} - \epsilon_j) \tag{1f}$$

From this classification, we are in position to prove the

Proposition 2.5. *Let $0 \leq j \leq n-1$. Let V be a simple infinite dimensional $(\mathfrak{g}, \mathfrak{l}_j)$-module of finite type and of degree 1. Then*

1. *If $j = 0$, then V or its contragredient is isomorphic to $N(a, 0, \dots, 0)$, for some $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$ or to $N(-1, m, 0, \dots, 0)$ for some $m \in \mathbb{Z}_{\geq 0}$.*
2. *If $j = n-1$, then V or its contragredient is isomorphic to $N(-1, \dots, -1, a)$, for some $a \in \mathbb{C} \setminus \mathbb{Z}_{<0}$ or to $N(-1, \dots, -1, -1-m, 0)$ for some $m \in \mathbb{Z}_{\geq 0}$.*
3. *If $0 < j < n-1$, then V or its contragredient is isomorphic to $N(\underbrace{-1, \dots, -1}_{j+1}, m, 0, \dots, 0)$ or $N(\underbrace{-1, \dots, -1}_{j}, -1-m, 0, \dots, 0)$, for some $m \in \mathbb{Z}_{\geq 0}$.*

Proof. From corollary 2.3, V or its contragredient is a simple highest weight module. Therefore we know that V or its contragredient is given by proposition 2.4. It thus remains to check whether or not the modules in proposition2.4 satisfy the restriction property.

Assume that $j = 0$. Let $0 < k < n-1$ and consider the module $V = N(-1, \dots, -1, a, 0, \dots, 0)$ where a is a complex number in position $k+1$. Then the highest weight x for this module satisfies:

$$H_{k-1} \cdot x = (-1-a)x, \; H_k \cdot x = ax.$$

If $k-1 \neq 0$, then by our assumption on V, x should generate a finite dimensional module. This imposes that the vectors $H_1, \dots, H_{n-1}$ of the Cartan subalgebra acts on x by non negative integers. Therefore a and $-1-a$ should be non negative integers, which is impossible. The same argument shows that for $k=1$ we must have $a \in \mathbb{Z}_{\geq 0}$.

Consider now the module $N(-1, \dots, -1, a)$ for $a \in \mathbb{C}$. Using once again the same argument, we show that necessarily a should be a negative integer. In this case, the module is finite dimensional.

We need to show now that $N(a, 0, \dots, 0)$ for $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$ and $N(-1, m, 0, \dots, 0)$ for $m \in \mathbb{Z}_{\geq 0}$ do satisfy the restriction property. Let us prove it for $N(a, 0 \dots, 0)$.

We want to find those linear combinations $\sum \mu_k x(k)$ which are highest weight vectors for the action of $\mathfrak{l}_0$. From the explicit action given by formulae (1), we conclude that the highest weight vectors are the linear combinations of the following linearly independent highest weight vectors:

$$x(k_1, 0, \dots, 0),\ k_1 \in \mathbb{Z}_{\geq 0}.$$

We shall prove now that the module $\mathcal{U}(\mathfrak{l}_0)x(k_1, 0, \dots, 0)$ is a simple highest weight module. Since it is a highest weight module, it is indecomposable. It is simple if and only if it does not contain a highest weight vector linearly independent of $x(k_1, 0, \dots, 0)$. But any such vector is a linear combination of $x(k', 0, \dots, 0)$ for some $k' \in \mathbb{Z}_{\geq 0}$. However the action of $nH_0 + (n-1)H_1 + \cdots + H_{n-1}$, vector generating the center of $\mathfrak{l}_0$, on $x(k', 0, \dots, 0)$ is:

$$(nH_0 + (n-1)H_1 + \cdots + H_{n-1}) \cdot x(k', 0, \dots, 0) = na - (n+1)k'.$$

Therefore the center acting as a scalar on $\mathcal{U}(\mathfrak{l}_0)x(k_1, 0, \dots, 0)$, we conclude that there is no highest weight vector in this module but the multiples of $x(k_1, 0, \dots, 0)$. Thus proving that the module is simple. Hence it is clear that we have the following branching:

$$N(a, 0, \dots, 0)_{|\mathfrak{l}_0} = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathcal{U}(\mathfrak{l}_0)x(k, 0, \dots, 0).$$

Therefore we proved that $N(a, 0, \dots, 0)$ is a $(\mathfrak{g}, \mathfrak{l}_0)$-module of finite type as asserted. The proof for $N(-1, m, 0, \dots, 0)$ is the same.

The case $j > 0$ is analogous. □

3. Type A case

In this section, we shall find which degree 1 $\mathfrak{sl}(n+1, \mathbb{C})$-module integrate to a continuous representation of some real Lie group whose complexified Lie algebra is $\mathfrak{sl}(n+1, \mathbb{C})$.

3.1. **A natural action of $SU(1, n)$.**

Let n be a positive integer. Let $SU(1, n)$ denote the subgroup of $GL(n+1, \mathbb{C})$ consisting of those matrices g such that

$$^t\overline{g} \times \begin{pmatrix} -1 & \\ & I_n \end{pmatrix} \times g = \begin{pmatrix} -1 & \\ & I_n \end{pmatrix}$$

and whose determinant is 1. We shall label rows and columns from 0 to n. This is a real Lie group. It acts on $\mathbb{S}_n := \{(z_j) \in \mathbb{C}^n \mid \sum_{j=1}^n |z_j|^2 = 1\}$ via

$$g \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \dfrac{\sum_{j=0}^n g_1^j z_j}{\sum_{j=0}^n g_0^j z_j} \\ \vdots \\ \dfrac{\sum_{j=0}^n g_n^j z_j}{\sum_{j=0}^n g_0^j z_j} \end{pmatrix},$$

where $g = (g_k^j)_{0\leq j,k\leq n} \in SU(1,n)$ and $z_0 = 1$. Since $SU(1,n)$ preserves the quadratic form $-|Z_0|^2 + |Z_1|^2 + \cdots + |Z_n|^2$, the denominator is never 0 and $g \cdot z$ is in $\mathbb{S}_n$ for any $z \in \mathbb{S}_n$. Denote by $d\sigma$ the measure on $\mathbb{S}_n$ induced from the Lebesgue measure of $\mathbb{C}^n$ and by Ω_n the volume of $\mathbb{S}_n$. It is well known that $\Omega_n = \frac{2\pi^n}{(n-1)!}$. We also denote by $\mathcal{H}(\mathbb{C}^n)$ the space of holomorphic functions from $\mathbb{C}^n$ to $\mathbb{C}$. Then the action of $SU(1,n)$ on $\mathbb{S}_n$ induces a natural continuous representation on $L^2(\mathbb{S}_n, \frac{d\sigma}{\Omega_n}) \cap \mathcal{H}(\mathbb{C}^n)$. We can further construct a unitary representation ρ on this space by

$$\rho(g)(\varphi)(z) := \left(\sum_{j=0}^n (g^{-1})_0^j z_j\right)^{-n} \times \varphi(g^{-1} \cdot z).$$

Let $k = (k_j) \in \mathbb{Z}_{\geq 0}^n$. Set $P(k)(z) := \prod_{j=1}^n z_j^{k_j}$. Then the family $(P(k))_{k\in\mathbb{Z}_{\geq 0}^n}$ is an orthogonal basis for the Hilbert space $L^2(\mathbb{S}_n, \frac{d\sigma}{\Omega_n}) \cap \mathcal{H}(\mathbb{C}^n)$. Moreover, we have $\|P(k)\|^2 = \dfrac{\prod_{j=1}^n k_j!}{\prod_{j=1}^{|k|} (j+n-1)}$, where $|k| := \sum_{j=1}^n k_j$.

Consider the following 1-parameter families:

$$e^{itH_1} := \begin{pmatrix} 1 & & & & & & \\ & e^{-it} & & & & & \\ & & e^{it} & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \end{pmatrix}, \ldots, e^{itH_{n-1}} := \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & e^{-it} & & & \\ & & & & e^{it} & & \end{pmatrix}.$$

$$e^{tX_1} := \begin{pmatrix} 1 & & & & & \\ & \cos t & -\sin t & & & \\ & \sin t & \cos t & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}, e^{tY_1} := \begin{pmatrix} 1 & & & & & \\ & \cos t & -i\sin t & & & \\ & -i\sin t & \cos t & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}, \dots,$$

$$e^{tX_{n-1}} := \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \cos t & -\sin t & \\ & & & \sin t & \cos t & \end{pmatrix}, e^{tY_{n-1}} := \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \cos t & -i\sin t & \\ & & & -i\sin t & \cos t & \end{pmatrix},$$

$$e^{tX_0} := \begin{pmatrix} \cosh t & -\sinh t & & & & \\ -\sinh t & \cosh t & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \end{pmatrix}, e^{tY_0} := \begin{pmatrix} \cosh t & -i\sinh t & & & & \\ i\sinh t & \cosh t & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \end{pmatrix},$$

$$e^{itH_0} := \begin{pmatrix} e^{-it} & & & & \\ & e^{it} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Then the Lie algebra $\mathfrak{su}(1,n)$ of $SU(1,n)$ is generated (as a Lie algebra) by

$$iH_0, \dots, iH_{n-1}, X_0, \dots, X_{n-1}, Y_0, \dots Y_{n-1}.$$

Set

$$E_0 := \frac{X_0 + iY_0}{2},\ F_0 := \frac{X_0 - iY_0}{2},$$
$$E_j := -\frac{X_j + iY_j}{2},\ F_j := \frac{X_j - iY_j}{2},\ 1 \le j \le n-1.$$

Then $(H_j, E_j, F_j)_{0 \le j \le n}$ generates a Lie algebra $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(n+1, \mathbb{C})$. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the subalgebra generated by $\{H_0, \dots, H_{n-1}\}$. We can compute as usual the action of $\mathfrak{g}$ on the basis $(P(k))_{k \in \mathbb{Z}^n_{\ge 0}}$. We get:

$$\text{(2a)} \qquad \begin{cases} H_0 \cdot P(k) &=& (-n - |k| - k_1)P(k) \\ E_0 \cdot P(k) &=& k_1 P(k - \epsilon_1) \\ F_0 \cdot P(k) &=& (-n - |k|)P(k + \epsilon_1) \end{cases},$$

$$\text{(2b)} \qquad \begin{cases} H_j \cdot P(k) &=& (k_j - k_{j+1})P(k) \\ E_j \cdot P(k) &=& k_{j+1} P(k - \epsilon_{j+1} + \epsilon_j) \\ F_j \cdot P(k) &=& k_j P(k + \epsilon_{j+1} - \epsilon_j) \end{cases},\ \forall\ 1 \le j \le n-1,$$

where ϵ_j is the vector in $\mathbb{Z}^n_{\ge 0}$ whose entries are all zero except the jth entry which is 1.

In the sequel we shall deform this infinitesimal representation and show the 1-parameter deformation thus constructed integrates to a continuous representation of the universal

cover of $SU(1,n)$. We shall further explicit those values of the parameter such that the representation is unitary.

3.2. **Deformation of the natural action of $\mathfrak{sl}(n+1,\mathbb{C})$.** To each $k \in \mathbb{Z}^n_{\geq 0}$, let us associate a vector $e(k)$. Let

$$\mathcal{H} := \left\{ u := \sum_{k\in\mathbb{Z}^n_{\geq 0}} u_k e(k) \;\middle|\; \sum_{k\in\mathbb{Z}^n_{\geq 0}} |u_k|^2 \frac{\prod_{j=1}^{n} k_j!}{\prod_{j=1}^{|k|}(j+n-1)} < \infty \right\}.$$

We define on $\mathcal{H}$ a Hilbert space structure by requiring that the basis $(e(k))$ is orthogonal and that $\|e(k)\|^2 = \dfrac{\prod_{j=1}^{n} k_j!}{\prod_{j=1}^{|k|}(j+n-1)}$. Denote by G the universal cover of $SU(1,n)$. According to the previous subsection, there is a continuous representation ρ of G (in fact, a unitary representation of $SU(1,n)$) corresponding to the representation of $\mathfrak{g}$ given on $\mathcal{H}$ by formulae (2).

Let $a \in \mathbb{C}\setminus\mathbb{Z}_{\geq 0}$. For any $l \in \mathbb{Z}_{\geq 0}$, set $\mu_a(l) := \sqrt{\prod_{j=1}^{l} \frac{j+n-1}{|j-a-1|}}$. Note that $\mu_a(l)$ is a well defined positive real number such that $\mu_{-n}(l) = 1$. We define now operators $(H_j(a), E_j(a), F_j(a))_{0\leq j\leq n-1}$ on $\mathcal{H}$ by their action on the basis $e(k)$ as follows:

$$\forall\ 1 \leq j \leq n-1, H_j(a) = H_j,\ E_j(a) = E_j,\ F_j(a) = F_j,$$

$$(H_0(a) - H_0)\cdot e(k) = (n+a)e(k), \tag{3a}$$

$$(E_0(a) - E_0)\cdot e(k) = k_1\left(\frac{\mu(|k|-1)}{\mu(|k|)} - 1\right)e(k-\epsilon_1), \tag{3b}$$

$$(F_0(a) - F_0)\cdot e(k) = \left((a-|k|)\frac{\mu(|k|+1)}{\mu(|k|)} + n + |k|\right)e(k+\epsilon_1). \tag{3c}$$

Notice that if $a = -n$ then all the new operators coincide with the corresponding undeformed operator. It is easy to check that the new operators give rise to another representation of $\mathfrak{sl}(n+1,\mathbb{C})$. We shall refer to the space of this representation as $\mathcal{H}_a$ (even though as a Hilbert space it is nothing but $\mathcal{H}$). In the sequel we will use

$$m_k^- := k_1\left(\frac{\mu(|k|-1)}{\mu(|k|)} - 1\right),\ m_k^+ := (a-|k|)\frac{\mu(|k|+1)}{\mu(|k|)} + n + |k|.$$

Remark 3.1. *Set* $x(k) := \mu(|k|)e(k)$. *Then* $\|x(k)\|^2 = \dfrac{\prod_{j=1}^{n} k_j!}{\prod_{j=1}^{|k|} |j-a-1|}$. *Moreover,* $x(k)$ *is also a basis for* $\mathcal{H}_a$ *and the deformed action of* $\mathfrak{sl}(n+1,\mathbb{C})$ *on* $\mathcal{H}_a$ *is precisely the one given in* [1].

3.3. **Integrability of the representation $\mathcal{H}_a$.** To prove the integrability of the representation $\mathcal{H}_a$ we shall use a criterion of Jørgensen and Moore [9]. Let us recall it. Let $\mathcal{H}'$ be a Hilbert space. Let D be a dense subspace. Denote by $\|\cdot\|_0$ the Hilbert norm. By $\mathcal{A}(D)$ we mean the set of all operators on D. Let $A_0 = Id$ and let $A_1, \dots, A_d$ be a basis for some (finite dimensional) Lie algebra included in $\mathcal{A}(D)$. We define inductively a norm $\|\cdot\|_l$ on D by setting $\|u\|_{l+1} := \max\{\|A_k u\|_l,\ 0 \leq k \leq d\}$. Denote by D_l the completion of D with respect to $\|\cdot\|_l$ and by L_j the space of continuous operators of D_j. Let $L^u(D_\infty) := \cap\{L_j,\ j \geq 0\}$ and $\mathcal{A}_\infty(D) := \mathcal{A}(D) \cap L^u(D_\infty)$. This is the set of operators on D bounded for all the norms $\|\cdot\|_l$.

Theorem 3.2 (Jørgensen-Moore)**.** *Let* G *be a connected simply-connected Lie group, whose corresponding Lie algebra is denoted* $\mathfrak{g}_\mathbb{R}$. *Let* π_0 *be a continuous representation of* G *on* $\mathcal{H}'$. *Set* $\mathcal{L}_0 := d\pi_0(\mathfrak{g}_\mathbb{R})$. *Let* $D := C^\infty(\pi_0)$. *Let* S_0 *be a set of Lie generator for* $\mathcal{L}_0$. *Let* $f : S_0 \to \mathcal{A}_\infty(D)$ *be such that* $S := \{A + f(A),\ A \in S_0\}$ *generates a finite dimensional Lie algebra* $\mathcal{L}$. *Then the representation* $\mathcal{L}$ *can be integrated into a continuous representation* π *of* G *such that* $d\pi(\mathfrak{g}_\mathbb{R}) = \mathcal{L}$.

To apply the theorem to our situation, we set $S_0 := \{iH_j, X_j, Y_j,\ 0 \leq j \leq n-1\}$. The Hilbert space is $\mathcal{H}$, the dense subset is

$$D := \left\{ u = \sum_{k \in \mathbb{Z}^n_{\geq 0}} u_k e(k) \in \mathcal{H} \;\middle|\; \sum_{k \in \mathbb{Z}^n_{\geq 0}} |k|^N u_k e(k) \in \mathcal{H},\ \forall\ N \in \mathbb{Z}_{\geq 0} \right\}.$$

The function f is given by the formulae (2). At this point, we need to check that the image of f is in $\mathcal{A}_\infty(D)$, i.e. to check that the operators defined on D by formulae (2) are bounded for all the norms $\|\cdot\|_l$. As $f(iH_j) = f(X_j) = f(Y_j) = 0$ for $j \geq 1$, we only need to consider the three operators $f(iH_0)$, $f(X_0)$ and $f(Y_0)$. As $f(iH_0)$ is a scalar operator, the boundedness is clear. We have to prove it for $f(X_0)$ and $f(Y_0)$. First we note the following:

Lemma 3.3. *The operators* $f(X_0)$ *and* $f(Y_0)$ *are bounded for all the norms* $\|\cdot\|_l$ *if and only if the operators* $f(E_0)$ *and* $f(F_0)$ *are.*

Proof. This is clear since X_0 and Y_0 are linear combination of E_0 and F_0 and vice-versa. □

Lemma 3.4. *The operators* $f(E_0)$ *and* $f(F_0)$ *are bounded for all the norms* $\|\cdot\|_l$ *if and only if they are bounded for all the norms constructed using the set* $\tilde{S}_0 := \{H_j, E_j, F_j,\ 0 \leq j \leq n-1\}$ *instead of* S_0.

Proof. Again, this follows from the fact that we can express the elements of S_0 as linear combinations of those in $\tilde{S}_0$ and vice-versa. □

Proposition 3.5. *The operators $f(E_0)$ and $f(F_0)$ are bounded for all the norms $\|\cdot\|_l$.*

Proof. From the lemma 3.4, we can use the norms $\|\cdot\|_l$ constructing from the set $\tilde{S}_0$. We prove the lemma by induction on l for $f(E_0)$. The proof for $f(F_0)$ is analogous. Let $u := \sum\ u_k e(k) \in D$. Then

$$\begin{aligned}
\|f(E_0)u\|^2 &= \sum_k\ |u_k|^2 |m_k^-|^2 \|e(k-\epsilon_1)\|^2 \\
&= \sum_{k|k_1>0}\ |u_k|^2 |m_k^-|^2 \|e(k)\|^2 \frac{|k|+n-2}{k_1} \\
&= \sum_{k|k_1>0}\ |u_k|^2 \|e(k)\|^2 \times k_1(|k|+n-2)\left(\frac{\mu(|k|-1)}{\mu(|k|)}-1\right)^2 \\
&\leq \sup_{k|k_1>0}\left\{k_1(|k|+n-2)\left(\frac{\mu(|k|-1)}{\mu(|k|)}-1\right)^2\right\} \times \sum_k\ |u_k|^2\|e(k)\|^2 \\
&= \sup_{k|k_1>0}\left\{k_1(|k|+n-2)\left(\frac{\mu(|k|-1)}{\mu(|k|)}-1\right)^2\right\} \times \|u\|^2.
\end{aligned}$$

Using an asymptotic development of $\left(\frac{\mu(|k|-1)}{\mu(|k|)}-1\right)^2$ we easily see that the above supremum is finite, thus proving that $f(E_0)$ is bounded for $\|\cdot\|_0$. Assume now that $f(E_0)$ is bounded for the norms $\|\cdot\|_l$ for $0 \leq l \leq M-1$.

Let $A_1, \ldots, A_M$ be elements in $\tilde{S}_0$. Let u ba as above and consider the expression $\|A_1 \cdots A_M f(E_0)u\|^2$. Since $A_1 \cdots A_M$ is a weight vector in the enveloping algebra, the vectors $A_1 \cdots A_M e(k)$ are mutually orthogonal. Therefore, we have

$$\|A_1 \cdots A_M f(E_0)u\|^2 = \sum_k\ |u_k|^2 |m_k^-|^2 \|A_1 \cdots A_M e(k-\epsilon_1)\|^2. \tag{4}$$

Now from formulae (2), it is clear that

$$A_1 \cdots A_M e(k) = P_{A_1 \cdots A_M}(k) e(k + l(A_1 \cdots A_M)),$$

where $P_{A_1 \cdots A_M}(k)$ is a polynomial in $(k_1, \ldots, k_n)$, product of M monomials of degree 1. For brevity, we shall denote it by $P(k)$ in the sequel. Moreover $l(A_1 \cdots A_M) \in \mathbb{Z}^n$ and $|l(A_1 \cdots A_M)| := \sum_{j=1}^{n} |l(A_1 \cdots A_M)_j|$ is a non negative integer smaller than $2M$, since for any $A \in \tilde{S}_0$, the vector $Ae(k)$ is either a multiple of $e(k)$ or a multiple of $e(k \pm \epsilon_1)$ or a multiple of $e(k \pm \epsilon_j \pm \epsilon_{j+1})$. In the sequel, we denote $l(A_1 \cdots A_M)$ simply by l.

Let k be such that $P(k-\epsilon_1) \neq 0$ and $k_1 \neq 0$. If $P(k) = 0$ then there is $1 \leq j \leq M$ such that $A_j = H_1$. This is proved by induction on M using the action of $\tilde{S}_0$ given by formulae (2). In particular, if $k_1 \neq 0$, $P(k) \neq 0$ and $P(k-\epsilon_1) \neq 0$, then $\frac{|P(k-\epsilon_1)|}{|P(k)|}$ is well-defined and is bounded by a number depending on M only.

Now we write

$$\|A_1\cdots A_M f(E_0)u\|^2 = \sum_{\substack{k \mid A_1\cdots A_M e(k)\neq 0,\\ A_1\cdots A_M e(k-\epsilon_1)\neq 0}} |u_k|^2|m_k^-|^2\|A_1\cdots A_M e(k-\epsilon_1)\|^2$$

$$+ \sum_{\substack{k \mid A_1\cdots A_M e(k)= 0,\\ A_1\cdots A_M e(k-\epsilon_1)\neq 0}} |u_k|^2|m_k^-|^2\|A_1\cdots A_M e(k-\epsilon_1)\|^2. \tag{5}$$

Let us work with the first sum. We can rewrite it in the following form:

$$\sum\ |u_k|^2|m_k^-|^2\frac{\|A_1\cdots A_M e(k-\epsilon_1)\|^2}{\|A_1\cdots A_M e(k)\|^2}\|A_1\cdots A_M e(k)\|^2.$$

This in turn is equal to:

$$\sum\ |m_k^-|^2\frac{|P(k-\epsilon_1)|^2}{|P(k)|^2}\frac{\|e(k+l-\epsilon_1\|^2}{\|e(k+l)\|^2}\times|u_k|^2\|A_1\cdots A_M e(k)\|^2.$$

Therefore we have:

$$\begin{aligned}
&\sum\ |m_k^-|^2\frac{|P(k-\epsilon_1)|^2}{|P(k)|^2}\frac{\|e(k+l-\epsilon_1\|^2}{\|e(k+l)\|^2}\times|u_k|^2\|A_1\cdots A_M e(k)\|^2\\
&\leq \sup\left\{|m_k^-|^2\frac{|P(k-\epsilon_1)|^2}{|P(k)|^2}\frac{\|e(k+l-\epsilon_1\|^2}{\|e(k+l)\|^2}\right\}\ \sum\|A_1\cdots A_M u_k e(k)\|^2\\
&\leq \sup\left\{|m_k^-|^2\frac{|P(k-\epsilon_1)|^2}{|P(k)|^2}\frac{\|e(k+l-\epsilon_1\|^2}{\|e(k+l)\|^2}\right\}\ \|A_1\cdots A_M u\|^2\\
&\leq \sup\left\{|m_k^-|^2\frac{|P(k-\epsilon_1)|^2}{|P(k)|^2}\frac{\|e(k+l-\epsilon_1\|^2}{\|e(k+l)\|^2}\right\}\ \times\|u\|_M^2.
\end{aligned}$$

We must now prove that $\sup\left\{|m_k^-|^2\frac{|P(k-\epsilon_1)|^2}{|P(k)|^2}\frac{\|e(k+l-\epsilon_1\|^2}{\|e(k+l)\|^2}\right\}$ is bounded by a number independent of l (but possibly depending on M). From previous remarks, it is sufficient to prove that $|m_k^-|^2\frac{\|e(k+l-\epsilon_1)\|^2}{\|e(k+l)\|^2}$ is bounded. Since $|l|\leq 2M$, this is an easy consequence of the explicit expression for m_k^-.

Let us now investigate the second sum in (5), assuming it is not empty. As we already mentioned, there is an index j such that $A_j = H_1$. Then using commutation relations in the enveloping algebra, we have:

$$A_1\cdots A_M = A_1'\cdots A_{M-1}' H_1 + A_1''\cdots A_{M-1}''.$$

Therefore, we have:

$$\begin{aligned}
&\sum_{k\ |A_1\cdots A_M e(k)=0,\ A_1\cdots A_M e(k-\epsilon_1)\neq 0} |u_k|^2|m_k^-|^2\|A_1\cdots A_M e(k-\epsilon_1)\|^2\\
&\leq \sum\ |u_k|^2|m_k^-|^2\left(\|A_1'\cdots A_{M-1}' H_1 e(k-\epsilon_1)\|^2 + \|A_1''\cdots A_{M-1}'' e(k-\epsilon_1)\|^2\right)\\
&\leq \Big(\sum\ |u_k|^2|m_k^-|^2(k_1-k_2-1)^2\|A_1'\cdots A_{M-1}' e(k-\epsilon_1)\|^2\Big)\\
&\qquad + \|A_1''\cdots A_{M-1}'' f(E_0)u\|^2\\
&\leq \sup\{(k_1-k_2-1)^2\}\times\|A_1'\cdots A_{M-1}' f(E_0)u\|^2 + \|A_1''\cdots A_{M-1}'' f(E_0)u\|^2
\end{aligned}$$

The second sum is by induction smaller than $\|f(E_0)\|^2_{M-1} \times \|u\|^2_{M-1}$. For the first sum, the induction shows that $\|A'_1 \cdots A'_{M-1} f(E_0)u\|^2 \leq \|f(E_0)\|^2_{M-1} \times \|u\|^2_{M-1}$. Thus, it suffices to prove that $(k_1 - k_2 - 1)^2$ is bounded. Since $A_1 \cdots A_M e(k) = 0$, there is an integer j such that

$$A_j = H_1,\ H_1 A_{j+1} \cdots A_M e(k) = 0 \text{ and } A_{j+1} \cdots A_M e(k) \neq 0.$$

But then $A_{j+1} \cdots A_M e(k) = C \times e(k + l')$ for some non zero constant C. As above $|l'| \leq 2M$. Thus $H_1 e(k + l') = 0$, which means that $(k + l')_1 = (k + l')_2$ or also $k_1 - k_2 - 1 = l'_1 - l'_2 - 1$. And we have $|l'_1 - l'_2 - 1| \leq |l'| + 1 \leq 2M + 1$, proving thus that $\sup\{(k_1 - k_2 - 1)^2\}$ is bounded by a number depending on M only.

Altogether, we have proved that:

$$\|A_1 \cdots A_M f(E_0)u\|^2 \leq C(M) \times \|u\|^2_M,$$

for some constant $C(M)$ depending on M only (Note here that $\|u\|_{M-1} \leq \|u\|_M$). As a consequence, we get

$$\|f(E_0)u\|_M \leq \sqrt{C(M)}\|u\|_M,$$

proving that $f(E_0)$ is bounded for the norm $\|\cdot\|_M$. □

Corollary 3.6. *Let $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$. Then the representation $\mathcal{H}_a$ of $\mathfrak{g}$ integrates into a continuous representation of G on the Hilbert space $\mathcal{H}$.*

Remark 3.7. *Let $a \in \mathbb{Z}_{\geq 0}$. Define a representation $\mathcal{H}_a$ as above by restricting the index set of k to those $k \in \mathbb{Z}^n_{\geq 0}$ such that $|k| \leq a$. Then $\mathcal{H}_a$ is indeed a representation and is finite dimensional. Therefore it also integrates into a continuous representation of G on some Hilbert space.*

3.4. **Unitarisability.** We now know a whole family of continuous representation of G. We should ask then which of these are unitary. If the representation $\mathcal{H}_a$ is unitary then the infinitesimal action given by the Lie basis $\{iH_j(a),\ X_j(a),\ Y_j(a)\}$ should be given by skew-symmetric operators. In other word, we should have $(iH_j(a))^* = -iH_j(a)$, $X_j(a)^* = -X_j(a)$ and $Y_j(a)^* = -Y_j(a)$. Using the expression of the $H_j(a), E_j(a), F_j(a)$ in term of this basis, it is equivalent to have $H_j(a)^* = H_j(a)$, $E_j(a)^* = F_j(a)$ for $j > 0$ and $E_0(a)^* = -F_0(a)$. As $H_j(a)$ is a diagonal operator, it is selfadjoint if and only if its eigenvalues are real. This imposes $a \in \mathbb{R}$. Remember that the representation that we started with is unitary. So it only remains to prove $f(E_j)^* = f(F_j)$ for $j > 0$ and $f(E_0)^* = -f(F_0)$. The first condition is trivial since $f(E_j) = f(F_j) = 0$. To check the second condition, we need to compare $\langle f(F_0)e(k), e(l)\rangle$ and $-\langle e(k), f(E_0)e(l)\rangle$. They are equal for all k and l if and only if $a \in \mathbb{R}_{<0}$.

Proposition 3.8. *The continuous representation $\mathcal{H}_a$ of G is unitary if and only if $a \in \mathbb{R}_{<0}$.*

Remark 3.9. *When $a \in \mathbb{Z}_{\geq 0}$, the representation $\mathcal{H}_a$ constructed in the remark 3.1 is not unitary (unless $a = 0$) since it has finite dimension greater than 1 and G is not a compact group.*

3.5. $SU(p,q)$ **case.** Let $1 \leq p \leq n$. Set $q = n+1-p$. Let $G_{p,q}$ denote the universal cover of $SU(p,q)$. The complex Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$ is the complexification of the Lie algebra of $G_{p,q}$. Moreover, $G_{p,q}$ contains a compact subgroup $K_{p,q}$ isomorphic to $SU(p) \times SU(q)$, whose complexified Lie algebra is isomorphic to the semisimple part of $\mathfrak{l}_{p-1}$. Let us now give the classification of all simple infinite dimensional degree 1 modules coming from a continuous representation of $G_{p,q}$ on some Hilbert space.

Theorem 3.10. *Let V be a simple infinite dimensional weight $\mathfrak{g}$-module of degree 1. Then V integrates into a continuous representation of $G_{p,q}$ on a Hilbert space if and only if*

1. *Either V or its contragredient is isomorphic to $N(a,0,\dots,0)$ (for $a \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$) or to $N(-1,m,0,\dots,0)$ (for $m \in \mathbb{Z}_{\geq 0}$), in case $p=1$.*
2. *Either V or its contragredient is isomorphic to $N(-1,\dots,-1,a)$ (for $a \in \mathbb{C} \setminus \mathbb{Z}_{<0}$) or to $N(-1,\dots,-1,-1-m,0)$ (for $m \in \mathbb{Z}_{\geq 0}$), in case $p=n$.*
3. *Either V or its contragredient is isomorphic to $N(\underbrace{-1,\dots,-1}_{p},m,0,\dots,0)$ (for $m \in \mathbb{Z}_{\geq 0}$) or to $N(\underbrace{-1,\dots,-1}_{p-1},-1-m,0,\dots 0)$ (for $m \in \mathbb{Z}_{\geq 0}$), in case $1<p<n$.*

Moreover, the corresponding representation of $G_{p,q}$ is unitary if and only if

1. *Either V or its contragredient is isomorphic to $N(a,0,\dots,0)$ (for $a \in \mathbb{R}_{<0}$) or to $N(-1,m,0,\dots,0)$ (for $m \in \mathbb{Z}_{\geq 0}$), in case $p=1$.*
2. *Either V or its contragredient is isomorphic to $N(-1,\dots,-1,a)$ (for $a \in \mathbb{R}_{>0}$) or to $N(-1,\dots,-1,-1-m,0)$ (for $m \in \mathbb{Z}_{\geq 0}$), in case $p=n$.*
3. *Either V or its contragredient is isomorphic to $N(\underbrace{-1,\dots,-1}_{p},m,0,\dots,0)$ (for $m \in \mathbb{Z}_{\geq 0}$) or to $N(\underbrace{-1,\dots,-1}_{p-1},-1-m,0,\dots 0)$ (for $m \in \mathbb{Z}_{\geq 0}$), in case $1<p<n$.*

Proof. First, remark that given any continuous representation π of $G_{p,q}$ on a Hilbert space, its restriction to $K_{p,q}$ splits into a direct sum of finite dimensional representations (possibly with infinite multiplicities). Therefore, the corresponding $\mathfrak{g}$-module should also split into a direct sum of finite dimensional $\mathfrak{l}_p$-modules. In other word, a necessary condition is that the underlying $\mathfrak{g}$-module is a $(\mathfrak{g},\mathfrak{l}_p)$-module of finite type. Therefore, by proposition 2.5, V or its contragredient should be isomorphic to the asserted modules.

1. In case $p=1$, we already know from corollary 3.6, that $N(a,0,\dots,0)$ does integrate into a continuous representation of G_1. Moreover from proposition 3.8, we know that this representation is unitary exactly when $a \in \mathbb{R}_{<0}$. Now, the module $N(-1,m,0,\dots,0)$ is a highest weight module with highest weight $\lambda_m = (-1-m,m,0,\dots,0) \in \mathbb{Z}^n$. This is clearly an analytically integral weight, and dominant with respect to the positive roots of $\mathfrak{l}_0$. It is then straightforward to check that it is the underlying $\mathfrak{g}$-module of the holomorphic discrete series of $SU(1,n)$ corresponding to the parameter λ_m.
2. The case $p=n$ is of course identical to the previous one up to a relabeling of the simple roots.

(3) The intermediate case $1 < p < n$ is easy, since the possible underlying $\mathfrak{g}$-modules all correspond to holomorphic discrete series, their parameter being $(0, \dots, 0, -1-m, m, 0, \dots,)$ or $(0, \dots, 0, m, -1-m, 0, \dots, 0)$.

□

3.6. $SL(n, \mathbb{R})$ **case.** Assume that $n \geq 3$. Let G_n denote the universal cover of $SL(n, \mathbb{R})$. Its complexified Lie algebra is also $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. The compact Lie group $K_n = SO(n)$ is a subgroup of G_n.

Theorem 3.11. *Let V be a simple weight $\mathfrak{g}$-module. Then V can be integrated into a continuous representation of G_n on a Hilbert space if and only if V is finite dimensional.*

Proof. Assume V can be integrated into a continuous representation of G_n in a Hilbert space. The complexified Lie algebra of K_n contains the vectors $E_j + F_j$. By lemma 2.2, these vectors should act locally finitely on V. Let V_λ be a weight space of V. Denote by α_j the weight of E_j. Then $E_j + F_j : V_\lambda \to V_{\lambda+\alpha_j} \oplus V_{\lambda-\alpha_j}$. Therefore $E_j + F_j$ is locally finite if and only if both E_j and F_j are. Then the module is finite dimensional as asserted. The converse is obvious. □

4. Type C case

Let n be a positive integer. Let p and q be positive integers such that $p + q = n$. In this section, we consider the groups $Sp(n, \mathbb{R})$ and $Sp(p, q)$ and their universal cover G_n and $G_{p,q}$. They contain the compact subgroup K_n and $K_{p,q}$ isomorphic to $SU(n)$ and $SP(p) \times Sp(q)$ respectively (see for instance [10]). Denote by $\mathfrak{g}$ the Lie algebra $\mathfrak{sp}(n, \mathbb{C})$.

Theorem 4.1. *Let V be a simple infinite dimensional weight $\mathfrak{g}$-module of degree 1. Then*

(1) V cannot integrate into a continuous representation of $G_{p,q}$ on a Hilbert space.

(2) V integrates into a continuous representation of G_n on a Hilbert space if and only if V or its contragredient is isomorphic to $M(-1, \dots, -1)$ or $M(-1, \dots, -1, -2)$. In this case, the corresponding representation of G_n is simple and unitary, and isomorphic to the even or odd part of the metaplectic representation or its contragredient.

Proof. The proof is analogous to the proof of theorem 3.10. In the case of $G_{p,q}$, the complexified Lie algebra of $K_{p,q}$ is the Lie algebra $\mathfrak{l}_p$ whose roots with respect to the standard Cartan subalgebra of $\mathfrak{g}$ are (see [10, Appendix C])

$$\{\pm 2\epsilon_l,\ \pm(\epsilon_j \pm \epsilon_k)\ :\ 1 \leq l \leq p,\ 1 \leq j \neq k \leq p\}$$
$$\sqcup\{\pm 2\epsilon_l,\ \pm(\epsilon_j \pm \epsilon_k)\ :\ p+1 \leq l \leq p+q,\ p+1 \leq j \neq k \leq p+q\}.$$

The module V should be a $(\mathfrak{g}, \mathfrak{l}_p)$-module of finite type. Using an analogue of proposition 2.5, we see that V or its contragredient should be a highest weight module. The simple infinite dimensional highest weight module of degree 1 have been classified by Benkart, Britten, Lemire [1, Proposition 3.6]. They are only two: their highest weights are $-\frac{1}{2}\omega_n$ and $\omega_{n-1} - \frac{3}{2}\omega_n$ respectively. It is then easy to check that the possible modules are not $(\mathfrak{g}, \mathfrak{l}_p)$-module of finite type.

The case of G_n is analogous. The complexified Lie algebra of K_n is the Lie algebra $\mathfrak{l}$ whose roots are

$$\{\pm(\epsilon_j \pm \epsilon_k)\ :\ 1 \leq j \neq k \leq n-1\}.$$

Once again, V or its contragredient should be isomorphic to a highest weight module. It is then well-known that these two highest weight modules correspond to the even and odd part of the metaplectic representation (see e.g. [11]). □

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