Bounds on Seshadri constants on surfaces with Picard number 1.

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Abstract

In this note we improve a result of Steffens [Ste] on the lower bound for Seshadri constants in very general points of a surface with 1–dimensional Néron-Severi space. We also show a multi-point counterpart of such a lower bound.

1 Introduction

Seshadri constants are interesting invariants of big and nef line bundles on algebraic varieties. They capture the so-called local positivity of a given line bundle. Seshadri constants were introduced by Demailly in [Dem]. As a nice introduction to this circle of ideas serves [PAG], an overview of recent result is given in [PSC]. Here we merely recall the basic definition.

Definition 1.1 *Let X be a smooth projective variety, L a big and nef line bundle on X and $x \in X$ a point on X. The number*

$$\varepsilon(L;x) := \inf_{C \ni x} \frac{L \cdot C}{\mathrm{mult}_x C}$$

is the Seshadri constant *of L at x.*

By $\varepsilon(L;1)$ we denote the maximum

$$\varepsilon(L;1) := \max_{x \in X} \varepsilon(L;x) \tag{1}$$

of Seshadri constants of L over all points $x \in X$. It is well known (see [PSC, Statement 2.2.8]) that the maximum is attained for very general points $x \in X$, i.e. away of a countable union of proper Zariski closed subsets of X. It is also well known (see [PSC, Proposition 2.1.1]) that there is an upper bound

$$\varepsilon(L;1) \leqslant \sqrt[n]{L^n}, \tag{2}$$

where n is the dimension of X.

As for lower bounds, Steffens in [Ste, Proposition 1] gave an interesting estimate on $\varepsilon(L;1)$ in case that X is a surface with Picard number 1.

Proposition 1.2 (Steffens) *Let X be a smooth projective surface with Picard number 1 and let L be the ample generator of the Néron-Severi group of X. Then*

$$\varepsilon(L;1) \geqslant \left\lfloor \sqrt{L^2} \right\rfloor. \tag{3}$$

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It is clear that if L^2 is a square, then there is actually an equality

$$\varepsilon(L;1) = \sqrt{L^2} \quad \text{if} \quad \sqrt{L^2} \in \mathbb{Z}.$$

Our first observation is that only under these circumstances (i.e. $\sqrt{L^2} \in \mathbb{Z}$) is the bound (3) sharp.

For the rest of the paper we write $N := L^2$ and $s := \left\lfloor \sqrt{L^2} \right\rfloor$.

Let as before X be a smooth projective surface with Picard number $\rho(X) = 1$ and let L be the ample generator of the Néron-Severi group.

Lemma 1.3 *If L^2 is not a square, then it is always*

$$\varepsilon(L;1) > s.$$

Proof. We have by assumption that $s < \sqrt{N}$, so that

$$s^2 + 1 \leqslant L^2. \tag{4}$$

Assume to the contrary that $\varepsilon(L;1) = s$. Then for a general point $x \in X$ there exists a curve $C \in |pL|$, for some integer p, such that

$$pL^2 = sm, \tag{5}$$

where m denotes as usually the multiplicity of C at x.

The curve C cannot be smooth at x because then $pL^2 = s$ could never be satisfied (by our assumption $L^2 > 1$). Hence $m \geqslant 2$ and we have by [KSS, Theorem A]

$$m(m-1) + 1 \leqslant C^2. \tag{6}$$

Combining (5) and (4) we get

$$ps^2 < ps^2 + p \leqslant pL^2 = sm,$$

which after dividing by s and using the fact that s and p are integers yields

$$ps \leqslant m - 1. \tag{7}$$

Now, combining (7) with (6) we get

$$m(m-1) + 1 \leqslant C^2 = p^2L^2 = psm \leqslant (m-1)m$$

a contradiction. □

With this fact established, it is natural to ask if there is a lower bound better than $\left\lfloor \sqrt{L^2} \right\rfloor$ if L^2 is not a square. It is not obvious that such a bound exists because there could be a sequence of polarized surfaces (X_n, L_n) with Picard number 1, such that $L_n^2 = N$ for all n and $\lim_{n\to\infty} \varepsilon(L_n;1) = \left\lfloor \sqrt{N} \right\rfloor$. We show that this cannot happen and that there exists a lower bound on $\varepsilon(L;1)$ improving that of Steffens in case L^2 is not a square.

2 A new lower bound

We introduce some more notation. We assume that $\sqrt{N}$ is irrational and denote its fractional part by β, thus $\beta := \sqrt{N} - s > 0$. We define p_0 as the least integer k such that $k \cdot \beta > \frac{1}{2}$, i.e.

$$p_0 := \left\lceil \frac{1}{2\beta} \right\rceil. \tag{8}$$

Further we set the number m_0 to be equal

$$m_0 := \left\lceil p_0 \cdot \sqrt{N} \right\rceil = p_0 s + \lceil p_0 \beta \rceil = p_0 s + 1. \tag{9}$$

The following theorem is the main result of this note.

Theorem 2.1 *Let X be a smooth projective surface with Picard number 1 and let L be the ample generator of the Néron-Severi space such that $N = L^2$ is not a square. Then*

$$\varepsilon(L;1) \geqslant \frac{p_0}{m_0} N.$$

Proof. Note that $s < \frac{p_0}{m_0} N < \sqrt{N}$. Indeed, as $N = (s+\beta)^2$, we have

$$\frac{p_0}{m_0} N > \frac{p_0}{p_0 s + 1} \cdot (s + 2\beta) \cdot s \geqslant s.$$

On the other hand

$$\frac{p_0}{m_0} N = \frac{p_0 \sqrt{N}}{\left\lceil p_0 \sqrt{N} \right\rceil} \cdot \sqrt{N} < \sqrt{N}.$$

Now, we assume to the contrary that $\varepsilon(L;1) < \frac{p_0}{m_0} N$. Then there exists an integer m such that for every point $x \in X$, there exists a curve C_x vanishing at x to order $\geqslant m$ (i.e. $\mathrm{mult}_x C_x \geqslant m$) and

$$\varepsilon(L;x) \leqslant \frac{L \cdot C}{m} < \frac{p_0}{m_0} N. \tag{10}$$

Such curves $\{C_x\}$ can be chosen to form an algebraic family and for its arbitrary member C we have

$$m(m-1) + 1 \leqslant C^2 \tag{11}$$

by Theorem A in [KSS].

On the other hand there must exist an integer p such that $C \in |pL|$. The condition (10) then translates into

$$\frac{p}{m} < \frac{p_0}{m_0},$$

whereas the inequality (11) requires

$$m(m-1) + 1 \leqslant p^2 \cdot N. \tag{12}$$

This contradicts Lemma 2.2, which we prove below. $\square$

We have the following numerical lemma.

Lemma 2.2 *Let N be a positive integer which is not a square. Let*

$$\Omega := \left\{(p,m) \in \mathbb{Z}^2_{>0} : \quad m(m-1)+1 \leqslant Np^2\right\}$$

and let $\varepsilon_0 := \min_{(p,m)\in\Omega} \frac{p}{m}$. Then

$$\varepsilon_0 = \frac{p_0}{m_0}$$

with p_0 and m_0 defined for N as in (8) and (9).

Proof. For the fixed p, the quotient in question is minimalized by the maximal integer m satisfying the inequality (12). This is

$$m_p := \left\lfloor \frac{1}{2} + \sqrt{Np^2 - \frac{3}{4}} \right\rfloor .$$

We need to show that

$$\frac{p_0}{m_0} \leqslant \frac{p}{m_p} \quad \text{for all } p.$$

We have certainly

$$\left\lfloor \frac{1}{2} + p\sqrt{N} \right\rfloor \geqslant \left\lfloor \frac{1}{2} + \sqrt{Np^2 - \frac{3}{4}} \right\rfloor = m_p,$$

so that it is enough to show

$$\frac{p_0}{m_0} \leqslant \frac{p}{\left\lfloor \frac{1}{2} + p\sqrt{N} \right\rfloor} \quad \text{for all } p. \tag{13}$$

Since

$$p_0ps + p_0 \left\lfloor \frac{1}{2} + p\beta \right\rfloor = p_0 \left\lfloor \frac{1}{2} + p\sqrt{N} \right\rfloor \quad \text{and} \quad p_0m_0 = p_0ps + p$$

inequality (13) would follow from

$$p \geqslant p_0 \left\lfloor \frac{1}{2} + p\beta \right\rfloor . \tag{14}$$

For $p < p_0$ the right hand side of (14) is zero, since then $p\beta < \frac{1}{2}$. For $p \geqslant p_0$ we write $p = qp_0 + r$ with $q \geqslant 1$ and $0 \leqslant r \leqslant p_0 - 1$. In particular $r\beta < \frac{1}{2}$, so that

$$p_0 \left\lfloor \frac{1}{2} + p\beta \right\rfloor = p_0 \left\lfloor \frac{1}{2} + r\beta + qp_0\beta \right\rfloor \leqslant p_0 \lfloor qp_0\beta + 1 \rfloor \leqslant p_0q \leqslant p.$$

The last but one inequality holds because $p_0\beta < 1$ and $q \geqslant 1$. This verifies (14) and the proof is finished. □

The next example shows that in some situations our bound is optimal.

Example 2.3 Let $N = 2d$ be an even integer such that $N + 1 = \ell^2$ is a square. A general abelian surface X with polarization L of type $(1,d)$ has Picard number equal 1, see e.g. [CAV, Section 9.9]. For such surfaces we know by [Bau, Theorem 6.1] that

$$\varepsilon(L;1) = \frac{2d}{\ell} = \frac{1}{\sqrt{N+1}}N.$$

On the other hand we have $p_0 = 1$ and $m_0 = \left\lfloor \sqrt{N} \right\rfloor + 1 = \ell$, so that

$$\varepsilon(L;1) = \frac{p_0}{m_0} N$$

in that case.

In general we expect however that $\varepsilon(L;1)$ on surfaces with Picard number 1 is subject to a much stronger numerical restriction.

Conjecture 2.4 *Let X be a smooth projective surface with Picard number* 1 *and let L be the ample generator of the Néron-Severi space with $N = L^2$. Then*

$$\varepsilon(L;1) \geqslant \begin{cases} \sqrt{N} & \text{if} \quad N \text{ is a square} \\ \frac{Nk_0}{\ell_0} & \text{if} \quad N \text{ is not a square} \end{cases}$$

and (ℓ_0, k_0) is the primitive solution of Pell's equation

$$\ell^2 - Nk^2 = 1.$$

The inequality in Theorem 2.1 can be viewed as the next step (after Steffens) towards approximating $\sqrt{N}$ by continued fractions.

3 Multi-point Seshadri constants

In the last paragraph we show that a lower bound of Steffens type can be given also for multi-point Seshadri constants. This is a variant of Definition 1.1 due to Xu, see [X94].

Definition 3.1 *Let X be a smooth projective variety, L a big and nef line bundle on X, $r \geqslant 1$ an integer and $x_1, \ldots, x_r$ distinct points on X. The real number*

$$\varepsilon(L; x_1, \ldots, x_r) := \inf_{C \cap \{x_1, \ldots, x_r\} \neq \varnothing} \frac{L \cdot C}{\operatorname{mult}_{x_1} C + \ldots + \operatorname{mult}_{x_r} C}$$

is the multi-point Seshadri constant *of L at poins $x_1, \ldots, x_r$.*

The interest in these numbers comes from the fact that, at least conjecturally, their behavior is more predictable than that of their one-point cousins. We refere again to [PSC, Sections 2 and 6] for introduction to that circle of ideas.

Similarly to (1) we set

$$\varepsilon(L; r) := \max_{\{x_1, \ldots, x_r\} \subset X} \varepsilon(L; x_1, \ldots, x_r).$$

The following result parallels Proposition 1.2.

Theorem 3.2 *Let X be a smooth projective surface with Picard number* 1 *and let L be the ample generator of the Néron-Severi group of X. Then*

$$\varepsilon(L; r) \geqslant \left\lfloor \sqrt{\frac{L^2}{r}} \right\rfloor.$$

Proof. We denote $s := \left\lfloor \sqrt{\frac{L^2}{r}} \right\rfloor$ and assume to the contrary that $\varepsilon(L; r) < s$. Then for arbitrary $x_1, \ldots, x_r$ there are irreducible curves $C_{x_1,\ldots,x_r}$ such that

$$\frac{L \cdot C_{x_1,\ldots,x_r}}{\sum_{i=1}^r \operatorname{mult}_{x_i} C_{x_1,\ldots,x_r}} < s. \tag{15}$$

One can choose these curves to move in an algebraic family. As the Picard number of X is 1, there is in fact an integer p such that this family is a subset of the linear series pL. If $m_1, \ldots, m_r$ are positive integers such that

$$\operatorname{mult}_{x_i} C_{x_1,\ldots,x_r} \geqslant m_i \quad \text{for all} \quad i = 1, \ldots, r,$$

then for any member C of the family we have by [X94, Lemma 1]

$$C^2 \geqslant m_1^2 + \ldots + m_{r-1}^2 + m_r(m_r - 1). \tag{16}$$

We can renumber the points so that $m_r \leqslant m_i$ for all $i = 1, \ldots, r$. The inequality (15) implies that

$$rps^2 \leqslant pL^2 < s \cdot \sum_{i=1}^r m_i.$$

Dividing by s and taking into account that all involved numbers are integers we obtain

$$rps \leqslant \sum_{i=1}^r m_i - 1. \tag{17}$$

On the other hand from (16), (15) and (17) we have

$$\sum_{i=1}^r m_i^2 - m_r \leqslant pL \cdot C < ps \sum_{i=1}^r m_i \leqslant \frac{1}{r} \sum_{i=1}^r m_i \left(\sum_{i=1}^r m_i - 1 \right) \leqslant \sum_{i=1}^r m_i^2 - m_r$$

which gives the desired contradiction. □

We have the following straightforward corollary.

Corollary 3.3 *Let X and L be as in Theorem 3.2. Assume that the degree of L is of the form $L^2 = rd^2$ for some positive integer d. Then we have the equality*

$$\varepsilon(L; r) = d.$$

Remark 3.4 The same statement was proved in [PSC, Theorem 6.1.10] for surfaces with arbitrary Picard number under the additional assumption that L is very ample. It is expected that the equality

$$\varepsilon(L; r) = \sqrt{\frac{L^2}{r}}$$

holds on arbitrary surfaces, provided r is sufficiently large. This is a natural generalization of Nagata Conjecture as explained in detail in [Sze].

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