ENDOMORPHISM ALGEBRAS OF MAXIMAL RIGID OBJECTS IN CLUSTER TUBES

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ABSTRACT. Given a maximal rigid object T of the cluster tube, we determine the objects finitely presented by T. We then use the method of Keller and Reiten to show that the endomorphism algebra of T is Gorenstein and of finite representation type, as first shown by Vatne. This algebra turns out to be the Jacobian algebra of a certain quiver with potential, when the characteristic of the base field is not 3. We study how this quiver with potential changes when T is mutated. We also provide a derived equivalence classification for the endomorphism algebras of maximal rigid objects.

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CONTENTS

1. INTRODUCTION

In the theory of additive categorification of Fomin–Zelevsinky's cluster algebras [11], 2-Calabi–Yau triangulated categories with cluster-tilting objects play a central role, see [19] for a nice survey of this topic. A cluster-tilting object is always a maximal rigid object, while the converse is generally not true. There exist 2-Calabi–Yau triangulated categories in which maximal rigid objects are not cluster-tilting. The first examples of such categories were given by Burban–Iyama–Keller–Reiten in [8].

Cluster tubes, introduced by Barot–Kussin–Lenzing in [6], are another family of 2-Calabi–Yau triangulated categories without cluster-tilting objects. In [5], Buan–Marsh–Vatne classified maximal rigid objects of cluster tubes, none of which is cluster-tilting. In [27], Vatne studied endomorphism algebras of maximal rigid objects. He gave an explicit description of these algebras in terms of quivers with relations, and showed that they are Gorenstein of Gorenstein dimension at most 1 (resembling 2-Calabi–Yau tilted algebras, *cf.* [22]) and are of finite representation type. One motivation to study these endomorphism algebras is to categorify cluster algebras of type B/C by cluster tubes, see [5, 29].

In Section 4 of this paper, we give a categorical explanation of the Gorenstein property and the representation-finiteness. This is based on a result analogous to the result of Keller–Reiten [22] (*cf.* also Koenig–Zhu [23]): for a maximal rigid object T, the functor $\mathsf{Hom}(T, ?)$ induces an equivalence between the module category of the endomorphism algebra of T and the additive quotient of a suitable subcategory of the cluster tube by the ideal generated by the shift of T. We determine this suitable subcategory (Proposition 4.5 and Proposition 4.8), and the Gorenstein property and the representation-finiteness follow as consequences. The Gorenstein property has recently been proved by Zhou–Zhu for endomorphism algebras of maximal rigid objects in any 2-Calabi–Yau triangulated categories [28].

In [5], maximal rigid objects of a cluster tube were shown to form a cluster structure with loops. A nice feature of the cluster structure is the existence of *mutation*. It is of interest to know the relation between the endomorphism algebras of two maximal rigid objects related by a mutation. When the mutation is simple, the two algebras are nearly Morita equivalent in the sense of Ringel [26] —- this follows from a more general result (Corollary 4.3); while when the mutation is not simple, it is not clear whether we can formulate an analogous statement. In Section 5, we study when the two neighbouring endomorphism algebras are derived equivalent. More generally, we prove that the endomorphism algebras of two maximal rigid objects (not necessarily related by a mutation) are derived equivalent if and only if their quivers have the same number of 3-cycles (Theorem 5.5). This derived equivalence classification is analogous to that of Buan–Vatne [7] for cluster-tilted algebras of type A.

In Section 6, we associate to each maximal rigid object a quiver with potential, whose Jacobian algebra is isomorphic to the endomorphism algebra of the maximal rigid object. This is a consequence of Vatne's explicit description of the endomorphism algebra. We study the change of the associated quivers with potential induced from the mutation of maximal rigid objects (Proposition 6.4). In particular, when two maximal rigid objects are related by a simple mutation, the two associated quivers with potential are related by the Derksen–Weyman–Zelevinsky mutation.

Necessary knowledge on cluster tubes, including the definition, the classification of maximal rigid objects, and the description of their endomorphism algebras, will be recalled in Section 2. Section 3 is devoted to the study of mutations of maximal rigid objects of cluster tubes.

Throughout, we fix a field k. All vector spaces, algebras, representations, modules, and categories will be over the field k. We identify a representation of a quiver with the corresponding right module over the path algebra of the opposite quiver.

Acknowledgments

2. Preliminaries on the cluster tube

Let n be a positive integer.

2.1. **The tube.** Let $\overrightarrow{\Delta}_n$ be the cyclic quiver with n vertices such that arrows are going from i to $i-1$ (taken modulo n).

The *tube of rank n* is the category of finite-dimensional nilpotent representations of the cyclic quiver $\overrightarrow{\Delta}_n$. It is a hereditary abelian category. Every indecomposable representation is uniserial, *i.e.* it has a unique composition series, and hence is determined by its socle and its length up to isomorphism. For $a = 1, \ldots, n$ and $b \in \mathbb{N}$, we will denote by (a, b) the unique (up to isomorphism) representation with socle the simple at the vertex a and of length b. When the first argument does not belong to the set $\{1, \ldots, n\}$, it should be read as modulo n.

The abelian category $\mathcal{T}_n$ has Auslander–Reiten sequences, and the Auslander–Reiten translation τ is an autoequivalence of $\mathcal{T}_n$ which takes the indecomposable representation (a, b) to the indecomposable representation $(a - 1, b)$.

For an indecomposable representation (a, b) and an arbitrary representation M of $\overrightarrow{\Delta}_n$, an extension of M by (a, b) factors through the canonical inclusion $(a, b) \hookrightarrow (a, b+l)$ for all $l \geq 1$.

The *Loewy length* of an object M of $\mathcal{T}_n$, denoted by $\ell\ell(M)$, is defined as the maximum of the lengths of indecomposable direct summands of M.

2.2. **The cluster tube.** Let $\mathcal{D}^b(\mathcal{T}_n)$ be the bounded derived category of the abelian category $\mathcal{T}_n$. It is triangulated with suspension functor Σ, the shift of complexes. It has Auslander–Reiten triangles, and the Auslander–Reiten translation is the derived functor of the Auslander–Reiten translation τ of $\mathcal{T}_n$. By abuse of notation, we also denote it by τ.

Definition 2.1 (Barot–Kussin–Lenzing [6]). *The* cluster tube *of rank n is defined as the orbit category*

$$\mathcal{C}_n := \mathcal{D}^b(\mathcal{T}_n)/\tau^{-1} \circ \Sigma \ .$$

Precisely, the objects of $\mathcal{C}_n$ are the same as those of $\mathcal{D}^b(\mathcal{T}_n)$, and for two objects M and N the morphism space is

$$\mathsf{Hom}_{\mathcal{C}_n}(M, N) = \bigoplus_{i \in \mathbb{Z}} \mathsf{Hom}_{\mathcal{D}^b(\mathcal{T}_n)}(M, (\tau^{-1} \circ \Sigma)^i N).$$

The category $\mathcal{C}_n$ has a triangulated structure such that the canonical projection functor $\pi : \mathcal{D}^b(\mathcal{T}_n) \to \mathcal{C}_n$ is triangulated, by Keller [21, Theorem 9.9]. It is 2-Calabi–Yau, *i.e.* there is a bifunctorial isomorphism $D \, \mathsf{Hom}_{\mathcal{C}_n}(M, N) \cong \mathsf{Hom}_{\mathcal{C}_n}(N, \Sigma^2 M)$ for objects M and N in $\mathcal{C}_n$, where $D = \mathsf{Hom}_k(?, k)$ is the k-dual. It has Auslander–Reiten triangles, and the Auslander–Reiten translation τ is naturally equivalent to the suspension functor Σ. The Auslander–Reiten quiver of $\mathcal{C}_n$ is depicted as
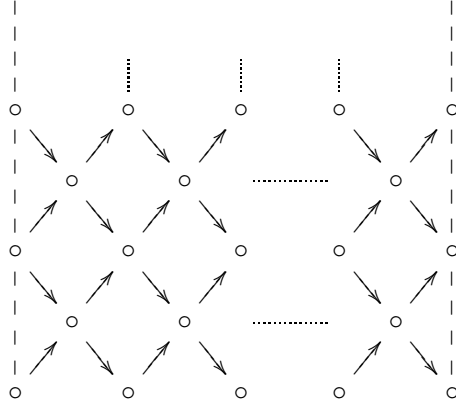
where the leftmost and rightmost columns are identified.

The following lemma is clear.

Lemma 2.2. *On isomorphism classes of objects, the composite functor $\mathcal{T}_n \to \mathcal{D}^b(\mathcal{T}_n) \xrightarrow{\pi} \mathcal{C}_n$ is bijective. We will identify objects of $\mathcal{T}_n$ and objects of $\mathcal{C}_n$ via this bijection. For M, N in $\mathcal{T}_n$,*

$$\mathsf{Hom}_{\mathcal{C}_n}(M, N) \ = \ \mathsf{Hom}_{\mathcal{T}_n}(M, N) \oplus \mathsf{Ext}^1_{\mathcal{T}_n}(M, \tau^{-1} N),$$

$$\mathsf{Hom}_{\mathcal{C}_n}(M, \Sigma N) \ = \ \mathsf{Ext}^1_{\mathcal{T}_n}(M, N) \oplus D \, \mathsf{Ext}^1_{\mathcal{T}_n}(N, M).$$

2.3. **Maximal rigid objects of the cluster tube.** Let $\mathcal{C}$ be a Krull–Schmidt 2-Calabi–Yau triangulated category with suspension functor Σ. An object M of $\mathcal{C}$ is *rigid* if $\mathsf{Hom}_{\mathcal{C}}(M, \Sigma M) = 0$. It is *maximal rigid* if it is rigid and $\mathsf{Hom}_{\mathcal{C}}(M \oplus N, \Sigma(M \oplus N)) = 0$ implies that $N \in \mathsf{add}_{\mathcal{C}}(M)$, the additive hull of M in $\mathcal{C}$. It is *cluster-tilting* if it is rigid and $\mathsf{Hom}_{\mathcal{C}}(M, \Sigma N) = 0$ implies that $N \in \mathsf{add}_{\mathcal{C}}(M)$. In view of the second formula in Lemma 2.2, an indecomposable object

of the cluster tube $\mathcal{C}_n$ is rigid if and only if it is rigid in $\mathcal{T}_n$ if and only if it has length smaller than or equal to $n-1$. In particular, the zero object is maximal rigid in $\mathcal{T}_1$. From now on, we assume $n \geq 2$.

Removing the vertex n in the cyclic quiver $\overrightarrow{\Delta}_n$, we obtain the quiver $\overrightarrow{A}_{n-1}$ of type A_{n-1} with linear orientation. This yields a linear functor from $\mathsf{rep}\, \overrightarrow{A}_{n-1}$ to $\mathcal{T}_n$ which preserves the Hom-spaces and the Ext^1-spaces. Composing this functor with the functor from $\mathcal{T}_n$ to $\mathcal{C}_n$ described in Lemma 2.2, we obtain a functor F from $\mathsf{rep}\, \overrightarrow{A}_{n-1}$ to $\mathcal{C}_n$. For $a = 1, \ldots, n$, let $F_a = \Sigma^{-a+1} \circ F$.

Proposition 2.3 (Buan–Marsh–Vatne [5] Proposition 2.6). *An object of $\mathcal{C}_n$ is maximal rigid if and only if it is the image under some F_a of a tilting module in $\mathsf{rep}\, \overrightarrow{A}_{n-1}$.*

Note that each tilting module in $\mathsf{rep}\, \overrightarrow{A}_{n-1}$ contains as a direct summand the unique projective-injective indecomposable module. Therefore, the Loewy length of a maximal rigid object is $n-1$.

Theorem 2.4 (Vatne [27] Theorem 2.1). *Fix $a = 1, \ldots, n$. Let T be a basic tilting module in $\mathsf{rep}\, \overrightarrow{A}_{n-1}$, $B \cong kQ/I$ be the corresponding cluster-tilted algebra, where I is an admissible ideal of kQ. Then the endomorphism algebra $\mathsf{End}_{\mathcal{C}_n}(F_aT)$ of F_aT in $\mathcal{C}_n$ is isomorphic to $k\tilde{Q}/\tilde{I}$, where $\tilde{Q}$ is the quiver obtained from Q by adding a loop φ at the vertex corresponding to the projective-injective indecomposable module in $\mathsf{rep}\, \overrightarrow{A}_{n-1}$ and $\tilde{I}$ is the ideal of $k\tilde{Q}$ generated by I and φ^2.*

Proof. We sketch a proof (for $a = 1$), see [27] for the details. By the first formula in Lemma 2.2, we have

$$\mathsf{End}_{\mathcal{C}}(FT) = \mathsf{Hom}_{\mathcal{T}_n}(FT, FT) \oplus \mathsf{Ext}^1_{\mathcal{T}_n}(FT, \tau^{-1}FT).$$

Suppose $T = T_0 \oplus T_1$, where T_1 is injective and T_0 has no injective direct summand. Then $\tau^{-1}FT_0 \cong F\tau^{-1}_{A_{n-1}}T_0$, where $\tau_{A_{n-1}}$ is the Auslander–Reiten translation of $\mathsf{rep}\, \overrightarrow{A}_{n-1}$, and hence

$$
\begin{aligned}
\mathsf{End}_{\mathcal{C}}(FT) &= \mathsf{Hom}_{\mathcal{T}_n}(FT, FT) \oplus \mathsf{Ext}^1_{\mathcal{T}_n}(FT, F\tau^{-1}_{A_{n-1}}T_0) \oplus \mathsf{Ext}^1_{\mathcal{T}_n}(FT, \tau^{-1}FT_1) \\
&\cong \mathsf{Hom}_{\mathsf{rep}\, \overrightarrow{A}_{n-1}}(T, T) \oplus \mathsf{Ext}^1_{\mathsf{rep}\, \overrightarrow{A}_{n-1}}(T, \tau^{-1}_{A_{n-1}}T_0) \oplus \mathsf{Ext}^1_{\mathcal{T}_n}(FT, \tau^{-1}FT_1) \\
&\cong \mathsf{Hom}_{\mathsf{rep}\, \overrightarrow{A}_{n-1}}(T, T) \oplus \mathsf{Ext}^1_{\mathsf{rep}\, \overrightarrow{A}_{n-1}}(T, \tau^{-1}_{A_{n-1}}T) \oplus \mathsf{Ext}^1_{\mathcal{T}_n}(FT, \tau^{-1}FT_1) \\
&= B \oplus \mathsf{Ext}^1_{\mathcal{T}_n}(FT, \tau^{-1}FT_1),
\end{aligned}
$$

where B is the cluster-tilted algebra corresponding to T, as in the statement of the theorem. Let φ be a nonzero element of the space $\mathsf{Ext}^1_{\mathcal{T}_n}((1, n), \tau^{-1}(1, n))$, which is a 1-dimensional subspace of $\mathsf{Ext}^1_{\mathcal{T}_n}(FT, \tau^{-1}FT_1)$. The square of φ is clearly zero. Finally one checks that the

space $\mathsf{Ext}^1_{\mathcal{T}_n}(FT, \tau^{-1}FT_1)$ has a basis each of which factors through φ, and that there are no more relations. $\sqrt{}$

The object $(1,1) \oplus \ldots \oplus (1, n-1)$ is a typical maximal rigid object. Its endomorphism algebra is the quotient of the path algebra of the quiver

$$\cdot \longrightarrow \cdot \longrightarrow \cdots \longrightarrow \cdot \circlearrowright \varphi$$

modulo the ideal generated by φ^2.

We define the *wing* determined by a rigid object (a, b) of $\mathcal{C}_n$ to be the additive hull of the indecomposable objects in the triangle with vertices (a, b), $(a, 1)$ and $(a+b-1, 1)$ of the Auslander–Reiten quiver. For $a = 1, \ldots, n$, the essential image of the functor $F_a : \mathsf{rep}\,\overrightarrow{A}_{n-1} \to \mathcal{C}_n$ is exactly the wing of $(a, n-1)$. It follows from Proposition 2.3 that each maximal rigid object of $\mathcal{C}_n$ is in the wing of some $(a, n-1)$, and in this case it has $(a, n-1)$ as a direct summand. The following lemma, which appears in the proof of [5, Corollary 2.7] (see also [27, Theorem 4.9]), shows that a maximal rigid object of $\mathcal{C}_n$ is not a cluster-tilting object but not too far from being one.

Lemma 2.5 (Buan–Marsh–Vatne [5])**.** *Let T be a maximal rigid object in the wing of $(a, n-1)$ for $a = 1, \ldots, n$. Then for an indecomposable object M of $\mathcal{C}_n$, the Hom-space $\mathsf{Hom}_{\mathcal{C}_n}(T, \Sigma M)$ vanishes if and only if either M is isomorphic to a direct summand of T or M is isomorphic to $(a, sn-1)$ for some $s \geq 2$.*

3. Mutations of maximal rigid objects

Let $\mathcal{C}$ be a 2-Calabi–Yau Krull–Schmidt triangulated category. Let T be a basic maximal rigid object. Let R be an indecomposable direct summand of T, and write $T = R \oplus \bar{T}$. By [2, Theorem I.1.10 (a)], there is a unique indecomposable object R' of $\mathcal{C}$ such that R' is not isomorphic to R and $T' = R' \oplus \bar{T}$ is maximal rigid. There are, up to isomorphism, unique triangles (called *exchange triangles*)

$$R' \xrightarrow{f} B \xrightarrow{g} R \longrightarrow \Sigma R' \quad \text{and} \quad R \xrightarrow{f'} B' \xrightarrow{g'} R' \longrightarrow \Sigma R$$

with f, f' being left $\mathsf{add}(\bar{T})$-approximations and g, g' being right $\mathsf{add}(\bar{T})$-approximations. The procedure of obtaining T' from T is called a *mutation*. It is a *simple mutation* if $\dim_k \mathsf{Hom}_{\mathcal{C}}(R, \Sigma R') = \dim_k \mathsf{Hom}_{\mathcal{C}}(R', \Sigma R) = 1$, *cf.* [19].

3.1. **Mutations of maximal rigid objects in the cluster tube.** Let $n \geq 2$ be an integer and let $\mathcal{C}_n$ be the cluster tube of rank n. In this subsection we will study mutations of maximal rigid objects of $\mathcal{C}_n$.

Recall that $\mathsf{rep}\,\vec{A}_{n-1}$ has a unique indecomposable projective-injective object (up to isomorphism). Let $\bar{M}$ be an almost complete basic tilting module in $\mathsf{rep}\,\vec{A}_{n-1}$. According to [14, Proposition 2.3], $\bar{M}$ has precisely two complements if it has the unique projective-injective module as a direct summand and one complement otherwise. In the former case, let N and N' be the two complements, then we say that $\bar{M} \oplus N$ and $\bar{M} \oplus N'$ are related by a mutation. Let $a = 1, \ldots, n$. By Proposition 2.3, $F_a(\bar{M})$ is an almost complete maximal rigid object in $\mathcal{C}_n$, and hence it has two complements in $\mathcal{C}_n$ (see the beginning of this section). Therefore we have

Lemma 3.1. *Fix $a = 1, \ldots, n$. Let M and M' be two basic tilting modules in $\mathsf{rep}\,\vec{A}_{n-1}$. Then M and M' are related by a mutation of tilting modules if and only if F_aM and F_aM' are related by a mutation of maximal rigid objects.*

Lemma 3.2. *Let T be a basic maximal rigid object in the wing of $(a, n-1)$ for $a = 1, \ldots, n$. Suppose that T' is a maximal rigid object in $\mathcal{C}_n$ related to T by a mutation. Write $T = R \oplus \bar{T}$ and $T' = R' \oplus \bar{T}$, where R, R' are non-isomorphic indecomposable objects. The following conditions are equivalent*

    i) *the mutation is simple,*

    ii) *the length of R is strictly smaller than $n-1$,*

    iii) *the length of R' is strictly smaller than $n-1$,*

    iv) *T' is in the wing of $(a, n-1)$.*

Proof. Recall that $(a, n-1)$ is a direct summand of T, and all other indecomposable direct summands of T have length strictly smaller than $n-1$. Thus, if the length of R is $n-1$, then $R = (a, n-1)$, and the length of R' has to be $n-1$ as well. Since R' is not isomorphic to R, it follows that $R' = (a', n-1)$ for some $a' = 1, \ldots, n$ with $a' \neq a$. In this case, $\mathsf{Hom}_{\mathcal{C}_n}(R, \Sigma R')$ is 2-dimensional (Lemma 2.2), and T' is not in the wing of $(a, n-1)$. If the length of R is strictly smaller than $n-1$, then $(a, n-1)$ is a direct summand of $\bar{T}$, and hence a direct summand of T'. In particular, T' is in the wing of $(a, n-1)$ and the length of R' is strictly smaller than $n-1$. Let M and M' be tilting modules in $\mathsf{rep}\,\vec{A}_{n-1}$ such that $T = F_aM$ and $T' = F_aM'$. By Lemma 3.1, the tilting modules M and M' are related by a mutation. By [14, Theorem 1.1], one of $\mathsf{Ext}^1_{\mathcal{T}_n}(R, R') = \mathsf{Ext}^1_{\vec{A}_{n-1}}(R, R')$ and $\mathsf{Ext}^1_{\mathcal{T}_n}(R', R) = \mathsf{Ext}^1_{\vec{A}_{n-1}}(R', R)$ is 1-dimensional and the other vanishes. Hence $\mathsf{Hom}_{\mathcal{C}_n}(R, \Sigma R')$ is 1-dimensional, that is, the mutation is simple. $\qquad\qquad \sqrt{}$

Lemma 3.3. *Fix $a = 1, \ldots, n$. A maximal rigid object of $\mathcal{C}_n$ in the wing of $(a, n-1)$ can be obtained from $(a, 1) \oplus \ldots \oplus (a, n-1)$ by a sequence of simple mutations.*

Proof. Suppose $T = F_a M$, where M is a tilting module in $\mathsf{rep}\,\overrightarrow{A}_{n-1}$. By [15, Corollary 2.2], M can be obtained by a sequence of mutations from P, the basic projective generator of $\mathsf{rep}\,\overrightarrow{A}_{n-1}$. It follows from Lemma 3.1 that $T = F_a M$ can be obtained from $F_a P = (1,1) \oplus \ldots \oplus (1, n-1)$ by a sequence of mutations such that each intermediate maximal rigid object is in the wing of $(a, n-1)$. By Lemma 3.2, each intermediate mutation is simple. $\qquad\qquad\surd$

In the above, we reduced the study of simple mutations to the study of tilting modules in $\mathsf{rep}\,\overrightarrow{A}_{n-1}$. When the mutation is not simple, we can explicitly describe the resulting maximal rigid object. Let $T = (a, n-1) \oplus \bar{T}$ be a basic maximal rigid object and T' be the maximal rigid object obtained from T by mutating at $(a, n-1)$. Let b be the maximal integer such that (a, b) is a direct summand of $\bar{T}$. Then $n - b - 2$ is the maximal integer b' such that $(a + n - 1 - b', b')$ is a direct summand of $\bar{T}$. Moreover, an indecomposable direct summand of $\bar{T}$ is either in the wing of (a, b) or in the wing of $(a + b + 1, n - b - 2)$. These follow from Proposition 2.3 and the corresponding results on tilting modules in $\mathsf{rep}\,\overrightarrow{A}_{n-1}$ (see [13, Section 4.1]). In [27], the triple $((a, n-1); (a, b), (a + b + 1, n - b - 2))$ is called a *subwing triple*. It is readily seen that $a + b + 1$ (taken modulo n) is the unique integer in $1, \ldots, n$ different from a such that the wing determined by $(a + b + 1, n - 1)$ contains (a, b) and $(a + b + 1, n - b - 2)$. It follows that T' is isomorphic to $(a + b + 1, n - 1) \oplus \bar{T}$. The subwing triple associated to T' is $((a + b + 1, n - 1); (a + b + 1, n - b - 2), (a, b))$. See the following picture:
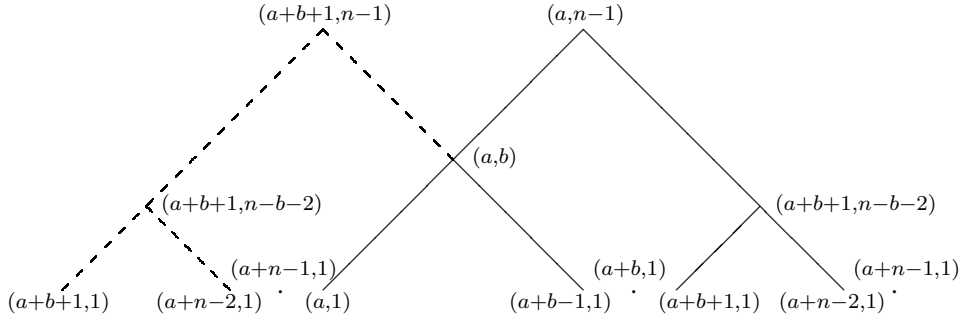
Here the vertices with the same label are identified.

4. Objects finitely presented by a maximal rigid object

Fix an integer $n \geq 2$ and let $\mathcal{C}_n$ be the cluster tube of rank n.

In [27], Vatne studied endomorphism algebras of maximal rigid objects of $\mathcal{C}_n$. Among other results, he proved that these algebras are Gorenstein of Gorenstein dimension 1 except when $n = 2$ in which case the algebras are symmetric, see [27, Proposition 3.3]. In fact, he showed that these algebras are gentle, and hence are Gorenstein by [12]. He described all indecomposable modules in terms of strings, and as a consequence he showed that these algebras are of finite representation type, see [27, Theorem 3.8]. In this section, we provide

a categorical explanation of the Gorenstein property and the representation-finiteness. Very recently, it has been proved by Zhou–Zhu in [28] that the endomorphism algebra of a maximal rigid object in *any* 2-Calabi–Yau triangulated category is Gorenstein of Gorenstein dimension at most 1.

4.1. **Objects finitely presented by a rigid object.** Let $\mathcal{C}$ be a Hom-finite Krull–Schmidt triangulated category over the field k with suspension functor Σ, and T a rigid object. An object M of $\mathcal{C}$ is *finitely presented by* T if there is a triangle in $\mathcal{C}$

$$T_1 \longrightarrow T_0 \xrightarrow{\ f\ } M \longrightarrow \Sigma T_1$$

with T_0, T_1 in $\mathsf{add}(T)$. The morphism f is necessarily a right $\mathsf{add}(T)$-approximation of M, and conversely, the cone of any $\mathsf{add}(T)$-approximation of an object M finitely presented by T belongs to $\mathsf{add}(\Sigma T)$. Let $\mathsf{pr}(T)$ denote the subcategory of $\mathcal{C}$ of objects finitely presented by T. Obviously, ΣT belongs to $\mathsf{pr}(T)$.

Let A be the endomorphism algebra of T. Let $\mathsf{mod}\,A$ denote the category of finite-dimensional right modules over A.

Lemma 4.1. *The functor* $\mathsf{Hom}_{\mathcal{C}}(T, ?) : \mathcal{C} \to \mathsf{mod}\,A$ *induces an equivalence of additive categories*

$$\mathsf{pr}(T)/(\Sigma T) \xrightarrow{\ \sim\ } \mathsf{mod}\,A,$$

where the category on the left is the additive quotient of $\mathsf{pr}(T)$ *by the ideal generated by* ΣT.

Proof. This is a special case of the first statement of [18, Proposition 6.2 (3)], *cf.* also [4, Theorem 2.2], [22, Proposition 2.1] and [25, Lemma 3.2]. $\sqrt{}$

Assume further that $\mathcal{C}$ is 2-Calabi–Yau. The following proposition is a special case of a more general result of Plamondon.

Proposition 4.2 (Plamondon [25] Proposition 2.7)**.** *Let* T *and* T' *be two maximal rigid objects of* $\mathcal{C}$ *related by a simple mutation. Then* $\mathsf{pr}(T) = \mathsf{pr}(T')$.

The following corollary of Proposition 4.2 generalizes [4, Theorem 4.2], [22, Proposition 2.2].

Corollary 4.3. *Let* T *and* T' *be two maximal rigid objects of* $\mathcal{C}$ *related by a simple mutation. Then the endomorphism algebras* $\mathsf{End}_{\mathcal{C}}(T)$ *and* $\mathsf{End}_{\mathcal{C}}(T')$ *are related by a nearly Morita equivalence in the sense of Ringel* [26].

Proof. The proof is similar to that of [4, Theorem 4.2]. For completeness we provide it here.

We have the following diagram

$$\begin{array}{ccc} \mathsf{pr}(T) & =\!\!=\!\!= & \mathsf{pr}(T') \\ {\scriptstyle G_T=\mathsf{Hom}_{\mathcal{C}}(T,?)} \downarrow & & \downarrow {\scriptstyle G_{T'}=\mathsf{Hom}_{\mathcal{C}}(T',?)} \\ \mathsf{mod}\,\mathsf{End}_{\mathcal{C}}(T) & & \mathsf{mod}\,\mathsf{End}_{\mathcal{C}}(T'). \end{array}$$

We assume that T and T' are basic and the mutation is at the indecomposable direct summand R of T. Suppose $T = R \oplus \bar{T}$ and $T' = R' \oplus \bar{T}$, where R' is indecomposable. Clearly $\Sigma T \oplus \Sigma R' = \Sigma T' \oplus \Sigma R$. The mutation being simple implies that $G_T(\Sigma R') \cong S$, the simple $\mathsf{End}_{\mathcal{C}}(T)$-module corresponding to R, and similarly, $G_{T'}(\Sigma R) \cong S'$, the simple $\mathsf{End}_{\mathcal{C}}(T')$-module corresponding to R'. Thus by Lemma 4.1 we obtain the following commutative diagram

$$\begin{array}{ccc} \mathsf{pr}(T)/(\Sigma T \oplus \Sigma R') & =\!\!=\!\!= & \mathsf{pr}(T')/(\Sigma T' \oplus \Sigma R) \\ \cong \downarrow & & \downarrow \cong \\ \mathsf{mod}\,\mathsf{End}_{\mathcal{C}}(T)/(S) & -\,-\,\succ & \mathsf{mod}\,\mathsf{End}_{\mathcal{C}}(T')/(S'), \end{array}$$

where the dashed arrow represents the desired nearly Morita equivalence. $\qquad\qquad \sqrt{}$

It is proved in [28] that the two algebras $\mathsf{End}_{\mathcal{C}}(T)$ and $\mathsf{End}_{\mathcal{C}}(T')$ as in Corollary 4.3 have the same representation type. Thus we have the following consequence of Lemma 3.3.

Corollary 4.4. *The endomorphism algebra of a maximal rigid object in the cluster tube of rank n is related to the algebra*

$$k(\;\cdot\;\longrightarrow\;\cdot\;\longrightarrow\;\cdots\;\longrightarrow\;\cdot\;\circlearrowright\;\varphi\;)/(\varphi^2)$$

by a sequence of nearly Morita equivalences. In particular, the two algebra have the same representation type.

4.2. **Rigid objects in the cluster tube and the Gorenstein property.** Let $\mathcal{C}_n$ be the cluster tube of rank n, and let T be a maximal rigid object. In this subsection we shall prove the following result.

Proposition 4.5. *Any indecomposable rigid object of $\mathcal{C}_n$ is finitely presented by T.*

This result still holds even if we replace $\mathcal{C}_n$ by any 2-Calabi–Yau triangulated category. This was first stated in [2], and a detailed proof can be found in [28]. Here we give a proof which relies on the features of the cluster tube $\mathcal{C}_n$.

We need some preparation. Let M, N be two objects of $\mathcal{T}_n$, viewed as objects of $\mathcal{C}_n$, and let $f : M \to N$ be a morphism in $\mathcal{C}_n$. According to the first formula in Lemma 2.2, f has two

components: $f_1 \in \mathsf{Hom}_{\mathcal{T}_n}(M, N)$ and $f_2 \in \mathsf{Ext}^1_{\mathcal{T}_n}(M, \tau^{-1}N)$. Form the triangle

$$C \longrightarrow M \overset{f}{\longrightarrow} N \longrightarrow \Sigma C \ .$$

As shown in Keller's proof of the main theorem (or rather of Theorem 9.9) in [21], we have a triangle in $\mathcal{D}(\mathcal{T}_n)$

$$\bigoplus (\tau^{-1} \circ \Sigma)^i C \longrightarrow \bigoplus (\tau^{-1} \circ \Sigma)^i M \overset{\oplus (\tau^{-1} \circ \Sigma)^i f}{\longrightarrow} \bigoplus (\tau^{-1} \circ \Sigma)^i N \longrightarrow$$

where all direct sums are over $\mathbb{Z}$. Taking cohomologies gives a long exact sequence

$$\tau^{-1}M \overset{\tau^{-1}f_1}{\longrightarrow} \tau^{-1}N \longrightarrow C \longrightarrow M \overset{f_1}{\longrightarrow} N,$$

where the short exact sequence

$$0 \longrightarrow \mathsf{cok}(\tau^{-1}f_1) \longrightarrow C \longrightarrow \mathsf{ker}(f_1) \longrightarrow 0$$

is induced from f_2 by the inclusion $\mathsf{ker}(f_1) \hookrightarrow M$ and the quotient $\tau^{-1}N \twoheadrightarrow \mathsf{cok}(\tau^{-1}f_1)$. Namely, we have the following commutative diagram

$$
\begin{array}{ccccccccc}
f_2: & 0 & \longrightarrow & \tau^{-1}N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
& & & \downarrow & & \downarrow & & \| & & \\
& 0 & \longrightarrow & \mathsf{cok}(\tau^{-1}f_1) & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0 \\
& & & \| & & \uparrow & & \uparrow & & \\
& 0 & \longrightarrow & \mathsf{cok}(\tau^{-1}f_1) & \longrightarrow & C & \longrightarrow & \mathsf{ker}(f_1) & \longrightarrow & 0 \ ,
\end{array}
$$

where the square in the upper-left corner is a pushout and the square in the lower-right corner is a pullback. As a consequence, the Loewy length of C is smaller than or equal to the sum of the Loewy lengths of M and N. Namely, we have proved the following lemma.

Lemma 4.6. *Let* $C \longrightarrow M \longrightarrow N \longrightarrow \Sigma C$ *be a triangle in* $\mathcal{C}_n$. *Then*

$$\ell\ell(C) \leq \ell\ell(M) + \ell\ell(N).$$

In particular, for an object M of $\mathsf{pr}(T)$ we have $\ell\ell(M) \leq 2(n-1)$.

Proof of Proposition 4.5. Let M be an indecomposable rigid object. Then $\ell\ell(M) \leq n-1$. Let $T_0 \overset{f}{\to} M$ be a right $\mathsf{add}_{\mathcal{C}_n}(T)$-approximation of M, and form the triangle

$$T_1 \longrightarrow T_0 \overset{f}{\longrightarrow} M \longrightarrow \Sigma T_1 \ .$$

Then $\mathsf{Hom}_{\mathcal{C}_n}(T, \Sigma T_1) = 0$ and $\ell\ell(T_1) \leq 2(n-1)$. Hence it follows from Lemma 2.5 that T_1 belongs to $\mathsf{add}_{\mathcal{C}_n}(T)$ since $\ell\ell(a, sn-1) = sn-1 > 2(n-1)$ for all $a = 1, \ldots, n$ and all $s \geq 2$. This finishes the proof of Proposition 4.5. $\qquad\qquad\surd$

As an application of Proposition 4.5, we have the following result which was first proved by Vatne in [27].

Corollary 4.7 (Vatne [27] Proposition 3.3). *The endomorphism algebra $A = \mathsf{End}_{\mathcal{C}_n}(T)$ of T is Gorenstein of Gorenstein dimension 1 unless $n = 2$, in which case A is symmetric.*

Proof. When $n = 2$, there are two basic maximal rigid objects up to isomorphism, namely, $(1, 1)$ and $(2, 1)$. Their endomorphism algebras are isomorphic to $k[x]/(x^2)$, which is symmetric.

We assume $n \geq 3$. The functor $\mathsf{Hom}_{\mathcal{C}_n}(T, ?)$ takes T to A and takes $\Sigma^2 T$ to $D(A)$ due to the 2-Calabi–Yau property. It follows from Proposition 4.5 that $\Sigma^2 T$ is finitely presented by T and T is finitely presented by ΣT. Therefore A has injective dimension at most 1 and $D(A)$ has projective dimension at most 1. It remains to show that $D(A)$ is not projective, or equivalently, A is not injective. It follows from Lemma 4.1 that for an object M of $\mathsf{pr}(T)$ the A-module $\mathsf{Hom}_{\mathcal{C}_n}(T, M)$ is projective if and only if M belongs to $\mathsf{add}_{\mathcal{C}_n}(\Sigma T \oplus T)$. Notice that $\mathsf{add}_{\mathcal{C}_n}(\Sigma T \oplus T)$ contains precisely two indecomposable objects of Loewy length $n - 1$: $(a, n - 1)$ and $(a - 1, n - 1)$. Therefore the direct summand $(a - 2, n - 1)$ of $\Sigma^2 T$, and hence $\Sigma^2 T$, does not belong to $\mathsf{add}_{\mathcal{C}_n}(\Sigma T \oplus T)$, showing that $D(A)$ is not projective. $\sqrt{}$

4.3. **Objects finitely presented by a maximal rigid object and representation-finiteness.** Fix a basic maximal rigid object T of the cluster tube $\mathcal{C}_n$. In the preceding subsection we showed that each indecomposable rigid object lies in $\mathsf{pr}(T)$. In this subsection, we will determine all the indecomposable objects in $\mathsf{pr}(T)$. In particular, we will show that there are only finitely many of them. In view of Lemma 4.1, we deduce that the endomorphism algebra of T is of finite representation type.

Following Vatne [27, Section 4], we define $\mathcal{F}$ to be the set of indecomposable objects (a, b) satisfying either (1) $b \leq n - 1$, i.e. (a, b) is rigid, or (2) $n \leq b \leq 2(n - 1)$ and $a + b \leq 2n - 1$. Here is the picture of $\mathcal{F}$ for $n = 4$ (the black vertices belong to $\mathcal{F}$ while the white ones do not), see also [27, Figure 6]:

$$(1,1) \quad (2,1) \quad (3,1) \quad (4,1) \quad (1,1)$$
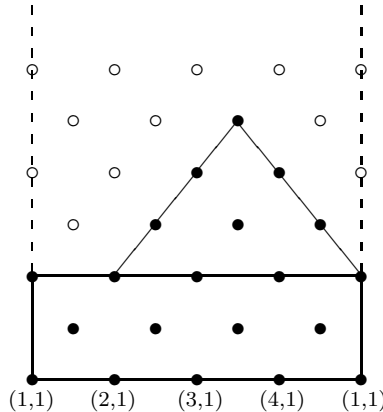
The main result of this subsection is

Proposition 4.8. *Suppose that T is in the wing of $(a, n-1)$ for $a = 1, \ldots, n$. Then an indecomposable object of $\mathcal{C}_n$ belongs to $\mathsf{pr}(T)$ if and only if it lies in $\Sigma^{-(a-1)}\mathcal{F}$.*

This shows that generally the category of finitely presented objects by a maximal rigid object is not preserved under mutation. Let $A = \mathsf{End}_{\mathcal{C}_n}(T)$ be the endomorphism algebra of T. We have the following results due to Vatne, who proved it using different methods.

Corollary 4.9.   a) (Vatne [27, Theorem 4.8 (2)]) *The functor $\mathsf{Hom}_{\mathcal{C}_n}(T, ?)$ induces a bijection between $\Sigma^{-(a-1)}\mathcal{F} \backslash \mathsf{add}(\Sigma T)$ and isoclasses of indecomposable A-modules.*
   b) (Vatne [27, Theorem 3.8]) *The algebra A is representation-finite. The number of indecomposable A-modules is $\frac{3}{2}n^2 - \frac{5}{2}n + 1$.*

Proof. a) follows from Lemma 4.1 and Proposition 4.8.
   b) The number of objects of $\Sigma^{-(a-1)}\mathcal{F}$ is

$$n(n-1) + \frac{n(n-1)}{2} = \frac{3n(n-1)}{2}.$$

Therefore, the number of indecomposable objects in $\mathsf{mod}\, A$ is

$$\frac{3n(n-1)}{2} - (n-1) = \frac{3}{2}n^2 - \frac{5}{2}n + 1.$$

$$\sqrt{}$$

The rest of this subsection is devoted to proving Proposition 4.8. By Proposition 4.2 and Lemma 3.3, we have $\mathsf{pr}(T) = \mathsf{pr}((a, 1) \oplus \ldots \oplus (a, n-1))$. Observing that the validity of the statement is preserved under the suspension functor Σ, we reduce the proof to the case $T = (1, 1) \oplus \ldots \oplus (1, n-1)$.

Suppose that $T = (1, 1) \oplus \ldots \oplus (1, n-1)$. In view of Lemma 4.6 and Proposition 4.5, we only need to consider the objects (a, b) with $n \le b \le 2(n-1)$. As pointed out in Section 4.1, (a, b) belongs to $\mathsf{pr}(T)$ if and only if the cone of a right $\mathsf{add}_{\mathcal{C}_n}(T)$-approximation of (a, b) belongs to $\mathsf{add}_{\mathcal{C}_n}(\Sigma T)$. We will find a right $\mathsf{add}_{\mathcal{C}_n}(T)$-approximation of (a, b) and determine whether its cone is in $\mathsf{add}_{\mathcal{C}_n}(\Sigma T)$. It is easy to see that any morphism from $(1, b')$ with $b' \le n-2$ to (a, b) factors through $(1, n-1)$, and hence a right $\mathsf{add}_{\mathcal{C}_n}((1, n-1))$-approximation of (a, b) is a right $\mathsf{add}_{\mathcal{C}_n}(T)$-approximation. We have

$$\mathsf{Hom}_{\mathcal{C}_n}((1, n-1), (a, b)) \;\; = \;\; \mathsf{Hom}_{\mathcal{T}_n}((1, n-1), (a, b)) \oplus \mathsf{Ext}^1_{\mathcal{T}_n}((1, n-1), (a+1, b)) \,,$$

where

$$\mathsf{Hom}_{\mathcal{T}_n}((1, n-1), (a, b)) = \begin{cases} k & \text{if } a \neq n \\ 0 & \text{if } a = n \end{cases},$$

$$\mathsf{Ext}^1_{\mathcal{T}_n}((1, n-1), (a+1, b)) = \begin{cases} k & \text{if } a + b \neq 2n - 1 \\ 0 & \text{if } a + b = 2n - 1 \end{cases}.$$

We divide the problem into four cases.

Case 1: $a = n$ and $a + b = 2n - 1$. This is not possible since $b \geq n$.

Case 2: $a = n$ and $a + b \neq 2n - 1$, i.e. $a = n$. In this case, $\tau^{-1}(n, b) = (1, b)$. Let f_2 be a basis element of $\mathsf{Ext}^1_{\mathcal{T}_n}((1, n-1), (1, b))$:

$$f_2: \quad 0 \longrightarrow (1, b) \longrightarrow (1, 2n-1) \oplus (1, b-n) \longrightarrow (1, n-1) \longrightarrow 0 \ .$$

Then f_2 is a right $\mathsf{add}_{\mathcal{C}_n}(T)$-approximation of (n, b). Form the triangle in $\mathcal{C}_n$

$$C \longrightarrow (1, n-1) \xrightarrow{\ f_2\ } (n, b) \longrightarrow \Sigma C \ .$$

As explained in the proof of Lemma 4.6, we have a long exact sequence

$$(2, n-1) \xrightarrow{\ 0\ } (1, b) \longrightarrow C \longrightarrow (1, n-1) \xrightarrow{\ 0\ } (n, b),$$

where the class of the short exact sequence in the middle is exactly f_2. Therefore $C \cong (1, 2n-1) \oplus (1, b-n)$ does not belong to $\mathsf{add}_{\mathcal{C}_n}(T)$, and hence (n, b) does not belong to $\mathsf{pr}(T)$.

Case 3: $a \neq n$ and $a + b = 2n - 1$. Let f_1 be a basis element of $\mathsf{Hom}_{\mathcal{T}_n}((1, n-1), (a, b))$. Then f_1 is a right $\mathsf{add}_{\mathcal{C}_n}(T)$-approximation of (a, b). Form the triangle in $\mathcal{C}_n$

$$C \longrightarrow (1, n-1) \xrightarrow{\ f_1\ } (a, b) \longrightarrow \Sigma C \ .$$

We obtain a long exact sequence

$$(2, n-1) \xrightarrow{\ \tau^{-1} f_1\ } (a+1, b) \longrightarrow C \longrightarrow (1, n-1) \xrightarrow{\ f_1\ } (a, b),$$

where the short exact sequence

$$0 \longrightarrow \mathsf{cok}(\tau^{-1} f_1) \longrightarrow C \longrightarrow \mathsf{ker}(f_1) \longrightarrow 0$$

splits. It follows that $C \cong (1, a-1) \oplus (1, n-1)$ belongs to $\mathsf{add}_{\mathcal{C}_n}(T)$, and hence (a, b) belongs to $\mathsf{pr}(T)$.

Case 4: $a \neq n$ and $a + b \neq 2n - 1$. Let f_1 be a basis element of $\mathsf{Hom}_{\mathcal{T}_n}((1, n-1), (a, b))$ and f_2 be a basis element of $\mathsf{Ext}^1_{\mathcal{T}_n}((1, n-1), (a+1, b))$:

$$f_2: \quad 0 \rightarrow (a+1, b) \rightarrow (a+1, n\lfloor \tfrac{a+b}{n} \rfloor + n - a - 1) \oplus (1, a+b - n\lfloor \tfrac{a+b}{n} \rfloor) \rightarrow (1, n-1) \rightarrow 0.$$

Then $f = (f_1, f_2) : (1, n-1) \oplus (1, n-1) \to (a, b)$ is a right $\mathsf{add}_{\mathcal{C}_n}(T)$-approximation of (a, b). Form the triangle in $\mathcal{C}_n$

$$C \longrightarrow (1, n-1) \oplus (1, n-1) \xrightarrow{\;f\;} (a, b) \longrightarrow \Sigma C \;.$$

We obtain a long exact sequence

$$(2, n-1) \oplus (2, n-1) \xrightarrow{(\tau^{-1} f_1, 0)} (a+1, b) \longrightarrow C \longrightarrow (1, n-1) \oplus (1, n-1) \xrightarrow{(f_1, 0)} (a, b),$$

where the short exact sequence

$$0 \longrightarrow \mathsf{cok}((\tau^{-1} f_1, 0)) \longrightarrow C \longrightarrow \mathsf{ker}((f_1, 0)) \longrightarrow 0$$

i.e.

$$0 \longrightarrow (1, a+b-n) \longrightarrow C \longrightarrow (1, a-1) \oplus (1, n-1) \longrightarrow 0$$

is induced from the short exact sequence

$$(0, f_2) : \; 0 \longrightarrow (a+1, b) \longrightarrow (1, n-1) \oplus (a+1, n\lfloor \tfrac{a+b}{n} \rfloor + n - a - 1)$$

$$\oplus (1, a+b-n\lfloor \tfrac{a+b}{n} \rfloor) \longrightarrow (1, n-1) \oplus (1, n-1) \longrightarrow 0.$$

It follows that $C \cong (1, a-1) \oplus (1, n\lfloor \tfrac{a+b}{n} \rfloor - 1) \oplus (1, a+b-n\lfloor \tfrac{a+b}{n} \rfloor)$. Therefore (a, b) belongs to $\mathsf{pr}(T)$ if and only if C belongs to $\mathsf{add}_{\mathcal{C}_n}(T)$ if and only if $\lfloor \tfrac{a+b}{n} \rfloor < 2$ if and only if $a+b < 2n-1$ (recall that in this case $a+b \neq 2n-1$). This completes the proof of Proposition 4.8.

5. Derived equivalence classification

Fix an integer $n \geq 2$, and let $\mathcal{C}_n$ be the cluster tube of rank n. In this section, we provide a derived equivalence classification for endomorphism algebras of maximal rigid objects of $\mathcal{C}_n$. This classification is analogous to that of Buan–Vatne [7] for cluster-tilted algebras of type A.

Let T be a basic maximal rigid object in $\mathcal{C}_n$. We also view T as a basic tilting module in $\mathsf{rep}\, \overrightarrow{A}_{n-1}$, *cf.* Proposition 2.3. Let $B \cong kQ/I$ be the cluster-tilted algebra corresponding to T, where I is an admissible ideal of kQ. Recall from Theorem 2.4 that the endomorphism algebra $A = \mathsf{End}_{\mathcal{C}_n}(T)$ of T in $\mathcal{C}_n$ is isomorphic to $k\tilde{Q}/\tilde{I}$, where $\tilde{Q}$ is the quiver obtained from Q by adding a loop φ at the vertex c corresponding to the projective-injective indecomposable module in $\mathsf{rep}\, \overrightarrow{A}_{n-1}$ and $\tilde{I}$ is the ideal of $k\tilde{Q}$ generated by I and φ^2. We denote $\gamma_c(Q) = \tilde{Q}$ and $\delta_c(\tilde{Q}) = Q$.

Following [27], we give a description of the quivers $\tilde{Q}$. The quivers $\tilde{Q}$ are exactly the quivers with $n-1$ vertices and satisfying the following

- all non-trivial minimal cycles of length at least 2 in the underlying graph are oriented and of length 3 (in particular, there are no multiple arrows or 2-cycles),

- any vertex has at most four neighbours,
- if a vertex has four neighbours, then two of its adjacent arrows belong to one 3-cycle, and the other two belong to another 3-cycle,
- if a vertex has three neighbours, then two of its adjacent arrows belong to a 3-cycle, and the third does not belong to any 3-cycle,
- there is precisely one loop φ, at a vertex c which has zero neighbour (this happens when and only when $n = 2$), or has one neighbour, or has two neighbours and is traversed by a 3-cycle.

Let $\widetilde{\mathcal{Q}}_{n-1}$ denote the set of such quivers. For a quiver $\tilde{Q} \in \widetilde{\mathcal{Q}}_{n-1}$, let $I_{\tilde{Q}}$ be the ideal of $k\tilde{Q}$ generated by the square of the unique loop and all paths of length 2 involved in a 3-cycle. The following is a corollary of Theorem 2.4 and the description of the relations of cluster-tilted algebras of type A (*cf.* [9], [7]).

Corollary 5.1. *An algebra is the endomorphism algebra of a maximal rigid object of $\mathcal{C}_n$ if and only if it is isomorphic to $k\tilde{Q}/I_{\tilde{Q}}$ for some $\tilde{Q} \in \widetilde{\mathcal{Q}}_{n-1}$. In particular, the endomorphism algebra of a maximal rigid object of $\mathcal{C}_n$ is determined by its quiver.*

For a quiver in $\widetilde{\mathcal{Q}}_{n-1}$, we will always denote by c the vertex where the unique loop lies.

Proposition 5.2. *Assume $n \geq 3$. Let T and T' be two basic maximal rigid objects of $\mathcal{C}_n$ related by a mutation. Let $\tilde{Q}$ and $\tilde{Q}'$ be the quivers of $\mathsf{End}_{\mathcal{C}_n}(T)$ and $\mathsf{End}_{\mathcal{C}_n}(T')$ respectively. Assume that T is in the wing of $(a, n-1)$.*

a) *If the mutation is simple, then the quivers $\tilde{Q}$ and $\tilde{Q}'$ are related by a Fomin–Zelevinsky mutation.*

b) *If the mutation is not simple, there are two cases*

   1) *if the vertex c has one neighbour, then $\tilde{Q}'$ is obtained from $\tilde{Q}$ by reversing the unique arrow adjacent to c;*

   2) *if the vertex c has two neighbours, then $\tilde{Q}'$ is obtained from $\tilde{Q}$ by reversing all arrows in the unique 3-cycle traversing c.*

Proof. a) Assume that the mutation is simple, and the mutation is performed at i. Then there are no loops or 2-cycles at i. The assertion follows from a local version of [2, Theorem I.1.6] (note that the proof of [2, Theorem I.1.6] is local).

b) Assume that the mutation is not simple. By Lemma 3.2, the mutation is performed at the direct summand $(a, n-1)$ of T, correspondingly, the vertex c of $\tilde{Q}$. In view of the exchange triangles, the arrows of $\tilde{Q}$ adjacent to c are reversed. As in the proof of [2, Theorem I.1.6], it remains to consider the situation where we have arrows $j \to c \to j'$. If c has one neighbour in $\tilde{Q}$, it also has only one neighbour in $\tilde{Q}'$ and there are no subquivers of the form $j \to c \to j'$.

In this case, to obtain $\tilde{Q}'$ from $\tilde{Q}$ we only need to reverse the unique arrow adjacent to c. If c has two neighbours, say, j and j', in $\tilde{Q}$, then j and j' are also the only neighbours of c in $\tilde{Q}'$. Moreover, in $\tilde{Q}$, there is precisely one subquiver of the form $j \to c \to j'$. Since c is traversed by a 3-cycle in $\tilde{Q}$, there is an arrow $j \leftarrow j'$. As explained above, in $\tilde{Q}'$ there is a subquiver of the form $j \leftarrow c \leftarrow j'$. Since $\tilde{Q}' \in \widetilde{\mathcal{Q}}_{n-1}$, it follows that in $\tilde{Q}'$, the vertex c is also traversed by a 3-cycle, and hence there is an arrow $j \to j'$. Therefore, to obtain $\tilde{Q}'$ from $\tilde{Q}$, all the three arrows in the 3-cycle traversing c are reversed. $\sqrt{}$

Proposition 5.2 shows that the quiver $\tilde{Q}'$ only depends on the quiver $\tilde{Q}$ and the vertex i at which the mutation is taken and does not depend on the choice of T. We will write $\tilde{Q}' = \mu_i(\tilde{Q})$, and by abuse of language we call it the *mutation* of $\tilde{Q}$ at i.

Lemma 5.3. *Let $\tilde{Q}$ be a quiver in $\widetilde{\mathcal{Q}}_{n-1}$ and i a vertex of $\tilde{Q}$. If i is different from c or $i = c$ has only one neighbour, then $\mu_i \gamma_c(Q) = \gamma_c \mu_i(Q)$, where $Q = \delta_c(\tilde{Q})$.*

Proof. When i is different from c, the statement follows from Proposition 5.2 a), the definition of Fomin–Zelevinsky mutation and the definition of γ_c. When $i = c$ has only one neighbour, the statement follows from Proposition 5.2 b) and the definition of γ_c. $\sqrt{}$

Next we give a sufficient condition for the endomorphism algebras of two neighbouring maximal rigid objects to be derived equivalent. It is worth noting that in general the nearly Morita equivalence in Corollary 4.3 is not a consequence of the derived equivalence.

Proposition 5.4. *Let T and T' be two basic maximal rigid objects of $\mathcal{C}_n$ related by a mutation. Then their endomorphism algebras are related by a derived equivalence if the corresponding mutation of quivers does not change the number of 3-cycles.*

Proof. Suppose $T = R \oplus \bar{T}$ and $T' = R' \oplus \bar{T}$ with R and R' indecomposable. We first assume that T and T' are related by a simple mutation. By Proposition 5.2, up to symmetry the quivers of $\mathsf{End}_{\mathcal{C}_n}(T)$ and $\mathsf{End}_{\mathcal{C}_n}(T')$ locally look like
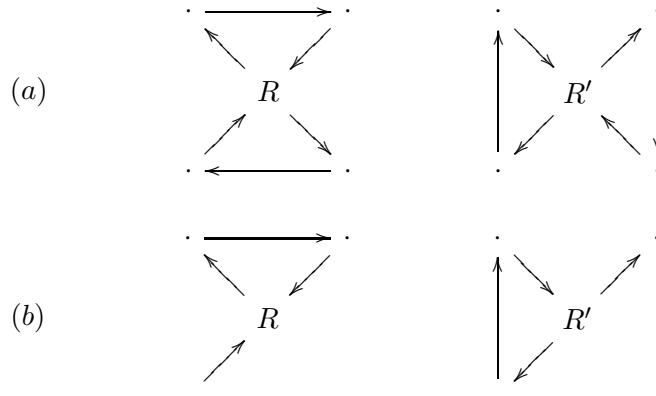
(c)

$$\mathsf{End}_{\mathcal{C}_n}(T) \qquad\qquad \mathsf{End}_{\mathcal{C}_n}(T')$$

It follows from Corollary 5.1 that a linear combination $\sum_j \lambda_j p_j$ of paths is nonzero if and only if there is some j such that p_j is nonzero and λ_j is nonzero. Moreover, in cases (a) (b) and (d), for any nonzero path p starting at R, there is an arrow β ending at R such that $p\beta$ is nonzero, and for any nonzero path p' ending at R', there is an arrow α' starting at R' such that $\alpha'p'$ is nonzero; while in case (c), the two quivers have different numbers of 3-cycles. It follows from [24, Proposition 2.3, Theorem 4.2] that in cases (a), (b) and (d) the algebras $\mathsf{End}_{\mathcal{C}_n}(T)$ and $\mathsf{End}_{\mathcal{C}_n}(T')$ are derived equivalent (via a BB-tilting module).

Now we assume that T and T' are related by a non-simple mutation. We claim that the endomorphism algebra of T is derived equivalent to that of T'. As in the proof of Proposition 5.2 b), let $((a, n-1); (a, b), (a+b+1, n-b-2))$ and $((a+b+1, n-1); (a+b+1, n-b-2), (a, b))$ be the subwing triples associated to T and T' respectively. There are three cases:

Case 1: $b = 0$. In this case, locally at $(a, n-1)$ and $(a+1, n-1)$ we have

The square of the loops φ^2 and $\varphi^{\star 2}$ are the (local) relations. The two exchange triangles are

$$(a, n-1) \xrightarrow{f=\binom{\alpha\varphi}{\alpha}} (a+1, n-2) \oplus (a+1, n-2) \xrightarrow{g=(\alpha^\star, \varphi^\star \alpha^\star)} (a+1, n-1) \longrightarrow \quad,$$

$$(a+1, n-1) \longrightarrow 0 \longrightarrow (a, n-1) \longrightarrow \quad.$$

A morphism in $\mathsf{Hom}_{\mathcal{C}_n}(T, (a, n-1))$ is a linear combination of paths ending at $(a, n-1)$, i.e. $\lambda_1 id_{(a,n-1)} + \lambda_2 \varphi$. Such a combination is sent by $\mathsf{Hom}_{\mathcal{C}_n}(T, f)$ to $\lambda_1 \binom{\alpha\varphi}{\alpha} + \lambda_2 \binom{0}{\alpha\varphi}$, which is zero if and only if $\lambda_1 = \lambda_2 = 0$. Therefore, $\mathsf{Hom}_{\mathcal{C}_n}(T, f)$ is injective. Dually, one shows that $\mathsf{Hom}_{\mathcal{C}_n}(g, T')$ is injective. It follows from [17, Lemma 3.4] that $\mathsf{End}_{\mathcal{C}_n}(T)$ and $\mathsf{End}_{\mathcal{C}_n}(T')$ are derived equivalent.

Case 2: $b = n - 2$. This case is dual to Case 1.

Case 3: $b \neq 0$ and $b \neq n - 2$. Locally we have
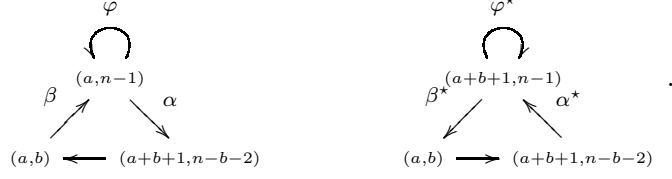
The (local) relations are the squares of the loops and the compositions of two consecutive arrows in the same 3-cycles. The exchange triangles are

$$(a + b + 1, n - 1) \xrightarrow{\binom{\beta^\star \varphi^\star}{\beta^\star}} (a, b) \oplus (a, b) \xrightarrow{(\beta, \varphi\beta)} (a, n - 1) \longrightarrow$$

$$(a, n - 1) \xrightarrow{f = \binom{\alpha\varphi}{\varphi}} (a + b + 1, n - b - 2) \oplus (a + b + 1, n - b - 2) \xrightarrow{g = (\alpha^\star, \varphi^\star \alpha^\star)} (a + b + 1, n - 1) \longrightarrow$$

A nonzero path ending at $(a, n-1)$ is of the form βp or $\varphi\beta p$, where p is a nonzero path ending at (a, b) and not passing through $(a+b+1, n-b-2)$. Thus a morphism of $\mathsf{Hom}_{\mathcal{C}_n}(T, (a, n-1))$ is of the form $\sum_p (\lambda_1^p \beta p + \lambda_2^p \varphi\beta p)$. Such a morphism is sent by $\mathsf{Hom}_{\mathcal{C}_n}(T, f)$ to $\sum_p \binom{\lambda_1^p \alpha\varphi\beta p}{\lambda_2^p \alpha\varphi\beta p}$, which is zero if and only if all coefficients are zero. Therefore, $\mathsf{Hom}_{\mathcal{C}_n}(T, f)$ is injective. Dually, one shows that $\mathsf{Hom}_{\mathcal{C}_n}(g, T')$ is injective. Again it follows from [17, Lemma 3.4] that $\mathsf{End}_{\mathcal{C}_n}(T)$ and $\mathsf{End}_{\mathcal{C}_n}(T')$ are derived equivalent.

Putting the above arguments together, we obtain the desired result. $\sqrt{}$

Theorem 5.5. *Let A and A' be the endomorphism algebras of two maximal rigid objects in $\mathcal{C}_n$. Then A and A' are derived equivalent if and only if their quivers have the same number of 3-cycles.*

Proof. If the number of 3-cycles of the quivers are different, then by Lemma 5.6 the determinants of the Cartan matrices of the two algebras are not equal and thus they are not derived equivalent, and the necessity follows.

It remains to prove the sufficiency. Let $\tilde{Q}$ be the quiver of A, c the vertex where the unique loop lies, and $Q = \delta_c(\tilde{Q})$, the quiver obtained from $\tilde{Q}$ by deleting the loop. By [7, Lemma 2.3], the quiver Q can be obtained by a sequence of mutations without changing the number of 3-cycles from the quiver


One proves by induction that in each intermediate quiver, the vertex c either has one neighbour, or has two neighbours and is traversed by a 3-cycle. Moreover, if an intermediate mutation is performed at c, then in the corresponding intermediate quiver c must have only one

neighbour; otherwise the mutation changes the number of 3-cycles. Therefore by Lemma 5.3, the operation γ_c commutes with the given sequence of mutations, and hence $\tilde{Q} = \gamma_c(Q)$ can be obtained by a sequence of mutations without changing the number of 3-cycles from the quiver
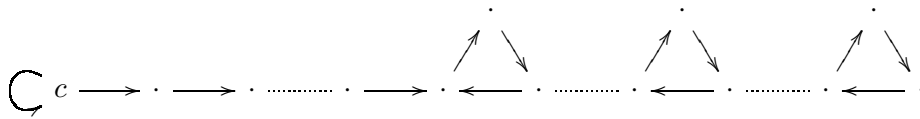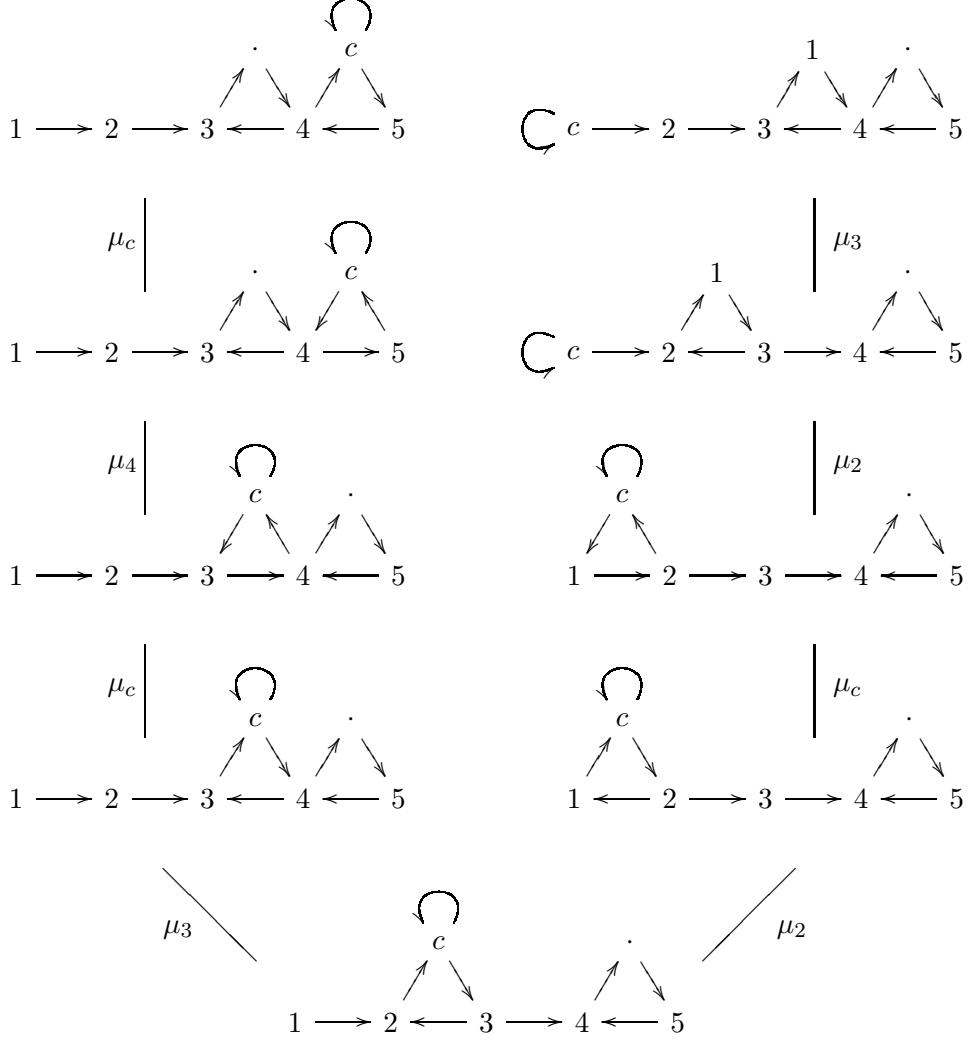

or the quiver
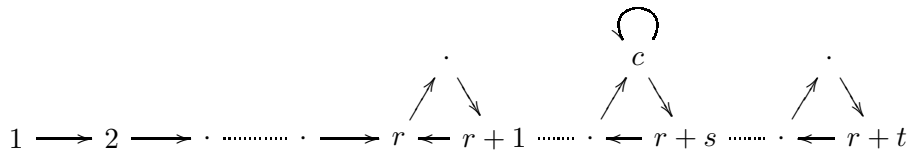
One checks by induction that successive mutations (from the right to the left) at the following sequence of vertices

$$(r, r - 1, \ldots, 2, c, 2, \ldots, r - 1, r; c, r + 1, \ldots, r + s - 1, c)$$

takes the second quiver to the first one. For example, when $r = 3$, $s = t = 2$ $(n = 8)$, the sequence of mutations at vertices $(3, 2, c, 2, 3; c, 4, c)$ yields the following sequence of quivers

$$
\begin{array}{ccc}
\text{(quiver 1)} & & \text{(quiver 2)} \\
\downarrow \mu_c & & \downarrow \mu_3 \\
\text{(quiver 3)} & & \text{(quiver 4)} \\
\downarrow \mu_4 & & \downarrow \mu_2 \\
\text{(quiver 5)} & & \text{(quiver 6)} \\
\downarrow \mu_c & & \downarrow \mu_c \\
\text{(quiver 7)} & & \text{(quiver 8)} \\
{}_{\mu_3} \searrow & & \nearrow {}_{\mu_2} \\
& \text{(quiver 9)} &
\end{array}
$$

One can check this example (and more examples) by using Keller's quiver mutation applet [20]: one draws the quiver without the loop and performs mutations, imagining that there is a loop at the vertex c and keeping in mind that each time mutating at the vertex c one has to add an arrow by hand (compare $\mu_c \gamma_c(Q)$ and $\gamma_c \mu_c(Q)$ when c is traversed by a 3-cycle).

Therefore the quiver of A' can be obtained from the quiver of A by a sequence $\underline{\mu}$ of mutations without changing the number of 3-cycles. Suppose $A = \mathsf{End}_{\mathcal{C}_n}(T)$ for a maximal rigid object T, and let $T' = \underline{\mu}(T)$. Then by Corollary 5.1 and Proposition 5.2, we have $A' = \mathsf{End}_{\mathcal{C}_n}(T')$. So the sufficiency follows from Proposition 5.4. $\qquad\qquad \sqrt{}$

Lemma 5.6. *Let A be the endomorphism algebra of a maximal rigid object of $\mathcal{C}_n$. Then the determinant of the Cartan matrix of A is 2^{t+1}, where t is the number of 3-cycles in the quiver of A.*

Proof. The algebra A is gentle, see [27, Theorem 3.1]. Moreover, all cycles are 3-cycles with full relations. Thus it follows from [16, Theorem 1] that the determinant of the Cartan matrix of A is 2^{t+1}. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \sqrt{}$

6. QUIVERS WITH POTENTIAL

In this section we study the quivers with potential associated to maximal rigid objects of cluster tubes. Assume the characteristic of the base field k is not 3.

6.1. **Quivers with potential and their mutations.** A *quiver with potential* is a pair (Q, W), where Q is a finite quiver and W is an infinite linear combination of nontrivial cycles of Q. To a quiver with potential is associated an algebra, called the *Jacobian algebra*, which is a certain quotient of the completed path algebra of Q (see [10] for the precise definition). Assume that in the expression of W all cycles have length ≥ 3. Given a vertex i of Q which is not involved in a loop or 2-cycle, one can extend Derksen–Weyman–Zelevinsky's mutation in [10] to (Q, W) at i (the quivers in [10] do not have loops). The mutation yields a new quiver with potential, denoted by $\mu_i(Q, W)$.

6.2. **Quivers with potential in cluster tubes.** Fix an integer $n \geq 2$ and let $\mathcal{C}_n$ be the cluster tube of rank n.

Let T be a basic maximal rigid object in $\mathcal{C}_n$. We associate to T a quiver with potential $(\tilde{Q}, \tilde{W})$ as follows. By Corollary 5.1 the endomorphism algebra $A = \mathsf{End}_{\mathcal{C}_n}(T)$ of T in $\mathcal{C}_n$ is isomorphic to $k\tilde{Q}/I_{\tilde{Q}}$, where $\tilde{Q}$ is a quiver in the set $\widetilde{\mathcal{Q}}_{n-1}$. Let $\tilde{W} = W_{\tilde{Q}}$ be the sum of the cube of the unique loop and all 3-cycles of $\tilde{Q}$. It follows from the definition of $I_{\tilde{Q}}$ that A is isomorphic to the Jacobian algebra of $(\tilde{Q}, \tilde{W})$.

The category $\mathcal{C}_n$ being Hom-finite, the algebra A is a finite-dimensional Jacobian algebra. Therefore it follows from [1, Theorem 3.6] that there is a 2-Calabi–Yau triangulated category $\mathcal{C}$ with a cluster-tilting object M whose endomorphism algebra is isomorphic to A. Applying [22, Proposition 2.1, Theorem 3.3] to the pair $(\mathcal{C}, M)$, we obtain that A is Gorenstein of Gorenstein dimension at most 1 and is stably 3-Calabi–Yau.

Let c be the vertex of $\tilde{Q}$ at which the unique loop lies. Let Q be the quiver obtained from $\tilde{Q}$ by deleting the loop, and W the potential obtained from $\tilde{W}$ by subtracting the cube of the loop. Then the Jacobian algebra of (Q, W) is isomorphic to the cluster-tilted algebra B of T viewed as a titling module in $\mathsf{rep}\,\overrightarrow{A}_{n-1}$ (see Theorem 2.4). Denote $(\tilde{Q}, \tilde{W}) = \gamma_c(Q, W)$, $(Q, W) = \delta_c(\tilde{Q}, \tilde{W})$.

Lemma 6.1. *For a vertex i of Q different from c, we have $\mu_i \gamma_c(Q, W) = \gamma_c \mu_i(Q, W)$.*

Proof. This follows from the definitions of μ_i and γ_c. $\qquad\qquad\qquad\qquad\qquad\qquad\quad \sqrt{}$
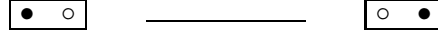
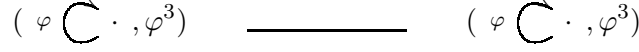FIGURE 1. The mutation graph of $\mathcal{C}_2$ – maximal rigid objects

FIGURE 2. The mutation graph of $\mathcal{C}_2$ – quivers with potential

Let $\widetilde{\mathcal{QP}}_{n-1}$ denote the set of quivers with potential $(\tilde{Q}, W_{\tilde{Q}})$ for $\tilde{Q} \in \tilde{\mathcal{Q}}_{n-1}$. Corollary 5.1 is reformulated as

Corollary 6.2. *An algebra is the endomorphism algebra of a maximal rigid object of $\mathcal{C}_n$ if and only if it is isomorphic to the Jacobian algebra of a quiver with potential in $\widetilde{\mathcal{QP}}_{n-1}$.*

Next we study the change of the quivers with potential in $\widetilde{\mathcal{QP}}_{n-1}$ induced from the mutation of maximal rigid objects. We first give some examples.

Example 6.3. *In this example we draw the exchange graph of mutations of maximal rigid objects in $\mathcal{C}_n$ for $n = 2, 3, 4$ and the corresponding graph with maximal rigid objects replaced by quivers with potential associated to them. In the former graph, (indecomposable direct summands of) maximal rigid objects are given as filled-in circles of the first $n - 1$ layers of the Auslander–Reiten quiver of $\mathcal{C}_n$.*

n=2: There are two basic maximal rigid objects up to isomorphism, which are indecomposable. The mutation graphs are shown in Figures 1 and 2.

n=3: There are six basic maximal rigid objects up to isomorphism, each of which has two indecomposable direct summands. The mutation graphs are shown in Figures 3 and 4.

n=4: Up to isomorphism there are twenty basic maximal rigid objects, each of which has three indecomposable direct summands. Figure 5 is the mutation graph for maximal rigid objects. In the graph, there are four pentagons, each of which corresponds to an integer $a = 1, 2, 3, 4$: the vertices of the pentagon corresponding to a are maximal rigid objects in the wing of $(a, 3)$. In Figure 6 we only give the mutation graph of the quivers (the potentials are uniquely determined by the quivers), where the wavy lines mean that there is a reordering of the vertices besides the mutation. [1]

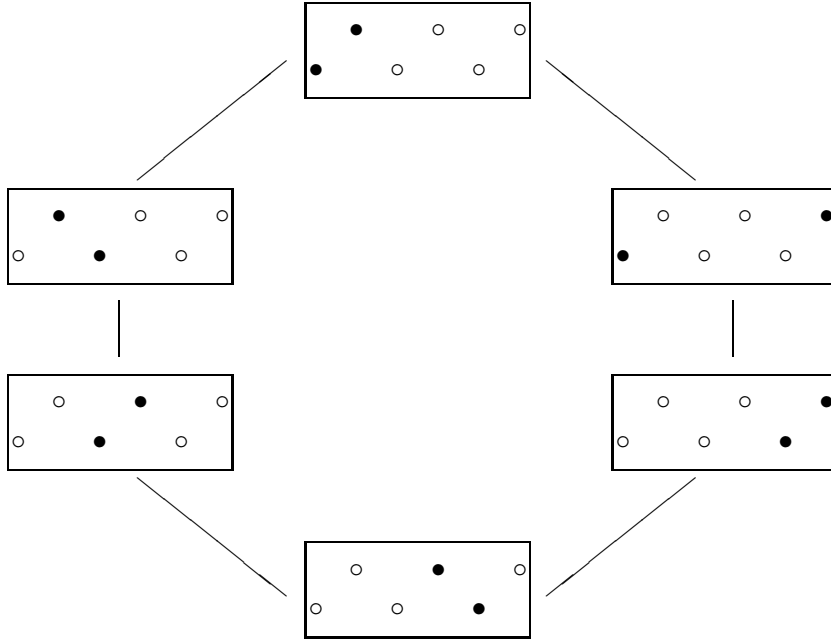[1] Many thanks to Laurent Demonet for much help in drawing these two graphs.

FIGURE 3. The mutation graph of $\mathcal{C}_3$ – maximal rigid objects
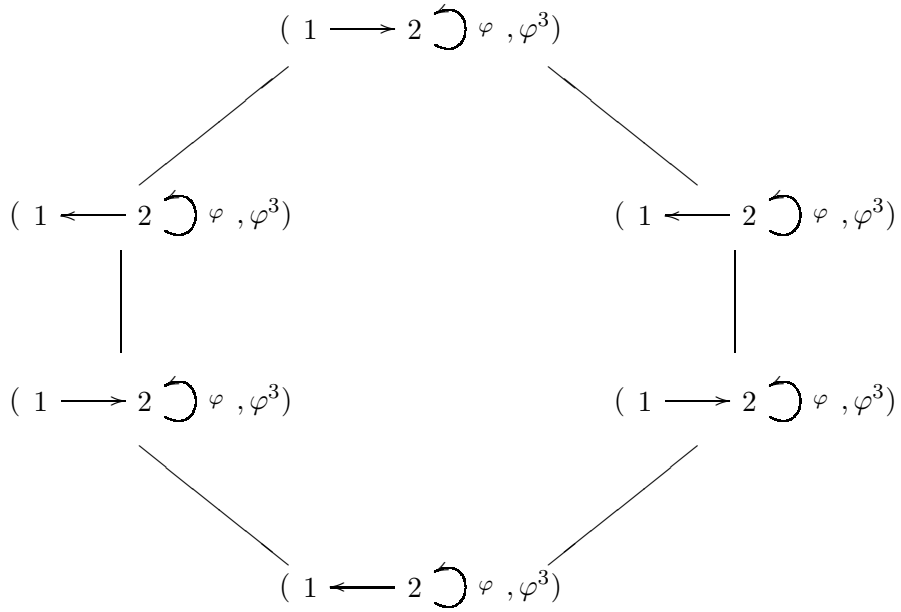

FIGURE 4. The mutation graph of $\mathcal{C}_3$ – quivers with potential

Generally we have a quiver with potential version of Proposition 5.2.

Proposition 6.4. *Assume $n \geq 3$. Let T and T' be two maximal rigid objects of $\mathcal{C}_n$ related by a mutation. Let $(\tilde{Q}, \tilde{W})$ and $(\tilde{Q}', \tilde{W}')$ be the quivers with potential associated to T and T' respectively. Assume that T is in the wing of $(a, n-1)$.*

  a) *If the mutation is simple, then $(\tilde{Q}, \tilde{W})$ and $(\tilde{Q}', \tilde{W}')$ are related by a Derksen–Weyman–Zelevinsky mutation.*
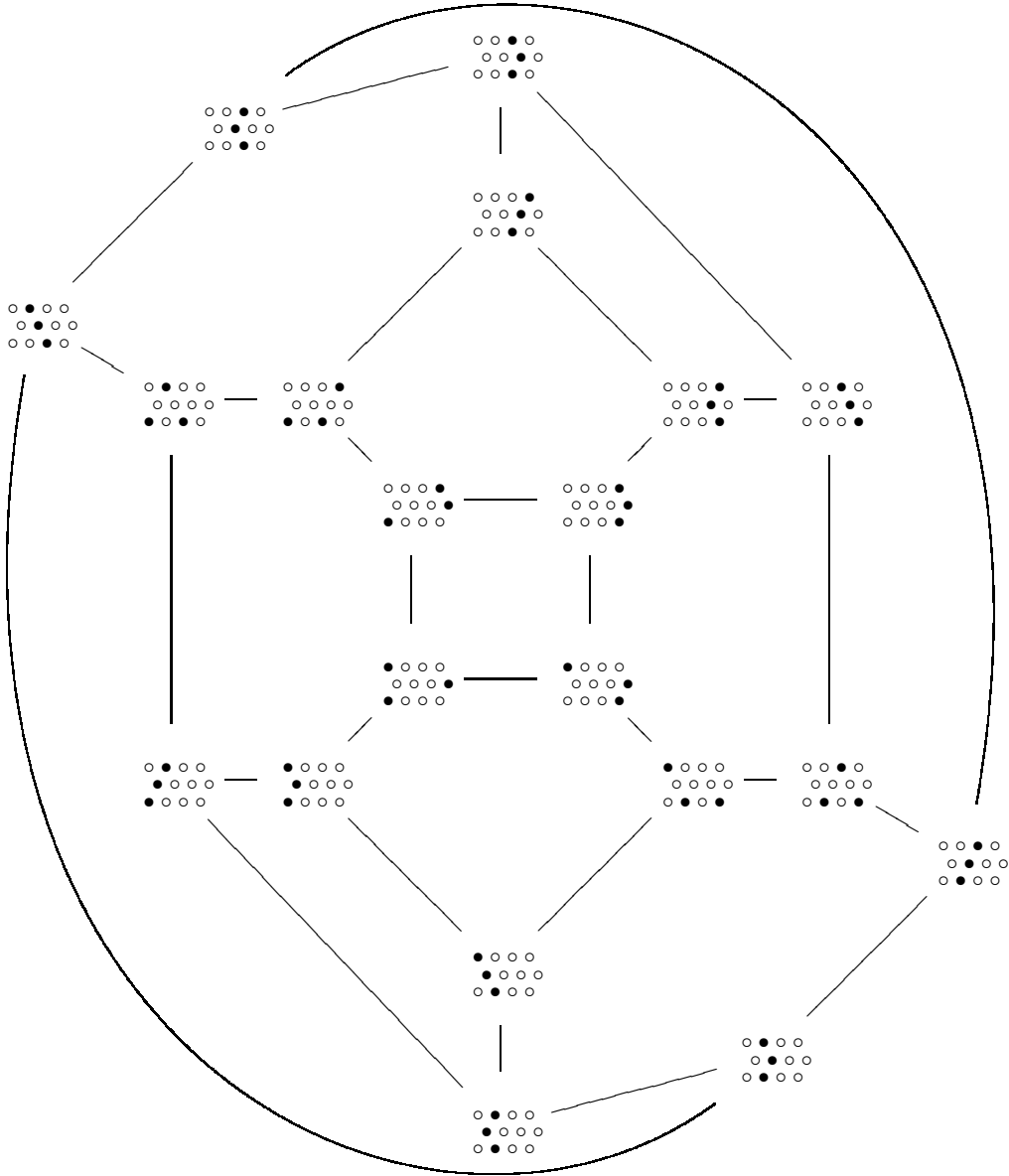
FIGURE 5. The mutation graph of $\mathcal{C}_4$ – maximal rigid objects

b) *If the mutation is not simple, there are two cases*

    1) *if the vertex c has one neighbour, then $\tilde{Q}'$ is obtained from $\tilde{Q}$ by reversing the unique arrow adjacent to c and $\tilde{W}' = \tilde{W}$;*

    2) *if the vertex c has two neighbours, then $\tilde{Q}'$ can be obtained from $\tilde{Q}$ by reversing all arrows in the unique 3-cycle C traversing c and $\tilde{W}'$ is obtained from $\tilde{W}$ by replacing C by the new 3-cycle.*

Proof. a) Suppose that T and T' are related by a simple mutation. Then by Lemma 3.2 they are in the wing of the same $(a, n-1)$ for some $a = 1, \ldots, n$. It follows from Lemma 3.1 that T
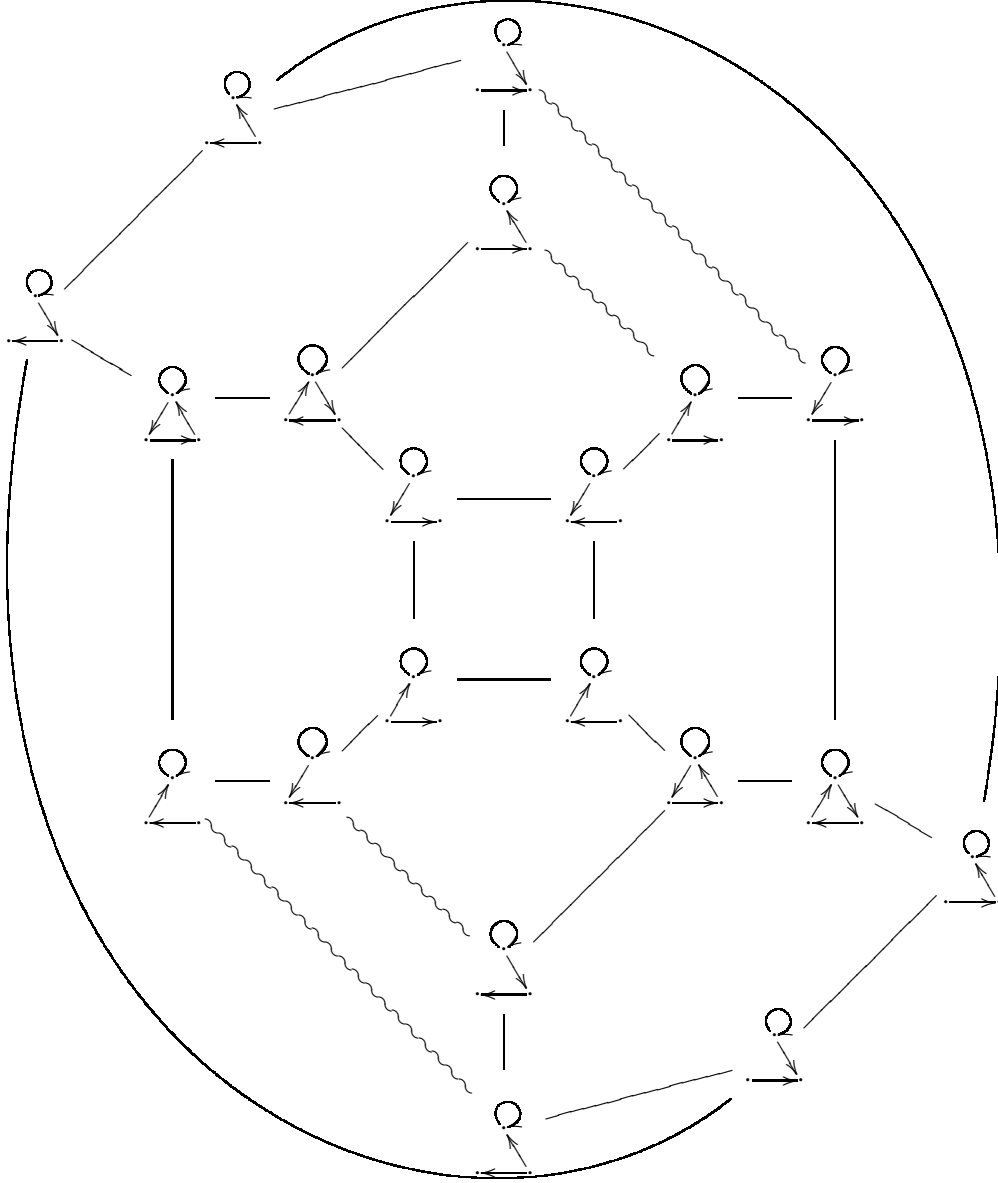
FIGURE 6. The mutation graph of $\mathcal{C}_4$ – quivers

and T', viewed as tilting modules in $\mathsf{rep}\, \overrightarrow{A}_{n-1}$, are related by a mutation. It follows that the quivers with potential (Q, W) and (Q', W') associated with the corresponding cluster-tilted algebras are related by a Derksen–Weyman–Zelevinsky mutation, *cf.* [3, Theorem 5.1], which is performed at a vertex $i \neq c$. By Lemma 5.3, we have

$$(\tilde{Q}', \tilde{W}') = \gamma_c(Q', W') = \gamma_c \mu_i(Q, W) = \mu_i \gamma_c(Q, W) = \mu_i(\tilde{Q}, \tilde{W}),$$

finishing the proof of a).

b) The assertion follows from Proposition 5.2 b) and Corollary 6.2. $\sqrt{}$

As in the quiver case, the quiver with potential $(\tilde{Q}', \tilde{W}')$ in Proposition 6.4 only depends on the quiver with potential $(\tilde{Q}, \tilde{W})$ and the vertex i at which the mutation is taken, and does not depend on the choice of the maximal rigid object T. We write $(\tilde{Q}', \tilde{W}') = \mu_i(\tilde{Q}, \tilde{W})$.

We reformulate Proposition 5.4 and Theorem 5.5 in terms of quivers with potential.

Theorem 6.5. *Let $(\tilde{Q}, \tilde{W})$ and $(\tilde{Q}', \tilde{W}')$ be two quivers with potential in $\widetilde{\mathcal{QP}}_{n-1}$. Then their Jacobian algebras are derived equivalent if and only if the quivers $\tilde{Q}$ and $\tilde{Q}'$ have the same number of 3-cycles. In particular, for a vertex i of $\tilde{Q}$, the Jacobian algebras of $(\tilde{Q}, \tilde{W})$ and $\mu_i(\tilde{Q}, \tilde{W})$ are derived equivalent if and only if the mutation does not change the number of 3-cycles of the quiver.*

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