Abstract commensurators of solvable Baumslag – Solitar groups

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Abstract

We prove that for any $n \geqslant 2$, the abstract commensurator group of the Baumslag – Solitar group $\mathrm{BS}(1,n)$ is isomorphic to the subgroup $\{\begin{pmatrix} 1 & q \\ 0 & p \end{pmatrix} \mid q \in \mathbb{Q}, p \in \mathbb{Q}^*\}$ of $\mathrm{GL}_2(\mathbb{Q})$.

We also prove that for any finitely generated group G with the unique root property the natural homomorphisms $\mathrm{Aut}(G) \to \mathrm{Comm}(G) \to \mathrm{QI}(G)$ are embeddings.

1 Introduction

For a group G, we denote by $\mathrm{Aut}(G)$ its automorphism group, by $\mathrm{Comm}(G)$ its abstract commensurator group, and by $\mathrm{QI}(G)$ its quasi-isometry group; see Definitions 2.1 and 5.1. For a finitely generated G, there are natural homomorphisms

$$\mathrm{Aut}(G) \to \mathrm{Comm}(G) \to \mathrm{QI}(G),$$

which became embeddings if G has the unique root property, i.e. if

$$\forall x, y \in G \ \forall n \in \mathbb{N} \ (x^n = y^n \Rightarrow x = y);$$

see Sections 2 and 5.

We are interested in computing of abstract commensurator groups of (solvable) Baumslag – Solitar groups. The Baumslag – Solitar groups $\mathrm{BS}(m,n)$, $1 \leqslant m \leqslant n$, are given by the presentation $\langle a, b \mid a^{-1}b^m a = b^n \rangle$. These groups have served as a proving ground for many new ideas in combinatorial and geometric group theory (see, for instance, [2, 5, 6]). The only solvable groups in this class are groups $\mathrm{BS}(1,n)$; the groups $\mathrm{BS}(m,n)$ with $1 < m \leqslant n$ contain a free nonabelian group.

The automorphism groups of $\mathrm{BS}(m,n)$ were described by Collins in [4]. It follows that the automorphism groups of $\mathrm{BS}(1,n)$ and $\mathrm{BS}(1,k)$ with $n,k \geqslant 1$ are isomorphic if and only if n and k have the same sets of prime divisors.

In [5], Farb and Mosher proved for $n \geqslant 2$ that $\mathrm{QI}(\mathrm{BS}(1,n)) \cong \mathrm{Bilip}(\mathbb{R}) \times \mathrm{Bilip}(\mathbb{Q}_n)$, where $\mathbb{Q}_n$ is the metric space of n-adic rationals with the usual metric and $\mathrm{Bilip}(Y)$ denotes the group of bilipschitz homeomorphisms of a metric space Y.
Moreover, they proved that $\mathrm{BS}(1,n)$ and $\mathrm{BS}(1,k)$ with $n,k \geqslant 1$ are quasi-isometric if and only if these groups are commensurable, that happens if and only if n and k have common powers. In [6], Whyte proved that groups $\mathrm{BS}(m,n)$ with $1 < m < n$ are quasi-isometric.

In this paper we compute the abstract commensurator groups of $\mathrm{BS}(1,n)$. It turns out that the abstract commensurator groups of all groups $\mathrm{BS}(1,n)$, $n \geqslant 2$, are isomorphic.

Main Theorem. (Theorem 4.5) *For every $n \geqslant 2$, $\mathrm{Comm}(\mathrm{BS}(1,n))$ is isomorphic to the subgroup $\{\begin{pmatrix} 1 & q \\ 0 & p \end{pmatrix} \mid q \in \mathbb{Q},\, p \in \mathbb{Q}^*\}$ of $\mathrm{GL}_2(\mathbb{Q})$.*

Note that $\mathrm{BS}(1,1) \cong \mathbb{Z}^2$, and it is well known that $\mathrm{Comm}(\mathbb{Z}^m) \cong \mathrm{GL}_m(\mathbb{Q})$ for $m \geqslant 1$.

2 General facts on commensurators

Definition 2.1 Let G be a group. Consider the set $\Omega(G)$ of all isomorphisms between subgroups of finite index of G. Two such isomorphisms $\varphi_1 : H_1 \to H_1'$ and $\varphi_2 : H_2 \to H_2'$ are called *equivalent*, written $\varphi_1 \sim \varphi_2$, if there exists a subgroup H of finite index in G such that both φ_1 and φ_2 are defined on H and $\varphi_1|_H = \varphi_2|_H$.

For any two isomorphisms $\alpha : G_1 \to G_1'$ and $\beta : G_2 \to G_2'$ in $\Omega(G)$, we define their product $\alpha\beta : \alpha^{-1}(G_1' \cap G_2) \to \beta(G_1' \cap G_2)$ in $\Omega(G)$. The factor-set $\Omega(G)/\!\sim$ inherits the multiplication $[\alpha][\beta] = [\alpha\beta]$ and is a group, called the *abstract commensurator group* of G and denoted $\mathrm{Comm}(G)$.

Definition 2.2 A group G has the *unique root property* if for any $x, y \in G$ and any positive integer n, the equality $x^n = y^n$ implies $x = y$.

For closeness, we reproduce here short proofs of the following two statements from [1].

Proposition 2.3 *Let G be a group with the unique root property. Then $\mathrm{Aut}(G)$ naturally embeds in $\mathrm{Comm}(G)$.*

Proof. There is a natural homomorphism $\mathrm{Aut}(G) \to \mathrm{Comm}(G)$. Suppose that some $\alpha \in \mathrm{Aut}(G)$ lies in its kernel. Then $\alpha|_H = \mathrm{id}$ for some subgroup H of finite index in G. If m is this index, then $g^{m!} \in H$ for every $g \in G$. Then $\alpha(g^{m!}) = g^{m!}$. Extracting roots, we get $\alpha(g) = g$, that is $\alpha = \mathrm{id}$. □

Lemma 2.4 *Let G be a group with the unique root property. Let $\varphi_1 : H_1 \to H_1'$ and $\varphi_2 : H_2 \to H_2'$ be two isomorphisms between subgroups of finite index in G. Suppose that $[\varphi_1] = [\varphi_2]$ in $\mathrm{Comm}(G)$. Then $\varphi_1|_{H_1 \cap H_2} = \varphi_2|_{H_1 \cap H_2}$.* □

Proof. The equality $[\varphi_1] = [\varphi_2]$ means that there exists a subgroup H of finite index in G such that both φ_1 and φ_2 are defined on H and $\varphi_1|_H = \varphi_2|_H$. Clearly $H \leqslant H_1 \cap H_2$. Denote $m = |(H_1 \cap H_2) : H|$. Let h be an arbitrary element of $H_1 \cap H_2$. Then $h^{m!} \in H$ and so $\varphi_1(h^{m!}) = \varphi_2(h^{m!})$. Since G is a group with the unique root property, we get $\varphi_1(h) = \varphi_2(h)$. □

Lemma 2.5 *The group* BS$(1,n)$ *has the unique root property. In particular,* Aut(BS$(1,n)$) *naturally embeds in* Comm(BS$(1,n)$).

Proof. The first claim follows by using matrix calculations in view of Lemma 4.1. The second claim follows from Proposition 2.3. ☐

3 A structure of finite index subgroups of BS$(1,n)$

Let $\mathrm{BS}(1,n) = \langle a, b \,|\, a^{-1}ba = b^n\rangle$, where $n \geqslant 2$. Denote $b_j = a^{-j}ba^j$, $j \in \mathbb{Z}$. Then

$$b_j^n = b_{j+1}, \quad a^{-1}b_j a = b_{j+1}, \quad b_i b_j = b_j b_i \quad (i, j \in \mathbb{Z}).$$

Consider the homomorphism

$$\begin{aligned} \psi:\ \ \mathrm{BS}(1,n) &\to \mathbb{Z} \\ a &\mapsto 1 \\ b &\mapsto 0. \end{aligned}$$

Lemma 3.1 1) *We have* $\mathrm{BS}(1,n) = U \rtimes V$, *where* $U = \ker\psi = \langle b_j \,|\, j \in \mathbb{Z}\rangle$, $V = \langle a\rangle$, *and* V *acts on* U *by the rule* $a^{-1}b_j a = b_{j+1}$.

2) *The subgroup* U *has the presentation* $\langle b_j \,|\, b_j^n = b_{j+1},\ j \in \mathbb{Z}\rangle$ *and so it can be identified with* $\mathbb{Z}[\frac{1}{n}]$.

3) $\mathrm{BS}(1,n) \cong \mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z}$, *where* $\mathbb{Z}$ *acts on* $\mathbb{Z}[\frac{1}{n}]$ *by multiplication by* n.

Proof. The first claim is obvious, the second follows by applying the Reidemeister – Schreier method, and the third claim follows from the first two. ☐

Lemma 3.2 *Every subgroup* H *of finite index in* BS$(1,n)$ *can be written as* $H = \langle a^k u, w\rangle$ *for some* $k > 0$, $u, w \in U$ *and* $w \neq 1$.

Proof. The subgroup H is finitely generated. Since the image of H under the epimorphism $\psi : \mathrm{BS}(1,n) \to \mathbb{Z}$ is generated by some $k > 0$, we can write $H = \langle a^k u, u_1, \dots, u_s\rangle$ for some $u, u_1, \dots, u_s \in U = \ker\psi$. Observe that every finitely generated subgroup of $U \cong \mathbb{Z}[\frac{1}{n}]$ is cyclic. So, $H = \langle a^k u, w\rangle$ for some $w \in U$. Clearly, $w \neq 1$, otherwise BS$(1,n)$ were virtually cyclic, that is impossible. ☐

Lemma 3.3 *Let* $H = \langle a^k b_q^r, b_p^s\rangle$ *with* $k > 0$. *Then* $H = \langle a^k b_q^r, b_i^s\rangle$ *for every* $i \in \mathbb{Z}$.

Proof. Since $(a^k b_q^r)^{-t} \cdot b_p^s \cdot (a^k b_q^r)^t = b_{p+tk}^s$ for every integer t, we have

$$H = \langle a^k b_q^r, b_{p+tk}^s\rangle = \langle a^k b_q^r, b_{p+(t+1)k}^s\rangle.$$

Given $i \in \mathbb{Z}$, we choose t such that $p + tk \leqslant i < p + (t+1)k$. Then $H = \langle a^k b_q^r, b_i^s\rangle$, since b_i is a power of b_{p+tk} and $b_{p+(t+1)k}$ is a power of b_i. ☐

Proposition 3.4 *Every subgroup* H *of finite index in* BS$(1,n)$ *can be written as* $H = \langle a^k b^l, b^m\rangle$ *for some integer* k, l, m, *where* $k, m > 0$ *and* $(m, n) = 1$. *The index of this subgroup is* km.

Proof. By Lemma 3.2, $H = \langle a^k b_q^r, b_p^s \rangle$ for some $k, s > 0$ and $r, q, p \in \mathbb{Z}$. Set $m = s/(n,s)$. Clearly, $(m,n) = 1$. We claim that $H = \langle a^k b_q^r, b_p^m \rangle$. Indeed, b_p^s is a power of b_p^m. On the other hand, $(a^k b_q^r) \cdot (b_p^s)^{\frac{n^k}{(n,s)}} \cdot (a^k b_q^r)^{-1} = a^k \cdot b_p^{mn^k} \cdot a^{-k} = b_p^m$.

By Lemma 3.3, $H = \langle a^k b_q^r, b^m \rangle$. We show that $H = \langle a^k b^l, b^m \rangle$ for some l. If $q \geqslant 0$, then $b_q = b^{n^q}$ and we can take $l = rn^q$. Let $q < 0$. Since $(m,n) = 1$, there exists an integer t, such that $mt \equiv r \mod (n^{-q})$. Denote $l = (r - mt)/n^{-q}$. Then, again with the help of Lemma 3.3, we have

$$H = \langle a^k b_q^r, b_q^m \rangle = \langle a^k b_q^{r-mt}, b_q^m \rangle = \langle a^k b_q^{ln^{-q}}, b_q^m \rangle = \langle a^k b^l, b^m \rangle.$$

To prove the last claim, one have to check, that $\{a^i b^j \mid 0 \leqslant i < k, 0 \leqslant j < m\}$ is the set of representatives of the left cosets of H in $\mathrm{BS}(1,n)$. We leave this to the reader. □

Proposition 3.5 *Let $H = \langle a^k b^l, b^m \rangle$ be a subgroup of $\mathrm{BS}(1,n)$ with $k, m > 0$ and $(n,m) = 1$. Then H has the presentation $\langle x, y \mid x^{-1} y x = y^{n^k} \rangle$ with generators $x = a^k b^l$, $y = b^m$.*

Proof. Consider the homomorphism $\psi : \mathrm{BS}(1,n) \to \mathbb{Z}$ introduced above. We have $\psi(x) = k$ and $H \cap \ker \psi = \langle x^{-i} y x^i \mid i \in \mathbb{Z} \rangle$. Thus, we have $H = \langle x^{-i} y x^i \mid i \in \mathbb{Z} \rangle \rtimes \langle x \rangle$.

Using the isomorphism $\mathrm{BS}(1,n) \cong \mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z}$ from Lemma 3.1, we can write $H \cong \mathbb{Z}[\frac{m}{n^k}] \rtimes k\mathbb{Z} \cong \mathbb{Z}[\frac{1}{n^k}] \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{Z}[\frac{1}{n^k}]$ by multiplication by n^k. By Claim 3) of Lemma 3.1 we have $H \cong \mathrm{BS}(1, n^k)$. □

Proposition 3.6 *Let $H_1 = \langle a^{k_1} b^{l_1}, b^{m_1} \rangle$ and $H_2 = \langle a^{k_2} b^{l_2}, b^{m_2} \rangle$ be two subgroups of $\mathrm{BS}(1,n)$ with $k_1, k_2, m_1, m_2 > 0$ and $(n, m_1) = (n, m_2) = 1$. Then H_1 is isomorphic to H_2 if and only if $k_1 = k_2$.*

Proof. If $k_1 = k_2$, then $H_1 \cong H_2$ by Proposition 3.5. This proposition also implies, that $H_i/[H_i, H_i] \cong \mathbb{Z} \times \mathbb{Z}_{n^{k_i}-1}$. So, if $k_1 \neq k_2$, then $H_1 \ncong H_2$. □

4 The proof of the Main Theorem

Let $\mathcal{G}$ be the subgroup of $\mathrm{GL}_2(\mathbb{Q})$, consisting of the matrices $A = \begin{pmatrix} 1 & A_{12} \\ 0 & A_{22} \end{pmatrix}$ with $A_{12} \in \mathbb{Q}$ and $A_{22} \in \mathbb{Q}^*$. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ denote the diagonal and the unipotent subgroups of $\mathcal{G}$, i.e.

$$\mathcal{G}_1 = \{A \in \mathcal{G} \mid A_{12} = 0\}, \quad \mathcal{G}_2 = \{A \in \mathcal{G} \mid A_{22} = 1\}.$$

Clearly, $\mathcal{G} = \mathcal{G}_2 \rtimes \mathcal{G}_1$.

For any natural n, let $\mathcal{H}_n$ be the subgroup of $\mathcal{G}$ consisting of the matrices A with $A_{12} \in \mathbb{Z}[\frac{1}{n}]$ and $A_{22} \in \{n^i \mid i \in \mathbb{Z}\}$.

Lemma 4.1 *For any natural $n \geqslant 2$, the map $a \mapsto A = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$, $b \mapsto B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ can be extended to an isomorphism $\theta : \mathrm{BS}(1,n) \to \mathcal{H}_n$.*

Proof. The proof is easy; see Exercise 5.5 in Chapter 2 in [3]. □

We will use the following theorem of D. Collins.

Theorem 4.2 ([4, Proposition A]) *Let* $G = \langle a, b \mid a^{-1}ba = b^s \rangle$ *where* $|s| \neq 1$. *Let*

$$s = \delta p_1^{e_1} p_2^{e_2} \dots p_f^{e_f},$$

where $\delta = \pm 1$ *and* $p_1, p_2, \dots, p_f$ *are distinct primes. Then* $\mathrm{Aut}(G)$ *has presentation:*

$$\begin{aligned}&\langle C, Q_1, Q_2, \dots, Q_f, T \mid \\ &Q_i^{-1} C Q_i = C^{p_i},\ Q_i Q_j = Q_j Q_i, \\ &T^2 = 1,\ TQ_i = Q_i T,\ T^{-1} C T = C^{-1} \rangle,\end{aligned}$$

where $i, j = 1, 2, \dots, f$. *In this presentation the automorphisms are defined by*

$$Q_i : \begin{cases} a \mapsto a \\ b \mapsto b^{p_i}, \end{cases} \quad C : \begin{cases} a \mapsto ab \\ b \mapsto b, \end{cases} \quad T : \begin{cases} a \mapsto a \\ b \mapsto b^{-1}. \end{cases}$$

Proposition 4.3 *Let* $n \geqslant 2$ *be a natural number. We identify* $\mathrm{BS}(1, n)$ *with* $\mathcal{H}_n$ *through the isomorphism described in Lemma 4.1. Let* H_1, H_2 *be two isomorphic subgroups of* $\mathrm{BS}(1, n)$, *both of finite index. Then for every isomorphism* $\varphi : H_1 \to H_2$, *there exists a unique matrix* $M = M(\varphi) \in \mathcal{G}$ *such that* $M^{-1} x M = \varphi(x)$ *for every* $x \in H_1$.

Proof. First we prove the existence of $M(\varphi)$. By Propositions 3.4 and 3.6, we can write $H_1 = \langle a^k b^{l_1}, b^{m_1} \rangle$ and $H_2 = \langle a^k b^{l_2}, b^{m_2} \rangle$ for some integer l_1, l_2, and $k, m_1, m_2 > 0$, where $(n, m_1) = (n, m_2) = 1$. By Proposition 3.5, H_j has the presentation $\langle x_j, y_j \mid x_j^{-1} y_j x_j = y_j^{n^k} \rangle$, where $x_j = a^k b^{l_j}$, $y_j = b^{m_j}$, $j = 1, 2$. After identification of elements of $\mathrm{BS}(1, n)$ with matrices, we have

$$x_j = \begin{pmatrix} 1 & l_j \\ 0 & n^k \end{pmatrix}, \quad y_j = \begin{pmatrix} 1 & m_j \\ 0 & 1 \end{pmatrix}. \tag{1}$$

Let $\varphi_0 : H_1 \to H_2$ be the isomorphism, such that $\varphi_0(x_1) = x_2$ and $\varphi_0(y_1) = y_2$. Then $\varphi = \varphi_1 \varphi_0$ for some $\varphi_1 \in \mathrm{Aut}(H_1)$. By Theorem 4.2, $\mathrm{Aut}(H_1)$ is generated by the automorphisms

$$\alpha_i : \begin{cases} x_1 \mapsto x_1 \\ y_1 \mapsto y_1^{p_i}, \end{cases} \quad \beta : \begin{cases} x_1 \mapsto x_1 y_1 \\ y_1 \mapsto y_1, \end{cases} \quad \gamma : \begin{cases} x_1 \mapsto x_1 \\ y_1 \mapsto y_1^{-1}, \end{cases}$$

$i = 1, 2, \dots, f$, where $p_1, p_2, \dots, p_f$ are all prime numbers dividing n. Thus, it is sufficient to show the existence of the matrices $M(\varphi_0)$, $M(\beta)$, $M(\gamma)$, and $M(\alpha_i)$, $i = 1, 2, \dots, f$.

First we prove the existence of $M(\varphi_0)$. We shall find $M(\varphi_0) \in \mathcal{G}$, such that

$$\begin{aligned} x_1 \cdot M(\varphi_0) &= M(\varphi_0) \cdot \varphi_0(x_1), \\ y_1 \cdot M(\varphi_0) &= M(\varphi_0) \cdot \varphi_0(y_1). \end{aligned}$$

Using (1), one can compute that

$$M(\varphi_0) = \begin{pmatrix} 1 & \frac{l_1 m_2 - l_2 m_1}{m_1 (n^k - 1)} \\ 0 & \frac{m_2}{m_1} \end{pmatrix}. \tag{2}$$

Similarly, we get

$$M(\alpha_i) = \begin{pmatrix} 1 & \frac{l_1(p_i-1)}{n^k-1} \\ 0 & p_i \end{pmatrix}, \quad M(\beta) = \begin{pmatrix} 1 & \frac{-m_1}{n^k-1} \\ 0 & 1 \end{pmatrix}, \quad M(\gamma) = \begin{pmatrix} 1 & \frac{-2l_1}{n^k-1} \\ 0 & -1 \end{pmatrix}. \tag{3}$$

The uniqueness of M follows from the triviality of the centralizer of H_1 in $\mathcal{G}$; the later is easy to check. $\square$

Lemma 4.4 1) *Let $\varphi : H \to H'$ be an isomorphism between subgroups of finite index in* $\mathrm{BS}(1, n)$ *and let K be a subgroup of finite index in H. Then $M(\varphi|_K) = M(\varphi)$.*

2) *Let $\varphi_1 : H_1 \to H_1'$ and $\varphi_2 : H_2 \to H_2'$ be two isomorphisms between subgroups of finite index in* $\mathrm{BS}(1, n)$. *Suppose that $[\varphi_1] = [\varphi_2]$ in* $\mathrm{Comm}(\mathrm{BS}(1, n))$. *Then $M(\varphi_1) = M(\varphi_2)$.*

Proof. 1) For every $x \in K$ we have $M(\varphi|_K)^{-1}xM(\varphi|_K) = \varphi|_K(x) = \varphi(x) = M(\varphi)^{-1}xM(\varphi)$ and the claim follows from the uniqueness of M.

2) By Lemmas 2.4 and 2.5, we have $\varphi_1|_{H_1 \cap H_2} = \varphi_2|_{H_1\cap H_2}$. Claim 1) implies that $M(\varphi_1) = M(\varphi_1|_{H_1\cap H_2}) = M(\varphi_2|_{H_1\cap H_2}) = M(\varphi_2)$. $\square$

This enables to define M on commensurator classes: $M([\varphi]) := M(\varphi)$.

Theorem 4.5 *For every natural $n \geqslant 2$, the map $\Psi : \mathrm{Comm}(\mathrm{BS}(1, n)) \to \mathcal{G}$ given by $[\varphi] \mapsto M([\varphi])$ is an isomorphism.*

Proof. 1) First we prove that Ψ is a homomorphism. Let $\varphi_1 : H_1 \to H_2$, $\varphi_2 : H_3 \to H_4$ be two isomorphisms between subgroups of finite index in $\mathrm{BS}(1, n)$. We shall show that $M([\varphi_1])M([\varphi_2]) = M([\varphi_1\varphi_2])$. Write $\varphi_1\varphi_2 = \sigma\tau$, where σ is the restriction of φ_1 to $\varphi_1^{-1}(H_2 \cap H_3)$ and τ is the restriction of φ_2 to $H_2 \cap H_3$:

$$\varphi_1^{-1}(H_2 \cap H_3) \xrightarrow{\sigma} (H_2 \cap H_3) \xrightarrow{\tau} \varphi_2(H_2 \cap H_3).$$

For $x \in \varphi_1^{-1}(H_2 \cap H_3)$ we have $(\varphi_1\varphi_2)(x) = \tau((\sigma(x))) = M(\tau)^{-1}M(\sigma)^{-1}xM(\sigma)M(\tau)$. Hence, $M(\varphi_1\varphi_2) = M(\sigma)M(\tau) = M(\varphi_1)M(\varphi_2)$ and the claim follows.

2) The injectivity of Ψ trivially follows from the definition of $M([\varphi])$.

3) Now we prove that Ψ is a surjection. By specializing parameters in (2) and (3), we will obtain some matrices in $\mathrm{im}\Psi$. Taking $l_1 = m_2$ and $l_2 = m_1$ in $M(\varphi_0)$, we get the matrix

$$D(\tfrac{m_1}{m_2}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{m_1}{m_2} \end{pmatrix}$$

with $m_1, m_2 > 0$, $(m_1, n) = (m_2, n) = 1$. Taking $l_1 = 0$ in $M(\alpha_i)$ and in $M(\gamma)$, and taking $m_1 = 1$ in $M(\beta)$, we get the matrices

$$D(p_i) = \begin{pmatrix} 1 & 0 \\ 0 & p_i, \end{pmatrix}, \quad D(-1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T(k) = \begin{pmatrix} 1 & \frac{1}{n^k-1} \\ 0 & 1 \end{pmatrix}, \ k > 0.$$

The matrices $D(\frac{m_1}{m_2})$, $D(p_i)$ and $D(-1)$ generate the subgroup $\mathcal{G}_1$ in the image of Ψ.

So, it suffices to show that $\mathcal{G}_2$ is contained in $\mathrm{im}\Psi$. Since the additive group of $\mathbb{Q}$ is generated by $\mathbb{Z}[\frac{1}{n}]$ and all numbers $\frac{1}{s}$ with $(s, n) = 1$, it suffices to show that the matrices

$\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$ with $q \in \mathbb{Z}[\frac{1}{n}]$ and the matrices $\begin{pmatrix} 1 & \frac{1}{s} \\ 0 & 1 \end{pmatrix}$ with (s,n)=1 are contained in the image of Ψ. The first follows from the fact that the group of the commensurator classes of inner automorphisms of $\mathrm{BS}(1,n)$ is mapped, under Ψ, onto $\mathcal{H}_n$. The second follows from the formula $\begin{pmatrix} 1 & \frac{1}{s} \\ 0 & 1 \end{pmatrix} = (T(\phi(s)))^t$, where ϕ is the Euler function and t is the natural number such that $n^{\phi(s)} - 1 = st$. □

5 Appendix: Commensurators and quasi-isometries

Let X and Y be two metric spaces. A map $f : X \to Y$ is called a (coarse) *quasi-isometry* between X and Y if there are some constants $K, C, C_0 > 0$, such that the following holds:

1. $K^{-1}d_X(x_1, x_2) - C \leqslant d_Y(f(x_1), f(x_2)) \leqslant Kd_X(x_1, x_2) + C$ for all $x_1, x_2 \in X$.
2. The C_0-neighborhood of $f(X)$ coincides with Y.

There is always a coarse inverse of f, a quasi-isometry $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are a bounded distance from the identity maps in the sup norm; these bounds, and the quasi-isometry constants for g, depend only on the quasi-isometry constants of f.

Definition 5.1 Let X be a metric space. Two quasi-isometries f and g from X to itself are considered equivalent if there exists a number $M > 0$ such that $d(f(x), g(x)) \leqslant M$ for all $x \in X$. Let $\mathrm{QI}(X)$ be the set of equivalence classes of quasi-isometries from X to itself. Composition of quasi-isometries gives a well-defined group structure on $\mathrm{QI}(X)$. The group $\mathrm{QI}(X)$ is called the *quasi-isometry group* of X.

Let G be a group with a finite generating set S. For $g \in G$ denote by $|g|$ the minimal k, such that $g = s_1s_2 \dots s_k$, where $s_1, s_2, \dots, s_k \in S \cup S^{-1}$. We consider G as a metric space with the word metric with respect to S: $d(x, y) = |x^{-1}y|$ for $x, y \in G$. For a finitely generated group G, the group $\mathrm{QI}(G)$ is well defined and does not depend on a choice of a finite generating set S.

It is well known that there is a natural homomorphism $\Lambda : \mathrm{Comm}(G) \to \mathrm{QI}(G)$. This homomorphism is defined by the following rule. Let $\varphi : H \to H'$ be an isomorphism between two finite index subgroups of G. We choose a right transversal T for H in G with $1 \in T$. First we define a map $f_\varphi : G \to G$ by the rule $f_\varphi(ht) := \varphi(h)$ for every $h \in H$ and $t \in T$. Clearly, f_φ is a quasi-isometry. Then we set $\Lambda([\varphi]) := [f_\varphi]$.

Proposition 5.2 *Let G be a finitely generated group with the unique root property. Then $\Lambda : \mathrm{Comm}(G) \to \mathrm{QI}(G)$ is an embedding.*

Proof. We will use notation introduced before this lemma. Suppose that $[f_\varphi] = [\mathrm{id}_{|G}]$. Then there is a constant $M > 0$, such that $d(f_\varphi(x), x) \leqslant M$ for every $x \in G$. Let $h \in H$. Then for every integer n holds: $|h^{-n}\varphi(h^n)| = d(\varphi(h^n), h^n) \leqslant M$. Since G is finitely generated, the M-ball in G centered at 1 is finite. Hence, there exist distinct n, m such that $h^{-n}\varphi(h^n) = h^{-m}\varphi(h^m)$. Then $h^{n-m} = (\varphi(h))^{n-m}$ and so $h = \varphi(h)$ by the unique root property. Hence $[\varphi] = 1$ and the injectivity of Λ is proved. □

Corollary 5.3 *The group* $\mathrm{Comm}(\mathrm{BS}(1,n))$ *naturally embeds in* $\mathrm{QI}(\mathrm{BS}(1,n))$.

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